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Analysis and Geometric Singularities

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May 6th – May 12th, 2012

Abstract. This is a report on the Oberwolfach conference “Analysis and Geometric Singularities”, May 6-12, 2012. The main themes discussed in the meeting include the interplay between the analytic, geometric and topological study of singular spaces and noncompact spaces highly regular structure at infinity and asymptotic analysis of global spectral invariants. More specific topics include index theory on such spaces, various nonlinear geometric problems, and in particular the asymptotic structure of natural metrics on geometric moduli spaces, and techniques from linear analysis to approach such problems.

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Introduction by the Organisers

The Conference “Analysis and Geometric Singularities” took place at Oberwolfach from May 6 to May 12, 2012. The general idea of this and past conferences in this series has been to bring together a disparate collection of researchers with some common interests, roughly centered around the theme of geometric analysis on singular spaces. Each one of these meetings has included a more specific theme, a field the organizers regard as ripe for further development and meriting special attention by this community. The theme this year was the study of special metrics on geometric moduli spaces. This fit comfortably within the broader topics of interest, which included spectral invariants on singular and noncompact spaces, new directions in index theory, scattering theory and related topics.

There is very strong interest in this collection of topics from researchers in a variety of disciplines, and the organizers were not able to include many who had expressed interest and had been placed on a waiting list. Particularly heartening was the participation of quite a few very talented and enthusiastic young mathematicians, whose presence testified to the vitality of this field.

Each day of the conference was loosely organized around a different theme, centered around a somewhat longer survey talk, which was the first talk of the morning, prearranged well before the meeting. Wednesday morning was reserved for several shorter talks by the younger participants. The 25 talks that took place during the week allowed participants to hear about some of the latest advances by the senior researchers as well as the research projects of the younger ones, but still left plenty of time for informal interaction. All indications are that new collaborations were formed, old collaborations were reinforced, and everyone got a chance to learn about some promising new directions.

Monday's survey talk, by Werner Müller, dealt with the asymptotic behaviour of analytic torsion on locally symmetric spaces, with respect to certain sequences of representations. This was related to the later talk by Jean-Michel Bismut, who studied the asymptotic behaviour of analytic torsion on increasingly high powers of a line bundle and the relationship of this to the theory of Toeplitz operators. Matthias Lesch reported on his new gluing theorem for analytic torsion on spaces with conic singularities. Tuesday's survey talk by Roger Bielawski, presented the current state of knowledge concerning the monopole moduli spaces, their compactifications, and the asymptotics of the natural hyperKähler metric on it. This led naturally to Sergey Cherkis' description of his far-reaching generalization in the theory of bow moduli spaces. Other talks with a distinctly differential geometric theme included Bernd Ammann's talk on the behaviour of the Yamabe invariant under surgery, Sergiu Moroianu's report on a joint Cauchy problem for Einstein metrics and parallel spinors and Gilles Carron's work on rigidity phenomena associated to metrics satisfying curvature pinching measured by integral inequalities. Also similarly themed were the talks by Frederic Rochon on a regularity theorem for Kähler-Einstein metrics and Spyros Alexakis' lecture on bubbling phenomena for a Willmore energy functional. Some of these lectures were also related to Thursday's survey talk, by Colin Guillarmou, describing the renormalized volume functional on Poincaré - Einstein spaces, with particular attention to the three-dimensional hyperbolic case which has many close connections with Teichmüller theory. Robin Graham's talk gave a new conceptual approach to Juhl's remarkable explicit formulas for Q-curvature in higher dimensions. Maxim Braverman's gave a talk on regularized cohomology of non-compact asymptotically Kähler G-manifolds. The final survey talk, on Friday morning, was given by Thomas Schick, who recalled the subject of coarse index theory and its many geometric applications. Bunke's presentation of differential algebraic K-theory outlined what seems to be a remarkably inclusive general framework for studying many index theorems. The talks by Albin and Banagl gave very different viewpoints, Albin's being analytic and Banagl's very topological, on the subject of signature theorems on stratified spaces that do not satisfy the Witt hypothesis. Mendoza brought the audience back to some classical singular elliptic problems. Xianzhe Dai closed this very successful meeting with a talk on the Bergman kernel on orbifolds. Mostly on Wednesday morning, but also interspersed through the week, the junior mathematicians Gell-Redman, Lapp, Kottke, Rowlett, Vertman and Waterstraat gave shorter talks on their current research.

The current running through all these talks is the fascinating way with which topology, analysis and geometry intertwine in this class of problems and how deep techniques in each of these areas are now being used to make significant progress in other areas.

Workshop: Analysis and Geometric Singularities

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Abstracts

Analytic torsion of locally symmetric spaces

Werner Müller
(joint work with Jonathan Pfaff)

The purpose of this talk is to discuss some recent results concerning the asymptotic behavior of the analytic torsion of compact locally symmetric spaces $\Gamma\backslash G/K$ with respect to sequences of representations of Γ which are obtained by restriction of irreducible representations of G to Γ. This is an extension of the work on which I reported on the last meeting of this conference. For all details we refer to [5]. The results discussed here are expected to have applications to the cohomology of arithmetic groups as in [6].

To begin with let me recall the definition of the analytic torsion. Let X be a compact Riemannian manifold of dimension n and let $\rho\colon \pi_1(X) \to \mathrm{GL}(V_\rho)$ be a finite-dimensional complex representation of the fundamental group $\pi_1(X)$ of X. Let $E_\rho \to X$ be the associated flat vector bundle. Choose a Hermitian metric in E_ρ. Let $\Delta_p(\rho)$ be the Laplace operator on the space of E_ρ-valued p-forms. Let $\zeta_p(s;\rho)$, $\mathrm{Re}(s) > n/2$, be the zeta function of $\Delta_p(\rho)$. It is well known that it admits a meromorphic extension to $\mathbb{C}$ which is regular at $s = 0$. Then the analytic torsion $T_X(\rho) \in \mathbb{R}^+$ is defined as

$$\log T_X(\rho) = \frac{1}{2}\sum_{p=0}^{n}(-1)^p p \frac{d}{ds}\zeta_p(s;\rho)\big|_{s=0}.$$

We note that if $\dim X$ is odd and $H^*(X, E_\rho) = 0$, then $T_X(\rho)$ is independent of the metrics on M and E_ρ.

Now let $\tilde{X} = G/K$ be a global Riemannian symmetric space of non-positive curvature. Thus G is a real connected semisimple Lie group with finite center and of non-compact type, and K is a maximal compact subgroup of G. Let $\Gamma \subset G$ be a co-compact lattice in G, i.e., Γ is a discrete subgroup of G such that $\Gamma\backslash G$ is compact. For simplicity we assume that Γ is torsion free. Then Γ acts properly discontinuously and fixpoint free on $\tilde{X}$. The quotient $X = \Gamma\backslash\tilde{X}$ is a compact locally symmetric manifold. Let $\tau\colon G \to \mathrm{GL}(V)$ be a finite-dimensional complex representation of G. Let E_τ be the flat vector bundle associated to $\rho := \tau|_\Gamma$. It is isomorphic to the locally homogeneous vector bundle attached $\tau|_K$ and can be equipped with a canonical Hermitian fibre metric. Let $T_X(\tau)$ denote the analytic torsion with respect to these choices of metrics on X and E_τ, respectively.

Let $\mathfrak{g}$ be the Lie algebra of G. Let $G_\mathbb{C}$ be the simply connected Lie group with Lie algebra $\mathfrak{g} \otimes \mathbb{C}$. We assume that $G \subset G_\mathbb{C}$. Let $\mathfrak{h} \subset \mathfrak{g}$ be a fundamental Cartan subalgebra. Let $\theta\colon G \to G$ denote the Cartan involution. Let τ be an irreducible representation of G with highest weight $\lambda \in \mathfrak{h}^*_\mathbb{C}$. Denote by λ_θ the highest weight of the representation $\tau \circ \theta$. Finally, let $\delta(\tilde{X}) = \mathrm{rank}_\mathbb{C}(G) - \mathrm{rank}_\mathbb{C}(K)$. Then our main result is the following theorem.

Theorem 1. (i) *Let $\widetilde{X}$ be even dimensional or let $\delta(\widetilde{X}) \neq 1$. Then $T_X(\tau) = 1$ for all finite-dimensional representations τ of G.*
(ii)*Let $\widetilde{X}$ be odd-dimensional with $\delta(\widetilde{X}) = 1$. Let $\lambda \in \mathfrak{h}_{\mathbb{C}}^*$ be a highest weight with $\lambda_\theta \neq \lambda$. For $m \in \mathbb{N}$ let $\tau_\lambda(m)$ be the irreducible representation of G with highest weight $m\lambda$. There exist constants $c > 0$ and $C_{\widetilde{X}} \neq 0$, which depends on $\widetilde{X}$, and a polynomial $P_\lambda(m)$, which depends on λ, such that*

$$\log T_X(\tau_\lambda(m)) = C_{\widetilde{X}} \operatorname{vol}(X) \cdot P_\lambda(m) + O\left(e^{-cm}\right) \tag{0.1}$$

as $m \to \infty$. Furthermore, there is a constant $C_\lambda > 0$ such that

$$P_\lambda(m) = C_\lambda \cdot m \dim(\tau_\lambda(m)) + R_\lambda(m), \tag{0.2}$$

where $R_\lambda(m)$ is a polynomial whose degree equals the degree of the polynomial $\dim(\tau_\lambda(m))$.

Note that (0.1) provides a complete asymptotic expansion for $\log T_X(\tau_\lambda(m))$. If one is only interested in the leading term, one can use (0.2) which implies that there exists a constant $C = C(\widetilde{X}, \lambda) \neq 0$, which depends on $\widetilde{X}$ and λ, such that

$$\log T_X(\tau_\lambda(m)) = C \operatorname{vol}(X) \cdot m \dim(\tau_\lambda(m)) + O\left(\dim(\tau_\lambda(m))\right) \tag{0.3}$$

as $m \to \infty$. Now the coefficient of the highest power can be determined by Weyl's dimension formula.

For hyperbolic manifolds, we proved the vanishing result (i) of Theorem 1 in [3, Proposition 1.7]. In general it was first proved by Bismut, Ma, and Zhang [1]. It extends a result of Moscovici and Stanton [2] who showed that $T_X(\rho) = 1$, if $\delta(\widetilde{X}) \geq 2$ and ρ is a unitary representation of Γ. Our proof is different from the previous proofs and, as we believe, also simpler. It does not rely on the use of orbital integrals or the Fourier inversion formula.

Part (ii) is a consequence of the following two propositions. The first one shows that the asymptotic behavior of the analytic torsion with respect to the representations $\tau_\lambda(m)$ is determined by the asymptotic behavior of the L^2-torsion.

Proposition 2. *Let $\widetilde{X}$ be odd-dimensional with $\delta(\widetilde{X}) = 1$. Let $\lambda \in \mathfrak{h}_{\mathbb{C}}^*$ be a highest weight. Assume that $\lambda_\theta \neq \lambda$. For $m \in \mathbb{N}$ let $\tau_\lambda(m)$ be the irreducible representation of G with highest weight $m\lambda$. Then there exists $c > 0$ such that*

$$\log T_X(\tau_\lambda(m)) = \log T_X^{(2)}(\tau_\lambda(m)) + O\left(e^{-cm}\right) \tag{0.4}$$

for all $m \in \mathbb{N}$.

This result was first proved in [3] for hyperbolic manifolds. It was also proved in [1] in the more general context of this paper (see Remark 7.8). Our method of proof of Proposition 2 follows the method developed in [3].

The key result on which part (ii) of Theorem 1 relies is the computation of the L^2-torsion. The computation is based on the Plancherel formula. It gives

Proposition 3. *Let the assumptions be as in Proposition 2. There exists a constant $C_{\widetilde{X}}$, which depends on $\widetilde{X}$, and a polynomial $P_\lambda(m)$, which depends on λ, such that*

$$\log T_X^{(2)}(\tau_\lambda(m)) = C_{\widetilde{X}} \operatorname{vol}(X) \cdot P_\lambda(m), \quad m \in \mathbb{N}. \tag{0.5}$$

Moreover there is a constant $C_\lambda > 0$ *such that*

$$P_\lambda(m) = C_\lambda \cdot m \cdot \dim(\tau_\lambda(m)) + O\left(\dim(\tau_\lambda(m))\right) \tag{0.6}$$

as $m \to \infty$.

Finally, we note that if one specializes the main result of [1], Theorem 1.1, to the case of analytic torsion of a locally symmetric space, one can also determine the leading term of the asymptotic expansion of (0.3). This has been carried out in [1] in the case of hyperbolic 3-manifolds.

If we consider one of the odd-dimensional irreducible symmetric spaces $\widetilde{X}$ with $\delta(\widetilde{X}) = 1$ and choose λ to be a fundamental weight, the statements can be made more explicit.

Let $\widetilde{X} = \mathrm{Spin}(p,q)/(\mathrm{Spin}(p) \times \mathrm{Spin}(q))$, p, q odd, and $\widetilde{X} = G/K$. Let $n := (p+q-2)/2$. There are two fundamental weight $\omega^{\pm}_{f,n}$ which are not invariant under θ. One has $\omega^{-}_{f,n} = (\omega^{+}_{f,n})_\theta$. It suffices to consider the weight $\omega^{+}_{f,n}$. For $m \in \mathbb{N}$ let $\tau(m)$ be the representation with highest weight $m\omega^{+}_{f,n}$. By Weyl's dimension formula there exists a constant $C > 0$ such that

$$\dim(\tau(m)) = Cm^{\frac{n(n+1)}{2}} + O\left(m^{\frac{n(n+1)}{2}-1}\right) \tag{0.7}$$

as $m \to \infty$. Let $\widetilde{X}_d$ be the compact dual of $\widetilde{X}$. Let

$$C_{p,q} = \frac{(-1)^{\frac{pq-1}{2}} 2\pi}{\mathrm{vol}(\widetilde{X}_d)} \binom{n}{\frac{p-1}{2}}. \tag{0.8}$$

Corollary 4. *Let* $\widetilde{X} = \mathrm{Spin}(p,q)/(\mathrm{Spin}(p) \times \mathrm{Spin}(q))$, p, q *odd, and* $X = \Gamma\backslash\widetilde{X}$. *With respect to the above notation we have*

$$\log T_X(\tau(m)) = C_{p,q} \,\mathrm{vol}(X) \cdot m \dim(\tau(m)) + O\left(m^{\frac{n(n+1)}{2}}\right)$$

as $m \to \infty$.

The case $q = 1$ was treated in [3] and the case $p = 3$, $q = 1$ in [4]. In the latter case we have $\mathrm{Spin}(3,1) \cong \mathrm{SL}(2,\mathbb{C})$. The irreducible representation of $\mathrm{Spin}(3,1)$ with highest weight $\frac{1}{2}(m,m)$ corresponds to the m-th symmetric power of the standard representation $\mathrm{SL}(2,\mathbb{C})$ on $\mathbb{C}^2$ and we have

$$-\log T_X(\tau(m)) = \frac{1}{4\pi} \,\mathrm{vol}(X) m^2 + O(m).$$

The remaining case is $\widetilde{X} = \mathrm{SL}(3,\mathbb{R})/\,\mathrm{SO}(3)$.

References

[1] J.-M. Bismut, X. Ma, W. Zhang, *Asymptotic torsion and Toeplitz operators*, Preprint, 2011.

[2] H. Moscovici, R. Stanton, *R-torsion and zeta functions for locally symmetric manifolds*, Inv. Math. **105** (1991), 185–216.

[3] W. Müller, J. Pfaff, *The asymptotics of the Ray-Singer analytic torsion for hyperbolic manifolds*, Intern. Math. Research Notices 2012, doi: 10.1093/imrn/rns130.

[4] W. Müller, *The asymptotics of the Ray-Singer analytic torsion of hyperbolic 3 manifolds*, in: "Metric and Differential Geometry", The Jeff Cheeger Anniversary Volume, Progress in Math. Vol. 297, pp. 317 - 352, Birkhäuser, 2012.

[5] W. Müller, J. Pfaff, *Analytic torsion and L^2-torsion of compact locally symmetric manifolds*, Preprint 2012, arXiv:1204.0659.
[6] S. Marshall, W. Müller, *On the torsion in the cohomology of arithmetic hyperbolic 3-manifolds*, Preprint 2011, arXiv:1103.2262.

The signature operator on Cheeger spaces

Pierre Albin
(joint work with Eric Leichtnam, Rafe Mazzeo, and Paolo Piazza)

The intersection homology theory of Mark Goresky and Robert MacPherson assigns to each perversity function $\overline{p}$ and stratified pseudomanifold $\hat{X}$ a sequence of groups

$$IH_*^{\overline{p}}(\hat{X})$$

that are invariant under stratified homotopy equivalences. There is a perversity function $\overline{q}$, dual to $\overline{p}$, and a non-degenerate intersection product

$$IH_*^{\overline{p}}(\hat{X}) \times IH_*^{\overline{q}}(\hat{X}) \longrightarrow \mathbb{Q}$$

that generalizes Poincaré-duality of smooth spaces. For a class of spaces, known as Witt spaces, the two 'closest' dual perversity functions $\overline{m}$, $\overline{n}$ give isomorphic intersection homology groups

$$IH_*^{\overline{m}}(\hat{X}) \cong IH_*^{\overline{n}}(\hat{X})$$

which then satisfy Poincaré duality.

Jeff Cheeger gave an analytic approach to these groups. Indeed, for an 'iterated incomplete edge' or *iie* metric on the regular part X of $\hat{X}$, there are two natural de Rham complexes: one involving the minimal domain of the exterior derivative,

$$\mathcal{D}_{\min}(d) = \{\omega \in L^2(X; \Lambda^* T^* X) : \\ \exists \eta \in L^2(X; \Lambda^* T^* X), (\omega_n) \subseteq \mathcal{C}_c^\infty(X; \Lambda^* T^* X) \text{ s. t. } \omega_n \to \omega, d\omega_n \to \eta\},$$

and one involving the maximal domain of the exterior derivative,

$$\mathcal{D}_{\max}(d) = \{\omega \in L^2(X; \Lambda^* T^* X) : d\omega \in L^2(X; \Lambda^* T^* X)\}.$$

Both of these complexes have finite dimensional cohomology and Cheeger showed that the latter is the dual of $IH_*^{\overline{m}}(\hat{X})$ while the former is the dual of $IH_*^{\overline{n}}(\hat{X})$. On a Witt space the exterior derivative has a unique closed extension from smooth compactly supported forms, so these domains coincide.

The purpose of this talk is to report on work in progress extending the results of Cheeger to non-Witt stratified pseudomanifolds. If a space $\hat{X}$ is not Witt then there is a stratum Y with even-dimensional link Z such that

$$IH_*^{\overline{n}}(Z) \neq \{0\}.$$

At every such stratum we choose a splitting

$$(IH_*^{\overline{n}}(Z))^* = W(Z) \oplus W(Z)'$$

that is orthogonal with respect to an *iie*-metric and parallel with respect to a natural flat connection. If there are two such strata, and their closures intersect, then we demand that the splittings be compatible at the intersection. We call a compatible collection of splittings a *flat system* or *mezzoperversity.* For spaces with isolated conic singularities, Cheeger showed that a flat system defined a natural domain for the de Rham operator and that Stokes' theorem holds for forms in this domain. We generalize this result to stratified pseudomanifolds.

Theorem (A.-Leichtnam-Mazzeo-Piazza) *Let $(\hat{X}, g)$ be a stratified pseudomanifold with a (suitably scaled) iie-metric. To each flat system $\mathcal{L}$ on $\hat{X}$ there is a naturally associated domain of the exterior derivative and of the de Rham operator*

$$\mathcal{D}_{\mathcal{L}}(d), \quad \mathcal{D}_{\mathcal{L}}(d+\delta).$$

The latter makes $d+\delta$ a self-adjoint operator with compact resolvent and induces a strong Kodaira decomposition on differential forms, the former induces a Fredholm complex whose cohomology

$$H^*_{\mathcal{L}}(\hat{X}) = H^*(d, \mathcal{D}_{\mathcal{L}}(d))$$

satisfies a Hodge theorem, is invariant under stratified homotopy equivalences, and is independent of the choice of iie-metric.

Every flat system has a dual flat system so that the corresponding cohomologies are in duality. We call a flat system that coincides with its dual a *Lagrangian structure.* Note that there are topological obstructions (e.g., the signatures of the links) to the existence of a Lagrangian structure on a stratified pseudomanifold. We call those spaces that carry Lagrangian structures *Cheeger spaces.*

Theorem (A.-Leichtnam-Mazzeo-Piazza) *Let $(\hat{X}, g)$ be a Cheeger space with a (suitably scaled) iie-metric. If $\mathcal{L}$ is a Lagrangian structure on $\hat{X}$ then the domains*

$$\mathcal{D}_{\mathcal{L}}(d), \quad \mathcal{D}_{\mathcal{L}}(d+\delta)$$

are invariant under the action of the Hodge star. Thus the cohomology

$$H^*_{\mathcal{L}}(\hat{X}) = H^*(d, \mathcal{D}_{\mathcal{L}}(d))$$

satisfies Poincaré duality, and its signature is the index of the de Rham operator $d+\delta$ with the involution induced by the Hodge star and the domain induced by $\mathcal{L}$.

Note that there is a topological approach to Lagrangian structures and Poincaré duality due to Markus Banagl. In work-in-progress with Banagl we are investigating the relationship between our analytic definition of Lagrangian structures and his topological definition.

In further work-in-progress, we establish the Novikov conjecture for Cheeger spaces whose fundamental groups satisfy the strong Novikov conjecture. We also show that the signature of a Cheeger space is independent of the choice of Lagrangian structure, following a proof by Markus Banagl of the corresponding topological statement.

A gluing formula for the analytic torsion on singular spaces

Matthias Lesch

The Cheeger–Müller Theorem [4, 8, 9] on the equality of the analytic and combinatorial torsion is one of the cornerstones of modern global analysis. To extend the theorem to certain singular manifolds is an intriguing open challenge.

The purpose of this project is to provide a framework for attacking the problem. The main technical achievement is an analytic proof of a gluing formula for the analytic torsion in the context of singular manifolds.

Let M be a riemannian manifold (boundaryless but not necessarily compact, also the *interior* of a manifold with boundary is allowed) and let P^0 be an elliptic differential operator acting on the sections $\Gamma^\infty(E)$ of the hermitian vector bundle E. Moreover, we assume P^0 to be bounded below; fix a bounded below self–adjoint extension $P \geq -C > -\infty$ in $L^2(M, E)$.

e^{-tP} is an integral operator with a smooth kernel $k_t(x, y)$ which on the diagonal has a pointwise asymptotic expansion $k_t(x, x) \sim_{t\searrow 0} \sum_{j=0}^\infty a_j(x)\, t^{\frac{j-\dim M}{\operatorname{ord} P}}$. This asymptotic expansion is *uniform on compact subsets of* M and hence may (only) be integrated over such subsets. It is therefore a fundamental problem to give criteria which ensure that e^{-tP} is of trace class and such that there is an asymptotic expansion

$$\operatorname{Tr}\bigl(e^{-tP}\bigr) \sim_{t\searrow 0} \sum_{\substack{\operatorname{Re}\alpha\to\infty\\ 0\leq k\leq k(\alpha)}} a_{\alpha k}\, t^\alpha \log^k t. \tag{0.1}$$

Operators with this property will, after Connes and Moscovici [5], be addressed as having *discrete dimension spectrum.*

A rather generic description of a singular manifold can be given as follows: suppose that there is a compact manifold $M_1 \subset M$ and a "well understood" model manifold U such that $M = U \cup_{\partial M_1} M_1$. Typical examples for U which are reasonably well understood (or geometrically significant) are cylinders, cones, cusps, or edges.

Without becoming too technical suppose that for $P_U = P \restriction U$ and $P_1 = P \restriction M_1$ (of course suitable extensions have to be chosen for P_U and P_1) we have proved expansions Eq. (0.1). Then in terms of a suitable cut-off function φ which is 1 in a neighborhood of M_1 one expects to hold:

Principle 1 (Duhamel's principle for heat asymptotics; informal version)**.** *If P_U and P_1 are discrete with trace-class heat kernels then so is P and*

$$\operatorname{Tr}\bigl(e^{-tP}\bigr) = \operatorname{Tr}\bigl(\varphi e^{-tP_1}\bigr) + \operatorname{Tr}\bigl((1-\varphi)e^{-tP_U}\bigr) + O(t^N), \quad \text{as } t \to 0+ \tag{0.2}$$

for all N.

Thus although the heat kernel is a *global operator* its short time asymptotic expansion behaves in a sense local. Principle 1 is a folklore theorem which appears in various versions in the literature. In [6, Cor. 3.7] we prove a fairly general rigorous version of it.

Once the asymptotic expansion Eq. (0.1) is established, the meromorphic continuation of the ζ–function $\zeta(P;s) := \sum_{\mathfrak{gl}\in \operatorname{spec}(P)\setminus\{0\}} \mathfrak{gl}^{-s}$ follows via a Mellin transform argument.

Let us specialize to the de Rham complex. So suppose that we have chosen an ideal boundary condition (essentially this means that we have chosen closed extensions for the exterior derivative) $(\mathcal{D}, D)$ for the de Rham complex such that the corresponding extensions $\Delta_j = D_j^* D_j + D_{j-1} D_{j-1}^*$ of the Laplace operators satisfy Eq. (0.1). Then we can form the analytic torsion of $(\mathcal{D}, D)$

$$\log T(\mathcal{D}, D) := \frac{1}{2} \sum_{j\geq 0} (-1)^j j \frac{d}{ds}\Big|_{s=0} \zeta(\Delta_j; s). \tag{0.3}$$

In terms of the decomposition $M = U \cup_{\partial M_1} M_1$ the problem of proving a CM type Theorem for the singular manifold M decomposes into the following steps.

(1) Prove that the analytic torsion exists for the model manifold U.
(2) Compare the analytic torsion with a suitable combinatorial torsion for U.
(3) Prove a gluing formula for the analytic and combinatorial torsion and apply the known Cheeger–Müller Theorem for the manifold with boundary M_1.

A gluing formula for the combinatorial torsion is more or less an algebraic fact due to Milnor [7, Thm. 3.1/3.2]; cf. [6, Appendix]. The following Theorem which follows from our gluing formula solves (3) under a product structure assumption:

Theorem 2. *Let M be a singular manifold $M = U \cup_{\partial M_1} M_1$ as described above and assume that near ∂M_1 all structures are product. Then for establishing a Cheeger-Müller Theorem for M it suffices to prove it for the model space U of the singularity.*

The Theorem basically says that, under product assumptions, one gets step (3) for free. Otherwise the specific form of U is completely irrelevant. We conjecture that the product assumption in Theorem 2 can be dispensed with.

The Theorem is less obvious than it sounds since torsion invariants are global in nature. However, we will show here that under minimal technical assumptions the analytic torsion satisfies a gluing formula.

To explain the gluing formula let X be a riemannian manifold (not necessarily compact or complete!). Furthermore, let (F, ∇) be a flat bundle with a (not necessarily flat) hermitian metric h^F. We assume furthermore, that X contains a compact separating hypersurface $Y \subset X$ such that in a collar neighborhood $W = (-c, c) \times Y$ all structures are product. In other words X is obtained by gluing two manifolds with boundary $X^{\pm}$ along their common boundary Y where all structures are product near Y.

We make the fundamental assumption that

(0.4) we are given ideal boundary conditions $(\mathcal{D}^{\pm}, D^{\pm})$ of the twisted de Rham complexes $(\Omega^\bullet(X^{\circ,\pm}; F), d)$ which have discrete dimension spectrum over $U^{\pm} := X^{\pm} \setminus W$. We put $X^{\mathrm{cut}} := X^- \coprod X^+$.

Using Duhamel's principle one then shows that we therefore have the following Hilbert complexes with discrete dimension spectrum: $\mathcal{D}^\bullet(X^{\pm}; F)$ (absolute boundary condition at Y), $\mathcal{D}^\bullet(X^{\pm}, Y; F)$ (relative boundary condition at Y), $\mathcal{D}(X; F)$ (continuous transmission condition at Y).

By construction we have the following exact sequences of Hilbert complexes

$$0 \longrightarrow \mathcal{D}^\bullet(X^-, Y; F) \xhookrightarrow{\alpha_-} \mathcal{D}^\bullet(X; F) \xrightarrow{\beta} \mathcal{D}^\bullet(X^+; F) \longrightarrow 0, \tag{0.5}$$

$$0 \longrightarrow \mathcal{D}^\bullet(X^{\pm}, Y; F) \xhookrightarrow{\gamma_\pm} \mathcal{D}^\bullet(X^{\pm}; F) \xrightarrow{i_\pm^*} \mathcal{D}^\bullet(Y; F) \longrightarrow 0. \tag{0.6}$$

Here $\alpha_\pm$ are extension by 0, β is pullback (i.e. restriction) to X^+, $\gamma_\pm$ is the natural inclusion of the complex $\mathcal{D}^\bullet(X^{\pm}, Y; F)$ with relative boundary condition at Y into the complex $\mathcal{D}^\bullet(X^{\pm}; F)$ with absolute boundary condition, and $i_\pm : Y \hookrightarrow X^{\pm}$ is the inclusion map.

Each of the complexes (0.5), (0.6) induces a long exact sequence in cohomology. We abbreviate these long exact cohomology sequences by $\mathcal{H}((X^-, Y), X, X^+; F)$, $\mathcal{H}((X^{\pm}, Y), X^{\pm}, Y; F)$, resp.

The Euler characteristics of the complexes in Eq. (0.5), (0.6) are denoted by $\chi(X^{\pm}, Y; F), \chi(X^{\pm}; F), \chi(X; F), \chi(Y; F)$ etc.

The following Theorem is our main result. It essentially says that Milnor's algebraic gluing formula holds for the exact sequences Eq. (0.5), (0.6) of *infinite–dimensional* Hilbert complexes.

Theorem 3. *For the analytic torsions of the Hilbert complexes $\mathcal{D}^\bullet(X^{\pm}, Y; F)$, $\mathcal{D}^\bullet(X^{\pm}; F)$, $\mathcal{D}^\bullet(X; F)$ we have the following formulas:*

$$\begin{aligned} \log T(\mathcal{D}^\bullet(X; F)) &= \log T(\mathcal{D}^\bullet(X^-, Y; F)) + \log T(\mathcal{D}^\bullet(X^+; F)) \\ &\quad + \log \tau\big(\mathcal{H}((X^-, Y), X, X^+; F)\big) - \frac{1}{2}\log 2 \cdot \chi(Y; F), \end{aligned} \tag{0.7}$$

$$\begin{aligned} \log T(\mathcal{D}^\bullet(X^-; F)) &= \log T(\mathcal{D}^\bullet(X^-, Y; F)) + \log T(\mathcal{D}^\bullet(Y; F)) \\ &\quad + \log \tau\big(\mathcal{H}((X^-, Y), X^-, Y; F)\big), \end{aligned} \tag{0.8}$$

The blueprint for our proof is a technique of moving boundary conditions due to Vishik [10] who applied it to prove the Cheeger-Müller Theorem for compact manifolds with smooth boundary. We emphasize, however, that the technical part of the present paper [6] is completely independent of (and in our slightly biased view simpler than) [10]. Also we work with the de Rham complex coupled to an arbitrary flat bundle F.

We note here that in the context of *closed* manifolds gluing formulas for the analytic torsion have been proved in [10], [1], and recently [2]. In contrast our method applies to a wide class of singular manifolds. The details of the material presented here have been published in [6].

References

[1] D. Burghelea, L. Friedlander, and T. Kappeler, *Torsions for manifolds with boundary and glueing formulas*, Math. Nachr. **208** (1999), 31–91. arXiv:9510010v1 [dg-ga], MR 1719799 (2000m:58053)

[2] J. Brüning and X. Ma, *On the gluing formula for the analytic torsion*, Preprint, downloaded from http://www.math.jussieu.fr/~ma/publi.html Nov 16, 2011.

[3] ———, *An anomaly formula for Ray-Singer metrics on manifolds with boundary*, Geom. Funct. Anal. **16** (2006), no. 4, 767–837. MR 2255381 (2007i:58042)

[4] J. Cheeger, *Analytic torsion and the heat equation*, Ann. of Math. (2) **109** (1979), no. 2, 259–322. MR 528965 (80j:58065a)

[5] A. Connes and H. Moscovici, *The local index formula in noncommutative geometry*, Geom. Funct. Anal. **5** (1995), no. 2, 174–243. MR 1334867 (96e:58149)

[6] M. Lesch, *A gluing formula for the analytic torsion on singular spaces,* arXiv:1203.2793 [math.SP]

[7] J. Milnor, *Whitehead torsion*, Bull. Amer. Math. Soc. **72** (1966), 358–426. MR 0196736 (33 #4922)

[8] W. Müller, *Analytic torsion and R-torsion of Riemannian manifolds*, Adv. in Math. **28** (1978), no. 3, 233–305. MR 498252 (80j:58065b)

[9] ———, *Analytic torsion and R-torsion for unimodular representations*, J. Amer. Math. Soc. **6** (1993), no. 3, 721–753. MR 1189689 (93m:58119)

[10] S. M. Vishik, *Generalized Ray-Singer conjecture. I. A manifold with a smooth boundary*, Comm. Math. Phys. **167** (1995), no. 1, 1–102. arXiv:hep-th/9305184v1, MR 1316501 (96f:58184)

The Cauchy problem for Einstein metrics and for parallel spinors

Sergiu Moroianu

(joint work with Bernd Ammann, Andrei Moroianu)

We investigate under what conditions a Riemannian manifold (M, g) can be locally embedded as a hypersurface in an Einstein manifold $(\mathcal{Z}, gz)$ with prescribed second fundamental form. A related problem is extending a spinor from M to a parallel spinor on $\mathcal{Z}$.

Necessary conditions for the first problem are obtained from the Gauss and Codazzi equations. Assume that $(M, g) \subset (\mathcal{Z}, g^{\mathcal{Z}})$ is a hypersurface in an Einstein manifold satisfying $\mathrm{Ric}^{\mathcal{Z}} = \lambda g^{\mathcal{Z}}$. Denote by ν the unit normal vector field along M and by $W \in \mathrm{End}(TM)$ the Weingarten tensor defined by

$$\nabla^{\mathcal{Z}}_X \nu = -W(X), \qquad \forall\, X \in TM.$$

Then after some manipulations we obtain

$$d\mathrm{Tr}(W) + \delta^g W = 0, \tag{0.1}$$

$$\mathrm{Scal}^g + \mathrm{Tr}(W^2) - \mathrm{Tr}^2(W) = (n-1)\lambda. \tag{0.2}$$

By using the normal geodesic flow from M, we can identify a neighborhood of M in $\mathcal{Z}$ with a neighborhood of $M \times \{0\}$ in the cylinder $M \times \mathbb{R}$ with metric $dt^2 + g_t$, where g_t is a family of metrics on M such that $g_0 = g$. The time derivative of the Weingarten tensor satisfies

$$\dot{W}_t = -g_t^{-1}\mathrm{Ric}^{g_t} + W_t\mathrm{Tr}(W_t) - 2\lambda\mathrm{Id}. \tag{0.3}$$

If we now start with M, g, W satisfying the initial conditions (0.1), (0.2), we can prove that g_t exists for small time such that $g^{\mathcal{Z}} = dt^2 + g_t$ is Einstein, under the hypothesis that the initial data M, g, W are real-analytic. The proof consists in solving the evolution equation (0.3) using the Cauchy-Kowalewskaya theorem, and checking that (0.1), (0.2) remain valid for all time. Note that although every Riemannian Einstein manifold is real-analytic, hypersurfaces threin clearly are not forced to be real-analytic.

In Lorentzian setting, the above extension result was known since the work of Choquet-Bruhat [4], and notably the real-analyticity hypothesis is not necessary.

In our Riemannian setting, the initial conditions (0.1), (0.2) are satisfied for g of constant scalar curvature and W a constant multiple of the identity. The existence of an Einstein metric $g^{\mathcal{Z}}$ would imply that g itself was analytic, since it would be a constant mean-curvature hypersurface in an analytic Riemannian manifold. Thus the Cauchy problem in this particular case has the striking property of having a solution *if and only if* the initial data is real-analytic. This phenomenon appears starting in dimension 3 since in dimension 2 every metric is real-analytic.

Let now $\mathcal{Z}$ be a spin manifold with a non-zero parallel spinor Ψ. Then $\mathcal{Z}$ is necessarily Ricci-flat, and the restriction of Ψ to M must be a generalized Killing spinor, in the sense that its restriction $\Phi := \Psi|_M$ satisfies

$$\nabla^M_X\Phi = \tfrac{1}{2}W(X)\cdot\Phi, \qquad \forall\ X \in TM. \tag{0.4}$$

We prove that conversely, every generalized Killing spinor on M with respect to some symmetric endomorphism field W can be extended to a parallel spinor on a Ricci-flat manifold $\mathcal{Z}$, with Weingarten tensor W, provided the initial data M, g, Φ are real-analytic. In particular, we note that (0.4) implies the initial constraints (0.1), (0.2) with $\lambda = 0$. This result had been obtained in several special cases: if the stress-energy tensor W of Φ is the identity [2], if W is parallel [5] and if W is a Codazzi tensor [3].

These two results are contained in a joint paper with Bernd Amman and Andrei Moroianu [1]. In a separate paper joint with Andrei Moroianu [6] we examine, in the particular case of Riemannian surfaces (M, g), the existence problem of tensors W satisfying the constraints (0.1), (0.2) for $\lambda = 0$. This question, asked by Schläfli in 1875, is not solved near points where the Gaussian curvature of g vanishes. We specialize to trace-free W, hence g should be embedded as a minimal surface in $\mathbb{R}^3$. A necessary condition for this is

$$K\Delta^g K + g(dK, dK) + 4K^3 = 0, \tag{0.5}$$

where K is the Gaussian curvature, and $\Delta = \delta^g d$ is the scalar Laplacian. At points where $K < 0$, this is equivalent to the fact that $\sqrt{-K}g$ is a flat metric. Riemannian surfaces whose Gaussian curvature satisfies (0.5) are called *Ricci surfaces.* In 1895 Ricci has proved that every point x of a Ricci surface has a neighborhood which embeds isometrically in $\mathbb{R}^3$ as a minimal surface, provided that $K(x) < 0$. We prove this result in full generality by showing that a Ricci surface can be locally isometrically embedded either in $\mathbb{R}^3$ as a minimal surface or in the Lorentz space $\mathbb{R}^{2,1}$ as a maximal surface. In particular, the curvature of Ricci surfaces has only isolated zeros.

As a corollary we show that every hyperelliptic Riemann surface of odd genus admits a metric which is locally isometrically embeddable in $\mathbb{R}^3$ as a minimal surface.

References

[1] B. Ammann, A. Moroianu, S. Moroianu, *The Cauchy problems for Einstein metrics and parallel spinors*, arXiv:1106.2066.

[2] C. Bär, *Real Killing spinors and holonomy*, Commun. Math. Phys. **154** (1993), 509–521.

[3] C. Bär, P. Gauduchon, A. Moroianu, *Generalized Cylinders in Semi-Riemannian and Spin Geometry,* Math. Z. **249** (2005), 545–580.

[4] Y. Fourès-Bruhat, *Résolution du problème de Cauchy pour des équations hyperboliques du second ordre non linéaires,* Bull. Soc. Math. France **81** (1953), 225–288.

[5] B. Morel, *The energy-momentum tensor as a second fundamental form*,

[6] A. Moroianu, S. Moroianu, *Ricci surfaces*, arXiv:1206.1620

Lower bounds for the Yamabe invariant

Bernd Ammann

(joint work with Mattias Dahl, Emmanuel Humbert)

In this talk I gave an overview over a series of articles [1], [2], [3], [4], joint work with M. Dahl and E. Humbert, about the (smooth) Yamabe invariant. We have seen for example that the Yamabe invariant of a simply connected compact manifold of dimension 5 is between 45 and 79. Similar estimates hold for simply connected compact manifolds of dimension 6, and for 2-connected compact manifolds of dimension at least 7 for which the KO_*-valued index of the Dirac operator vanishes.

Let us give some more details. The conformal Yamabe constant of a compact n-dimensional Riemannian manifold (M, g_0) is defined as

$$Y(M,[g_0]) := \inf_{g\in[g_0]} \frac{\int_M \mathrm{scal}^g\, dv^g}{\mathrm{vol}(M,g)^{\frac{n-2}{n}}} = \inf_{u\geq 0, u\not\equiv 0} \frac{\int_M 4\frac{n-1}{n-2}|du|^2_{g_0} + \mathrm{scal}^{g_0}\, u^2\, dv^{g_0}}{\|u\|^2_{L^p(M,g_0)}}$$

with $p = 2n/(n-2)$. The second characterization also makes sense for non-compact M, which will be used below. The smooth Yamabe invariant of a compact manifold M is then defined as

$$\sigma(M) := \sup Y(M,[g_0])$$

where the supremum runs over all conformal classes $[g_0]$ on M.

The invariant $\sigma(M)$ is positive iff M carries a metric of positive scalar curvature. On the other hand one sees that $Y(M,[g_0])$ is bounded from above by $Y(\mathbb{S}^n) = n(n-1)\mathrm{vol}(\mathbb{S}^n)^{2/n}$, the value of the standard sphere, and thus the same upper bounds holds for $\sigma(M)$. One knows the value for $\sigma(M)$ for $M = \mathbb{R}P^3$, $M = \mathbb{C}P^2$, compact quotients of hyperbolic 3-space, and few other spaces. The proofs use Seiberg-Witten theory, the Penrose inequality from general relativity and the Ricci-flow. However, in general it is hard to calculate $\sigma(M)$. One even does not know any compact manifold M of dimension n at least 5 for which one can prove $0 < \sigma(M) < \sigma(S^n)$, although there are plenty of candidates for which such a value is expected.

In [1] we have proven a formula that estimates the behaviour of $\sigma(M)$ under performing surgery at M, namely if N is obtained by surgery of dimension $k \le n-3$ from M, then

$$\sigma(N) \ge \min\{\sigma(M), \Lambda_{n,k}\},$$

where $\Lambda_{n,k} > 0$ only depends on $n = \dim M$ and k.

If $k \le n-4$ or if $n \le 6$, then the constants $\Lambda_{n,k} > 0$ can be defined as

$$\Lambda_{n,k} = \inf_{c\in[0,1]} Y(\mathbb{M}_c), \qquad \mathbb{M}_c = H_c^{k+1} \times \mathbb{S}^{n-k-1}$$

where H_c^{k+1} is the simply connected complete Riemannian manifold with sectional curvature $-c^2$, thus it is a rescaled hyperbolic space or Euclidean space. The definition is a bit more involved in the remaining case $k+3 = n \ge 7$. Such surgery formulas immediately imply that $\min\{\sigma(M), \Lambda_{n,k}\}$ is a bordism invariant in a certain sense. In the case $k = 0$ the result holds for $\Lambda_{n,0} = Y(\mathbb{S}^n)$, i.e. $\sigma(N) \ge \sigma(M)$ in this case.

In order to get much information about $\sigma(M)$ there are several things to carry out: at first one has to obtain explicit lower estimates for $\Lambda_{n,k}$ for as many pairs (n,k) as possible. This is mainly a complicated analytical problem. As a second step one has to find lower bounds for the smooth Yamabe invariant or conformal Yamabe constant of certain building blocks, see below. Then as a third step one has use bordism theory to prove that a large class of manifolds is obtained from these building blocks by a finite number of surgeries of dimension k for which we know a lower bound for $\Lambda_{n,k}$.

In [2] we found an efficient method for a lower bound for $Y(M_1 \times M_2, [g_1+g_2])$ in terms of $Y(M_1,[g_1])$ and $(M_2,[g_2])$ provided that both factors (M_i, g_i) have dimensions $n_i \ge 3$ and have positive conformal Yamabe constant. Namely

$$Y(M_1 \times M_2, [g_1+g_2])^{n_1+n_2} \ge c_{n_1,n_2} Y(M_1,[g_1])^{n_1} Y(M_2,[g_2])^{n_2}$$

where c_{n_1,n_2} is a constant which is “close” to optimal in a sense that we will not explain here to be short. This result yields a strong explicit positive lower bound for $\Lambda_{n,k}$ if $2 \le k \le n-4$.

Another method was developed in [4] which provides a lower bound for $Y(\mathbb{M}_c)$ in terms of $Y(\mathbb{M}_0) = Y(\mathbb{R}^{k+1} \times \mathbb{S}^{n-k-1})$. It relies on the volume preserving diffeomorphism $H_c^{k+1} \to \mathbb{R}^{k+1}$ which has in polar coordinates the form $(r,\phi) \mapsto (f(r),\phi)$. This diffeomorphism allows us to use the Yamabe minimizer of $\mathbb{M}_c$ —

which exists due to [6] — as a test function defined on $\mathbb{R}^{k+1} \times \mathbb{S}^{n-k-1}$. As result we obtain an explicit positive lower bound for $\Lambda_{n,k}$ as soon as there is an explicit positive lower bound for $Y(\mathbb{R}^{k+1} \times \mathbb{S}^{n-k-1})$. Luckily such a lower bound was calculated in [7] for $(n, k) = (4, 1)$, and we are very grateful to Petean and Ruiz, that they adapted in [8] their method for $(n, k) \in \{(5, 1), (5, 2), (9, 1), (10, 1)\}$. Their method can probably be adapted to all dimensions, but these dimensions play an important role when we apply bordism theory to draw conclusions.

In the second step one should find lower bounds for the smooth Yamabe invariant of simple building blocks. Important buidling blocks are total spaces T of bundles whose fibers are quaternionic projective planes $\mathbb{H}P^2$. Assume that we have a sequence of metrics g_i on such a fixed T, such that all fibers are isometric for all basepoints and all i. We also assume that the bundle map $(T, g_i) \to (B, a_i h)$ is a Riemannian submersion, where h is a metric on the base B and where $a_i \in \mathbb{R}^+$ tend to ∞. This is often called an adiabatic limit. Then the conformal Yamabe constants $Y(T, [g_i])$ converge to $Y(\mathbb{H}P^2 \times \mathbb{R}^{n-8})$ and the latter quantity is positively bounded from below by Obata's theorem for $n = 8$, by further work of Petean in cases $n = 9$ and $n = 10$ and by the product formula explained above for $n \geq 11$. These fiber bundles are important as they generate the spin bordism classes of simply connected manifolds with vanishing index, see [9]. For simply connected oriented non-spin manifold, $\mathbb{C}P^2$-bundles play a similar role, see [5]. Another example of an important building block is SU(3)/SO(3) whose bi-invariant metric g_{bi} is Einstein, and thus Obata's theorem determines the value of $Y(\mathrm{SU}(3)/\mathrm{SO}(3), [g_{bi}]) > 0$, As a consequence Obata's theorem yields an explicit positive lower bound for $\sigma(\mathrm{SU}(3)/\mathrm{SO}(3))$. This manifold is important as it represents the only non-trivial class in the oriented bordism group in dimension 5. Combinded with further arguments, in particular results about possible decompositions of bordisms into elementary bordisms which correspond to surgeries, we finally obtain for instance that $\sigma(M) > 45$ for any simply connected manifold M of dimension 5. In current work we prove an analogous bound for simply-connected manifolds of dimension 6.

A similar statement holds for 2-connected manifolds with vanishing KO-index of dimension different from 4, see [4] for details, in particular Table 2 gives explicit numbers for dimension at most 11.

There are many other conclusions, for example:

- The set $\{\sigma(\Sigma) \,|\, \Sigma \text{ is a homotopy-}S^7\}$ has at most 4 values.
- For many fundamental groups Γ one can show that there is no strictly decreasing sequence in
$$\{\sigma(M) \,|\, \pi_1(M) = \Gamma, \dim M = n\} \cap [0, \min\{\Lambda_{n,1}, \ldots, \Lambda_{n,n-3}\}].$$
- Assume that M is a 5-dimensional manifold with $0 \leq \sigma(M) < 45$. Then for relatively prime natural numbers r_1 and r_2 we have $\sigma(\#^{r_1} M) = \sigma(M)$ or $\sigma(\#^{r_2} M) = \sigma(M)$.

The last application raises the question whether such an M exists, obviously it would be not simply-connected. As the explicit calculation of $\sigma(M)$ is very difficult and only possible in very rare cases, no such examples are known. It is conjectured that $M = S^5/(\mathbb{Z}/_m)$, $m \geq 5$, provides such an example, namely it is conjectured that the supremum in the definition of $\sigma(M)$ is attained by the round metric which would imply $\sigma(M) = \sigma(S^5)/m^{2/5} \approx 78.996/m^{2/5}$.

References

[1] B. Ammann, M. Dahl, and E. Humbert, *Smooth Yamabe invariant and surgery*, Preprint, 2008, http://arxiv.org/abs/0804.1418, to appear in J. Diff. Geom.

[2] ———, *The conformal Yamabe constant of product manifolds*, Preprint, 2011, http://arxiv.org/abs/1103.1826, to appear in Proc. AMS.

[3] ———, *Square-integrability of solutions of the Yamabe equation*, Preprint, 2011, http://arxiv.org/abs/1111.2780.

[4] ———, *Low-dimensional surgery and the Yamabe invariant*, Preprint, 2012, http://arxiv.org/abs/1204.1197.

[5] S. Führing, *Bordismus und $\mathbb{C}P^2$-Bündel*, Diplomarbeit, TU München, 2008.

[6] N. Große, *The Yamabe equation on manifolds of bounded geometry*, (2009), http://arxiv.org/abs/0912.4398.

[7] J. Petean and J. M. Ruiz, *Isoperimetric profile comparisons and Yamabe constants*, Ann. Global Anal. Geom. **40** (2011), 177–189.

[8] ———, *On the Yamabe constants of $S^2 \times \mathbb{R}^3$ and $S^3 \times \mathbb{R}^2$*, Preprint, 2012, http://arxiv.org/abs/1202.1022.

[9] S. Stolz, *Simply connected manifolds of positive scalar curvature*, Ann. of Math. (2) **136** (1992), no. 3, 511–540.

Compactifications of monopole moduli spaces

Roger Bielawski

Many gauge-theoretic and algebro-geometric moduli spaces come equipped with natural Riemannian metrics. Examples include moduli spaces of instantons and monopoles on asymptotically flat manifolds, moduli spaces of Higgs bundles on a Riemann surface, moduli spaces of representations of quivers, and many others. These natural metrics often have connections to physics and lead to interesting questions connected to Hodge theory. In the case of magnetic monopoles on $\mathbb{R}^3$, the unsolved problem is the Sen conjecture, which predicts the dimension of L^2-cohomology of the moduli space of *strongly centred* $SU(2)$-monopoles.

We recall that a magnetic monopole on $\mathbb{R}^3$ for a gauge group G is given by a connection A on a principal G-bundle P over $\mathbb{R}^3$ and a section Φ of $\operatorname{ad} P$ (the Higgs field), which is a local minimum of the Yang-Mills-Higgs energy

$$\int_{\mathbb{R}^3} |F_A|^2 + |D_A\Phi|^2.$$

The finiteness of the energy implies that the Higgs field at infinity takes values in a fixed adjoint orbit O of G and the induced homology class $\Phi_*[S^2_\infty] \in H_2(O)$ gives the *magnetic charges* of a monopole. In the case of $SU(2)$, the magnetic charge is a single positive integer, while in the case of $SU(N)$-monopoles with *maximal symmetry breaking* (i.e. when O is a regular orbit), there are $N-1$ magnetic charges.

Quotienting the space of monopoles by gauge transformations equal to 1 at infinity yields the moduli space of framed monopoles. For $SU(N)$-monopoles with maximal symmetry breaking, this moduli space carries a natural hyperkähler metric given by the L^2-norm of infinitesimal variations of fields orthogonal to gauge directions [1]. Topologically, these spaces are spaces of based rational maps from $\mathbb{CP}^1$ to the orbit O. In the case of $SU(2)$-monopoles of charge n, the asymptotic region of the moduli space $\mathcal{M}_n$ has the following description, due to Atiyah and Hitchin [1], building on an earlier work of Taubes:

Given an infinite sequence of points of $\mathcal{M}_n$, there exists a subsequence m_r, a partition $n = \sum_{i=1}^{s} n_i$ with $n_i > 0$, a sequence of points $x_r^i \in \mathbb{R}^3$, $i = 1, \ldots, s$, such that

(i) *the sequence m_r^i of monopoles translated by $-x_r^i$ converges weakly to a monopole of charge n_i with centre at the origin;*
(ii) *as $r \to \infty$, the distances between any pair of points x_r^i, x_r^j tend to ∞ and the direction of the line $x_r^i x_r^j$ converges to a fixed direction.*

We can think of clusters of charge n_i with centres at x_r^i receding from one another in definite directions. This description remain valid for $SU(N)$-monopoles with maximal symmetry breaking, if we take into account the fact that we now have $N-1$ magnetic charges (and consequently $N-1$ "centres").

In this talk I shall describe a natural compactification $\overline{\mathcal{M}}_n$ of $\mathcal{M}_n$, corresponding to the above description of the asymptotics.

In the first instance, an approximation to $\partial\overline{\mathcal{M}}_n$ is the quotient of a fibration over the unit sphere $S^{3n-1} \subset \mathbb{R}^n \otimes \mathbb{R}^3$ by the symmetric group Σ_n. The fibre over a point of S^{3n-1} where the action of Σ_n is free is the torus T^n (corresponding to phases of particles), while over a point with a nontrivial isotropy, it is a torus of smaller dimension. We construct $\partial\overline{\mathcal{M}}_n$ by successively blowing up fixed point sets in S^{3n-1} and glueing in moduli spaces of centred monopoles of lower charges, in a way that the phases match. The resulting manifold with corners can be smoothed out and the compactification $\overline{\mathcal{M}}_n$ is a manifold with boundary. Keeping track of the corners results in a stratification of $\partial\overline{M}_n$. The different strata correspond to boundaries of regions of $\mathcal{M}_n$, where there centres x_r^i in (ii) above recede from each other at different rates of speed. We then show that in each such region the monopole metric is quasi-isometric (modulo coverings) to the product metric of moduli spaces of monopoles of lower charge. Our hope is that such an approximation of the monopole metric will be enough to prove the Sen conjecture.

We have a similar compactification of the moduli space of $SU(N)$-monopoles with maximal symmetry breaking and magnetic charges $m_1, \ldots, m_{N_1}$. This time we look at the quotient of a fibration over $S^{3(m_1+\cdots+m_{N_1})}$ by $\Sigma_{m_1} \times \ldots \Sigma_{m_{N-1}}$ and proceed similarly, blowing up and glueing in moduli spaces of $SU(N)$-monopoles with lower charges. The quasi-isometric behaviour of the monopole metric in this case is a work in progress, but we expect that it does not differ greatly from the $SU(2)$-case.

References

[1] M.F. Atiyah and N.J. Hitchin, *The geometry and dynamics of magnetic monopoles*, Princeton University Press, Princeton (1988).

Asymptotic torsion and Toeplitz operators

Jean-Michel Bismut

(joint work with Xiaonan Ma, Weiping Zhang)

The purpose of the talk is to explain results obtained jointly with Xiaonan Ma and Weiping Zhang [4, 5] on the asymptotic analytic torsion of a compact manifold, and on the proper use of Toeplitz operators in this context.

1. Introduction

Let X be a compact manifold, let F be a complex flat vector bundle on X, and let $\left(\Omega^{\cdot}(X,F), d^X\right)$ be the de Rham complex with coefficients in F, with cohomology $H^{\cdot}(X,F)$. Let $\lambda = \det H^{\cdot}(X,F)$ be the determinant of the cohomology of F. Let g^{TX}, g^F be smooth metrics on TX, F. Let $\| \|$ be the Ray-Singer metric on the complex line λ, that one obtains via a spectral invariant of the associated Hodge Laplacian $\square^X$, the Ray-Singer analytic torsion [17], which we will also call de Rham torsion.

I will limit myself to the case where X is odd dimensional. Then $\| \|_\lambda$ does not depend on g^{TX}, g^F. Given a triangulation K of X, there is an associated metric $\| \|_\lambda^K$ on λ, that is called the Reidemeister metric. If F is unitarily flat, the metric $\| \|_\lambda$ does not depend on K. The Ray-Singer conjecture, established by Cheeger [10] and Müller [14] says that if F is unitarily flat, the Ray-Singer and Reidemeister metrics coincide. When F is exact, i.e., $H^{\cdot}(X,F) = 0$, this gives the equality of two real numbers, the Ray-Singer torsion and the Reidemeister torsion. This result was extended by Müller [15] to the case where F is only unimodular. In the general case, $\| \|_\lambda^K$ is no longer an invariant. A formula relating $\| \|_\lambda$ and $\| \|_\lambda^K$ was given in Bismut-Zhang [8].

Flat vector bundles have odd Chern classes valued in $\mathbf{C}/\mathbf{Z} = \mathbf{R} \oplus \mathbf{R}/\mathbf{Z}$. Here, we shall be concerned with the $\mathbf{R}$ components of such classes. Let F be a flat vector bundle, and let ∇^F be the flat connection. If g^F is a Hermitian metric on F, and if $\omega\left(\nabla^F, g^F\right) = \left(g^F\right)^{-1} \nabla^F g^F$, for $k \in \mathbf{N}$, k odd, $\mathbf{Tr}\left[\omega^k\left(\nabla^F, g^F\right)\right]$ is a closed form whose cohomology class does not depend on g^F. Such forms can be put together in a series of forms $c\left(F, g^F\right)$, whose cohomology class is denoted $c(F)$. As explained in [3], the classes $c(F)$ are the odd Chern classes that were described above.

If $\pi : M \to S$ is a proper submersion with fibre X, and if F is a complex vector bundle on M, then $\pi_! F = H^{\cdot}(X, F|_X)$ is a $\mathbf{Z}$-graded flat vector bundle on S.

Let $e(TX)$ be the Euler class of TX. In [3], Lott and I proved a Riemann-Roch-Grothendieck formula for the above odd classes,

$$c(\pi_! F) = \pi_* [e(TX) c(F)] . \tag{1.1}$$

By transgression, given metrics g^{TX}, g^F on TX, F, in [3], even forms $\mathcal{T}$ on S were constructed in [3] such that

$$d\mathcal{T} = \pi_* \left[e\left(TX, \nabla^{TX}\right) c\left(F, g^F\right)\right] - c\left(H^{\cdot}(X, F|_X), g^{H^{\cdot}(X,F|_X)}\right). \tag{1.2}$$

In (1.2), ∇^{TX} is a metric connection on TX that is canonically attached to the geometric data, $e\left(TX, \nabla^{TX}\right)$ is the associated Euler form, and $g^{H^{\cdot}(X,F|_X)}$ is the metric on $H^{\cdot}(X, F|_X)$ that is obtained by identification with the corresponding fibrewise harmonic forms. The remarkable fact established in [3] is that the component $\mathcal{T}^{(0)}$ of degree 0 is just the fibrewise Ray-Singer analytic torsion. If X is odd dimensional, and $H^{\cdot}(X, F|_X) = 0$, equation (1.2) says that $\mathcal{T}$ is a closed form, whose cohomology class is independent of the metric data.

It follows from the above that analytic torsion, and the analytic torsion forms are part of a theory of the Euler class. Contrary to the Todd class, which is stable, the Euler class is unstable, which makes de Rham analytic torsion very different from holomorphic torsion.

2. A family of flat vector bundles

In [6, 7], Vasserot and myself have studied the asymptotics as $p \to +\infty$ of the holomorphic torsion of L^p, where L is a positive line bundle, and also of the symmetric powers of a positive vector bundle F. This last example was treated by the same techniques as in the case of a line bundle, by viewing the symmetric powers as the direct image of the powers of the canonical line bundle on the projectivization $\mathbf{P}(F^*)$.

We will adopt the same point of view as in [7] in the context of de Rham torsion. For simplicity, we will only consider the case of one single odd dimensional compact fibre X, but the results of [4, 5] also apply to the case of analytic torsion forms.

Let G be a Lie group, let $p : P_G \to X$ be a flat G-bundle. Let N be a compact Kähler manifold, let L be a positive line bundle on N. We assume that G acts holomorphically on (N, L). Let $\mathcal{N} = P \times_G N$ be the obvious bundle with fibre N on X, and let $q : \mathcal{N} \to X$ be the corresponding projection. The line bundle L on N induces a corresponding line bundle on $\mathcal{N}$, that is still denoted L.

For $p \in \mathbf{N}$, G acts on $H^{(0,0)}(N, L^p)$. Set

$$F_p = P_G \times_G H^{(0,0)}(N, L^p) . \tag{2.1}$$

For any $p \in \mathbf{N}$, F_p is a flat vector bundle on X.

Let g^L be a Hermitian metric on L over $\mathcal{N}$, so that the corresponding fibrewise $(1,1)$ form $c_1\left(L, g^L\right)$ is positive. The flat connection on P_G induces a corresponding flat covariant differentiation operator ∇^L_H on L in horizontal directions with respect to the flat connection on P_G. Let ω be the section of $q^* T^* X$ that is given by $\omega = \left(g^L\right)^{-1} \nabla^L_H g^L$. In the rest of the talk, we will say that g^L is nondegenerate if ω does not vanish on $\mathcal{N}$.

When G is a reductive group, the nondegeneracy assumption takes a simple form. Let K be a maximal compact subgroup, let $\mathfrak{g}$ be the Lie algebra of G, and let $\mathfrak{g} = \mathfrak{p} \oplus \mathfrak{k}$ be a Cartan decomposition. Let G/K the corresponding symmetric space. There exists a smooth section of the bundle $P \times_G G/K$ on X. Let P_K be the corresponding reduction of P to K. Let $\theta^{\mathfrak{g}}$ denote the $\mathfrak{g}$-valued connection 1-form on P_G, and let $\theta^{\mathfrak{g}} = \theta^{\mathfrak{p}} + \theta^{\mathfrak{k}}$ be its restriction to P_K, where $\theta^{\mathfrak{p}}, \theta^{\mathfrak{k}}$ are the components of $\theta^{\mathfrak{g}}$ in $\mathfrak{p}, \mathfrak{k}$. Then $\theta^{\mathfrak{k}}$ is a connection form on P_K.

Let U be the compact form of G, and let $\mathfrak{u} = i\mathfrak{p} \oplus \mathfrak{k}$ be its Lie algebra. We assume that U acts holomorphically on N, L and preserves metrics g^{TN}, g^L on TN, L. Let $\mu : N \to \mathfrak{u}^*$ be the associated moment map. If $u \in \mathfrak{u}$, if L_U is the corresponding Lie derivative operator acting on $C^\infty(N, L)$, if ∇^L is the holomorphic Hermitian connection on L, then

$$L_U = \nabla^L_U - 2i\pi \langle \mu, u \rangle . \tag{2.2}$$

Under the above assumptions, the line bundle L descends to a Hermitian line bundle on $\mathcal{N}$, and μ descends to a map $\mu : \mathcal{N} \to P_K \times_K \mathfrak{u}$. Then $\langle \mu, i\theta^{\mathfrak{p}} \rangle$ is a section of $q^* T^* X$. Then g^L is nondegenerate if and only if this section does not vanish on $\mathcal{N}$.

3. Toeplitz operators

Let g^{TN}, g^L be Hermitian metrics on TN, L. We still assume that $c_1\left(L, g^L\right)$ is positive on N. By Kodaira's vanishing, for $p \in \mathbf{N}$ large enough, $H^{(0,i)}(N, L^p)$ vanishes for $i > 0$. We equip $C^\infty(N, L^p)$ with the obvious L_2 Hermitian product induced by g^{TN}, g^L. Let P_p be the orthogonal projection operator from $C^\infty(N, L^p)$ on $H^{(0,0)}(N, L^p)$. If $f \in C^\infty(N, \mathbf{C})$, set

$$T_{f,p} = P_p f P_p. \tag{3.1}$$

Operators like $T_{f,p}$ are called Toeplitz operators. The theory of Toeplitz operators was developed by Boutet de Monvel and Guillemin in [9].

We equip F_p with the metric g^{F_p} induced by the Hermitian metric $C^\infty(N, L^p)$. The following important result was established in [5].

Proposition 3.1. *If g^L is nondegenerate, then for $p \in \mathbf{N}$ large enough, $H^{\cdot}(X, F_p) = 0$. There exist $C > 0, C' > 0$ such that if λ_p is the lowest eigenvalue of the Hodge Laplacian $\square^X_p$ acting on $\Omega^{\cdot}(X, F_p)$ that is associated with g^{TX}, g^{F_p}, then*

$$\lambda_p \geq C p^2 - C'. \tag{3.2}$$

Proof. We give an idea of the proof, which uses Toeplitz operators. The Weitzenböck formula for $\square^X_p$ can be written in the form

$$\square^X_p = -\Delta^{X,u}_p + \frac{1}{4} \left| \omega\left(\nabla^{F_p}, g^{F_p}\right) \right|^2 + \dots . \tag{3.3}$$

In (3.3), if $e_1, \ldots, e_n$ is an orthonormal basis of TX, then

$$\left|\omega\left(\nabla^{F_p}, g^{F_p}\right)\right|^2 = \sum_{i=1}^{n} \omega^2\left(\nabla^{F_p}, g^{F_p}\right)(e_i). \tag{3.4}$$

The idea is to express $\omega\left(\nabla^{F_p}, g^{F_p}\right)$ as a Toeplitz operator. Observe that $C^{\infty}(N, L^p)$ is a flat Hermitian vector bundle on X. Then $\omega\left(\nabla^{F_p}, g^{F_p}\right)$ can be easily expressed in terms of ω. Also, one has the easy equation,

$$\omega\left(\nabla^{F_p}, g^{F_p}\right) = T_{\omega\left(\nabla^{C^{\infty}(N,L^p)}, g^{C^{\infty}(N,L^p)}\right),p}. \tag{3.5}$$

When G is reductive, in [5], explicit formulas are given for the composition of two such operators. In general, results of Ma-Marinescu [11] are used to show that as $p \to +\infty$,

$$\frac{1}{p^2}\left|\omega\left(\nabla^{F_p}, g^{F_p}\right)\right|^2 = T_{|\omega|^2,p} + \mathcal{O}(1/p), \tag{3.6}$$

from which the proposition is easily derived. □

4. Asymptotic torsion

Bergeron and Venkatesh [1] have initiated the study of the asymptotics of analytic de Rham torsion under a tower of coverings of a locally symmetric space. Müller [16] studied the asymptotic torsion of a 3-dimensional locally symmetric space $\Gamma \setminus G/K$ associated with the reductive group $G = \mathrm{SL}_2(\mathbf{C})$, equipped with the family of flat vector bundles $\Gamma \setminus \left(G \times_K S^p\left(\mathbf{C}^2\right)\right)$, where $S^p\left(\mathbf{C}^2\right)$ denote the p-*th* symmetric power of $\mathbf{C}^2$.

Now we will explain the results obtained in [4, 5] on the asymptotics of the analytic torsion of X, under the same assumptions as in section 3. Also we assume the metric g^L to be nondegenerate. Set $n = \dim_{\mathbf{C}} N$. Let $o(TX)$ be the orientation bundle of TX. Let $\mathcal{T}_p$ be the analytic torsion for $\Omega^{\cdot}(X, F_p)$. The main result of [4, 5] is as follows.

Theorem 4.1. *There exists an explicitly locally computable smooth section W of $\Omega^n(o(TX))$ such that as $p \to +\infty$,*

$$p^{-n-1}\mathcal{T}_p = \int_X W + \mathcal{O}(1/p). \tag{4.1}$$

Observe that since analytic torsion is not sensitive to orientation, W is necessarily a section of $o(TX)$, and has to be related to the Euler class in some way.

Let us give a formula for W. Let q be the projection $\mathcal{N} \to X$. Set $\theta = -\frac{1}{2}\omega$. Let ∇^{TX} be the Levi-Civita connection on TX. Let $\widehat{TX}$ be another copy of TX, and let $\widehat{\theta}$ be the section of $q^*\widehat{TX}$ that corresponds to θ. If $m = \dim X$, let ψ be the $m-1$-form on the total space of $\widehat{TX} \setminus 0$ that transgresses the Euler form of $\widehat{TX}$. In the sequel, we view θ as a 1-form on $\mathcal{N}$, and $\widehat{\theta}^*\psi$ as a $m-1$ form on $\mathcal{N}$. Let ∇^L be the unitary connection on L over $\mathcal{N}$ that coincides with the holomorphic Hermitian connection along the fibres N, and with the flat connection on L made unitary in horizontal directions of $\mathcal{N}$ with respect to the flat connection. Let $c_1\left(L, g^L\right)$ denote the corresponding first Chern form of L on $\mathcal{N}$.

Then, by [5], we have the identity

$$W = q_*\left[\theta\left(\widehat{\theta}^*\psi\right)\exp\left(c_1\left(L, g^L\right)\right)\right]. \tag{4.2}$$

Equation (4.2) for W does not reveal explicitly the many hidden properties of $\int_X W$, which makes this expression compatible to the functorial properties of analytic torsion. In [5], W is obtained by a direct construction, and some of the hidden properties are verified directly.

In [5], when $X = \Gamma \setminus G/K$ is a locally symmetric space associated with a reductive group G, and the vector bundles F_p come from representations of G, another approach to the above asymptotics is obtained by evaluating the orbital integrals that are part of the definition of analytic torsion via heat kernels using the explicit formulas of [2]. In particular the vanishing results of Moscovici-Stanton [13] are shown to be still valid for such vector bundles. Moreover, in [5], we show that the asymptotics of the analytic torsion of X is essentially the same as the asymptotics of the L_2 analytic torsion of G/K with coefficients in vector bundles like F_p (corrected by the volume of X) up to exponentially small terms. From the point of orbital integrals, only the orbit of the identity in G is asymptotically relevant.

Let U be the compact form of G, let λ be a weight for an irreducible representation of U, and let N be the coadjoint orbit of λ. Let L be the canonical line bundle on N such that $H^{(0,0}\left(N, L^p\right)$ is just the irreducible representation of U with maximal weight $p\lambda$. Let T, T_U be maximal tori in K, U such that $T \subset T_U$, and let $\mathfrak{t}, \mathfrak{t}_U$ be their Lie algebras. We may and we will assume that $\lambda \in \mathfrak{t}_U^*$. Let W_U be the Weyl group of U. It is shown in [5] that g^L is nondegenerate if and only if the image of λ by W_U does not intersect $\mathfrak{t}^*$.

5. The analysis of asymptotic torsion

The analysis which is needed in [5] to study the asymptotic analytic torsion reveals some form of fight between the base X and the fibre N. In this fight, *the fibre wins*. While the base has essentially nothing to offer except the Euler form (which vanishes here...) and its transgression, the fibre N has all the richness associated with the Todd class, Borel-Weil theory and geometric quantization. Ultimately Getzler rescaling on the basis introduces the sort of Berezin integrals that appear in Mathai-Quillen's construction of the form ψ [12], while the asymptotic analysis along the fibre ultimately involves the algebra of Toeplitz operators, and a form of coordinate rescaling along the fibre G of the principal bundle P_G. These two kinds of rescaling fit together nicely.

References

[1] N. Bergeron and A. Venkatesh, *The asymptotic growth of torsion homology for arithmetic groups*, http://arxiv.org/abs/1004.1083 (2010).

[2] J.-M. Bismut, *Hypoelliptic Laplacian and orbital integrals*, Annals of Mathematics Studies, vol. 177, Princeton University Press, Princeton, NJ, 2011. MR 2828080

[3] J.-M. Bismut and J. Lott, *Flat vector bundles, direct images and higher real analytic torsion*, J. Amer. Math. Soc. **8** (1995), no. 2, 291–363. MR 96g:58202

[4] J.-.M. Bismut, X. Ma, and W. Zhang *Opérateurs de Toeplitz et torsion analytique asymptotique*, C. R. Math. Acad. Sci. Paris **349** (2011), no. 17–18, 977–981. MR 2838248

[5] ———, *Asymptotic torsion and Toeplitz operators*, To appear (2011). http://www.math.u-psud.fr/~bismut/BismutMaZhangglob.pdf

[6] J.-M. Bismut and É. Vasserot, *The asymptotics of the Ray-Singer analytic torsion associated with high powers of a positive line bundle*, Comm. Math. Phys. **125** (1989), no. 2, 355–367. MR 91c:58141

[7] ———, *The asymptotics of the Ray-Singer analytic torsion of the symmetric powers of a positive vector bundle*, Ann. Inst. Fourier (Grenoble) **40** (1990), no. 4, 835–848 (1991). MR 92b:58237

[8] J.-M. Bismut and W. Zhang, *An extension of a theorem by Cheeger and Müller*, Astérisque (1992), no. 205, 235, With an appendix by François Laudenbach. MR 93j:58138

[9] L. Boutet de Monvel and V. Guillemin, *The spectral theory of Toeplitz operators*, Annals of Mathematics Studies, vol. 99, Princeton University Press, Princeton, NJ, 1981. MR 620794 (85j:58141)

[10] J. Cheeger, *Analytic torsion and the heat equation*, Ann. of Math. (2) **109** (1979), no. 2, 259–322. MR 80j:58065a

[11] X. Ma and G. Marinescu, *Holomorphic Morse inequalities and Bergman kernels*, Progress in Mathematics, vol. 254, Birkhäuser Verlag, Basel, 2007. MR MR2339952 (2008g:32030)

[12] V. Mathai and D. Quillen, *Superconnections, Thom classes, and equivariant differential forms*, Topology **25** (1986), no. 1, 85–110. MR 87k:58006

[13] H. Moscovici and R. J. Stanton, *R-torsion and zeta functions for locally symmetric manifolds*, Invent. Math. **105** (1991), no. 1, 185–216. MR 92i:58199

[14] W. Müller, *Analytic torsion and R-torsion of Riemannian manifolds*, Adv. in Math. **28** (1978), no. 3, 233–305. MR 80j:58065b

[15] ———, *Analytic torsion and R-torsion for unimodular representations*, J. Amer. Math. Soc. **6** (1993), no. 3, 721–753. MR 93m:58119

[16] ———, *The asymptotics of the Ray-Singer analytic torsion of hyperbolic 3-manifolds*, http://arxiv.org/abs/1003.5168v1 (2010).

[17] D. B. Ray and I. M. Singer, *R-torsion and the Laplacian on Riemannian manifolds*, Advances in Math. **7** (1971), 145–210. MR 45 #4447

Optimal integral curvature pinching

Gilles Carron

(joint work with Vincent Bour)

Our paper [3] is motivated by the following nice result of M. Gursky ([7]):

Theorem : *Let (M^4, g) be an oriented 4 manifold with positive Yamabe invariant:*

i) *If $\int_M |\mathring{Ric}|^2 dv_g \le \frac{1}{12}\int_M \mathrm{scal}_g^2 dv_g$ then either the first Betti number $b_1(M^n)$ of M vanishes or (M,g) is conformally equivalent to a quotient of the Riemannian product $\mathbb{R}\times\mathbb{S}^3$.*

ii) *If $\int_M |W_g|^2 dv_g \le \frac{1}{24}\int_M \mathrm{scal}_g^2 dv_g$ then either the second Betti number $b_2(M^n)$ of M vanishes or (M,g) is conformally equivalent to the complex projective plane $\mathbb{P}^2(\mathbb{C})$.*

If (M^n, g) is a Riemannian manifold of dimension $n > 2$, then its *Yamabe invariant* is defined by :

$$Y(M,g) := \inf_{\varphi \in C_0^\infty(M)} \frac{\int_M \left[\frac{4(n-1)}{n-2}|d\varphi|^2 + \mathrm{scal}_g \varphi^2\right] dv_g}{\left(\int_M \varphi^{\frac{2n}{n-2}} dv_g\right)^{\frac{n-2}{n}}}$$

When $u \in C^\infty(M)$, then it is well known that $Y(M,g) = Y\left(M, e^{2u}g\right)$, hence the Yamabe invariant only depends on the conformal class of the metric: $[g] = \{e^u g,\ u \in C^\infty(M)\}$. When M is closed, the Yamabe invariant has been introduced in order to search for a metric with constant scalar curvature in the conformal class of g. The conclusion of the Yamabe program, completed by H. Yamabe, N. Trudinger, T. Aubin and R. Schoen, is that we can always find a metric $\tilde{g} \in [g]$ conformally equivalent to g such that:

$$\mathrm{scal}_{\tilde{g}} = \frac{Y(M^n,[g])}{\mathrm{vol}(M^n,g)^{\frac{2}{n}}}.$$

and is called a *Yamabe minimizer*.

In fact in dimension 4, the condition [1]

$$\int_M |\mathring{Ric}|^2 dv_g \leq \frac{1}{12}\int_M \mathrm{scal}_g^2 dv_g$$

is conformally invariant. Indeed in dimension 4, the Q-curvature of Paneitz is defined by :

$$Q_g = \frac{1}{2}\left(\frac{1}{3}\Delta_g \mathrm{scal}_g + \frac{1}{12}\mathrm{scal}_g^2 - |\mathring{Ric}|^2\right)$$

satisfies the following conformal change law: if $\tilde{g} = e^{2u}g$: $P_g u + Q_g = Q_{\tilde{g}} e^{4u}$; where $P_g = \Delta_g^2 + \delta\left(\frac{2}{3}\mathrm{scal}_g - \mathrm{Ric}\right) d$ is the Paneitz operator. Because P_g is self adjoint and $P_g(1) = 0$ we easily get :

$$\int_M Q_g dv_g = \int_M Q_{\tilde{g}} dv_{\tilde{g}}\ ,$$

hence the curvature condition i) is equivalent to $\int_M Q_g dv_g \geq 0$.

We have found another proof of this result that does not used the study of the Paneitz curvature. We give now the main argument of our new proof of the above result i): assume that the Yamabe invariant is positive, then we can choose a Yamabe minimizer metric in the conformal class i.e. we suppose that the scalar curvature is constant

$$\mathrm{scal}_g = Y(M,[g])$$

and the volume is one, in particular ,we get the Sobolev inequality : $\forall \varphi \in C^\infty(M)$:

$$Y(M,[g])\|\varphi\|_{L^4}^2 \leq \int_M \left[6\varphi\Delta\varphi + \mathrm{scal}_g\varphi^2\right] dv_g$$

If the first Betti number of M is not zero $b_1(M) \neq 0$, then we can find a non zero harmonic 1 form ξ: $d\xi = \delta\xi = 0$. Using the Bochner inequality and the

[1] Recall that the Ricci trace free tensor $\mathring{Ric}$ is defined by $\mathring{Ric} = \mathrm{Ricci} - \frac{1}{n}\mathrm{scal}_g g$.

refined Kato inequality ([2],[4]), we know that the function $\varphi := |\xi|^{\frac{2}{3}}$ satisfies the inequation

$$\Delta\varphi \leq -\frac{2}{3}\mathrm{Ric}_{-}\varphi$$

Where Ric_{-} is the lowest eigenvalue of the Ricci tensor, by definition

$$\mathrm{Ric}_{-} = \frac{1}{4}\mathrm{scal}_g + \lambda$$

where λ is the lowest eigenvalue of the Trace less Ricci tensor $\overset{\circ}{Ric}$, and it is easy to show that

$$|\lambda|^2 \leq \frac{3}{4}|\overset{\circ}{Ric}|^2$$

Hence we get

$$\Delta\varphi + \frac{1}{6}\mathrm{scal}_g\varphi \leq \frac{1}{\sqrt{3}}|\overset{\circ}{Ric}|\varphi$$

If we enter this estimate in the above Sobolev inequality, we obtain

$$Y(M,[g])\,\|\varphi\|^2_{L^4} \leq \frac{6}{\sqrt{3}}\int_M |\overset{\circ}{Ric}|\varphi^2 dv_g \leq 2\sqrt{3}\left(\int_M |\overset{\circ}{Ric}|^2 dv_g\right)^{\frac{1}{2}} \|\varphi\|^2_{L^4}$$

But $\|\varphi\|^2_{L^4} \neq 0$ by hypothesis and because g is a Yamabe minimizer with unit volume, we also have $Y(M,[g]) = \|\mathrm{scal}_g\|_{L^2}$ eventually we have obtained :

$$\left(\int_M \mathrm{scal}_g^2 dv_g\right)^{\frac{1}{2}} \leq \sqrt{12}\left(\int_M |\overset{\circ}{Ric}|^2 dv_g\right)^{\frac{1}{2}}.$$

That is to say we have shown that if (M^4, g) is a closed Riemannian manifold with positive Yamabe invariant and non vanishing first Betti number then

$$\int_M Q_g dv_g \leq 0.$$

Analyzing the equality case in all the above inequalities, we can show that if moreover $\int_M Q_g dv_g = 0$ then (M, g) is conformally equivalent to a quotient of the Riemannian product $\mathbb{R} \times \mathbb{S}^3$. In fact our argument is quite general and for example, we obtained the following analogue of the second part of the theorem of Gursky in dimension 6:

Theorem : *If (M^6, g) is a compact Riemannian manifold with positive Yamabe invariant $Y(M^6,[g]) > 0$ and such that*

$$\|W_g\|_{L^3} \leq \frac{1}{2\sqrt{10}}Y(M^6,[g]),$$

then

- *either its third Betti number $b_3(M^6)$ vanishes,*
- *or M^6 is conformally equivalent to a quotient of the product of two 3-dimensional round spheres: $a\mathrm{S}^3 \times b\mathrm{S}^3$.*

In fact, A. Chang, M. Gursky and P. Yang proved in [5, 6] that when

$$\int_M |W_g|^2 dv_g + \frac{1}{2}\int_M |\mathring{Ric}_g|^2 dv_g < \frac{1}{24}\int_M \mathrm{scal}_g^2 dv_g,$$

then the manifold is diffeomorphic to a quotient of the round sphere $\mathbb{S}^4$. The result of M. Gursky already implied that such a manifold is a rational homology sphere. In [1], the first author has been able to recover part of this beautiful result by using the gradient flow of some quadratic curvature functional. In order to analyze the formation of singularities for such gradient flows, it is necessary to have integral pinching results for non-compact manifolds. We have obtained the following extension of the first part of Gursky's theorem to non-compact manifolds and we hope that it should be useful to analyze these singularities

Theorem : *Let (M^4, g) be a complete non-compact Riemannian manifold with positive Yamabe invariant. Assume that for some $p > 2$, the lowest eigenvalue of the Ricci curvature satisfies $Ric_- \in L^p$. If*

$$\int_M |\mathring{Ric}|^2 dv_g \leq \frac{1}{12} Y^2(M, [g]),$$

then

- *either $H^1_c(M, \mathbb{Z}) = \{0\}$ and in particular M has only one end.*
- *Or (M^n, g) or one of its two-fold covering is isometric to:*

$$\left(\mathbb{S}^3 \times \mathbb{R},\, \alpha\, \cosh^2(t)\left(h + (dt)^2\right)\right).$$

References

[1] V. Bour: Fourth order curvature flows and geometric applications, preprint (2010) ArXiv mathDG:1012.0342.

[2] T. Branson: Kato constants in Riemannian geometry, *Math. Res. Lett.* **7** (2000), no. 2-3, 245–261.

[3] V. Bour and G. Carron *Optimal Integral Pinching Results.*, Preprint, Arxiv 1203.0384.

[4] D. M. J. Calderbank, P. Gauduchon and M. Herzlich: Refined Kato inequalities and conformal weights in Riemannian geometry, *J. Funct. Anal.* **173** (2000) 214–255.

[5] A. Chang, M. Gursky, P. Yang: An equation of Monge-Ampère type in conformal geometry, and four-manifolds of positive Ricci curvature, *Ann. of Math.* (2) **155** (2002), no. 3, 709–787.

[6] A. Chang, M. Gursky, P. Yang: A conformally invariant sphere theorem in four dimensions, *Publ. Math. Inst. Hautes Études Sci.* **98** (2003) 105–143.

[7] M. Gursky: The Weyl functional, de Rham cohomology, and Kähler-Einstein metrics, *Ann. of Math.* (2) **148** (1998), no. 1, 315–337.

Asymptotics of complete Kähler metrics

Frédéric Rochon

(joint work with Zhou Zhang)

This report is on a joint work with Zhou Zhang [6] on the asymptotic behavior at infinity of complete Kähler metrics on quasiprojective manifolds. We will restrict our attention to quasiprojective manifolds X taking the form $X = \overline{X} \setminus \overline{D}$, where $\overline{X}$ is a smooth projective manifold of dimension n and $\overline{D}$ is a divisor with normal crossings, that is, the irreducible components $\overline{D}_1, \ldots, \overline{D}_\ell$ of $\overline{D}$ are smooth and intersect transversely. Let $L \to \overline{X}$ be a positive holomorphic line bundle and h_L be a choice of Hermitian metric inducing a positive curvature form. For $i = 1, \ldots, \ell$, let also $s_i \in H^0(\overline{X}; [\overline{D}_i])$ be a choice of section such that $s_i^{-1}(0) = \overline{D}_i$ and let $\| \cdot \|_{\overline{D}_i}$ be a choice of Hermitian metric for $[\overline{D}_i]$. From an idea of Carlson and Griffiths [3, Proposition 2.1], we then know that for $\epsilon > 0$ sufficiently small, the $(1,1)$-form

$$\begin{aligned} \omega &= \sqrt{-1}\Theta_L + \sqrt{-1}\,\overline{\partial}\partial \log\left(\prod_{i=1}^{\ell}(-\log \epsilon \|s_i\|^2_{\overline{D}_i})^2\right) \\ &= \sqrt{-1}\Theta_L + 2\sqrt{-1}\sum_{i=1}^{\ell}\left(\frac{\Theta_{\overline{D}_i}}{\log \epsilon \|s_i\|^2_{\overline{D}_i}}\right) \\ &\quad + 2\sqrt{-1}\sum_{i=1}^{\ell}\left(\frac{(\partial \log \epsilon \|s_i\|^2_{\overline{D}_i}) \wedge (\overline{\partial} \log \epsilon \|s_i\|^2_{\overline{D}_i})}{(\log \epsilon \|s_i\|^2_{\overline{D}_i})^2}\right). \end{aligned} \tag{0.1}$$

is the Kähler form of a complete Kähler metric g_ω of finite volume on X. The metric g_ω is the prototypical example of an asymptotically tame polyfibred cusp metric. When $K_{\overline{X}} + [\overline{D}] > 0$, we can take $L = K_{\overline{X}} + [\overline{D}]$ to be our positive holomorphic line bundle. From the work of Yau, Cheng and Yau, Kobayashi, Tsuji, Tian and Yau and Bando [2], we know that there exists a Kähler-Einstein metric g_{KE} bi-Lipschitz to g_ω. The Kähler form of the Kähler-Einstein metric g_{KE} is of the form $\omega_{KE} = \omega + \sqrt{-1}\,\partial\overline{\partial}u$ with the function u obtained by solving the complex Monge-Ampère equation

$$\log\left(\frac{(\omega + \sqrt{-1}\partial\overline{\partial}u)^n}{\omega^n}\right) - u = F, \tag{0.2}$$

for some appropriate function F. Alternatively, the Kähler-Einstein metric can be obtained by using the Ricci flow. Indeed, as shown in [4] or [5, Example 6.18], the Ricci flow with initial metric g_ω exists for all time and converges to the Kähler-Einstein metric g_{KE}.

To study the asymptotic behavior of such metrics, we introduce a compactification of X by a manifold with corner. When the divisor $\overline{D}$ is smooth, we obtain such a compactification by blowing up $\overline{D}$ in $\overline{X}$ in the sense of Melrose,

$$\hat{X} = [\overline{X}; \overline{D}] = \overline{X} \setminus \overline{D} \bigsqcup S(N\overline{D}), \tag{0.3}$$

where $S(N\overline{D})$ is the unit normal bundle of $\overline{D}$ in $\overline{X}$. The set $\hat{X}$ is naturally a manifold with boundary $\partial\hat{X} = S(N\overline{D})$. In particular, its boundary has an induced circle fibration $\hat{\Phi} : \partial\hat{X} \to \overline{D}$. If $\rho \in \mathcal{C}^\infty(\hat{X})$ is a choice of boundary defining function for $\partial\hat{X}$, that is, $\rho^{-1}(0) = \partial\hat{X}$, ρ is positive on $\hat{X} \setminus \partial\hat{X}$ and the differential $d\rho$ is nowhere zero on $\partial\hat{X}$, then the compactification we are looking for, which we call the **logarithmic compactification** $\widetilde{X}$ of X, is obtained by declaring it homeomorphic to $\hat{X}$ with ring of smooth functions $\mathcal{C}^\infty(\widetilde{X})$ generated by $\mathcal{C}^\infty(\hat{X})$ and the function $x = \frac{-1}{\log \rho}$, assuming without loss of generality that

ρ is always less than 1. The function $x \in \mathcal{C}^\infty(widetildeX)$ is then a boundary defining function for $\widetilde{X}$.

On X, we can consider two natural spaces of smooth functions, one being the restriction of $\mathcal{C}^\infty(\widetilde{X})$ to X, and the other being the Cheng-Yau Hölder ring $\mathcal{C}^\infty_{fc}(X)$ of bounded smooth functions having their covariant derivatives with respect to the metric g_ω bounded on X. Since neither of them is contained in the other, we can also consider the space given by their intersection,

$$\mathcal{C}^\infty_{fc}(\widetilde{X}) = \mathcal{C}^\infty(\widetilde{X}) \cap \mathcal{C}^\infty_{fc}(X). \tag{0.4}$$

Since a function in $\mathcal{C}^\infty_{fc}(\widetilde{X})$ is in particular in $\mathcal{C}^\infty(\widetilde{X})$, it has a Taylor series at $\partial\widetilde{X}$. The key fact motivating the introduction of the space $\mathcal{C}^\infty_{fc}(\widetilde{X})$ is that requiring the function to be in $\mathcal{C}^\infty_{fc}(X)$ forces the Taylor series of f at $\partial\widetilde{X}$ to be of the form

$$f \sim \sum_{k=0}^{\infty} \widetilde{\Phi}^*(a_k)x^k, \quad a_k \in \mathcal{C}^\infty(D), \tag{0.5}$$

where x is choice of boundary defining function for $\partial\widetilde{X}$ in $\widetilde{X}$. As a consequence, a function $f \in \mathcal{C}^\infty_{fc}(\widetilde{X})$ has a well-defined restriction to D and its full asymptotic behavior at infinity is completely described by its Taylor series at $\partial\widetilde{X}$. The space $\mathcal{C}^\infty_{fc}(\widetilde{X})$ provides the right framework to describe the asymptotic behavior of complete Kähler metrics on X bi-Lipschitz to g_ω.

Using the logarithmic compactification $\widetilde{X}$ and the space $\mathcal{C}^\infty_{fc}(\widetilde{X})$, our first result concerns the evolution of the asymptotic behavior of Kähler metrics under the Ricci flow, answering a question of [5]. A similar result holds when the divisor D has normal crossings.

Theorem 4. *If g_ω is an asymptotically tame polyfibred cusp Kähler metric on X and $\widetilde{\omega}_t = \omega_t + \sqrt{-1}\partial\overline{\partial}u(t,\cdot)$, with $\omega_t = -\operatorname{Ric}(\omega) + e^{-t}(\omega + \operatorname{Ric}(\omega))$, is the solution to the normalized Ricci flow for $t \in [0,T)$ with*

$$\frac{\partial u}{\partial t} = \log\left(\frac{(\omega_t + \sqrt{-1}\partial\overline{\partial}u)^n}{\omega_0^n}\right) - u, \quad u(0,\cdot) = 0,$$

then $g_{\widetilde{\omega}_t}$ is an asymptotically tame polyfibred cusp Kähler metric and $u(t,\cdot) \in \mathcal{C}^\infty_{fc}(\widetilde{X})$ for all $t \in [0,T)$.

Our general strategy to prove this result is to restrict the evolution equation of the potential function to D and solve it to get a candidate u_D for what should be the restriction of u to D. Using a suitable decay estimate proved using a barrier function and the maximum principle, we then check u_D is indeed the restriction of u to D. We can then proceed recursively in the same fashion to build up the whole Taylor series of u at D.

From Theorem 4, we would naively expect the Kähler-Einstein metric to be also an asymptotically tame polyfibred cusp Kähler metric. When $\dim_{\mathbb{C}} X = 1$, this is indeed the case as described in [1]. However, when $\dim_{\mathbb{C}} X > 1$ and the divisor $\overline{D}$ is smooth, this is no longer the case as our next result shows.

Theorem 5. *Suppose the divisor* $\overline{D}$ *is smooth and* $K_{\overline{X}} + [\overline{D}] > 0$. *Let* u *be the solution to the complex Monge-Ampère equation* (0.2) *so that* $\omega_{KE} = \omega + \sqrt{-1}\partial\overline{\partial}u$ *is the Kähler form of the Kähler-Einstein metric* g_{KE} *bi-Lipschitz to* g_ω. *Then there exists an index set* $E \subset [0,\infty) \times \mathbb{N}_0$ *such that* u *has an asymptotic expansion at* $\partial\widetilde{X}$ *of the form*

$$u \sim \sum_{(z,k)\in E} \widetilde{\Phi}^*(a_{z,k})x^z(\log x)^k, \quad a_{z,k} \in \mathcal{C}^\infty(\overline{D}).$$

Moreover, the index set E *is such that*

$$(z,k) \in E,\ z \leq 1 \quad \Longrightarrow \quad (z,k) \in \{(0,0),(1,0),(1,1)\}.$$

We also have a topological criterion determining when such logarithmic term actually occurs. In complex dimension 2, this criterion is particularly simple to describe: there is a term $x\log x$ in the asymptotic expansion of u if and only if the (complex) normal bundle of $\overline{D}$ in $\overline{X}$ is non-trivial. Thus, an easy example of a Kähler-Einstein metric with such a logarithmic term is obtained by taking $\overline{X} = \mathbb{CP}_2$ with $\overline{D} \subset \mathbb{CP}_2$ a smooth curve of degree at least 4. This can also be used to construct an example where $\overline{D}$ has normal crossings and the solution u to the complex Monge-Ampère equation is not in $\mathcal{C}^\infty_{fc}(\widetilde{X})$.

References

[1] P. Albin, C.L. Aldana, and F. Rochon, *Ricci flow and the determinant of the Laplacian on non-compact surfaces*, to appear in Comm. in PDE.

[2] S. Bando, *Einstein Kähler metrics of negative Ricci curvature on open Kähler manifolds. Kähler metrics and moduli spaces*, Adv. Stud. in Pure Math **18** (1990), no. 2, 105–136.

[3] J. Carlson and P. Griffiths, *A defect relation for equidimensional holomorphic mappings between algebraic varieties*, Ann. of Math. (2) **95** (1972), 557–584.

[4] A. Chau, *Convergence of the Kähler-Ricci flow on noncompact Kähler manifolds*, J. Differential Geom. **66** (2004), 211–232.

[5] John Lott and Zhou Zhang, *Ricci flow on quasiprojective manifolds*, Duke Math. J. **156** (2011), no. 1, 87–123.

[6] Z. Zhang and F. Rochon, *Asymptotics of complete Kähler metrics of finite volume on quasiprojective manifolds*, preprint, arXiv:1106.0873v1.

Geometry of Bow Moduli Spaces

Sergey Cherkis

Kronheimer and Nakajima [1] constructed Yang-Mills instantons on Asymptotically Locally Euclidean (ALE) spaces in terms of quivers. The moduli spaces of such instantons are quiver varieties. We formulate a generalization of this construction [2] delivering Yang-Mills instantons on Asymptotically Locally Flat (ALF) spaces. Our construction is formulated in terms of bows. Bows generalize quivers and provide convenient description of instanton moduli spaces. Each bow representation has a moduli space. We claim it to be isomorphic to the corresponding moduli space of instantons.

All ALE spaces were shown by Kronheimer to be hyperkähler resolutions of orbifolds $\mathbb{R}^4/\Gamma$ of the flat four-space (with the finite group $\Gamma \subset SU(2)$). These are complete hyperkähler spaces with quartic volume growth. ALF space, on the other hand, are complete kyperkähler manifolds with cubic volume growth. A prototypical example of an ALE space is $\mathbb{R}^4$, while a prototypical example of an ALF space is the Taub-NUT space, which has the metric

$$ds^2 = \frac{1}{4}\left(\left(l + \frac{1}{|\vec{x}|}\right)d\vec{x}^2 + \frac{1}{\left(l + \frac{1}{|\vec{x}|}\right)}(d\tau + \omega)^2\right). \tag{0.1}$$

The Taub-NUT space here is viewed as a circle fibration over $\mathbb{R}^3 \ni \vec{x}$, with $\tau \sim \tau + 4\pi$ being the coordinate along the fiber. The one form ω on the base satisfies $d\omega = *_3 d\frac{1}{|\vec{x}|}$, and the positive real parameter l determines the asymptotic size of the fiber. (For $l = 0$ the metric of Eq. (0.1) is the metric on flat $\mathbb{R}^4$.)

We associate to the Taub-NUT space a simplest bow of Fig. 1.

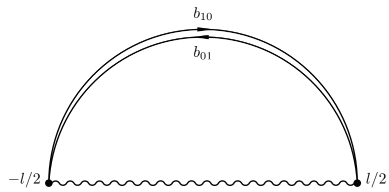

Figure 1. A Taub-NUT Bow Diagram.

Instantons on this space are connections on Hermitian bundles with self-dual curvature and finite action. Each topological type of instantons on the Taub-NUT space corresponds to a bow representation. An $SU(2)$ instanton, for example, corresponds to the bow representation of Fig. 2. This bow representation was

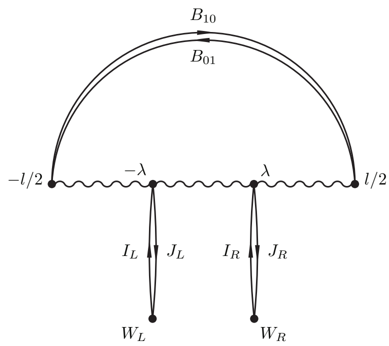

Figure 2. The bow representation for an $SU(2)$ Instanton on the Taub-NUT.

used to construct all self-dual connections with instanton number one in [3] and to compute the metric on the instanton moduli space in [4]. This moduli space metric in fact has a rather simple form:

$$ds^2 = \left(l + \frac{1}{2R_1}\right) d\vec{R}_1^2 - 4\lambda d\vec{R}_1 d\vec{r} + + \left(2\lambda + \frac{1}{r}\right) d\vec{r}^2 \tag{0.2}$$

$$+ \frac{\left(\frac{1}{2}d\theta - \frac{1}{4}\omega_{R_1}\right)^2}{l - 2\lambda + 1/r + 1/(2R_1)} + \frac{1}{4}\frac{(d\alpha + \omega_r)^2}{2\lambda + 1/r}, \tag{0.3}$$

where $\theta \sim \theta + 4\pi, \alpha \sim \alpha + 4\pi$ and the one-forms ω_{R_1} and ω_r on the three-spaces parameterized, respectively, by $\vec{R}_1$ and $\vec{r}$, satisfy $d\omega_{R_1} = *d(1/R_1), d\omega_r = *d(1/r)$.

For a most general instanton moduli space, we employ [2] the bow formulation to compute the asymptotic metric on it using the techniques developed by Bielawski [5].

Using the (conjectural) geometric Langlands correspondence for complex surfaces [6, 7, 8] one can organize of L^2 cohomology of the moduli spaces of instantons with gauge group G into integrable highest weight representations of the Langlands dual affine group ${}^L G_{\mathrm{aff}}$. For instantons on ALF spaces such correspondence follows from the existence of six-dimensional super-conformal (0,2) field theories [9]. This gives a prediction for the number of L^2 harmonic forms on the instanton moduli spaces. We verify this prediction for low instanton number moduli spaces.

References

[1] P. B. Kronheimer and H. Nakajima, "Yang-Mills Instantons on ALE Gravitational Instantons," Math. Ann. **288**, no. 2, 263–307 (1990).

[2] S. A. Cherkis, "Instantons on Gravitons," Commun. Math. Phys. **306**, 449 (2011) [arXiv:1007.0044 [hep-th]].

[3] S. A. Cherkis, "Instantons on the Taub-NUT Space," Adv. Theor. Math. Phys. **14**, 609 (2010) [arXiv:0902.4724 [hep-th]].

[4] S. A. Cherkis, "Moduli Spaces of Instantons on the Taub-NUT Space," Commun. Math. Phys. **290**, 719 (2009) [arXiv:0805.1245 [hep-th]].

[5] R. Bielawski, "Asymptotic metrics for SU(N) monopoles with maximal symmetry breaking," Commun. Math. Phys. **199**, 297 (1998) [hep-th/9801092].

[6] A. Braverman and M. Finkelberg, "Pursuing the Double Affine Grassmannian I: Transversal Slices via Instantons on A_k Singularities," Duke Math. J. **152** (2010), no. 2, 175-206 [arXiv:0711.2083].

[7] A. Licata, "Framed Rank r Torsion-free Sheaves on $\mathbb{C}P^2$ and Representations of the Affine Lie Algebra $\widehat{gl(r)}$," arXiv:math/0607690;
A. Licata, "Framed Torsion-free Sheaves on $\mathbb{C}P^2$, Hilbert Schemes, and Representations of Infinite Dimensional Lie Algebras," Adv. Math. **226** (2011), no. 2, 1057-1095.

[8] H. Nakajima, "Quiver Varieties and Branching," SIGMA **5**, 3, (2009) arXiv:0809.2605.

[9] E. Witten, "Geometric Langlands from Six Dimensions," arXiv:0905.2720 [hep-th].

Zeta-regularized Determinants of Laplacians on Polygons

Julie Rowlett
(joint work with Clara Aldana and Werner Mueller)

The main result presented here is Theorem 6 which gives an explicit formula for the variation of the derivative of the spectral zeta function at zero for any convex polygonal domain. In forthcoming work [1], we shall use this to derive an explicit formula for the zeta-regularized determinant of the Laplacian. Let $\Omega \subset \mathbb{R}^2$ be a convex polygonal domain with n sides. The Euclidean Laplacian Δ_Ω on Ω with Dirichlet boundary condition has eigenvalues

$$0 < \lambda_1 < \lambda_2 \leq \lambda_3 \leq \dots$$

By Weyl's Law, the spectral zeta function

$$\zeta_\Omega(s) := \sum_{k=1}^{\infty} \lambda_k^{-s}$$

is holomorphic on the half plane $\{\Re s > 1\}$. The heat trace

$$TrH_\Omega(t) = \sum_{k=1}^{\infty} e^{-\lambda_k t},$$

is related to the zeta function by

$$\zeta_\Omega(s) = \frac{1}{\Gamma(s)} \int_0^\infty t^{s-1} TrH_\Omega(t) dt. \tag{0.1}$$

The heat trace admits an asymptotic expansion as time tends to zero computed in [3]

$$TrH_\Omega(t) \sim \frac{|\Omega|}{4\pi t} - \frac{|\partial\Omega|}{8\sqrt{\pi t}} + \sum_{i=1}^{n} \frac{\pi^2 - \alpha_i^2}{24\pi\alpha_i} + O(e^{-c/t}). \tag{0.2}$$

Above, α_i is the interior angle at the i^{th} vertex, and $|\Omega|$, $|\partial\Omega|$ denote respectively the area of Ω and the length of the boundary $\partial\Omega$. We note that the constant c is bounded below by a constant computed in [5].

It follows from (0.1), (0.2), and the meromorphic continuation of the Gamma function that ζ admits a meromorphic continuation to the complex plane which is regular at 0. The *zeta-regularized determinant* is defined to be

$$\det(\Delta_\Omega) = e^{-\zeta'_\Omega(0)}.$$

It is straightforward to compute that $\zeta(0)$ is the coefficient of t^0 in (0.2). Consequently, for $\lambda \in (0, \infty)$, the zeta function transforms under scaling of the domain by λ as follows

$$\zeta'_{\lambda\Omega}(0) = \zeta_\Omega(0) \log \lambda + \zeta'_\Omega(0).$$

The determinant therefore scales by

$$\det(\Delta_{c\Omega}) = c^{-\zeta_\Omega(0)} \det(\Delta_\Omega).$$

For *smoothly* bounded domains, the coefficient of t^0 in the small-time asymptotic expansion of the heat trace is a *topological invariant,* namely one sixth of the Euler characteristic [3]. Therefore, the extrema of the determinant are well defined on convex smoothly bounded domains of fixed area. For polygons, this is no longer the case.

Lemma 1. *Let R be a convex n-gon with all angles equal, and let P be a convex n-gon whose angles are not all equal. Assume R and P both have unit area. Then,*

(1) $\exists\, a > 0$ *such that* $\det(\Delta_{aR}) > \det(\Delta_{aP})$,
(2) $\exists\, b > 0$ *such that* $\det(\Delta_{bR}) < \det(\Delta_{bP})$,
(3) $\exists\, c > 0$ *such that* $\det(\Delta_{cR}) = \det(\Delta_{cP})$.

The proof is a straightforward calculation and is left to the reader. □

We are therefore motivated to define a spectral invariant which is *well-defined* on the moduli space of convex n-gons.

Proposition 0.1. *Let $\mathbb{M}_n$ be the moduli space of convex n-gons, which is the space of all similarity classes of convex n-gons. Then, the following function is well defined on $\mathbb{M}_n$.*

$$f(P) = Z_P'(0) - \frac{1}{2} Z_P(0) \log Area(P), \quad P \in \mathbb{M}_n.$$

For more details, see [1].

1. Preliminary variational formulae

Consider a conformal variation of the Euclidean metric $g \mapsto e^{2\sigma} g$, where σ is a smooth function. A computation analogous to those in [4] gives the following variational formula for $\zeta'(0)$,

$$\delta\zeta'(0) = -\gamma\delta\zeta(0) + C(\sigma), \tag{1.1}$$

where γ is Euler's constant, $\gamma = \Gamma'(1)$, and $C(\sigma)$ is the constant term in the trace of $2\delta\sigma H$. The coefficients in the short-time asymptotic expansion of the heat trace (0.2) can be computed by integrating a corresponding *local expansion* defined by the curvature and its derivatives [3]. To compute $C(\sigma)$, we may integrate the product of $2\delta\sigma$ with the *local heat trace expansion.* Since the curvature vanishes identically away from the corners, only the corners contribute to $C(\sigma)$. We may therefore compute the contribution to $C(\sigma)$ from each vertex and sum over the vertices.

A fixed half-strip with the standard Euclidean metric,

$$T = (-\infty, 0]_x \times [0,1]_y, \quad g_{Eucl} = dx^2 + dy^2,$$

can be conformally mapped onto the sector

$$S = (0, \alpha^{-1} e^{\lambda}]_r \times [0, \alpha]_\phi, \quad g_{Eucl} = dr^2 + r^2 d\phi^2 = e^{2\sigma}(dx^2 + dy^2),$$

where the coordinates and conformal factor σ are

$$y = \frac{\phi}{\alpha}, \quad x = \frac{\log(r\alpha) - \lambda}{\alpha}, \quad \sigma = x\alpha + \lambda.$$

The conformal factor σ is a smooth function of x which depends on the two parameters, α and λ. The total differential of the conformal factor σ with respect to the parameters α and λ,

$$\delta\sigma = x\ \mathbf{d}\alpha + \ \mathbf{d}\lambda = \frac{\log r + \log\alpha - \lambda}{\alpha}\ \mathbf{d}\alpha + \ \mathbf{d}\lambda.$$

Formula (2.5) in [5] gives the Green's function for a sector of opening angle α (see also [3] p. 44). The inverse Laplace transform, denoted L^{-1}, of the Green's function is the heat kernel. Let

$$C(\alpha) = \int_0^\infty r\log r dr \int_0^\alpha d\alpha L^{-1}\left\{\frac{1}{\pi^2}\int_0^\infty K_{ix}^2(r\sqrt{s})\frac{\sinh(\pi-\alpha)x}{\sinh\alpha x}dx\right\},$$

and let

$$A(\alpha) := -\frac{\pi\log\alpha}{12\alpha} - \frac{\pi}{12\alpha} - \frac{\alpha\log\alpha}{12\pi} + \frac{\alpha}{12\pi} - \gamma\frac{\pi^2-\alpha^2}{24\pi\alpha}.$$

The angles and side lengths of a convex polygonal domain P cannot be varied independently, but must satisfy certain *constraints.* The interior angles $\{\alpha_i\}_{i=1}^n$ must sum to $\pi(n-2)$, and the scale factors at each vertex $\{\lambda_i\}_{i=1}^n$ together with a global scale factor λ_0 are related by

$$\lambda_i = \lambda_0 - \sum_{j\neq i}(\pi-\alpha_i)\log|p_i - p_j|,$$

where the points $\{p_j\}_{j=1}^n$ lie on the unit circle. We then have the following.

Theorem 6. *Let P be a convex n-gon in the plane with interior angles $\{\alpha_i\}_{i=1}^n$, and let $A(\alpha_i)$, $C(\alpha_i)$, and λ_i be defined as above. For a conformal variation of P which maps P onto a Euclidean n-gon, the conformal variation of $\zeta'_P(0)$ is*

$$\sum_{i=1}^n \frac{C(\alpha_i)}{\alpha_i}\ \boldsymbol{d}\alpha + \delta\left(\sum_{i=1}^n A(\alpha_i) + \left(\frac{\pi\lambda_i}{12\alpha_i}\right) - \frac{1}{12\pi}(\alpha_i\lambda_i) + \left(\frac{1}{12\pi}(1-\alpha_i)(\lambda_i-\lambda_0)\right)\right).$$

In forthcoming work, we use Theorem 1 to compute an explicit formula for the function f defined in Proposition 1 and study the extrema of this spectral invariant in the spirit of [4]. We shall also apply our results to surfaces with conical singularities. Useful ideas for this work were inspired by [2]; the above result is a correction of a similar formula in [2].

References

[1] C. Aldana, W. Müller, and J. Rowlett, *Zeta-regularized determinants on moduli spaces of polygons,* in preparation.

[2] E. Aurell and P. Salomonson, *On functional determinants of laplacians in polygons and simplicial complexes,* Commun. Math. Phys. 165, (1994), 233–259.

[3] H. P. McKean and I. M. Singer, *Curvature and the eigenvalues of the Laplacian,* J. Diff. Geom. I, (1967), 43–69.

[4] B. Osgood, R. Phillips, and P. Sarnak, *Extremals of Determinants of Laplacians,* Journ. Funct. Anal., **80**, (1988), 148–211.

[5] M. van den Berg and S. Srisatkunarajah, *Heat equation for a region in $\mathbb{R}^2$ with a polygonal boundary,* J. London Math. Soc. vol. 37, no. 2, (1988), 119–127.

The index of the Dirac operator on manifolds with perturbed metric horns

Frank Lapp

We look at the following setting: let (M, g) be a closed manifold that consists of a compact manifold M_1 with boundary N and a smoothly attached singularity neighborhood U that isometric to

$$\big((0,\varepsilon)_r \times N, dr^2 \oplus g_N(r)\big)$$

First we look at an example: If $g_N(r) = r^{2\beta} g_N$, the singularities are called cones ($\beta = 1$) or metric horns ($\beta > 1$). If M is an even-dimensional Spin manifold, $S = S^+ \oplus S^-$ the Spin bundle split by the complex volume element and $D : C^\infty(S) \to C^\infty(S)$ the appropriate Spin Dirac operator, the closed extensions D_V^+ of the split Dirac operator $D^+ = D|_{S^+}$ are in a one-to-one correspondence to the subspaces

$$V \subset \begin{cases} \bigoplus\limits_{\substack{\lambda \in \operatorname{spec} D_N \\ |\lambda| < \frac{1}{2}}} \ker(D_N - \lambda) & \text{cone} \\ \ker D_N & \text{metric horn} \end{cases}$$

where D_N is the Spin Dirac operator over N. They are all Fredholm and

$$\operatorname{ind} D_V^+ = \int_M \hat{A} - \frac{1}{2}\left(\eta_{D_N}(0) + \dim\ker D_N\right) + \dim V.$$

These results were proved in [Cho85] (cone) and [LP98] (horn).

Using separation of variables it can be shown that a general Dirac operator over a manifold with singularity as defined above has the form

$$D^+ \simeq \frac{\partial}{\partial r} + S(r),$$

where $S(r)$ is a family of self-adjoint operators on a fixed domain. In [Brü92] it has been shown that if $S(r) = r^{-1} S_1(r)$ and $S_1(r)$ satisfies certain assumptions concerning it's spectrum and it's strong derivative it extends the index formula in the cone case. The methods applied there can be used again to prove an analogous extension of the index formula in the metric horn case.

References

[Brü92] Jochen Brüning, *On L^2-index theorems for complete manifolds of rank-one-type*, Duke Math. J. **66** (1992), no. 2, 257–309.

[Cho85] A. W. Chou, *The Dirac operator on spaces with conical singularities and positve scalar curvatures*, Transactions of the American Mathematical Society **289** (1985), 1–40.

[LP98] Matthias Lesch and Norbert Peyerimhoff, *On index formulas for manifolds with metric horns*, Comm. Partial Differential Equations **23** (1998), no. 3-4, 649–684.

A framework for Witten deformation

Chris Kottke
(joint work with Pierre Albin, Rafe Mazzeo)

In [Wit82] Witten suggested an approach to Morse theory through deformation of the de Rham complex and spectral analysis of the resulting Laplacian. First proved by [HS85], the result has been extended to Morse-Bott functions (having critical submanifolds instead of isolated critical points) in [Bis86], [BZ92], [BF97], [Pro97] and admits many other generalizations. Here I describe the construction of a uniform resolvent for such a deformed Laplacian which includes the case of Morse-Bott functions as well as so-called 'generalized Morse functions' which allow for degenerate critical points of birth-death type. Our method produces a detailed picture of the spectral asymptotics of the Witten Laplacian and should extend to other interesting situations such as smoothly stratified spaces and group actions.

Recall the basic idea behind Witten's version of Morse theory. Let (M, g) be a compact Riemannian manifold, $f \in C^\infty(M)$ a Morse function, and consider the deformed complex

$$(L^2\Omega^*(M), d_t := e^{-tf} de^{tf} = d + t\, df \wedge \cdot).$$

For any $t \geq 0$ this complex is isomorphic to the L^2 de Rham complex and therefore has cohomology isomorphic to $H^*(M)$. On the other hand, Hodge theory applies to each $\Delta_t := (d_t + d_t^*)^2$: eigenforms are smooth, span L^2, and form subcomplexes for each eigenvalue. Moreover only the harmonic forms (the subcomplex for $\lambda = 0$) contribute to the cohomology; the other subcomplexes are acyclic.

As $t \to \infty$, the spectrum of Δ_t decomposes into *small eigenvalues* with $\mathcal{O}(e^{-ct})$ decay, and *large eigenvalues* with $\mathcal{O}(t)$ growth. Furthermore, there is a finite basis of *small eigenforms* spanning the small eigenspace, with a unique form of degree k concentrating at each critical point of index k. The small eigenforms therefore form a finite dimensional subcomplex whose cohomology is isomorphic to $H^*(M)$, and the dimensions of whose chain spaces are determined by the number of critical points of f with given indices. An immediate consequence is the *Morse inequalities*, which estimate the dimensions of the cohomology groups in terms of these dimensions:

$$M_f(t) - P(t) = (1 + t)R(t), \quad R(t) \text{ has} \geq 0 \text{ coefficients} \tag{0.1}$$

where $P(t) = \sum \beta_k t^k$, $\beta_k = \dim H^k(M; \mathbb{R})$ is the Poincaré polynomial and $M_f(t) = \sum \nu_k t^k$, $\nu_k = \dim \Omega^k_{\text{small}}(M) = \# \{p \in M : df_p = 0,\ \text{ind}(p) = k\}$ is the *Morse polynomial.*

Recall that a Morse function f is one whose *critical set*

$$C := \{p \in M : df_p = 0\} \subset M$$

consists of discrete isolated points at which the Hessian $\partial^2 f_p$ is nondegenerate. In contrast, a *Morse-Bott function* is one for which $C = \sqcup_j C_j$ is a disjoint union of smooth submanifolds over which which $\partial^2 f$ restricts to a nondegenerate quadratic form on the normal bundle $NC = N^+C \oplus N^-C$ (which splits according to the signature of $\partial^2 f$), with the index of a component C_j defined to be the rank of N^-C_j. As a further extension, a *generalized Morse function* is one for which C is again discrete and isolated, with both nondegenerate and degenerate critical points of *birth-death type*, meaning that f has the local form

$$f(x) - f(0) = 1/2\left(-x_1^2 - \cdots - x_k^2 + x_{k+1}^2 + \cdots + x_{n-1}^2\right) + x_n^3/3$$

The theory of such functions goes back to Cerf [Cer70], who proved that any two Morse functions may be connected through a one-dimensional family of generalized Morse functions, birth-death critical points occuring as nearby nondegenerate critical points meet and annihilate one another.

Witten deformation in the classical and Morse-Bott cases has been considered in [HS85], [Bis86], [BZ92], [BF97] and [Pro97]. In the latter case, the coefficients of the Morse polynomial in (0.1) are replaced by

$$(0.2)\quad \nu_k = \dim \Omega^k_{\mathrm{small}}(M) = \dim H^k_c(N^-C) = \sum_j \dim H^{k-\mathrm{ind}(C_j)}\big(C_j; o(N^-)\big),$$

where $H^*(C_j; o(N^-))$ denotes cohomology of C_j twisted by the orientation bundle of N^-C. (Observe that this recovers the classical case in which each C_j is an isolated point.) In the case of generalized Morse functions, some results have been announced by Goette, Connes, Burghelea and Wai, and it is well-known that only nondegenerate critical points contribute to $\Omega^k_{\mathrm{small}}(M)$. In these generalizations, the small and large eigenvalues of the Witten Laplacian Δ_t are supplemented by a countable collection of 'medium' eigenvalues with $\mathcal{O}(1)$ asymptotic behavior in the Morse-Bott case, and 'medium large' eigenvalues with $\mathcal{O}(t^{2/3})$ behavior in the generalized Morse case (the latter coming from the degenerate critical pjjoints).

Part of our project is to build a framework for the construction of a resolvent $(\Delta_{1/h} - \lambda)^{-1}$ which is 'uniform' in an appropriate sense as $h := 1/t \to 0$, resolving the singular behavior at $h = 0$. This framework should extend to more interesting situations including stratified spaces and group actions. In particular, we prove

Theorem 0.1 (Albin-K-Mazzeo). *Given a generalized Morse-Bott function f, the resolvent family $(\Delta_{1/h} - \lambda)^{-1}$ is an element of an algebra $\Psi^*_W(M_W; \Lambda^*M)$ of pseudodifferential operators, where M_W is obtained from $M \times [0,\infty)_h$ by blow-up. Furthermore, the 'limit' $(\Delta_{1/h} - \lambda)^{-1}$ as $h \to 0$ is given by*

$$N_{\mathrm{cf}}\big((\Delta_{1/h} - \lambda)^{-1}\big) = (\Delta_{C_n \otimes o(N^-)} - \lambda)^{-1}.$$

Here $N_{\mathrm{cf}}(\cdot)$ denotes the normal operator at a particular boundary face of M_W associated to the critical set at $h = 0$, C_n denotes the nondegenerate components of C, and $\Delta_{C_n \otimes o(N^-)}$ is the induced Laplacian on C_n twisted by a flat connection on the line bundle $o(N^-)$.

An immediate consequence of this theorem is a precise characterization of the small and medium eigenvalues. Indeed, the small eigenvalues are those that limit to $0 \in \mathrm{Spec}(\Delta_{C_n \otimes o(N^-)})$, with corresponding eigenspace consisting of the harmonic forms on C_n (twisted by $o(N^-)$, and with a degree shift by the index), from which (0.2) follows. The medium eigenvalues limit to the nonzero spectrum of

$\Delta_{C_n \otimes o(N^-)}$, and can in principle be computed with multiplicity. The large (resp. medium large) eigenvalues may be similarly computed by considering $\Delta_{1/h} - \lambda/h$ (resp. $\Delta_{1/h} - \lambda/h^{2/3}$).

Our method of proof is to build a pseudodifferential operator calculus associated to a geometric resolution M_W of $M \times [0, \infty)_h$ (by inhomogeneous blow-up of $C \times 0$) and construct the resolvent within this calculus. In contrast to previous methods, the analysis is greatly simplified by the need to consider only certain 'model' operators with much simpler behavior than the original, with the need for delicate estimates completely obviated by the pseudodifferential machinery.

References

[BF97] M. Braverman and M. Farber, *Novikov type inequalities for differential forms with non-isolated zeros*, Mathematical Proceedings of the Cambridge Philosophical Society, vol. 122, Cambridge University Press, 1997, pp. 357–375.

[Bis86] J.M. Bismut, *The witten complex and the degenerate morse inequalities*, Journal of Differential Geometry **23** (1986), no. 3, 207–240.

[BZ92] J.M. Bismut and W. Zhang, *An extension of a theorem by cheeger and müller. with an appendix by françois laudenbach*, Astérisque **205** (1992), 235.

[Cer70] J. Cerf, *La stratification naturelle des espaces de fonctions différentiables réelles et le théoreme de la pseudo-isotopie*, Publ. Math. IHES **39** (1970), 5–173.

[HS85] B. Helffer and J. Sjöstrand, *Puits multiples en limite semi-classique IV-Etude du complexe de Witten*, Comm. Partial Differential Equations **10** (1985), no. 3, 245–340.

[Pro97] I. Prokhorenkov, *Morse-Bott functions and the Witten Laplacian*, Ph.D. thesis, Rice University, 1997.

[Wit82] E. Witten, *Supersymmetry and Morse theory*, J. Differential Geometry **17** (1982), no. 4, 661–692.

Yamabe flow on Manifolds with Edges

Boris Vertman

On a compact Riemannian manifold, the Yamabe problem asserts that every conformal class of metrics contains a representative of constant scalar curvature. There are now several proofs of this fact. The first proof, commenced by Yamabe and continued by Trudinger, Aubin and Schoen used the calculus of variations and elliptic partial differential equations. Another proof uses the geometric Yamabe flow:

$$\begin{cases} \partial_t g &= -\mathrm{scal}(g(t)) \cdot g, \\ g(0) &= g_0, \end{cases} \tag{0.1}$$

(and appropriate normalizations) to converge to constant scalar curvature metrics.

It is natural to wonder to what extent the Yamabe problem holds in other settings. There has been recent work in understanding this problem on singular manifolds with conic and more general incomplete edge metrics. See the work of Akutagawa-Botvinnik for the conic case, and Akutagawa-Carron-Mazzeo for the

Yamabe problem on very general stratified spaces, which attack the problem from an elliptic PDE point of view.

In a joint work with Eric Bahuaud, we are interested in the Yamabe flow on spaces with incomplete edge singularities that preserves the singular structure. We mention that in the setup of isolated conical singularities, existence and regularity of solutions to the inhomogeneous heat equation has recently been addressed in a recent preprint of Behrndt. Another approach to estimates for conical singularities is given in the forthcoming paper by Mazzeo-Rubinstein-Sesum.

Our main result is as follows.

Theorem 0.2. *There exists a solution to the Yamabe flow starting within a class of compact Riemannian spaces with admissible simple edge singularities that remains asymptotically admissible for a short-time.*

To make the statement precise, consider a (feasible) incomplete edge space (M, g) with a singular neighborhood $\mathscr{U} \cong (0,1] \times Y$, where

$$\phi : (Y, \phi^* g^B + \kappa) \to (B, g^B)$$

is a Riemannian submersion with fibres F and κ being a symmetric tensor on Y, restricting to (isospectral) Riemannian metrics on fibres. Then g is said to be an admissible edge metric if $g = g_0 + h$ where

$$g_0|_{\mathscr{U}} = dx^2 + \phi^* g^B + x^2 \kappa,$$
$$\text{and } |h|_{g_0} = O(x), x \to 0.$$

We consider the Friedrichs extension Δ of the Laplacian on such a feasible edge space (M, g). We want to understand mapping properties of $e^{-t\Delta}$ on (M, g) acting via convolution in time on the following Hölder spaces.

$$\Lambda^{\alpha,\frac{\alpha}{2}} := \{u \in C(\overline{M} \times [0,T]) \mid \sup \left(\frac{|u(p,t) - u(p',t')|}{d_g(p,p')^\alpha + |t-t'|^{\frac{\alpha}{2}}} \right) < \infty\},$$
$$\Lambda^{2+\alpha,1+\frac{\alpha}{2}} := \{u \in \Lambda^{\alpha,\frac{\alpha}{2}} \mid \partial_t u,\ \partial_x u,\ \partial_y u,\ x^{-1}\partial_z u,\ \Delta_g u \in \Lambda^{\alpha,\frac{\alpha}{2}}\},$$

where y and z are local coordinates on the base and the fibre, respectively. These non-standard Hölder spaces have recently also appeared in the analysis of Kähler-Einstein metrics by Donaldson on cones and Jeffres-Mazzeo-Rubinstein on edges.

We then have the following theorem

Theorem 0.3. *The Friedrichs heat kernel on an admissible incomplete edge space (M, g), convolving in time variable, is a bounded map ($\alpha \in (0,1)$)*

$$e^{-t\Delta_{\mathscr{F}}} : \Lambda^{\alpha,\frac{\alpha}{2}} \to \sqrt{t}\Lambda^{2+\alpha,1+\frac{\alpha}{2}},$$
$$e^{-t\Delta_{\mathscr{F}}} : \Lambda^{2+\alpha,1+\frac{\alpha}{2}} \to \sqrt{t}\Lambda^{2+\alpha,1+\frac{\alpha}{2}},$$

provided the following three conditions on the edge metric g:

(1) $|h|_{g_0} = O(x^2)$ *as* $x \to 0$.

(2) *Consider the Laplacian* Δ_Y *on* $(Y, \phi^*\mathfrak{h} + \kappa)$. *Then for any* $u \in C^\infty(B)$, *the function* $\Delta_Y \phi^* u$ *is a lift of a function on* B.
(3) *Any* $\lambda \in \operatorname{Spec} \Delta_{\kappa,y} \backslash \{0\}$ *satisfies* $\lambda_0 > \dim F$.

These parabolic Schauder estimates apply to estalish local extence of certain quasi-linear evolution equations, including the Yamabe flow. Our previous parabolic Schauder estimates allow us to set up a contraction mapping on $\Lambda^{2+\alpha,1+\frac{\alpha}{2}}$ and prove existence of a fixed point. We obtain the following

Theorem 0.4. *Let* (M^m, g_0) *be a feasible edge space such that* $\operatorname{scal}(g_0) \in \Lambda^{2+\alpha}$. *Then the (transformed) Yamabe flow equation admits a unique solution* $u \in \Lambda^{2+\alpha,1+\frac{\alpha}{2}}$ *for some* $T > 0$ *and* $\alpha \in (0, \alpha_0)$.

The index bundle for gap-continuous families and the spectral flow

Nils Waterstraat

Every Fredholm operator L acting on a Hilbert space has an integer valued index, which is invariant under deformations of the operator. Atiyah and Jänich constructed a natural extension of this index to families of bounded Fredholm operators which is an element in the K-theory of the parameter space. Both indices vanish if the considered operators are in addition selfadjoint. Later Atiyah and Singer showed that a natural index for families of selfadjoint Fredholm operators can be obtained in the odd K-theory of the parameter space.
The aim of my talk is to introduce a new construction of the index bundle for families of generally unbounded Fredholm operators and to discuss some applications. Let X be a compact topological space, H a Hilbert space and $\mathcal{A} : X \to \mathcal{C}(H)$ be a gap-continuous family of Fredholm operators. Our definition of the index bundle can be sketched along the following lines:

i) Since our operators have varying domains $\mathcal{D}(\mathcal{A}_x)$, $x \in X$, we turn the disjoint union of these spaces into a Hilbert bundle $\mathfrak{D}(\mathcal{A})$ over X.
ii) Next, one can show that the family $\mathcal{A}$ defines a Hilbert bundle morphism $\mathcal{A} : \mathfrak{D}(\mathcal{A}) \to X \times H$ which is a bounded Fredholm operator in each fibre.
iii) Finally, we extend the construction of the index bundle by Atiyah and Jänich to Hilbert bundle morphisms (in fact, even to Banach bundle morphisms [4]).

If $Y \subset X$ is a closed subspace such that all operators $\mathcal{A}_x$, $x \in Y$, have a bounded inverse, then the final result of this construction is an element $\operatorname{ind} \mathcal{A} \in K(X, Y)$. If all operators $\mathcal{A}_x$, $x \in X$, are bounded, then $\operatorname{ind} \mathcal{A}$ coincides with the classical definition of Atiyah and Jänich.
Along the same lines we associate to a family $\mathcal{A}$ of unbounded selfadjoint Fredholm operators which is invertible on Y an element $\text{s-ind}\, \mathcal{A} \in K^{-1}(X, Y)$. In the case $X = [0, 1]$ and $Y = \{0, 1\}$, we have an identification $K^{-1}(X, Y) \cong \mathbb{Z}$ given by the first Chern number. One now can show that the integer corresponding to $\text{s-ind}\, \mathcal{A}$ is the spectral flow of the path $\mathcal{A}$ as defined in this generality in [1].

We now want to present two applications of our index bundle in the selfadjoint case. The first application is an index theorem that computes the index bundle of families of symmetric second order ordinary differential operators. More precisely, let again X be a compact topological space and let $S : [0,1] \times X \to M(n;\mathbb{R})$ be a continuous family of symmetric matrices such that for each fixed $x \in X$, $S_x(\cdot)$ is smooth. Then for any diagonal matrix $J = \operatorname{diag}(1,\ldots,1,-1,\ldots,-1)$ we obtain a family of selfadjoint Fredholm operators on $L^2([0,1],\mathbb{C}^n)$ by

$$\mathcal{A}_x u = Ju'' + S_x(\cdot)u, \quad u \in \mathcal{D}(\mathcal{A}_x) = H^2([0,1],\mathbb{C}^n) \cap H^1_0([0,1],\mathbb{C}^n)$$

and it is easily seen that $\mathcal{A}$ is actually gap continuous. Hence for any choice of a closed subspace $Y \subset X$ such that the operators $\mathcal{A}_x$, $x \in Y$, are invertible, we can consider the index bundle $\text{s-ind}\,\mathcal{A} \in K^{-1}(X,Y)$. Our index theorem now states that this odd K-theory class coincides with $[\Theta(\mathbb{C}^n),\Theta(\mathbb{C}^n),b]$, where $\Theta(\mathbb{C}^n)$ is the product bundle with fibre $\mathbb{C}^n$ over $X \times \mathbb{R}$ and $b : X \times \mathbb{R} \to M(n;\mathbb{C})$ is a matrix family constructed from the fundamental solutions of the ordinary differential equations $\mathcal{A}_x u + is \cdot u = 0$, where $x \in X$, $s \in \mathbb{R}$ and i denotes the imaginary unit. Let us consider again the case $X = [0,1]$, $Y = \{0,1\}$ in which our indices can be identified with integers. Then the index theorem is equivalent to the semi-Riemannian Morse index theorem [2] which is a generalisation of the classical Morse index theorem from Riemannian geometry to geodesics in semi-Riemannian manifolds. A proof of our index theorem in this special case is published in [6] and the other results mentioned so far can be found in [5].

A more recent application of the selfadjoint index bundle concerns periodic Hamiltonian systems. Here we consider families of first order operators

$$\mathcal{A}_x u = \sigma u' + S_x(\cdot)u, \quad u \in \mathcal{D}(\mathcal{A}_x) = H^{\frac{1}{2}}(S^1,\mathbb{R}^{2n}),$$

where σ is the standard symplectic matrix on $\mathbb{R}^{2n}$ and $S : [0,1] \times X \to M(2n;\mathbb{R})$ is a family of symmetric matrices such that $S_x(0) = S_x(1)$ for all $x \in X$. Then for any closed subspace $Y \subset X$ as above, the index bundle $\text{s-ind}\,\mathcal{A} \in K^{-1}(X,Y)$ is defined. And we can again prove an index theorem which now computes this odd K-theory class in terms of the monodromy matrices of the family of ordinary differential equations $\mathcal{A}_x u + is \cdot u = 0$. Let us finally once more consider the special case $X = [0,1]$, $Y = \{0,1\}$. Salamon and Zehnder proved in [3] that the spectral flow of $\mathcal{A}$ is given by the Conley-Zehnder index of the path of monodromy matrices of the homogeneous equations $\mathcal{A}_x = 0$. Our index theorem now introduces in particular a new integer, which is constructed differently than the Conley-Zehnder index, but coincides with it.

References

[1] B. Booß-Bavnbek, M. Lesch, J. Phillips, *Unbounded Fredholm Operators and Spectral Flow*, Canad. J. Math. **57** (2005), 225–250.

[2] M. Musso, A. Portaluri, J. Pejsachowicz, *A Morse Index Theorem for Perturbed Geodesics on Semi-Riemannian Manifolds*, Topol. Methods Nonlinear Anal. **25** (2005), 69–99.

[3] D. Salamon, E. Zehnder, *Morse Theory for Periodic Solutions of Hamiltonian Systems and the Maslov Index*, Comm. Pure Appl. Math. **45** (1992), 1303–1360.
[4] N. Waterstraat, *The index bundle for Fredholm morphisms*, Rend. Semin. Mat. Univ. Politec. Torino **69** (2011), 299–315.
[5] N. Waterstraat, *The Index Bundle for Gap-Continuous Families, Morse-Type Index Theorems and Bifurcation*, PhD–thesis, University of Göttingen (2011).
[6] N. Waterstraat, *A K-theoretic proof of the Morse index theorem in semi-Riemannian geometry*, Proc. Amer. Math. Soc. **140** (2012), 337–349.

Hodge cohomology of some foliated boundary and foliated cusp metrics

Jesse Gell-Redman
(joint work with Frédéric Rochon)

The Hodge theorem states that for a complete, compact Riemannian manifold without boundary, the space of L^2 harmonic forms – the 'Hodge cohomology' – is isomorphic to the de Rham cohomology. For manifolds that are either not complete or not compact, no general relationship between the Hodge cohomology and a topological invariant is known, but there is a wealth of Hodge type theorems in various settings: on manifolds with cylindrical ends [2], on singular algebraic varieties [6], on locally symmetric spaces [18], [16], on asymptotically geometrically finite hyperbolic quotients [11], [12], and the well-known work of Cheeger [5] and Nagase [13] (see also [4]), which relates the Hodge cohomology of manifolds with iterated conical singularities with the intersection cohomology groups of Goresky and Macpherson, [8], [9]. These same intersection cohomology groups appear in the work of Hausel, Hunsicker, and Mazzeo [10] on the Hodge cohomology of fibred boundary and fibred cusp metrics, two natural geometries defined on a smooth manifold whose boundary is diffeomorphic to a fibration.

In this talk, we discuss an extension of the results of Hausel, Hunsicker, and Mazzeo to the more general case where the boundary is diffeomorphic to a Seifert fibration. A Seifert fibration is, loosely speaking, a foliation whose space of leaves is an orbifold. On such manifolds, fibred boundary and fibred cusp metrics have natural analogues: *foliated* boundary and *foliated* cusp metrics as introduced in [15], and it is these metrics whose Hodge cohomology we study here. Furthermore, we assume that the Seifert fibration is *good*, meaning that the space of leaves of the boundary foliation is the quotient of a smooth compact manifold by a smooth, properly discontinuous action of a finite group.

Thus, let M be a non-compact manifold inside a compact, smooth manifold with boundary $\overline{M}$, whose boundary $\partial M = \overline{M} - M$ is the total space of a Seifert fibration $\mathcal{F}$, for the moment not necessarily good. Let B denote the space of leaves of $\mathcal{F}$, and let $\pi\colon \partial M \longrightarrow B$ be the associated projection. Let x be a boundary defining function (b.d.f.) for ∂M, i.e. $x \in C^\infty(\overline{M})$ with $x^{-1}(0) = \partial M$, $dx|_{\partial M} \neq 0$ and $x > 0$ on M. For small $\epsilon > 0$, the set $x^{-1}([0, \epsilon))$ is diffeomorphic to $\partial M \times [0, \epsilon)_x$, and we extend the projection π to this neighborhood of the boundary in the obvious way. Then an exact **foliated boundary metric** is a Riemannian metric which on $\partial M \times [0,\epsilon)_x$ takes the form

$$g_{\mathcal{F}} = \frac{dx^2}{x^4} + \frac{\pi^* h}{x^2} + k, \tag{0.1}$$

where h is an orbifold Riemannian metric on B and k is a $(0,2)$-tensor which restricts to a Riemannian metric on each leaf of the foliation $\mathcal{F}$. Similarly, an exact **foliated cusp metric** is a metric of the form

$$g_{\mathcal{F}-c} := \frac{dx^2}{x^2} + h + x^2 k, \tag{0.2}$$

with h and k as above. General (i.e. non-exact) foliated metrics are permitted to have "cross-terms". For a choice of foliated boundary or foliated cusp metric g on M, let $L^2\mathcal{H}^k(M,g)$ denote the space of L^2 harmonic forms of degree k. To relate $L^2\mathcal{H}^k(M,g)$ with some topological data, we consider, instead of M, the space X obtained by collapsing the leaves of the foliation on ∂M onto the space of leaves B. To be precise, let

$$X := \overline{M}/\sim, \text{ where } p \sim q \iff p = q \text{ or } p,q \in \partial M \text{ with } \pi(p) = \pi(q). \tag{0.3}$$

There is a corresponding collapsing map $c_\pi : \overline{M} \to X$ which is the identity on M and is given by the projection π on ∂M. We identify B with $c_\pi(\partial M) \subset X$. The space X is a (smoothly) stratified space (see for instance [1] or [7] for a definition), and as such, the intersection homology and cohomology groups of Goresky and MacPherson [8] can be defined thereon. These groups are not homotopically invariant like singular homology and cohomology groups, but they are topological invariants. They are defined in terms of a perversity function, i.e. a map $\mathfrak{p}\colon \{0,1,\dots,n\} \longrightarrow \mathbb{N}$ satisfying $\mathfrak{p}(\ell) \le \mathfrak{p}(\ell+1) \le \mathfrak{p}(\ell)+1$, and a stratification, i.e. is a nested sequence $X \supset X_{n-2} \supset \cdots \supset X_j \supset \cdots \supset X_0$, where $X - X_{n-2}$ is a smooth manifold and $X_j - X_{j-1}$ is either empty or a manifold of dimension j. The group $IH^k_{\mathfrak{p}}(X)$ is the k^{th} cohomology group of the complex of cochains defined on chains which intersect each stratum of codimension ℓ in a set of dimension at most $k - \ell + \mathfrak{p}(\ell)$. The original intersection homology theory was developed for **standard** perversities, i.e. those satisfying the additional condition $\mathfrak{p}(0) = \mathfrak{p}(1) = 0$, but general perversities are now in common use, see e.g. [10], [14]. For standard perversities, the group $IH^k_{\mathfrak{p}}(X)$ turns out to be independent of the stratification. For more on intersection cohomology, see [3].

Let

$$\begin{aligned} n &:= \dim M \\ b &:= \dim B \\ f &:= n - b - 1, \end{aligned} \tag{0.4}$$

so f is the dimension of a typical leaf of $\mathcal{F}$. X carries a natural stratification induced by the orbifold structure of the space of leaves, B. In particular, the spaces $X_{n-2},\dots,X_{n-(f+1)}$ are all equal to B. In fact, *we show that the intersection* *cohomology of X depends only on $\mathfrak{p}(f+1)$.* Thus, the following definition makes sense. For $j \in \mathbb{N}$, let

$$IH^k_j(X,B) := \begin{cases} H^k(X-B) & \text{if} \quad j \le -1 \\ IH^k_{\mathfrak{p}}(X) \text{ where } \mathfrak{p}(f+1) = j. & \text{if} \quad 0 \le j \le f-1 \\ H^k(X,B) & \text{if} \quad f \le j, \end{cases} \tag{0.5}$$

c.f. Section 2.2, equation (9) of [10].

Our main theorems are the following

Theorem 7. *Let $\overline{M}$ be a manifold with boundary, $\partial M = \overline{M} - M$. Let $\mathcal{F}$ be a good Seifert fibration on ∂M, and let $g_{\mathcal{F}}$ be a foliated boundary metric on M. Then for any degree $0 \le k \le n$, there are natural isomorphisms*

$$L^2\mathcal{H}^k(M, g_{\mathcal{F}}) \longrightarrow \begin{cases} \operatorname{Im}\left(IH^k_{f+\frac{b+1}{2}-k}(X,B) \longrightarrow IH^k_{f+\frac{b-1}{2}-k}(X,B)\right) & b \text{ odd} \\ IH^k_{f+\frac{b}{2}-k}(X,B) & b \text{ even}, \end{cases}$$

with B as in (?) and X as in (0.3).

Theorem 8. *Notation and assumptions as in Theorem 7, let $g_{\mathcal{F}-c}$ be a foliated cusp metric on M. Then for $0 \le k \le n$, there is a natural isomorphism*

$$L^2\mathcal{H}^k(M, g_{\mathcal{F}-c}) \longrightarrow \operatorname{Im}\left(IH^k_{\underline{\mathfrak{m}}}(X,B) \longrightarrow IH^k_{\overline{\mathfrak{m}}}(X,B)\right)$$

where $\underline{\mathfrak{m}}$ and $\overline{\mathfrak{m}}$ are the lower middle and upper middle perversities.

References

[1] P. Albin, E. Leichtnam, R. Mazzeo, and P. Piazza, *The signature package on witt spaces*, available online at arXiv:1112.0989, 2011.

[2] M. F. Atiyah, V. K. Patodi, and I. M. Singer, *Spectral asymmetry and Riemannian geometry. I*, Math. Proc. Cambridge Philos. Soc. **77** (1975), 43–69.

[3] A. Borel and et al., *Intersection cohomology*, Modern Birkhäuser Classics, Birkhäuser Boston Inc., Boston, MA, 2008, Notes on the seminar held at the University of Bern, Bern, 1983, Reprint of the 1984 edition.

[4] J. Brasselet, G. Hector, M. Saralegi, *L^2−cohomologie des espaces statifiés*, Manuscripta Math, 76 (1992), 21-32.

[5] Jeff Cheeger, *On the Hodge theory of Riemannian pseudomanifolds*, Geometry of the Laplace operator (Proc. Sympos. Pure Math., Univ. Hawaii, Honolulu, Hawaii, 1979), Proc. Sympos. Pure Math., XXXVI, Amer. Math. Soc., Providence, R.I., 1980, pp. 91–146.

[6] Jeff Cheeger, Mark Goresky, and Robert MacPherson, *L^2-cohomology and intersection homology of singular algebraic varieties*, Seminar on Differential Geometry, Ann. of Math. Stud., vol. 102, Princeton Univ. Press, Princeton, N.J., 1982, pp. 303–340.

[7] C. Debord, J.-M. Lescure, and F. Rochon, *Pseudodifferential operators on manifolds with fibred corners*, available online at arXiv:1112.4575, 2011.

[8] Mark Goresky and Robert MacPherson, *Intersection homology theory*, Topology **19** (1980), no. 2, 135–162.

[9] ———, *Intersection homology. II*, Invent. Math. **72** (1983), no. 1, 77–129.

[10] Tamás Hausel, Eugenie Hunsicker, and Rafe Mazzeo, *Hodge cohomology of gravitational instantons*, Duke Math. J. **122** (2004), no. 3, 485–548.

[11] Rafe Mazzeo, *The Hodge cohomology of a conformally compact metric*, J. Differential Geom. **28** (1988), no. 2, 309–339.

[12] Rafe Mazzeo and Ralph S. Phillips, *Hodge theory on hyperbolic manifolds*, Duke Math. J. **60** (1990), no. 2, 509–559.

[13] Masayoshi Nagase, *Sheaf theoretic L^2-cohomology*, Complex analytic singularities, Adv. Stud. Pure Math., vol. 8, North-Holland, Amsterdam, 1987, pp. 273–279. MR 894298 (88g:58009)

[14] Arvind Nair, *Weighted cohomology of arithmetic groups*, Ann. of Math. (2) **150** (1999), no. 1, 1–31.

[15] Frédéric Rochon, *Pseudodidfferential operators on manifolds with foliated boundaries*, J. Funct. Anal. **262** (2012), 1309–1362.

[16] Leslie Saper and Mark Stern, *L_2-cohomology of arithmetic varieties*, Ann. of Math. (2) **132** (1990), no. 1, 1–69.

[17] Boris Vaillant, *Index and spectral theory for manifolds with generalized fibred cusps*, available online at arXiv: math/0102072v1, 2001.

[18] Steven Zucker, *L_2 cohomology of warped products and arithmetic groups*, Invent. Math. **70** (1982/83), no. 2, 169–218.

Poincaré Duality on non-Witt Spaces

Markus Banagl

Our talk reviewed some fundamental results of [1] and [2] on both local and global Poincaré duality for general singular spaces. In particular, these spaces do not have to satisfy Paul Siegel's Witt condition. Our results are valid on any oriented topological stratified pseudomanifold X — no assumptions on smoothness of pure strata, tubular neighborhoods of strata or structure groups are needed. We use a sheaf-theoretic framework and view differential complexes as objects of the derived category. Let $\mathbf{IC}^\bullet_{\bar{p}}(X)$ be Goresky-MacPherson's intersection chain sheaf for perversity $\bar{p}$ and real coefficients. The hypercohomology groups of this complex are the intersection homology groups $IH^{\bar{p}}_*$. If X is compact and has no boundary, there is a nondegenerate intersection pairing $IH^{\bar{p}}_*(X) \otimes IH^{\bar{q}}_{\dim X - *}(X) \to \mathbb{R}$, for complementary perversities $\bar{p}, \bar{q}$. The pairing is induced by a duality isomorphism $\mathbf{IC}^\bullet_{\bar{p}} \cong \mathcal{D}(\mathbf{IC}^\bullet_{\bar{q}})[\dim X]$, where $\mathcal{D}$ denotes the Verdier dualizing functor. There are two complementary middle perversities, the lower middle ($\bar{m}$) and the upper middle ($\bar{n}$) perversity. Siegel [5] calls X a *Witt space*, if the middle dimensional, lower middle perversity rational intersection homology of the links of all odd codimensional strata vanishes. In this case, the canonical morphism $\mathbf{IC}^\bullet_{\bar{m}} \to \mathbf{IC}^\bullet_{\bar{n}}$ is a quasi-isomorphism, so $\mathbf{IC}^\bullet_{\bar{m}}$ is self-dual. Thus one obtains an adequate generalization of Poincaré duality to such spaces, and in particular (provided $\dim X$ is even) a signature $\sigma(X)$ of X, and following the Thom-Milnor program, Hirzebruch L-classes.

In [1], we have developed methods to extend the above approach to obtaining generalized Poincaré duality and producing invariants for pseudomanifolds to spaces more general than Witt spaces. Consideration of examples such as the cone on complex projective space $\mathbb{CP}^2$ in the context of bordism invariance of the signature shows that an attempt to carry out this program will not be meaningful on the class of all pseudomanifolds. However, note that in this example, the link of the singularity has nonzero signature. This observation suggests that one should look at spaces having the property that the links of all singular strata of odd codimension "have signature zero." This last statement is not well-defined if the link is itself non-Witt, as it is not initially clear, in which theory such a signature has to be computed. Our results give an answer to this question.

If a space X has just one singular stratum, the links are all contained in the regular part of X, and it is clear what the signature of the links is. If we assume this signature to be zero, the middle dimensional homology of the links will contain a Lagrangian subspace (which need not be unique). The idea is to make a continuous choice of Lagrangian subspaces along the singular stratum, and to use this data in constructing a self-dual perverse sheaf $\mathbf{IC}^\bullet_L$ that interpolates between $\mathbf{IC}^\bullet_{\bar{m}}$ and $\mathbf{IC}^\bullet_{\bar{n}}$ via morphisms $\mathbf{IC}^\bullet_{\bar{m}} \to \mathbf{IC}^\bullet_L \to \mathbf{IC}^\bullet_{\bar{n}}$. We call such a Lagrangian subsheaf a *Lagrangian structure* along a stratum of odd codimension. (The definition that we actually work with is phrased in terms of an algebraic notion of nullcobordism of sheaves as introduced in [3], as this is better adapted to the triangulated structure of the derived category and to Verdier duality.) Lagrangian structures can be conveniently organized into a morphism category. For smoothly triangulated pseudomanifolds equipped with suitable conical metrics, the idea of trying to employ Lagrangian subspaces in order to obtain self-duality is present in an L^2-cohomology setting as J. Cheeger's "$*$-invariant boundary conditions," see e.g. [4]; also cf. P. Albin's talk at this workshop on his recent joint work with E. Leichtnam, R. Mazzeo and P. Piazza. The idea is also invoked in unpublished work of J. Morgan on the characteristic variety theorem.

In [1], we start out by defining the category $SD(X)$ of self-dual perverse sheaves on X. Our first main theorem gives a Postnikov-tower type decomposition of $SD(X)$ into a fibered product of categories of Lagrangian structures along the various strata of odd codimension. More precisely, we construct a functor that assigns Lagrangian structures to a self-dual sheaf, and another functor that builds self-dual sheaves from Lagrangian structures. We prove that these functors set up an equivalence of categories.

The self-duality isomorphism of a given $\mathbf{S}^\bullet \in \mathrm{Ob}SD(X)$ induces a signature $\sigma(\mathbf{S}^\bullet)$ (the signature of the quadratic form that Borel-Moore duality places on middle dimensional hypercohomology), but moreover, by a theorem of Cappell and Shaneson [3], characteristic L-classes $L_i(\mathbf{S}^\bullet) \in H_i(X;\mathbb{Q})$. In [2] we obtain the second main theorem on non-Witt spaces: the invariants $\sigma(\mathbf{S}^\bullet)$ and $L_*(\mathbf{S}^\bullet)$ are independent of the choice of $\mathbf{S}^\bullet \in \mathrm{Ob}SD(X)$, that is, independent of the choice of Lagrangian structures. Consequently, we obtain for pseudomanifolds X with $SD(X)$ not empty a well-defined bordism invariant signature $\sigma(X)$ and L-classes $L_*(X)$. For Witt spaces, these coincide with the previous constructions by Goresky, MacPherson and Siegel.

References

[1] M. Banagl, *Extending intersection homology type invariants to non-Witt spaces*, Memoirs Amer. Math. Soc. **160** (2002), no. 760, 1 – 83.

[2] M. Banagl, *The L-class of non-Witt spaces*, Annals of Math. **163** (2006), no. 3, 743 – 766.

[3] S. E. Cappell and J. L. Shaneson, *Stratifiable maps and topological invariants*, J. Amer. Math. Soc. **4** (1991), 521–551.

[4] J. Cheeger, *On the Hodge theory of Riemannian pseudomanifolds*, Proc. Sympos. Pure Math. **36** (1980), 91–146.
[5] P. H. Siegel, *Witt spaces: A geometric cycle theory for KO-homology at odd primes*, Amer. J. Math. **105** (1983), 1067–1105.

Differential algebraic K-theory

Ulrich Bunke
(joint work with David Gepner)

This is a report on the paper [BG] which is in preparation.

1. Differential algebraic K-theory

We consider a number ring R and form its algebraic K-theory spectrum $KR \in \mathtt{Sp}_\infty$. It represents a sheaf of spectra $\mathtt{Sm}(KR) \in \mathtt{Fun}^{desc}(\mathtt{Mf}^{op}, \mathtt{Sp}_\infty)$ on the site of smooth manifolds $\mathtt{Mf}$ with the open covering topology (see [Lur09] for the set-up). We further define the complex of sheaves $\Omega^{KR} \in \mathtt{Fun}^{desc}(\mathtt{Mf}^{op}, \mathtt{Ch}_\infty)$ of differential forms with coefficients in the graded group $\pi_*(KR) \otimes \mathbb{R}$. Using Borel's calculation [Bor74] of the latter we identify $\Omega^{KR,0}(M)$ with the subspace of $\Omega(M \times \mathtt{Spec}(R)(\mathbb{C}))^{\mathtt{Gal}(\mathbb{C}/\mathbb{R})}$ of those forms which satisfy:

(1) the degree zero component is constant along $\mathtt{Spec}(R)(\mathbb{C})$,
(2) the average over $\mathtt{Spec}(R)(\mathbb{C})/\mathtt{Gal}(\mathbb{C}/\mathbb{R})$ of the degree-one component vanishes.

The Eilenberg-MacLane equivalence $H : \mathtt{Ch}_\infty \xrightarrow{\sim} \mathtt{Mod}(H\mathcal{Z})$ (see [Shi07]) and the forgetful map $\mathtt{Mod}(H\mathcal{Z}) \to \mathtt{Sp}_\infty$ induce a map $\mathtt{Ch}_\infty \to \mathtt{Sp}_\infty$ (also denoted with H). The Borel regulator $b : \mathtt{Sm}(KR) \to H(\Omega^{KR})$ is a morphism of sheaves of spectra which induces an equivalence after realification of KR. In the following we make b explicit.

A locally constant sheaf $\mathcal{V} \in \mathtt{Loc}_R(M)$ of finitely generated R-modules naturally induces a class $[\mathcal{V}] \in KR^0(M)$. We consider $\mathcal{V}$ as a sheaf of locally free $\mathrm{pr}^*_{\mathtt{Spec}(R)} \mathcal{O}_{\mathtt{Spec}(R)}$-modules on $M \times \mathtt{Spec}(R)$. Its pull-back (over $\mathtt{Spec}(\mathcal{Z})$) is the sheaf of parallel sections of a $\mathtt{Gal}(\mathbb{C}/\mathbb{R})$-equivariant flat complex vector bundle (write ∇ for the connection) on $M \times \mathtt{Spec}(R)(\mathbb{C})$. By definition, a geometry on $\mathcal{V}$ is the choice of an invariant hermitean metric $h^{\mathcal{V}}$ on that bundle. Using the transgression of the Chern character form $\tilde{\mathbf{ch}}$ we define a form

$$\tilde{\omega}(h^{\mathcal{V}}) := \dim(\mathcal{V}) + \tilde{\mathbf{ch}}(\nabla^*, \nabla) \in \Omega^{KR,0}(M)$$

(in the explicit description of the latter given above). The Borel regulator is then fixed by the condition that $b([\mathcal{V}]) = [\omega(h^{\mathcal{V}})]$ in $H^0(\Omega^{KR}(M))$.

Let $\Omega^{KR,0}_{cl} \in \mathtt{Fun}^{desc}(\mathtt{Mf}^{op}, \mathtt{Ch}_\infty)$ denote the sheaf of zero cycles.

Definition: *We define the differential algebraic K-theory sheaf*

$$\widehat{KR} \in \mathtt{Fun}^{desc}(\mathtt{Mf}^{op}, \mathtt{Sp}_\infty)$$

as the homotopy pull-back

$$\begin{array}{ccc} \widehat{KR} & \xrightarrow{R} & H(\Omega^{KR,0}_{cl}) \\ \downarrow I & & \downarrow \\ KR & \xrightarrow{b} & H(\Omega^{KR}) \end{array}$$

and set $\widehat{KR}^0 := \pi_0(\widehat{KR}) \in \mathtt{Fun}(\mathtt{Mf}^{op}, \mathtt{Ab})$.

This is an instance of a general theory of differential extensions of generalized cohomology theories. The maps R and I induces the usual structure maps of differential cohomology. Furthermore, we have a transformation $a : \Omega^{KR,-1} \to \widehat{KR}^0$.

We let $\mathtt{Loc}^{geom}_R \in \mathtt{Fun}(\mathtt{Mf}^{op}, \mathtt{Mon})$ be the functor which associates to a manifold the monoid under direct sum of pairs $[\mathcal{V}, h^{\mathcal{V}}]$.

Theorem: *There exists an additive cycle map*

$$\widehat{\mathtt{cycl}} : \mathtt{Loc}^{geom}_R \to \widehat{KR}^0$$

such that $I(\widehat{\mathtt{cycl}}[\mathcal{V}, h^{\mathcal{V}}]) = [\mathcal{V}]$ *and* $R(\widehat{\mathtt{cycl}}[\mathcal{V}, h^{\mathcal{V}}]) = \omega(h^{\mathcal{V}})$.

2. Differential Transfer

Let $\pi : W \to B$ be a proper submersion. Then we have a Becker-Gottlieb transfer [BG75]

$$\pi_! : KR^*(W) \to KR^*(B) \ .$$

A Riemannian structure g on π consists of a vertical metric and a horizontal distribution. It gives rise to an Euler form $\chi(g) \in \Omega^n(W, \Lambda_{W/B})$, where $\Lambda_{W/B}$ is the relative orientation bundle and $n := \dim(W) - \dim(B)$ (see [BGV92], Sec.1.6 and Ch.9). The following works for any differential extension of a generalized cohomology theory.

Theorem: *For a proper submersion equipped with a Riemannian structure g there exists a differential transfer*

$$\hat{\pi}_! : \widehat{KR}^0(W) \to \widehat{KR}^0(B)$$

such that

$$R(\pi_!(x)) = \int_{W/B} \chi(g) \wedge R(x) \ , \quad I(\pi_!(x)) = \pi_!(I(x)) \ ,$$

and

$$a(\int_{W/B} \chi(g) \wedge \alpha) = \hat{\pi}_!(a(\alpha))$$

for $x \in \widehat{KR}^0(W)$ and $\alpha \in \Omega^{KR,-1}(W)$. The differential transfer is compatible with pull-backs and compositions in a natural way.

Definition: *We define the topological index* $\mathrm{ind}^{top} : \mathtt{Loc}_R^{geom}(W) \to \widehat{KR}^0(B)$ *by*

$$\mathrm{ind}^{top} := \hat{\pi}_! \circ \widehat{\mathtt{cycl}} : \mathtt{Loc}_R^{geom}(W) \to \widehat{KR}^0(B) \ .$$

3. The analytic index and the transfer index conjecture

The metrics g and $h^{\mathcal{V}}$ induce, via Hodge theory, L^2-metrics $h^{R^i\pi_*(\mathcal{V})}$ on $R^i\pi_*(\mathcal{V}) \in \mathtt{Loc}_R(B)$ for all $i \geq 0$. Furthermore, we have a Bismut-Lott analytic torsion form $\mathcal{T}(\pi, g, \mathcal{V}, h^{\mathcal{V}}) \in \Omega^{KR,-1}(B)$ (see [BL95]) which satisfies

$$d\mathcal{T}(\pi, g, \mathcal{V}, h^{\mathcal{V}}) = \int_{W/B} \chi(g) \wedge \omega(h^{\mathcal{V}}) - \sum_{i \geq 0} (-1)^i \omega(h^{R^i\pi_*(\mathcal{V})}) \ .$$

Definition: *We define the analytical index* $\mathrm{ind}^{an} : \mathtt{Loc}_R^{geom}(W) \to \widehat{KR}^0(B)$ *by*

$$\mathrm{ind}^{an}[\mathcal{V}, h^{\mathcal{V}}] := \sum_{i \geq 0} \widehat{\mathtt{cycl}}[R^i\pi_*(\mathcal{V}), h^{R^i\pi_*(\mathcal{V})}] + a(\mathcal{T}(\pi, g, \mathcal{V}, h^{\mathcal{V}}))$$

We can now state the transfer index conjecture (TIC):

Conjecture: *We have the equality* $\mathrm{ind}^{top} = \mathrm{ind}^{an}$*.*

There are many consequences and special cases which can be verified independently. All this supports the validity of the TIC.

A consequence of the Dwyer-Weiss-Williams theorem [DWW03] is

Theorem: *We have the equality* $I \circ \mathrm{ind}^{top} = I \circ \mathrm{ind}^{an}$*.*

The following is a reformulation of the Bismut-Lott index theorem [BL95]:

Theorem: *We have the equality* $R \circ \mathrm{ind}^{top} = R \circ \mathrm{ind}^{an}$*.*

The Cheeger-Müller theorem [Che79], [Mül78] implies

Theorem: *If* $B = *$ *and* $h^{\mathcal{V}}$ *is parallel, then* $\mathrm{ind}^{top}[\mathcal{V}, h^{\mathcal{V}}] = \mathrm{ind}^{an}[\mathcal{V}, h^{\mathcal{V}}]$*.*

The following is a consequence of the construction by Beilinson of elements with known regulators in the algebraic K-theory of cyclotomic fields.

Theorem: *If* $\pi : W \to \mathbb{CP}^n$ *is a* $U(1)$*-principal bundle with Chern class* pc_1*,* $p \in \mathcal{Z}^+$ *a prime,* $R = \mathcal{Z}[\xi]/(1+\xi+\cdots+\xi^{p-1})$*,* $\mathcal{V}$ *is one-dimensional with holonomy* ξ*, and* $h^{\mathcal{V}}$ *is parallel, then* $\mathrm{ind}^{top}[\mathcal{V}, h^{\mathcal{V}}] - \mathrm{ind}^{an}[\mathcal{V}, h^{\mathcal{V}}]$ *is a torsion element.*

References

[BG75] J. C. Becker and D. H. Gottlieb. The transfer map and fiber bundles. *Topology*, 14:1–12, 1975.

[Bei86] A. A. Beilinson. Higher regulators of modular curves. In *Applications of algebraic K-theory to algebraic geometry and number theory, Part I, II (Boulder, Colo., 1983)*, volume 55 of *Contemp. Math.*, pages 1–34. Amer. Math. Soc., Providence, RI, 1986.

[BGV92] Nicole Berline, Ezra Getzler, and Michèle Vergne. *Heat kernels and Dirac operators*, volume 298 of *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer-Verlag, Berlin, 1992.

[BL95] Jean-Michel Bismut and John Lott. Flat vector bundles, direct images and higher real analytic torsion. *J. Amer. Math. Soc.*, 8(2):291–363, 1995.

[Bor74] Armand Borel. Stable real cohomology of arithmetic groups. *Ann. Sci. École Norm. Sup. (4)*, 7:235–272 (1975), 1974.

[BG] U. Bunke and D. Gepner. Differential function spectra and differential becker-gottlieb transfer with application to differential algebraic k-theory. in preparation.

[Che79] Jeff Cheeger. Analytic torsion and the heat equation. *Ann. of Math. (2)*, 109(2):259–322, 1979.

[DWW03] W. Dwyer, M. Weiss, and B. Williams. A parametrized index theorem for the algebraic K-theory Euler class. *Acta Math.*, 190(1):1–104, 2003.

[Lur09] Jacob Lurie. *Higher topos theory*, volume 170 of *Annals of Mathematics Studies*. Princeton University Press, Princeton, NJ, 2009.

[Mül78] Werner Müller. Analytic torsion and R-torsion of Riemannian manifolds. *Adv. in Math.*, 28(3):233–305, 1978.

[Shi07] Brooke Shipley. $H\mathbb{Z}$-algebra spectra are differential graded algebras. *Amer. J. Math.*, 129(2):351–379, 2007.

The renormalized volume

Colin Guillarmou

1. Abstract

The notion of volume for closed 3-dimensional hyperbolic manifolds is important since it carries some topological information. In the case of Einstein manifold, it turns out that the volume has also some interesting content, we shall discuss in particular the case of Poincaré-Einstein manifolds appearing in AdS-CFT correspondance.

Let us start with Poincaré-Einstein manifolds: an $(n+1)$-dimensional open manifold (M, g) is Poincaré-Einstein if there exists a smooth manifold with boundary $\overline{M}$ such that M is the interior of $\overline{M}$, the metric g is such that for any smooth boundary defining function x of $\partial\overline{M}$, $x^2 g$ extends as a smooth metric on $\overline{M}$, and finally

$$\mathrm{Ric}(g) = -ng.$$

It turns out that such metrics are complete and have sectional curvature converging to -1 at $\partial\overline{M}$. Particular cases of those are given by hyperbolic space $\mathbb{H}^{n+1}$ or its quotient by convex co-compact groups of isometries (for instance Schottky or quasi-fuchsian groups in $\mathrm{PSL}_2(\mathbb{C})$ when $n = 2$). We now assume n even. Certain boundary defining functions, called geodesic boundary defining functions, have interesting properties: they are those which satisfy $|d\log(x)|_g = 1$ near the boundary, and they parametrize the conformal class of $h_0 := (x^2 g)|_{T\partial\overline{M}}$ (called conformal infinity of (M, g)) in the sense that for any choice in the conformal class $[h_0]$ on $\partial\overline{M}$, there is a (unique near $\partial\overline{M}$) geodesic boundary defining function x such that $x^2 g|_{T\partial\overline{M}} = h_0$. For such x, the metric has an expansion in normal coordinates near the boundary induced by x of the form

$$g = \frac{dx^2 + h(x)}{x^2}, \quad h(x) = h_0 + x^2 h_2 + \cdots + x^n \log(x) K + x^n h_n + \mathcal{O}(x^{n+1})$$

where h_j, K are tensors on the boundary. The tensors h_j, K are all determined formally and locally by h_0 if $j < n$ but h_n is formally undetermined when solving the Einstein equation by Taylor expansion in x at ∂M. In a way, h_0 is the Dirichlet data for Einstein equation and h_n is Neumann data.

The renormalized volume was introduced by Henningson-Skenderis [5] and Graham [1] and is defined by a Hadamard finite part regularization (the volume is infinite for (M, g))

$$\mathrm{Vol}_R(M) = \mathrm{FP}_{\varepsilon\to 0} \int_{x>\varepsilon} 1 \ \mathrm{dvol}_g.$$

This volume is not intrinsic in the sense that it depends on the choice of x, or equivalently the choice of conformal representative in the conformal infinity $[h_0]$.

In dimension $n + 1 = 3$, we can check that as a function on the conformal class $[h_0]$, the renormalized volume satisfies the Polyakov formula

$$\mathrm{Vol}_R(M, e^{2\omega} h_0) - \mathrm{Vol}_R(M, h_0) = -\frac{1}{4} \int_{\partial\overline{M}} (|\nabla\omega|^2_{h_0} + \omega.\mathrm{Scal}_{h_0}) \mathrm{dvol}_{h_0}$$

for $\omega \in C^\infty(\partial\overline{M})$. For genus > 1 surfaces, there is a maximizer in the conformal class given by the -1 curvature metric (uniformization theorem on surfaces), it is then a natural choice in dimension $n + 1 = 3$ to choose define the renormalized volume with this maximized conformal representative. That way, if we know that there is a map Φ from conformal classes on $\partial\overline{M}$ (which amounts essentially to consider Teichmüller space after moding out by action of diffeos) to Poincaré-Einstein metrics on a given manifold M (in dim $n + 1 = 3$), then the renormalized volume can be seen as a function on Teichmüller space. From Ahflors-Bers theorem, it is known that for M a topological cylinder (ie. quasi-fuchsian 3-manifolds), such map Φ exists, and in fact by an extension of Marden, such map Φ exists as long as M carries one Poincaré-Einstein (here hyperbolic in dimension 3) metric. It was shown for the quasi-fuchsian case and Schottky cases (M is a handle-body) by Takhtajan-Teo [8] and Krasnov-Schlenker [6] that $\mathrm{Vol}_R(M)$ is a Kähler potential for the Weil-Petersson form on Teichmüller space. With S.Moroianu [3], we extended this result to all Poincaré-Einstein 3-manifolds using Chern-Simons theory: for each 3-manifold M admitting a Poincaré-Einstein, there is a Φ given by Marden from Teichmüller space of the boundary $\mathcal{T}(\partial\overline{M})$ to Poincaré-Einstein metrics on M, $\mathcal{PE}(M)$, and the renormalized volume is a Kähler potential for Weil-Petersson form. Therefore they all agree modulo pluri-harmonic functions on $\mathcal{T}(\partial\overline{M})$. Another feature here is that ends of Poincaré-Einstein manifolds, parametrized by

$(h_0, h_2 - \frac{1}{2}h_0)$ in dimension $n+1 = 3$, can be seen as cotangent vectors to $\mathcal{T}(\partial\overline{M})$: h_0 is a hyperbolic metric on $\partial\overline{M}$ thus a point on $\mathcal{T}(\partial\overline{M})$, and $h_2 - \frac{1}{2}h_0$ turns out to be trace free and divergence free with respect to h_0, thus a tangent vector to $\mathcal{T}(\partial\overline{M})$ at h_0. Now $h_0 \mapsto \Phi(h_0) \mapsto h_2 - \frac{1}{2}h_0$ is a section of $T^*\mathcal{T}(\partial\overline{M})$, and Mc-Mullen [7], Krasnov-Schlenker [6] proved that the graph of this section is a Lagrangian submanifold in $T^*\mathcal{T}(\partial\overline{M})$ (the symplectic form is the Liouville one), and generated by the function $\mathrm{Vol}_R(M)$.

In forthcoming work with S.Moroianu and J-M.Schlenker [4], we extend to a certain point this theory to higher dimensional cases, where the replacement of Teichmüller space is the set of conformal classes on a given manifold. This becomes of course infinite dimensional, but certain properties remain true. For instance the set of Poincaré-Einstein ends parametrized by pairs $(h_0, h_n - F(h_0))$ for a certain functional F can still be identified with the cotangent space of conformal structures, and the renormalized volume, when defined by a particular choice of conformal representative, is a generator of the Lagrangian space formed by pairs $(h_0, h_n - F(h_0))$ such that (h_0, h_n) correspond to a Cauchy data at infinity for Einstein equation. The choice in the conformal class has to be with $v_n =$ cst where v_n is the n-th term in the expansion of the volume form $\mathrm{dvol}_{h(x)} = \mathrm{dvol}_{h_0}(1 + \sum_j x^{2j} v_{2j} + o(x^n))$. Notice that v_n is the Q-curvature of Branson modulo a divergence term (see [2]).

References

[1] C.R.Graham, *Volume and area renormalizations for conformally compact Einstein metrics.* The Proceedings of the 19th Winter School "Geometry and Physics" (Srn, 1999). Rend. Circ. Mat. Palermo (2) Suppl. No. 63 (2000), 31-42.

[2] C.R. Graham, A. Juhl, *Holographic formula for Q-curvature.* Adv. Math. **216** (2007), no. 2, 841–853.

[3] C. Guillarmou, S.Moroianu, *Chern-Simons line bundle on Teichmüller space*, arXiv:1102.1981.

[4] C. Guillarmou, S.Moroianu, J-M.Schlenker, *Renormalized volume in even dimension*, work in progress.

[5] M.Henningson, K. Skenderis, *The holographic Weyl anomaly* J. High Energy Phys. **7**, (1998), Paper 23.

[6] K. Krasnov, J.-M. Schlenker, *On the renormalized volume of hyperbolic 3-manifolds*, Comm. Math. Phys. **279** (2008), no. 3, 637–668.

[7] C. T. McMullen, *The moduli space of Riemann surfaces is Kähler hyperbolic*, Ann. Math. (2) **151** (2000), no. 1, 327–357.

[8] L. A. Takhtajan, L.-P. Teo, *Liouville action and Weil-Petersson metric on deformation spaces, global Kleinian reciprocity and holography.* Comm. Math. Phys. **239** (2003), no. 1–2, 183–240.

Regularized cohomology of a non-compact asymptotically Kähler *G*-manifold

Maxim Braverman

If E is a holomorphic vector bundle over a campact Kähler manifold, the Dobeault cohomology $H^{0,\bullet}(M,E)$ is finite dimensional and has a lot of nice properties. If M is non-compact, $H^{0,\bullet}(M,E)$ is an infinite dimensional space and much less is known about it. In this talk we consider a Hamiltonian action of a compact Lie group G on a non-compact manifold M and assume that E is a G-equivariant holomorphic vector bundle over M. If the moment map μ for this action is proper and the vector field induced by μ does not vanish outside of a compact subset of M, we construct a new regularized Dolbeault cohomology space $H^{0,\bullet}_{\mathrm{reg}}(M,E)$. It is still infinite dimensional. But as a representation of G it decomposes into a direct sum of irreducible components and each component appears in this decomposition finitely many times:

$$H^{0,p}_{\mathrm{reg}}(M,E) \;=\; \sum_{V\in \mathrm{Irr}\ G} \beta^{p}_{\mathrm{reg},V}\cdot V, \qquad p=0,\dots,n. \tag{0.1}$$

The alternating sum of the regularized cohomology is equal to the regularized index of the pair (E,μ) which was introduced in [1] (see also [4]).

The regularized cohomology (0.1) behaves in many respects as the Dolbeault cohomology of a compact manifold. In [3] we prove an analogue of the Kodaira vanishing theorem for the regularized cohomology. In [2] we specialize to the case when $G=S^1$ is a circle group. In this case we prove an analogue of the holomorphic Morse inequalities of Witten [5] (see also [6]).

0.1. **The assumptions.** The construction of the regularized cohomology is done under the following two assumptions:

(1) The moment map μ is proper;
(2) Via a G-invariant scalar product on the Lie algebra $\mathfrak{g}$ of G, μ induces a map $\mathbf{v}: M\to\mathfrak{g}$. Let v denote the vector field on M associated to this map. We assume that this vector field does not vanish outside of a compact subset K of M.

The assumption (1) above is rather restrictive. It excludes, for example, the action of the circle group S^1 on $\mathbb{C}^n$, which has both positive and negative weights. Unfortunately it is not clear how to define the regularized cohomology without this condition for the general compact Lie group G. However, in [2] we consider the case when $G=S^1$ is a circle group and in this case extend the definition of the regularized cohomology to the situation when the moment map is not necessarily proper.

0.2. **The construction of the regularized cohomology.** We define the regularized cohomology as the reduced cohomology of a certain deformation of the Dolbeault differential $\bar\partial$. To construct this deformation we first choose a smooth strictly increasing function $s:[0,\infty)\to[0,\infty)$. We call such a function *admissible* if it satisfies a rather technical growth condition at infinity. Luckily the cohomology of deformed Dolbeault operators constructed using different choices of s are naturally isomorphic. So the regularized cohomology is essentially independent of the choice of s. It is important however to know that at least one admissible function s exists.

We now explain the construction of the regularized cohomology in more details. Let $s:[0,\infty)\to[0,\infty)$ be an admissible function. Set

$$\phi(x) \;:=\; s\big(\,|\mu(x)|^2/2\,\big) \qquad x\in M,$$

and consider the *deformed Dolbeault differential*

$$\bar\partial_s = \; e^{-\phi}\circ\bar\partial\circ e^{\phi}.$$

We view $\bar\partial_s$ as a densely defined operator on the space $L_2\Omega^{0,p}(M,E)$ of square-integrable differential forms with values in E and we define the *deformed Dolbeault cohomology* $H^{0,\bullet}_s(M,E)$ as the reduced cohomology of $\bar\partial_s$:

$$H^{0,p}_s(M,E) \;=\; \frac{\operatorname{Ker}\big(\,\bar\partial_s : L_2\Omega^{0,p}(M,E)\to L_2\Omega^{0,p+1}(M,E)\,\big)}{\overline{\operatorname{Im}\big(\,\bar\partial_s : L_2\Omega^{0,p-1}(M,E)\to L_2\Omega^{0,p}(M,E)\,\big)}}.$$

We show that for any two admissible functions s_1 and s_2 the deformed cohomology is naturally isomorphic. Thus we can define the *regularized cohomology* $H^{0,p}_{\rm reg}(M,E)$ as $H^{0,p}_s(M,E)$ for some admissible function s.

The space $H^{0,p}_{\rm reg}(M,E)$ decomposes as a sum of irreducible representations of G. We show that each irreducible representation of G appears in $H^{0,p}_{\rm reg}(M,E)$ with finite multiplicity:

$$H^{0,p}_{\rm reg}(M,E) \;=\; \sum_{V\in\operatorname{Irr}\,G}\beta^p_{{\rm reg},V}\cdot V.$$

0.3. **Kodaira-type vanishing theorem.** Let L be a positive G-equivariant line bundle over M. We prove the following extension of the Kodaira vanishing theorem to our non-compact setting: for every irreducible representation V of G there exists a integer $k_0>0$, such that for all $k\ge k_0$ the V-component of the background cohomology

$$H^{0,p}_{{\rm reg},V}(M,E\otimes L^{\otimes k}) \;=\; 0,$$

for all $p>0$.

References

[1] M. Braverman, *Index theorem for equivariant Dirac operators on noncompact manifolds*, K-Theory **27** (2002), no. 1, 61–101.

[2] ——, *Background cohomology of an S^1-equivariant holomorphic vector bundle over a non-compact Kähler manifold*, (In preparation).

[3] ——, *Regularized cohomology of a non-compact asymptotically Kähler G-manifold*, (Preprint).

[4] P.-É. Paradan, Spin^c*-quantization and the K-multiplicities of the discrete series*, Ann. Sci. École Norm. Sup. (4) **36** (2003).

[5] E. Witten, *Holomorphic Morse inequalities*, Algebraic and differential topology, Grund. Math. Wiss. 188, Teubner-Texte Math., 70, ed. G. Rassias, Teubner, Leipzig, 1984, pp. 318–333.

[6] S. Wu and W. Zhang, *Equivariant holomorphic Morse inequalities. III. Non-isolated fixed points*, Geom. Funct. Anal. **8** (1998), 149–178.

Minimal surfaces in $\mathbb{H}^3$: Willmore energy and bubbles.

Spyros Alexakis
(joint work with R. Mazzeo)

This lecture presented the recent joint work of the author with R. Mazzeo, [2]. This study focused on the Willmore energy considered on the space of minimal suraces on $\mathbb{H}^3$. We specificaly investiagate how the finiteness or smallness of this energy controls the boundary regularity of such surfaces. The optimal classical norm that can be controlled by this energy turns out to be the C^1 norm. We also investigate the mechanism for loss of convergence in C^1 for sequences of surfaces with energy uniformly bounded above.

The space of minimal surfaces in $\mathbb{H}^3$ with a prescribed boundary curve at infinity $\gamma \subset \partial_\infty \mathbb{H}^3$ was first studied by M. Anderson [3]. Important early results on regularity estimates up to the bounday for this Dirichlet problem were obtained starting with the work of Lin, [5]. Following this early work, a natural notion of renormalized area for such minimal suraces with smooth enough boundary at infinity was rigorously defined by Graham and Witten [4]. Our work commences with an observation in [1] that links the renormalized area Ren.Area$[Y]$ of a minimal surface $Y \subset \mathbb{H}^3$ with its total curvature, or Wilmore energy $E[Y] := \int_Y |A|^2 dV_Y$:

$$\text{Ren.Area}[Y] = -2\pi\chi(Y) - \frac{1}{2}\int_Y |A|^2 dV_Y.$$

The main question we wish to study here is the question of compactness of the space $M_{k,g}$ of complete minimal surfaces in $\mathbb{H}^3$ with k well-separated asymptotic ends and genus g, whose Willmore energy is bounded above uniformly. Specificaly, consider a sequence of suitable normalized surfaces Y_n with $E[Y_n] \leq M < \infty$. We study the possibility of (sub)convergence of such sequences and examine the mechanism responsible for lack of convergence in the C^1 norm.

The first main result in [2] shows that the Willmore energy of a minimal surface controls its regularity up to the boundary in the C^1 norm: Specifically we prove that given a portion of a minimal surface which lies in a half ball centered at $P \in \partial_\infty \mathbb{H}^3$ and with radius $R > 0$, if Y is graphical (via a function $u(x,y)$) over a vertical half-plane, then the $|u|_{C^1}$ can be made arbitrarily small provided the energy $E^{B(P,R)}[Y]$ of Y in that half-ball is correspondingly small. This has an interesting analytic content: Thought of in terms of the graph function $u(x,y)$, the condition of small energy for Y is slightly weaker than then smallness of the $W^{2,2}$ norm of u. Now, the Sobolev embedding guarantees that $W^{2,2}(D(0,1))$ embeds in C^α, for all $\alpha < 1$; however it is well-known that it *does not* embed into either C^1 or $C^{0,1}$. It is thus interesting that our minimal surfaces (whose graph functions satisfy a non-linear gedenerate elliptic PDE) exhibit *better* regularity up to the boundary than the Sobolev embedding guarantees. It is difficult to determine what lies at the heart of this extra regularity. It seems however to be a geometric non-linear phenomenon, closely tied to the fact that the boundary of our surfaces lies at infinity.

This culminates in two theorems concerning our main question: A suitable "global" measure of $C^{0,1}$ regularity for the boundary curve γ of a given minimal surface $Y \in M_{k,g}$ is the ζ-Lipschitz radius $\mathrm{LipRad}^{\zeta}_{\gamma}(Q)$ defined at each point $Q \in \gamma$; this is properly defined in [2]. We first show

Theorem 9. *There is a $\zeta_0, 0 < \zeta_0 < 1/20$ with the property that if $\zeta \in (0, \zeta_0)$, then there exists an $\epsilon(\zeta) > 0$ such that if $Y \in M_{k,g}$ and $E^{B(P,R)}(Y) < \epsilon(\zeta)$ for some $P \in \gamma = \partial_\infty Y$ and $R \leq 1$, then*

$$\mathrm{LipRad}^{\zeta}_{\gamma}(Q) \geq \zeta \cdot \frac{R - |PQ|}{10}$$

for all $Q \in \gamma'_{B(P,R)}$.

Our second result concerns a sequence of suitable normalized surfaces Y_n with $E[Y_n] \leq M < \infty$ and addresses the second part of the main question. The mechanism responsible for lack of convergence in the C^1 norm at a given point turns out to be the invariance of the Willmore energy under Möbius transformations. We can construct examples of lack of cinvergence as follows: The easiest examples are Möbius transoformations of a *fixed* minimal surface, such that the transformed surfaces converge (in the C^α norms but not in C^1) to a totally geodesic halfsphere: The Möbius transformations "push" the surface towards a chosen point at infinity, making all its energy disappear in the limit. One can use this idea and a gluing construction to give exaples of sequences of this loss of energy in the limit at any finite number of points for suitably constructed sequence of surfaces.

Our second theorem shows that this is the *only* reason for lack of convergence of Y_n to the limit Y_* in the C^1 norm: At any point in $P \in \partial_\infty Y_*$ where the convergence is not in C^1, there exists a sequence of blow-ups Ψ_n such that $\Psi_n(Y_n)$ converge to a surface $Y^\sharp$ with non-zero energy. In other words *prior* to the blow-p, the surfaces Y_n (locally near P) resembled a slightly perturbed $Y^\sharp$ "shrinking down" towards the pont P. Specificaly our result is as follows:

Theorem 10. *Let Y_j be a sequence of minimal surfaces in $M_{k,g}$ with $E(Y_j) \leq M < \infty$ and such that $Y_j \to Y_*$ where Y_* is C^1 up to $\gamma_* \setminus \{P_1, \ldots, P_N\}$. After rotation and translation, Y_j is a horizontal graph $z = u_j(x, y)$ over the half-disc $\{x^2 + y^2 \leq \delta^2\}$, with $|\nabla u_j| \leq 2\zeta$ and $u_j \to u_*$ in C^∞ away from $\{x = 0\}$ and in $C^{0,\alpha}$ up to $\{x = 0\}$. Finally, suppose that for some $y_0 \in (-\delta, \delta)$, $\lim_{j\to\infty} \partial_y u_j(y_0, 0) \neq \partial_y u_*(y_0, 0)$. Setting $P_j = (0, y_0, u_j(y_0, 0))$, then there exists a sequence of interior points $Q_j \in Y'_{j,B(0,\delta)}$ with $Q_j \to P_j$ such that if Ψ_j is a hyperbolic isometry which maps Q_j to $(1, 0, 0)$, then $\Psi_j(Y_j) \to Y'_*$ for some complete minimal surface Y'_* with $E(Y'_*) > 0$.*

We close by presenting the main idea in the proof of Theorem 9. We argue by contradiction: If this were to fail, then one could construct a sequence of minimal surfaces, the energies of which vanish in the limit, but such that there is a jump in the tangent lines in the limit, for a given point on the boundary curves. To reach our contradiction, we wish to relate the slope of the tangent line at the boundary to information on a parallel curve in the interior of the surface and then use the known C^∞ convergence in the interior.

The relationship between derivative information in the interior and at the boundary, i.e. the difference between the 'horizontal' derivatives at height 0 and 1, say, is given by integrating the mixed second derivative of the graph function along a vertical line and showing that this is controlled by the energy. To do this we must use a choice of 'gauge', which is a special isothermal coordinate system for which we have explicit pointwise control of the conformal factor. Using some deep results in harmonic analysis, such coordinate systems have been obtained for related problems, most importantly in a very influential paper by Müller and Sverak [6], We must modify those arguments to our setting, which requires a 'preparation' of our surface in a couple of ways being careful that none of the alterations we perform change the fact that there is a jump in first derivatives at the origin. Finally, in these isothermal coordinates, we use the fact that the mixed component A_{12} of the second fundamental form is a harmonic function; this is a special feature of minimal surfaces in constant curvature three-manifolds. This harmonicity leads, via a monotonicity formula, to the fact that this component decays somewhat more quickly than was known before, which leads eventually to partial control of the line integral mentioned above. In this argument there is a second line integral which it is necessary to control in terms of the energy of Y_j in a half-ball. This second line integral plays a crucial role in the later analysis of bubbling in the proof of theorem 10.

References

[1] S. Alexakis, R. Mazzeo *Renormalized area and properly embedded minimal surfaces in hyperbolic 3-manifolds* Comm. Math. Phys. **297** (2010), no. 3, 621–651.

[2] S. Alexakis, R. Mazzeo *The Willmore functional on complete minimal surfaces in H3: boundary regularity and bubbling*, arXiv:1204.4955 .

[3] M. Anderson *Complete Minimal Varieties in hyperbolic space*, Invent. Math. **69**, (1982), 477–494.

[4] C.R. Graham, E. Witten *Conformal Anomaly Of Submanifold Observables In AdS/CFT Correspondence* Nucl.Phys. B **546** (1999) 52–64.

[5] F.H. Lin *On the Dirichlet problem for the minimal graphs in hyperbolic space,* Invent. Math. **96**, 593-612 (1989).

[6] S. Müller, V. Sverák *On surfaces of finite total curvature* J. Differential Geom. **42** (1995), no. 2, 229–258.

Coarse index theory

Thomas Schick

(joint work with Bernhard Hanke, Paolo Piazza)

Coarse index theory uses methods from operator algebras and K-theory to study index problems in particular on non-compact manifolds and to analyse the geometry of such manifolds in this way. It is a beautiful theory, introduced in particular by John Roe, compare [5].

Given a complete Riemannian spin manifold X, the main players are:

- the pair of *coarse C^*-algebras* $C^*X \subset D^*X$ (C^*X is an ideal in D^*X), where both algebras are bounded operators on $L^2(X)$ of finite propagation (i.e. operators increasing the support only by a bounded distance). In C^*, the operators T in addition have to satisfy that $T\phi$ and ϕT are compact for each compactly supported multiplication operator $\phi \in C_c(X)$, whereas in D^*X the weaker condition of *pseudolocality* is required, i.e. $\phi T \psi$ has to be compact whenever $\phi, \psi \in C_c(X)$ have disjoint support.
- the second main player is the *coarse fundamental class*

  $$[D] \in K_{m+1}(D^*X/C^*X).$$

  Here, D is the Dirac operator. If m is odd, this is simply defined to be the class of the projector $(\chi(D)+1)/2$ for any odd function $\chi\colon \mathbb{R} \to \mathbb{R}$ with $\chi(t) \xrightarrow{t\to\pm\infty} \pm 1$ (by homotopy invariance, the resulting K-theory class does not depend on this choice). By finite propagation speed, $\chi(D)$ is an operator of finite propagation, elliptic regularity implies that $\chi(D) \in D^*(X)$ and that indeed $\chi(D)^2 - 1 \in C^*(X)$ which implies that $[D]$ really is a projection in the quotient algebra. For even m, one has to incorporate the splitting into even and odd spinors to define an interesting unitary representing $[D] \in K_{m+1}(D^*X/C^*X)$.
- The *coarse index* $ind_c(D) \in K_m(C^*X)$ is the image of $[D]$ under the boundary map of the six-term exact sequence for the pair $C^*X \subset D^*X$.
- If X has positive scalar curvature, then by the Weitzenböck formula, the spectrum of D does not contain an interval around zero. One can therefore choose χ to be constantly equal to ± 1 on the spectrum of D, and therefore $\chi^2(D) = 1 \in D^*X$. Consequently, the formula for the class $[D]$ canonically defines a class, the coarse ρ-invariant, $\rho_c(D) \in K_{m+1}(D^*X)$.
- If a group Γ acts isometrically on X, one restricts to the Γ-invariant parts C^*X^Γ and D^*X^Γ and everything above has corresponding generalizations.
- C^*X and D^*X are functorial for coarse continuous maps, in particular for proper uniformly Lipschitz maps.

Example. If X is compact, $C^*X \cong \mathcal{K}(L^2(X))$, $K_0(C^*X) = \mathbb{Z}$ and the above index is the usual Fredholm index.

If Γ acts freely and cocompactly on X, then $C^*X^\Gamma \cong C^*_{red}\Gamma \otimes \mathcal{K}$ and $K_*(C^*X^\Gamma) = K_*(C^*_{red}\Gamma)$. In this case, $ind_c(D) \in K_m(C^*_{red}\Gamma)$ is the Mishchenko-Fomenko index of the Dirac operator on the compact quotient, twisted by the Mishchenko line bundle.

Note that this gives an interesting homomorphism out of $C^*_{red}\Gamma$, namely to coarse algebra C^*X.

There are many tools for the calculation of the K-theory of the coarse C^*-algebras, in particular there is a Mayer-Vietoris sequence. Moreover, these groups vanish for spaces of the form $X\times[0,\infty)$. It follows that $K_m(C^*\mathbb{R}^n)\cong K_{m-n}(\mathbb{C})=\mathbb{Z}$ if $m-n$ even (and zero otherwise).

The talk was designed as an introductory and survey talk. Most of the time was spent on explaining this introduction.

Finally, two applications of coarse index theory were presented.

Theorem[Main Theorem of [1]] Assume that M is a closed spin manifold of dimension n such that the universal covering admits for each $\epsilon>0$ an ϵ-Lipschitz map to S^n which is constant outside a compact subset of $\tilde M$ and which has non-zero degree (then M is called *enlargeable*, this concept was introduced by Gromov and Lawson).

Then the Mishchenko-Fomenko index $ind(M)\in K_n(C^*_{red}\pi_1(M))$ is non-zero.

For the proof, uses the picture above and shows that even $ind_c(M)\in K_n(C^*\tilde M)$ is non-zero. This, indeed is obtained by manufacturing out of the maps $f_{1/k}$ one coarse map f to a special coarse space B_n, the *balloon space*. Using Mayer-Vietoris arguments one shows that $K_n(C^*B_n)=\prod_{k\in\mathbb{N}}\mathbb{Z}/\bigoplus_{k\in\mathbb{N}}\mathbb{Z}$. Moreover, using a refinement of Atiyah's L^2-index theorem and a proof of the coarse Baum-Connes conjecture for the space B_n one shows that the image of $ind_c(M)$ under f_* in $K_n(C^*B_n)$ is represented by the sequences of degrees of the maps $f_{1/n}$, therefore is non-zero i $K_n(C^*B_n)$.

The second application are certain coarse index theorems, established in [3]. In particular, one has a **delocalized coarse APS-index theorem**: If X is a spin manifold of even dimension $m+1$ with boundary Y and Y admits a metric of positive scalar curvature then (using the invertibility at the boundary) one can define $ind_c(X)\in K_{m+1}(C^*X)$. Its image in $K_{m+1}(D^*X)$ can be though of as a delocalized part of the index.

On the other hand, the positive scalar curvature metric on the boundary defines a coarse rho-class $\rho_c(Y)\in K_{m+1}(D^*Y)$.

The theorem now states that the delocalized coarse index is exactly the image of $\rho(Y)$ in $K_{m+1}(D^*X)$, using the inclusion $Y\to X$.

The similar statement holds with Γ-actions and for the Γ-invariant algebras.

This is **proved** in two steps: using "soft" methods from K-theory of operator algebras and properties of the functional calculus, one can reduce to a model situation involving only Y and $Y\times\mathbb{R}$. For this model situation, one has to carry out the calculation explicitly. This is strongly connected to a secondary partitioned manifold index theorem (a ρ_c variant of the partitioned manifold index theorem proved e.g. in [2] or [4], for the Γ-invariant case in [6]. The model calculation is highly non-trivial, indeed so far only achieved if m is odd. Of course, we expect that the theorem holds in general.

References

[1] Bernhard Hanke, Dieter Kotschick, John Roe, and Thomas Schick. Coarse topology, enlargeability, and essentialness. *Ann. Sci. Éc. Norm. Supér. (4)*, 41(3): 471–493, 2008.
[2] Nigel Higson. A note on the cobordism invariance of the index. *Topology*, 30(3):439–443, 1991.
[3] Paolo Piazza and Thomas Schick. Index theorems for ρ-classes and Stolz' positive scalar curvature sequence in preparation.
[4] John Roe. Partitioning noncompact manifolds and the dual Toeplitz problem. In *Operator algebras and applications, Vol. 1*, volume 135 of *London Math. Soc. Lecture Note Ser.*, pages 187–228. Cambridge Univ. Press, Cambridge, 1988.
[5] John Roe. *Index theory, coarse geometry, and topology of manifolds*, volume 90 of *CBMS Regional Conference Series in Mathematics*. Published for the Conference Board of the Mathematical Sciences, Washington, DC, 1996.
[6] Mostafa Esfahani Zadeh. Index theory and partitioning by enlargeable hypersurfaces. *J. Noncommut. Geom.*, 4(3):459–473, 2010.

The trace bundle of an elliptic wedge operator

Gerardo A. Mendoza
(joint work with Thomas Krainer)

Let $\mathrm{Diff}^m_e(\mathcal{M};E,F)$ be the class of edge differential operators of order m of Mazzeo [3], associated to a manifold $\mathcal{M}$ with boundary $\mathcal{N}=\partial\mathcal{M}$ and fibration $\wp:\mathcal{N}\to\mathcal{Y}$ with typical (compact) fiber $\mathcal{Z}$; E and F are vector bundles over $\mathcal{M}$. Let x be a defining function for $\mathcal{N}$, positive in $\mathring{\mathcal{M}}$. Let $A\in x^{-m}\mathrm{Diff}^m_e(\mathcal{M};E,F)$, $m>0$, be a wedge differential operator. The operator A has a natural principal symbol ${}^w\boldsymbol{\sigma}(A)$ off the zero section of the w-cotangent bundle ${}^w\pi:{}^wT^*\mathcal{M}\to\mathcal{M}$ of $\mathcal{M}$, a section of $\mathrm{Hom}({}^w\pi^*E,{}^w\pi^*F)$, see [1]. Suppose throughout the rest of this note that A is w-elliptic, which of course means that ${}^w\boldsymbol{\sigma}(A)$ is invertible everywhere. Equivalently, we assume that the principal symbol of x^mA is elliptic in the sense of [3].

Denote by $\pi_\wedge:\mathcal{N}^\wedge\to\mathcal{N}$ the inward pointing normal bundle of $\mathcal{N}$ including the zero section and by $\wp_\wedge$ the composition $\mathcal{N}^\wedge\to\mathcal{N}\to\mathcal{Y}$, so $\wp_\wedge:\mathcal{N}^\wedge\to\mathcal{Y}$ is a fibration with typical fiber $\mathcal{Z}\times\overline{\mathbb{R}}_+$. We write x also for the function $dx:\mathcal{N}^\wedge\to\mathbb{R}$ determined by the defining function for $\mathcal{N}$. This x is of course a defining function for $\partial\mathcal{N}^\wedge$, the zero section of $\mathcal{N}^\wedge$.

The indicial operator of $P=x^mA$, an operator in $\mathrm{Diff}^m_e(\mathcal{N}^\wedge;\pi_\wedge^*E,\pi_\wedge^*F)$, commutes with multiplication by elements of $\wp_\wedge^*C^\infty(\mathcal{Y})$ so it can be viewed as a family of elliptic b-operators ${}^bP_y$ on the fibers $\mathcal{N}^\wedge_y$ of $\wp_\wedge$. See [5] for the meaning of b-operators and the concept of ellipticity in that context, as well as the notion of boundary spectrum, $\mathrm{spec}_b({}^bP_y)$, to be used in the statement of the theorem below.

Pick $\mu \in \mathbb{R}$ arbitrarily. For $y \in \mathcal{Y}$ let $\mathcal{T}_y$ be the space whose elements u are functions on $\mathring{\mathcal{N}}^\wedge_y$ of the form

$$u = \sum_{\substack{\sigma \in \operatorname{spec}_b({}^b P_y) \\ \mu - m < \operatorname{Im}\sigma < \mu}} \sum_{\ell=0}^{N_\sigma} a_{\sigma,\ell} x^{i\sigma} \log^\ell x, \quad a_{\sigma,\ell} \in C^\infty(\wp^{-1}(y); E|_{\wp^{-1}(y)})$$

and satisfy ${}^b P_y u = 0$. Define

$$\mathcal{T} = \bigsqcup_{y \in \mathcal{Y}} \mathcal{T}_y, \quad \pi : \mathcal{T} \to \mathcal{Y} \text{ the natural map.}$$

If $U \subset \mathcal{Y}$ is open and u is a section of $\mathcal{T}$ over U (the meaning of which is clear), then u can be viewed as a section of $\pi_\wedge^* E$ over $\mathring{\wp}_\wedge^{-1}(U) \subset \mathcal{N}^\wedge \backslash \partial \mathcal{N}^\wedge$. Define $\mathcal{B}^\infty(U;\mathcal{T})$ as the space of sections of $\mathcal{T}$ which viewed thus are smooth over $\mathring{\wp}_\wedge^{-1}(U)$. Then $\mathcal{B}^\infty(U;\mathcal{T})$ is a module over $C^\infty(U)$, in particular, $\mathcal{B}^\infty(\mathcal{Y};\mathcal{T})$ is a module over $C^\infty(\mathcal{Y})$.

Theorem 11. *Suppose that the set*

$$\operatorname{spec}_e(A) = \{(y,\sigma) \in \mathcal{Y} \times \mathbb{C} : \sigma \in \operatorname{spec}_b({}^b P_y)\}.$$

is disjoint from $\{(y,\sigma) \in \mathcal{Y} \times \mathbb{C} : \operatorname{Im}\sigma = \mu,\ \mu - m\}$. *Then* $\mathcal{T} \to \mathcal{Y}$ *is a smooth vector bundle whose space of* C^∞ *sections is* $\mathcal{B}^\infty(\mathcal{Y};\mathcal{T})$.

The proof is given elsewhere (see [2]). The relevancy of the theorem lies in the well known interpretation of the indicial roots of the ${}^b P_y$ as being (after multiplication by the imaginary unit i) the leading powers of the (generalized) Taylor expansion at $x = 0$ of solutions of $Au = f$ when $u \in x^{-\mu} L^2_b(\mathcal{M};E)$ and $f \in x^{-\mu} L^2_b(\mathcal{M};F)$. Of course the difficulty of implementing this interpretation in a general setting lies in the possibility that the indicial roots in the range $\mu - m < \operatorname{Im}\sigma < \mu$ vary with y without constant multiplicity (this may even be the case if the location of the indicial roots is constant). Our theorem, or rather its proof, is one of the tools we use to handle this difficulty.

Complementing Theorem 11, we also show in [2], under the assumption that the normal family of A is invertible on its minimal domain (see [1] for the effect of this hypothesis on the nature of $\mathcal{D}_{\min}(A)$), how to construct a continuous operator $\mathcal{P} : H^m(\mathcal{Y};\mathcal{T}) \to \mathcal{D}_{\max}(A)$ with range complementary to $\mathcal{D}_{\min}(A)$ that admits a left inverse γ modulo smoothing a smoothing operator. These two tools allow us to give solid meaning to boundary value problems of the form

$$\begin{cases} Au = f \in x^{-\mu} L^2_b(\mathcal{M};F), & u \in H^m_A, \\ B\gamma u = g \end{cases}$$

where $H^m_A = \mathcal{D}_{\min}(A) \oplus \mathcal{P}(H^m(\mathcal{Y};\mathcal{T}))$, g is a section of some other given vector bundle over $\mathcal{Y}$, B is, for instance, a pseudodifferential operator acting on sections of $\mathcal{T}$, and of course the unkown is u.

We point out that Mazzeo and Vertman [4] have results starting with a slightly different set-up concerning boundary value problems for elliptic edge operators but assuming constancy of the indicial roots.

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References

[1] J. B. Gil, T. Krainer, and G. A. Mendoza, *On the closure of elliptic wedge operators*, to appear in J. Geom. Anal.
[2] T. Krainer and G. A. Mendoza, *The trace bundle of an elliptic wedge operator*, in preparation.
[3] R. Mazzeo, *Elliptic theory of differential edge operators I*, Comm. Partial Differential Equations **16** (1991), 1615–1664.
[4] R. Mazzeo and B. Vertman, personal communication.
[5] R. B. Melrose, The Atiyah-Patodi-Singer index theorem. Research Notes in Mathematics, 4. A K Peters, Ltd., Wellesley, MA, 1993.

New Proof of Juhl's Formulae for GJMS Operators and Q-curvatures

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(joint work with Charles Fefferman)

This is a report on joint work with Charles Fefferman in [2] giving a new proof of Juhl's formulae for GJMS operators and Q-curvatures. These remarkable formulae were discovered and proved by Andreas Juhl in [5], [6], building on previous work beginning with [4]. Juhl's approach was based on his theory of residue families and their factorization identities. The new proof proceeds directly from the original construction of [3]. It provides an explanation for the previously mysterious appearance of both Juhl's generating function $\mathcal{M}(r)$ for his building-block second order differential operators $\mathcal{M}_{2N}$, and the square root $W(r)$ of the volume ratio $V(r)$.

Juhl's formulae are expressed in terms of quantities arising in the expansion of the Poincaré metric in normal form determined by a given pseudo-Riemannian metric. Let g be a pseudo-Riemannian metric of signature (p,q), $p+q=n\geq 3$, on an n-dimensional manifold M, and let g_+ be a metric on $M\times(0,\epsilon)$ of the form

$$g_+ = r^{-2}\left(dr^2 + h_r\right),$$

where h_r is a smooth 1-parameter family of metrics on M satisfying $h_0 = g$. g_+ is required to satisfy $\mathrm{Ric}(g_+) + ng_+ = 0$ asymptotically in the sense that if n is odd, then $\mathrm{Ric}(g_+) + ng_+ = O(r^\infty)$, while if n is even, then $\mathrm{Ric}(g_+) + ng_+ = O(r^{n-2})$ and the tangential trace of $r^{2-n}\left(\mathrm{Ric}(g_+) + ng_+\right)$ vanishes at $r=0$. (See [1].) Set

$$V(r) = \sqrt{\frac{\det h_r}{\det h_0}}$$

and $W(r) = \sqrt{V(r)}$. Let δ denote the divergence operator on vector fields with respect to g, given by $\delta\varphi = \nabla_i\varphi^i$. Define a 1-parameter family $\mathcal{M}(r)$ of second order differential operators on M by

$$\mathcal{M}(r) = \delta(h_r^{-1}d) - U(r),$$

where

$$U(r) = \frac{\left[\partial_r^2 - (n-1)r^{-1}\partial_r + \delta(h_r^{-1}d)\right] W(r)}{W(r)}$$

acts as a zeroth order term. Use $\mathcal{M}(r)$ as a generating function for second order differential operators $\mathcal{M}_{2N}$ on M defined for $N \geq 1$ (and $N \leq n/2$ if n is even) by

$$\mathcal{M}(r) = \sum_{N\geq 1} \mathcal{M}_{2N} \frac{1}{(N-1)!^2} \left(\frac{r^2}{4}\right)^{N-1}.$$

The $\mathcal{M}_{2N}$ are natural scalar differential operators. Natural scalar invariants W_{2N} are defined by

$$W(r) = 1 + \sum_{N\geq 1} W_{2N} r^{2N}$$

for $N \geq 1$ (and $N \leq n/2$ if n is even).

Juhl's formulae involve constants n_I, m_I which are parametrized by ordered lists $I = (I_1, \ldots, I_r)$ of positive integers. I is referred to as a composition of the sum $|I| = I_1 + I_2 + \cdots + I_r$. The constants are:

$$n_I = (|I|-1)!^2 \prod_{j=1}^{r} \frac{1}{(I_j-1)!^2} \prod_{j=1}^{r-1} \frac{1}{\left(\sum_{k=1}^{j} I_k\right)\left(\sum_{k=j+1}^{r} I_k\right)}$$

$$m_I = (-1)^{r+1}|I|!(|I|-1)! \prod_{j=1}^{r} \frac{1}{I_j!(I_j-1)!} \prod_{j=1}^{r-1} \frac{1}{I_j + I_{j+1}}.$$

Empty products are interpreted as 1.

Let P_{2N} denote the GJMS operators, with sign convention determined by $P_{2N} = \Delta^N + \ldots$ with $\Delta = \delta(g^{-1}d)$. The Q-curvatures are defined in terms of the zeroth order terms of the GJMS operators: $P_{2N}(1) = (-1)^N\left(\frac{n}{2} - N\right)Q_{2N}$. The P_{2N} and Q_{2N} are defined for all $N \geq 1$ if n is odd and for $1 \leq N \leq n/2$ if n is even (an analytic continuation is involved for Q_{2N} for n even and $N = n/2$). Iterated compositions of the P_{2N} and the $\mathcal{M}_{2N}$ are denoted by $P_{2I} = P_{2I_1} \circ \cdots \circ P_{2I_r}$ and $\mathcal{M}_{2I} = \mathcal{M}_{2I_1} \circ \cdots \circ \mathcal{M}_{2I_r}$.

There are four formulae: an explicit formula and a recursive formula each for GJMS operators and for Q-curvatures. All four formulae hold for $N \geq 1$ (and $N \leq n/2$ if n is even), and are universal in the dimension.

Explicit formula for GJMS operators:

$$P_{2N} = \sum_{|I|=N} n_I \mathcal{M}_{2I}.$$

Recursive formula for GJMS operators:

$$P_{2N} = -\sum_{\substack{|I|=N\\ I\neq(N)}} m_I P_{2I} + \mathcal{M}_{2N}.$$

Explicit formula for Q-curvatures:

$$(-1)^N Q_{2N} = \sum_{|(I,a)|=N} n_{(I,a)} a!(a-1)!2^{2a}\mathcal{M}_{2I}(W_{2a}).$$

Recursive formula for Q-curvatures:

$$(-1)^N Q_{2N} = -\sum_{\substack{|(I,a)|=N\\ a<N}} m_{(I,a)}(-1)^a P_{2I}(Q_{2a}) + N!(N-1)!2^{2N}W_{2N}.$$

In the formulae for Q-curvatures, the convention for compositions written in the form (I,a) is that I is allowed to be empty but $a>0$.

The derivation of these formulae in [2] proceeds by first giving direct proofs of the explicit formulae for GJMS operators and Q-curvatures. This is discussed below. The recursive formula for GJMS operators follows from the explicit formula for GJMS operators using an inversion argument due to Krattenthaler presented in [6]. The recursive formula for Q-curvatures is derived from the explicit formula for Q-curvatures and the recusive formula for GJMS operators by a more complicated analogue of this inversion argument.

The explicit formulae are derived via the ambient metric in normal form determined by g, which is equivalent to the Poincaré metric. The original construction of $P_{2N}f$ in [3] applies a power of the ambient Laplacian $\widetilde{\Delta}$ to a homogeneous ambient extension of f. It is shown in [2] that in this construction, $\widetilde{\Delta}$ can be replaced by $\widetilde{\Delta}_v := v^{1/2}\circ\widetilde{\Delta}\circ v^{-1/2}$, where $v(\rho)=V(r)$ with $\rho=-r^2/2$. The introduction of $\widetilde{\Delta}_v$ is motivated by an attempt to find a direct proof in terms of the GJMS construction of the self-adjointness of the P_{2N}. It turns out that the generating function $\mathcal{M}(r)$ appears naturally when $\widetilde{\Delta}_v$ is applied to a homogeneous function on the ambient space. It is an immediate consequence that P_{2N} can be written as a linear combination of the compositions $\mathcal{M}_{2I}$. The identification of the coefficients in the linear combination as the n_I reduces to a rather nontrivial combinatorial identity which is proved in [2]. A direct derivation of the explicit formula for Q_{2N} is given using a variant of the argument used to derive the explicit formula for the P_{2N} together with a variant of the combinatorial identity.

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References

[1] C. Fefferman and C. R. Graham, *The Ambient Metric*, Annals of Mathematics Studies 178, Princeton University Press, 2012, arXiv:0710.0919.

[2] C. Fefferman and C. R. Graham, *Juhl's formulae for GJMS operators and Q-curvatures*, arXiv:1203.0360.

[3] C. R. Graham, R. Jenne, L.J. Mason and G. A. J. Sparling, *Conformally invariant powers of the Laplacian, I: Existence*, J. London Math. Soc. **46** (1992), 557–565.
[4] A. Juhl, *Families of Conformally Covariant Differential Operators, Q-Curvature and Holography*, Progress in Mathematics 275, Birkhäuser, 2009.
[5] A. Juhl, *On the recursive structure of Branson's Q-curvature*, arXiv:1004.1784.
[6] A. Juhl, *Explicit formulas for GJMS-operators and Q-curvatures*, arXiv:1108.0273.

Asymptotic Expansion of Bergman Kernel for Orbifolds

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(joint work with Kefeng Liu, Xiaonan Ma)

§1. Bergman Kernel and Donaldson Theorem

The Bergman kernel in the context of several complex variables (i.e. for pseudoconvex domains) has long been an important subject. Its analogue for complex projective manifolds is studied by Tian, Zelditch, Catlin, Lu, etc., establishing the diagonal asymptotic expansion for high powers of an ample line bundle. Moreover, the coefficients in the asymptotic expansion encode geometric information of the underlying complex projective manifolds. This asymptotic expansion plays a crucial role in the seminal work of Donaldson where the existence of Kähler metrics with constant scalar curvature is shown to be closely related to Chow-Mumford stability.

More precisely, let (X, L) be compact polarized Kähler manifold. That is, X is a compact Kähler manifold with Kähler form ω and $L \longrightarrow X$ a holomorphic line bundle whose curvature is $-2\pi\sqrt{-1}\omega$. For any $p \in \mathbb{N}$, choose an orthonormal basis $s_\alpha(x)$ of $H^0(X, L^p)$, the (diagonal) Bergman kernel is the smooth function

$$B_p(x) = \sum_\alpha |s_\alpha(x)|^2. \tag{0.1}$$

Donaldson Theorem relates the existence of Kähler metrics with constant scalar curvature to that of balanced metric, i.e., those with Bergman kernel $B_p(x) = c$ a constant, which in turn relates to the GIT notion of stability. It is an interesting question whether analog of Donaldson's result holds for orbifolds.

§2. Asymptotic Expansion of Bergman Kernel—Agmon Type Estimate

Playing a crucial role in the proof of Donaldson's theorem is the Tian-Yau-Zelditch asymptotic expansion for the Bergman kernel. Namely, as $p \to \infty$,

$$B_p(x) \sim p^n + b_1(x)p^{n-1} + \cdots , \tag{0.2}$$

where the coefficient b_1 has been identified by Lu: $b_1 = \frac{1}{8}S(\omega)$ is given by the scalar curvature of ω.

In [1] we have established a global Agmon type estimate for the off-diagonal behavior of the Bergman kernel. As a consequence, we derive an asymptotic expansion for Bergman kernel on orbifolds. Moreover, we gave an explicit description of the singular behavior as one approaches the singularity.

Our global Agmon estimate is best described in local coordinates $Z \in X$. Let $D_p = \sqrt{2}(\overline{\partial} + \overline{\partial}^*) = \sum c(e_i)\nabla_{e_i} : \Omega^{0,*}(X, L^p) \longrightarrow \Omega^{0,*}(X, L^p)$. Let P_p be the orthogonal projection from $\Omega^{0,*}(X, L^p)$ onto $\ker D_p$, and $P_p(Z, Z')$ $(Z, Z' \in X)$ be the smooth kernels of P_p with respect to the Riemannian volume form.

Theorem 1 (Dai-Liu-Ma). *For any $k, m \in \mathbb{N}$, there exist $N \in \mathbb{N}, C > 0, C' > 0$ such that for $\alpha, \alpha' \in \mathbb{N}^n$, $|\alpha| + |\alpha'| \le m$,*

$$\left| \frac{\partial^{|\alpha|+|\alpha'|}}{\partial Z^{\alpha} \partial Z'^{\alpha'}} \left(\frac{1}{p^n} P_p(Z, Z') - \sum_{r=0}^{k} P^{(r)}(\sqrt{p}Z, \sqrt{p}Z')\kappa^{-1}(Z')p^{-r/2} \right) \right| \\ \le Cp^{-(k+1-m)/2}(1 + |\sqrt{p}Z| + |\sqrt{p}Z'|)^N \exp(-C'\sqrt{p}|Z - Z'|),$$

where κ is the ratio of the Riemannian volume element and the Euclidean volume element, and $P^{(r)}$ is explicitly computable.

Here $P^{(r)}$ is computed via the following rescaling analysis of the local problem. For a smooth section $s(Z)$, set

$$(S_t s)(Z) = s(Z/t), \quad L_2^t = S_t^{-1} t^2 D_p^2 S_t.$$

Then $L_2^t = L_2^0 + tQ_1 + t^2 Q_2 + \cdots$, with Q_i second order differential operators. As L_2^0 is essentially a generalized harmonic oscillator, its heat kernel is given by

$$e^{-uL_2^0}(Z, Z') = \frac{1}{(1 - e^{-4\pi u})^n} \exp\Big(- \frac{\pi(|Z|^2 + |Z'|^2)}{2\tanh(2\pi u)} + \frac{\pi \left\langle e^{-2\sqrt{-1}\pi u J} Z, Z' \right\rangle}{\sinh(2\pi u)} \Big) e^{-2u\omega_{d,0}},$$

where $\omega_{d,0} = -\sum_{l,m} R_0^L(w_l, \overline{w}_m)\, \overline{w}^m \wedge i_{\overline{w}_l}$. Taking u to infinity, we obtain the Bergman kernel for the model operator

$$\mathcal{P}(Z, Z') = e^{-\frac{\pi}{2}\sum(|z_i|^2 + |z_i'|^2 - 2z_i \bar{z}_i')}.$$

Now set

$$J_{r,u} = \sum (-1)^j \int_{u\Delta_j} e^{-(u-u_j)L_2^0} Q_{r_j} e^{-(u_j - u_{j-1})L_2^0} \cdots Q_{r_1} e^{-u_1 L_2^0} du_1 \cdots du_j \tag{0.3}$$

where the summation runs over $\sum_{i=1}^j r_i = r$, $r_i \ge 1$. Then

$$J_{r,u}(Z, Z') = \sum_{|\beta|+|\beta'| \le 3r} J_{r,\beta,\beta'}(u) Z^{\beta} Z'^{\beta'} e^{-uL_{2,\mathbb{C}}^0}(Z, Z'). \tag{0.4}$$

Finally,

$$P^{(r)}(Z, Z') = \sum_{\beta,\beta'} J_{r,\beta,\beta'}(\infty) Z^{\beta} Z'^{\beta'} \mathcal{P}(Z, Z'). \tag{0.5}$$

Now let (X, ω) be a compact Kähler orbifold of real dimension $2n$. By definition, for any $x \in X$, there exists a small neighborhood $U_x \subset X$, a finite group $G_x \subset$

$GL(n,\mathbb{C})$, and $\widetilde{U}_x \subset \mathbb{C}^n$ an G_x-open set such that $\widetilde{U}_x \overset{\tau_x}{\to} \widetilde{U}_x/G_x = U_x$ and $\{0\} = \tau_x^{-1}(x) \in \widetilde{U}_x$.

For $g \neq id \in G$ we denote $\widetilde{U}^g$ the fixed point set of g. Then $\widetilde{Z} = \widetilde{Z}_{1,g} + \widetilde{Z}_{2,g}$ with $\widetilde{Z}_{1,g} \in T\widetilde{U}^g$, $\widetilde{Z}_{2,g} \in N_g$ (the normal bundle to $\widetilde{U}^g$ in $\widetilde{U}$). Further, the g-action on L is given by multiplication by $e^{\sqrt{-1}\theta_g}$, with θ_g being locally constant on $\widetilde{U}^g$.

Then our Agmon estimate implies the following precise singular behavior of the Bergman kernel near the orbifold singularity.

$$(0.6)\quad \sup_{|\alpha|\leq m'} \left| \frac{\partial^{|\alpha|}}{\partial \widetilde{Z}^\alpha} \Big(\frac{1}{p^n} P_p(\widetilde{Z},\widetilde{Z}) - \sum_{r=0}^{k} b_r(\widetilde{Z}) p^{-r} - \sum_{r=0}^{2k} p^{-\frac{r}{2}} \sum_{1\neq g\in G} e^{\sqrt{-1}\theta_g p} K^{(r)}_{\widetilde{Z}_{1,g}}(\sqrt{p}\widetilde{Z}_{2,g}) e^{-2\pi p \langle (1-g^{-1})\tilde{z}_{2,g}, \bar{\tilde{z}}_{2,g}\rangle} \Big) \right|$$
$$\leq C\left(p^{-k-1} + p^{-k+\frac{m'-1}{2}} \left(1+\sqrt{p}d(Z,X')\right)^N \exp\left(-C'\sqrt{p}d(Z,X')\right)\right).$$

Here $K^{(r)}_{\widetilde{Z}_{1,g}}$ is a polynomial in $\widetilde{Z}_{2,g}$ of degree $\leq 3r$ and $K^{(0)}_{\widetilde{Z}_{1,g}}(\sqrt{p}\widetilde{Z}_{2,g}) = 1$.

§3. Recent Work of Ross-Thomas

In very interesting recent work [5, 6], Ross-Thomas describes a notion of ampleness for line bundles on Kähler orbifolds with cyclic quotient singularities which is related to embeddings in weighted projective spaces. In this case, Ross-Thomas is able to prove an orbifold version of Donaldson Theorem. They introduced weighted Bergman kernels on orbifolds, which plays an important role there.

A weighted Bergman kernel is

$$B_p^{orb} = \sum_i c_i B_{p+i},$$

where c_i's are positive constants and i runs over some fixed finite index set of nonnegative integers.

Let (X,ω) be a compact n-dimensional Kähler orbifold with cyclic quotient singularities, i.e., the stabilizer group G_x is a cyclic group for any $x \in X$. Let L be an ample orbifold line bundle on X such that for any $x \in X$, the stabilizer group G_x acts on $L_{\tilde{x}}$ as $\mathbb{Z}_{|G_x|}$-order cyclic group.

Let m be the lowest common multiple of the orders of the stabilizer groups of all points of X. Fix $N, r \geq 0$. Consider the c_i which are defined by

$$(z^{m-1} + z^{m-2} + \cdots + 1)^{N+r+1} = \sum_i c_i z^i.$$

Theorem 2 (Ross-Thomas, Dai-Liu-Ma)**.** *For the above choice of c_i, the weighted Bergman kernel B_p^{orb} admits a global C^{2r}-expansion of order N:*

$$B_p^{orb} = \sum_{j=0}^{N} b_j p^{n-j} + O_{C^{2r}}(p^{n-N-1}).$$

Furthermore $b_0 = \sum_i c_i, \quad b_1 = \sum_i c_i \left(ni + \frac{1}{8\pi} S(\omega)\right).$

This result was first proved by Ross-Thomas [5] and rederived in [2] with a slight better regularity. The proof of [2] comes directly from (0.6). We would also like to remark that one can use (0.6) to establish the Berezin-Toeplitz quantization for orbifolds.

References

[1] X. Dai, K. Liu, and X. Ma, On the asymptotic expansion of Bergman kernel, J. Differential Geom. 72(2006), 1–41.

[2] X. Dai, K. Liu, and X. Ma, A remark on weighted Bergman kernels on orbifolds, Math. Research Letters. 19(2012), 143-148.

[3] X. Ma and G. Marinescu, Holomorphic Morse inequalities and Bergman kernels, Progress in Mathematics, vol. 254, Birkhäuser Boston Inc., Boston, MA, 2007, 422 pp.

[4] X. Ma and G. Marinescu, Generalized Bergman kernels on symplectic manifolds, Adv. Math. 217 (2008), no. 4, 1756–1815.

[5] J. Ross and R. Thomas, Weighted Bergman kernels on orbifolds. J. Differential Geom. 88(2011), 87–108.

[6] J. Ross and R. Thomas, Weighted projective embeddings, stability of orbifolds, and constant scalar curvature Kähler metrics, J. Differential Geom. 88(2011), 109–160.

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