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Reductions of Shimura Varieties

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Abstract. The workshop brought together leading experts in the theory of reductions of Shimura varieties. The talks presented new methods and results that intertwine a multitude of topics such as geometry and cohomology of moduli spaces of abelian varieties, p-divisible groups and Drinfeld shtukas, p-adic Hodge theory, arithmetic intersections of cycles on Shimura varieties, Bruhat-Tits buildings, and p-adic automorphic forms.

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Introduction by the Organisers

The workshop *Reduction of Shimura varieties* was organized by Laurent Fargues (Strasbourg), Ulrich Görtz (Essen), Eva Viehmann (München) and Torsten Wedhorn (Paderborn). It was attended by 27 participants with broad geographic representation, including a number of young participants. The programme included 18 talks of 60 minutes each.

Arithmetic properties of Shimura varieties which are encoded in their reduction to positive characteristic are an exciting topic which has contributed to some of the most spectacular developments in number theory and arithmetic geometry in the last fifteen years.

It is closely related to the Langlands program (classical as well as p-adic). A particular case is given by moduli spaces of abelian varieties, which are a classical object of study in algebraic geometry. Via automorphic forms (again classical as well as p-adically), there is an important connection to number theory. Moduli spaces of p-divisible groups are also closely related. All these topics were present in talks of the workshop.

All participants immensely enjoyed the unique environment provided by the Mathematisches Forschungsinstitut Oberwolfach. The organizers intend to propose another workshop on the same topic for 2015.

In the last few years reductions of PEL Shimura varieties attached to unitary groups of signature $(1, n-1)$ have been studied intensely. One main motivation is the Kudla program which predicts a relation between the intersection numbers of certain arithmetic cycles and Fourier coefficients of the derivative of certain incoherent Eisenstein series. Here the supersingular locus of these Shimura varieties is of particular interest. It is uniformized by the corresponding Rapoport-Zink space. For inert (unramified) primes Kudla and Rapoport proved such a relation with the derivative of certain incoherent Eisenstein series if the intersection of the cycles is non-degenerate. U. Terstiege presented in his talk *On the regularity of special difference divisors* his progress in the degenerate case. He explained that it suffices to consider so-called special difference divisors and proved that these divisors are regular. It was also explained how these techniques can be applied t o the arithmetic fundamental lemma conjecture of W. Zhang in the minuscule case.

At inert primes the combinatorial structure of that Rapoport-Zink space is controlled by a certain Bruhat-Tits building and their irreducible components are certain Deligne-Lusztig varieties. For ramified primes M. Rapoport reported in his talk *On the supersingular locus of the Shimura variety for* $\mathrm{GU}(1, n-1)$ *in the ramified case* analogous results obtained together with U. Terstiege and S. Wilson. The calculation of arithmetic intersection numbers of special cycles in the ramified case was the topic of B. Howards talk *Special cycles on unitary Shimura varieties* in which he explained results obtained together with J. Bruinier and T.-H. Yang.

The last two authors also formulated a conjecture about the finite part of arithmetic intersecion numbers of special divisors (also called Heegner divisors by Borcherds) for Shimura varieties of orthogonal type. E. Goren reported in his talk *On a Conjecture of Bruinier-Yang* on significant progress obtained together with F. Andreatta.

On a more foundational level P. Scholze presented his work on *p-adic Hodge theory for rigid analytic varieties.* He showed how his theory of perfectoid spaces together with the introduction of a pro-étale site for locally noetherian schemes or adic spaces can be used to prove a deRham comparison isomorphism with coefficients for rigid-analytic varieties.

An important tool to study the reduction modulo p of PEL Shimura varieties is the Newton stratification, i.e. the decomposition according to the isogeny class of the p-divisible groups of the abelian varieties. A foundational result on this stratification is Rapoport and Zink's uniformization theorem showing that PEL Shimura varieties are uniformized along Newton strata by certain moduli spaces of p-divisible groups associated with the fixed isogeny class. In his talk on *Period spaces for Hodge structures in equal characteristics* Hartl presented the recent work of his student E. Arasteh Rad who proved an extended analog of this result in the function field case. There, for any reductive group G moduli spaces of local G-shtukas associated with a given isogeny class uniformize the corresponding moduli spaces of global G-shtukas along the associated Newton stratum.

The reduced subschemes underlying moduli spaces of p-divisible groups or local G-shtukas are called affine Deligne-Lusztig varieties. They can (as sets) be defined in a purely group-theoretic way generalizing Deligne and Lusztig's classical construction. In her talk on *Connected components of minuscule affine Deligne-Lusztig varieties*, M. Chen presented recent results with M. Kisin and E. Viehmann determining the sets of connected components of these varieties, and applications to the local Langlands correspondence. X. He (in his talk *Affine Weyl group, affine Hecke algebra, and affine Deligne-Lusztig variety*) reported on new group-theoretic methods and important new results on the questions of non-emptiness and dimension of affine Deligne-Lusztig varieties in affine flag varieties. In particular, this completely proves a conjecture by Görtz, Haines, Kottwitz and Reuman on non-emptiness of these varieties that has been studied by many people in the past f ew years. The presented methods provide a new approach to these arithmetic questions which hopefully lead to even more geometric applications in the near future.

A central tool to understand p-divisible group over p-adic rings is the display theory developed by T. Zink. In his talk *Truncated displays*, E. Lau presented his work to extend these techniques to truncated p-divisible groups. He introduced the notion of a truncated display and showed how to attach to every truncated p-divisible group over an arbitary base scheme of characteristic p such a truncated display. Then he studied the induced morphism Φ_n from the algebraic stack of truncated p-divisible groups to the algebraic stack of truncted displays. He explained that Φ_n is smooth and an equivalence on geometric points und he described its inertia.

The theory of p-adic automorphic forms plays a very important role in the recent developments of the Langlands program by p-adically interpolating between the known cases of Langlands correspondences.

In his talk Farid Mokrane explained a new approach and a genereralization of Hida's theory (the case of ordinary p-adic modular forms). This is a joint work with Jacques Tilouine. It relies heavily on the work by Brinon and Mokrane on the overconvergence of the Iugsa monodromy representation. Mokrane and Tilouine use this overconvergence to construct p-adic automorphic forms in infinite level on an overconvergent version of the Igusa tower.

In another direction, Stroh announced an important result about new cases of Artin conjectures, generalizing the work of Buzzard and Taylor from the case of ordinary modular forms to the case of Hilbert modular forms. This is a joint work in common with Kassaei, Pilloni, Tian and Sasaki. Recently, Pilloni and Stroh and independently Kassaei, Tian and Sasaky have given a generalization of the result by Buzzard and Taylor. But their results needed a ramification hypothesis. In his talk Stroh explained how to remove this hypothesis by a more detailed study of the action of Hecke operators on p-adic Hilbert modular varieties. They use this to prove a classicity criterion à la Coleman for p-adic Hibert modular forms. The result about Artin conjecture is then obtained by using the theory of companion forms due to Gee.

Shen Xu explained a new result obtained in his PHD about the structure the action of Hecke operators on some unitary Rapoport-Zink spaces and unitary type PEL Shimura varieties with signature $(1, n-1)$ at a non split prime. Rapoport-Zink spaces are p-adic rigid analytic spaces that p-adically uniformize some parts of Shimura varities. The Shimura varities Shen Xu considers are the same as the one studied by Harris and Taylor in their work on the local Langlands correspondence but instead of considering a prime at which the unitary groups becomes linear (as in Harris-Taylor), Shen Xu considers a prime at which the unitary group is a p-adic unitary group. The involved p-adic geometry is much more complicated. The mod p geometry of those spaces has been studied in details by Vollard and Wedhorn, linking this geometry to the one of a Bruhat-Tits building. Shen-Xu shows one can lift this to the p-adic geometry by showing the involved Rapoport-Zink spaces have a "good" cellular decomposition under the action of Hecke operators. This uses the theory of Harder-Narasimhan of finite flat group schemes (Fargues) and gives another example of such type of cellular decompositions after the work of Fargues (the linear case).

Pascal Boyer explained his work on the torsion in the ℓ-adic cohomology of Lubin-Tate spaces. Lubin-Tate spaces are the Rapoport-Zink spaces showing up in the work of Harris and Taylor in their proof of the local Langlands correspondence. Boyer has proved that in a lot of cases the ℓ-adic cohomology of those spaces has no torsion. For this he proceeds to a very detailed study of the perverse sheaf of vanishing cycles on the corresponding Shimura varities, the ℓ-adic cohomology of Lubin-Tate spaces being the fiber a a geometric point of those vanising cycles. He gives a description of this perverse sheaf together with the action of the Hecke operators and the monodromy operator. One of the main difficulties is that there are two t-structures switched by Verdier duality in the context if integral perverse sheaves and he has to play with them. Since the study of the torsion in the cohomology of Shimura varieties is now an active domain (see the recent work of Bergeron and Venkatesh for example), the techniques introduced by Boyer may be useful in the future.

In his talk on *Local models for Shimura varieties* G. Pappas gave an overview of his recent results with X. Zhu on a general method of constructing local models, i.e. schemes defined in terms of linear algebra (more precisely, affine Grassmannians and affine flag varieties), which are expected to model, étale-locally, the singularities of suitable models of Shimura varieties over the ring of integers of the reflex field. For PEL-type Shimura varieties, a general framework of such models of Shimura varieties and corresponding local models had been proposed by Rapoport and Zink, and has been further investigated and improved upon by several other people. But with the work of Pappas and Zhu a new way of constructing the local model is now available, which is on the one hand completely general. On the other hand the construction is independent of the Shimura variety. This opens a new approach to studying the local structure of arithmetic models of Shimura varieties; for i nstance, Kisin and Pappas are working on applications to Shimura varieties which are not of PEL type. The talk of Pappas was complemented by X. Zhu's talk on *Nearby cycles for local models*, where he explained the proof of the Kottwitz conjecture by Pappas and Zhu, which gives a description of the trace of Frobenius on the sheaf of nearby cycles in terms of a suitable Hecke algebra.

The work of Pappas and Zhu relays heavily on their results about reductive groups over 2-dimensional base schemes, and on Bruhat-Tits theory. A different look on Bruhat-Tits theory was explained by T. Haines, in his talk *A Tannakian approach to Bruhat-Tits theory and parahoric group schemes*. The Tannakian approach aims at defining the building of a group G in terms of all of its representations, thus reducing the problem to the general linear group.

C.-F. Yu gave a talk on *Shuffle structures on KR strata*; these shuffle structures allow to reduce many questions about the structure of the loci of p-rank ≥ 1 in Siegel modular varieties with Iwahori level structure to questions about the p-rank 0 locus in Siegel modular varieties of lower genus. P. Hartwig's talk *p-rank strata and Kottwitz-Rapoport strata in Shimura varieties of PEL type* showed that for Iwahori level structure the p-rank is constant on KR strata (as previously proved by Ngô and Genestier in the Siegel case) and discussed how to actually compute the p-rank on a given stratum.

Workshop: Reductions of Shimura Varieties

Table of Contents

Farid Mokrane (joint with Olivier Brinon and Jacques Tilouine)
Eigenvarieties via Igusa towers 1969

Peter Scholze
p-adic Hodge theory for rigid-analytic varieties 1974

Eike Lau
Truncated displays 1975

Urs Hartl (joint with M. Esmail Arasteh Rad)
Rapoport-Zink uniformization over function fields 1977

Pascal Boyer
Filtrations of stratification of some simple Shimura varieties 1977

Eyal Z. Goren (joint with Fabrizio Andreatta)
On a Conjecture of Bruinier-Yang 1981

Thomas J. Haines (joint with Kevin Wilson)
A Tannakian approach to Bruhat-Tits buildings and parahoric group schemes 1984

Ulrich Terstiege
On the regularity of special difference divisors on unitary Rapoport-Zink spaces of signature $(1, n-1)$ 1987

Xu Shen
Cell decomposition of some unitary group Rapoport-Zink spaces 1989

George Pappas (joint with X. Zhu)
On local models of Shimura varieties 1991

Miaofen Chen (joint with Mark Kisin, Eva Viehmann)
Connected components of minuscule affine Deligne-Lusztig varieties 1992

Chia-Fu Yu (joint with Ulrich Görtz)
Shuffle structures on KR strata 1995

Xinwen Zhu (joint with George Pappas)
Nearby Cycles on Local Models 1998

Philipp Hartwig
Kottwitz-Rapoport and p-rank strata in the reduction of Shimura varieties of PEL type 1999

Xuhua He
Affine Weyl group, affine Hecke algebra and affine Deligne-Lusztig variety 2001

Michael Rapoport (joint with U. Terstiege and S. Wilson)
On the supersingular locus of the Shimura variety for $GU(1, n-1)$ in the ramified case 2002

Benjamin Howard (joint with Jan Bruinier, Tonghai Yang)
Height pairings on unitary Shimura varieties 2005

Benoît Stroh (joint with Payman Kassaei, Vincent Pilloni, Shu Sasaki, Yichao Tian)
Overconvergence, classicity and ramification 2007

Abstracts

Eigenvarieties via Igusa towers

Farid Mokrane
(joint work with Olivier Brinon and Jacques Tilouine)

1. Introduction

Our work in progress ([1], [2]) concerns the construction of the holomorphic eigenvariety of $\mathcal{A}_{g,N}$, the moduli space of principally polarized abelian varieties of dimension g and level structure N (constructed also independently by Andreatta-Iovita-Pilloni using a different technique). The method is based on the study of the p-adic monodromy following Katz's point of view [6]. The result should be generalized to all PEL Shimura varieties with dense ordinary locus and perhaps non PEL type (like $SO(2,19)$). In this report we present the elliptic case. More precisely we consider the Legendre family $\mathcal{E} = (\mathcal{E}_\lambda) : y^2 = x(x-1)(x-\lambda)$. Before the study of the p-adic monodromy, we review the complex monodromy of this family.

1.1. **Complex monodromy (Euler).** Let $\pi : \mathcal{E} \to S := \mathbb{C} - \{0,1\}$ be the structural morphism defining the Legendre family over $\mathbb{C}$. The locally constant sheaf $\mathcal{L} = R^1\pi_*\mathbb{Z}$ induces the monodromy representation :

$$\rho : \pi_1(S, \lambda_0) \to Aut(H^1(\mathcal{E}_{\lambda_0}, \mathbb{Z})) \simeq GL_2(\mathbb{Z})$$

ρ is injective with image conjugate to a subgroup of index 2 of $\Gamma_2(\mathbb{Z})$. The relation with differential equations is given by de Rham cohomology $H^1_{dR}(\mathcal{E}/S)$ which is a free $\mathcal{O}_S$-module of rank 2 equipped with Gauß-Manin connection ∇. The classes of the differential forms of the second kind :

$$\omega = \frac{dx}{y}, \omega' = \nabla(\frac{\partial}{\partial\lambda})(\omega) = \frac{dx}{\sqrt{x(x-1)(x-\lambda)^3}}$$

form a basis of $H^1_{dR}(\mathcal{E}/S)$. The class of ω is a solution of the Picard-Fuchs differential equation :

$$\lambda(\lambda-1)\omega'' + (2\lambda-1)\omega' + \frac{1}{4}\omega = 0$$

We have a de Rham-Betti comparison theorem:

$$\mathcal{L} \otimes \mathbb{C} \simeq H^1_{dR}(\mathcal{E}/S)^{\nabla=0} = \{\lambda(\lambda-1)(G'(\lambda)\omega - G(\lambda)\omega')\}$$

where $G(\lambda) \in \mathcal{O}_S$ runs over all the solutions of the Picard-Fuchs differential equation. The Gauß hypergeometric series

$$G(\lambda) = \sum_{n=0}^{+\infty} \Big(\frac{\frac{1}{2}(\frac{1}{2}+1)\dots(\frac{1}{2}+n-1)}{n!}\Big)^2 \lambda^n$$

is a particular solution.

1.2. **p-adic monodromy.** Let p be a prime number > 2, $k = \overline{\mathbb{F}}_p$ and consider the Legendre family and its ordinary part :

$$\mathcal{E} \to \mathbb{P}^1_k - \{0,1,\infty\} = Y_0 \supset X_0 = Spec\Big(k\Big[\lambda, \frac{1}{\lambda(\lambda-1)h(\lambda)}\Big]\Big)$$

where $h(\lambda)$ is the Hasse polynomial :

$$h(\lambda) = (-1)^{\frac{p-1}{2}} \sum_{n=0}^{\frac{p-1}{2}} \binom{\frac{p-1}{2}}{n}^2 \lambda^n$$

The étale quotient $\mathcal{L}$ of the Barsotti-Tate group $\mathcal{E}_{|X_0}[p^\infty]$ induces the monodromy representation

$$\rho : \pi_1(X_0, \lambda_0) \to \mathbb{Z}_p^\times$$

We know that ρ is surjective (Igusa).

The relation with differential equations is given by Dwork's theory of unit F-crystals [5]. Let $W = W(k)$ be the ring of Witt vectors with coefficients in k and $K = Frac(W) = \widehat{\mathbb{Q}}_p^{nr}$. Consider the formal affine schemes :

$$Y = Spf\Big(W\Big\{\lambda, \frac{1}{\lambda(\lambda-1)}\Big\}\Big) \supset X = Spf\Big(W\Big\{\lambda, \frac{1}{\lambda(\lambda-1)h(\lambda)}\Big\}\Big)$$

Let $R = W\Big\{\lambda, \frac{1}{\lambda(\lambda-1)h(\lambda)}\Big\}$ and $\varphi : R \to R$ a lift of the absolute Frobenius of $k\Big[\lambda, \frac{1}{\lambda(\lambda-1)h(\lambda)}\Big]$ (For example $\varphi(\lambda) = \lambda^p$). Then $H^1_{dR}(\mathcal{E}/X)$ is an F-crystal equipped with an integrable connection ∇ and a horizontal φ-semilinear Frobenius Φ.

Let $Fil^1 = \pi_*\Omega_{\mathcal{E}/X}$ (the Hodge filtration), there exists a unit sub-F-crystal $\mathcal{U}$ of rank 1 such that $H^1_{dR}(\mathcal{E}/X) = \mathcal{U} \oplus Fil^1$. Let $\widehat{R}^{nr}$ be the p-adic completion of the union of all étale R-algebras in a fixed algebraic closure of the field of fractions of R. We have a canonical isomorphism $\mathcal{L}^\vee \simeq (\mathcal{U} \otimes \widehat{R}^{nr})^{\nabla=0,\Phi=1}$. Moreover, there exists a basis e of $\mathcal{U}$ such that

$$\Phi(e) = \frac{G(\lambda)}{G(\varphi(\lambda))}e \quad \text{and} \quad \nabla(\frac{\partial}{\partial\lambda})e = \frac{G'(\lambda)}{G(\lambda)}e$$

2. Overconvergence

2.1. **Overconvergence of the p-adic monodromy.** Dwork showed that $\frac{G'(\lambda)}{G(\lambda)}$ and $\frac{G(\lambda)}{G(\lambda^p)}$ do not overconverge around the supersingular discs. But if we consider the "excellent lifting" φ defined by $j(\varphi(\lambda)) = \sum a_n q^{pn}$ with $j(\lambda) = \sum a_n q^n$ the j invariant, then $\frac{G(\lambda)}{G(\varphi(\lambda))}$ overconverges.

By passing to the generic fiber, the p-adic monodromy ρ induces a representation

$$\rho_\eta : \pi_1(X^{rig}, \lambda_0) \to \mathbb{Z}_p^\times$$

We say that ρ *overconverges* if ρ_η extends to $\pi_1(V, \lambda_0)$ where V is a strict neighborhood of X^{rig} in Y^{rig}.

Theorem 1. *(Brinon-M) The p-adic monodromy overconverges.*

2.2. **Overconvergent p-adic modular forms.** Let $v = \frac{a}{b} \in \mathbb{Q}^+$, p^v a root of the polynomial $T^b - p^a$ and $R_v = R[p^{1/b}]\{T\}/(hT - p^v)$. R_v is a normal domain, not smooth over $\mathcal{O}_K$ the ring of integers of $K = K_0[p^v]$ if $v > 0$, and not semi-stable if $a > 1$. Let $X_v = Spf(R_v)$, if $v > 0$, X_v^{rig} is a strict neighborhood of X consisting of points $x \in Y$ such that $|h(x)| \geq p^{-v}$. If $0 < v < \frac{p-1}{p^2}$, we have a tower of Galois étale covers $(V_{n,v})$ of X_v^{rig} with Galois group $(\mathbb{Z}/p^n\mathbb{Z})$ (precise version of Theorem 1).

Set $T_{n,v} = Isom_{X_v^{rig}}(V_{n,v}, \mathbb{Z}/p^n\mathbb{Z})$ the torsor of bases of $V_{n,v}$ over X_v^{rig} and $T_{\infty,v} = \varprojlim_n T_{n,v}$, $\pi : T_{\infty,v} \to X_v^{rig}$. Let $M_{v,w}$ be the Banach space over K of locally analytic vectors (wrt the action of $\mathbb{Z}_p^\times$) of $H^0(X_v^{rig}, \pi_*\mathcal{O}_{T_{\infty,v}})^\wedge$ with a radius of convergence $\geq p^{-1/w}$.

$M^\dagger = \lim_{v,w} M_{v,w}$ is by definition the space of p-adic overconvergent modular forms.

Let $\mathcal{W} = Hom_{cont}(\mathbb{Z}_p^\times, \mathbb{C}_p^\times)$ the group of continuous characters of $\mathbb{Z}_p^\times$. $\mathcal{W} \simeq \widehat{(\mathbb{Z}/p\mathbb{Z})}^\times \times B(1, 1^-)$ has a natural structure of a rigid space (finite union of open unit discs).

Let $\mathcal{W}_w = \{\kappa \in \mathcal{W}$ such that κ extends to $T_w = \mathbb{Z}_p^\times(1 + p^{1/w}\mathcal{O}_{\mathbb{C}_p})$, $\mathcal{W}_w$ is an affinoïde domain and $(\mathcal{W}_w)_w$ is an admissible covering of $\mathcal{W}$. Let $M_{v,w}^\kappa$ be the subspace of $M_{v,w}\hat{\otimes}\mathcal{O}_{\mathcal{W}_w}$ of eigenforms wrt the universal character $T_w \to \mathcal{O}_{\mathcal{W}_w}$. $M_{v,w}^\kappa$ is an orthonormalizable Banach $\mathcal{O}_{\mathcal{W}_w}$-module. Let $\mathcal{T} \xrightarrow{\pi} Y$ be the $R^\times$-torsor of bases of the locally free sheaf $\omega = \pi_*\Omega_{\mathcal{E}/Y}$. The space of classical modular forms is

$$M = H^0(X, \pi_*\mathcal{O}_{\mathcal{T}}) = \bigoplus_{k \in \mathbb{Z}} H^0(X, \omega^k)$$

By considering the site "Zprofét" of profinite étale covers of Zariski open rigid scheme of X_v (introduced independently by Scholze), the relation with classical forms comes from the following theorem :

Theorem 2. *(Brinon-M-Tilouine) We have a canonical X_v-morphism of sheaves on the site Zprofét, $\mathbb{Z}_p^\times$-equivariant :*

$$\underline{HTI} : T_{\infty,v} \to \mathcal{T}_{|X_v}$$

We call $\underline{HTI}$ the Hodge-Tate-Igusa map.

Corollary 3. *For every classical weight $k \in X^*(T) \subset \mathcal{W}(K)$, $\underline{HTI}$ induces an injection of K-vector spaces $\underline{HTI}^* : M^k(\Gamma_N; K) \to M_{v,w}^k$ compatible with cuspidality : $\underline{HTI}^* : S^k(\Gamma_N; K) \to S_{v,w}^k$.*

By the relative spectral theory of the U_p-operator over $M_{v,w}^\kappa$ over $\mathcal{O}_{\mathcal{W}_w}$-Banach modules, there exists a sub-$\mathcal{O}_{\mathcal{W}_w}$-projective module $(M_{v,w}^\kappa)^{\leq \alpha}$ of $\mathcal{O}_{\mathcal{W}_w}$ of forms with slope $\leq \alpha$. We have also a cuspidal part $(S_{v,w}^\kappa)^{\leq \alpha}$ of $S_{v,w}^\kappa$.

Corollary 4. *(Coleman) For any $\kappa \in \mathcal{W}_w(L)$, $S_{v,w}^\kappa \otimes_{\mathcal{O}_{\mathcal{W}_w},\kappa} L = S(X_v, \kappa, L)$. For any $\kappa \in \mathcal{W}$ and $\alpha > 0$, there exists an affinoïde neighbourhood $\mathcal{W}(\kappa)$ of κ such that for every $\kappa' \geq 2$ with $\alpha < \kappa' - 1$, we have $(S_{v,w})^{\kappa \leq \alpha} \otimes_{\mathcal{O}_{\mathcal{W}_w},\kappa'} K =$*

$S(X_v, \kappa', K) = S_{\kappa'}^{\leq\alpha}$ *(the space of classical cuspidal forms of weight* κ*, level* $\Gamma(p)$ *and slope* α*).*

We obtain the Eigenvariety from the data above using standard techniques (Coleman-Mazur, Buzzard). Proofs of the theorems above are based on p-adic Hodge theory. We give some indications in the following section.

3. Periods of overconvergent Hodge crystals

3.1. **The structure of H^1_{dR} over X_v.** Let $\mathcal{E}_v \to X_v$ be the Legendre family, we have the crystal $H^1_{dR}(\mathcal{E}_v/X_v) \simeq H^1_{cris}(\mathcal{E}_v/X_v)$ over X_v. We fix the excellent lifting of the Frobenius $\varphi : R_v \to R_{v/p}$, a φ-crystal M over R_v is a crystal endowed with an isogeny : $\Phi : \varphi^* M \to M \otimes_{R_v} R_{v/p}$. We have a structure of φ-cystal on $H^1_{cris}(\mathcal{E}_v/X_v)$. The theory of canonical subgroup shows that $\Phi(Fil^1) \subset p^{1-v}M$. We call such a triple (M, Φ, Fil^1) an "Overconvergent Hodge F-crystal". We want associate to it a p-adic lisse sheaf over X_v extending the Dwork-Katz recipe over the ordinary locus.

3.2. **Relative periods.** The algebraic fundamental group of a rigid space Y classifies finite étale covers of Y. If Y is the generic fiber of an affine formal scheme $Y^f = \operatorname{Spf} R$ over $\mathcal{O}_K$ with R a normal domain, $\pi_1(Y, y)$ classifies normal finite extensions S of R in some fixed algebraic closure $\overline{Fr(R)}$ of $Fr(R)$, such that $S[\frac{1}{p}]$ is étale over $R[\frac{1}{p}]$. Let $\overline{R} \subset \overline{Fr(R)}$ be the union of all such S. Let $\mathcal{R} = \operatorname{projlim} \overline{R}/p\overline{R}$ and $W(\mathcal{R})$ the ring of Witt vectors. We have a lifting ϕ of the Frobenius $x \mapsto x^p$. $\operatorname{Gal}(\overline{R}/R)$ acts on $W(\mathcal{R})$ and the action commutes with the Frobenius ϕ. We have also a natural Galois-equivariant map $\theta : W(\mathcal{R}) \to \widehat{\overline{R}}$.

Let $A^{\nabla}_{cris}(R)$ be the p-adic completion of the PD-envelope of $W(\mathcal{R})$ wrt $\operatorname{Ker}\theta$. On $A^{\nabla}_{cris}(R)$ acts the group $\operatorname{Gal}(\overline{R}/R)$ and the Frobenius ϕ.

The ideal $\operatorname{Ker}\theta$ is principal in $W(\mathcal{R})$, generated by $\xi = [\widetilde{p}] - p$ where $[\widetilde{p}] \in W(\mathcal{R})$ is the Teichmüller representative of a projective system $\widetilde{p} = (p, p^{1/p}, p^{1/p^2}, \ldots)$. Let $u : T \twoheadrightarrow R$ be a smooth presentation. Let $\theta_u = \theta \otimes u : W(\mathcal{R}) \otimes_W T \to \widehat{\overline{R}}$ and $A_{cris}(u)$ be the p-adic completion of the PD-envelope of $W(\mathcal{R}) \otimes_W T$ wrt $\operatorname{Ker}\theta_u$. $A_{cris}(u)$ is a $D(u)$-algebra and $\operatorname{Gal}(\overline{R}/R)$ acts. Moreover there is a natural connection acting on it and the set of horizontal sections is $A^{\nabla}_{cris}(R)$. We can put also a Frobenius structure.

3.3. **Overconvergence of the Unit F-crystal.** Let (M, Φ, Fil^1) be an overconvergent Hodge F-crystal over R_v. $\mathcal{M}_u = (M_u \otimes_{D(u)} A_{cris}(u))^{\nabla=0}$ is an $A^{\nabla}_{cris}(R_v)$-module free of rank the rank of M over R_v endowed with an action of Φ and $\operatorname{Gal}(\overline{R}/R)$. Fil_u induces not canonically a filtration $\mathcal{F}il_u$ on $\mathcal{M}_u$. Let $\mathcal{M} = H^0_{cris}(\operatorname{Spec}(\overline{R}_v/p\overline{R}_v), M)$, We have a canonical isomorphism $\mathcal{M} \simeq \mathcal{M}_u$

Theorem 5. *(Brinon-M) If* $v < \frac{p-1}{p^2}$*, there is a unique sub-*$A^{\nabla}_{cris}(R_v)$*-module* $\mathcal{U}$ *of* $\mathcal{M}$ *of rank* 1 *stable by* Φ *and* $\operatorname{Gal}(\overline{R}/R)$ *and such that :*

- $\mathcal{U} \oplus \mathcal{F}il_u = [\widetilde{p}]^v \mathcal{M}_u + \mathcal{F}il_u$ *for any presentation* u *and filtration* $\mathcal{F}il_u$ *lifting* Fil^1.

- $[\widetilde{p}]^{pv} \in \det \Phi_{|\mathcal{U}}$

3.4. **The comparison theorem.** Let $\Lambda = W(\mathcal{R})/([\zeta]-1)^{p-1}$ where $[\zeta]$ is the Teichmüller lifting of a basis of the Tate module of the multilplicative group. We have :

$$\Lambda_0(R_v) \cong A^{\nabla}_{cris}(R_v)/I^{[p-1]}A^{\nabla}_{cris}(R_v)$$

$$\text{with} \quad I^{[r]}A^{\nabla}_{cris}(R_v) = \{x \in A^{\nabla}_{cris}(R_v); \forall m \geq 0,\ \phi^m(x) \in J^{[r]}A^{\nabla}_{cris}(R_v)\}$$

and $J^{[r]}A^{\nabla}_{cris}(R_v)$ is the closure for the p-adic topology of the r-th divided power of $J^{[1]}$. Let $\mathcal{U}$ be an $A^{\nabla}_{cris}(R_v)$-module, $\widetilde{\mathcal{U}}$ a lift of $\mathcal{U} \otimes \Lambda$ to $W(\mathcal{R})$ and

$$\mathbf{V}(\mathcal{U}) = \mathrm{Ker}(\varphi/p \otimes \Phi : \xi W(\mathcal{R})\{[\widetilde{p}]^{-1}]\} \otimes \widetilde{\mathcal{U}} \to W(\mathcal{R})\{[\widetilde{p}]^{-1}]\} \otimes \widetilde{\mathcal{U}})$$

Theorem 6. *(Brinon-M) Let (M, Φ, Fil) be an overconvergent Hodge F-crystal such that $p^v \in \det(\Phi_{M/Fil})$. If $v < \frac{p-1}{p^2}$ then $\mathbf{V}(\mathcal{U})$ is a free $\mathbb{Z}_p$-module of rank the rank of $\mathcal{U}$ over R_v.*

Theorem 7. *(Brinon-M-Tilouine) The natural map*

$$\mathbf{V}(\mathcal{U}) \otimes_{\mathbb{Z}_p} A^{\nabla}_{cris}(R_v)[\frac{1}{p}] \longrightarrow \xi\mathcal{U}[\frac{1}{p}]$$

is an isomorphism.

3.5. **The Hodge-Tate-Igusa map.** From Theorem 7, we have an isomorphism

$$\mathbf{V}(\mathcal{U}) \otimes_{\mathbb{Z}_p} \widehat{\overline{R}}_v[\frac{1}{p}] \longrightarrow \omega^\vee \otimes_{R_v} \widehat{\overline{R}}_v[\frac{1}{p}]$$

Let R_∞ be the fixed field of $\ker(\rho)$, taking invariants by $\mathrm{Gal}\,(\overline{R}_v/R_\infty)$ and using the purity theorem (Faltings, Scholze), we obtain the Hodge-Tate-Igusa map

$$\mathbf{V}(\mathcal{U})[\frac{1}{p}] \quad \hookrightarrow \quad \omega^\vee \otimes_{R_v} \widehat{R}_\infty[\frac{1}{p}].$$

We deduce a map of sheaves on the site Zprofét :

$$\mathbf{V}(\mathcal{U})[\frac{1}{p}] \quad \hookrightarrow \quad \omega^\vee[\frac{1}{p}]$$

4. The higher genus case

In the Siegel case, the Igusa tower $T_{\infty,v}$ is a $GL_g(\mathbb{Z}_p)$-torsor over X_v. Let $T_{I,v}$ be the quotient of $T_{1,v}$ by $B^+(\mathbb{Z}/p\mathbb{Z})$. $T_{\infty,v}/T_{I,v}$ is a Galois cover with group the Iwahori group $I = N_1^- \times T \times N^+$ where N^+ is the upper unipotent subgroup and N_1^- is the subgoup of the lower unipotent matrices congruent to the identity mod p. For $u \geq w$, we define a "Banach sheaf" $M_{v,w,u}$ of sections of the Igusa tower over $T_{I,v}$ invariant under N^+ and locally analytic wrt T and N_1^- with order w and u. We then obtain similar results that described above in the elliptic case.

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p-adic Hodge theory for rigid-analytic varieties

Peter Scholze

In this talk, we explained parts of the proof of the following result.

Theorem 1. *Let k be a p-adic field, i.e. a complete nonarchimedean extension of $\mathbb{Q}_p$ with perfect residue field. Let X be a proper smooth rigid-analytic variety over k, and let $\mathbb{L}$ be a $\mathbb{Z}_p$-local system on X. Let $C = \hat{\bar{k}}$ be the completion of an algebraic closure of k.*

(i) *The étale cohomology groups $H^i_{\text{ét}}(X_C, \mathbb{L})$ are finitely generated $\mathbb{Z}_p$-modules for all $i \geq 0$, and vanish for $i > 2\dim X$.*

(ii) *Assume that $\mathbb{L}$ is de Rham. Then there is an associated module with integrable connection $(\mathcal{E}, \nabla)$ on X, with a separated and exhaustive decreasing filtration $\mathrm{Fil}^\bullet\mathcal{E} \subset \mathcal{E}$ by locally direct summands, satisfying Griffiths transversality. There is a $\mathrm{Gal}(\bar{k}/k)$-equivariant isomorphism*

$$H^i_{\text{ét}}(X_C, \mathbb{L}) \otimes_{\mathbb{Z}_p} B_{\mathrm{dR}} \cong H^i_{\mathrm{dR}}(X, (\mathcal{E}, \nabla, \mathrm{Fil}^\bullet)) \otimes_k B_{\mathrm{dR}} ,$$

where B_{dR} is Fontaine's field of p-adic periods. This isomorphism preserves filtrations.

(iii) *In the situation of (ii), the Hodge-de Rham spectral sequence*

$$H^{i,j}_{\mathrm{Hodge}}(X, (\mathcal{E}, \nabla, \mathrm{Fil}^\bullet)) \Rightarrow H^{i+j}_{\mathrm{dR}}(X, (\mathcal{E}, \nabla, \mathrm{Fil}^\bullet))$$

degenerates.

Here, we define

$$H^{i,j}_{\mathrm{Hodge}}(X, (\mathcal{E}, \nabla, \mathrm{Fil}^\bullet)) = \mathbb{H}^{i+j}(X, \mathrm{gr}^i \mathrm{DR}(\mathcal{E}, \nabla, \mathrm{Fil}^\bullet)) .$$

In particular, for $\mathbb{L} = \mathbb{Z}_p$, the theorem says that the usual Hodge-de Rham spectral sequence degenerates for any proper smooth rigid-analytic variety. Moreover, its étale cohomology groups $H^i_{\text{ét}}(X_C, \mathbb{Q}_p)$ are de Rham representations of

$\mathrm{Gal}(\bar{k}/k)$ with associated filtered k-vector space $H^i_{\mathrm{dR}}(X)$. Also, there is a Hodge-Tate decomposition

$$H^i_{\text{ét}}(X_C, \mathbb{Q}_p) \otimes_{\mathbb{Q}_p} C \cong \bigoplus_{j=0}^{i} H^{i-j}(X, \Omega^j_X) \otimes_k C(-j) \ ,$$

answering a question of Tate.

The proof of the theorem follows ideas of Faltings, amplified by the theory of perfectoid spaces.

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Truncated displays

Eike Lau

It is known that formal p-divisible groups over p-adic rings are equivalent to nilpotent displays [Zi, L1]. There is a natural notion of truncated displays of level n, and these objects form an Artin stack $\mathcal{D}isp_n$ over $Spec\,\mathbb{F}_p$. Let $\mathcal{BT}_n$ be the stack of truncated p-divisible group of level n. The central result of [L2] is that the Dieudonné crystal of a (truncated) p-divisible group can be endowed with a natural display structure:

Theorem 1. *There is a functor from p-divisible groups over p-adic rings to displays, $\phi : \mathcal{BT} \to \mathcal{D}isp$. It induces a morphism of Artin algebraic stacks*

$$\phi_n : \overline{\mathcal{BT}_n} = \mathcal{BT}_n \times Spec\,\mathbb{F}_p \to \mathcal{D}isp_n,$$

which is smooth of relative dimension zero and an equivalence on geometric points.

The stack $\mathcal{D}isp_n$ can be described as follows. Let I_R be the kernel of the first projection $W(R) \to R$. The inverse of the Verschiebung is a σ-linear homomorphism $\sigma_1 : I_R \to W(R)$. Let $K(R) = GL_h(W(R))$, and for fixed non-negative integers d, c let $K_\mu(R) \subset K(R)$ be the subgroup of block matrices $\left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right)$ of size (d, c) where C has coefficients in I_R. Define a homomorphism

$$\sigma_\mu : K_\mu \to K, \qquad \begin{pmatrix} A & B \\ C & D \end{pmatrix} \mapsto \begin{pmatrix} \sigma(A) & p\sigma(B) \\ \sigma_1(C) & \sigma(D) \end{pmatrix}$$

and let $g \in K_\mu$ act on K by $h \mapsto g^{-1}h\sigma_\mu(g)$. Then the quotient stack $[K/K_\mu]$ is the space of displays of height $c + d$ and dimension d. If R is a ring of characteristic p, the truncated Witt ring $W_n(R)$ carries a Frobenius. This allows to define a truncated variant $\sigma_{\mu,n} : K_{\mu,n} \to K_n$ of σ_μ where $K_n(R) = GL_h(W_n(R))$ and where $K_{\mu,n}(R)$ is the group of invertible block matrices $\left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right)$ of size (d, c) such that A, B, D have coefficients in $W_n(R)$ and C has coefficients in $I_{n+1,R}$, the kernel of the first projection $W_{n+1}(R) \to R$. Then $[K/K_\mu] \subset \mathcal{D}isp_n$ is the open and closed substack of truncated displays of dimension d and height $c + d$.

The diagonal

$$\Delta : \overline{\mathcal{BT}_n} \to \overline{\mathcal{BT}_n} \times_{\mathcal{D}isp_n} \overline{\mathcal{BT}_n}$$

measures the failure of ϕ_n to be an isomorphism. For a truncated p-divisible group G over an $\mathbb{F}_p$-algebra let $\underline{Aut}^o(G)$ be the sheaf of automorphisms of G which become trivial on the associated truncated display.

Theorem 2. *The morphism Δ is finite flat and surjective. More precisely, for truncated p-divisible groups G, H of level n, dimension d, and height $c+d$ over an $\mathbb{F}_p$-algebra R, the morphism*

$$\pi : \underline{Isom}(G, H) \to \underline{Isom}(\phi_n G, \phi_n H)$$

induced by ϕ_n is a torsor under $\underline{Aut}^o(G)$, and $\underline{Aut}^o(G)$ is an infinitesimal commutative finite flat group scheme of rank p^{ncd}.

Here we see no difference between truncated p-divisible groups which are infinitesimal or not, but a difference appears in the limit over n:

Theorem 3. (*a*) *For a p-divisible group G over an $\mathbb{F}_p$-algebra, the affine group scheme*

$$\underline{Aut}^o(G) = \varprojlim_n \underline{Aut}^o(G_n)$$

is trivial if and only if all fibres of G are infinitesimal or unipotent.

(*b*) *Over p-adic rings, the functor ϕ induces an equivalence between infinitesimal p-divisible groups and nilpotent displays.*

The situation in (a) can be described quite explicitly; see [LZ]: If G is an infinitesimal p-divisible group of dimension d and height $c + d$ over a reduced $\mathbb{F}_p$-algebra, the reduction

$$\underline{Aut}^o(G_n) \to \underline{Aut}^o(G_m)$$

is trivial as soon as $n \geq (c + 1)m$. It follows that the limit is trivial as required. The equivalence in (b) is known, but the proof is new.

As another application of Theorems 1 and 2 we get the following.

Corollary 4. *Let R be a perfect ring of characteristic p.*

(*a*) *The functor ϕ_n induces an equivalence between truncated p-divisible groups over R and truncated displays over R.*

(*b*) *The category of finite flat commutative p-group schemes over R is equivalent to the category of triples (M, F, V) where M is a finitely presented $W(R)$-module annihilated by a power of p and of projective dimension at most one with a σ-linear endomorphism F and a σ^{-1}-linear endomorphism V such that $FV = VF = p$.*

Part (b) was proved earlier by Gabber by a reduction to the case of perfect valuation rings, which is due to Berthelot [Be].

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Rapoport-Zink uniformization over function fields

Urs Hartl
(joint work with M. Esmail Arasteh Rad)

There is a remarkable parallel between the arithmetic of number fields and the arithmetic of function fields under which abelian varieties and p-divisible groups have, as function field counterparts, global and local G-shtukas. Here G is a parahoric group scheme over a smooth projective geometrically irreducible curve C. For a fixed $r \in \mathbb{N}_{>0}$ a *global G-shtuka* over an $\mathbb{F}_q$-scheme S consists of r *characteristic morphisms* $c_1, \ldots, c_r : S \to C$, a G-torsor $\mathcal{G}$ over $C_S := C \times_{\mathbb{F}_q} S$ and an isomorphism $\phi : \sigma^*\mathcal{G}|_{C_S \setminus \Gamma} \xrightarrow{\sim} \mathcal{G}|_{C_S \setminus \Gamma}$ of G-torsors outside the union $\Gamma = \Gamma_{c_1} \cup \ldots \cup \Gamma_{c_r}$ of the graphs of the c_i, where $\sigma = \mathrm{id}_C \times \mathrm{Frob}_{q,S}$ is the Frobenius on C_S.

Local G-shtukas arise as the completion of global G-shtukas at the places (closed points) of C. Local G-shtukas with bounded Hodge-polygon possess Rapoport-Zink spaces, that is, deformation spaces by isogenies, which are formal schemes locally formally of finite type. The Rapoport-Zink spaces can be used to partially uniformize Newton strata in the algebraic moduli stacks of global G-shtukas with the same bounds on their Hodge polygons.

Filtrations of stratification of some simple Shimura varieties

Pascal Boyer

For $l \neq p$ two distinct prime numbers, in [1], we described the $\bar{\mathbb{Q}}_l$-perverse sheaf of vanishing cycles of some simple unitary Shimura variety studied in [2]. In this sort resume of my talk, I want to explain how to attack the problem of studying the $\bar{\mathbb{Z}}_l$-version of these results.

1. Filtration of stratification

1.1. **Torsion theories.** A torsion theory on a abelian category $\mathcal{A}$ is a couple $(\mathcal{T}, \mathcal{F})$ of full subcategories such that:

- for all objects T in $\mathcal{T}$ and F in $\mathcal{F}$, we have

$$\mathrm{Hom}_{\mathcal{A}}(T, F) = 0;$$

- for all objects A of $\mathcal{A}$, there exist objects T and F of $\mathcal{T}$ and $\mathcal{F}$ respectively, and a short exact sequence
$$0 \to T \longrightarrow A \longrightarrow F \to 0.$$

Remark: If $\mathcal{A}$ is $\mathbb{Z}_l$-linear, we say that T (resp. F) is of torsion (resp. free) if $l^N 1_A$ is null for some integer N (resp. $l.1_A$ is a monomorphism). If we note $\mathcal{T}$ (resp. $\mathcal{F}$) the set of such objects, then $(\mathcal{T}, \mathcal{F})$ is a torsion theory.

Proposition 1. *(cf.* [3] *1.3.6)*
If $\mathcal{C} = \mathcal{D}^{\leq 0} \cap \mathcal{D}^{\leq 0}$ is the heart of a perverse t-structure with a torsion theory $(\mathcal{T}, \mathcal{F})$ then
$$\begin{aligned} {}^{+}\mathcal{D}^{\leq 0} &:= \{A \in \mathcal{D}^{\leq 1} :\ h^1(A) \in \mathcal{T}\} \\ {}^{+}\mathcal{D}^{\geq 0} &:= \{A \in \mathcal{D}^{\geq 0} :\ h^0(A) \in \mathcal{F}\} \end{aligned}$$
define a new t-structure with heart ${}^{+}\mathcal{C}$ with $(\mathcal{F}, \mathcal{T}[-1])$ as torsion theory.

Notation 2. *For $T : \mathcal{D}_1 \longrightarrow \mathcal{D}_2$ a triangulated functor, we note ${}^{p+}h^0 T$ for ${}^{+}h^0 \circ T \circ \epsilon_1$.*

1.2. **Saturation.**

Notation 3. *For $A, B \in \mathcal{C} \cap {}^{+}\mathcal{C}$, we note*
$$\begin{tikzcd} A \arrow[r, hook, two heads] & B \end{tikzcd}$$
a monomorphism $A \to B$ in $\mathcal{C}$ and a epimorphism in ${}^{+}\mathcal{C}$.

Proposition 4. *Let $i : A \hookrightarrow P$ a monomorphism in $\mathcal{C}$ with A and P free in $\mathcal{C}$; then there exists an unique factorisation $A \to B \to P$ of i in $\mathcal{C}$ such that:*
- *B is free in $\mathcal{C}$;*
- *$\begin{tikzcd} A \arrow[r, hook, two heads] & B \end{tikzcd}$;*
- *the cokernel of $B \to P$ in $\mathcal{C}$ is free.*

We call B the saturation of i, it is in fact the image of i in ${}^{+}\mathcal{C}$.

1.3. **Filtrations.** Let $P_\Lambda \in \mathcal{C} \cap {}^{+}\mathcal{C}$, and let us consider the commutative diagram in $\mathcal{C}$:
$$\begin{tikzcd} & P_\Lambda \arrow[rr] & & {}^{p}j_*j^*P_\Lambda \arrow[r] & {}^{p+}j_*j^*P_\Lambda \\ {}^{p}j_!j^*P_\Lambda \arrow[r] \arrow[ur] & {}^{p+}j_!j^*P_\Lambda \arrow[r] \arrow[u] & {}^{p}j_{!*}j^*P_\Lambda \arrow[r] & {}^{p+}j_{!*}j^*P \arrow[u] \arrow[ur] & \end{tikzcd}$$

Definition 5. *We define*
- *$\mathrm{Fil}^1_U(P_\Lambda)$ the image in ${}^{+}\mathcal{C}$ of ${}^{p+}j_!j^*P_\Lambda \to P_\Lambda$;*
- *$\mathrm{Fil}^0_U(P_\Lambda)$ the image in ${}^{+}\mathcal{C}$ of*
$${}^{p}h^{-1}_{free} i^* j_* j^* P_\Lambda \to {}^{p+}j_!j^*P_\Lambda \hookrightarrow \mathrm{Fil}^1_U(P_\Lambda).$$

Lemma 6. *The perverse sheaves $\mathrm{Fil}^0_U(P_\Lambda)$ and $\mathrm{Fil}^1_U(P_\Lambda)$ define in $\mathcal{C}$ a filtration*
$$\mathrm{Fil}^0_U(P_\Lambda) \subset \mathrm{Fil}^1_U(P_\Lambda) \subset P_\Lambda$$
such that:

- *the graded pieces are free in $\mathcal{C}$;*
- *we have a natural epimorphism in ${}^+\mathcal{C}$:*

$$ {}^{p+}j_{!*}j^*P_\Lambda \twoheadrightarrow \mathrm{Fil}^1_U(P_\Lambda)/\mathrm{Fil}^0_U(P_\Lambda); $$

- $P_\Lambda/\mathrm{Fil}^1_U(P_\Lambda) \simeq i_*{}^{p+}i^*P_\Lambda$.

Dually, using the short exact sequence in $\mathcal{C}$

$$ 0 \to {}^{p+}j_{!*}j^*P_\Lambda \longrightarrow {}^pj_*j^*P_\Lambda \longrightarrow {}^ph^0_{libre}i^*j_*j^*P_\Lambda \to 0 $$

and the fact that ${}^pj_*j^*P_\Lambda$ is free in $\mathcal{C}$, we define the following cofiltration.

Definition 7. *One defines*

- $\mathrm{CoFil}_{U,-1}(P_\Lambda)$ *the image in $\mathcal{C}$ of $P_\Lambda \to {}^pj_*j^*P_\Lambda$;*
- $\mathrm{CoFil}_{U,0}(P_\Lambda)$ *the image in $\mathcal{C}$ of*

$$ P_\Lambda \to {}^pj_*j^*P_\Lambda \twoheadrightarrow {}^pi_*{}^ph^0_{libre}i^*j_*P_\Lambda. $$

Remark: If the scheme X is stratified, we can iterate the preceding construction to obtain a filtration or a cofiltration such that the graded pieces gr are of the form

$$ {}^pj_{!*}\mathcal{L} \hookrightarrow gr \hookrightarrow {}^{p+}j_{!*}\mathcal{L} $$

for some local system $\mathcal{L}$ shifted by its dimension.

2. The case of some simple unitary Shimura varieties

2.1. **Definitions.** Let $F = F^+E$ a CM field with $E/\mathbb{Q}$ imaginary quadratic. In [2], they proove the existence of a unitary group G such that:

- $G(\mathbb{R}) \simeq U(1, d-1) \times U(0, d)^{r-1}$;
- $G(\mathbb{Q}_p) \simeq (\mathbb{Q}_p)^\times \times \prod_{i=1}^r (B^{op}_{v_i})^\times$ where $v = v_1, v_2, \cdots, v_r$ are the places of F above the place u of E such that $p = u^cu$ and where B is a central division algebra on F of dimension d^2 with certain properties, for example it is split or a division algebra in each place and split at the place v.

Notation 8. *We note X the tower of the Shimura variety associated to the group G. The Newton stratification of the special fiber X_s is noted $X_s^{\geq h}$.*

To each irreducible representation τ of the group of invertibles $D^\times_{v,h}$ of the central division algebra $D_{v,h}$ on F_v with invariant $1/h$, the authors of [2] define a local system $\mathcal{F}_\tau$ called a Harris-Taylor local system. The intermediate extension associated to these local system is called a Harris-Taylor perverse sheaf.

Notation 9. *By the Jacquet-Langlands correspondence, each irreducible representation τ of $D^\times_{v,h}$ is associated to an irreducible cuspidal representation π_v of $GL_g(F_v)$ with $h = tg$. The corresponding Harris-Taylor perverse sheaf of weight n is noted $\mathcal{P}(\pi_v, t)(-n/2)$.*

2.2. **Filtrations of the perverse sheaf of vanishing cycles.** We note Ψ_v the perverse sheaf of vanishing cycles on the special fiber X_s of X at the place v. Using the monodromy we can decompose Ψ_v in a direct sum:

$$\Psi_{\mathcal{I}} = \bigoplus_{\substack{1 \leq g \leq d \\ \pi_v \in \mathrm{Cusp}_v(g)}} \Psi_{\mathcal{I},\pi_v}$$

where $\mathrm{Cusp}_v(g)$ is the set of inertial equivalence classes of irreducible cuspidal representations of $GL_g(F_v)$ with $1 \leq g \leq d$.

Proposition 10. *(cf.* [1] *corollaire 5.4.2)*
In some Grothendieck group we have the following equality:

$$[\Psi_{\mathcal{I},\pi_v}] = \sum_{k=1-s_g}^{s_g-1} \sum_{\substack{|k|<t\leq s_g \\ t\equiv k-1 \mod 2}} \mathcal{P}(t,\pi_v)(-\frac{k}{2}).$$

Proposition 11. *Let*

$$0 = \mathrm{Fil}^0_{\mathfrak{S}}(\Psi_{\mathcal{I},\pi_v}) \subset \mathrm{Fil}^1_{\mathfrak{S}}(\Psi_{\mathcal{I},\pi_v}) \subset \cdots \subset \mathrm{Fil}^s_{\mathfrak{S}}(\Psi_{\mathcal{I},\pi_v}) = \Psi_{\mathcal{I},\pi_v}$$

be the filtration of stratification of Ψ_{π_v}. For all $1 \leq k \leq s$, the surjection

$$j_!^{\geq kg} HT(\pi_v, [\overleftarrow{k-1}]_{\pi_v}) \otimes L_g(\pi_v)(\frac{1-k}{2}) \twoheadrightarrow \mathrm{Fil}^k_{\mathfrak{S}}(\Psi_{\mathcal{I},\pi_v})/\mathrm{Fil}^{k-1}_{\mathfrak{S}}(\Psi_{\mathcal{I},\pi_v})$$

has image in some Grothendieck group

$$\sum_{i=k}^{s} \mathcal{P}(i,\pi_v)(\frac{k-i}{2}).$$

Remark: This construction explains the maps in the spectral sequence which calculate the sheaves of cohomology of Ψ_{π_v}.

2.3. **Entire version.**

Conjecture 12. *Let π_v be a irreducible cuspidal representation of $GL_g(F_v)$ such that its modular reduction is supercuspidal, then for all $1 \leq t \leq s = \lfloor \frac{d}{g} \rfloor$, we have*

$${}^p j_{!*}^{\geq tg} \mathcal{F}_{\bar{\mathbb{Z}}_l}(\pi_v,t)[d-tg] \simeq {}^{p+} j_{!*}^{\geq tg} \mathcal{F}_{\bar{\mathbb{Z}}_l}(\pi_v,t)[d-tg].$$

Remark: We can easily prove that this result can't be true if the reduction modulo l of π_v is cuspidal but not supercuspidal; in fact we can describe precisely the quotient of these two intermediate extensions in terms of the modular reduction of the Steinberg representation $\mathrm{St}_t(\pi_v)$.

Proposition 13. *The conjecture is true for π_v a character.*

Remark: From this result we can prove the following result.

Corollary 14. *In prime dimension, the cohomology groups of the Lubin-Tate spaces are free.*

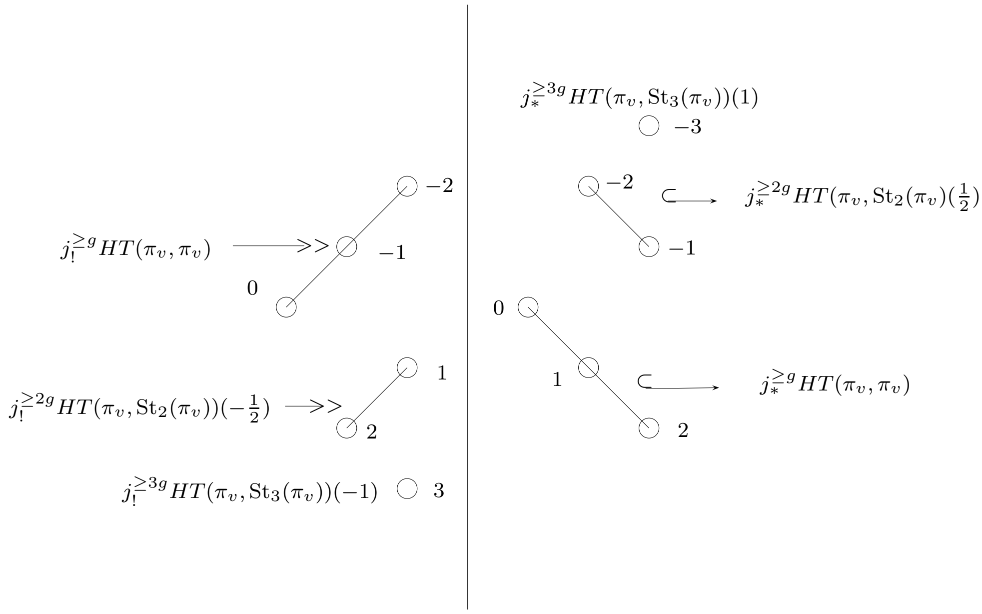

FIGURE 1. Filtration (fig. on the left) and cofiltration (fig. on the right) of stratification of Ψ_{π_v} with $d = 3g$.

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On a Conjecture of Bruinier-Yang

EYAL Z. GOREN
(joint work with Fabrizio Andreatta)

This talk announces major progress towards the proof of a conjecture of Bruinier-Yang concerning the arithmetic intersection number of the so-called small CM points and Heegner, or special, divisors on Shimura varieties of orthogonal type. Although at this time the proof is not written yet in complete detail, we believe that the ideas presented in this talk suffice for a complete proof. Similar results, using similar methods, were obtained independently by Ben Howard and Keerthi Madapusi-Pera.

Let (L,q) be a lattice with quadratic form q of signature $(n+,2-)$. Assume that $L = P \oplus^{\perp} N$, where N is a totally negative two-dimensional sublattice. The even Clifford algebra $\mathfrak{k} = C^{+}(N \otimes_{\mathbb{Z}} \mathbb{Q})$ is then a quadratic imaginary field with associated torus $T = \mathfrak{k}^{\times}$. The map $T(\mathbb{Q})\backslash\{\pm 1\} \times T(\mathbb{A}_f)/K' \to \mathrm{CSpin}(\mathbb{Q})\backslash X \times \mathrm{CSpin}(\mathbb{A}_f)/K =: X_K$, where X is the symmetric space of CSpin, K a compact open subgroup of $\mathrm{CSpin}(\mathbb{A}_f)$ and K' its intersection with $T(\mathbb{A}_f)$, supplies one with a zero cycle $Z(N)$ on X_K, whose points are called small CM points.

By the work of Kisin [5] and Vasiu [10], the varieties X_K have canonical integral models over $\mathbb{Z}$ after inverting 2, the primes p dividing the order of the discriminant group $|L^{\vee}/L|$ and the primes at which K is not hyperspecial. One lets $\mathscr{Z}(N)$ be the normalization of the flat closure of $Z(N)$ in that model. By the work of Borcherds, one has divisors $Z_{\mu,n},\ \mu \in L^{\vee}/L, n < 0$, on X_K; Borcherds called them *Heegner divisors* and also *rational quadratic divisors*, but here we use the terminology *special divisors*. Likewise, they have a flat closure $\mathscr{Z}_{\mu,n}$. Bruinier-Yang [2] made the following conjecture concerning the finite part of the Arakelov intersection number:

$$(\star)\ \ \langle \mathscr{Z}_{m,\mu}, \mathscr{Z}(N)\rangle_{\mathrm{fin}} = \\ -\frac{\deg(Z(N))}{2} \sum_{\substack{\mu_1 \in N^{\vee}/N, \mu_2 \in P^{\vee}/P \\ \mu = \mu_1 + \mu_2}} \ \sum_{\substack{m_i \in \mathbb{Q}_{\geq 0} \\ m = m_1 + m_2}} r(m_1,\mu_1)\cdot \kappa(m_2,\mu_2),$$

where the coefficients $r(m,\mu)$ are coming from a theta series associated to the quadratic imaginary field $\mathfrak{k}$, and the coefficients $\kappa(m,\mu)$ are coming from a theta function associated to P. In a sense, this is an explicit formula. A similar conjecture was formulated for "big" CM points by Bruinier, Kudla and Yang in [1]. Both conjectures are best understood in the context of a far-reaching program due to Kudla, see, e.g., [6]. For lack of space, we cannot review here later important developments.

By virtue of results of Bruinier and Funke, we allow ourselves here to assume that $Z_{\mu,n}$ is the Borcherds lift $\Psi(f)$ of a modular form $f = f_{\mu,n}$. Conjecture $(\star)$ is intertwined with a formula obtained in [2] for the value $\Phi(f; Z(N))$ - the product of the logarithmic Borcherds lift $\Phi(f) = \log \|\Psi(f)\|^2_{\mathrm{Pet}}$ over the points of $Z(N)$.

The usual scenario is that $Z(N) \cap Z_{\mu,n} \neq \emptyset$. This immediately raises two obstacles:

(a) Since $\Psi(f)$ is not defined on $Z_{\mu,n}$ the value of $\Phi(f)$, although well-defined by [9], has mysterious nature;
(b) The intersection of $\mathscr{Z}(N)$ and $\mathscr{Z}_{\mu,n}$ is not proper and that needs to be dealt with in the context of Arakelov intersection theory.

Assume that $L^{\vee}/L$ has square free order and that K is hyperspecial at all odd primes. The prime 2 is excluded from the discussion in this report. Our proof is based on several ingredients:

(1) Let G be the reductive algebraic group over $\mathbb{Q}$ given by $\{x \in C^+(V) : x\cdot\iota(x) \text{ is a scalar}\}$. It defines a Shimura variety of PEL-type, which admits an integral model over $\mathbb{Z}[1/2]$ by Rapoport-Zink [8]. The endomorphism structure is, essentially, multiplication by $C^+(L)$ - a maximal order away from 2. Assuming "K comes from G", we get arithmetic varieties $\mathscr{S}_K$. Relative to $\mathscr{S}_K$ we can describe very explicitly the Shimura variety X_K as PEL + a single Hodge class.

(2) We can prove that the special divisor $Z_{\mu,n}$ is described inside the Shimura variety X_K by a relative PEL property. Given the modular interpretation of points obtained from $\mathscr{S}_K$ as parameterizing abelian varieties with additional structure, the special divisor is - roughly speaking - the locus of abelian varieties with an additional endomorphism that satisfies various properties.

(3) The small CM points have a moduli interpretation. The initial moduli interpretation of the points on the Shimura variety associated to T is that of elliptic curves E with additional structure. We show that the image in X_K are points that parameterize (via the modular description provided by $\mathscr{S}_K$) abelian varieties of the form $C^+(L) \otimes_{C^+(N)} E$ with additional structure.

(4) The analysis of non-proper intersection is handled via deformation to the normal cone in Arakelov geometry developed in Hu's thesis [4]. This method clarifies at the same time the meaning of the function $\Phi(f)$ at points on $Z_{\mu,n}$. One ingredient is the calculation of the intersection of a component D_λ of $\mathscr{Z}_{\mu,n}$ with $\widehat{D}_\lambda$ - the divisor D_λ equipped with a particular Green function, proving that it is proportional to the tautological meterized line bundle $\widehat{\omega}$ on D_λ.

(5) Using the theory of local models for the lattices $L \subset L^\vee$ we explicitly describe the normal cone of a small CM point inside the normalization of the schematic closure of X_K inside $\mathscr{S}_K$. At least at primes not dividing $|L^\vee/L|$, it gives the local ring at the given point for the integral model of X_K [5, 10]; using recent work of Madapusi-Pera, it should possible to extend this to all primes.

(6) The analysis of components of $\mathscr{Z}_{\mu,n}$ having proper intersection with points of $\mathscr{Z}(N)$ is then translated, making use of the relative PEL nature of the special divisor and the interpretation of the points in $\mathscr{Z}(N)$ as arising via a Serre tensor construction, to a problem about special endomorphisms (in the sense of Kudla) of elliptic curves with CM. This is handled very much in the same way as in [7] via reduction to calculation of lengths of certain artinian rings; a calculation that rests on Gross's work [3].

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A Tannakian approach to Bruhat-Tits buildings and parahoric group schemes

Thomas J. Haines
(joint work with Kevin Wilson)

1. Introduction

For the general linear group, the Bruhat-Tits building can be described explicitly in terms of periodic lattice chains in the standard representation. Furthermore, any parahoric group scheme may be described as a certain automorphism group of such chains. This talk will explain joint work in progress with Kevin Wilson, in which we give a Tannakian description of buildings and parahoric group schemes for general connected reductive groups over complete discretely valued fields. This project was inspired by the study of Rapoport-Zink local models.

2. Basic objects for GL_n

Let K denote a complete discretely valued field, with ring of integers $\mathcal{O}$ and uniformizer π. Let G be a connected reductive group over K. Then Bruhat-Tits theory [BT1], [BT2] defines a building $\mathcal{B}(G,K)$. It decomposes, essentially canonically, as a product $\mathcal{B}(G_{\mathrm{der}},K) \times \mathcal{B}(Z(G)^\circ, K)$. Note the second factor is simply a Euclidean space of dimension dim $Z(G)$. For example, the building for GL_2 is the product of a tree with the Euclidean space $\mathbb{R}$.

Given a facet $\mathbf{F} \subset \mathcal{B}(G,K)$, Bruhat-Tits theory constructs a smooth affine group scheme $\mathcal{G}_{\mathbf{F}}$ over $\mathcal{O}$.

Consider the example $G = \mathrm{GL}_n$. The building has maximal simplices (a.k.a *alcoves* $\mathbf{a}$) in 1-1 correspondence with complete periodic $\mathcal{O}$-lattice chains

$$\mathbf{a} = (\Lambda_0 \subset \Lambda_1 \subset \cdots \subset \Lambda_n = \pi^{-1}\Lambda_0) \subset K^n.$$

Given such $\mathbf{a} = \Lambda_\bullet$, we can define a group functor (designated RZ for *Rapoport-Zink*) on commutative $\mathcal{O}$-algebras R by

$$\mathcal{G}_{\mathbf{a}}^{\mathrm{RZ}}(R) = \{(g_0, \dots, g_{n-1}) \in \prod_i \mathrm{GL}_R(\Lambda_{i,R}) \mid \text{condition (Comm) holds}\}$$

(Comm) Writing $\Lambda_{i,R}$ for $\Lambda_i \otimes_{\mathcal{O}} R$, the following diagram commutes

$$\begin{tikzcd}
\Lambda_{0,R} \arrow[r] \arrow[d, "g_0"'] & \cdots \arrow[r] & \Lambda_{n-1,R} \arrow[r, "\pi"] \arrow[d, "g_{n-1}"'] & \Lambda_{0,R} \arrow[d, "g_0"'] \\
\Lambda_{0,R} \arrow[r] & \cdots \arrow[r] & \Lambda_{n-1,R} \arrow[r, "\pi"] & \Lambda_{0,R}.
\end{tikzcd}$$

A similar definition works for parahoric group schemes.

Proposition 1 ([RZ])**.** *$\mathcal{G}_{\mathbf{a}}^{\mathrm{RZ}}$ is formally smooth over $\mathcal{O}$.*

Corollary 2. $\mathcal{G}_{\mathbf{a}}^{\mathrm{RZ}} = \mathcal{G}_{\mathbf{a}}$.

Proof. It is easy to see that an affine, finite-type, and formally smooth group scheme $\mathcal{H}$ over $\mathcal{O}$ with fixed generic fiber H/K is uniquely determined (up to a unique isomorphism) by the group $\mathcal{H}(\mathcal{O}_L)$, where $\mathcal{O}_L$ is the ring of integers in the field $L := \widehat{K^{\mathrm{un}}}$. The two group schemes $\mathcal{G}_{\mathbf{a}}$ and $\mathcal{G}_{\mathbf{a}}^{\mathrm{RZ}}$ have these properties and have as $\mathcal{O}_L$-points the fixer of $\mathbf{a}$ in $G(L)$. $\square$

3. General case: Moy-Prasad filtrations

It is simpler (and a first step) to construct objects over the field $L = \widehat{K^{\mathrm{un}}}$. Henceforth, we therefore assume $K = L$. For simplicity we also assume that G is K-split, and that $G_{\mathrm{der}} = G_{\mathrm{sc}}$.

We need to work integrally; thus we fix a special maximal vertex $o \in \mathcal{B}(G, K)$. We denote by $\mathcal{G} := \mathcal{G}_o$ the Chevalley group scheme over $\mathcal{O}$ associated to o. We will construct lattice chains in $V_K := V \otimes_{\mathcal{O}} K$ for all $V \in \mathrm{Rep}^\circ_{\mathcal{O}}(\mathcal{G})$, the category of representations of $\mathcal{G}$ on finite-type projective $\mathcal{O}$-modules V.

What kind of lattice chains will we consider? For $V = \mathrm{Lie}(G) =: \mathfrak{g}$, Moy and Prasad [MP] construct $\mathcal{O}$-lattices $\mathfrak{g}_{x,r}$ ($x \in \mathcal{B}(G, K)$, $r \in \mathbb{R}$). We define similar lattices $V_{x,r}$ in V_K for every V and every $r \in \mathbb{R}$.

There is a 1-1 correspondence $\mathbf{A} \leftrightarrow A$ between apartments $\mathbf{A}$ containing o and maximal $\mathcal{O}$-split tori $A \subset \mathcal{G}$ (we also write $A \subset G$ for its generic fiber). Fix $\mathbf{A}$. For $x \in \mathbf{A}$, we set

$$V_{x,r}^{\mathbf{A}} = \bigoplus_{\lambda \in X^*(A)} V_\lambda^A \otimes_{\mathcal{O}} \mathcal{O}\pi^{\lceil r - \langle \lambda, x - o \rangle \rceil}. \tag{1}$$

Here V_λ^A is the λ-weight space for the restriction of V to A.

For any $y \in \mathcal{B}(G, K)$, write $y = gx$ for some $g \in G(K)$ and $x \in \mathbf{A}$, and set $V_{y,r}^{\mathbf{A}} = g(V_{x,r}^{\mathbf{A}})$. A lemma shows that $V_{y,r}^{\mathbf{A}}$ is independent of the choices for g and x. Another lemma shows that $V_{x,r}^{\mathbf{A}}$ is independent of the choice of $\mathbf{A}$. Hence we may define, for any x, r,

$$V_{x,r} := V_{x,r}^{\mathbf{A}}$$

and this agrees with (1) whenever $x \in \mathbf{A}$.

The family of lattices $V_{x,r}$ satisfies some obvious compatibilities as V and r vary. We call the collection of all such *abstract* families $(V_r)_{V,r}$ *Moy-Prasad filtrations on the category* $\mathrm{Rep}^\circ_{\mathcal{O}}(\mathcal{G})$ and denote the set of such by $\mathcal{M}P(G,K,o)$. This notion yields a Tannakian description of the building.

Theorem 3. *(In progress) The map $x \mapsto (V_{x,r})_{V,r}$ gives a $G(K)$-equivariant bijection $\mathcal{B}(G,K) \xrightarrow{\sim} \mathcal{M}P(G,K,o)$.*

4. Construction of $\mathcal{G}_{\mathbf{F}}$

Decompose $o = o' \times o''$ and $\mathbf{F} = \mathbf{F}' \times \mathcal{B}(Z(G)^\circ, K)$ in $\mathcal{B}(G_{\mathrm{der}}, K) \times \mathcal{B}(Z(G)^\circ, K)$, and identify $\mathbf{F}'$ with $\mathbf{F}' \times o'' \subset \mathbf{F}$.

Definition 4. *Let $\mathrm{Aut}_{\mathbf{F}}$ be the group-valued functor on the category $\underline{\mathrm{Alg}}_{\mathcal{O}}$ defined as follows:*

$$\mathrm{Aut}_{\mathbf{F}}(R) = \Big\{ (g^R_{V_{x,r}}) \in \prod_{\substack{V \in \mathrm{Rep}^\circ_{\mathcal{O}}(\mathcal{G}) \\ x \in \mathbf{F}',\, r \in \mathbb{R}}} \mathrm{GL}_R(V_{x,r} \otimes_{\mathcal{O}} R) \mid \textit{conditions (Aut0)-(Aut3) hold} \Big\}$$

(Aut0) *For all pairs $(x,r),(y,s) \in \mathbf{F}' \times \mathbb{R}$ and integers $n \in \mathbb{N}$ such that $V_{x,r+n} \subset V_{y,s} \subset V_{x,r}$ the following diagram commutes:*

$$\begin{CD}
V_{x,r}\otimes_{\mathcal{O}} R @>{\cong \cdot \pi^n}>> V_{x,r+n}\otimes_{\mathcal{O}} R @>>> V_{y,s}\otimes_{\mathcal{O}} R @>>> V_{x,r}\otimes_{\mathcal{O}} R \\
@V{g^R_{V_{x,r}}}VV @V{g^R_{V_{x,r+n}}}VV @V{g^R_{V_{y,s}}}VV @V{g^R_{V_{x,r}}}VV \\
V_{x,r}\otimes_{\mathcal{O}} R @>{\cong \cdot \pi^n}>> V_{x,r+n}\otimes_{\mathcal{O}} R @>>> V_{y,s}\otimes_{\mathcal{O}} R @>>> V_{x,r}\otimes_{\mathcal{O}} R.
\end{CD}$$

(Aut1) *Let $\mathbf{1}$ denote the trivial representation on $\mathcal{O}$. For all $(x,r) \in \mathbf{F}' \times \mathbb{R}$ we have $g^R_{\mathbf{1}_{x,r}} = \mathrm{id}_R$.*

(Aut2) *For every morphism $U \xrightarrow{\phi} V$ in $\mathrm{Rep}^\circ_{\mathcal{O}}(\mathcal{G})$ and every $(x,r) \in \mathbf{F}' \times \mathbb{R}$ the following diagram commutes:*

$$\begin{CD}
U_{x,r} \otimes_{\mathcal{O}} R @>{\phi^R}>> V_{x,r} \otimes_{\mathcal{O}} R \\
@V{g^R_{U_{x,r}}}VV @V{g^R_{V_{x,r}}}VV \\
U_{x,r} \otimes_{\mathcal{O}} R @>{\phi^R}>> V_{x,r} \otimes_{\mathcal{O}} R.
\end{CD}$$

(Aut3) *For $V, W \in \mathrm{Rep}^\circ_{\mathcal{O}}(\mathcal{G})$ and $(x,r) \in \mathbf{F}' \times \mathbb{R}$ and $s,t \in \mathbb{R}$ with $s+t=r$, the following diagram commutes:*

$$\begin{CD}
(V_{x,s} \otimes_{\mathcal{O}} R) \otimes_R (W_{x,t} \otimes_{\mathcal{O}} R) @>>> (V \otimes_{\mathcal{O}} W)_{x,r} \otimes_{\mathcal{O}} R \\
@V{g^R_{V_{x,s}} \otimes g^R_{W_{x,t}}}VV @V{g^R_{(V\otimes W)_{x,r}}}VV \\
(V_{x,s} \otimes_{\mathcal{O}} R) \otimes_R (W_{x,t} \otimes_{\mathcal{O}} R) @>>> (V \otimes_{\mathcal{O}} W)_{x,r} \otimes_{\mathcal{O}} R.
\end{CD}$$

Using the same characterization ideas as for GL_n, this leads to our Tannakian description of the group schemes $\mathcal{G}_{\mathbf{F}}$ in this situation.

Theorem 5. $\mathrm{Aut}_{\mathbf{F}} = \mathcal{G}_{\mathbf{F}}$.

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On the regularity of special difference divisors on unitary Rapoport-Zink spaces of signature $(1, n-1)$

Ulrich Terstiege

In this talk it was explained that special difference divisors on unitary Rapoport-Zink spaces of signature $(1, n-1)$ in the unramified case are always regular. It was also explained how one can apply some methods of the proof of this statement to the arithmetic fundamental lemma conjecture in the minuscule case.

Let us recall from [1] the definition of the Rapoport-Zink space and of special cycles in that case. Let $n \geq 1$ be an integer and let $p \geq 3$ be a prime. Let $\mathbb{F} = \overline{\mathbb{F}}_p$ and $W = W(\mathbb{F})$. We consider the Rapoport-Zink space $\mathcal{N} := \mathcal{N}_n := \mathcal{N}(1, n-1)$ over W parameterizing tuples $(X, \iota, \lambda, \rho)$ over W-schemes S where p is locally nilpotent in $\mathcal{O}_S$ and where a tuple $(X, \iota, \lambda, \rho)$ over S consists of the following objects. First, X is a p-divisible group of dimension n and height $2n$ over S, $\iota : \mathbb{Z}_{p^2} \to \mathrm{End}(X)$ is a homomorphism satisfying the determinant condition of signature $(1, n-1)$, further λ is a principal polarization of X such that $\iota^*(a) = \iota(\overline{a})$ for the Rosati involution and for all $a \in \mathbb{Z}_{p^2}$, and

$$\rho : X \times_S \overline{S} \to \mathbb{X} \times_{\mathrm{Spec}\ \mathbb{F}} \overline{S}$$

is a $\mathbb{Z}_{p^2}$-linear quasi-isogeny of height 0. Here $\overline{S} = S \times_{\mathrm{Spec}\ W} \mathrm{Spec}\ \mathbb{F}$ and $(\mathbb{X}, \iota_{\mathbb{X}}, \lambda_{\mathbb{X}})$ is a fixed triple over $\mathrm{Spec}\ \mathbb{F}$ as before and where $\mathbb{X}$ is also required to be supersingular. We also require that locally up to a scalar in $\mathbb{Z}_p^\times$ we have the identity $\hat{\rho} \circ \lambda_{\mathbb{X}} \circ \rho = \lambda$. Denote by $(\mathbb{Y}, \iota_{\mathbb{Y}}, \lambda_{\mathbb{Y}})$ over $\mathbb{F}$ the fixed supersingular object for $n = 1$ and denote by $\overline{\mathbb{Y}}$ the same object but the $\mathbb{Z}_{p^2}$-action replaced by its conjugate. It has a canonical lift $\overline{Y}$ over W. The space of special homomorphisms is the n-dimensional hermitian $\mathbb{Q}_{p^2}$-space $\mathbb{V} = \mathrm{Hom}_{\mathbb{Z}_{p^2}}(\overline{\mathbb{Y}}, \mathbb{X}) \otimes \mathbb{Q}$ with hermitian form h given by

$$h(x, y) = \lambda_{\overline{\mathbb{Y}}}^{-1} \circ \hat{y} \circ \lambda_{\mathbb{X}} \circ x \in \mathrm{End}_{\mathbb{Z}_{p^2}}(\overline{\mathbb{Y}}) \otimes \mathbb{Q} \cong \mathbb{Q}_{p^2},$$

where the last isomorphism is via $\iota_{\overline{\mathbb{Y}}}^{-1}$, and where $\hat{y}$ is the dual of y. For $j \in \mathbb{V}$ the special cycle $\mathcal{Z}(j)$ is the closed formal subscheme of $\mathcal{N}$ such that $\mathcal{Z}(S)$ is the set of all $(X, \iota, \lambda, \rho)$ over S such that the quasi-homomorphism

$$\overline{\mathbb{Y}} \times_{\mathbb{F}} \overline{S} \xrightarrow{j} \mathbb{X} \times_{\mathbb{F}} \overline{S} \xrightarrow{\varrho^{-1}} X \times_S \overline{S}$$

lifts to a homomorphism $\overline{Y} \times_{\mathrm{Spec}\ W} S \to X$. Further we define as in [3] the special difference divisor $\mathcal{D}(j)$ as $\mathcal{D}(j) = \mathcal{Z}(j) - \mathcal{Z}(j/p)$.

A conjecture of Kudla and Rapoport connects intersection multiplicities of special cycles with derivatives of certain representaion densities. It was proved for non-degenerate intersections (which can be reduced to the case $n = 2$) by Kudla and Rapoport (see [1]). For $n = 3$ (which is the first case with degenerate intersections) it was proved in [3]. An important ingredient of the proof for $n = 3$ is the statement that the $\mathcal{D}(j)$ are regular. The main theorem discussed in this talk is that this is true in arbitrary dimension:

Theorem 1. *Let j be a special homomorphism. Then the special difference divisor $\mathcal{D}(j)$ is regular.*

An important ingredient of the proof is the following statement proved in [2]:

Theorem 2. *Let $x \in \mathcal{N}(\mathbb{F})$ be a point such that there is no special homomorphism j_0 of valuation 0 with $x \in \mathcal{Z}(j_0)(\mathbb{F})$ and let j be a special homomorphism such that $x \in \mathcal{Z}(j)(\mathbb{F})$ but $x \notin \mathcal{Z}(j/p)(\mathbb{F})$. Then the special fiber $\mathcal{Z}(j)_p$ of $\mathcal{Z}(j)$ is regular at x.*

If j_0 is a special homomorphism of valuation 0 then $\mathcal{Z}(j_0)$ can be identified with $\mathcal{N}_{n-1}$. This allows an inductive approach to the proof of Theorem 1 using Theorem 2 (at least at points x such that $x \in \mathcal{Z}(j)(\mathbb{F})$ but $x \notin \mathcal{Z}(j/p)(\mathbb{F})$).

It was also explained how these techniques can be applied to the arithmetic fundamental lemma conjecture of W. Zhang in the minuscule case (cf. [2]). To this end one can use Theorem 2 so show the following theorem.

Theorem 3. *Let $j_1, ..., j_n$ be special homomorphisms such that their fundamental matrix $T(j_1, ..., j_n)$ is equivalent to a matrix of the form $\mathrm{diag}(1, ..., 1, p, ..., p)$. Then the intersection $\bigcap_{i=1}^n \mathcal{Z}(j_i)$ lies in the special fiber.*

The arithmetic fundamental lemma compares the derivative of a certain orbital integral with an intersection number on $\mathcal{N}$. In the so called minuscule case one can use theorem 3 to show that this intersection lies in the special fiber. This allows an explicit calculation of the intersection number (at least for p large enough) and can be used for a proof of the arithmetic fundamental lemma in the minuscule case (for $F = \mathbb{Q}_p$ and $n \leq 2p$), see [2].

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Cell decomposition of some unitary group Rapoport-Zink spaces

Xu Shen

Let $p > 2$ be a fixed prime, $\mathbb{Q}_{p^2}|\mathbb{Q}_p$ be a quadratic unramified extension. Let $(V, \langle,\rangle)$ be a hermitian space over $\mathbb{Q}_{p^2}$, and $G = GU(V, \langle,\rangle)$ be the associated unitary similitude group over $\mathbb{Q}_p$. Denote by $n = \dim_{\mathbb{Q}_{p^2}} V$, and assume there exists an autodual $\mathbb{Z}_{p^2}$-lattice in V. This implies G is unramified. Let $\overline{\mathbb{Q}}_p$ be an algebraic closure, then after fixing a basis of V we have an isomorphism $G_{\overline{\mathbb{Q}}_p} \simeq GL_{n\overline{\mathbb{Q}}_p} \times \mathbb{G}_{m\overline{\mathbb{Q}}_p}$. Consider the cocharacter $\mu : \mathbb{G}_{m\overline{\mathbb{Q}}_p} \to G_{\overline{\mathbb{Q}}_p}$, such that under the above isomorphism it is given by $z \mapsto (diag(z, \cdots, z, 1), z)$. Let $b = b_0 \in B(G, \mu)$ (the Kottwitz set) be the basic element, J_b be the associated inner form of G. We remark that if n is odd, then $J_b \simeq G$, and if n is even J_b is up to isomorphism the unique non quasi-split inner form of G.

Let $W = W(\overline{\mathbb{F}}_p), L = W_{\mathbb{Q}}$. Consider the associated Rapoport-Zink space $\widehat{\mathcal{M}}$ over $Spf\, W$: for any $S \in \mathrm{Nilp}\, W$, $\widehat{\mathcal{M}}(S) = \{(H, \iota, \lambda, \rho)\}/\simeq$, where H is a p-divisible group over S, ι is a $\mathbb{Z}_{p^2}$-action on H satisfying the determinant condition corresponding to μ, λ is a polarization which is compatible with ι, and $\rho : \mathbb{H}_{\overline{S}} \to H_{\overline{S}}$ is a quasi-isogeny (cf. [8] for more details). Here $\mathbb{H}$ is the standard unitary p-divisible group over $\overline{\mathbb{F}}_p$. We consider the Berkovich analytic generic fiber $\mathcal{M} = \widehat{\mathcal{M}}^{an}$ over L. As usual, there is in fact a tower of L-analytic spaces $(\mathcal{M}_K)_{K\subset G(\mathbb{Z}_p)}$, where the index set is the open compact subgroups K of $G(\mathbb{Z}_p)$ and $\mathcal{M}_{G(\mathbb{Z}_p)} = \mathcal{M}$. $J_b(\mathbb{Q}_p)$ acts naturally on each space $\mathcal{M}_K$ by modifying the quasi-isogeny, and moreover, $G(\mathbb{Q}_p)$ acts on the tower $(\mathcal{M}_K)_{K\subset G(\mathbb{Z}_p)}$ by Hecke correspondences. N ote we have the decompositions (cf.[8])

$$\widehat{\mathcal{M}} = \coprod_{i\in\mathbb{Z},\, ni\ even} \widehat{\mathcal{M}}^i, \quad \mathcal{M} = \coprod_{i\in\mathbb{Z},\, ni\ even} \mathcal{M}^i.$$

To state the theorem, we should fix some data. If n is even, fix an element $g_1 \in J_b(\mathbb{Q}_p)$ such that it induces an isomorphism $\mathcal{M}^0 \to \mathcal{M}^1$. We fix also a $\Lambda \in \mathcal{B}(J_b^{der}, \mathbb{Q}_p)$, the set of vertices of the Bruhat-Tits building of the derived subgroup J_b^{der} of J_b, such that $t(\Lambda)$ is maximal (cf. [8] for the precise meaning of the function t). Let $Stab(\Lambda)$ be the stabilizer of Λ in $J_b^{der}(\mathbb{Q}_p)$.

Theorem 1. *There exists a relatively compact analytic domain $\mathcal{D} \subset \mathcal{M}^0$, such that we have a locally finite covering*

$$\mathcal{M} = \bigcup_{\substack{T\in G(\mathbb{Z}_p)\backslash G(\mathbb{Q}_p)/G(\mathbb{Z}_p)\\ g\in J_b^{der}(\mathbb{Q}_p)/Stab(\Lambda)}} T.g\mathcal{D}$$

if n is odd, and

$$\mathcal{M} = \bigcup_{\substack{T\in G(\mathbb{Z}_p)\backslash G(\mathbb{Q}_p)/G(\mathbb{Z}_p)\\ g\in J_b^{der}(\mathbb{Q}_p)/Stab(\Lambda)\\ j=0,1}} T.gg_1^j\mathcal{D}$$

if n is even.

The proof of this theorem is based some ideas developed in [3] and [4]. In particular we use the theory of Harder-Narasimhan filtrations of finite flat group schemes to study the p-analytic geometry of $\mathcal{M}$. The fundamental inequality between Harder-Narasimhan polygon and Newton polygon (Théorème 21 of [4]) can be easily generalized to our case. But we have to modify Fargues's algorithm in [4] a little to produce totally isotropic finite flat group schemes to be compatible with Hecke correspondences. The analytic domain $\mathcal{D}$ is defined as following. Let $\mathcal{M}^{ss}$ be the semi-stable locus in $\mathcal{M}$ (cf. Définition 4 of [4]). Consider

$$\mathcal{C} = \{x \in \mathcal{M} |\, \exists \,\text{some finite extension}\, K'|\mathcal{H}(x), \text{ and a finite flat}\, \mathbb{Z}_{p^2} - \text{subgroup}\\ \text{scheme}\, G \subset H_x[p] \,\text{over}\, O_{K'}, \text{such that}\, H_x/G \,\text{is semi-stable over}\, O_{K'}\}.$$

Then one can prove that $\mathcal{C}$ is a closed analytic domain of $\mathcal{M}$. Note $\mathcal{M}^{ss} \subset \mathcal{C}$. Let Λ be as above, and $\mathcal{M}_\Lambda \subset \mathcal{M}^0_{red}$ be the associated projective subvariety of the reduced special fiber of $\widehat{\mathcal{M}}^0$ defined by Vollaard-Wedhorn in [8]. Consider the specialization map $sp : \mathcal{M}^0 \to \mathcal{M}^0_{red}$, then $sp^{-1}(\mathcal{M}_\Lambda)$ is an open subspace of $\mathcal{M}^0$. The analytic domain $\mathcal{D}$ is defined by $\mathcal{D} := \mathcal{C} \bigcap sp^{-1}(\mathcal{M}_\Lambda)$. The relatively compactness of $\mathcal{D}$ is proved by introducing some special unitary Shimura varieties (cf. [1] and [8]), and the fact that their Harder-Narasimhan stratification and Newton stratification coincide (cf. [6] and [7]). We remark that our methods of proof of the above theorem in some other places are also different from that of [4].

This theorem has many useful applications. First, we have corresponding coverings of the associated p-adic period domain and Shimura varieties. Second, we have the locally finite coverings for all Rapoport-Zink spaces $\mathcal{M}_K$ for any open compact subgroup $K \subset G(\mathbb{Z}_p)$. By studying the action of regular semi-simple elliptic elements on the coverings of the later, we can verify easily that the conditions of Theorem 3.13 in [5] hold. Thus we can establish a Lefschetz trace formula for some sufficiently large subspaces. For more details, see section 11 of [7]. This formula should be useful for proving the realization of local Jacquet-Langlands correspondence in our case.

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On local models of Shimura varieties

George Pappas

(joint work with X. Zhu)

We give a group theoretic definition of "local models" as sought after in the theory of Shimura varieties [2]. These are projective schemes over the integers of a p-adic local field that are expected to model the singularities of integral models of Shimura varieties with parahoric level structure. Our local models are certain mixed characteristic degenerations of Grassmannian varieties; they are obtained by extending constructions of Beilinson, Drinfeld, Gaitsgory to mixed characteristics and to the case of general (tamely ramified) reductive groups.

Suppose that $(G, K_p, \{\mu\})$ are the "local Shimura data" at the prime p obtained from the triple $(\mathbb{G}, \{h\}, K)$ defining a Shimura variety: Then $G = \mathbb{G} \otimes_{\mathbb{Q}} \mathbb{Q}_p$, $K_p \subset G(\mathbb{Q}_p)$ is the level subgroup at p and $\{\mu\}$ is the conjugacy class (defined over the local reflex field E) of the miniscule coweight μ obtained from $\{h\}$. We assume that K_p is a parahoric subgroup in the sense of Bruhat-Tits and that G splits over a tamely ramified extension of $\mathbb{Q}_p$. Recall that Bruhat and Tits define a canonical smooth affine connected group scheme $\mathcal{K}$ over $\mathbb{Z}_p$ (a "parahoric group scheme") with generic fiber G and $\mathcal{K}(\mathbb{Z}_p) = K_p$. Our construction now proceeds as follows:

Step 1. We construct a certain (smooth, affine, connected) group scheme $\mathcal{G}$ over the affine line $\mathbb{Z}_p[u]$ such that $\mathcal{G} \otimes_{\mathbb{Z}_p[u]} \mathbb{Z}_p[u, u^{-1}]$ is reductive and with

$$\mathcal{G} \otimes_{\mathbb{Z}_p[u]} \mathbb{Q}_p((u)) = G \otimes_{Q_p} \mathbb{Q}_p((u)), \qquad \mathcal{G} \otimes_{\mathbb{Z}_p[u], u \mapsto p} \mathbb{Z}_p = \mathcal{K}.$$

In addition, the base change $\mathcal{K}' := \mathcal{G} \otimes_{\mathbb{Z}_p[u]} \mathbb{F}_p[[u]]$ is a parahoric group scheme for the reductive group $G' := \mathcal{G} \otimes_{\mathbb{Z}_p[u]} \mathbb{F}_p((u))$ over $\mathbb{F}_p((u))$.

Step 2. We show that the global affine Grassmannian $\mathrm{Gr}_{\mathcal{G}} \to \mathbb{A}^1_{\mathbb{Z}_p}$ for $\mathcal{G}$ is representable by an ind-scheme which is ind-projective over $\mathbb{A}^1_{\mathbb{Z}_p}$. Here, $\mathrm{Gr}_{\mathcal{G}}$ is the functor which to the $\mathbb{A}^1_{\mathbb{Z}_p}$-scheme $y : S \to \mathbb{A}^1_{\mathbb{Z}_p}$ associates the set of isomorphism classes of $\mathcal{G}$-torsors over $\mathbb{A}^1_{\mathbb{Z}_p} \times_{\mathbb{Z}_p} S$ together with a trivialization on the complement of the graph of y.

Step 3. We show that the base change $\mathrm{Gr}_{\mathcal{G}} \otimes_{\mathbb{Z}_p[u], u \mapsto p} \mathbb{Q}_p$ of $\mathrm{Gr}_{\mathcal{G}} \to \mathbb{A}^1_{\mathbb{Z}_p}$ along $\mathbb{Z}_p[u] \to \mathbb{Q}_p$ can be identified with the loop Grassmannian $G(\mathbb{Q}_p((u)))/G(\mathbb{Q}_p[[u]])$ of G and that the G-homogeneous space $X_\mu = G/P_\mu$ associated to the orbit of minuscule coweights $\{\mu\}$ can be embedded equivariantly (as a closed smooth subvariety) in the base change of $G(\mathbb{Q}_p((u)))/G(\mathbb{Q}_p[[u]])$ to E.

Step 4. Finally, the local model $M^{\rm loc}$ associated to $(G, K_p, \{\mu\})$ is by definition the Zariski closure of $X_\mu \subset (G(\mathbb{Q}_p((u)))/G(\mathbb{Q}_p[[u]]))_E$ in $\mathrm{Gr}_{\mathcal{G}} \otimes_{\mathbb{Z}_p[u], u\mapsto p} \mathcal{O}_E$. By definition, it supports an action of $\mathcal{K} \otimes_{\mathbb{Z}_p} \mathcal{O}_E$ and its special fiber is a closed subscheme of the base change of the "twisted" affine flag variety for G' and $\mathcal{K}'$ to the residue field of E.

Although, this is an involved definition, it has several advantages and we can use it to obtain structural results on $M^{\rm loc}$ and its special fiber: For example, under the additional assumption that p does not divide the order of the fundamental group of the derived group of G, we can show that the geometric special fiber of $M^{\rm loc}$ is reduced and can be stratified with strata parametrized by the μ-admissible set of Kottwitz-Rapoport. The closures of these strata are affine Schubert varieties and therefore, by results of Kumar, Mathieu, Faltings and Pappas-Rapoport, are normal and Cohen-Macaulay.

Finally, we explain that in most cases of Shimura varieties of PEL type our "abstract" local models can be identified with the flat closures of the local models defined by Rapoport and Zink [3]. Therefore, in these cases, the local models fit in a diagram

$$\mathcal{S}_K \xleftarrow{\pi} \tilde{\mathcal{S}}_K \xrightarrow{\psi} M^{\rm loc}$$

where $\mathcal{S}_K$ is a flat $\mathcal{O}_E$-model of the Shimura variety, π is a $\mathcal{K}$-torsor and ψ is a $\mathcal{K}$-equivariant *smooth* morphism. As a consequence, our structure results on the special fiber of $M^{\rm loc}$ imply similar results for the corresponding integral model $\mathcal{S}_K$ of the Shimura variety. We conjecture the existence of a similar diagram for the general Shimura variety with parahoric level subgroup. For more details the reader is referred to [1].

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Connected components of minuscule affine Deligne-Lusztig varieties

Miaofen Chen

(joint work with Mark Kisin, Eva Viehmann)

Let k be a finite field with $q = p^r$ elements and let $\overline{k}$ be an algebraic closure of k. We consider both the equal characteristic case and the mixed characteristic case. In the equal (resp. mixed) characteristic case, let $F = k((t))$ (resp. $F = W(k)[1/p]$ be the fraction field of the Witt vector ring of k) and let accordingly $L = \overline{k}((t))$

(resp. $L = W(\overline{k})[1/p]$). Let $\mathcal{O}_F$ and $\mathcal{O}_L$ be the valuation rings. We denote by ϵ the uniformizer t or p. Let $\sigma : x \mapsto x^q$ be the Frobenius of $\overline{k}$ over k and also the induced Frobenius of L over F.

Let G be a connected unramified group over $\mathcal{O}_F$. Let G_F be the generic fiber of G. Let $B \subset G_F$ be a Borel subgroup and $T \subset B$ the centralizer of a maximal split torus in B. We denote by $X_*(T)$ the set of cocharacters of T defined over L. The Galois group acts on $X_*(T)$. As G is unramified, the Galois action factors through the quotient $\Gamma = \langle \sigma \rangle$.

For $b \in G(L)$ and a minuscule dominant coweight $\mu \in X_*(T)$, the affine Deligne-Lusztig variety $X_\mu^G(b) = X_\mu(b)$ is defined as

$$X_\mu(b)(\overline{k}) = \{g \in G(L)/G(\mathcal{O}_L) \mid g^{-1}b\sigma(g) \in G(\mathcal{O}_L)\mu(\epsilon)G(\mathcal{O}_L)\}.$$

For non minuscule coweights, we can also define the corresponding (closed) affine Deligne-Lusztig varieties. However, for the applications to Shimura varieties and the moduli spaces of p-divisible groups, the coweight is always minuscule. So we restrict ourselves to the minuscule situation.

A priori, the affine Deligne-Lusztig varieties are just sets of points, and do not have the structure of an algebraic variety. If F is a function field, these sets are the sets of $\overline{k}$-valued points of closed subschemes locally of finite type of the affine Grassmannian LG/K where LG denotes the loop group of G (compare [R], [GHKR]). If F is of mixed characteristic, in general there is no known variety structure on the affine Deligne-Lusztig varieties. However, we can still define a meaningful notion of a set of connected components $\pi_0(X_\mu(b))$. For the particular cases when $F = \mathbb{Q}_p$ and G is the restriction of scalars of a linear group or similitude unitary/symplectic group, $X_\mu(b)$ are the sets of $\overline{k}$-valued points of moduli spaces of p-divisible groups as defined by Rapoport and Zink [RZ] which are formal schemes locally formally of finite type over $\mathrm{Spf}\mathcal{O}_L$. In this case, $\pi_0(X_\mu(b))$ coincides naturally with the set of connected components of the corresponding Rapoport-Zink space.

Let $J = J_b = \{g \in G(L) | g^{-1}b\sigma g = b\}$. The group J acts on $X_\mu(b)(\overline{k})$ by multiplication on the left.

By $[x]$ we denote the σ-conjugacy class of an element $x \in G(L)$. The isomorphism class of the affine Deligne-Lusztig variety only depends on $[b]$ and not on b. We denote by $B(G)$ the set of σ-conjugacy classes of elements in $G(L)$.

We write $\pi_1(G)$ for the quotient of $X_*(T)$ by the coroot lattice of G. In [K2], Kottwitz defines a homomorphism $w_G : G(L) \to \pi_1(G)$. The homomorphism w_G induces a homomorphism $\kappa_G : B(G) \to \pi_1(G)_\Gamma$.

We have a criterion for $X_\mu(b)$ to be nonempty (see [KR], [GHKR], [Ga]). In particular, if $X_\mu(b)(\overline{k}) \neq \emptyset$, then $\kappa_G(b) = \mu$ in $\pi_1(G)_\Gamma$.

To $[b] \in B(G)$, we may associate a Levi subgroup $M_b \subset G_F$ which is an inner form of J. After replacing b by some representative in the same σ-conjugay class, we may further assume that $b \in M_b$. If $X_\mu(b)(\overline{k}) \neq \emptyset$, we call the pair (b, μ) indecomposable with respect to the Hodge-Newton decomposition if for all standard Levi subgroups M with $M_b \subseteq M \subsetneq G$, we have $\kappa_M(b) \neq \mu$ in $\pi_1(M)_\Gamma$.

If the couple (b,μ) is not indecomposable with respect to the Hodge-Newton decomposition, then there exists a standard Levi subgroup M with $M_b \subset M \subsetneq G$ such that $\kappa_M(b) = \mu$ in $\pi_1(M)_\Gamma$. By [MaV], Theorem 6, the natural inclusion $M(L)/M(\mathcal{O}_L) \hookrightarrow G(L)/G(\mathcal{O}_L)$ induces a bijection $X_\mu^M(b) \xrightarrow{\sim} X_\mu^G(b)$.

¿From now on, we focus on the (b,μ)-indecomposable case. Let G_{ad} be the adjoint group of G. We denote the images of b and μ in G_{ad} also by b and μ. Then the sets of connected components of $X_\mu^G(b)$ and $X_\mu^{G_{\mathrm{ad}}}(b)$ are closely related. In order to describe the set of connected components of $X_\mu^G(b)$, it is enough to describe $\pi_0(X_\mu^{G_{\mathrm{ad}}}(b))$. So we may assume that G is simple.

Consider the short exact sequence

$$0 \to \pi_1(G)/\pi_1(G)^\Gamma \xrightarrow{\alpha} \pi_1(G) \to \pi_1(G)_\Gamma \to 0$$

where $\alpha(x \mod \pi_1(G)^\Gamma) = \sigma(x) - x$. If $X_\mu(b) \neq \emptyset$, let $c_{b,\mu}\pi_1(G)^\Gamma \in \pi_1(G)/\pi_1(G)^\Gamma$ such that $\alpha(c_{b,\mu}\pi_1(G)^\Gamma) = \kappa_G(b) - \mu$.

Theorem 1 (in progress)**.** *Let G be a classical group and let (b,μ) be indecomposable with respect to the Hodge-Newton decomposition.*

(1) *If $\kappa_M(b_0) \neq \mu$ for all proper standard Levi subgroups M of G with $M \cap [b] \neq \emptyset$ and all $b_0 \in M \cap [b]$, then w_G induces a bijection $\pi_0(X_\mu(b)) \cong c_{b,\mu}\pi_1(G)^\Gamma$.*

(2) *Assume furthermore that G is simple. If the hypothesis in the above statement does not hold then $[b] = [\mu(\epsilon)]$ with μ central and*
$$X_\mu(b) = X_{\preceq\mu}(b) \cong J/(J \cap K) \cong G(F)/G(\mathcal{O}_F)$$
is discrete.

When F is a function field and G is split, this theorem is proved by Viehmann [V2]. We use a generalization of Viehmann's methods. We expect to be able to generalize the same method to all unramified groups.

Corollary 2. *Let G be a classical group and let μ be minuscule, then J acts transitively on $\pi_0(X_\mu(b))$.*

One direct application of this theorem is the following. Let $F = \mathbb{Q}_p$. Let (G,b,μ) be as above such that G is the restriction of scalars of a linear group or similitude unitary/symplectic group. We may associate to this triple a Rapoport-Zink space of EL or PEL type $\breve{\mathcal{M}} = \breve{\mathcal{M}}(G,b,\mu)$ which is the moduli space of p-divisible groups with additional structures (cf. [RZ]). There exists a locally constant function $\varkappa : \breve{\mathcal{M}} \to \Delta := \mathrm{Hom}(X^*_{\mathbb{Q}_p}(G), \mathbb{Z})$ which is the height of the quasi-isogeny involved in the moduli space up to scalar. In fact, group theoretically, $\varkappa$ is induced by the morphism κ_G modulo torsion.

Corollary 3. *Same assumption as in the statement (1) of the theorem. Then $\varkappa : \breve{\mathcal{M}} \to \Delta$ induces an injection $\pi_0(\breve{\mathcal{M}}) \hookrightarrow \Delta$.*

In my thesis, by using this corollary, we can describe the set of geometrically connected components of the tower of Rapoport-Zink spaces on the generic fiber. This realizes the local Langlands correspondence between the 1-dimensional automorphic representations of G and the characters of the Galois group given by the local class field theory. Moreover, Theorem 1 is also needed in Kisin's work on the mod p points in Shimura varieties of Hodge type.

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Shuffle structures on KR strata

Chia-Fu Yu

(joint work with Ulrich Görtz)

In this talk we explain how to reduce the geometry of positive p-rank strata in the Siegel moduli spaces with Iwahori level structure to that of p-rank zero strata in those with lower genera. The construction holds for general PEL-type moduli spaces.

1. KR stratification

Let p be a prime and $g \geq 1$ be an integer. Let $\mathcal{A}_{g,I}$ be the moduli space over $\overline{\mathbb{F}}_p$ of g-dimensional principally polarized abelian varieties with Iwahori level structure at p. It parametrizes the isomorphism classes of chains of abelian varieties $(A_0 \overset{\alpha}{\to} A_1, \ldots, \overset{\alpha}{\to} A_g, \lambda_0, \lambda_g, \eta)$ with compatible conditions. This moduli space also parametrizes the isomorphism classes of objects $(A, \lambda, \eta, H_\bullet)$, where

$$H_\bullet : H_1 \subset \ldots, \subset H_g \subset A[p]$$

is a chain of finite flat subgroup schemes such that H_g is maxmally isotropic. We have the KR stratification on $\mathcal{A}_{g,I}$:

$$\mathcal{A}_{g,I} = \coprod_{x \in \mathrm{Adm}(\mu_g)} \mathcal{A}_{I,x},$$

where $\mu_g = (1^g, 0^g)$ is the minuscule dominant coweight of GSp_{2g}, and $\mathrm{Adm}(\mu_g) \subset \widetilde{W}$ is the μ_g-admissible set.

For a scheme X of finite type over an algebraically closed field, denote by $\pi_0(X)$ the set of (geometrically) connected components of X and $\Pi_0(X)$ the set of (geometrically) irreducible components of X.

We know the following results about the KR strata:

Theorem 1.

(1) *Each $\mathcal{A}_{I,x}$ is smooth, quasi-affine, equi-dimensional of dimension $\ell(x)$, where $\ell(x)$ is the length of x.*
(2) *If $\mathcal{A}_{I,x}$ is not supersingular, then $\mathcal{A}_{I,x}$ is irreducible. If $\mathcal{A}_{I,x}$ is supersingular, then the number $|\pi_0(\mathcal{A}_{I,x})|$ of connected components of $\mathcal{A}_{I,x}$ is a class number.*
(3) *There is a simple criterion to determine whether $\mathcal{A}_{I,x}$ is supersingular or not.*
(4) *Each supersingular KR stratum is a disjoint union of copies of certain Deligne-Lusztig varieties.*

For $f \in \mathbb{Z}$ with $0 \leq f \leq g$, let $\mathcal{A}_{g,I}^{(f)}$ be the p-rank f stratum of $\mathcal{A}_{g,I}$. One has

$$\mathcal{A}_{g,I}^{(f)} = \coprod_{x \in \mathrm{Adm}(\mu_g)^{(f)}} \mathcal{A}_{I,x},$$

where $\mathrm{Adm}(\mu_g)^{(f)} \subset \mathrm{Adm}(\mu_g)$ is the subset of elements of "p-rank" f. Let $\mathrm{Adm}(\mu_g)^{(f)}_{\max}$ be the maximal elements.

2. The shuffle construction

Let $h \geq 1$ be a positive integer. Let $\mathrm{BT}^1_{h,I}$ be the category of groupoids of objects $(G, \lambda, H_\bullet)$, where

- G is a truncated Barsotti-Tate group of level one, or a BT^1, of height $2h$,
- $\lambda : G \to G^D$ is a principal polarization, where G^D is the Cartier dual of G, and
- $H_\bullet :\ H_1 \subset \cdots \subset H_h \subset G$ is a chain of finite flat subgroup schemes such that H_h is maximally isotroptic.

Let $[p] : \mathcal{A}_{g,I} \to \mathrm{BT}^1_{g,I}$ be the functor which sends objects $(A, \lambda, \iota, H_\bullet)$ to $(A[p], \lambda, H_\bullet)$. The KR map $KR : \mathcal{A}_{g,I} \to \mathrm{Adm}(\mu_g)$ factors through a surjective map which we still denote by $KR : \mathrm{BT}^1_{g,I} \to \mathrm{Adm}(\mu_g)$.

For two positive integers $s \geq 1$ and $t \geq 1$ with $s + t = g$, denote by $\mathrm{Sh}(s,t)$ the set of maps $\varphi : \{0, 1, \dots, g\} \to \{0, 1, \dots, s\}$ such that

$$\varphi(0) = 0,\ \varphi(g) = s,\ \text{and}\ \ \varphi(i) \leq \varphi(i+1) \leq \varphi(i) + 1,\ \forall\, i = 0, \dots, g-1.$$

Elements in $\mathrm{Sh}(s,t)$ are called shuffle maps of s letters and t letters.

For any may $\varphi \in \mathrm{Sh}(s,t)$, define a map $\varphi' : \{0, 1, \dots, g\} \to \{0, 1, \dots, t\}$ as follows:

$$\varphi'(0) = 0, \quad \text{and}\ \ \varphi'(i+1) + \varphi(i+1) = \varphi'(i) + \varphi(i) + 1, \quad \forall\, i = 0, \dots, g-1.$$

The map $\varphi' \in \mathrm{Sh}(t,s)$ is called the complement of φ. With the information above, we construct a functor $\varphi_* : \mathrm{BT}^1_{s,I} \times \mathrm{BT}^1_{t,I} \to \mathrm{BT}^1_{g,I}$ by $\big((G,\lambda,H_\bullet),(G',\lambda' H'_\bullet)\big) \mapsto (G\times G', \lambda\times\lambda', \varphi(H_\bullet,H'_\bullet))$, where

$$\varphi(H_\bullet,H'_\bullet):\ K_1 \subset K_2 \subset \cdots \subset K_g \subset G\times G', \quad K_i := H_{\varphi(i)} \times H'_{\varphi'(i)}.$$

The shuffle map φ_* descends to a map φ_* on the μ-admissible sets $\mathrm{Adm}(\mu)$: $\varphi_*\mathrm{Adm}(\mu_s)\times\mathrm{Adm}(\mu_t) \to \mathrm{Adm}(\mu_g)$.

We have the following properties:

- The map φ_* preserves the Bruhat order.
- For each f the restriction $\varphi_* : \mathrm{Adm}(\mu_{g-f})^{(0)} \times \mathrm{Adm}(\mu_f)^{(f)} \to \mathrm{Adm}(\mu_g)^{(f)}$ is injective, and
$$\mathrm{Adm}(\mu_g)^{(f)} = \coprod_{\varphi\in\mathrm{Sh}(g-f,f)} \varphi_*(\mathrm{Adm}(\mu_{g-f})^{(0)} \times \mathrm{Adm}(\mu_f)^{(f)}).$$

Corollary 2. *For any integer f with $0 \le f \le g$, there is a natural bijection*
$$\mathrm{Adm}(\mu_{g-f})^{(0)}_{\max} \times \mathrm{Adm}(\mu_f)^{(f)} \times \mathrm{Sh}(g-f,f) \xrightarrow{\sim} \mathrm{Adm}(\mu_g)^{(f)}_{\max}.$$
Consequently, we have $|\mathrm{Adm}(\mu_g)^{(f)}_{\max}| = |\mathrm{Adm}(\mu_{g-f})^{(0)}_{\max}| \cdot 2^f \binom{g}{f}$.

Proposition 3. *For any elements $y \in \mathrm{Adm}(\mu_f)^{(f)}$ and $\varphi \in \mathrm{Sh}(g-f,f)$, the map $\varphi_*(\cdot,y) : \mathrm{Adm}(\mu_{g-f}) \to \mathrm{Adm}(\mu_g)$ is injective and ranked, that is, there is an integer d such that $\ell(\varphi_*(x,y)) = \ell(x)+d$ for all $x\in\mathrm{Adm}(\mu_{g-f})$.*

3. Geometry of p-rank strata

Theorem 4. *If $f\ge 1$, then we have*
$$|\Pi_0(\mathrm{Adm}(\mu_g)^{(f)}| = |\mathrm{Adm}(\mu_{g-f})^{(0)}_{\max}| \binom{g}{f} 2^f.$$

If $f=0$, the set $\mathrm{Adm}(\mu_g)^{(0)}_{\max}$ consists of the supersingular part $\mathrm{Adm}(\mu_g)^{(0),\mathrm{ss}}_{\max}$ and the non-supersingular part $\mathrm{Adm}(\mu_g)^{(0),\mathrm{ns}}_{\max}$. We have
$$|\Pi_0(\mathrm{Adm}(\mu_g)^{(0)}| = |\mathrm{Adm}(\mu_g)^{(0),\mathrm{ns}}_{\max}| + \sum_{x\in\mathrm{Adm}(\mu_g)^{(0),\mathrm{ss}}_{\max}} |\pi_0(\mathcal{A}_{I,x})|.$$

Theorem 5. *For any integer f with $0\le f\le g$, we have*
$$\mathrm{Codim}(\mathcal{A}^{(f)}_{g,I},\mathcal{A}_{g,I}) = \mathrm{Codim}(\mathcal{A}^{(0)}_{g-f,I},\mathcal{A}_{g-f,I}) = \lceil \frac{g-f}{2} \rceil.$$

Theorem 5 is also due to P. Hamacher [1].

Theorem 6. *Assume $f\ge 1$, or $f=0$ and $g\ge 3$. The map*
$$(\varphi,y) \mapsto \bigcup_{x\in\mathrm{Adm}(\mu_{g-f})^{(0)}} \mathcal{A}_{I,\varphi_*(x,y)}$$
induces a bijection $\mathrm{Sh}(g-f,f)\times\mathrm{Adm}(\mu_f)^{(f)} \xrightarrow{\sim} \pi_0(\mathcal{A}^{(f)}_{g,I})$. Consequently, we have $|\pi_0(\mathcal{A}^{(f)}_{g,I})| = \binom{g}{f}2^f$.

Finally we make two remarks on the set $\mathrm{Adm}(\mu_g)^{(0)}_{\max}$:

- There is a upper bound: $|\mathrm{Adm}(\mu_g)^{(0)}_{\max}| \le g!$.
- $|\mathrm{Adm}(\mu_g)^{(0)}_{\max}| = 1, 2, 5, 12$ if $g = 1, 2, 3, 4$, respectively.

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Nearby Cycles on Local Models

Xinwen Zhu

(joint work with George Pappas)

Given certain group theoretical data $(G, \{\mu\}, K)$, one can define a Shimura variety S_K over a number field E, whose ℓ-adic cohomology $\mathrm{H}^* := \mathrm{H}^*(S_K \otimes \bar{E}, \overline{\mathbb{Q}}_\ell)$ carries on the commuting actions of the Hecke algebra $\mathcal{H}(G, K)$ and the Galois group $\mathrm{Gal}(\bar{E}/E)$. To understand the pattern of the decomposition of H^* under these actions is of fundamental importance in arithmetic geometry and number theory as it is expected to realize a large part of Langlands correspondence.

Under some reasonable assumptions, a possible way to attack the above question is to study for each finite place v of E and each integer $r \ge 1$, the sum

$$\sum_{x \in \mathcal{S}_K(k_r)} \mathrm{tr}^{ss}(\Phi_x, R\Psi_{\bar{x}})$$

where

(1) k_r is the degree r extension of the residue field k of E at v;
(2) $\mathcal{S}_K$ is an integral model of S_K over the ring of integers $\mathcal{O}_{E_v}$ of E_v;
(3) $R\Psi$ is the sheaf of nearby cycles on $\mathcal{S}_K \otimes \bar{k}$, with the action of $\mathrm{Gal}(\bar{E}_v/E_v)$, compatible of the action of $\mathrm{Gal}(\bar{k}/k)$ on $\mathcal{S}_K \otimes \bar{k}$;
(4) x ranges over all k_r points of $\mathcal{S}_K$, $\bar{x}$ a geometrical point lying over x, Φ_x the geometrical Frobenius element in $\mathrm{Gal}(\bar{x}/x)$ and tr^{ss} is the semisimple trace, a notion introduced by Rapoport.

In general, it is not clear how to give a reasonable definition of $\mathcal{S}_K$ nor known how to calculate the nearby cycles. However, in the case $K = K_pK^p$ ($p = \mathrm{char} k$), where $K_p = \mathcal{G}(\mathbb{Z}_p)$ and $\mathcal{G}$ is an parahoric groupscheme of $G_{\mathbb{Q}_p}$, we calculated $\mathrm{tr}^{ss}(\Phi_x, R\Psi_{\bar{x}})$ via local models and therefore confirms the Kottwitz conjecture (the case $G_{\mathbb{Q}_p} = \mathrm{GL}_n$ or GSp_n was previously proved by Haines and Ngô [3]) and more generally some cases of a conjecture of Haines and Kottwitz.

Our definition of local models M_{K_p} is group theoretical. It is defined over $\mathcal{O}_{E_v}$, with an action of $\mathcal{G}_{\mathcal{O}_{E_v}}$, and is embedded into the base change to $\mathcal{O}_{E_v}$ of an ind-scheme $\mathrm{Gr}_{\mathcal{G}}$ over $\mathbb{Z}_p$, which is the mixed characteristic analogue (and generalization) of the deformation from affine Grassmannian to the affine flag variety appearing in the geometrical Langlands program ([1]). Then the theory of local models for Shimura varieties asserts that there is a local model diagram

$$\pi : \mathcal{S}_K \to [\mathcal{G}_{\mathcal{O}_{E_v}} \setminus M_{K_p}],$$

which is smooth and of relative dimension $\dim G$. Therefore, we reduce the problem to the calculation of the semisimple trace on M_{K_p}, which can be done in a way analogous to the function field case [1, 5].

We also obtained some results on the monodromy of $R\Psi$. For example, we showed that the monodromy of $R\Psi \otimes \mathcal{O}_{F_w}$ is purely unipotent, where $E_v \subset F_w$ is a splitting field of $G_{\mathbb{Q}_p}$ (the case $G_{\mathbb{Q}_p} = \mathrm{GL}_n$ or GSp_n was proved by Görtz and Haines [2]). In certain cases, we determine the monodromy of $R\Psi$ completely, which requires results from [6] if $F_w/\mathbb{Q}_p$ is ramified.

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Kottwitz-Rapoport and p-rank strata in the reduction of Shimura varieties of PEL type

Philipp Hartwig

This is a report on some results of [2]. Fix a rational prime $p \neq 2$ and a PEL datum $\mathcal{B} = (B, *, V, (\cdot,\cdot), J)$ with auxiliary data $\mathcal{B}_p = (\mathcal{O}_B, \mathcal{L})$ at p. The datum $\mathcal{B}$ gives rise to a reductive group G over $\mathbb{Q}$ and a conjugacy class h of homomorphisms $\mathrm{Res}_{\mathbb{C}/\mathbb{R}} \mathbb{G}_m \to G_{\mathbb{R}}$. Fix a compact open subgroup $C^p \subset G(\mathbb{A}_f^p)$. From C^p and $\mathcal{B}_p$ one obtains a compact open subgroup $C \subset G(\mathbb{A}_f)$ and thus a Shimura datum (G, h, C).

In [4], Rapoport and Zink construct from $\mathcal{B}$, $\mathcal{B}_p$ and C^p an integral model $\mathcal{A}_{C^p}$ of the Shimura variety associated with (G, h, C). They further show that this integral model fits into a *local model diagram*

$$\begin{tikzcd}
& \widetilde{\mathcal{A}}_{C^p} \arrow[dl, "\widetilde{\varphi}"'] \arrow[dr, "\widetilde{\psi}"] & \\
\mathcal{A}_{C^p} & & M^{\mathrm{loc}}.
\end{tikzcd}$$

Here M^{loc} is a scheme defined purely in terms of linear algebra, having the same étale local structure as $\mathcal{A}_{C^p}$.

The group $\mathcal{G} = \mathrm{Aut}(\mathcal{L})$ of automorphisms of the self-dual multichain $\mathcal{L}$ is a smooth group scheme over $\mathbb{Z}_p$, and it acts on both M^{loc} and $\widetilde{\mathcal{A}}_{C^p}$. The map $\widetilde{\varphi}$ is an $\mathcal{G}$-torsor, while the map $\widetilde{\psi}$ is $\mathcal{G}$-equivariant. Denote by $\mathbb{F}$ an algebraic closure of $\mathbb{F}_p$. Via the local model diagram, the decomposition of $M^{\mathrm{loc}}(\mathbb{F})$ into $\mathcal{G}(\mathbb{F})$-orbits induces the *Kottwitz-Rapoport* (or KR) *stratification*

$$\mathcal{A}_{C^p}(\mathbb{F}) = \coprod_{x \in \mathcal{G}(\mathbb{F}) \backslash M^{\mathrm{loc}}(\mathbb{F})} \mathcal{A}_{C^p,x}.$$

It was first introduced in [3] in the case of the Siegel moduli space with Iwahori level structure $\mathcal{A}_I$.

We are interested in the relationship of the KR stratification and the p-rank stratification on $\mathcal{A}_{C^p}$. In the case of $\mathcal{A}_I$, this relationship has been determined by Ngô and Genestier in [3]. They show that the KR stratification is a refinement of the p-rank stratification and they determine an explicit formula for the p-rank on a given KR stratum.

As a first step, we show the following result.

Theorem 1. *Let $\mathcal{B}$ be an arbitrary PEL datum. If $\mathcal{L}$ is* complete*, the p-rank is constant on a KR stratum.*

The assumption on $\mathcal{L}$ in Theorem 1 corresponds to the assumption that the level structure at p in the definition of $\mathcal{A}_{C^p}$ is an *Iwahori* level structure, as opposed to a more general parahoric level structure. Our expectation is that the p-rank should be constant on all KR strata if *and only if* the multichain $\mathcal{L}$ is complete.

As a second step, we prove a group theoretic formula for the p-rank on a given KR stratum. We only do this under the assumption that the reductive $\mathbb{Q}_p$-group $\mathrm{GL}_{B\otimes\mathbb{Q}_p}(V \otimes \mathbb{Q}_p)$ is quasi-split, but this assumption should be unnecessary. However this abstract formula seems to be of limited use when it comes to actual applications. We therefore proceed to establish more concrete formulas in the following two cases.

(1) The group G is essentially the restriction of scalars $\mathrm{Res}_{F/\mathbb{Q}}\,\mathrm{GSp}_{2n,F}$ of the group of symplectic similitudes over a totally real field $F/\mathbb{Q}$.
(2) The group G is essentially the restriction of scalars $\mathrm{Res}_{F_0/\mathbb{Q}}\,\mathrm{GU}$ of a group of unitary similitudes over an imaginary quadratic extension F/F_0 of a totally real field $F_0/\mathbb{Q}$.

Let us emphasize that in both cases we allow arbitrary ramification of p in the occuring extension $F/\mathbb{Q}$.

Let us state the obtained formula in an easy special case and mention an interesting consequence. Let $n, r, s \in \mathbb{N}$ with $r + s = n$, and assume that the group G is the group of unitary similitudes of signature (r, s) over an imaginary quadratic extension $F/\mathbb{Q}$ in which p *splits*. In this case the KR stratification is indexed by a subset $\mathrm{Perm}_r \subset S_n \ltimes \mathbb{Z}^n$, and our result on the p-rank on a KR stratum reads as follows.

Theorem 2. *Let $x \in \mathrm{Perm}_r$, say $x = (w, \lambda)$ in $S_n \ltimes \mathbb{Z}^n$ for some $w \in S_n, \lambda \in \mathbb{Z}^n$. Then the p-rank on $\mathcal{A}_{C^p,x}$ is constant with value $|\mathrm{Fix}(w)|$, where $\mathrm{Fix}(w) = \{i \in \{1, \ldots, n\} \mid w(i) = i\}$.*

Copying the approach of Görtz and Yu in [1, §8], we use Theorem 2 to compute the dimension of the p-rank zero locus $\mathcal{A}^{(0)}_{C^p} \subset \mathcal{A}_{C^p}$ in this special case.

Theorem 3. $\dim \mathcal{A}^{(0)}_{C^p} = \min\big((r-1)(n-r), r(n-r-1)\big)$.

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Affine Weyl group, affine Hecke algebra and affine Deligne-Lusztig variety

Xuhua He

In Lie theory, the following topics are closely related: Weyl groups, Hecke algebras and algebraic groups. In this talk, we discuss some new relations among

- Combinatorics of affine Weyl groups;
- Representations of affine Hecke algebras;
- Structure of loop groups.

We then use them to study affine Deligne-Lusztig varieties. The work we present is:

(1) *Some combinatorial properties of affine Weyl groups.*

For finite Weyl groups, Geck and Pfeiffer in [2] show that minimal length elements in any conjugacy class have some remarkable properties. Their approach was based on a case-by-case analysis and relied on computer for exceptional types. Recently, in joint work with Nie [4], [5], we give a general proof that minimal length elements in any conjugacy class of finite or affine Weyl groups have nice properties.

(2) *Class polynomials of affine Hecke algebras.*

Let H be the affine Hecke algebra of the affine Weyl group W over $\mathbb{Z}[v, v^{-1}]$. Based on the combinatorial properties in (1), in [5] we show that for any $w \in W$ and conjugacy class $\mathcal{O}$ of W, there exists a unique polynomial $f_{w,\mathcal{O}} \in \mathbb{N}[v - v^{-1}]$ such that for any finite dimensional representation V of H,

$$Tr(T_w, V) = \sum_{\mathcal{O}} f_{w,\mathcal{O}} Tr(T_{w_{\mathcal{O}}}, V).$$

Here $w_{\mathcal{O}}$ is a minimal length element in $\mathcal{O}$.

(3) *Affine Deligne-Lusztig varieties.*

Let $L = \bar{\mathbb{F}}_q((t))$ and $G(L)$ be a split loop group. Let $\sigma : G(L) \to G(L)$ the Frobenius morphism and I the standard Iwahori subgroup of $G(L)$. An affine Deligne-Lusztig variety (in affine flag) is defined as follows. For $w \in W$ and $b \in G(L)$, set

$$X_w(b) = \{gI; g^{-1}b\sigma(g) \in IwI\} \subset G(L)/I.$$

Affine Deligne-Lusztig varieties play an important role in the study of Newton strata (indexed by b) and Kottwitz-Rapoport strata (indexed by w) on the special fiber of Shimura variety with Iwahori level structure.

One of the main theorems I proved in [3] is the “Dimension=Degree” theorem, which relates the dimension of $X_w(b)$ to the degree of the class polynomial $f_{w,\mathcal{O}}$. As an application, I proved the Görtz-Haines-Kottwitz-Reuman conjecture [1] on the dimension of $X_w(b)$ for b basic and w in the Shrunken Weyl chamber.

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On the supersingular locus of the Shimura variety for $GU(1, n-1)$ in the ramified case

Michael Rapoport

(joint work with U. Terstiege and S. Wilson)

Let $S(G, X)_K$ be a Shimura variety with canonical model over its reflex field E. We assume that the open compact subgroup K is of the form $K = K^p K_p$, where $K^p \subset G(\mathbb{A}^{\infty,p})$ and where $K_p \subset G(\mathbb{Q}_p)$ is a parahoric subgroup. Let $\mathcal{S}$ be a ‘good’ integral model over the ring of integers of the completion of E at a prime ideal over p. Experience has shown that it should always be possible to give a ‘synthetic’ description of the *basic locus* of the special fiber of $\mathcal{S}$. By the uniformization theorem [2], this is essentially equivalent to asking for a description of the underlying reduced scheme of an RZ-space in the *basic case*.

More precisely, it should be possible to describe the set of irreducible components of the underlying reduced scheme in terms of the Bruhat-Tits building of a reductive group over $\mathbb{Q}_p$, and the individual irreducible components should be describable in terms of Deligne-Lusztig varieties (although they will only rarely be equal to Deligne-Lusztig varieties).

In various special cases, this has been done by Drinfeld, by Kaiser, by Oort, and others. Vollaard and Wedhorn [3, 4] have treated the case of the Shimura variety for $GU(1, n-1)$ for an unramified prime of E. In the talk I reported on analogous results in the ramified case.

1. The moduli space

We denote by E a ramified quadratic extension of $\mathbb{Q}_p$. We fix a uniformizer π of E such that $\pi_0 = \pi^2 \in \mathbb{Q}_p$ is a uniformizer. Since we assume $p \neq 2$, this is always possible. We denote by $\mathbb{F}$ an algebraic closure of $\mathbb{F}_p$, and by $W = W(\mathbb{F})$ its ring of Witt vectors and by $W_{\mathbb{Q}}$ its fraction field. We denote by σ the Frobenius on $\mathbb{F}$, on W, and on $W_{\mathbb{Q}}$.

Let $\breve{E} = W_{\mathbb{Q}} \otimes_{\mathbb{Q}_p} E$ and let $\mathcal{O}_{\breve{E}} = W \otimes_{\mathbb{Z}_p} \mathcal{O}_E$ be its ring of integers. Let $\sigma = \sigma \otimes \mathrm{id}$ on $\breve{E}$. We denote by $\psi_0 : E \longrightarrow \breve{E}$ the natural embedding, and by $\psi_1 = \psi_0 \circ {}^-$ its conjugate.

Let Nilp be the category of $\mathcal{O}_{\breve{E}}$-schemes S such that $\pi \cdot \mathcal{O}_S$ is a locally nilpotent ideal sheaf. For $S \in \mathrm{Nilp}$, we denote by $\bar{S} = S \times_{\operatorname{Spec} \mathcal{O}_{\breve{E}}} \operatorname{Spec} \mathbb{F}$ its reduction modulo π.

Let $(\mathbb{X}, \iota)$ be a fixed supersingular p-divisible group of dimension n and height $2n$ over $\mathbb{F}$ with an action $\iota : \mathcal{O}_E \longrightarrow \mathrm{End}(\mathbb{X})$. Let $\lambda_{\mathbb{X}}$ be a principal quasi-polarization such that its Rosati involution induces on $\mathcal{O}_E$ the non-trivial automorphism over $\mathbb{Q}_p$. The triple $(\mathbb{X}, \iota, \lambda_{\mathbb{X}})$ is unique up to isogeny.

Fix $n \geq 2$. Let $\mathcal{N}$ be the set-valued functor on Nilp which associates to $S \in \mathrm{Nilp}$ the set of isomorphism classes of quadruples $(X, \iota, \lambda, \varrho)$. Here X is a p-divisible group over S, and $\iota : \mathcal{O}_E \longrightarrow \mathrm{End}(X)$ is a homomorphism satisfying the following two conditions (the *Kottwitz condition* and the *Pappas condition*).

$$\begin{gathered} \mathrm{char}(\iota(a) | \mathrm{Lie}\, X) = (T - \psi_0(a)) \cdot (T - \psi_1(a))^{n-1} \, . \\ \bigwedge^2 (\iota(x) - \pi | \mathrm{Lie}\, X) = 0 \, , \quad \bigwedge^n (\iota(\pi) + \pi | \mathrm{Lie}\, X) = 0 \, , \text{ if } n \geq 3. \end{gathered} \tag{1}$$

Furthermore, $\lambda : X \longrightarrow X^\vee$ is a principal quasi-polarization whose associated Rosati involution induces on $\mathcal{O}_E$ the non-trivial automorphism over $\mathbb{Q}_p$. Finally, $\varrho : X \times_S \bar{S} \longrightarrow \mathbb{X} \times_{\operatorname{Spec} \mathbb{F}} \bar{S}$ is a $\mathcal{O}_E$-linear quasi-isogeny such that λ and $\varrho^*(\lambda_{\mathbb{X}})$ differ locally on $\bar{S}$ by a factor in $\mathbb{Q}_p^\times$. An isomorphism between two quadruples $(X, \iota, \lambda, \varrho)$ and $(X', \iota', \lambda', \varrho')$ is given by an $\mathcal{O}_E$-linear isomorphism $\alpha : X \longrightarrow X'$ such that $\varrho' \circ (\alpha \times_S \bar{S}) = \varrho$ and such that $\alpha^*(\lambda')$ is a $\mathbb{Z}_p^\times$-multiple of λ.

Proposition 1. *The functor $\mathcal{N}$ is representable by a separated formal scheme $\mathcal{N}$, locally formally of finite type over* $\mathrm{Spf}\,\mathcal{O}_{\breve{E}}$. *Furthermore, $\mathcal{N}$ is flat over $\mathcal{O}_{\breve{E}}$. It is formally smooth over $\mathcal{O}_{\breve{E}}$ in all points of the special fiber except those corresponding to $(X, \iota, \lambda, \varrho) \in \mathcal{N}(\mathbb{F})$, where* $\mathrm{Lie}(\iota(\pi)) = 0$ (these form an isolated set of points).

Proof. The representability follows from [2]. The assertions concerning flatness and formal smoothness follow from [1], 4.5. □

We denote by $\mathcal{N}^0$ the open and closed formal subscheme of $\mathcal{N}$ where the height of ρ is zero. Let $\bar{\mathcal{N}}^0$ be the reduction modulo π of $\mathcal{N}^0$. Hence $\bar{\mathcal{N}}^0$ parametrizes isomorphism classes of quadruples $(X, \iota, \lambda, \varrho)$, where X is a p-divisible group of height $2n$ and dimension n, and where $\iota : \mathcal{O}_E \longrightarrow \mathrm{End}\,(X)$ is an action of $\mathcal{O}_E$ on X, such that for $n \geq 3$

$$\bigwedge^2 (\iota(\pi)|\mathrm{Lie}\,\mathrm{X}) = 0\,, \tag{2}$$

and where λ is a principal quasi-polarization whose Rosati involution induces on $\mathcal{O}_E$ the non-trivial automorphism over $\mathbb{Q}_p$, and where $\varrho : X \longrightarrow \mathbb{X} \times_{\mathrm{Spec}\,\mathbb{F}} S$ is a quasi-isogeny of height 0 which is $\mathcal{O}_E$-linear and such that $\varrho^*(\lambda_{\mathbb{X}})$ and λ differ locally on S by a factor in $\mathbb{Z}_p^\times$ (we note that ψ_0 and ψ_1 are identical modulo π).

2. The structure theorem

Let N be the rational Dieudonné module of $\mathbb{X}$. Then $\tau := \iota_{\mathbb{X}}(\pi)V^{-1}$ is a σ-linear endomorphism of N which is isoclinic with all slopes equal to zero. Let C be the fixed space of τ. Then C is a hermitian vector space of dimension n over E. A *vertex lattice* in C is a lattice Λ with

$$\pi\Lambda \subset \Lambda^\vee \subset \Lambda. \tag{3}$$

The dimension of the $\mathbb{F}_p$-vector space $\Lambda/\Lambda^\vee$ is called the *type* of Λ and denoted by $t(\Lambda)$. This is always an *even* integer between 0 and n, and all these integers occur as types of suitable vertex lattices.

Theorem 2. *To every vertex lattice Λ there is associated a closed irreducible subscheme $\mathcal{N}_\Lambda$ of $(\bar{\mathcal{N}}^0)_{\mathrm{red}}$. As Λ ranges over all vertex lattices, the $\mathcal{N}_\Lambda$ form a stratification of $(\bar{\mathcal{N}}^0)_{\mathrm{red}}$ (i.e., their union is $(\bar{\mathcal{N}}^0)_{\mathrm{red}}$, and the non-empty intersection of strata is a stratum). Furthermore,*

- $\mathcal{N}_{\Lambda_1} \subset \mathcal{N}_{\Lambda_2} \iff \Lambda_1 \subset \Lambda_2$.
- *If $\Lambda_1 \cap \Lambda_2$ is a vertex lattice, then $\mathcal{N}_{\Lambda_1 \cap \Lambda_2} = \mathcal{N}_{\Lambda_1} \cap \mathcal{N}_{\Lambda_2}$. Otherwise $\mathcal{N}_{\Lambda_1} \cap \mathcal{N}_{\Lambda_2} = \emptyset$.*
- $\dim \mathcal{N}_\Lambda = \frac{1}{2} t(\Lambda)$.
- *Let $\mathcal{N}_\Lambda^o = \mathcal{N}_\Lambda \setminus \bigcup_{\Lambda' \subsetneq \Lambda} \mathcal{N}_{\Lambda'}$. Then $\mathcal{N}_\Lambda^o$ is isomorphic to a Deligne-Lusztig variety to a symplectic group of size $t(\Lambda)$ over $\mathbb{F}_p$ and a Coxeter element.*

In particular, $(\bar{\mathcal{N}}^0)_{\mathrm{red}}$ is connected and $\dim\,(\bar{\mathcal{N}}^0)_{\mathrm{red}} = [\frac{n}{2}]$.

In contrast to the unramified case, the strata $\mathcal{N}_\Lambda$ are singular (with isolated singularities) as soon as $t(\Lambda) \geq 4$. There exists a Demazure style nonsingular resolution of $\mathcal{N}_\Lambda$.

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Height pairings on unitary Shimura varieties

Benjamin Howard

(joint work with Jan Bruinier, Tonghai Yang)

Let $k \subset \mathbb{C}$ be a quadratic imaginary field, which for simplicity we assume to have class number one and odd discriminant. For any $m > 1$, denote by $M_{(m,0)}$ the moduli space of abelian schemes, B, of dimension m over $\mathcal{O}_k$-schemes, S, equipped with a principal polarization and an action of $\mathcal{O}_k$. We demand that the action of $\mathcal{O}_k$ satisfy the following signature $(m, 0)$ condition: the induced action on the $\mathcal{O}_S$-module $\mathrm{Lie}(B)$ is through the structure map $\mathcal{O}_k \to \mathcal{O}_S$. The stack $M_{(m,0)}$ is smooth of relative dimension 0 over $\mathcal{O}_k$. Note that $M_{(1,0)}$ is simply the moduli space of elliptic curves with complex multiplication by $\mathcal{O}_k$, with the action of $\mathcal{O}_k$ on the Lie algebra suitable normalized. For any $A_0 \in M_{(1,0)}(S)$ denote by $\overline{A}_0$ the elliptic curve A_0, but the $\mathcal{O}_k$-action replaced by its complex conjugate.

For a fixed integer $n > 1$ there is a unitary Shimura variety $M_{(n-1,1)}$ parametrizing abelian schemes A of dimension n (over $\mathcal{O}_k$-schemes) equipped with a principal polarization and an action of $\mathcal{O}_k$ satisfying a suitable signature $(n-1, 1)$ condition. Over the complex numbers this condition is easy to describe: the maximal subspace of $\mathrm{Lie}(A)$ on which $\mathcal{O}_k$ acts through the fixed embedding $\mathcal{O}_k \to \mathbb{C}$ has dimension $n-1$, while the maximal subspace on which $\mathcal{O}_k$ acts through the conjugate embedding has dimension 1. Over an arbitrary $\mathcal{O}_k$-scheme the definition of the signature condition is slightly subtle.

Consider the product space $M = M_{(1,0)} \times M_{(n-1,1)}$, a regular algebraic stack over $\mathcal{O}_k$, flat of relative dimension $n-1$. The stack M carries a natural family of divisors $Z(m)$, the *Kudla-Rapoport divisors*, indexed by positive integers. For each pair $(A_0, A) \in M$ the space $\mathrm{Hom}_{\mathcal{O}_k}(A_0, A)$ is endowed with the Hermitian form $\langle x_1, x_2 \rangle = x_2^\vee \circ x_1$ (the right hand side is an element of $\mathrm{End}_{\mathcal{O}_k}(A_0) \cong \mathcal{O}_k$), and $Z(m)$ is the moduli space of triples (A_0, A, x) where $(A_0, A) \in M$, and $x \in \mathrm{Hom}_{\mathcal{O}_k}(A_0, A)$ has Hermitian norm m. Using regularized theta lifts, one can construct a Green function $\Phi(m)$ for each divisor $Z(m)$, and so define an element

$$\widehat{Z}(m) = (Z(m), \Phi(m)) \in \widehat{\mathrm{Pic}}(M)$$

in the group of isomorphism classes of metrized line bundles on M. There is also a *tautological bundle* T on M, which is essentially the line bundle of weight one modular forms for a certain unitary group, and endowing the tautological bundle with a particular metric defines the class $\widehat{T} \in \widehat{\mathrm{Pic}}(M)$. The *modified Kudla-Rapoport divisor* is

$$\widehat{\Theta}(m) = \widehat{Z}(m) + c(m)\widehat{T} \in \widehat{\mathrm{Pic}}(M)$$

where each $c(m)$ is the constant term of a certain weak harmonic Maass form of weight $2-n$ whose holomorphic component has q-expansion $q^{-m} + O(1)$.

The are natural cycles on M of dimension one. Define $Y = M_{(1,0)} \times M_{(n-1,0)}$. This stack can be viewed as a cycle on M by the map $Y \to M$ defined by $(A_0, B) \mapsto (A_0, \overline{A}_0 \times B)$. Moreover, the connected components of Y are indexed in a natural way by the set of self-dual Hermitian $\mathcal{O}_k$-lattices of signature $(n-1,0)$. Fix one such lattice Λ, and let Y_Λ be the corresponding component of Y. It is smooth of relative dimension 0 over $\mathcal{O}_k$, and so is an arithmetic curve. As such, every metrized line bundle on Y_Λ has an *arithmetic degree*, and composing this degree with the restriction of metrized line bundles from M to Y_Λ defines the linear function *arithmetic intersection against* Y_Λ:

$$[- : Y_\Lambda] : \widehat{\mathrm{Pic}}(M) \to \mathbb{R}.$$

The main theorem of [1] asserts that the generating series of arithmetic intersections

$$\sum_{m>0} [\widehat{\Theta}(m) : Y_\Lambda] \cdot q^m$$

is a modular form of weight n, level $\Gamma_0(\mathrm{disc}(k))$, and character χ^n, where χ is the quadratic Dirichlet character determined by k. In fact, this generating series is the holomorphic projection of the product $E'(\tau, 0)\theta(\tau)$, where $\theta(\tau)$ is the weight $n-1$ theta series associated to Λ, and $E(\tau, s)$ is a weight one Eisenstein series vanishing at $s = 0$. When $n = 2$ these results essentially reduce to results of Gross-Zagier.

Conjecturally, the formal power series

$$\widehat{\Theta}(\tau) = \sum_{m>0} \widehat{\Theta}(m) \in \widehat{\mathrm{Pic}}(M)[[q]]$$

is a vector-valued modular form. Assuming that this is true, for any weight n cuspform f we may form the Petersson inner product

$$\widehat{\Theta}(f) = \langle f, \widehat{\Theta} \rangle \in \widehat{\mathrm{Pic}}(M).$$

The usual Rankin-Selberg unfolding method then shows that

$$[\widehat{\Theta}(f) : Y_\Lambda] = L'(f, \theta_\Lambda, n-1),$$

where $L(f, \theta_\Lambda, s)$ is the convolution L-function of f with θ_Λ, and the equality is up to some simple constants and local factors.

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Overconvergence, classicity and ramification

Benoît Stroh
(joint work with Payman Kassaei, Vincent Pilloni, Shu Sasaki, Yichao Tian)

We study classicity questions for overconvergent Hilbert modular forms of parallel weight one in a ramified situation. Let F be a totally real number field and p a prime number. Denote by $p \cdot \mathcal{O}_F = \prod_{i=1}^{r} \pi_i^{e_i}$ the decomposition of p in distinct prime ideals of F. Denote by f_i the residue degree of π_i for $1 \leq i \leq r$. Let $X_0(\pi)$ be the Hilbert modular variety parametrizing polarized abelian schemes endowed with an $\mathcal{O}_F$-action and a finite flat subgroup scheme of the $\prod_{i=1}^{r} \pi_i$-torsion. Let ω denote the line bundle of weight one forms on $X_0(\pi)$. Then an overconvergent Hilbert modular form of weight one is a section of ω which is defined in a strict neighborhood of the multiplicative-ordinary locus in the rigid variety $X_0(\pi)^{rig}$ associated to $X_0(\pi)$. It is an interesting question in number theory to decide whether or not such a form is classical, *ie* can be extended to a section of ω on the whole $X_0(\pi)^{rig}$. Recall that a notion of "companion form" is defined in [2].

Theorem 1. *Suppose $f_i \geq 3$ and $e_i \geq 2$. Let f be an overconvergent modular form of weight 1 which has a companion form. Suppose that f is of finite slope. Then f is classical.*

Recall that having a companion form for f can be read on the Galois representation $\rho_f : Gal(\bar{F}/F) \to \mathrm{GL}_2(\bar{\mathbb{Q}}_p)$ restricted to the decomposition subgroups of the π_i. It implies in particular that this local Galois representation is de Rham.

The restriction $f_i \geq 3$ and $e_i \geq 2$ is of technical nature and could be removed by further computations in p-adic Hodge theory. Neverthless, as we are finally interested in Galois representations we can use solvable base change to achieve such conditions.

The theorem was proven in [1] if $F = \mathbb{Q}$. When $F \neq \mathbb{Q}$, it was proven in [3] if $e_i = 1$ for all i and in [4] if $e_i < p - 1$ for all i.

We use basically the same technique as Buzzard and Taylor : automatic extension of f to a big region of $X_0(\pi)^{rig}$ and gluing with its companion form. However the automatic extension is delicate to show in the case where $e_i \geq p-1$. It requires the use of Berkovich spaces and study of Barsotti-Tate groups over non discrete valuation rings.

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