Mathematisches Forschungsinstitut Oberwolfach

Report No. 33/2012

DOI: 10.4171/OWR/2012/33

Geometric Aspects of Spectral Theory

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July 1st – July 7th, 2012

Abstract. The workshop "Geometric Aspects of Spectral Theory" brought together leading researchers working in various areas of this vast field of mathematics. The meeting featured presentations on some of the most fascinating recent developments in the subject, including five survey talks given by top experts, as well as reports on the progress made by graduate students and postdocs. A number of new stimulating questions were formulated during the open problem session.

Mathematics Subject Classification (2000): 35J05, 35J10, 35J20, 58J05, 58J32, 58J50, 35P15, 35P20, 35P99.

Introduction by the Organisers

Geometric spectral theory is a rapidly developing area of mathematics with connections to Riemannian geometry, mathematical physics, calculus of variations and other fields. The talks presented at the meeting covered a broad variety of topics, already well-established as well as relatively unexplored. The workshop featured daily survey talks given by Brian Davies, Bernard Helffer, Bruno Colbois, Aldo Pratelli and Rupert Frank. On Wednesday evening, Timo Weidl gave a lecture aimed at a general mathematical audience, attended by the participants of both workshops that were held at the Institute during the week.

Main topics of the meeting included optimisation problems for eigenvalues (talks by Aldo Pratelli, Dorin Bucur, Pedro Freitas, Mette Iversen, Richard Laugesen, and a related talk by Michael Loss), spectral properties of Dirac and Schrödinger operators (talks by Rupert Frank, Anna Dall'Acqua, Ari Laptev, Michael Levitin and Timo Weidl), geometric estimates for Laplace and Steklov eigenvalues on Riemannian manifolds (talks by Bruno Colbois, Alexandre Girouard and Alessandro Savo), geometric features of nodal domains and spectral minimal partitions (talks by Bernard Helffer and Uzy Smilansky). Brian Davies presented an overview of recent developments in the spectral theory of non-self-adjoint operators, an exciting subject with lots of open questions. The talks of Vadim Kostrykin and Karsten Fritzsch focussed on applications of spectral theory to new problems arising in physics and engineering, such as the study of plasmons and metamaterials. Alexander Strohmaier discussed new results on precise numerical computations of spectral quantities on Riemann surfaces. Emily Dryden reported on her recent work lying on the interface of spectral theory and symplectic geometry. In several talks the use of numerical computation for creating mathematical conjectures was emphasized (talks by Brian Davies, Alexander Strohmaier, Bernard Helffer, Uzy Smilansky and Pedro Freitas).

The talks presented at the workshop stimulated numerous fruitful interactions between the participants. The group included a large number of young researchers, in particular several Ph.D. students and postdocs, who benefitted from discussions with the renowned experts in the field.

One of the highlights of the workshop was the open problem session chaired by Michiel van den Berg. A list of open problems formulated at this session and discussed during the Workshop can be found below.

Workshop: Geometric Aspects of Spectral Theory

Table of Contents

Abstracts

Recent progress in the spectral theory of non-self-adjoint operators

E. Brian Davies

Since about 1990 there have been major advances in understanding the spectral theory of a variety of non-self-adjoint operators, based on geometric insights provided by numerical computations. These have led to new spectral theorems that are in no sense generalizations of results for self-adjoint operators.

A general theory does not yet exist, but there are many exciting recent developments. This review describes two particular problems, but they are typical in the sense that their solutions involve special techniques, often involving theorems from complex analysis. A range of methods that have been found useful in other non-self-adjoint (NSA) contexts may be found in [1, 2, 5].

General problems

Let A be a closed linear operator acting on a dense domain $\mathcal{D}$ in a Hilbert space $\mathcal{H}$.

If $Ae_n = \lambda_n e_n$ for all $n \in \mathbf{N}$, the density of the linear span of $\{e_n\}$ in $\mathcal{H}$ does not imply that one can use the e_n to expand a general vector $f \in \mathcal{H}$ in the form

$$f = \sum_{n=1}^{\infty} \langle f, u_n \rangle e_n.$$

where e_n, u_n is a biorthogonal system, i.e. $\langle e_m, u_n \rangle = \delta_{m,n}$ for all m, $n \in \mathbf{N}$. It may not help to weaken the requirement to Cesaro or Abel convergence for all $f \in \mathcal{H}$ or to assume that A has a compact resolvent.

There are are at least five inequivalent definitions of essential spectrum for operators in $\mathcal{H}$ that are not self-adjoint. (All coincide for self-adjoint operators.) The following two definitions are both important in applications. We say $\lambda \in \mathrm{Ess}(A)$ if $\lambda I - A$ is not Fredholm; equivalently $\lambda I - A$ is invertible modulo compact operators or its image in the Calkin algebra is invertible. We say $\lambda \in \mathrm{Stab}(A)$ if $\lambda I - A$ is not Fredholm or it is Fredholm with a non-zero index; equivalently $\lambda \in \mathrm{Spec}(A + K)$ for all compact perturbations K of A.

If N is a normal operator acting in $\mathcal{H}$ and $\lambda \notin \mathrm{Spec}(N)$ then

$$\|(\lambda I - N)^{-1}\| = [\mathrm{dist}(\lambda, \mathrm{Spec}(N))]^{-1}.$$

For non-normal operators the resolvent norm may be far larger than the quantity on the right hand side. Therefore the unique solubility of the equation $\lambda f - Af = g$ by $f = (\lambda I - A)^{-1} g$ does not imply its stable solubility, in the numerical sense. This issue was investigated by Trefethen in the 1990s using the notion of pseudospectra; see [5]. He provided many examples that demonstrated that this problem is so common, even for matrices of quite small sizes, that it cannot be regarded as pathological.

The NSA harmonic oscillator

Consider

$$H = P^2 + aQ^2 = -\frac{\mathrm{d}^2}{\mathrm{d}x^2} + ax^2$$

acting in $L^2(\mathrm{R})$, where a is not real. The eigenvalues of H are $\lambda_n = (2n+1)\sqrt{a}$, where $n = 0, 1, 2, \ldots$. This operator arises in the study of a single mode laser with decay caused by losses in the cavity. It illustrates the first of the general problems described above, but the same operator also exhibits the phenomenon described in the third problem.

Theorem 1 ([3])**.** *Let P_n be the rank 1 spectral projections of the NSA harmonic oscillator H. If $a = \mathrm{e}^{i\theta}$ where $0 < \theta < \pi$ then*

$$\lim_{n\to\infty} n^{-1} \log(\|P_n\|) = 2\Re\left\{f(r(\theta)\mathrm{e}^{i\theta/4})\right\}$$

where

$$f(z) := \log(z + (z^2-1)^{1/2}) - z(z^2-1)^{1/2}$$

and

$$r(\theta) := (2\cos(\theta/2))^{-1/2}.$$

Corollary 2. *The expansion*

$$\mathrm{e}^{-Ht} := \sum_{n=0}^{\infty} \mathrm{e}^{-\lambda_n t} P_n$$

is norm convergent if

$$t > t_a := \frac{\Re\left\{f(r(\theta)\mathrm{e}^{i\theta/4})\right\}}{\cos(\theta/2)}$$

and norm divergent if $0 < t < t_a$.

The very precise results above are only available because the eigenfunctions of the operator H can be written down in closed form, as Hermite functions. Similar but weaker bounds can be proved for $H = P^2 + p(Q)$ where p is a polynomial of even order with complex highest order coefficient. They may be extended to a wide class of NSA pseudodifferential operators. See [1, p. 426] and [6].

NSA perturbations of self-adjoint operators

Consider

$$A_\gamma = A + \gamma B$$

acting on $\mathcal{H}$ where $\dim(\mathcal{H}) = N < \infty$, $A = A^*$, $0 \le B = B^*$ has rank M and $\gamma \in \mathbf{C}_+ = \{z \in \mathbf{C} : \Im(z) > 0\}$. Such operators have been studied in great detail when γ is real, but several new phenomena arise for complex γ, of which we only mention one. The following theorem may be extended to sectorial perturbations B.

Theorem 3 ([4]). *Under certain genericity conditions, let $g(t) = te^{i\theta}$ where $0 \leq t < \infty$ and $0 < \theta < \pi$. Then there exist N real-analytic curves $\lambda_r(t)$ lying in $\mathbf{C}_+$ such that $\mathrm{Spec}(A_{g(t)}) = \{\lambda_1(t), \ldots, \lambda_N(t)\}$ for all $t \in [0, \infty)$. One can choose the ordering of these so that $\lambda_r(0) = \alpha_r$ for all $r \in \{1, \ldots, N\}$, where α_r are the eigenvalues of A written in increasing order. Assuming this is done, there exists a θ-dependent permutation π on $\{1, \ldots, N\}$ such that*

$$\lim_{t\to\infty} \frac{\lambda_{\pi(r)}(t)}{g(t)} = \beta_r$$

for $1 \leq r \leq M$, where β_r are the non-zero eigenvalues of B written in increasing order, and

$$\lim_{t\to\infty} \lambda_{\pi(M+r)}(t) = \delta_r$$

for $1 \leq r \leq N - M$, where δ_r are the non-zero eigenvalues of the truncation of A to $\mathrm{Ker}(B)$ written in increasing order.

The theorem states that the permutation π depends on the angle θ. In fact π is constant as θ increases except for discrete changes at certain non-generic values of θ for which two of the curves $\lambda_r(t)$ meet. This happens at a value of γ for which A_γ has a multiple eigenvalue.

The situation changes substantially if one allows $N = \infty$ while keeping $M < \infty$. The number of curves $\lambda_r(t)$ may be finite, and the curves that do exist may not appear until t reaches a positive critical value, which depends on r and θ.

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Eigenvalues of a graphene operator pencil

Michael Levitin

(joint work with Daniel M Elton, Iosif Polterovich)

Let

$$\mathrm{T} = \begin{pmatrix} k & -\mathrm{i}D \\ \mathrm{i}D & -k \end{pmatrix} = \sigma_2 D + k\sigma_3$$

(where $k \in \mathbb{R} \setminus \{0\}$ is a given parameter, $D = -\mathrm{i}\frac{\mathrm{d}}{\mathrm{d}x}$, and σ_j are Pauli matrices) be a formally self-adjoint one-dimensional Dirac operator. The study of the zero-energy modes for two-dimensional electron waveguides in graphene, see e.g. [3], leads after the separation of variables to the operator pencil problem,

$$(\mathrm{T} + \gamma V)\boldsymbol{\psi} = \mathbf{0}.$$

Here $\boldsymbol{\psi} = \begin{pmatrix} \psi_1(x) \\ \psi_2(x) \end{pmatrix}$, $V(x)$ is a given scalar potential decaying at $\pm\infty$ (always assumed to be multiplied by the 2×2 identity matrix), and γ plays a rôle of the spectral parameter. In other words, we want to study the γ-spectrum associated to a potential V, which we define as

$$\Sigma_V = \left\{ \gamma \in \mathbb{C} : 0 \in \operatorname{spec}(\mathrm{T} + \gamma V) \right\}.$$

Our aim is to determine the key properties of Σ_V under minimal assumptions on the potential V; in particular, we consider symmetries, estimates for the distribution of points in Σ_V, and asymptotics for these points.

To state the precise results we need to make some basic restrictions on the local regularity and global decay of the potential V. We shall assume that all the potentials are real-valued and locally L^2. Let $\mathbb{V}_0$ denote the class of such potentials which additionally satisfy

$$V_{L^2(x-1,x+1)} \to 0 \qquad \text{as } |x| \to \infty;$$

We shall also use $\mathbb{V}_1 = \mathbb{V}_0 \cap L^1$ to denote the class of locally L^2 and integrable potentials.

The following results are obtained using standard techniques by showing that any $V \in \mathbb{V}_0$ is a relatively compact perturbation of T.

Theorem 1. *Suppose $V \in \mathbb{V}_0$. Then $\mathrm{T} + V$ is an unbounded self-adjoint operator on $(L^2(\mathbb{R}))^2$. Its essential spectrum $\operatorname{spec}_{\mathrm{ess}}(\mathrm{T} + V) = \operatorname{spec}_{\mathrm{ess}}(\mathrm{T}) = E_k := \mathbb{R} \setminus (-|k|, |k|)$. The operator $\mathrm{T} + V$ may have eigenvalues outside E_k if $V \not\equiv 0$ but these must be isolated and of finite multiplicity. The operator pencil spectrum $\Sigma_V = \{\gamma : 0 \in \operatorname{spec}(\mathrm{T} + \gamma V)\}$ is a discrete subset of $\mathbb{C}$.*

In common with other Dirac operators, $\mathrm{T}+V$ possesses a number of elementary symmetries which lead to the symmetries for the set Σ_V. In particular, if $V \in \mathbb{V}_0$ then $-\Sigma_V = \Sigma_V = \overline{\Sigma_V}$, while Σ_V is unchanged if we replace k with $-k$ in the definition of T. With this last symmetry in mind we shall henceforth assume $k > 0$.

Even though the operator $\mathrm{T} + V$ is self-adjoint (recall that V is real-valued), the operator pencil $\mathrm{T} + \gamma V$ is not! Extra conditions are required to ensure that Σ_V contains only real points. In particular we have the following.

Proposition 2. *If $V \in \mathbb{V}_0$ is single-signed then $\Sigma_V \subset \mathbb{R}$.*

For variable sign V the set Σ_V may contain non-real points; we give in the talk some examples of explicit potentials which illustrate various possible behaviours for complex points in Σ_V. As a general result illustrating the contrast with single-signed potentials we have the following (which is proved using a fairly direct argument based on symmetry considerations).

Proposition 3. *If $V \in \mathbb{V}_0$ is odd (that is, $V(-x) = -V(x)$ for all $x \in \mathbb{R}$) then $\Sigma_V \cap \mathbb{R} = \emptyset$.*

To obtain estimates for the distribution of points in Σ_V we impose extra global decay conditions on the potential V, requiring $V \in \mathbb{V}_1$.

Firstly we consider the number of points of Σ_V lying inside the disc $\{z \in \mathbb{C} : |z| \le R\}$ of radius $R > 0$.

Theorem 4. *Suppose $V \in \mathbb{V}_1$. Then*

$$\#\big(\Sigma_V \cap \{z \in \mathbb{C} : |z| \le R\}\big) \le C \left\|V\right\|_{L^1} R$$

for any $R \ge 0$, where C is a universal constant (we can take $C = 4\mathrm{e}/\pi$).

Lower bounds which complement the upper bounds given by the previous Theorem can also be obtained. Restricting our attention to real points we have the following.

Theorem 5. *Suppose $V \in \mathbb{V}_1$. Then*

$$\#(\Sigma_V \cap [0, R]) \ge \frac{R}{\pi} \left| \int_{\mathbb{R}} V(x)\mathrm{d}x \right| + o(R)$$

as $R \to \infty$. The same estimate holds for $\#(\Sigma_V \cap [-R, 0])$ (by symmetry).

In particular, $\Sigma_V \cap \mathbb{R}$ contains infinitely many points if $\int_{\mathbb{R}} V(x)\mathrm{d}x \ne 0$. On the other hand, the fact that $\Sigma_V \cap \mathbb{R} = \emptyset$ for odd potentials clearly limits the possible improvements to the previous Theorem.

For single-signed potentials we can improve the asymptotic upper bound to obtain the same leading order term as for the lower bound. This results in an asymptotic formula for the points in Σ_V in this case.

Theorem 6. *Suppose $V \in \mathbb{V}_1$ is single-signed and non-trivial. Let $\{\gamma_n\}$ denote the sequence of positive points in Σ_V, arranged in order of increasing size. Then*

$$\gamma_n = \frac{\pi}{\left\|V\right\|_{L^1}} n + o(n)$$

as $n \to \infty$.

We obtain this result via the Prüfer angle technique, but similar results can be obtained by using the Birman-Schwinger principle, cf., e.g., [1, 4].

While we have not obtained a general asymptotic formulae for the points in Σ_V when V is not single-signed, it is reasonable to conjecture that the leading order asymptotic for the number of points in $\Sigma_V \cap [0, R]$ is between

$$\frac{R}{\pi} \left| \int_{\mathbb{R}} V(x)\mathrm{d}x \right| \quad \text{and} \quad \frac{R}{\pi} \int_{\mathbb{R}} |V(x)|\mathrm{d}x = \frac{1}{\pi} \left\|V\right\|_{L^1} R. \tag{1}$$

The former is certainly a lower bound but also a possible asymptotics (in the case of odd potentials, for example). On the other hand, we have examples of potentials V with compact support and $\int_{\mathbb{R}} V(x)\mathrm{d}x = 0$ which satisfy the latter asymptotics. Modifying these examples a bit it is probably possible to construct $V \in \mathbb{V}_1$ which gives asymptotic behaviour anywhere between the above extremes. For somewhat related results involving sign-indefinite potentials cf. [5].

The full account will be published in [2].

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Eigenvalue inclusions and spectral determinants of hyperbolic surfaces

Alexander Strohmaier
(joint work with Ville Uski)

Suppose that X is a compact oriented surface of genus $g > 1$ equipped with a metric of constant negative curvature. We will be interested in the spectrum of the Laplace operator Δ_X on functions and its derived spectral quantities. Such a surface may be specified by $6g - 6$ Fenchel-Nielsen coordinates

$$(\ell_1, t_1; \ell_2, t_2; \dots; \ell_{3g-3}, t_{3g-3})$$

together with a three valent graph and a labelling scheme, that fixes a decomposition of X into pairs of hyperbolic pants. Such a decomposition allows to decompose the surface into subsets Z_i of hyperbolic cylinders with piecewise geodesic boundaries ∂Z_i. On the disjoint union Z of all these cylinders there exists a special linearly independent set of functions $F_j^{(\lambda)}$ that satisfy

$$(\Delta - \lambda) F_j^{(\lambda)} = 0,$$
$$\|F_j^{(\lambda)}\|_{L^2(Z)} \geq 1.$$

These functions are obtained by separation of variables on each of the cylinders and can be expressed in terms of hypergeometric functions and exponentials. Now choose an equidistant set of points $\{x_1, x_2, \dots, x_M\} \subset \partial Z$ that partition the boundary into intervals of length δ. These points correspond to another set of points $\{\tilde{x}_1, \tilde{x}_2, \dots, \tilde{x}_M\}$ on the boundary by the glueing map, i.e. on the surface the point x_i is identified with the point $\tilde{x}_i$. Now define the following $M \times N'$

matrices.

$$A_\lambda = \begin{pmatrix} \sqrt{a_m}(F_j^{(\lambda)}(x_m) - F_j^{(\lambda)}(\tilde{x}_m))) \\ \sqrt{a_m}(\mathbf{n}F_j^{(\lambda)}(x_m) + \mathbf{n}F_j^{(\lambda)}(\tilde{x}_m))) \end{pmatrix}_{m=1,\dots,M;\; j=1,\dots,N'},$$

$$B_\lambda = \begin{pmatrix} \sqrt{a_m}F_j^{(\lambda)}(x_m) \\ \sqrt{a_m}\mathbf{n}F_j^{(\lambda)}(x_m) \end{pmatrix}_{m=1,\dots,M;\; j=1,\dots,N'},$$

where $\mathbf{n}$ is the exterior normal derivative and the coefficients a_m are the coefficients of the Simpson rule for integrating a function over the boundary ∂Z. Here N' is the number of basis functions with Fourier modes appearing in the separation of variables between $-N$ and $+N$ and we take only these basis functions to form the matrices. Thus, $N' = 2b(2N+1)$, where b is the number of connected components of Z. Now let $\sigma_1(A,B)$ be the smallest relative singular value of the pair of matrices (A,B) and $\sigma_1(A)$ be the smallest singular value for A. We have proved the following theorems.

Theorem 1. *There exists a constant $C > 0$ independent of N and M, and a quadratic form q depending only on N such that for a singular vector v with $\|v\| = 1$ and $\|A_\lambda v\| = \sigma_1(A_\lambda)\|v\|$ the inequality*

$$\epsilon := C\left(\sigma_1(A_\lambda)^2 + \delta^4 q(v,v)\right)^{1/2} < 1$$

implies

$$\mathrm{dist}(\mathrm{spec}(\Delta_X), \lambda) \leq \frac{\epsilon(\lambda+1)}{1-\epsilon}.$$

Theorem 2. *There exists $\tilde{N}(\lambda) > 0$ such that for every $N > \tilde{N}$ there exists an $\tilde{M} > 0$ such that for all $M > \tilde{M}$ we have*

$$\sigma_1(A_\lambda, B_\lambda) \geq \beta_N(\lambda) + C'(\lambda)\mathrm{dist}(\mathrm{spec}(\Delta_X), \lambda),$$

where β_N decays exponentially fast in N for fixed $\lambda > 0$, and $C'(\lambda)$ is a constant depending only on λ and the geometry of the problem.

It is important here that all the constants in the theorems can be made explicit. Therefore, these theorems can be used to prove eigenvalue inclusions by employing interval arithmetics. They also provide a powerful numerical algorithm for the numerical computation of eigenvalues on hyperbolic surfaces. A computer program that (non-rigorously) computes eigenvalues for given Fenchel-Nielsen coordinates of genus two hyperbolic surfaces was published under GPLv3 and can be downloaded from `www-staff.lboro.ac.uk/~maas3/hyperbolic-surfaces/hypermodes.html`.

Once a list of eigenvalues is found bounds on the heat kernel and on the counting function can be used to establish error bounds for values of spectral zeta functions and spectral determinants. We were able to compute spectral zeta functions and spectral determinants with high accuracy for genus two hyperbolic surfaces and study their behavior in Teichmüller space. For the Bolza surface it appears that both the first non-zero eigenvalue and the spectral determinants are maximized. The values we find for the Bolza surface are

$$\lambda_1 \approx 3.8388872588421995185866224504354645970819 15$$

and

$$\det_\zeta(\Delta) \approx 4.72273280444557,$$
$$\zeta_\Delta(-1/2) \approx -0.65000636917383,$$

for spectral determinant and Casimir energy. In order to obtain explicit estimates for the error bounds we derived explicit bounds on the heat kernel, counting function and C^1-norms of eigenfunctions. The details as well as the precise statements of the theorems can be found in [1].

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A Spectral Problem on Two Almost Touching Domains

Karsten Fritzsch

Introduction. Consider the following setting: Let $n \geq 3$ and $\Omega_\pm \subset \mathbb{R}^n$ be open, bounded and connected sets with smooth boundary such that $\mathbb{R}^n \setminus \overline{\Omega_- \cup \Omega_+}$ is connected and $\overline{\Omega_-} \cap \overline{\Omega_+} = \{0\} \subset \mathbb{R}^n$. Suppose $\partial\Omega_-$, $\partial\Omega_+$ are tangent to second order at the origin. By translation of $\Omega_\pm$, perpendicular to their boundaries at 0, a parameter dependent setting is obtained: $\Omega_a = \Omega_-(a) \cup \Omega_+(a)$, where $a = \mathrm{dist}\big(\Omega_-(a), \Omega_+(a)\big)$.

Then, for $a > 0$, consider the following boundary value problem:

(1) $\quad \Delta u_\pm = 0 \quad \text{and} \quad \Delta u_0 = 0 \qquad \text{in} \quad \Omega_\pm(a) \text{ resp. } \mathbb{R}^n \setminus \overline{\Omega_a}$

(2) $\quad u_\pm - u_0 = 0 \qquad \text{at} \quad \partial\Omega_\pm(a)$

(3) $\quad \varepsilon\partial_{\mathrm{n}} u_\pm + \partial_{\mathrm{n}} u_0 = 0 \qquad \text{at} \quad \partial\Omega_\pm(a)$

(4) $\quad u_0(x) = \mathcal{O}(|x|^{2-n}) \qquad \text{as} \quad |x| \to \infty$

where $u_\pm : \Omega_\pm(a) \to \mathbb{R}$ and $u_0 : \mathbb{R}^n \setminus \overline{\Omega_a} \to \mathbb{R}$ are smooth up to the boundary and $\varepsilon \in \mathbb{R}$; ∂_{n} denotes the outward normal derivative with respect to $\partial\Omega_a$.

Now let

$$\mathcal{H} = \Big\{ u : \mathbb{R}^n \to \mathbb{R} \,\Big|\, u\big|_{\Omega_\pm(a)}, u\big|_{\mathbb{R}^n \setminus \overline{\Omega_a}} \text{ solve (1), (2) and (4) } \Big\}$$

and $N_\pm$ denote the exterior respectively interior Dirichlet-to-Neumann operators, then:

$$u \in \mathcal{H} \text{ solves (3)}$$
$$\Longleftrightarrow g = u\big|_{\partial\Omega_a} \text{ satisfies } (-N_+^{-1} N_-) g = \frac{1}{\varepsilon} g$$

Note that $-N_+^{-1}N_-$ is an elliptic zeroth order pseudodifferential operator on $H^{\frac{1}{2}}(\partial\Omega_a)$, self-adjoint with respect to the inner product $(g,f)_+ = (g, N_+ f)_{L^2(\partial\Omega_a)}$.

If $u \in \mathcal{H} \setminus \{0\}$ solves (3) for ε, then ε is called an *plasmonic eigenvalue* for Ω_a and u a corresponding *plasmon*.

Again, for fixed $a > 0$, in [4] and [1] this reformulation as a genuine eigenvalue problem and other techniques are used to show (including, but not limited to; cf. [2] as well):

- plasmonic eigenvalues form a sequence of positive numbers $\varepsilon_1, \varepsilon_2, \dots \to 1$ and the corresponding eigenspaces are finite dimensional as long as $\varepsilon \neq 1$
- regularity results
- explicit formulae for the concerning the behaviour of solutions (ε, u) under perturbation of $\partial\Omega_a$

Questions. Now, as $a \to 0$ certain interesting questions arise:

- Are there solutions for $a = 0$? How should the equations be modified in this singular case?
- Are there branches of solutions with qualitatively different behaviour? In particular, solutions which...
  - ..."localise" (in a sense still to be defined) between the domains? Following numerical computations of [5] for two spheres in $\mathbb{R}^3$ this is to be expected.
  - ...are linear combinations of solutions to the respective single-domain problems? (This corresponds to the "hybridisation" method used in physics, cf. [6].)

These questions are of particular importance in the study of or in applications using plasmon surface resonances, see [1, 2, 5] and [6] for further references.

Ansatz. The following general ansatz is proposed to answer these questions: After introducing $a \geq 0$ as an additional coordinate, a quasi-homogeneous blow-up of the origin is used to resolve the singularity of $M_a = \bigcup_{0 \leq a} \partial\Omega_a$ at 0. (Compare [8] and [3] for a background on blow-ups.) This leads to a manifold with corners with two boundary hypersurfaces, H_{int} corresponding to the half-sphere introduced by the blow-up, and H_{ext}, corresponding to the 'old boundary' at $a = 0$.

Then, assume that solutions (ε, u) have asymptotic expansions

$$\varepsilon(a) \sim \sum_i \varepsilon_i a^{\alpha_i} \quad \text{and} \quad u(x,a) \sim \sum_j u_j^H a^{\alpha_j}$$

where $\varepsilon_i \in \mathbb{R}$ and the u_j^H are functions on the two boundary hypersurfaces. Moreover, assume that equations (1)-(4) are solved locally uniformly in a with all derivatives and that the expansions are compatible at the corner. (More precisely, assume that u is *polyhomogeneous*.) Expanding the equations in suitable powers of a and comparing coefficients then leads to two *model problems*, i.e. systems of equations similar to (1)-(4), one for each boundary hypersurface, and matching conditions coupling these problems.

The geometry of the model problem on H_{int} is completely determined by curvature information and can be solved recursively. The model problem on H_{ext} leads to a manifold with fibred boundary M_ϕ which corresponds to the boundaries of the touching domains with resolved singularity. In this case the ϕ-calculus developed in [7] and extended in [3] can be used: The interior and exterior Dirichlet-to-Neumann operators on functions on M_ϕ belong to the full ϕ-calculus; they are first-order, elliptic ϕ-pseudodifferential operators. As in [2, 4] this leads to a discrete spectrum for the model problem on H_{ext}.

Conjectures. Combining explicit calculations on H_{int} and the results on H_{ext} obtained via the ϕ-calculus, it should be possible to show the following:

- There exist formal solutions, i.e. sets of asymptotic data $(\varepsilon_i, u_j^H)_{i,j,H}$ which solve the model problems and satisfy the matching conditions.
- In particular, there exist formal solutions which are *localised* and properties concerning *localisation* and *hybridisation* are decoded in the leading order terms of the asymptotic data.
- Each formal solution leads to an a-dependent solution $(\varepsilon(a), u(a))$ (defined for short distances a) with the corresponding asymptotic behaviour.

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On nodal domains and spectral minimal partitions: a survey

Bernard Helffer
(joint work with T. Hoffmann-Ostenhof, V. Bonnaillie-Noel, G. Vial, P. Bérard)

Given a bounded open set Ω in $\mathbb{R}^n$ (or in a Riemannian manifold) and a partition of Ω by k open sets D_j, we can consider the quantity $\max_j \lambda(D_j)$ (which is called the energy of the partition) where $\lambda(D_j)$ is the ground state energy of the Dirichlet realization of the Laplacian in D_j. If we denote by $\mathfrak{L}_k(\Omega)$ the infimum over all the k-partitions of $\max_j \lambda(D_j)$, a minimal k-partition is then a partition which realizes the infimum. Although the analysis is rather standard when $k = 2$ (we find the nodal domains of a second eigenfunction), the analysis of higher k's becomes non trivial and quite interesting. In this talk, we consider the two-dimensional case and present the state of the art. The existence and the regularity of these spectral minimal partitions were proved in [11, 12, 13, 22] (see also [8, 10]). These minimal partitions are spectral equipartitions (all the $\lambda(D_j)$ are equal). It is actually enough to analyze regular minimal partitions. In this case the boundary set $N(\mathcal{D})$ of the partition consists of singular points $\{y_i\}_{i=1}^a$ inside Ω, of singular points $\{z_i\}_{i=1}^b$ on $\partial\Omega$, of C^1 arcs $\{\gamma_i\}_{i=1}^c$ which bound two adjacent domains of the partition, and of arcs $\{\delta_i\}_{i=1}^d$ contained in $\partial\Omega$.

There is a nice criterion conjectured in [9] and proven in [22] permitting to determine when minimal partitions are nodal, i.e. consisting of nodal domains of an eigenfunction of the Dirichlet Laplacian in Ω. The eigenfunction has to be Courant sharp (situation corresponding to the equality in Courant's theorem, which is usually not true according to [25]). Many examples like the square, the disk, the rectangle or the sector have been analyzed, sometimes rigorously and sometimes only numerically [3, 6, 4, 14, 19]. A magnetic characterization of these minimal partitions is proven relating them to nodal partitions for suitable Aharonov-Bohm operators [17, 1, 20, 24]. Two cases of 2-dimensional compact manifolds have been analyzed: the sphere, where we are able to determine the minimal 3-partitions [23] and the anisotropic torus [21]. Some lower bounds on the length of these regular minimal partitions have been obtained with P. Bérard [2] in the spirit of Brüning-Gromes [5] using the techniques of [27, 15, 16]. Finally we will discuss below with more details the behavior for k large of $\mathfrak{L}_k(\Omega)$, describing the efforts to prove the hexagonal conjecture in connection with the proof of the honeycomb conjecture by Hales [16].

Zoom on hexagonal conjectures

Let us describe this hexagonal conjecture and what is known at the moment. The Faber-Krahn inequality implies that $\mathfrak{L}_k(\Omega) \geq k\,\lambda(\mathrm{Disk}_1)A(\Omega)^{-1}$, where $A(\Omega)$ denotes the area of Ω and Disk_1 denotes the disk of area 1. We recall that $\lambda(\mathrm{Disk}_1) = \pi \mathbf{j}^2$, where $\mathbf{j}$ is a zero of some Bessel function. Using an hexagonal tiling, it is easy to see that:

$$\limsup_{k\to+\infty} \frac{\mathfrak{L}_k(\Omega)}{k} \leq \lambda(\mathrm{Hexa}_1)A(\Omega)^{-1}\,.$$

where Hexa_1 denotes the regular hexagon of area 1. The hexagonal conjecture for minimal partitions (transmitted to us by M. Van den Berg, see also [10] for a variant involving the sum instead of the max in the definition of the energy of a partition) says that

$$\lim_{k\to+\infty}\frac{\mathfrak{L}_k(\Omega)}{k}=\lambda(\text{Hexa}_1)A(\Omega)^{-1}\,.$$

There are various controls of the conjecture using numerics directly or indirectly on theoretical consequences of this conjecture [7, 6]. There is a corresponding (proved by Hales [16]) conjecture for k- partitions of equal area and minimal length called the honeycomb conjecture. We define the length of the boundary set $N(\mathcal{D})$ by:

$$P(\mathcal{D}):=\sum_{i=1}^{c}\ell(\gamma_i)+\frac{1}{2}\ell(\partial\Omega),$$

where ℓ denotes the length of the curves. Of course the hexagonal conjecture leads to a natural conjecture for the length of a minimal partition:

$$\lim_{k\to+\infty}(P(\mathcal{D}_k)/\sqrt{k})=\frac{1}{2}\ell(\text{Hexa}_1)A(\Omega)^{\frac{1}{2}}\,, \tag{1}$$

where $\ell(\text{Hexa}_1)$ is the perimeter of Hexa_1. We obtain in [2] an asymptotic lower bound for the length:

$$\liminf_{k\to+\infty}(P(\mathcal{D}_k)/\sqrt{k})\geq\frac{1}{2\mathbf{j}}\sqrt{\liminf_{k\to+\infty}(\frac{\mathcal{L}_k(\Omega)}{k})}\,. \tag{2}$$

Assuming that the elements of the minimal partitions have no hole, we could apply Polya's inequality [26, 27] and get the sharper estimate:

$$\liminf_{k\to+\infty}(P(\mathcal{D}_k)/\sqrt{k})\geq\frac{\mathbf{j}}{\sqrt{\pi}}A(\Omega)^{\frac{1}{2}}\,. \tag{3}$$

The techniques used here give more generally an information for spectral k-equipartitions. Implementing an inequality of Hales [16] obtained in his proof of the honeycomb conjecture, we get:

$$\liminf_{k\to+\infty}\frac{P(\mathcal{D}_k)}{\sqrt{k}}\geq\frac{1}{2}\ell(\text{Hexa}_1)\ \left(\lambda(\text{Disk}_1)/\lambda(\text{Hexa}_1)\right)^{\frac{1}{2}}\ A(\Omega)^{\frac{1}{2}}\,. \tag{4}$$

For a domain Ω with at most one hole, one can actually obtain a universal estimate for the length of a regular spectral k-equipartition $\mathcal{D}_k$ independent of the energy:

$$P(\mathcal{D}_k)+\frac{1}{2}\ell(\partial\Omega)\geq k^{\frac{1}{2}}\,12^{\frac{1}{8}}\,(\frac{\pi}{4})^{\frac{1}{4}}\,A(\Omega)^{\frac{1}{2}}\,. \tag{5}$$

Asymptotically this inequality is weaker than (4) but universal.

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Oscillations and vibrations – The Sturm and Courant theorems – revisited

Uzy Smilansky

Consider the Laplace (Schrödinger) operator on a domain $\Omega \in R^d$. Dirichlet boundary conditions are assumed on $\partial\Omega$. Arrange the spectrum as a non-decreasing sequence, and consider the n'th eigenfunction $f^{(n)}$. The *number of nodal domains* ν_n is the number of maximally connected subdomains where the eigenfunction has a constant sign.

For $d = 1$ Sturm's oscillation theory states that the number of sign changes of the wave function $\phi_n = n - 1$ and consequently $n = \nu_n$.

For $d > 1$ ϕ_n is not defined, and Courant's theorem states that $n \geq \nu_n$.

In the present talk I shall discuss the *nodal deficiency* $= n - \nu_n$, and show that it contains valuable information on the geometry of Ω. In particular I shall present recent result which show that the nodal deficiency can be derived from a new variational approach, and that it is equal to the Morse index (the number of unstabe directions) of a properly defined Energy functional at its critical points.

The generalization of Sturm's oscillation theorem will be presented for the discrete Schrödinger operator on graphs, where the variational principle mentioned above takes a simple form.

Spectral gap and involutions

Alessandro Savo
(joint work with Bruno Colbois)

Let (M, g) be a closed Riemannian manifold of dimension n and D a Laplace-type operator acting on sections of a vector bundle over M. Examples of Laplace-type operators are given by the Laplacian acting on differential p-forms (in particular, the usual Laplacian acting on functions) and by the Schrödinger operator acting on functions. Let

$$\lambda_1(D) \leq \lambda_2(D) \leq \dots$$

be the sequence of the eigenvalues of D. The scope of this talk is to explain some upper bounds for the gap $\Gamma_{k+1} = \lambda_{k+1}(D) - \lambda_1(D)$ obtained in [1] and [2].

In the first part we assume that, for a fixed integer k, the gap Γ_{k+1} is large. Fix any eigensection ψ_1 associated to $\lambda_1(D)$. We then show that, under some mild metric conditions, the measure

$$\mu = |\psi_1|^2 d\mathrm{vol}_g$$

must concentrate around a certain set of k points. Let us give a precise formulation when $D = \Delta$ is the usual Laplacian acting on functions. Then $\lambda_1 = 0$, the gap coincides with the $(k+1)$-th eigenvalue λ_{k+1}, ψ_1 is a constant function and μ is just a multiple of the Riemannian measure.

Theorem. *Let λ_j be the j-th eigenvalue of the Laplacian on functions. Fix an integer k and assume that $\lambda_{k+1} \geq e$. Then, there exist a constant c_k and a set of k points $S = \{x_1, \ldots, x_k\}$ such that, if*

$$r = c_k C(M,d)^2 \cdot \frac{\log \lambda_{k+1}}{\sqrt{\lambda_{k+1}}},$$

then

$$\frac{\mathrm{Vol}(S^r)}{\mathrm{Vol}(M)} \geq 1 - r.$$

Here S^r is the r-tubular neighborhood of the set S, for the Riemannian distance d.

$C(M,d)$ is called the *packing constant* of M; it depends only on the distance d and is defined as follows. Fix $r > 0$ and let $C_r(M,d)$ be the minimal number of balls of radius r needed to cover a ball of radius $2r$. We now set:

$$C(M,d) = \sup_{r \in (0,1]} C_r(M,d).$$

The theorem is interesting only when, for a fixed k, the eigenvalue λ_{k+1} is very large: in that case one sees that, if the packing constant is uniformly bounded above, then r is small and consequently almost all the relative volume of the manifold lies in a small tubular neighborhood of a carefully chosen set of k points.

The scale $\log \lambda/\sqrt{\lambda}$ is sharp. The corresponding statement for a general Laplace-type operator D is sligthly weaker: it is obtained replacing $\log \lambda_{k+1}/\sqrt{\lambda_{k+1}}$ by $(\lambda_{k+1} - \lambda_1)^{-1/3}$ and the Riemannian measure $d\mathrm{vol}_g$ by the measure μ (this is due to technical reasons, caused by the fact that the eigensection ψ_1 might vanish somewhere). We refer to [1] for complete statements and for estimates of the packing constant in terms of curvature.

In the second part we assume that (M,g) admits an involutive isometry γ with no fixed points and derive some upper bounds for the gap (see [2]). Define the *smallest displacement* of γ by:

$$\beta(\gamma, d) = \inf\{d(x, \gamma(x)) : x \in M\},$$

so that $\beta(\gamma, d) > 0$. We consider the gap $\Gamma_2 = \lambda_2(D) - \lambda_1(D)$ for a γ-invariant Laplace-type operator D and assume that $\lambda_1(D)$ is simple (otherwise Γ_2 is simply zero). An easy argument shows that the measure μ is γ-invariant and, as the dislacement is bounded below by a positive number, μ cannot concentrate too much near one single point. The theorem above suggests that then it should be possible to bound Γ_2 from above simply in terms of $\beta(\gamma, d)$ and the packing constant. In fact, one has the following estimate.

Theorem. *For any γ-invariant Laplace-type operator D on M one has the inequality*

$$\lambda_2(D) - \lambda_1(D) \leq \frac{16C(M,d)}{\min\{1, \beta(\gamma,d)^2\}}.$$

Note that the upper bound is purely metric; in particular it does not depend on the operator D.

Finally, we focus on the particular case where M is a submanifold of $\mathbf{R}^N$, or $\mathbf{S}^N$, invariant under the antipodal map. In that case, using the classical barycenter method, we are able to find a sharp upper bound which depends only on the dimension; moreover, when D is the Laplacian acting on p-forms, the equality case leads to a rigidity theorem.

Theorem. *Let M^n be a compact submanifold of $\mathbf{S}^N$, invariant under the antipodal isometry γ. Let D be any γ-invariant Laplace-type operator on M. Then:*

$$\lambda_2(D) - \lambda_1(D) \leq n.$$

If equality holds, then M is minimal in $\mathbf{S}^N$ and any eigensection associated to $\lambda_1(D)$ must be parallel.

Now assume that M^n is an antipodal invariant hypersurface of $\mathbf{S}^{n+1}$ and let Δ_p be the p-form Laplacian. Then Δ_p is γ-invariant and the above inequality applies. Moreover, it is possible to show that the only compact, minimal hypersurface of $\mathbf{S}^{n+1}$ supporting a parallel p-form is the (minimal) Clifford torus

$$\mathrm{CL}_{n,p} = \mathbf{S}^p\Big(\sqrt{\frac{p}{n}}\Big) \times \mathbf{S}^{n-p}\Big(\sqrt{\frac{n-p}{n}}\Big).$$

This has the following consequence.

Theorem. *Let M^n be a compact hypersurface of $\mathbf{S}^{n+1}$, invariant under the antipodal isometry γ. Let Δ_p be the Laplacian acting on p-forms and $p = 1, \dots, n-1$. Then:*

$$\lambda_2(\Delta_p) - \lambda_1(\Delta_p) \leq n.$$

Moreover:
a) If $p = n/2$ equality never holds;
b) If $p \neq n/2$ equality holds if and only if M is isometric to the Clifford torus $\mathrm{CL}_{n,p}$.

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Constructing Polytopes from Spectral Data

Emily B. Dryden
(joint work with Victor W. Guillemin and Rosa Sena-Dias)

In [1], Miguel Abreu asked, "Can one hear the shape of a Delzant polytope?" A convex polytope is Delzant if it is simple, rational, and smooth; these polytopes are of interest due to a theorem by Delzant which says that the Delzant, or moment, polytope of a compact symplectic toric manifold M determines M up to symplectomorphism. (For background on symplectic toric manifolds and Delzant polytopes, see [3].) In fact, Delzant proved that the moment polytope independently determines both a symplectic and a complex structure. We may take either the symplectic or the complex perspective and generate a set of torus-invariant Riemannian metrics that we call *toric metrics*; the set of metrics generated is the same regardless of the perspective we choose. Thus Abreu's question can be stated more technically as follows: Let M be a compact symplectic toric manifold equipped with a toric metric g. Does the spectrum of the Laplacian with respect to g determine the moment polytope, and hence the symplectomorphism type, of M?

Abreu's original question remains open, although there has been progress in the setting of toric orbifolds (e.g., [2, 7]). A Riemannian orbifold is like a Riemannian manifold except that it is allowed to have a well-behaved singular set in which local neighborhoods are modelled by a quotient of $\mathbb{R}^n$ by a finite group. It was proved by Lerman and Tolman [8] that compact symplectic toric orbifolds are classified by rational simple polytopes with a positive integer attached to each open facet. Rational simple polytopes are more general than Delzant polytopes in that they do not have to be smooth; instead, the outward normals corresponding to the facets meeting at a vertex must form a basis for $\mathbb{Q}^n$. This added flexibility is one of the main reasons that Abreu's question seems more tractable in the orbifold setting.

In addition to moving to the orbifold setting, we consider additional spectral data. Namely, to the usual list of all eigenvalues of the Laplacian, we add the weights of the representation of the torus on each eigenspace; we call this collection of data the *equivariant spectrum*. Our question, then, is whether the equivariant spectrum of a symplectic toric orbifold equipped with a toric metric determines the labelled moment polytope of the orbifold. The main tool we use to answer this question is the asymptotic expansion of the heat kernel on an orbifold in the presence of an isometry. This generalizes work of Donnelly [4] and follows the spirit of the construction in [5]. From this expansion, we see that the equivariant spectrum associated to a toric orbifold whose moment polytope has no parallel facets determines the (unsigned) normal directions to the facets, the volumes of the corresponding facets, and the labels of the facets. We would like to know that this data determines our labeled moment polytope uniquely, but that is too optimistic. The troublemakers in our data are subpolytopes and parallel facets, where a subpolytope is a proper subcollection of normals and associated volumes that themselves form a convex polytope. Parallel facets introduce indeterminants, in that we only know the *sum* of the volumes of the facets in a parallel pair and we do not know which normal directions are repeated. However, thanks to the flexibility of the moment polytopes associated to toric orbifolds, we can show that there is a moment polytope without subpolytopes and without parallel facets "close" to any rational simple polytope in $\mathbb{R}^n$. Then, for these "generic" toric orbifolds, we can conclude that the equivariant spectrum determines the labelled moment polytope, up to two choices and up to translation. For more details, see [6].

We do not know if the genericity assumption in our theorem is necessary, i.e., are there examples of non-generic non-isometric symplectic toric orbifolds with the same equivariant spectrum? Since our theorem holds for any toric metric on the symplectic toric orbifold, one is also tempted to look for metric results; for instance, one might ask whether the equivariant spectrum of a symplectic toric orbifold determines the toric metric. A positive answer to this question seems unlikely, but the equivariant spectrum might determine certain properties of the metric. In particular, we can show that the equivariant spectrum determines whether the metric has constant scalar curvature. Of course, the full force of the equivariant spectrum may not be needed for any of our results, and one could still hope to prove spectral uniqueness using just the usual Laplace spectrum.

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Upper bounds for the spectrum of Riemannian manifolds

Bruno Colbois

This is a survey talk explaining how to get geometric and metric upper bounds for the spectrum of the Laplacian on a compact Riemannian manifolds (M, g) of dimension m. The idea behind these estimates is to construct a family of disjointly supported test functions, and then to use the classical min-max characterization of the spectrum.

So we are lead to construct the support of these test functions. Originally, a rather powerful idea was to use metric balls, but for different kinds of problems, we need more elaborated constructions.

We first describe a construction coming from [Ko] and which is described in a very understandable way in [GNY]. This allows to prove the following theorem ([Ko]):

Theorem 1. *(Korevaar) Let (M^m, g_0) be a compact Riemannian manifold. Then, there exist a constant $C(g_0)$ depending on g_0 such that for any Riemannian metric $g \in [g_0]$, where $[g_0]$ denotes the conformal class of g_0, we have*

$$\lambda_k(M,g)Vol(M,g)^{2/m} \leq C(g_0)k^{2/m}.$$

Moreover, if the Ricci curvature of g_0 is nonegative, we can replace the constant $C(g_0)$ by a constant depending only on the dimension m.

In the special case of surfaces, we have a bound depending only on the topology.

Theorem 2. *Let S be an oriented surface of genus γ. Then, there exist a universal constant C such that for any Riemannian metric g on S*

$$\lambda_k(S,g)Vol(S) \leq C(\gamma+1)k.$$

We then describe another construction given in [CM] and [CEG], particularly useful for submanifolds. It allows, for example, to get the following result given in [CEG]:

Theorem 3. *For any bounded domain $\Omega \subset \mathbb{R}^{m+1}$ with smooth boundary $\Sigma = \partial\Omega$, and all $k \geq 1$,*

$$\lambda_k(\Sigma)Vol(\Sigma)^{2/m} \leq \gamma_m I(\Omega)^{1+2/m}k^{2/m} \tag{1}$$

with γ_m explicit constant depending on m and $I(\Omega)$ denotes the isoperimetric ratio of Ω,

$$I(\Omega) = \frac{Vol(\Sigma)}{Vol(\Omega)^{m/(m+1)}}.$$

These theorems are compatible with the Weyl law, but the coefficient term of $k^{2/m}$ depends on the geometry and not only on the dimension, as we can expect.

We end with a construction mixing the two above methods, see [Ha]. It allows to improve the result of [Ko] in the following way:

Theorem 4. *There exist two constant A_m, B_m depending only on m such that, if (M^m, g_0) is a compact Riemannian manifold, we have the following: for any $g \in [g_0]$,*

$$\lambda_k(M,g)Vol(M,g)^{2/m} \leq A_m V([g_0]) + B_m k^{2/m}.$$

where $V([g_0])$ is a constant depending on the conformal class of g_0 (the min-conformal volume) defined by

$$V([g_0]) = inf\{Vol_g(M) : g \in [g_0], Ric(M,g) \geq -(m-1)\}.$$

In particular, if there exist a metric $g \in [g_0]$ with Ricci$(M, g) \geq 0$, then $V([g_0]) = 0$.

In the case of surfaces, we have the following:

There exist two universal constant A, B such that, if S is a compact surface of genus γ, then

$$\lambda_k(S)Vol(S) \leq A(\gamma + 1) + Bk.$$

Open question: In Theorem 3, is it possible to get an estimate of the type

$$\lambda_k(\Sigma)Vol(\Sigma)^{2/m} \leq A_m I(\Omega)^{1+2/m} + B_m k^{2/m}$$

for constants A_m, B_m depending only on the dimension m?

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The ionization conjecture for pseudorelativistic atoms in Hartree-Fock theory

Anna Dall'Acqua

(joint work with Jan Philip Solovej)

The ionization conjecture can be formulated as follows. Considering atoms of arbitrarly large nuclear charge, the number of electrons that these atoms can bind is bounded by the nuclear charge plus a universal constant and moreover the atomic radius remain bounded. Indeed, in nature only negative ions of charge -1 are present and looking at the periodic table of elements one sees that the atomic radius changes at most by a factor of two.

As a model for an atom with nuclear charge Z and N electrons we consider (in units where $\hbar = m = e = 1$) the operator

$$H = \sum_{i=1}^{N} \alpha^{-1}\left(\sqrt{-\Delta_i + \alpha^{-2}} - \alpha^{-1} - \frac{Z\alpha}{|\mathbf{x}_i|}\right) + \sum_{1 \leq i < j \leq N} \frac{1}{|\mathbf{x}_i - \mathbf{x}_j|}, \tag{1}$$

where α is Sommerfeld's fine structure constant. The operator H acts on a dense subset of the N body Hilbert space $\mathcal{H}_F := \wedge_{i=1}^{N} L^2(\mathbb{R}^3; \mathbb{C}^q)$ of antisymmetric wave functions, where q is the number of spin states. For simplicity we fix here $q = 1$. The operator H is bounded from below on this subspace if $Z\alpha \leq 2/\pi$. Here we will consider the sub-critical case $Z\alpha < 2/\pi$. This choice of the kinetic energy takes into account some relativistic effects and that is the reason for speaking of pseudo-relativistic atoms.

The quantum ground state energy is the infimum of the spectrum of H considered as an operator acting on $\mathcal{H}_F$. In the Hartree-Fock approximation one restricts to wave-functions ψ which are pure wedge products, also called Slater determinants:

$$\psi(\mathbf{x}_1, \mathbf{x}_2, , \dots, \mathbf{x}_N) = \tfrac{1}{\sqrt{N!}} \det(u_i(\mathbf{x}_j))_{i,j=1}^N, \tag{2}$$

with $\{u_i\}_{i=1}^N$ orthonormal in $L^2(\mathbb{R}^3; \mathbb{C})$. The Hartree-Fock ground state energy is

$$E^{\mathrm{HF}}(N, Z, \alpha) := \inf\{\mathrm{q}(\psi, \psi) | \psi \in \mathcal{Q}(H) \text{ and } \psi \text{ a Slater determinant}\},$$

with q the quadratic form defined by H and $\mathcal{Q}(H)$ the corresponding form domain. The one-particle density associated to a Slater determinant as in (2) is given by

$$\rho^{\mathrm{HF}}(\mathbf{x}) = \sum_{i=1}^N |u_i(\mathbf{x})|^2 .$$

For $\nu \in (0, N)$, we define the HF-radius $R_{N,Z}^{\mathrm{HF}}(\nu)$ to the ν last electrons by

$$\int_{|\mathbf{x}| \geq R_{N,Z}^{\mathrm{HF}}(\nu)} \rho^{\mathrm{HF}}(\mathbf{x})\, d\mathbf{x} = \nu .$$

The main result presented in the talk is the following.

Theorem 1. *(See* [1]*) Let $Z \geq 1$ and $\alpha > 0$. Let $Z\alpha = \kappa$ and assume that $0 \leq \kappa < 2/\pi$. There is a constant $Q > 0$ depending only on κ such that if N is such that a Hartree-Fock minimizer exists then $N \leq Z + Q$. Moreover, both $\liminf_{Z\to\infty} R_{Z,Z}^{\mathrm{HF}}(\nu)$ and $\limsup_{Z\to\infty} R_{Z,Z}^{\mathrm{HF}}(\nu)$ are bounded and behave asymptotically as*

$$3^{\frac{4}{3}} \frac{2^{\frac{1}{2}\pi^{\frac{2}{3}}}}{q^{\frac{2}{3}}} \nu^{-\frac{1}{3}} + o(\nu^{-\frac{1}{3}}) \text{ as } \nu \to \infty .$$

This extends the result obtained by Solovej in [2] for classical atoms, that is in the case when the kinetic energy is given by $-\Delta$.

In the proof one considers a number of electrons N such that a HF-minimiser exists. Let denote its density by ρ^{HF}. One considers the *HF screened nuclear potential*

$$\Phi_R^{\mathrm{HF}}(\mathbf{x}) = \frac{Z}{|\mathbf{x}|} - \int_{|\mathbf{y}|<R} \frac{\rho^{\mathrm{HF}}(\mathbf{y})}{|\mathbf{x} - \mathbf{y}|}\, d\mathbf{y} . \tag{3}$$

This describe the charge that an electron at distance from the nucleus bigger than R feels, indeed one has

$$\frac{1}{4\pi R} \int_{|\mathbf{x}|=R} \Phi_R^{\mathrm{HF}}(\mathbf{x})\, d\omega(\mathbf{x}) = Z - \int_{|\mathbf{x}|<R} \rho^{\mathrm{HF}}(\mathbf{x})\, d\mathbf{x} .$$

Further one considers the Thomas-Fermi minimiser ρ^{TF} for the neutral atom of nuclear charge Z and associates to it the TF screened nuclear potential defined as in (3) replacing the HF density with the TF density. Theorem 1 follows from the following key estimate.

Let $Z\alpha = \kappa$ for $0 \le \kappa < 2/\pi$. Then there exists universal constants α_0, $\epsilon \in (0,4)$, C_M and C_Φ such that for all $\alpha \le \alpha_0$

$$\left|\Phi^{\mathrm{TF}}_{|\mathbf{x}|}(\mathbf{x}) - \Phi^{\mathrm{HF}}_{|\mathbf{x}|}(\mathbf{x})\right| \le C_\Phi |\mathbf{x}|^{-4+\epsilon} + C_M \,.$$

This estimate is proven by an iterative procedure. We first prove the estimate for small $|\mathbf{x}|$ (i.e. $|\mathbf{x}| \le Z^{-1/3}$) by directly comparing the HF screened nuclear potential with the TF screened nuclear potential. At an intermediate distance the TF minimiser is not a good approximation of the HF minimiser, for this reason one defines an *outside Thomas Fermi* functional, that is a Thomas Fermi variational problem that takes into account that we are at a certain distance from the nucleus and also of the screening due to the other electrons present. The estimate is then obtained at intermediate distance by comparing the HF minimiser with the outside Thomas Fermi minimiser and at big distance via localisation.

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Isoperimetric control of the Steklov spectrum

Alexandre Girouard

(joint work with Iosif Polterovich, Bruno Colbois, Ahmad El Soufi)

The goal of this talk is to survey isoperimetric upper bounds for the Steklov eigenvalues.

Let Ω be a smooth compact connected Riemannian manifold with boundary $\Sigma = \partial\Omega$. The *Dirichlet-to-Neumann* map (DtN map for short) $\Lambda : C^\infty(\Sigma) \to C^\infty(\Sigma)$ is defined by $\Lambda f = \partial_n \tilde{f}$, where ∂_n is the outward normal derivative along the boundary Σ, and $\tilde{f}$ is the harmonic extension of f to Ω. The DtN map is a first order pseudodifferential elliptic operator. The spectrum of the DtN map, also called [12] the *Steklov spectrum of* Ω is non-negative, discrete and unbounded. It is denoted $0 = \sigma_0 < \sigma_1 \le \sigma_2 \le \sigma_3 \le \cdots \nearrow \infty$. It satisfies the Weyl type asymptotic formula [11]

$$\sigma_k \sim c(n) \left(\frac{k}{|\Sigma|}\right)^{1/n} \qquad \text{as } k \nearrow \infty,$$

where $c(n)$ depends only on the dimension $n = \dim \Sigma$. Of course the Steklov eigenvalues can also be directly represented by the eigenvalue problem

$$\Delta f = 0, \quad \partial_n f = \sigma f.$$

The DtN map is closely related to the Calderón problem of determining the anisotropic conductivity of a body from measurements on its surface. It can be thought of as a *voltage–to–current map*. The Steklov eigenvalues and eigenfunctions are used in heat transmission, vibration problems, and sloshing problems in hydrodynamics [2, 10].

How large can σ_k be ?

For a fixed $k \in \mathbb{N}$, the problem is to maximize the functional σ_k on the set $\mathcal{R}$ of Riemannian metrics g on Ω constrained by $|\Sigma|_g = 1$, or equivalently to maximize the scaling invariant functional $g \mapsto \sigma_k(g)|\Sigma|_g^{1/n}$ on the space of all smooth Riemannian metrics.

Simply connected planar domains. Let Ω be a simply connected bounded planar domain. For simplicity, assume its boundary to be smooth. In 1954, Weinstock [13] proved $\sigma_1(\Omega)|\partial\Omega| \leq 2\pi$ with equality if and only if Ω is a disk. In 1974, Hersch, Payne and Schiffer [9] extended this result to higher eigenvalues. In particular, they proved that for each $k \in \mathbb{N}$,

$$\sigma_k(\Omega)|\partial\Omega| \leq 2k\pi. \tag{1}$$

In [6], I. Polterovich and I proved that inequality (1) is sharp by constructing a family of simply connected domains Ω_ϵ such that

$$\lim_{\epsilon \to 0} \sigma_k(\Omega_\epsilon)|\partial\Omega_\epsilon| = 2k\pi.$$

We also proved in [7] that inequality (1) is strict for $k = 2$.

Surfaces. Let Ω be a compact surface with boundary. It was proved by Ahlfors [1] that there exists a proper conformal branched cover $\psi : \Omega \to \mathbb{D}$ with degree bounded above in terms of the genus γ of Ω and the number l of connected components of its boundary. Fraser and Schoen [4] used an improvement of this theorem due to A. Gabard [5] to prove $\sigma_1(\Omega)|\Sigma| \leq 2\pi(\gamma + l)$. For higher eigenvalues, one recovers [8] the natural analogue of the Hersch–Payne–Schiffer bound (1); that is, $\sigma_k(\Omega)|\Sigma| \leq 2\pi(\gamma + l)k$. Fraser and Schoen have initiated a beautiful study of the maximizers for σ_1 in terms of minimal surfaces in Euclidean balls with free boundary conditions. In particular, they have proved that for $\gamma = 0$ and $l = 2$, the first eigenvalue σ_1 is maximized by the so-called *critical catenoid*. In [4], this result is presented as a conjecture, but they have since announced a complete proof.

Higher dimension. Let N be a Riemannian manifold of dimension $n + 1$ which is conformally equivalent to a complete Riemannian manifold with non-negative Ricci curvature. The *isoperimetric ratio* of a bounded domain $\Omega \subset N$ is

$$I(\Omega) = \frac{|\Sigma|}{|\Omega|^{n/(n+1)}}.$$

In collaboration with B. Colbois and A. El Soufi, we proved [3] that for any bounded domain $\Omega \subset N$, and for each $k \in \mathbb{N}$,

$$\sigma_k(\Omega)|\Sigma|_g^{1/n} \leq \frac{c(n)}{I(\Omega)^{(1-1/n)}} k^{2/(n+1)}. \tag{2}$$

In particular, it follows from known isoperimetric inequalities that for any domain Ω in $\mathbb{R}^{n+1}$, $\mathbb{H}^{n+1}$ or in an hemisphere of the sphere $\mathbb{S}^{n+1}$,

$$\sigma_k(\Omega)|\Sigma|_g^{1/n} \leq c'(n)k^{2/(n+1)}.$$

It is perhaps surprising that the isoperimetric ratio $I(\Omega)$ appears in the denominator of inequality (2). For $n = \dim(\Sigma) \geq 2$, this implies that if a sequence of domains Ω_l is such that $\lim_{l\to\infty} I(\Omega_l) \to \infty$, then for each $k \in \mathbb{N}$, $\lim_{l\to\infty} \sigma_k(\Omega_l) = 0$.

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A survey on spectral minimization problems: existence, regularity and stability

Aldo Pratelli

The aim of this survey is to give a general overview on some problems of spectral minimization, putting together the main classical and recent results, as well as some important open questions.

It happens often for several reasons, both coming from abstract investigation or from many applications, that one is led to minimize a shape functional which depends on the eigenvalues. To be more precise, let us start fixing some notation: for any given open set $\Omega \subseteq \mathbb{R}^N$, we will denote by

$$0 < \lambda_1(\Omega) \leq \lambda_2(\Omega) \leq \cdots \qquad \text{and} \qquad 0 = \mu_1(\Omega) \leq \mu_2(\Omega) \leq \cdots$$

the eigenvalues of the standard Laplacian, with Dirichlet or Neumann boundary conditions respectively; recall that, by trivial rescaling, one obtains that

$$\lambda_i(t\Omega) = t^{-2/N}\lambda_i(\Omega)\,, \qquad \mu_j(t\Omega) = t^{-2/N}\mu_j(\Omega)\,, \tag{1}$$

for any $t > 0$ and any $i, j \in \mathbb{N}$. Then, a *spectral minimization problem* is the problem to minimize, among a class of sets $\Omega \subseteq \mathbb{R}^N$, a certain functional depending on some of the eigenvalues of Ω. Usually, in view of (1), the class of sets which are considered is given by all the open sets of fixed measure.

The simplest problem of this kind is to minimize λ_1 among the sets of given (say, unite) volume. It is classically known (see for instance [10]) that the solution of this problem is given by any unit ball: this is the Faber–Krahn inequality. Analogously, the minimizer of λ_2 among the sets of unit volume is any disjoint union of two balls of volume 1/2: this is the Krahn–Szego inequality (see for instance [11, 14]). Concerning the third Dirichlet eigenvalue, it is still presently not known which is the minimizer: one only knows that some minimizer exists, and that it is connected in dimension $N = 2, 3$ (see [6]). A major conjecture is to show that the minimizer for λ_3 in the plane is given by the unit ball; however, even if many numerical computations suggest this result, the problem is still completely open. On the other hand, concerning the Neumann eigenvalues, there is nothing to say concerning μ_1, since $\mu_1(\Omega) = 0$ for every Ω; more in general, $\mu_j(\Omega) = 0$ for any set Ω having at least j distinct connected components. Therefore, it becomes interesting the question to maximize the eigenvalues: for instance, the classical Szegő–Weinberger inequality (see for instance [15, 16]) tells that the maximum μ_2, among the sets of unit volume, is again attained by the unit ball. Another interesting problem, of which the solution is again the ball, is the minimization of the ratio λ_1/λ_2, proved in [1].

Except for those classical examples, very little is known about the exact shapes of the solutions of some spectral problems. The following very important result, due to Buttazzo and Dal Maso [8], ensures at least the existence for a wide class of problems, dealing with the first k eigenvalues (being k any given positive number).

Theorem 1. *Let $F : \mathbb{R}^k \to \mathbb{R}$ be a l.s.c. and increasing function, and for any quasi-open set $\Omega \subseteq \mathbb{R}^N$ let us write $\mathfrak{F}(\Omega) := F\big(\lambda_1(\Omega), \lambda_2(\Omega), \dots, \lambda_k(\Omega)\big)$. Let also $D \subseteq \mathbb{R}^N$ be a given bounded, open set. Then there exists Ω which minimizes $\mathfrak{F}$ among all the quasi-open sets of given volume contained in D.*

In this result, one considers a bounded ambient space D instead of the whole $\mathbb{R}^N$ in order to overcome the possible concentration-compactness problems for a minimizing sequence. Moreover, one considers the class of the quasi-open sets, instead of that of the open sets, because it allows to obtain compactness results which otherwise would be missing. Hence, the main questions left after the Buttazzo–Dal Maso Theorem are whether one can consider the problem in the natural ambient space of $\mathbb{R}^N$, and whether one can show that the minimizers are actually regular.

We start with the first question, which has now a completely positive answer, thanks to the following two contemporary results.

Theorem 2 (Mazzoleni–Pratelli, [13])**.** *Let $F : \mathbb{R}^k \to \mathbb{R}$ be a l.s.c. and increasing function, and for any quasi-open set $\Omega \subseteq \mathbb{R}^N$ let us write $\mathfrak{F}(\Omega) := F\big(\lambda_1(\Omega), \lambda_2(\Omega), \dots, \lambda_k(\Omega)\big)$. Then there exists Ω which minimizes $\mathfrak{F}$ among all the quasi-open sets of given volume contained in $\mathbb{R}^N$. Moreover, such a minimizer is bounded.*

Theorem 3 (Bucur, [5])**.** *For any natural number k, there exists a set $\Omega \subseteq \mathbb{R}^N$ which minimizes λ_k among the quasi-open sets of given volume contained in $\mathbb{R}^N$. Moreover, such a minimizer has finite perimeter.*

It is to be mentioned that, in the paper [13], the existence of *some* bounded minimizer for the functional $\mathfrak{F}$ is shown: the fact that *any* minimizer is bounded has been shown later in [12].

Let us now briefly discuss the regularity issue. Notice that, on one hand, in the few examples in which the actual minimizers are known, they are smooth sets. On the other hand, the available existence results are only able to find minimizers among the wide and unpleasant class of the quasi-open sets. Therefore, one would be happy to show at least that the optimal sets are open. In fact, Bucur and Velichkov have recently shown [7] that this is the case when one considers a functional depending only on the first two eigenvalues.

Let us now pass to the last problem that we want to address, namely, the stability of the main spectral inequalities. The question is very easy to explain; as said above, the Faber–Krahn inequality ensures that, if a set $\Omega \subseteq \mathbb{R}^N$ minimizes the first eigenvalue of the Dirichlet Laplacian among the sets of unit volume, then it must be some unit ball. The stability problem is the following: if a set Ω is *almost* minimizing the first eigenvalue, is it true in some suitable sense that it must be *close* to a unit ball? The most natural way to measure the distance between Ω and a ball is the so-called "Fraenkel asymmetry", given by

$$\mathcal{A}(\Omega) = \inf \Big\{ \big|\Omega \Delta B\big| : B \text{ is a unit ball} \Big\} ,$$

where $\big|\Omega\Delta B\big|$ is the volume of the symmetric difference $\Omega\Delta B = \big(\Omega \setminus B\big) \cup \big(B \setminus \Omega\big)$ between the set Ω and the ball B. In words, the Fraenkel asymmetry measures how far the set Ω is from being a ball, and it is of course always strictly positive, unless Ω is a ball itself. The main results in this direction are the following.

Theorem 4 (Bhattacharya [2])**.** *For any open set $\Omega \subseteq \mathbb{R}^2$ one has*

$$\lambda_1(\Omega) - \lambda_1(B) \geq C\mathcal{A}(\Omega)^3 .$$

Theorem 5 (Fusco–Maggi–Pratelli [9])**.** *For any open set $\Omega \subseteq \mathbb{R}^N$ one has*

$$\lambda_1(\Omega) - \lambda_1(B) \geq C(N)\mathcal{A}(\Omega)^4 . \tag{2}$$

In fact, it is very reasonable to guess that the correct exponent, for any dimension N, should always be 2. This is another important open problem. It is also to be mentioned that, in the paper [9], the authors consider the more general question of the $p-$Laplacian λ^p; in fact, for any $p \in (1, \infty)$, one can show that

$\lambda^p(\Omega) - \lambda^p(B) \geq C(N,p)\mathcal{A}(\Omega)^{2+p}$, of which of course the inequality (2) is nothing but the particular case corresponding to $p = 2$.

Other results of the same flavour, concerning the stability versions of the Krahn–Szego inequality and of the Szegő–Weinberger inequality, have been obtained in [4], while the extension to boundary conditions of Robin type has been studied in [3].

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Torsional Rigidity of a Radially Perturbed Ball

Mette Iversen

The goal of this talk is to discuss bounds for the torsional rigidity of a radially perturbed ball in $\mathbb{R}^n$. This problem was motivated by work in [6] minimising convex combinations of the first three eigenvalues of the Dirichlet Laplacian, and an analogous result for the first eigenvalue of the Dirichlet Laplacian given in [2].

For an open set $\Omega \subseteq \mathbb{R}^n$ the torsional rigidity is defined by

$$P(\Omega) := 4 \sup_{u \in H_0^1(\Omega)} \frac{\left(\int_\Omega u\right)^2}{\int_\Omega |\nabla u|^2}. \tag{1}$$

The function u for which the supremum is achieved is known as the torsion function. If Ω is a simply connected planar set then $P(\Omega)$ gives the torsional rigidity of a cylindrical beam of cross section Ω, and more generally the torsion function plays a role in a variety of areas including gamma convergence [3] and Brownian motion [1]. The torsion function is the unique weak solution of

$$\begin{cases} -\Delta u = 2 \text{ in } \Omega, \\ \quad u = 0 \text{ on } \partial\Omega. \end{cases}$$

It follows from (1) that P satisfies the following scaling property: For an open set $\Omega \subseteq \mathbb{R}^n$ and a homothety of ratio $t > 0$

$$P(t\Omega) = t^{n+2} P(\Omega).$$

For the ball $B = B(0,1) \subset \mathbb{R}^n$ the maximum in (1) is achieved by the function

$$u(x) = \frac{1}{n}(1 - |x|^2),$$

with $P(B) = \frac{4\omega_n}{n(n+2)}$, where ω_n denotes the volume of the ball of radius 1 in $\mathbb{R}^n$.

An important property is the analogue for the torsional rigidity of the Faber-Krahn inequality for the first Dirichlet eigenvalue [5]. It is proved in [4] that the ball in $\mathbb{R}^n$ maximises the torsional rigidity over sets of fixed volume. More precisely, for any open, bounded C^1 set $\Omega \subset \mathbb{R}^n$ and any ball $\Omega^* \subset \mathbb{R}^n$ with the same volume as Ω, we have

$$P(\Omega^*) \geq P(\Omega),$$

with equality if and only if Ω is a ball. Taking scaling into account shows that this is equivalent to

$$P(\mathcal{B}) \left(\frac{|\Omega|}{|\mathcal{B}|} \right)^{\frac{n+2}{n}} \geq P(\Omega), \tag{2}$$

if $\mathcal{B}$ is any ball.

We will consider the torsional rigidity for a class of radial perturbations of the unit ball in $\mathbb{R}^n$ to give an idea of the correction terms when moving away from the optimal set given by the analogue of the Faber-Krahn inequality. This is done by making an appropriate choice of test function in the variational characterisation (1), and is an improvement on that given in [7].

Let $\mathcal{S}^{n-1}$ be the unit sphere in $\mathbb{R}^n$ with ∇_θ denoting the gradient on $\mathcal{S}^{n-1}$, and let $0 \leq \varepsilon \leq \frac{1}{4}$. Assume that $R : \mathcal{S}^{n-1} \longrightarrow [1 - \varepsilon, 1]$ is continuous, and that $\nabla_\theta R$ exists θ-almost everywhere. Let

$$B_R = \{(r, \theta) : 0 \leq r < R(\theta), \theta \in \mathcal{S}^{n-1}\}.$$

Theorem 1. *We have*

$$(i)\ P(B_R) \leq P(B)\left(1 - \frac{n+2}{n\omega_n}\int_{\mathcal{S}^{n-1}}(1-R)\right) + C_1(n)\int_{\mathcal{S}^{n-1}}(1-R)^2,$$

$$(ii)\ P(B_R) \geq P(B)\left(1 - \frac{n+2}{n\omega_n}\int_{\mathcal{S}^{n-1}}(1-R)\right) - C_2(n)\left(\varepsilon\int_{\mathcal{S}^{n-1}}|\nabla_\theta R|^2 + \int_{\mathcal{S}^{n-1}}(1-R)^2\right) - C_2(n)\left(\int_{\mathcal{S}^{n-1}}(1-R)^2\int_{\mathcal{S}^{n-1}}|\nabla_\theta R|^2\right)^{1/2},$$

for positive constants $C_1(n)$ and $C_2(n)$ depending on n only.

The result is analogous to that given in Theorem 1 in [2] for the first Dirichlet eigenvalue, and the proof follows the method described there.

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Sharp Spectral Bounds on Starlike Domains

Richard S. Laugesen

(joint work with B. A. Siudeja)

Write $\lambda_j(\Omega)$ for the eigenvalues of the Dirichlet Laplacian on a bounded plane domain Ω. Write u_j for a corresponding orthonormal basis of eigenfunctions. Then

$$\begin{cases} -\Delta u_j = \lambda_j u_j \ \text{ in } \Omega \\ \qquad u_j = 0 \ \text{ on } \partial\Omega \end{cases}$$

and

$$0 < \lambda_1 < \lambda_2 \leq \lambda_3 \leq \ldots$$

How does the shape of the domain affect the eigenvalues?

This problem is long-standing and difficult. Our contribution is to establish geometrically sharp upper bounds on eigenvalues for the class of starlike domains (see Figure 1). The normalization on the starlike domain Ω involves the scale-invariant geometric factor $G = \max\{G_0, G_1\}$, where

$$G_0 = \frac{1}{2\pi} \int_{\partial\Omega} \frac{1}{x \cdot N(x)} \, ds(x), \qquad G_1 = \frac{2\pi I_{\text{origin}}}{A^2},$$

and $N(x)$ is the outward unit normal vector, A is the area of Ω and $I_{\text{origin}} = \int_\Omega |x|^2 \, dA$ is its polar moment of inertia about the origin. Note that $x \cdot N(x) > 0$ because the domain is starlike.

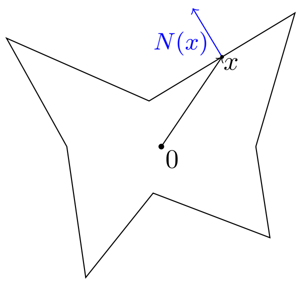

FIGURE 1. A starlike domain with outer normal $N(x)$.

One has $G \geq 1$ for all starlike domains, with equality for centered disks. Thus one may regard the value of G as measuring the deviation of the domain from roundness.

Here is our main result, in 2 dimensions.

Theorem 1 (Dirichlet eigenvalues)**.** *Suppose the function $R(\theta)$ is 2π-periodic, positive, and Lipschitz continuous, and consider the starlike plane domain $\Omega = \{re^{i\theta} : 0 \leq r < R(\theta)\}$. Let $\Phi : \mathbb{R}_+ \to \mathbb{R}$ be concave and increasing.*

Then for each $n \geq 1$, the eigenvalue functional $\sum_{j=1}^n \Phi(\lambda_j A/G)$ is maximal when Ω is a disk centered at the origin.

In particular, the following spectral functionals are maximal for the centered disk: the first eigenvalue (fundamental tone), sums of arbitrarily many eigenvalues or roots of eigenvalues, and products of eigenvalues. That is, the value of each of the following functionals is less than or equal to the value attained when Ω is a centered disk:

$$\lambda_1 A/G_0, \qquad (\lambda_1^s + \cdots + \lambda_n^s)^{1/s} \, A/G, \qquad \sqrt[n]{\lambda_1 \lambda_2 \cdots \lambda_n} \, A/G,$$

for each exponent $0 < s \leq 1$.

Further, the partial sums of the spectral zeta function and trace of the heat kernel are minimal when Ω *is a centered disk. That is, the functionals*

$$\sum_{j=1}^{n} (\lambda_j A/G)^s \qquad \textit{and} \qquad \sum_{j=1}^{n} \exp(-\lambda_j At/G)$$

attain their smallest value when Ω *is a centered disk.*

The theorem extends to all dimensions, and to the Neumann and Robin spectra too.

Our theorem significantly strengthens the only known result of its type, which is the case of the fundamental tone λ_1 with Dirichlet boundary conditions. That case is due to Pólya and Szegő in 2 dimensions [2] and Freitas and Krejčiřík in higher dimensions [1]. Those results rely crucially on the fundamental Dirichlet mode of the disk and ball being radial functions, and so this approach is limited to the first eigenvalue.

Methods. To treat higher eigenvalues, we need to transform Ω into a ball while preserving *angular* information in the Rayleigh quotients of the eigenvalues. Any such transformation will change the Rayleigh quotients substantially, and so we must devise a scheme for extracting the geometric effect and leaving behind the portion of the Rayleigh quotient that corresponds to the disk.

Our new transformation maps linearly on rays and has constant Jacobian, as indicated in Figure 2. Wherever it stretches radially it must compress in the angular directions, and so on. We construct trial functions on the disk by composing eigenfunctions of the disk with T and with a rotation. Then to extract the geometric contribution to the Rayleigh quotient, we average over all possible rotations.

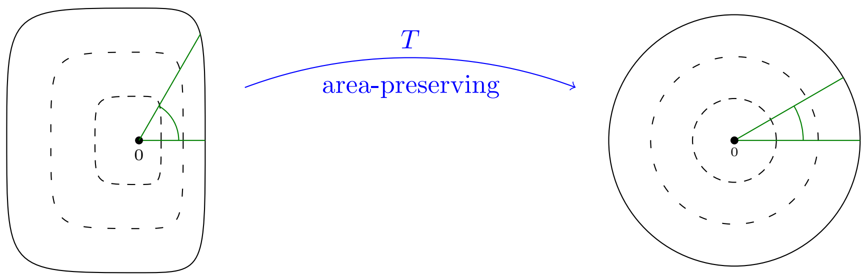

FIGURE 2. An area-preserving, linear-on-rays transplanation from a domain Ω to the disk.

Reverse inequalities. The Faber–Krahn inequality says that $\lambda_1 A$ is minimal for the disk. Luttinger's generalization asserts maximality for the disk of the heat trace $\sum_j \exp(-\lambda_j At)$, for each $t > 0$. Notice that Theorem 1 produces inequalities in the reverse direction, after introducing the geometric factor G. The resulting theorem improves in two respects on Faber–Krahn and Luttinger's results, namely that it holds even forl partial sums of the heat trace (and many other functionals besides), and that it handles all three of Dirichlet, Neumann and Robin boundary conditions.

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Minimization of eigenvalues by free boundary - free discontinuity methods

Dorin Bucur

I this talk I reported on new methods based on free boundary - free discontinuity techniques that can be used in order to obtain qualitative information on the optimal domains minimizing a shape functional. In particular, I considered functionals depending on the spectrum of the Laplace operator with Dirichlet or Robin boundary condition.

Dirichlet boundary conditions. Let $F : \mathbb{R}^k \to \mathbb{R}$ be a lower semicontinuous function, non-decreasing in each variable. For every quasi-open set $\Omega \subseteq \mathbb{R}^N$ ($N \geq 2$) we denote

$$\lambda_1(\Omega) \leq \lambda_2(\Omega) \leq ... \leq \lambda_k(\Omega)$$

the first k eigenvalues of the Laplacian with Dirichlet boundary conditions. Concerning the definition of quasi-open sets and of the Laplace operator on quasi-open sets, we refer to [3, Chapters 4, 5]. Denoting by $|\Omega|$ the measure of Ω, for some $m > 0$ we consider the following problems:

$$(P_1) \qquad \min_{\Omega \text{ quasi-open}, |\Omega|=m} F(\lambda_1(\Omega), ..., \lambda_k(\Omega));$$

$$(P_2) \qquad \min_{\Omega \text{ quasi-open}} F(\lambda_1(\Omega), ..., \lambda_k(\Omega)) + |\Omega|.$$

Below is a compilation, presented in a unified framework, of two recent existence-qualitative results obtained independently and using different methods by Mazzoleni and Pratelli [8] and D. B. [2].

a) Under the hypotheses above, problems (P_1) and (P_2) have at least one solution Ω, which is a bounded set.
b) If F is moreover locally Lipschitz, then every solution Ω of (P_2) is a bounded set with finite perimeter.
c) If in addition F is locally bi-Lipschitz in at least one variable, then every solution Ω of (P_1) is a bounded set with finite perimeter.

The existence result for (P_1) and (P_2) under the additional constraint that the competing domains are contained in a prescribed bounded open set was proved by Buttazzo and Dal Maso in [6]. Point a) is proved in [8] and is based on a geometric argument allowing to cut small (unbounded) regions of a set while keeping control on the spectrum. Points b) and c) are a consequence of the analysis of the shape subsolutions by free boundary methods (see [2]) and can be complemented with more information related to the inner density of the optimal shapes. In all cases, the diameter of the optimal sets are controlled.

Robin boundary conditions. Let $\beta > 0$ be fixed. For every bounded set with Lipschitz boundary $\Omega \subseteq \mathbb{R}^N$ we consider the Robin-torsional rigidity defined by $P(\Omega) = \int_\Omega u dx$, where u solves the problem

$$\begin{cases} -\Delta u = & 1 \text{ in } \Omega \\ \frac{\partial u}{\partial n} + \beta u = & 0 \;\; \partial\Omega \end{cases}$$

In a joint work with A. Giacomini (in progress) we prove that

$$P(B) \geq P(\Omega),$$

where B is the ball of the same measure as Ω. Moreover, equality holds if and only if Ω is the ball.

The method is based on a free discontinuity approach, via the analysis of the relaxed free discontinuity problem

$$\min_{u^2 \in SBV(\mathbb{R}^N), |\{u>0\}=m|} \frac{1}{2}\int_{\mathbb{R}^N} |\nabla u|^2 dx + \frac{\beta}{2}\int_{J_u} (|u^+|^2 + |u^-|^2) d\mathcal{H}^{N-1} - \int_{\mathbb{R}^N} u dx,$$

for which we prove existence of a solution and Ahlfors regularity of the jump set. Of crucial importance in the proof of the Ahlfors regularity is the monotonicity lemma obtained in a joint work with S. Luckhaus [4], as well as a regularity result for free discontinuity problems with Robin boundary conditions obtained in the same paper, asserting the existence of a positive constant $\alpha > 0$ such that the solution of the free discontinuity problem above satisfies

$$u(x) \geq \alpha \quad \text{a.e.} \quad x \in \{u > 0\}.$$

This results would be a consequence of the Hopf maximum principle if the boundary of the positivity region were smooth.

Consequently, one proves that the minimizer of the relaxed free discontinuity problem above consists on a couple: an open set with finite perimeter Ω and the Robin torsion function on Ω extended by zero on its complement. Radial symmetry of the set Ω and its torsion function is obtained by a cut and reflect argument.

This method can be extended to semi-linear eigenvalue Robin problems

$$\min_{|\Omega|=m} \min_{u \in H^1(\Omega)} \frac{\int_\Omega |\nabla u|^2 dx + \beta \int_{\partial\Omega} |u|^2 d\mathcal{H}^{N-1}}{\left(\int_\Omega |u|^q dx\right)^{\frac{2}{q}}},$$

for $1 \leq q < \frac{2N}{N-2}$. The first Robin eigenvalue of the Laplacian corresponds to the case $q = 2$ and the torsional rigidity to $q = 1$. In the literature, only the case $q = 2$ was known (Bossel [1] in $\mathbb{R}^2$, Daners [7] in $\mathbb{R}^N$ within a Lipschitz setting and B.-Giacomini [5] for the general SBV framework). The proof of the case $q = 2$ uses a key idea of Bossel related to the notion of extremal lengths which does not seem (easily) extendable to other values of q.

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Asymptotic behaviour of some eigenvalue optimisation problems

Pedro Freitas

(joint work with P.R.S. Antunes, J.B. Kennedy)

The isoperimetric structure of low eigenvalues of the Laplacian is a classical problem in spectral theory which has received a lot of the attention of mathematicians in the intervening 135 years since Rayleigh conjectured that, among all fixed membranes with a given area, the disk minimises the first frequency [R]. Although there are still some gaps in the theory waiting to be closed (i.e. the proof that two equal balls maximise the second nontrivial eigenvalue of the Neumann Laplacian among open sets in any dimension), the picture is now fairly clear regarding what happens to the first two eigenvalues for Dirichlet, Neumann and Robin boundary conditions [Bo, D, F, GNP, Ke, K1, K2, R, S, W].

In spite of this and several extensions in other directions (manifolds, higher order operators, etc), no progress whatsoever has been made regarding higher eigenvalues. As an example, we mention the conjecture that the third eigenvalue with Dirichlet boundary conditions is minimised by the disk in two dimensions, which has resisted all attempts to prove it so far. On the positive side, Bucur [Bu] and Mazzoleni and Pratelli [MP] have recently proved existence of optimisers for Dirichlet boundary conditions among quasi-open sets. The actual regularity of optimisers and even the existence for other boundary conditions remain open problems.

Maybe because of this, extensive numerical studies of optimal domains for higher eigenvalues have began appearing in the literature [O, AF, AFK]. Although the picture is still far from clear, it is by now apparent that in general and at least in two dimensions,

i. optimal sets will not be disks or unions of disks
ii. the boundary of optimal sets cannot be described in terms of *known* functions
iii. optimisers will be different for different boundary conditions
iv. optimal sets may not be unique

To the above it should also be added that there seem to exist optimisers which do not possess any symmetry, the lowest example being that of the 13^{th} eigenvalue with Dirichlet boundary conditions [AF].

On the other hand, there is the possibility that some structure is present with respect to the multiplicity of optimisers and also regarding high-order Robin minimisers [AF, AFK].

In some sense, and with the possible exception of the lack of symmetry of some optimal sets which in any case still needs to be confirmed, the above is not really unexpected, since we are discussing intermediate frequencies.

With all of the above in mind, we have turned our attention to the other end of the spectrum, namely, what happens as k goes to infinity. Again even here there are some surprises, such as the fact that, for Robin boundary conditions, the asymptotic behaviour of optimal sets does not agree with the corresponding Weyl asymptotics not even to first term. More precisely, we have

Theorem 1. [AFK] *Consider the eigenvalue problem*

$$\begin{aligned} -\Delta u &= \lambda u \qquad \text{in } \Omega, \\ \frac{\partial u}{\partial \nu} + \alpha u &= 0 \qquad \text{on } \partial\Omega \end{aligned} \tag{1}$$

where Ω is a bounded open set in $\mathbb{R}^n$, ν is the outer unit normal to Ω and the boundary parameter $\alpha > 0$ is constant. Given $V > 0$ and $k \geq 1$, let B_k denote the domain of volume V consisting of k equal balls of radius $r = (V/k\omega_n)^{1/n}$, where ω_n denotes the volume of the ball of unit radius in $\mathbb{R}^n$. Then, for every $\alpha > 0$, the optimal eigenvalue $\lambda_k^*(V,\alpha)$ satisfies

$$\lambda_k^*(V,\alpha) \leq \lambda_k(B_k,\alpha) \leq n\alpha \left(\frac{k\omega_n}{V}\right)^{\frac{1}{n}}. \tag{2}$$

We thus see that while the first term in Weyl's law is of order $k^{2/n}$, optimal eigenvalues will grow at most with $k^{1/n}$, and it is also possible to show that the gap between optimisers goes to zero as k goes to infinity – see [AFK] for more details.

On the other hand, Dirichlet eigenvalues satisfy an added constraint in the form of the Berezin [B]–Li-Yau [LY] inequalities, namely,

$$\lambda_k(\Omega) \geq \frac{4\pi^2 n}{n+2}\left(\frac{k}{|\Omega|\omega_n}\right)^{2/n}.$$

It follows that optimal eigenvalues in this case must grow at least with the same power as the first term in the Weyl asymptotics, where whether the constant in front of $k^{2/n}$ is the same will depend on whether the famous Pólya inequality holds for general domains or not [P].

In this full generality, this problem as a whole is probably quite difficult and, if nothing else, Theorem 1 above reminds us that not only is it not correct to use arguments based on asymptotics for problems of this type, but moreover the result in itself might not hold.

In order to gain further understanding of the issues at stake, we shall now turn to a simpler situation which still retains some of the ingredients of the problem above, namely, minimising the k^{th} eigenvalue of the Dirichlet Laplacian on rectangles of fixed area. More precisely, and denoting by $\mathcal{R}_a$ a rectangle with sides a and $1/a$ ($a \geq 1$) and its corresponding Dirichlet eigenvalues by $\lambda_k = \lambda_k(a)$, $k = 1, 2, \ldots$ we want to understand the behaviour of

$$\lambda_k^* = \min_{a \geq 1} \lambda_k(a), \quad k = 1, 2, \ldots. \tag{3}$$

as k goes to infinity. Since these eigenvalues are given by

$$\lambda_k(a) = \pi^2 \left(\frac{p^2}{a^2} + a^2 q^2 \right), \quad p, q = 1, \cdots,$$

this problem is related to the problem of counting integer lattice points and may be reformulated as

> Among all ellipses centred at the origin with horizontal and vertical axes, determine the one with the least area which contains k integer lattice points in the first quadrant (excluding the axes)

This problem poses a computational challenge even for finite k, which we believe to be interesting in its own right – see, for instance, the graph of the function $\lambda_{100000}(a)$ in Figure 1. In [AF2] we introduced an algorithm for this purpose and

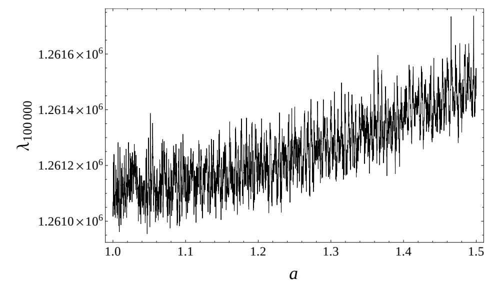

FIGURE 1. *The function* $\lambda_{100\ 000}(a)$ *for* $a \in [1, 1.5]$

in Figure 2–right we show the results for the first 50 000 values of a_k^*. From this it is not very conclusive if any sort of convergence is to be expected, which prompted us to look further up in the spectrum. The results for k of the order of 10^8 are shown in Figure 2–left, and here we see, by comparison with the previous results, that the values of a_k^* do seem to be decreasing, although very slowly. In fact, the

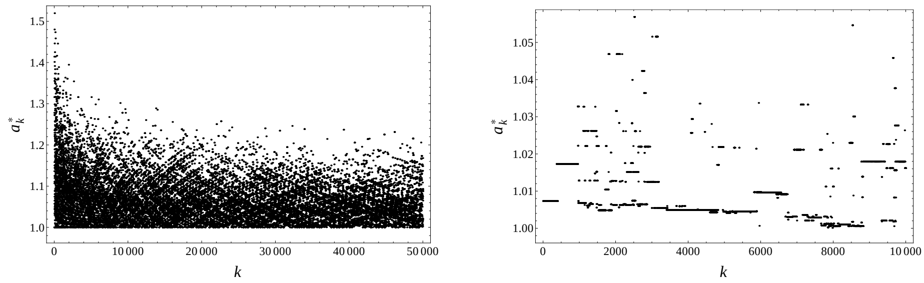

FIGURE 2. *The first* 50 000 *values of* a_k^* *(left) and the first* 10 000 *after* 10^8 *(right).*

optimum set does converge to the square (and thus the optimal ellipse converges to the circle), which is our main result in this direction.

Theorem 2. [AF2]

$$\lim_{k\to\infty} a_k^* = 1.$$

The crucial step of the proof is to show that optimal rectangles remain bounded uniformly in k, which is basically achieved by an improvement upon Pólya's inequalities for the case of rectangles. Once this is done, we can use existing results for the integer lattice problem to show that the limit of the sequence of optimal perimeters does converge to the perimeter of the square.

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Lieb–Thirring inequalities: A review and some recent results

Rupert L. Frank
(joint work with Elliott H. Lieb, Mathieu Lewin, Robert Seiringer)

The classical Lieb–Thirring (LT) inequality [4] is a generalization of the Sobolev inequality to systems of orthonormal functions. It states that there is a constant $K_d > 0$, depending only on the dimension d, such that for any $N \in \mathbb{N}$, any functions $\psi_1, \dots, \psi_N \in H^1(\mathbb{R}^d)$ satisfying $\int_{\mathbb{R}^d} \overline{\psi_j}\psi_k \, dx = \delta_{jk}$ and any numbers $\lambda_1, \dots, \lambda_N \in [0,1]$ one has

$$\sum_{j=1}^N \lambda_j \int_{\mathbb{R}^d} |\nabla\psi_j|^2 \, dx \geq K_d \int_{\mathbb{R}^d} \left(\sum_{j=1}^N \lambda_j |\psi_j|^2 \right)^{1+2/d} dx \,. \tag{1}$$

The important feature of this inequality is that the constant K_d is independent of N. For $N = 1$, the inequality is an easy consequence of Sobolev inequalities.

The inequality can be written in a more concise way as

$$\operatorname{Tr}(-\Delta)\gamma \geq K_d \int_{\mathbb{R}^d} \rho_\gamma^{1+2/d} \, dx \tag{2}$$

for any operator γ on $L^2(\mathbb{R}^d)$ satisfying $0 \leq \gamma \leq 1$. Here $\rho_\gamma(x)$ is (at least formally) $\gamma(x,x)$. The link between both formulations is the expansion $\gamma = \sum_{j=1}^N \lambda_j |\psi_j\rangle\langle\psi_j|$.

The physical meaning of (2) is as follows. Let $\rho \geq 0$ be a given function on $\mathbb{R}^d$ which is sufficiently regular and vanishes sufficiently fast at infinity. We interpret ρ as a density of electrons. Typically, we require $\int_{\mathbb{R}^d} \rho \, dx \in \mathbb{N}$ and interpret this as the number of electrons, but this is not necessary in what follows. The question we are interested in is the minimal amount of kinetic energy it costs to create a system of electrons with the given density ρ. By 'a system of electrons with the given density ρ' we mean an operator γ on $L^2(\mathbb{R}^d)$, a *one-body density matrix*, with $0 \leq \gamma \leq 1$ and $\rho_\gamma = \rho$. Thus, we are looking for a lower bound on

$$T_\rho = \inf \{ \mathrm{Tr}(-\Delta)\gamma : \ 0 \leq \gamma \leq 1 \,, \ \rho_\gamma = \rho \} \,.$$

(Here we ignore spin and use units where $\hbar = 2m = 1$.) The Pauli exclusion principle is encoded in the requirement $\gamma \leq 1$. Now (2) provides the lower bound

$$T_\rho \geq K_d \int_{\mathbb{R}^d} \rho^{1+2/d} \, dx \,.$$

The right side is (up to the value of the constant) the approximation used in Thomas–Fermi and density functional theory. An important feature of this bound is that the right side is additive over position space in the sense that, if ρ consists of two pieces with disjoint support, then the integral on the right side is the sum of the integrals of the two pieces. This is an advantage of this bound over other possible bounds like, e.g., $T_\rho \geq S_d \left(\int_{\mathbb{R}^d} \rho^{d/(d-2)} \, dx \right)^{(d-2)/d}$ in $d \geq 3$, which follows from the Hoffmann-Ostenhof and the Sobolev inequalities. Inequality (1) plays a crucial role in Lieb–Thirring's proof of the stability of matter.

Despite considerable efforts the optimal constant in (1) is not known. For a conjecture, see [4], and for the best possible value at the moment, see [1].

The original proof of [4] used a duality argument, based on the observation that (1) is equivalent to a bound on the sum of negative eigenvalues of Schrödinger operators,

$$\mathrm{Tr}(-\Delta + V)_- \leq L_d \int_{\mathbb{R}^d} V_-^{1+d/2} \, dx \tag{3}$$

for all V. Here $a_- = \max\{0, -a\}$ denotes the negative part. There is a one-to-one correspondence between optimal constants in K_d in (1) and L_d in (3). Recently, Rumin [5, 6] found a direct proof of (1) without going via (3).

The motivation behind our recent work [3] (see also [2]) was to extend the classical LT inequalities to the case of a positive background density and to answer the question: How much energy does it cost to make a hole in the Fermi sea? More specifically, we are interested in local perturbations of a system of electrons which has a constant background density $\rho_0 > 0$. We think of the Fermi sea with density ρ_0 as being described by the operator $\Pi_0 = \chi_{\{-\Delta < \mu\}}$, where μ is defined by

$$\rho_0 = (2\pi)^{-d} \, |\{ p \in \mathbb{R}^d : \ |p| < 1 \}| \, \mu^{d/2} \,. \tag{4}$$

This equality means that $\Pi_0(x,x) = \rho_0$. Since this state has infinite kinetic energy, we are lead to estimating the *change* in kinetic energy. Our main result in [3] reads

Theorem 1. *Let $d \geq 2$, $\rho_0 > 0$ and define μ by (4). Then for any operator γ in $L^2(\mathbb{R}^d)$ satisfying $0 \leq \gamma \leq 1$ one has*

$$\mathrm{Tr}\left((-\Delta - \mu)(\gamma - \Pi_0) \right) \geq K_d' \int_{\mathbb{R}^d} \left(\rho_\gamma^{1+d/2} - \rho_0^{1+d/2} - \left(1 + \tfrac{d}{2}\right) \rho_0^{d/2} (\rho_\gamma - \rho_0) \right) dx \,.$$

Remarks. (1) The right side is again additive as in the classical LT inequality and as desired in density functional theory. The integrand is non-negative. It behaves like $\rho_\gamma^{1+d/2}$ if $\rho_\gamma \gg \rho_0$ (as in the classical LT bound) and like ρ_γ^2 if $\rho_\gamma \ll \rho_0$.
(2) The left side is somewhat formal. Strictly speaking, it should read

$$(5) \qquad \operatorname{Tr} |-\Delta-\mu|^{1/2} \left(\Pi_0^\perp(\gamma-\Pi_0)\Pi_0^\perp - \Pi_0(\gamma-\Pi_0)\Pi_0\right) |-\Delta-\mu|^{1/2} \,,$$

which coincides with the left side if $\gamma - \Pi_0$ is finite rank, say. Here, $\Pi_0^\perp = 1 - \Pi_0$. Note that $\Pi_0^\perp(\gamma-\Pi_0)\Pi_0^\perp \geq 0 \geq \Pi_0(\gamma-\Pi_0)\Pi_0$. Thus the left side is non-negative as well. We argue that (5) is a natural definition for the kinetic energy shift, since it minimizes a variational principle and since it can be obtained by considering the thermodynamic limit of corresponding expressions on large tori.
(3) We really prove four inequality, one for each of the operators $\Pi_0(\gamma-\Pi_0)\Pi_0$, $\Pi_0(\gamma-\Pi_0)\Pi_0^\perp$, $\Pi_0^\perp(\gamma-\Pi_0)\Pi_0$ and $\Pi_0^\perp(\gamma-\Pi_0)\Pi_0^\perp$. The proof for the 'diagonal pieces' uses the technique introduced in [5].
(4) The corresponding inequality in $d = 1$ is *not* true. We prove a replacement, however, which involves a logarithmic correction term.
(5) There is an equivalent inequality for Schrödinger operators. It involves both the discrete and the continuous spectrum of $-\Delta + V$.
(6) There are similar inequalities for the pressure at positive temperature.
(7) It would be desirable to obtain good estimates on the constant K_d'.

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Sharp constants in the Caffarelli-Kohn-Nirenberg inequalities

Michael Loss
(joint work with Jean Dolbeault, Maria Esteban)

The Caffarelli-Kohn-Nirenberg inequalities [3]

$$C_{a,b}^N \left(\int_{\mathbb{R}^N} \frac{|w|^p}{|x|^{b\,p}}\,dx \right)^{2/p} \le \int_{\mathbb{R}^N} \frac{|\nabla w|^2}{|x|^{2\,a}}\,dx \quad \forall\, w \in \mathcal{D}_{a,b} \tag{1}$$

form a natural extension of Hardy and Sobolev inequalities. Here $a \le b \le a+1$ if $N \ge 3$ and $a < b \le a+1$ if $N = 2$, and the exponent p is given by

$$p = \frac{2N}{N-2+2(b-a)}\ . \tag{2}$$

We restrict our attention to the case where $a \le a_c = \frac{N-2}{2}$ in which case the domain $\mathcal{D}_{a,b}$ is given by

$$\mathcal{D}_{a,b} := \Big\{ w \in \mathrm{L}^p(\mathbb{R}^N, |x|^{-b}\,dx)\ :\ |x|^{-a}\,|\nabla w| \in \mathrm{L}^2(\mathbb{R}^N, dx) \Big\}\ .$$

In what follows, $C_{a,b}^N$ denotes the optimal constant in the above inequality. It is straightforward to compute the extremals among *radial* functions. Employing rearrangement inequalities the extremals and hence the sharp constants have been explicitly computed for the cases where $a \ge 0$ (see the references in [5]). For $a \le 0$ the the situation is more complicated. Rearrangement inequalities are not readily available and, more interestingly, radial symmetry can be broken. In fact it was shown in [4] and in [6] that in the region $a \le 0$ and $b < b_{\mathrm{FS}}(a)$ where

$$b_{\mathrm{FS}}(a) := \frac{2\,N\,(a_c - a)}{\sqrt{(a_c-a)^2 + (N-1)}} + a - a_c\ ,$$

the extremal functions are not radial. This report is about joint work with Jean Dolbeault and Maria Esteban concerning new symmetry results for the extremals for $a \le 0, b \ge b_{\mathrm{FS}}(a)$. In [5] it is shown that for $b \le b_\star(a)$ where

$$b_\star(a) := \frac{N\,(N-1) + 4\,N\,(a-a_c)^2}{6\,(N-1) + 8\,(a-a_c)^2} + a - a_c\ .$$

the extremals are radial. Nothing is known in the region $b_{\mathrm{FS}}(a) \le b \le b_\star(a)$, although it is natural to expect radial symmetry there too. The situation is depicted in the figure at the end of this report for $N = 3$. The proof proceeds by first writing the problem in logarithmic variables

$$s = \log|x|\ , \quad \omega = \frac{x}{|x|} \in \mathbb{S}^{N-1}\ , \quad u(s,\omega) = |x|^{a_c - a}\, w(x)\ ,$$

$$C_{a,b}^N \|u\|_{L^p(\mathcal{C})}^2 \le \ \left(\|\nabla u\|_{\mathrm{L}^2(\mathcal{C})}^2 + \Lambda\, \|u\|_{\mathrm{L}^2(\mathcal{C})}^2 \right)\ , \tag{3}$$

where $\mathcal{C} = \mathbb{R} \times \mathbb{S}^{N-1}$ and $d\omega$ is the *normalized* uniform surface measure on the sphere. Here $\Lambda = (a_c - a)^2$ and the parameters a, b are uniquely specified by Λ

and p. Thus, there is symmetry for all $\Lambda \le \Lambda_\star$ and symmetry breaking for all $\Lambda > \Lambda_{\rm FS}$ where

$$\Lambda_\star = \frac{(N-1)(6-p)}{4(p-2)} < \Lambda_{\rm FS} = 4\frac{N-1}{p^2-4} \,. \tag{4}$$

In this language, the radial extremals solve the equation

$$-u_*'' + \Lambda\, u_* = u_*^{p-1} \quad \text{in} \quad \mathbb{R}\,, \tag{5}$$

and are given by

$$u_*(s) := \frac{1}{2}\,\frac{\Lambda}{\big[\cosh(\frac{1}{2}\sqrt{\Lambda}\,(p-2)\,s)\big]^{\frac{2}{p-2}}} \quad \forall\, s \in \mathbb{R}\,. \tag{6}$$

It was shown in [4] that the extremals for the sharp constant in (1) and thus in (3) exist for $a < b < a+1$. The following theorem about the solutions of the variational equation is proved in [5].

Theorem 1. *Let $N \ge 2$ and let u be a non-negative function on $\mathcal{C} = \mathbb{R} \times \mathbb{S}^{N-1}$ that satisfies*

$$-\partial_s^2 u - \Delta_{\mathbb{S}^{N-1}} u + \Lambda\, u = u^{p-1} \quad on \quad \mathcal{C}\,, \tag{7}$$

and consider the solution u_* given by (6)*. Assume that*

$$\int_{\mathcal{C}} |u(s,\omega)|^p \,ds\,d\omega \le \int_{\mathbb{R}} |u_*(s)|^p \,ds\,, \tag{8}$$

for some $p > 2$. If $a_c^2 < \Lambda \le \Lambda_\star(p)$, then for a.e. $\omega \in \mathbb{S}^{N-1}$ and $s \in \mathbb{R}$, we have $u(s,\omega) = u_*(s-C)$ for some constant C.

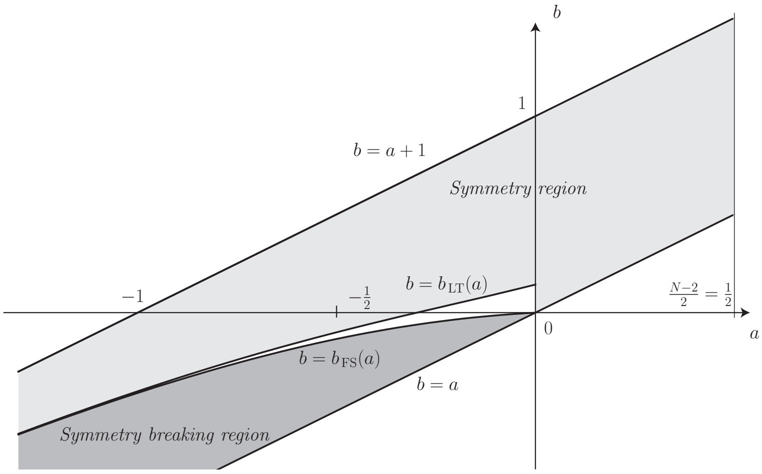
The symmetry results are an immediate consequence of this theorem. If u is an extremal, then after multiplying by a constant one can assume that it satisfies (7). One sees that the condition (8) is satisfied by multiplying (7) by u and integrating. This yields

$$\frac{\Big(\|\nabla u\|^2_{\mathrm{L}^2(\mathcal{C})} + \Lambda\,\|u\|^2_{\mathrm{L}^2(\mathcal{C})}\Big)}{\|u\|^2_{L^p(\mathcal{C})}} = C^N_{a,b} = \left(\int_{\mathcal{C}} |u(s,\omega)|^p\,ds\right)^{1-2/p}\,,$$

which is less or equal $\big(\int_{\mathcal{C}} |u_*(s)|^p\,ds\big)^{1-2/p}$ since u is an extremal. The proof of Theorem 1 relies on the sharp Poincaré type inequality

$$\int_{\mathbb{S}^{N-1}} |\nabla F(\omega)|^2 d\omega \ge \frac{N-1}{q-2}\left[\|F\|_q^2 - \|F\|_2^2\right]$$

valid for all $2 < q \le \frac{2(N-1)}{N-3}$. This inequality is due to Bidaut-Véron – Véron [2] and, independently, Beckner [1]. In fact, the results in [2] allow to extend the above theorem to the case where $\mathbb{S}^{N-1}$ is replaced by a compact closed manifold with bounded Ricci curvature. For details the reader may consult [5].



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Spectrum of 1D non-selfadjoint Dirac operators; sharp CLR inequalities for radial potentials

Ari Laptev

We show that the non-embedded eigenvalues of the Dirac operator on the real line with non-Hermitian potential V lie in the disjoint union of two disks in the right and left half plane, respectively, provided that the L^1-norm of V is bounded from above by the speed of light times the reduced Planck constant. An analogous result for the Schrödinger operator, originally proved by Abramov, Aslanyan and Davies, emerges in the nonrelativistic limit. For massless Dirac operators, the condition on V implies the absence of nonreal eigenvalues. Our results are further generalized to potentials with slower decay at infinity. As an application, we determine bounds on resonances and embedded eigenvalues of Dirac operators with Hermitian dilation-analytic potentials.

From the colour of the sun to modern spectral estimates

Timo Weidl

From the colour of stars one can determine their surface temerature. The base for this is Planck's law, which states that the radiation intensity per volume of a cavity radiator at the frequency ν is given by

$$\frac{8\pi\nu^2}{c^3} \times \frac{h\nu}{e^{\frac{h\nu}{kT}}-1}\,.$$

This formula is threefold remarkable:

- Firstly, in its entity it describes a universal law of interaction between heat and radiation.
- Secondly, the second factor in this expression is the initial spark for the development of quantum mechanics.
- Thirdly, as I outline with a few historic remarks, the first factor is the outset for the modern theory of spectral asymptotics (see also [4].

Indeed, motivated by this formula H. Weyl showed 100 years ago that [10]

$$\begin{aligned} N(\Lambda,\Omega) = \#\{\lambda_j < \Lambda\} &= (1+o(1)) \int\int_{x\in\Omega, |\xi|^2<\Lambda} \frac{dx d\xi}{(2\pi)^d} \\ &= (1+o(1)) \frac{\tau_d}{(2\pi)^d} \Lambda^{d/2} \mathrm{vol}(\Omega)\,, \end{aligned}$$

as $\Lambda \to +\infty$, where λ_j denote the ordered eigenvalue of the Dirichlet Laplacian $-\Delta_D^\Omega$ including multiplicities on an open domain $\Omega \subset \mathbb{R}^d$. About fifty years ago G. Pólya [8] realized that the phase space expression on the r.h.s. serves also as an upper bound on the counting function

$$N(\Lambda,\Omega) \le \int\int_{x\in\Omega, |\xi|^2<\Lambda} \frac{dx d\xi}{(2\pi)^d} = \frac{\tau_d}{(2\pi)^d} \Lambda^{d/2} \mathrm{vol}(\Omega) \quad \text{for all} \quad \Lambda > 0\,,$$

if Ω is tiling. He conjectured this bound to be true for all open Ω of finite volume, but this hypothesis remains unresolved so far. Recalling related results on eigenvalue sums by Berezin [1] and Li and Yau [6] I outline, how the phase space bounds on moments of the negative eigenvalues of Schrödinger operators by Lieb and Thirring [7] played a crucial role in the proof of stability of matter (which is in some sense more a proof of the stability of quantum theory).

Finally, I include in this survey also some of my own recent results, namely

(a) the violation of Pólya's conjecture in the presence of magnetic fields [2] (jointly with R. Frank and M. Loss)
(b) an improvement of the Li-Yau bound with remainder terms of almost correct order (jointly with S. Vugalter and H. Kovařík) [5]
(c) an improvement of Berezin's bound for higher moments [9]
(d) a more geometric version of this improvement of Berezin's bound for higher moments (jointly with A. Laptev and L. Geisinger) [3].

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The div(A grad) operator without ellipticity. Self-adjointness and spectrum

Vadim Kostrykin

(joint work with Amru Hussein, David Krejcirik, Stepan Schmitz)

Artificial optical metamaterials with negative refractive index is a subject of active research due to their unusual properties. In particular, they provide a possibility of constructing suitable devices making an object invisible, like a fabulous 'invisibility cloak'.

A mathematical model of invisibility (see [1] and references quoted therein) is described by a Dirichlet boundary value problem for the differential equation

$$-\div A(x)\,\mathrm{grad}\,u = 0$$

with indefinite coefficient matrix $A(x)$. The regions where A is positive definite correspond to the 'normal' material and those where A is negative definite - to the metamaterial with negative refraction index. The main difficulty to treat boundary value problems of this type is the absence of coercivity of the corresponding quadratic form $\langle \mathrm{grad}\,u, A\,\mathrm{grad}\,u\rangle$.

In the present work using the representation theorem for indefinite quadratic forms (see [2] and references quoted therein) we prove the existence of a unique self-adjoint, boundedly invertible operator $\mathcal{L}$, associated with this form for a wide class of coefficient matrices. A simplest example is the operator $\mathcal{L} = -\frac{d}{dx}\,\mathrm{sign}(x)\,\frac{d}{dx}$ on a bounded interval with Dirichlet boundary conditions at the endpoints.

The subspace

$$\mathcal{H} := \left\{ v \in L^2(\Omega)^n \mid v = \operatorname{grad}\varphi, \quad \varphi \in H_0^1(\Omega) \right\}$$

is closed in $L^2(\Omega)^n$, the space of vector valued square integrable functions on Ω. Here $H_0^1(\Omega)$ denote the closure of $C_0^\infty(\Omega)$ with respect to the Sobolev norm $\|\cdot\|_{2,1}$. Let $Q : L^2(\Omega)^n \to \mathcal{H}$ denote the partial isometry defined by

$$Qu := \begin{cases} u, & u \in \operatorname{Ran} D, \\ 0, & u \perp \operatorname{Ran} D. \end{cases}$$

The adjoint operator $Q^* : \mathcal{H} \to L^2(\Omega)^n$ is the embedding of $\mathcal{H}$ in $L^2(\Omega)^n$.

Theorem 1. *Let $\Omega \subset \mathbb{R}^n$, $n \geq 2$, be a bounded Lipschitz domain or a bounded open interval if $n = 1$. Let $A \in L^\infty(\Omega; \mathbb{C})^{n\times n}$ be such that*

(a) *$A(x)$ is Hermitian for almost all $x \in \Omega$,*
(b) *the operator $QM_AQ^* : \mathcal{H} \to \mathcal{H}$ is boundedly invertible, where M_A is the multiplication operator by $A(x)$ on $L^2(\Omega)^n$.*

Then

(i) *there exist a unique self-adjoint operator $\mathcal{L}$ with $\mathrm{Dom}(\mathcal{L}) \subset H_0^1(\Omega)$ such that*
$$\langle v, \mathcal{L}u\rangle_{L^2(\Omega)} = \langle \operatorname{grad} v, A \operatorname{grad} u\rangle_{L^2(\Omega)^n}$$
holds for all $v \in H_0^1(\Omega)$ and all $u \in \mathrm{Dom}(\mathcal{L})$. Its domain is given by
$$\mathrm{Dom}(\mathcal{L}) = \{u \in H_0^1(\Omega) \mid M_A D u \in E^2(\Omega)\},$$
where
$$E^2(\Omega) := \{v \in L^2(\Omega)^n \mid \div v \in L^2(\Omega)\}.$$
For any $u \in \mathrm{Dom}(\mathcal{L})$ one has $\mathcal{L}u = D^* M_A D u$, the domain $\mathrm{Dom}(\mathcal{L})$ is a core for the gradient operator D;
(ii) *the operator $\mathcal{L}$ is semibounded if and only if QM_AQ^* is sign definite;*
(iii) *the open interval $(-\alpha\mu, \alpha\mu)$ with*
$$\alpha := \|(QM_AQ^*)^{-1}\|^{-1}$$
and $\mu > 0$ the smallest eigenvalue of the Dirichlet Laplacian $-\Delta_D$ in $L^2(\Omega)$, belongs to the resolvent set of $\mathcal{L}$. In particular, $\mathcal{L}$ is boundedly invertible with $\|\mathcal{L}^{-1}\| \leq 1/\alpha\mu$;
(iv) *the inverse $\mathcal{L}^{-1}$ is compact. In particular, the spectrum of $\mathcal{L}$ is purely discrete.*

Let λ_j^+ respectively $-\lambda_j^-$ denote the positive respectively negative eigenvalues of $\mathcal{L}$ enumerated in the increasing respectively decreasing order counting their multiplicities. Denote by $N^\pm(\lambda)$ the counting functions for positive and negative eigenvalues, that is,

$$N^\pm(\lambda) = \#\{\lambda_j^\pm \leq \lambda\}.$$

Theorem 2. *In addition to the assumption of Theorem 1 suppose that* $A(\cdot)^{-1} \in L^\infty(\Omega;\mathbb{C})^{n\times n}$ *and*

(c) Ω *either is convex or has a* $C^{1,1}$ *boundary, that is,* $\partial\Omega$ *is locally the graph of a differentiable function with Lipschitz continuous derivative.*

Then the eigenvalue counting functions of the operator $\mathcal{L}$ *defined in Theorem 1 have the asymptotics*

$$N^\pm(\lambda) \sim \frac{(2\pi)^{-n}}{n}\omega^\pm \lambda^{n/2}, \quad \lambda \to \infty, \tag{1}$$

where

$$\omega^\pm = \int_\Omega \int_{|\xi|=1} \left(\langle \xi, A(x)^{-1}\xi\rangle_{\mathbb{C}^n}\right)_\pm^{n/2} d\sigma(\xi)dx,$$

with $(t)_\pm := (|t| \pm t)/2$ *and* σ *the Lebesgue measure on the unit sphere in* $\mathbb{R}^n$*. In particular, if there are nonempty open sets* $\Omega_\pm \subset \Omega$ *such that* $\Omega_- \cap \Omega_+ = \emptyset$, $\overline{\Omega_-} \cup \overline{\Omega_+} = \overline{\Omega}$*, and the matrix* $\pm A(x)$ *is positive definite for almost all* $x \in \Omega_\pm$*, then*

$$N_\pm(\lambda) \sim N_{\Omega_\pm}(\lambda), \quad \lambda \to \infty, \tag{2}$$

where $N_{\Omega_\pm}(\lambda)$ *are the eigenvalue counting functions of the elliptic differential operators* $\mp \div A(x)\,\mathrm{grad}$ *in* $L^2(\Omega_\pm)$ *with Dirichlet boundary conditions on* $\partial\Omega_\pm$*.*

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Open Problems Session

(i) **Some inequalities involving harmonic functions and Dirichlet eigenfunctions** by Michael Levitin

Let $m \geq 2$, and let $\Omega \subset \mathbb{R}^m$, be a bounded open set with a sufficiently smooth boundary. Denote by λ_j, u_j the eigenvalues (ordered non-decreasingly with account of multiplicity) and the orthonormal eigenfunctions of the Dirichlet Laplacian on Ω. Denote also by $\mathcal{H}(\lambda)$ the set $\{v \in L_2(\Omega) : \Delta v + \lambda v = 0\}$ of the λ-harmonic functions in Ω (they are just harmonic when $\lambda = 0$). Let $\langle \cdot, \cdot \rangle$ stand for the standard scalar product in $L^2(\Omega)$.

Conjecture 1. *For every* $\lambda > \lambda_1$ *there exists a harmonic function* $h \in \mathcal{H}(0)$ *such that*

$$\sum_{j=1}^\infty \frac{\lambda_j}{\lambda_j - \lambda} |\langle h, u_j\rangle|^2 < 0.$$

An equivalent form of this conjecture is

Conjecture 2. *For every $\lambda > \lambda_1$ there exists a λ-harmonic function $g \in \mathcal{H}(\lambda)$ such that*

$$\sum_{j=1}^{\infty} \frac{\lambda_j - \lambda}{\lambda_j} |\langle g, u_j \rangle|^2 < 0.$$

The conjectures can be also re-formulated in terms of Neumann or Steklov spectra.

Very roughly, proving the conjecture for $\lambda \to \lambda_2 - 0$ would mean that there always exists a harmonic function which is "almost parallel" to the first Dirichlet eigenfunction.

It is easy to check that the conjectures fail in the one-dimensional setting.

(ii) **Bounds for eigenvalues and the barycentric method** by Bruno Colbois
The barycentric method has been used to obtain upper bounds for the first eigenvalue of the Laplace operator. A typical example is the following. Fix a compact Riemannian manifold (M, g) and consider a Schrödinger operator

$$L = \Delta + q$$

on $L^2(M)$, where q is a continuous potential. An upper bound for the second eigenvalue λ_1 can be deduced from Theorem 2.2 of [14], depending on the mean of the potential $\int_M q dvol_g$:

$$\lambda_1(L) \leq m \left(\frac{V_c(g)}{Vol_g(M)} \right)^{2/m} + \frac{\int_M q dvol_g}{Vol_g(M)}.$$

where $V_c(g)$ denotes the conformal volume. *Problem:* Can we get similar upper bounds depending also on k for the other eigenvalues ? Grigor'yan-Netrusov-Yau [19] and Hassannezhad [20] have obtained partial results under additional hypothesis on q. For example that it has an uniform lower bound. Hassannezhad also obtained estimates for λ_k with respect to k, $Vol_g(M)$ and λ_0.

(iii) **Bounds for Dirichlet eigenvalues** by Bruno Colbois
Recall the following result by Colbois, El Soufi, and Girouard [9]. For any bounded domain $\Omega \subset \mathbb{R}^{m+1}$ with smooth boundary $\Sigma = \partial\Omega$, and all $k \geq 1$,

$$\lambda_k(\Sigma) Vol(\Sigma)^{2/m} \leq \gamma_m I(\Omega)^{1+2/m} k^{2/m} \tag{1}$$

with γ_m explicit constant depending on m and $I(\Omega)$ denotes the isoperimetric ratio. *Problem:* Is it possible to get a bound of the type

$$\lambda_k(\Sigma) Vol(\Sigma)^{2/m} \leq A_m I(\Omega) + B_m k^{2/m} \text{ ?}$$

(iv) **Eigenvalue inequalities for convex hypersurfaces** by Bruno Colbois
Let $\Sigma^m \subset \mathbb{R}^{m+1}$ be a convex hypersurface.
If $m \geq 3$ is it true that the round sphere maximizes $\lambda_1(\Sigma)|\Sigma|^{2/m}$ (here $\lambda_1(\Sigma)$ denotes the first non trivial eigenvalue of the Laplace-Beltrami operator)? The result is known for $m = 2$, see J. Hersch.
Problem: If $m = 2$ what is the domain which maximizes $\lambda_2(\Sigma)|\Sigma|$?

(v) L^p **bounds for Dirichlet eigenfunctions** by Michiel van den Berg

A simple proof, using the monotonicity of the heat kernel with respect to inclusion [12], shows that the j'th Dirichlet eigenfunction ϕ_j of any open set Ω in $\mathbb{R}^m$ (for which the Dirichlet Laplacian has compact resolvent) satisfies

(2)
$$\|\phi_j\|_\infty \leq \left(\frac{e}{2\pi m}\right)^{m/4} \lambda_j(\Omega)^{m/4}.$$

Problem: Find the sharp constant $C_{j,m}$ in

(3)
$$\|\phi_j\|_\infty \leq C_{j,m}\lambda_j(\Omega)^{m/4}.$$

Inequality (2) shows that $\phi_j \in L^p(\Omega)$ for all $2 \leq p \leq \infty$. *Problem:* Find an open set Ω such that the Dirichlet Laplacian acting in $L^2(\Omega)$ has compact resolvent, and such that $\phi_j \notin L^1(\Omega)$.

(vi) **Sharp constants for the torsion function with Dirichlet boundary conditions** by Michiel van den Berg
Let u_Ω be the torsion function of an open set Ω in $\mathbb{R}^m$. M. van den Berg and T. Carroll [6] showed that u_Ω is bounded if and only if the bottom of the spectrum of the Dirichlet Laplacian λ_Ω is strictly positive, and that in that case

$$\lambda_\Omega^{-1} \leq \|u_\Omega\|_\infty \leq (4 + 3m\log 2)\lambda_\Omega^{-1}.$$

The question is to find the sharp constants on the left (infinite strip ?) and on the right (ball ?) for the above inequalities.

(vii) **Eigenvalue inequalities for the magnetic Laplacian** by Richard Laugesen
For β a real number, and Ω a bounded open set in $\mathbb{R}^3$, let us consider the first eigenvalue E_1 of the magnetic Laplacian:

$$\left(i\nabla + \frac{\beta}{2}(-x_2, x_1, 0)\right)^2 u = E_1 u \quad \text{in } \Omega,$$
$$u = 0 \qquad \text{on } \partial\Omega.$$

Note that the eigenfunction $u(x)$ is complex-valued. The magnetic field $\nabla \times \frac{\beta}{2}(-x_2, x_1, 0) = (0, 0, \beta)$ is uniform, and points in the vertical direction.

Problem: Can one find Ω of fixed volume which minimizes E_1? The result is known in two dimensions, where the minimizer is the disk, see [15]. But in three dimensions the minimizer is presumably not the ball, due to the breaking of symmetry by the magnetic field.

(viii) **Nodal count for eigenfunctions of the Dirichlet-to-Neumann operator** by Iosif Polterovich

Let M be a compact Riemannian manifold with boundary Σ. Consider the Dirichlet-to-Neumann operator $D : C^\infty(\Sigma) \to C^\infty(\Sigma)$ which maps a function u on Σ to the outward normal derivative of its harmonic extension to M. The operator D is a first order elliptic pseudo-differential operator. It has discrete spectrum and its eigenfunctions u_n, $n = 1, 2, \dots$ form a basis in $L^2(\Sigma)$. Let ν_n be the number of nodal domains of an eigenfunction u_n. *Problem:* Find an upper bound on ν_n.

If M is two–dimensional, such a bound follows from the Courant nodal domain theorem for eigenfunctions of the Steklov eigenvalue problem: $\Delta f = 0$ in M, $\partial_n f = \sigma f$ on Σ (here ∂_n denotes the outward normal derivative). The eigenvalues of the Dirichlet-to-Neumann operator coincide with the Steklov eigenvalues of M, and the eigenfunctions u_n of D are the restrictions of the Steklov eigenfunctions ϕ_n to the boundary: $u_n = \phi_n|_\Sigma$. A simple topological argument following [1, Lemma 3.4] shows that for surfaces, the bound on the number of interior nodal domains of ϕ_n implies an estimate on the number of boundary nodal domains of ϕ_n, which are precisely the nodal domains of u_n. However, it is easy to see that in higher dimensions this argument fails, and while Courant's theorem holds for ϕ_n, it does not imply any upper bound on ν_n.

Note that if $M = \mathbb{B}^m \subset \mathbb{R}^m$ is a Euclidean ball, then the Courant nodal domain theorem holds for the Dirichlet-to-Neumann eigenfunctions u_n, because they coincide with the spherical harmonics (see [17, Example 2.1]). In general, the principal symbol of D is the square root of the principal symbol of the Laplace-Beltrami operator on Σ. Since Courant's theorem holds for the eigenfunctions of the Laplacian on Σ, one may hope to have at least a "semiclassical" (i.e. as $n \to \infty$) bound for the number of nodal domains ν_n. A related problem is discussed in (xvi).

(ix) **Do there exist non-isometric planar domains that are Steklov isospectral?** by Iosif Polterovich

This variation of M. Kac's famous question "Can one hear the shape of a drum?" has been around for quite a while. It follows from Weyl's law for Steklov eigenvalues that the Steklov spectrum determines the perimeter of a planar domain. Moreover, if two smooth simply-connected planar domains Ω_1 and Ω_2 have the same perimeter, the difference between the corresponding Steklov eigenvalues satisfies $|\sigma_n(\Omega_1) - \sigma_n(\Omega_2)| = o(n^{-\infty})$. This result was proved independently by Rozenblium [28] and Guillemin–Melrose (see [13]). It is also known that the disk is uniquely determined by its Steklov spectrum among all simply–connected planar domains [29, 13]. If one considers a more general setting, namely Riemannian manifolds with boundary, there are easy ways to construct manifolds that are Steklov isospectral but not isometric. For instance, a conformal change of metric by a factor that is equal to one near the boundary does not change the

Steklov spectrum of a surface. Also, if Σ_1 and Σ_2 are two Laplace isospectral compact closed Riemannian manifolds, than the cylinders $\Sigma_1 \times L$ and $\Sigma_2 \times L$ are Steklov isospectral for any $L > 0$ (see [9, Lemma 6.1]).

(x) **Does a finite number of Neumann eigenvalues determine a disk?** by Iosif Polterovich
It is well-known that a Euclidean ball is uniquely determined by either its Dirichlet or Neumann spectrum. Indeed, the volume can be obtained from the main term in Weyl's law. At the same time, it follows from the Faber–Krahn inequality that the ball is the unique minimizer of the first Dirichlet eigenvalue among all Euclidean domains of the same volume. Similarly, the Szegő–Weinberger inequality states that the ball is the unique maximizer of the first nonzero Neumann eigenvalue among all domains of the same volume.

The above argument requires the knowledge of *infinitely* many Dirichlet or Neumann eigenvalues. *Problem:* Is it possible to determine a ball using only a finite number of eigenvalues ? In the Dirichlet case the answer is positive — in fact, one needs to know only the first two eigenvalues. Indeed, by a theorem of Ashbaugh–Benguria [2], the ratio of the first two Dirichlet eigenvalues of a Euclidean domain attains its maximum if and only if the domain is a ball.

In the Neumann case the situation is drastically different. It follows from the results of Colin de Verdière [11] that in dimensions ≥ 3 no finite number of Neumann eigenvalues allows to determine whether a domain is a ball. In dimension two the question is open. Note that [11, Theorem 1.4] is not applicable in this case because Neumann spectrum of a disk contains multiple eigenvalues.

(xi) **Optimal power in upper bounds for the Steklov spectrum** by Alexandre Girouard
The *isoperimetric ratio* of a bounded Euclidean domain $\Omega \subset \mathbb{R}^{m+1}$ with smooth boundary $\Sigma = \partial\Omega$ is

$$I(\Omega) = \frac{|\Sigma|}{|\Omega|^{m/(m+1)}}.$$

In [9] we proved that the Steklov eigenvalues of Ω satisfy for each $k \in \mathbb{N}$,

$$\sigma_k(\Omega)|\Sigma|^{1/m} \leq \frac{c(m)}{I(\Omega)^{(1-1/m)}} k^{2/(m+1)}. \tag{4}$$

In particular, using the classical isoperimetric inequality, for any bounded Euclidean domain $\Omega \subset \mathbb{R}^{m+1}$,

$$\sigma_k(\Omega)|\Sigma|^{1/m} \leq c'(m) k^{2/(m+1)}. \tag{5}$$

The Weyl type asymptotic formula for the Steklov spectrum σ_k is

$$\sigma_k(\Omega)|\partial\Omega|^{1/m} \sim c''(m) k^{1/m} \quad \text{as} \quad k \nearrow \infty. \tag{6}$$

Problem: Does inequality (5) hold with the optimal exponent $k^{1/m}$? For $m = 1$, the exponent is already optimal. For $m \geq 2$, it is impossible to replace $k^{2/(m+1)}$ by $k^{1/m}$ in inequality (4) because this would imply, using the asymptotic formula (6), an upper bound for the isoperimetric ratio $I(\Omega)$.

(xii) **Large isoperimetric ratio for planar domains** by Alexandre Girouard
For $m \geq 2$, inequality (4) implies that if a sequence of domains Ω_l is such that $\lim_{l\to\infty} I(\Omega_l) \to \infty$, then for each $k \in \mathbb{N}$, $\lim_{l\to\infty} \sigma_k(\Omega_l) = 0$. Let Ω_l be a sequence of bounded planar domains ($m = 1$) such that $\lim_{l\to\infty} I(\Omega_l) \to \infty$. Does $\lim_{l\to\infty} \sigma_k(\Omega_l) = 0$? This really is a "planar question". On any compact surface with boundary, there exists a sequence of Riemannian metrics g_l such that the corresponding Steklov eigenvalues do not depend on l, but the isoperimetric ratios tend to infinity.

(xiii) **Surfaces with large Steklov eigenvalues** by Alexandre Girouard
Let Ω be a compact surface of genus γ, with $l > 0$ boundary components. Fraser and Schoen [17] proved $\sigma_1(\Omega)|\Sigma| \leq 2\pi(\gamma + l)$. In [18], we extended this result to higher eigenvalues and proved

$$\sigma_k(\Omega)|\Sigma| \leq 2\pi(\gamma + l)k. \tag{7}$$

For $k = 1$, $\gamma = 0$ and $l = 2$, Fraser and Schoen have identified a maximizer for $\sigma_1(\Omega)|\Sigma|$, which shows in particular that inequality (7) is not sharp in this case. In [9] we proved the existence of a universal constant $C > 0$ such that $\sigma_k(\Omega)|\Sigma| \leq C(\gamma + 1)k$. See also Theorem A_1 and Example 1.3 in [25], where G. Kokarev proved that one can choose $C = 8\pi$ for $k = 1$. Altogether, this means that the number l of boundary components is not essential in inequality (7), and that this inequality is never sharp if l is large enough. This raises the following problem. Does there exist a sequence Ω_l of compact Riemannian surfaces such that

$$\lim_{l\to\infty} \sigma_1(\Omega_l)|\partial\Omega_l| = +\infty \ ?$$

Of course, if such a sequence exists, the genera $\gamma(\Omega_l)$ must tend to infinity.

(xiv) **Problems for the number of nodal domains** by Thomas Hoffmann-Ostenhof

Consider $-\Delta$ on a bounded domain $\Omega \subset \mathbb{R}^d$, say, with Dirichlet boundary conditions. We could also consider a Schrödinger operator $H = -\Delta + V$ with, say, real valued bounded potential V and assume also other homogenous boundary conditions. Instead of the Laplacian we could have also a strictly elliptic operator in divergence form, also the Laplacian on a manifold would be possible. H has compact resolvent and the eigenvalues in increasing order so that

$$\lambda_1 < \lambda_2 \leq \lambda_3 \leq \lambda_4 \leq \cdots \leq \lambda_k \leq \ldots$$

tend to infinity. Without loss we can assume the eigenfunctions u_k to be real valued. We define the nodal set of a u_k by

$$N(u_k) = \overline{\{x \in \Omega \mid u_k(x) = 0\}}.$$

The nodal domains of u_k are then the connected components of $\Omega \setminus N(u_k)$. The number of nodal domains of u_k will be denoted by μ_k. One of the classical results in spectral theory is Courant's nodal theorem:

$$\mu_k \leq k. \tag{8}$$

This is in fact an extension of Sturm's oscillation theorem which says for a Schrödinger operator on an interval, i.e. a Sturm Liouville problem, that $\mu_k = k$. In 1956 Pleijel [27] showed that in (8) equality holds only finitely many times. In 1976, H. Lewy [26] showed for spherical harmonics that

$$\liminf_{k\to\infty} \mu_k = 2.$$

Later on, [5], related results were obtained for eigenfunctions of Laplacians on specific manifolds. Lewy's result came quite as a surprise since for the typical examples, rectangles, or other problems which can be constructed from one-dimensional problems, the number of nodal domains tends to infinity.
Problem: Is it possible that

$$\limsup_{k\to\infty} \mu_k < \infty$$

for $-\Delta$ on some $\Omega \subset \mathbb{R}^d$? Of course one might pose the same problem for Schrödinger operators on compact manifolds. We have a more detailed problem for the 2-dimensional case. Consider $-\Delta$ on $\Omega \subset \mathbb{R}^2$ with homogenous boundary conditions. *Problem:* Is there, for each positive integer n, a finite constant $K(n)$ such that in the collection of the eigenspaces of $\lambda_1, \lambda_2, \ldots, \lambda_{K(n)}$, call it $\mathcal{U}(K(n))$, there is a $u \in U(K(n))$ with the property that

$$\mu(u) \geq n? \tag{9}$$

Note that in higher dimension (9) is probably not true. One can, see Colin de Verdière [10], for a d-dimensional bounded manifold, $d \geq 3$, for each finite integer k construct a metric such that the second eigenvalue (with the Laplace operator with this metric) has multiplicity k.

(xv) **Spectral problems on the flat two-dimensional torus** by Thomas Hoffmann-Ostenhof
In [21] spectral minimal partitions where considered for the torus [1]. Hence we consider $-\Delta$ on the flat torus $T(a,b)$. (A rectangle $Q(a,b) = (0,a) \times (0,b)$ with periodic boundary conditions.)

[1] see e.g.[22] for the definitions of those minimal partitions

One problem with an isoperimetric touch is the following: Determine the minimiser of the following variational problem

$$\inf\{\lambda_1(\Omega) : \Omega \subset T(a,b), \Omega \text{ open, simply connected}\},$$

where $\lambda_1(\Omega)$ is the first eigenvalue of the Dirichlet Laplacian acting in $L^2(\Omega)$.
We were also interested in the number of nodal domains of the eigenfunctions. A simple calculation shows that the eigenvalues are given by $\lambda_{m,n} = 4\pi^2(\frac{m^2}{a^2} + \frac{n^2}{b^2})$ where m, n are non-negative integers. If a^2/b^2 is irrational and $(m,n) \geq 1$ then the eigenvalues have multiplicity 4, if for instance $m = 0$ and $n > 0$ then the multiplicity is 2. Furthermore the eigenfunctions are given by $\cos(2\pi m\frac{x}{a} + \theta_1)\cos(2\pi n\frac{y}{b} + \theta_2)$ and have an even number of nodal domains. This holds for the irrational case. For the case that $a^2/b^2 \in \mathbb{Q}$ additional multiplicities occur.

Is there a flat torus so that for some eigenfunction in the eigenspace $U(\lambda_{m,n})$ for some $\lambda_{m,n}$ there is an eigenfunction u with and odd number ≥ 3 of nodal domains ? The higher multiplicities for the rational case must play a role. But if one tries to construct an explicit example one runs into trigonometric and number theoretical problems. Also attempts to show that there are no examples with and odd number of nodal domains probably leads to such problems. It is not clear whether this problem is easy or hard. But there is a lot of work on eigenfunctions and their zero sets for the torus, mostly for problems with some "generic" touch, see e.g. [7], [8].

A related problem can be formulated for the Neumann case for rectangles, actually the square will suffice. Hence we look $\mathcal{Q} = (0,1)^2$ with Neumann boundary conditions for the Laplacian.
Consider Neumann eigenfunctions $\{u_k\}$,

$$-\Delta u_k = \lambda u_k \text{ on } \mathcal{Q}. \tag{10}$$

Problem: Is there a non-constant u satisfying (10) that satisfies $u \geq c > 0$ at the boundary? It is easy to see that for rectangles $R(a,b)$ with a^2/b^2 irrational the nodal set will always hit the boundary and an eigenfunction u, if it exists, whose nodal set does not hit the boundary must be in the eigenspace of an eigenvalue that is not simple. As for the case of a torus the construction of a possible example would lead also to trigonometric and number theoretical problems. Attempts to show that such an example does not exist would probably also lead to such problems.

(xvi) **Nodal domains of fractional Schrödinger operators** by Rupert Frank
We consider fractional Schrödinger operators $(-\Delta)^s + V$ in $L^2(\mathbb{R}^m)$ with $0 < s < 1$. The potential V is assumed to be real-valued and such that the operator is lower semi-bounded. Both the case where V decays to zero at infinity (in which case inf ess-spec $((-\Delta)^s + V) = 0$) and where V diverges

to infinity at infinity (in which case $\inf \text{ess-spec}\,((-\Delta)^s + V) = +\infty$) are interesting. Below we also consider the case where $\{V = +\infty\}$ is non-empty.

Let $\lambda_k < \inf \text{ess-spec}\,((-\Delta)^s + V)$ be the k-th eigenvalue of $(-\Delta)^s + V$ (counting multiplicities) and let ψ_k be a corresponding real-valued eigenfunction. If ψ_k is continuous (which it is under suitable assumptions on V), we call the connected components of $\{\psi_k \neq 0\}$ the nodal domains of ψ_k. *Problem:* What can one say about the number of nodal domains of ψ_k in terms of the index k?

Let us recall the classical case $s = 1$. Let $\lambda_k < \inf \text{ess-spec}\,(-\Delta + V)$ be the k-th eigenvalue (counting multiplicities) of $-\Delta + V$ with a corresponding real-valued eigenfunction ψ_k. If $m = 1$, Sturm's oscillation theorem says that ψ_k has exactly k nodal domains. For general dimensions $m \geq 1$, Courant's nodal theorem says that ψ_k has at most k nodal domains. The proofs of both theorems are essentially based on the locality of the equation.

Returning to the fractional case $0 < s < 1$, we first note that the ground state (i.e., ψ_1 corresponding to the lowest eigenvalue λ_1) has a definite sign and is unique. But already for an eigenfunction corresponding to the second eigenvalue we do not know how many nodal domains it could have. To prove that the second eigenfunction only has two nodal domains would settle a problem in non-linear analysis [16].

In [16] we have proved that if $m = 1$ and if ψ_k corresponds to the k-th eigenvalue $\lambda_k < \inf \text{ess-spec}\,((-\Delta)^s + V)$, then ψ_k has at most $2k - 1$ nodal domains. The same method leads to (unfortunately worse) bounds for radial potentials in higher dimensions.

Of considerable interest is also the case of the fractional Dirichlet Laplacian on a domain $\Omega \subset \mathbb{R}^m$, which is included in the above set-up with $V \equiv 0$ in Ω and $V \equiv +\infty$ in $\mathbb{R}^m \setminus \Omega$. If $m = 1$ and Ω is an interval, Bañuelos and Kulczycki [4] have shown that λ_2 and λ_3 are simple and the corresponding eigenfunctions ψ_2 and ψ_3 have two and three nodal domains, respectively. We are not aware of any further results even for the fractional Dirichlet Laplacian.

(xvii) **Do bubbles tend to corners** by Michael Loss
Let D be a bounded convex domain in $\mathbb{R}^m, m \geq 2$ with smooth boundary and consider the operator

$$H = -\Delta + q(\cdot - a) \tag{11}$$

on D with *Neumann boundary condition* on ∂D. Here $q(x)$ is a smooth potential with compact support and $a \in \mathbb{R}^m$ is such that the support of $q(\cdot - a)$ is contained in D. Denote by

$$G = \{a \in \mathbb{R}^m : \text{supp}\, q(\cdot - a) \subset D\}\ . \tag{12}$$

Denote the smallest eigenvalue of H by $E_0(a)$ and by $u_0(x; a)$ the corresponding eigenfunction. It was shown in [3] that $E_0(a)$ obeys a strong minimum principle in the sense that if $E_0(a)$ attains its minimum in G then $E_0(a) \equiv 0$ on G and the eigenfunction $u_0(x; a)$ is constant in the complement of the support of $q(x - a)$. We assume that the support of q has no holes.

An important example is the case where D is a cube and the potential q has the same reflection symmetries as the cube. In this case the region G is also a cube, $G = (-d, d)^m$ and one can show that either the function $E_0(a) \equiv 0$ or otherwise $E_0(a)$ is, in each variable, strictly increasing on $(-d, 0)$ and strictly decreasing on $(0, d)$ and the partial derivative can only vanish at 0. In particular this implies that the function $E_0(a)$ has its minima not only at the boundary of G but in fact in the *corners* of G. For a precise enunciation of this theorem the reader may consult [23, 24]. This fact is an important ingredient for the proof of localization in the random displacement model [23, 24].

Problem: Let D be an equilateral triangle and q a potential with the same reflection symmetries, e.g., radial symmetry. Thus, the domain G is also an equilateral triangle centered at the origin. Prove that $E_0(a)$ is either identically zero or has strict minima in the corners of G. If in addition the gradient vanishes only at the origin, then this fact would immediately yield localization of the random displacement model where the underlying two dimensional lattice is triangular.

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