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Geometric Group Theory, Hyperbolic Dynamics and Symplectic Geometry

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Abstract. The workshop brought together experts in the young and rapidly developing mathematical fields of Geometric group theory, hyperbolic dynamics and symplectic geometry. The activities included talks on recent achievements and breakthroughs by world renowned mathematicians as well as graduate students and postdocs. The workshop witnessed many informal discussions between the participants, exchange of questions and conjectures and as one of the highlights an open problem session. A list of open problems is, besides of the extended abstracts of talks, contained in the present report.

Mathematics Subject Classification (2000): 20Fxx, 22Fxx, 37Cxx, 53Cxx, 53Dxx.

Introduction by the Organisers

This workshop was the 4th round of the series "Geometric group theory, hyperbolic dynamics and symplectic geometry". The workshop demonstrated an impressive number of significant results. The list of topics included

- Geometric group theory (Belolipetskii, Fujiwara, I. Kapovich, Ledrappier);
- Algebra, geometry and dynamics in the context of diffeomorphism groups (Calegari, M. Kapovich, Karlsson, Kedra, Sandon, Usher, Witte Morris);
- Hamiltonian dynamics and symplectic rigidity (Abbondandolo, Bramham, Butler, Cornea, Humiliere, Schlenk);
- Riemannian geometry (Bangert, Burns, Courtois, Sambusetti);
- Restrictions on fundamental groups of manifolds in various categories (Panov/Petrunin, Py, Sapir);

- Applications of group theory, dynamics and geometry to mathematical physics (Knauf, Nemirovskii, Siburg).

A speciality of the workshop justifying its "multi-disciplinary" nature was the interaction between the speedily developing fields of mathematics mentioned in its title. Many of the talks manifested this interaction, often in a quite unexpected way. To indicate some of them, Panov (with Petrunin) and M. Kapovich reported on striking applications of hyperbolic geometry and geometric group theory to symplectic topology, Bramham talked on applications of powerful methods of modern symplectic topology to classical problems of two-dimensional dynamics, while the talk of Py linked together topology of Kähler manifolds, lattices and infinite-dimensional hyperbolic geometry. Several talks had a flavor of a fusion between a research talk and a survey which was greatly appreciated by other participants and was indispensable for the success of the workshop. For instance, Humilière gave a survey of C^0-symplectic topology, while the talk of Courtois contained an extensive introduction to Poincare inequalities on Riemannian manifolds accessible for non-experts.

The workshop witnessed many informal discussions between the participants and exchange by questions and conjectures. In particular we continued the tradition of running open problem sessions which was moderated this time by Danny Calegari. A remarkable feature of the workshop which reflects another facet of the above-mentioned interaction was that a number of questions posed on previous conferences of this series have been eventually resolved by participants. In particular, Ledrappier reported on a solution of a question raised by Erschler in 2010, while M. Kapovich talked on the solution of a problem posed in 2006.

The workshop caused a considerable interest among mathematicians all over the world working in all three fields entering the title of the conference. The list of participants included world renowned mathematicians as well as graduate students and postdocs. The young generation made a significant contribution to discussions and informal talks.

Let us finally mention that about one third of talks were delivered by the participants who never took part in the previous workshops of this series. They brought new fresh ideas, insights and research directions of high common interest.

Workshop: Geometric Group Theory, Hyperbolic Dynamics and Symplectic Geometry

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Abstracts

The Spectrum of an Adelic Markov Operator

Andreas Knauf

The Dirichlet series and Euler product

$$Z(s) := \frac{\zeta(s-1)}{\zeta(s)} = \prod_{p\in\mathbb{P}} \frac{1-p^{-s}}{1-p^{1-s}} = \sum_{n=1}^{\infty} \varphi(n) n^{-s} \tag{1}$$

(with Euler's φ–function) converge in the half-plane $\Re(s) > 2$. On the abelian group $G := \bigoplus_{\mathbb{N}}(\mathbb{Z}/2\mathbb{Z})$ one defines $h : G \to \mathbb{N}$ by $h(0) := 1$ and

$$h(g_1, \ldots, g_{n-1}, 1, 0, \ldots) := h(g_1, \ldots, g_{n-1}, 0, \ldots) + h(1-g_1, \ldots, 1-g_{n-1}, 0, \ldots).$$

As an example, one obtains for the subgroup $(\mathbb{Z}/2\mathbb{Z})^3$

g	000	001	010	011	100	101	110	111
$h(g)$	1	4	3	5	2	5	3	4

Then $Z(s) = \sum_{g\in G} h(g)^{-s}$.

We twist the Euler product of Z in (1) to obtain a Dirichlet series

$$\tilde{Z}(s) := \prod_{p\in\mathbb{P}} \frac{1+p^{-s}}{1+p^{1-s}} = \sum_{n=1}^{\infty} \lambda(n)\,\varphi(n)\,n^{-s} = \frac{\zeta(s)\,\zeta\big(2(s-1)\big)}{\zeta(s-1)\,\zeta(2s)} \qquad \big(\Re(s) > 2\big) \tag{2}$$

with the Liouville function, given by $\lambda(p_1^{a_1} \cdot \ldots \cdot p_k^{a_k}) = (-1)^{a_1+\ldots+a_k}$ for $p_i \in \mathbb{P}$. This has the following properties:

- Of the four zeta functions appearing in (2), only $\zeta(s-1)$ is not absolutely convergent for $\Re(s) > 3/2$.
- The pole of ζ at $s = 1$ gives rise to $\tilde{Z}(2) = 0$.
- The non-trivial zeros of ζ, shifted by 1 for Z, now turn into poles of $\tilde{Z}$.
- $\tilde{Z}$ has an additional pole at $3/2$.

So the Dirichlet series $\tilde{Z}$ converges in the half-plane $\{s \in \mathbb{C} \mid \Re(s) > s_0 + 1\}$ if and only if there are no zeros of ζ with real part larger than $s_0 \geq 1/2$. We look at the convergence for $k \to \infty$ of

$$\tilde{Z}_k(s) := \sum_{g\in(\mathbb{Z}/2\mathbb{Z})^k} \lambda \circ h(g)\; h(g)^{-s} \qquad (k \in \mathbb{N}). \tag{3}$$

A heuristic reason for such a supposed convergence is to compare the terms $\lambda \circ h(g)$ appearing in (3) to i.i.d. random variables which take the values ± 1 with equal probability $\frac{1}{2}$. For the case of $1/\zeta(s) = \sum_{n=1}^{\infty} \mu(n) n^{-s}$ a similar heuristic goes back to Denjoy (1931), and is described in Section 12.3 of Edwards [Ed].

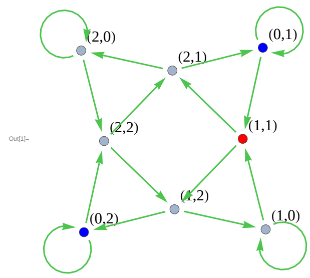
Although that is obviously absurd in the literal sense, we show in the forthcoming article that there is some truth to the argument.

Example. To convey the idea, we ask about the divisibility properties of the ensemble $h_k(g)$ $\quad (g \in (\mathbb{Z}/2\mathbb{Z})^k)$ of integers. For division by 3 a statistic is as follows.

k	0	1	2	3	4	5	6	7	8	9	10
$\#$	0	0	2	2	2	10	18	26	66	138	242
$\# - \mathbb{E}$	$-\frac{1}{4}$	$-\frac{1}{2}$	1	0	-2	2	2	-6	2	10	-14
$2^{k/2}$	1	$\sqrt{2}$	2	$2\sqrt{2}$	4	$4\sqrt{2}$	8	$8\sqrt{2}$	16	$16\sqrt{2}$	32

Here $\#$ denotes the number of $g \in (\mathbf{Z}/2\mathbf{Z})^k$ with $3|h_k(g)$.
In the third row the expectation value $\mathbb{E} = \frac{1}{4}2^k$ was subtracted.
Last row, for comparison: $2^{k/2}$, the square root of $|(\mathbf{Z}/2\mathbf{Z})^k|$.

Both the expectation value $\mathbb{E}$ and the scaling of $\# - \mathbb{E}$ are explained by the following Markov chain for addition (mod 3):



The spectrum of Markov transition matrix equals

$$\left\{1\,,\, \tfrac{1}{4}\big(-1+i\sqrt{7}\big)\,,\, \tfrac{1}{4}\big(-1-i\sqrt{7}\big)\,,\, \tfrac{1}{2}\,,\, \tfrac{1}{2}\,,\, \tfrac{1}{2}\,,\, 0\,,\, 0\right\}.$$

So the spectral radius (except Perron-Frobenius eigenvalue 1) is $1/\sqrt{2}$, and we get

$$\left|\left|\left\{g \in (\mathbb{Z}/2\mathbb{Z})^k \mid h_k(g) = 0\,(\text{mod } 3)\right\}\right| - \tfrac{1}{4}2^k\right| \;\leq\; c\,2^{k/2} \qquad (k \in \mathbb{N})$$

for the deviation from the mean. $\Diamond$

We widely generalize that kind of Markov estimate, to gain control on *joint* divisibility properties of the values of h. Such a control is clearly necessary for estimating the function $\lambda \circ h_k$ appearing in (3).

The natural language for this question is the one of adeles.

Given a unitary representation of $\mathrm{SL}(2,\mathbb{Z})$ on a Hilbert space $\mathcal{H}$, left and right addition

$$L, R \in \mathrm{SL}(2,\mathbb{Z}) \quad , \quad L(\ell, r) := (\ell + r, r)\;, \quad R(\ell, r) := (\ell, \ell + r),$$

give rise to unitary operators $\mathbf{L}$ and $\mathbf{R}$ on $\mathcal{H}$. We are to analyze the operators

$$\mathbf{T} \in \mathbf{B}(\mathcal{H}) \quad , \quad \mathbf{T} := \tfrac{1}{2}(\mathbf{L} + \mathbf{R}).$$

Independent of the representation, their restriction $\mathbf{T}^+$ to the (relevant) inversion-symmetric subspace satisfies the following

Proposition 1. $\mathrm{spec}\big(\mathbf{T}^+\big) \subseteq C \cup I$, with the circle $C := \big\{c \in \mathbb{C} \mid |c| = 1/\sqrt{2}\big\}$ and $I := [-1, -\frac{1}{2}] \cup [\frac{1}{2}, 1]$.

The radius $1/\sqrt{2}$ of C corresponds to the above-mentioned probabilistic paradigm. Whether there is additional spectrum in the real intervals I, depends on the representation. For example, we have for the regular representation

Proposition 2. $\mathrm{spec}(\mathbf{T}^+_{\mathrm{SL}}) = \{-\frac{1}{2}, \frac{1}{2}\} \cup C$. The spectrum of $\mathbf{T}^+_{\mathrm{SL}}$ on the circle C is absolutely continuous.

The estimates for adelic representations are a bit too involved to be presented here, but in all cases lead to spectral radii strictly smaller than one. In the proof we relate $\mathbf{T}^+$ to expander graphs.

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Some implications of weak integrability & symmetry

Leo T. Butler

(joint work with Alfonso Sorrentino)

This talk reports on work in [2].

Tonelli Hamiltonians are naturally encountered in Riemannian geometry (kinetic energy), Finsler geometry ($\frac{1}{2}F^2$ where F is the Finsler norm), mechanics (kinetic plus potential energies) and many other areas. Solutions to Hamilton's equations naturally obey the Least-Action Principle over short time intervals. One goal, or result, of Mather theory has been to clarify the applicability of the Least-Action Principle over infinite time intervals.

Let M be a closed (i.e. compact and boundary-less) smooth manifold and $H \in C^2(T^*M)$ be a twice-differentiable Hamiltonian on the cotangent bundle of M.

Definition 1. H is a Tonelli Hamiltonian iff

(1) H is fibre-wise strictly convex;
(2) H grows super-linearly along each fibre.

Condition (i) is equivalent to the positive-definiteness of the matrix $\frac{\partial^2 H}{\partial p_i \partial p_j}$, where $p_1, \ldots, p_n$ is a system of local linear coordinates on the fibres of T^*M. Condition (ii) is equivalent to the hypothesis that $\lim_{\lambda\to\infty} H(x, \lambda p)/\lambda = \infty$ for all $p \neq 0$.

In addition to the Least-Action Principle, Tonelli Hamiltonians enjoy the geometric property that their flows satisfy Hamilton's equations:

$$\dot{q}_i = \{H, q_i\} = \frac{\partial H}{\partial p_i} \qquad \dot{p}_i = \{H, p_i\} = -\frac{\partial H}{\partial q_i},$$

where $\{,\}$ is the canonical Poisson structure on T^*M and $(q_1, \ldots, q_n, p_1, \ldots, p_n)$ is a canonical local system of coordinates.

A classical theorem in Hamiltonian mechanics characterizes the typical dynamics of Tonelli Hamiltonian which has sufficiently many scalar conserved quantities that are in involution:

Theorem 1.[Liouville-Arnol′d] Let $F = (H = f_1, \ldots, f_n) \in C^\infty(T^*M^n)$. Assume

(1) $\{f_i, f_j\} \equiv 0$ for all i, j;
(2) $df_1 \wedge \cdots \wedge df_n \neq 0$ a.e.
Then, if $T \subset T^*M$ is a component of a regular level of F:
(1) T is a Lagrangian n-torus $\mathbf{T}^n$;
(2) There is a neighbourhood $U \supset T$ with coordinates $(\theta, c) : U \to \mathbf{T}^n \times \mathcal{O}$ where $\mathcal{O} \subset H^1(\mathbf{T}^n)$ such that

$$\omega = \sum_{i=1}^n dc_i \wedge d\theta^i, \qquad f_i = f_i(c), \text{ and}$$
$$X_{f_i} = \sum_{i=1}^n \frac{\partial f_i}{\partial c_i} \frac{\partial}{\partial \theta^i}.$$

The coordinates θ^i are classically called "angle" coordinates, the c_i are called "actions" and the theorem is summarised by saying that a completely integrable system possesses angle-action coordinates. The flow of the Hamiltonian vector field X_H of H is by 1-parameter subgroups of the torus $\mathbf{T}^n$.

Sorrentino [4] introduced the notion of weak-integrability, which generalizes the definition of complete integrability from the Liouville-Arnol′d Theorem.

Definition 2. A Tonelli Hamiltonian $H = f_1$ is *weakly integrable* if it enjoys n a.e. independent first integrals $f_1, \ldots, f_n$.

We prove the following analogue to the Liouville-Arnol′d Theorem:

Theorem 2. ([2]) Let $H \in C^\infty(T^*M)$ be a weakly-integrable Tonelli Hamiltonian with first integral map F. Assume that for some $c \in H^1(M)$, $\mathcal{M}_c \subset \mathrm{Reg}(F)$. Then

(1) $\mathcal{M}_c = \Lambda_c$ is a smooth Lagrangian graph;

(2) There is an open neighbourhood $\mathcal{O} \ni c$ such that for all $c' \in \mathcal{O}$, $\mathcal{M}_{c'} = \Lambda_{c'}$ is a smooth Lagrangian graph;
(3) Λ_c is a $\mathbf{T}^d$-bundle over a compact parallelisable base B^{n-d} ($d = \dim H^1(M)$);
(4) The Hamiltonian flow of H on Λ_c acts as a 1-parameter subgroup of $\mathbf{T}^d$;
(5) Mather's α function is C^1 on $\mathcal{O}$.

The set $\mathcal{M}_c$ is the so-called Mather set associated to the 1-dimensional cohomology class c. Roughly speaking, this set consists of orbits which globally minimize the shifted action functional. Mather proved that this set is a Lipschitz graph over its projection to M [3].

We also obtain a second analogue. The setup is as follows

(1) G is a connected, simply-connected Lie group;
(2) $\Gamma < G$ a lattice subgroup;
(3) For a bi-invariant 1-form ϕ on G, let $\Lambda_\phi = \Gamma \cdot \mathrm{Graph}(\phi) \subset T^*(\Gamma\backslash G)$.

Theorem 3. Let G be amenable and H a left-invariant Tonelli Hamiltonian on T^*G, $M = \Gamma\backslash G$. Then

(1) For all $c \in H^1(M;\mathbf{R})$, $\mathcal{M}_c = \Lambda_\phi$ where $[\phi] = c$
(2) The flow of $X_H|\Lambda_c$ is generated by a 1-parameter subgroup of G acting on the right
(3) Mather's α function is as smooth as H.

In this theorem, one can think of the cohomology group $H^1(M;\mathbf{R})$ as being the space of action coordinates and the group G provides the "angle" coordinates. An unusual aspect here is that right-translations on solvmanifolds can have positive topological entropy. The geodesic flow in [1] is an example of this: the positive-entropy subsystem Bolsinov & Tauĭmanov discover is an instance of the above theorem.

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Two extensions of Thurston's spectral theorem for surface diffeomorphisms

Anders Karlsson

Let M be an oriented closed surface of genus $g \geq 2$. Let $\mathcal{S}$ denote the isotopy classes of simple closed curves on M not isotopically trivial. For a Riemannian metric ρ on M, let $l_\rho(\beta)$ be the infimum of the length of curves isotopic to β.

In a seminal preprint from 1976, Thurston classified surface diffeomorphisms as being isotopic either to a periodic one, or else reducible or pseudo-Anosov. Using the theory of foliation of surfaces, this lead him to the following non-linear spectral theorem; the proof is worked out in exposé 11 of [FLP79]:

Theorem 1. ([T88, Theorem 5])*For any diffeomorphism f of M, there is a finite set $1 \leq \lambda_1 < \lambda_2 < \ldots < \lambda_K$ of algebraic integers such that for any $\alpha \in \mathcal{S}$ there is a λ_i such that for any Riemannian metric ρ,*

$$\lim_{n\to\infty} l_\rho(f^n\alpha)^{1/n} = \lambda_i.$$

The map f is isotopic to a pseudo-Anosov map iff $K = 1$ and $\lambda_1 > 1$.

This statement is analogous to the dynamical behaviour of linear maps A of finite dimensional vector spaces: the limits $\lim \|A^n v\|^{1/n}$ exist for every vector v. We obtain two extensions of Theorem 1. First:

Theorem 2. *Let $f_n = g_n g_{n-1} \ldots g_1$ be a product of random diffeomorphisms of M (more precisely, f_n is an integrable ergodic cocycle). Then almost surely there are a constant $\lambda \geq 1$ and a (random) measured foliation μ such that for any Riemannian metric ρ,*

$$\lim_{n\to\infty} l_\rho(f_n\alpha)^{1/n} = \lambda$$

for any $\alpha \in \mathcal{S}$ such that $i(\mu, \alpha) > 0$.

Kaimanovich-Masur [KM96] studied the case of random walks on mapping class groups. Using their work we get:

Corollary. *Let $f_n = g_n g_{n-1} \ldots g_1$ be a random product of diffeomorphism where g_i are chosen independently and distributed with a measure that generates a subgroup that contains two independent pseudo-Anosov maps. Then there is a number $\lambda > 1$ such that a.s. for any $\alpha \in \mathcal{S}$ and metric ρ*

$$\lim_{n\to\infty} l_\rho(f_n\alpha)^{1/n} = \lambda.$$

This can be viewed as analogous to a well-known theorem of Furstenberg and to Oseledets multiplicative ergodic theorem for random products of matrices.

Let $\mathcal{T}$ be the Teichmuller space of M, and let $Ext_x(\alpha)$ denote the extremal length of the curve α for $x \in \mathcal{T}$. The mapping class group, with some lower genus exceptions, is isomorphic to the complex automorphism group of $\mathcal{T}$(M). Thus the following provides a second extension of Theorem 1:

Theorem 3. *Let $f : \mathcal{T} \to \mathcal{T}$ be a holomorphic map and $x \in \mathcal{T}$. Then there is a number $\lambda \geq 1$ and a point P in the Gardiner-Masur compactification such that for all $n \geq 1$ and any curve $\beta \in \mathcal{S}$,*

$$Ext_{f^n x}(\beta) \geq \left(\inf_\alpha \frac{Ext_x^{1/2}(\alpha)}{E_P(\alpha)} \right)^2 E_P(\beta)^2 \lambda^n$$

and, provided that the extremal length $E_P(\beta) > 0$,

$$Ext_{f^n x_0}(\beta)^{1/n} \to \lambda.$$

The following can be seen as a weak extension of the Nielsen-Thurston classification of mapping classes to general holomorphic self-maps of Teichmuller spaces:

Theorem 4. *Let $f : \mathcal{T} \to \mathcal{T}$ be a holomorphic map. Then either every orbit in $\mathcal{T}$ is bounded, or every orbit leaves every compact set and there are associated points P in the Gardiner-Masur boundary . If P is uniquely ergodic, then it is unique and every orbit converges to this point, and for some $\lambda \geq 1$ and any $x \in \mathcal{T}(M)$*

$$\inf_\alpha \frac{Ext_{f(x)}^{1/2}(\alpha)}{E_P(\alpha)} \geq \lambda \inf_\alpha \frac{Ext_x^{1/2}(\alpha)}{E_P(\alpha)}.$$

This is reminiscent of the Wolff-Denjoy theorem in complex dynamics that, together with a theorem of Fatou, classifies holomorphic self-maps of the unit disk. Examples of important holomorphic self-maps of $\mathcal{T}$ beyond the automorphisms are the Thurston skinning map in three-dimensional topology and the Thurston pull-back maps in complex dynamics, see e.g. [M90].

Ingredients in the proofs are Thurston's asymmetric metric and Teichmuller's metric, as well as Thurston's and Gardiner-Masur's respective compactifications. For the connection between the metrics and the compactifications we use recent works of Cormac Walsh and H.Miyachi, Lixin Liu & Weixu Su. A crucial part for the proof of Theorem 2 is to verify that the proof of a general noncommutative ergodic theorem of Ledrappier and myself [KaL06] works also for asymmetric metrics.

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On hyperbolicity of the free group analogues of the curve complex

Ilya Kapovich
(joint work with Kasra Rafi)

The notion of a *curve complex* plays a key role in the study of hyperbolic surfaces, mapping class group and the Teichmüller space. If S is a closed oriented hyperbolic surface, the *curve complex* $\mathcal{C}(S)$ of S is a simplicial complex whose vertices are isotopy classes of essential non-peripheral simple closed curves. A collection $[\alpha_0], \dots, [\alpha_n]$ of $(n+1)$ distinct vertices of $\mathcal{C}(S)$ spans an n–simplex in $\mathcal{C}(S)$ if there exist representatives $\alpha_0, \dots, \alpha_n$ of these isotopy classes such that for all $i \neq j$ the curves α_i and α_j are disjoint. The complex $\mathcal{C}(S)$ is finite-dimensional but not locally finite, and it comes equipped with a natural action of the mapping class group $Mod(S)$ by simplicial automorphisms. The geometry of $\mathcal{C}(S)$ is closely related to the geometry of the Teichmüller space $\mathcal{T}(S)$ and also of the mapping class group itself. The curve complex is a basic tool in modern Teichmüller theory, and has also found numerous applications in the study of 3-manifolds and of Kleinian groups. A key general result of Masur and Minsky [10] says that the curve complex $\mathcal{C}(S)$, equipped with the simplicial metric, is a Gromov-hyperbolic space.

The outer automorphism group $\mathrm{Out}(F_N)$ of a free group F_N is a cousin of the mapping class group. However the group $\mathrm{Out}(F_N)$ is much less well understood and, in general, more difficult to study than the mapping class group.

Several possible analogs of the curve complex for the case of F_N have been suggested in recent years. The first of these is the *free splitting complex* FS_N. The vertices of FS_N are nontrivial splittings of the type $F_N = \pi_1(\mathbb{A})$ where $\mathbb{A}$ is a graph of groups with a single edge (possibly a loop edge) and the trivial edge group; two such splittings are considered to be the same if their Bass-Serre covering trees are F_N–equivariantly isometric. Two distinct vertices $\mathbb{A}$ and $\mathbb{B}$ of FS_N are joined by an edge if these splittings admit a common refinement, that is, a splitting $F_N = \pi_1(\mathbb{D})$ where $\mathbb{D}$ is a graph of groups with two edges and trivial edge groups, such that collapsing one edge gives the splitting $\mathbb{A}$ and collapsing the other edge produces the splitting $\mathbb{B}$. Higher-dimensional simplices are defined in a similar way. For example, if $F_N = A * B * C$, where A, B, C are nontrivial, then the splittings $F_N = (A * B) * C$ and $F_N = A * (B * C)$ are adjacent in FS_N. There is a natural action of $\mathrm{Out}(F_N)$ on FS_N by simplicial automorphisms.

A different free group analog of the curve complex is the *free factor complex* FF_N, originally introduced by Hatcher and Vogtmann [7]. The vertices of FF_N are conjugacy classes $[A]$ of proper free factors A of F_N. Two distinct vertices $[A], [B]$ are joined by an edge in FF_N if there exist representatives A, B of $[A], [B]$ such that $A \le B$ or $B \le A$. Higher-dimensional simplices are defined similarly.

The free factor complex FF_N and the free splitting complex FS_N are rather different objects geometrically. There is a natural $\mathrm{Out}(F_N)$-equivariant Lipschitz "multi-function" projection from $\tau : FS_N \to FF_N$. The map τ sends a free splitting of F_N to the set of conjugacy classes of the vertex groups for that splitting. Thus if the graph of groups $v = \mathbb{A}$ is a non-loop edge, $\tau(v)$ consists of two vertices of FF_N, and if $\mathbb{A}$ is a loop-edge, $\tau(v)$ consists of a single vertex of FF_N. In any case it is easy to check that $\tau(v)$ has diameter ≤ 3 in FF_N and that τ is Lipschitz. In general, the distance $d_{FF_N}(\tau(x), \tau(y))$ may be much smaller than the distance $d_{FS_N}(x, y)$ for vertices $x, y \in FS_N$.

Until recently, little was known about the geometry of the above complexes. Several years ago Kapovich-Lustig [8] and Behrstock-Bestvina-Clay [1] showed that for $N \ge 3$ the complex FF_N has infinite diameter. Since the multi-map τ above is Lipschitz, this implies that FS_N has infinite diameter as well. A subsequent result of Bestvina-Feighn [2] implies that every fully irreducible element $\phi \in \mathrm{Out}(F_N)$ acts on FF_N with positive asymptotic translation length (hence the same is true for the action of ϕ on FS_N). It is easy to see from the definitions that if $\phi \in \mathrm{Out}(F_N)$ is not fully irreducible then some positive power of ϕ fixes a vertex of FF_N, so that ϕ acts on FF_N with bounded orbits.

In 2011 two significant advances occurred. First, Bestvina and Feighn [3] proved that for $N \ge 2$ the free splitting complex is Gromov-hyperbolic (as noted above, for $N = 2$ this essentially follows from the definition of FF_2, so the main case of the Bestvina-Feign result is for $N \ge 3$). Then Handel and Mosher [6] proved that for all $N \ge 2$ the free splitting complex FS_N is also Gromov-hyperbolic. The two proofs are rather different in nature, although both are quite complicated. However, it does appear that the Handel-Mosher proof admits significant simplifications.

Our main result shows how to derive hyperbolicity of the free factor complex from the Handel-Mosher proof of hyperbolicity of the free splitting complex. This gives a new proof of the Bestvina-Feighn result [3] about hyperbolicity of FF_N.

We prove:

Theorem 1. Let $N \ge 3$. Then the free factor complex FF_N is Gromov-hyperbolic. Moreover, there exists a constant $C > 0$ such that for any two vertices x, y of FS_N and any geodesic $[x, y]$ in $FS_N^{(1)}$ the path $\tau([x, y])$ is C–Hausdorff close to a geodesic $[\tau(x), \tau(y)]$ in $FF_N^{(1)}$.

To prove Theorem 1, we first introduce a new object, called the *free bases graph*, and denoted FB_N. The vertices of FB_N are free bases of F_N, up to some natural equivalence. We then prove that the natural map from FB_N and FF_N is a quasi-isometry. Thus to show that FF_N is hyperbolic it suffices to establish hyperbolicity of FB_N. We use a hyperbolicity criterion for graphs due to Bowditch [4] and obtain a new "bounded projection criterion" of hyperbolicity. Roughly, it says that if X, Y are connected graphs, with X hyperbolic and if $f : X \to Y$ is a surjective Lipschitz graph-map with the property that if $d(f(x), f(y))$ is small then $f([x,y])$ has bounded diameter, then Y is also hyperbolic. Moreover, in this case $f([x,y])$ is uniformly Hausdorff-close to any geodesic $[f(x), f(y)]$ in Y.

We then construct a surjective Lipschitz map $f : FS'_N \to FB_N$, where FS'_N is the barycentric subdivision of FS_N. The map f restricts to a natural bijection from a subset S of $V(FS'_N)$, corresponding to N–roses, to the set $V(FB_N)$ of vertices of FB_N. In [6] Handel and Mosher constructed nice paths $g_{x,y}$ given by "folding sequences" between arbitrary vertices x and y of FS'_N, and proved that these paths are quasigeodesics in FS'_N. To apply the "bounded projection criterion" to the map $f : FS'_N \to FB_N$ it turns out to be enough to show that $f(g_{x,y})$ has bounded diameter if $x, y \in S$ and $d(f(x), f(y)) \leq 1$ in FB_N. To do that we analyze the properties of the Handel-Mosher folding sequences in this situation. The construction of $g_{x,y}$ for arbitrary $x, y \in V(FS'_N)$ is fairly complicated. However, we have $x, y \in S$, so that x, y correspond to free bases of F_N. In this case the construction of $g_{x,y}$ becomes much easier and boils down to using standard Stallings foldings (in the sense of [9, 11]) to get from x to y. Verifying that $f(g_{x,y})$ has bounded diameter in FB_N, assuming $d(f(x), f(y)) \leq 1$, becomes a much simpler task. Thus we are able to conclude that FB_N is Gromov-hyperbolic, and, moreover, that $f([x,y])$ is uniformly Hausdorff-close to any geodesic $[f(x), f(y)]$ in FB_N. Using the quasi-isometry between FB_N and FF_N, we then obtain the conclusion of Theorem 1. Our proof of Theorem 1 also provides a fairly explicit description of certain reparametrized quasigeodesics joining arbitrary vertices (i.e. free bases) in FB_N in terms of Stallings foldings.

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Regularity of the entropy for random walks on hyperbolic groups

François Ledrappier

Let p be a finitely supported probability measure on a group G and define inductively, with $p^{(0)}$ being the Dirac measure at the identity e,

$$p^{(n)}(x) \ = \ [p^{(n-1)} \star p](x) \ = \ \sum_{y \in F} p^{(n-1)}(xy^{-1})p(y).$$

Define the entropy h_p by

$$h_p := \lim_n -\frac{1}{n} \sum_{x \in F} p^{(n)}(x) \ln p^{(n)}(x).$$

The entropy h_p was introduced by Avez ([**Av**]) and is related to bounded solutions of the equation on G $f(x) = \sum_{y \in G} f(xy)p(y)$. Erschler and Kaimanovich have shown that, on Gromov hyperbolic groups, the entropy depends continuously on the probability p with finite first moment ([**EK**]). Here we are looking for a stronger regularity on a more restricted family of probability measures. We fix a finite set $F \subset G$ such that $\cup_n F^n = G$ and we consider probability measures in $\mathcal{P}(F)$, where $\mathcal{P}(F)$ is the set of probability measures p such that $p(x) > 0$ if, and only if, $x \in F$. The set $\mathcal{P}(F)$ is naturally identified with an open subset of the probabilities on F, which is a contractible open polygonal bounded convex domain in $\mathbb{R}^{|F|-1}$. We show:

Theorem 1. Assume G is a Gromov hyperbolic group and F is a finite subset of G such that $\cup_n F^n = G$. Then, with the above notation, the function $p \mapsto h_p$ is Lipschitz continuous on $\mathcal{P}(F)$.

If G is the free group, with the same hypotheses, the function $p \mapsto h_p$ is real analytic on $\mathcal{P}(F)$ ([**L1**]). If F is symmetric and $\mathcal{P}_S(F)$ is the set of symmetric probabilities with support F, then $p \mapsto h_p$ is a C^1 function on $\mathcal{P}_S(F)$ ([**M**]).

The proof will use a formula (see below) for the entropy h_p of the random walk directed by p which is due to Kaimanovich ([**K1**]). Let $\Omega = F^{\mathbb{N}}$ be the space of sequences of elements of F, M the product probability $p^{\mathbb{N}}$. The random walk is described by the probability P on the space of paths Ω, the image of M by the mapping:

$$(\omega_n)_{n \in \mathbb{Z}} \mapsto (X_n)_{n \geq 0}, \text{ where } X_0 = e \text{ and } X_n = X_{n-1}\omega_n \text{ for } n > 0.$$

In particular, the distribution of X_n is the convolution $p^{(n)}$. We have:

Proposition 1. [**An**] There is a mapping $X_\infty : \Omega \to \partial G$ such that for M-a.e. ω,

$$\lim_n X_n(\omega) \;=\; X_\infty(\omega).$$

The action of G over itself by left multiplication extends to ∂G and naturally to probability measures on ∂G. We say that the measure ν on ∂G is stationary if $\sum_{x\in F}(x_*\nu)p(x) = \nu$. The image measure $p^\infty := (X_\infty)_* M$ is the only stationary probability measure on ∂G and the entropy h_p is given by the following formula:

$$h_p \;=\; -\sum_{x\in F}\left(\int_{\partial G} \ln \frac{dx_*^{-1}p^\infty}{dp^\infty}(\xi)dp^\infty(\xi)\right)p(x). \tag{1}$$

The idea of the proof is to use formula (1) and to show that both mappings $p \mapsto -\frac{dx_*^{-1}p^\infty}{dp^\infty}(\xi)$ and $p \mapsto p^\infty$ are Lipschitz from a neighbourhood $\O_p$ of p in $\mathcal{P}(F)$ into respectively a space of Hölder continuous functions on ∂G and its dual (the metric on ∂G is the Gromov metric). The function $\frac{dx_*^{-1}p^\infty}{dp^\infty}(\xi)$ will be identified with the Martin kernel of the random walk, defined as follows: the Green function $G(x)$ associated to (G,p) is

$$G(x) \;:=\; \sum_{n=0}^{\infty} p^{(n)}(x).$$

For $y \in G$, the Martin kernel K_y is defined by

$$K_y(x) = \frac{G(x^{-1}y)}{G(y)}.$$

Ancona ([**An**]) showed that $y_n \to \xi \in \partial G$ if, and only if, the Martin kernels K_{y_n} converge towards a function K_ξ called the Martin kernel at ξ. We have

$$\frac{dx_* p^\infty}{dp^\infty}(\xi) \;=\; K_\xi(x). \tag{2}$$

Consider the space Γ_κ of functions ϕ on ∂G such that there is a constant C_κ with the property that $|\phi(\xi) - \phi(\eta)| \le C_\kappa(d(\xi,\eta))^\kappa$. For $\phi \in \Gamma_\kappa$, denote $\|\phi\|_\kappa$ the best constant C_κ in this definition. The space Γ_κ is a Banach space for the norm $\|\phi\| := \|\phi\|_\kappa + \max_{\partial G}|\phi|$. It is known ([**INO**]) that for $p \in \mathcal{P}(B)$, $x \in F$ and κ small enough, the function $\Phi_p(\xi) = -\ln K_\xi(x)$ belongs to Γ_κ. Our main technical result is:

Proposition 2. Fix $x \in F$. The mapping $p \mapsto \Phi(\xi) = -\ln K_\xi(x)$ is Lipschitz continuous from $\mathcal{P}(F)$ into Γ_κ.

In order to prove Proposition 2, we replace $\dfrac{G(x^{-1}y)}{G(y)}$ by $\dfrac{u(x,y)}{u(e,y)}$, where $u(x,y)$ is the probability that the random walk ever hits y when starting from x. Then, $u(x,y)$ is estimates by an iterative procedure, using that a path from x to y has cross a number of "walls" and "obstacles". The probabilities attached to these geometric objects depend C^∞ on $p \in \mathcal{P}(F)$. Details of the proof are in [**L2**].

By using the coding of the boundary ∂G by a subshift of finite type ([**CP**]) and thermodynamical formalism, we deduce from proposition 2 that the mapping $p \mapsto p^\infty$ is Lipschitz continuous from $\mathcal{P}(F)$ into $(\Gamma_\kappa)^*$. The regularity of the entropy follows from Formula (1).

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$\mathbf{SL}(n, \mathbb{Q})$ has no volume-preserving actions on $(n-1)$-dimensional compact manifolds

Dave Witte Morris

(joint work with Robert J. Zimmer)

We show that $\mathrm{SL}(n, \mathbb{Q})$ has no nontrivial, C^∞, volume-preserving action on any compact manifold M of dimension strictly less than n. To prove this, let

$$\Gamma_m = \mathrm{SL}\big(n, \mathbb{Z}[1/m]\big) \subset \mathrm{SL}(n, \mathbb{Q}).$$

The proof has two main ingredients:

(1) For each m, a theorem of R. J. Zimmer [3] tells us that the action of Γ_m extends (a.e.) to a measurable action of the profinite completion $\widehat{\Gamma}_m$.

(2) For each m, the Congruence Subgroup Property [1] tells us

$$\widehat{\Gamma}_m = \mathop{\times}_{p \nmid m} \mathrm{SL}(n, \mathbb{Z}_p).$$

The inclusion $\Gamma_1 \hookrightarrow \Gamma_m$ induces a homomorphism $\widehat{\Gamma}_1 \to \widehat{\Gamma}_m$, whose kernel is

$$\mathop{\times}_{p \mid m} \mathrm{SL}(n, \mathbb{Z}_p).$$

This kernel acts trivially on M (a.e.). Since the union of these kernels is dense in $\widehat{\Gamma}_1$, we conclude that $\widehat{\Gamma}_1$ acts trivially on M (a.e.). However, the subgroup

Γ_1 acts continuously, so it must act trivially on all of M (not just a.e.). Since $\mathrm{SL}(n, \mathbb{Q})$ has no infinite, proper, normal subgroups, this implies that the entire group $\mathrm{SL}(n, \mathbb{Q})$ acts trivially.

More generally, suppose $\mathbf{G}$ is a connected, isotropic, almost-simple algebraic group over $\mathbb{Q}$, such that the simple factors of every localization of $\mathbf{G}$ have rank ≥ 2. A similar proof shows that if there does not exist a nontrivial homomorphism from $\mathbf{G}(\mathbb{R})^\circ$ to $\mathrm{GL}(d, \mathbb{C})$, then every C^∞, volume-preserving action of $\mathbf{G}(\mathbb{Q})$ on any compact d-dimensional manifold must factor through a finite group. The proof also applies to most anisotropic groups, but, in that setting, the Congruence Subgroup Property is not yet known to be true in all cases.

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A brief introduction to C^0-symplectic topology

Vincent Humilière

The goal of the talk was to give an idea of what is "*C^0-symplectic topology*". It is impossible to speak about all known results and point of views on this subject in 50 minutes so I decided to concentrate on three particular theorems, to motivate them and to give an idea of their proof.

We will denote by (M, ω) a symplectic manifold. A Hamiltonian is a smooth compactly supported map $H : [0, 1] \times M \to \mathbb{R}$. Its symplectic gradient X_H generates a flow denoted ϕ_H^t. The poisson bracket of two smooth functions H and K is given by the formula $\{H, K\} = \omega(X_H, X_K)$. The following theorems hold on any symplectic manifold.

Theorem 1.[Gromov-Elishberg , see [6]] Let ϕ_k be a sequence of symplectic diffeomorphisms. Suppose that it converges in the C^0-sense to some diffeomorphism ϕ. Then, ϕ is symplectic.

Theorem 2.[Hofer [3], Lalonde-McDuff[5]] Let H_k be a sequence of Hamiltonians. Suppose that

(1) $\phi_{H_k}^1$ C^0-converges to some homeomorphism h,
(2) H_k C^0-converges to 0.

Then $h = \mathrm{Id}$.

Theorem 3.[Cardin-Viterbo [2]] Let F_k, G_k be sequences of Hamiltonians. Suppose that

(1) F_k and G_k C^0-converge to some smooth functions F and G,
(2) the Poisson bracket $\{F_k, G_k\}$ C^0-converges to 0.

Then, $\{F, G\} = 0$.

Comments, motivations, applications.

(1) First note that these results are surprising! Indeed, in Theorem 1, being a symplectic diffeomorphism is a condition on the differential of the diffeomorphism. So their should be no such C^0-rigidity. Similarly, the Poisson bracket is defined only in terms of the derivatives of the functions, so in Theorem 3 the Poisson bracket should not behave well with respect to the C^0-topology.
(2) Once we have these results, it is natural to wonder whether the right objects of symplectic topology are actually the smooth ones or whether they are less regular. More concretely, can one define C^0 counterparts to the classical smooth symplectic objects? For example, Theorem 1 allows to give a definition of what could be a *symplectic homeomorphism*: a homeomorphism which a C^0-limit of symplectic diffeomorphism. The question of defining a C^0-Hamiltonian dynamics is more subtle. We will discuss it later on.
(3) The C^0-rigidity results can also help to understand better the smooth objects themselves. The best example of this is the recent story of the Poisson bracket. After Theorem 3 was discovered, many papers have been published to understand the phenomenon and improve this result. In the end, this has lead Buhovsky, Entov and Polterovich to define new symplectic invariants [1] and derive nice results in (smooth!) Hamiltonian dynamics.

A word on the proofs. Amazingly the three theorems above can all be deduced from the following well known result. As defined by Hofer, the energy of a Hamiltonian diffeomorphism is:

$$\|\phi\| = \inf\{\int_0^1 (\max H_t - \min H_t) dt \,|\, \phi = \phi_H^1\}.$$

Theorem 4.[Hofer [3], Lalonde-McDuff[5]] For any symplectic ball B of radius r, if a Hamiltonian diffeomorphism ϕ satisfies $\phi(B) \cap B = \emptyset$, then $\|\phi\| \geq \pi r^2$.

Theorems 2 and 3 follows from that after some elementary differential calculus. To prove Theorem 1, a method is to define the notion of a symplectic capacity which is a way to measure the "symplectic size" of a subset of a symplectic manifold. The existence of symplectic capacities follows for example from Theorem 4. Then, one proves that a diffeomorphism is (anti-)symplectic if and only if it preserves symplectic capacities. Since the property of preserving a capacity is C^0-closed, Theorem 1 follows. This proof is nicely exposed in [6].

Attempts to define a continuous Hamiltonian dynamics. A first attempt has been proposed by Müller and Oh [7]. They define a *continuous Hamiltonian isotopy* as a path of homeomorphisms h^t with $h^0 = \mathrm{Id}$ and such that there exists a sequence of Hamiltonians H_k such that

(1) $\phi^t_{H_k}$ C^0-converges to h^t,
(2) H_k C^0-converges to some continuous function H.

A Hamiltonian homeomorphism is then any element of such an isotopy h^t. It follows from Theorem 2 that given a continuous H there is at most ont isotopy h^t such that the definition above is fullfilled. Therefore we can say that H "generates" h^t. Conversely, it is known (this is due independently to Viterbo and Buhovsky-Seyfaddini) that given a continuous Hamiltonian isotopy h^t there is a unique possible H up to constant. These uniqueness results show that this framework is a good generalization of what happens in the smooth case. Nevertheless, the existence problem is very hard. It is unknown which continuous functions actually generate a continuous Hamiltonian isotopy.

Another attempt (that would avoid this problem but create others) would be to work inside the completion of the Hamiltonian group for Hofer's distance. It is by definition given by $d(\phi,\psi) := \|\psi^{-1}\circ\phi\|$. The map between metric spaces $(C^\infty([0,1]\times M), \|\cdot\|_{C^0}) \to (\mathrm{Ham}(M,\omega), d)$, $H \mapsto \phi^t_H$ is Lipschitz. Thus, it extends to completions giving rise to a map $C^0([0,1]\times M) \to \overline{\mathrm{Ham}(M,\omega)}$. Hence, any continuous function has a flow in the completion. As before we can wonder whether the continuous Hamiltonian is unique up to constant. This question is answered positively on rationnal symplectic manifolds by a joint work with R. Leclercq and S. Seyfaddini [4].

Some open problems. There are many open interesting problems in this subjects. Here my favorite ones:

(1) Is the group of Hamiltonian diffeomorphisms C^0-closed in the group of symplectic diffeomorphisms? This is only known for surfaces, for the standard 2n-torus (Hermann 83) and for a few more examples (Lalonde-McDuff-Polterovich 97).
(2) Is the group of area preserving and compactly supported homeomorphisms of the 2-disk a simple group? The group of Hamiltonian homeomorphisms defined by Oh and Müller is a normal subgroup but so far no one has been able to prove that it is proper.
(3) Which symplectic invariants are invariant under conjugation by a symplectic homeomorphism? For example in the case of the Calabi invariant it has been established by Gambaudo and Ghys that two Hamiltonian diffeomorphisms of the 2-disk that are conjugated by an area preserving homeomorphism have the same Calabi invariant. The analogous problem in higher dimension is open.
(4) Understand "symplectically" Le Calvez's theory of area-preserving homeomorphisms of surfaces.

(5) Extend Aubry-Math theory to general non-convex Hamiltonians. It is likely that one needs to consider symplectic objects (e.g., Lagrangian submanifolds) having low regularity to develop such an extension.

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Asymptotically harmonic manifolds of negative curvature

A. Sambusetti

(joint work with P. Castillon)

The *harmonic manifolds* are Riemannian manifolds X whose geodesic spheres have constant mean curvature; equivalently, such that the volume density function, in normal coordinates around any point x, only depends on the distance $d(x, \cdot)$. Another equivalent condition is that the mean-value property

$$f(x_0) = \frac{1}{volS(x_0, r)} \int_{S(x_0,r)} f(x) dv_{S(x_0,R)}$$

holds for all harmonic functions f on M (cf. [3]). It is not difficult to show that these are Einstein spaces, hence harmonic manifolds have constant curvature in dimensions 2 and 3.

The history of harmonic manifolds begins in the thirties, when E.T. Copson and H.S. Ruse([5], [12]) investigated the problem of finding radial solutions of the Laplace equation $\Delta f = 0$ on a general Riemannian manifold, and showed that this is equivalent to asking that the density function is radial. In 1944, A. Lichnerowicz [11] conjectured that harmonic manifolds are all flat or ROSS (Rank One Symmetric Spaces), which was fully proved in dimension 4 by A.G. Walker some years later [16]. A major step forward in the solution of the conjecture in any dimension took place in 1990, when Z.I. Szabo proved it for *compact simply connected manifolds* X, cf. [15]. In 1992 the first, unexpected, counterexample arrived: E. Damek and F. Ricci showed that there exist, in dimension greater than six, some homogeneous manifolds (two-step nilpotent groups), which are harmonic but neither flat nor symmetric.

Since then, several authors worked to the solution of the problem in different settings. An asymptotic version of harmonicity was introduced in [9] by F. Ledrappier for negatively curved spaces: a Cartan-Hadamard (CH, for short) manifold X is *asymptotically harmonic* if its horospheres –which can be seen as metric limits of spheres– have constant mean curvature h. This notion was mainly studied in the *cocompact* case (i.e. for CH-spaces X admitting a compact quotient $X_0 = X/\Gamma$), and it turned out that this definition captures lot of the information on the dynamics of the harmonic manifolds, which was useful to approach the Lichnerowicz conjecture from a new point of view. In particular, F. Ledrappier and C. Yue showed that, for CH-manifolds X admitting cocompact quotients, asymptotical harmonicity is equivalent to the fact that the three natural families of measures (visual, harmonic and Patterson-Sullivan) on the boundary at infinity ∂X coincide; asymptotical harmonicity is also equivalent to asking that the harmonic measures for the weak stable foliations W^s or for the horospherical foliation coincide with one (and then all) of the three natural measures on the unitary tangent bundle SX_0 (Liouville, harmonic and Bowen-Margulis), see [10], [17]. In 1995, G. Besson, G. Courtois and S. Gallot [1], also using work of P. Foulon, Y. Benoist and F. Labourie on the geodesic flow of manifolds having smooth horospherical distribution (cf. [2], [6]), settled the problem in this class of spaces: every harmonic or asymptotically harmonic CH-manifold of negative curvature, admitting cocompact quotients, is necessarily a ROSS. In a different – orthogonal – setting, J. Heber in [7] proved that flat, ROSS and Damek-Ricci spaces are the only *homogeneous* harmonic (or asymptotically harmonic) manifolds.

Further steps in determining which spaces are harmonic or asymptotically harmonic, without any curvature assumption, have been recently done by G. Knieper (cf. [8] for manifolds without focal points), and by V. Schroeder and H. Shah (for 3-manifolds without conjugate points [14], [13]). However, in full generality, without any homogeneity or cocompactness assumption, the Lichnerowicz Conjecture (as modified by the counterexemples of Damek and Ricci) still holds open, starting from dimensions 5 and 4, respectively, for harmonic manifolds and asymptotically harmonic manifolds.

In the talk I presented some results from my recent work in collaboration with P. Castillon (University of Montpellier, France). We study asymptotically harmonic manifolds X of negative curvature, without any cocompactness or homogeneity assumption, and we show that this condition still gives a lot of information on the geometry. We determine the volume entropy, the spectrum and the relative densities of visual and harmonic measures on the ideal boundary of X. Then, we prove the asymptotic analogue of the characterization of harmonic manifolds by radiality of the density function, and a "mean value property" for horospheres:

Theorem 1. Let X be a CH-manifold with $K_X \leq -a^2 < 0$ and entropy E. X is asymptotically harmonic if and only if there exists a strictly positive function $\tau : SM \to \mathbb{R}$ such that the density function $\theta(u, r)$ is uniformly equivalent to $\tau(u)e^{-Er}$ for $r \to \infty$. Moreover, if $|DR_X| < \infty$ then the function τ is constant.

(Here, "uniformly equivalent" means that there exists a function $\epsilon(r)$, tending to zero for $r \to +\infty$, such that $\left|\frac{\theta(u,r)}{\tau(u)e^{Er}} - 1\right| < \epsilon(r)$ for all $u \in SX$.)

Theorem 2. Let X be a CH-manifold with $K_X \leq -a^2 < 0$.
If X is asymptotically harmonic, then for any horosphere H_ξ centered at ξ, for any harmonic function on X which extends continuously on a neighbourhood of $\xi \in X \cup \partial X$, and for any $x \in H_\xi$ we have

$$f(\xi) = \lim_{r_k \to \infty} \frac{1}{Vol B_{H_\xi}(x, r_k)} \int_{B_{H_\xi}(x,r_k)} f(y) dv_{H_\xi}(y)$$

for a sequence of balls $B_{H_\xi}(x, r_k)$ in H_ξ with radii $r_k \to \infty$.

Finally, we deduce the existence of a *harmonic* Margulis function, for all asymptotically harmonic manifolds:

Theorem 3. Let X be a CH-manifold with $-b^2 \leq K_X \leq -a^2 < 0$ and entropy E.
If X is asymptotically harmonic, then there exists a bounded, strictly positive, harmonic function $m : M \to \mathbb{R}$ such that $vol\, S(x, r) \asymp m(x)e^{Er}$ for any $x \in X$ (where $\asymp$ always means "uniformly equivalent to").

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How large is the shadow of a symplectic ball?

Alberto Abbondandolo
(joint work with Slava Matveyev)

Let $\omega = \sum_{j=1}^n dp_j \wedge dq_j$ be the standard symplectic form on $\mathbb{R}^{2n}$, which is endowed with coordinates $(p_1, q_1, \dots, p_n, q_n)$. A symplectomorphism between open subsets of $\mathbb{R}^{2n}$ is a diffeomorphism which preserves the form ω. Symplectomorphisms are the natural transformations of classical mechanics. Since ω^n is a multiple of the Euclidean volume form of $\mathbb{R}^{2n}$, symplectomorphisms preserve volume.

The celebrated non-squeezing theorem of Gromov states that if $0 < s < r$, then no symplectomorphism can map B_r, the ball of radius r, into the cylinder consisting of those points $(p_1, q_1, \dots, p_n, q_n)$ such that $p_1^2 + q_1^2 < s^2$. This theorem describes a two-dimensional rigidity which is obviously not shared by volume-preserving diffeomorphisms and which is reminiscent of the Heisenberg principle: if the initial status of a system is known with precision r, one cannot let the system evolve so that the knowledge of a pair of conjugate variables is simultaneously improved, even if one is willing to loose any information on all the other variables. Since symplectomorphisms also preserve ω^k, for $1 \leq k \leq n$, it is natural to ask whether symplectomorphisms must satisfy also some middle-dimensional non-squeezing property.

A question in this direction concerns symplectic embeddings of polydiscs. We denote by $P(r_1, \dots, r_n)$ the polydisk consisting of points $(p_1, q_1, \dots, p_n, q_n)$ such that $p_j^2 + q_j^2 < r_j^2$ for every j. If the symplectomorphism φ maps $P(r_1, \dots, r_n)$ into $P(s_1, \dots, s_n)$, the conservation of volume implies that $r_1 r_2 \dots r_n \leq s_1 s_2 \dots s_n$, while Gromov's non-squeezing theorem implies that $\min r_j \leq \min s_j$. Therefore, it is natural to ask whether similar inequalities for other products of the radii hold. In the large scale, the answer to this question turns out to be no, as shown by L. Guth in [3]: for every $\epsilon > 0$ there exists a symplectomorphism which maps $P(\epsilon, 1, 1)$ into $P(2\epsilon, 10\epsilon, \infty)$.

In this talk we address the question of the middle-dimensional squeezing versus non-squeezing behavior of symplectomorphisms from a different point of view, by keeping the ball as the domain of our maps. We start by noticing that the non-squeezing theorem can be restated as the following inequality:

$$\operatorname{area}\big(P_1\varphi(B_r)\big) \geq \pi r^2,$$

where P_1 is the orthogonal projector onto $\mathbb{R}^2$, that is the plane spanned by the vectors $\frac{\partial}{\partial_{p_1}}$ and $\frac{\partial}{\partial_{q_1}}$. This reformulation raises the following question: if P_k is the orthogonal projection onto $\mathbb{R}^{2k}$, that is the subspace spanned by the vectors $\frac{\partial}{\partial_{p_1}}, \frac{\partial}{\partial_{q_1}}, \dots, \frac{\partial}{\partial_{p_k}}, \frac{\partial}{\partial_{q_k}}$, is it true that

$$\operatorname{vol}_{2k}\big(P_k\varphi(B_r)\big) \geq \omega_{2k} r^{2k},$$

for every symplectomorphism φ? Here ω_{2k} denotes the volume of the unit ball in dimension $2k$. Equivalently, P_k can be replaced by the orthogonal projector onto any complex subspace of $\mathbb{R}^{2n} \cong \mathbb{C}^n$ $((p,q) \mapsto p+iq)$, or, replacing the Euclidean $2k$-volume by the integral of $\omega^k/k!$, by the projection onto any symplectic $2k$-dimensional subspace along its symplectic orthogonal.

Our first result is that the answer is yes in the linear category:

Theorem 1. *If Φ is a linear symplectomorphism, then $\operatorname{vol}_{2k}\big(P_k\Phi(B_1)\big) \geq \omega_{2k}$. Moreover, the equality holds if and only if the subspace $\Phi^T\mathbb{R}^{2k}$ is complex.*

However, the answer is no in the nonlinear category:

Theorem 2. *If $1 < k < n$ and $\epsilon > 0$, then there exists a symplectomorphism $\varphi : B_1 \hookrightarrow \mathbb{R}^{2n}$ such that $\operatorname{vol}_{2k}\big(P_k\varphi(B_1)\big) < \epsilon$.*

The construction of this counterexample uses some lemmas from the above mentioned paper of Guth.

Having a rigidity which holds for linear maps and does not hold for nonlinear ones, it is natural to ask at what scale this rigidity breaks down. For instance, one can ask whether the middle-dimensional non-squeezing property holds locally, in the following two senses:

(i) Let $\varphi : \mathbb{R}^{2n} \to \mathbb{R}^{2n}$ be a symplectomorphism and let $x \in \mathbb{R}^{2n}$. Is it true that

$$\operatorname{vol}_{2k}\big(P_k\varphi(B_r(x))\big) \geq \omega_{2k} r^{2k}$$

for every $r > 0$ small enough?

(ii) Let $\{\varphi_t\}_{t\in[0,1]}$ be a smooth family of symplectomorphisms such that φ_0 is linear. Is it true that

$$\operatorname{vol}_{2k}\big(P_k\varphi_t(B_1)\big) \geq \omega_{2k}$$

for every $t \geq 0$ small enough?

Notice that a positive answer to the second question would imply a positive answer to the first one, by rescaling. We do not have a definite answer to this questions yet, but we strongly believe that the answer is in both cases positive. The two following results corroborate this conjecture.

The first result is about the first "local in space" formulation:

Theorem 3. Let $\varphi : \mathbb{R}^{2n} \to \mathbb{R}^{2n}$ be a symplectomorphism. Then there exists an open and dense set $A \subset \mathbb{R}^{2n}$ and a function $\rho : A \to (0, +\infty)$ such that for every x in A there holds

$$\mathrm{vol}_{2k}\big(P_k \varphi(B_r(x))\big) \geq \omega_{2k} r^{2k}$$

for every $0 < r \leq \rho(x)$.

The proof of this result uses some non-trivial facts from geometric measure theory. The argument suggests a relationship between the middle-dimensional non-squeezing property and the integrability of a certain multi-valued distribution.

In order to state the second result, which concerns the second "local in time" formulation, we need to introduce some notation.

The symbol $\mathrm{Gr}_1(\mathbb{C}^k)$ denotes the Grassmannian of complex lines in $\mathbb{C}^k \subset \mathbb{C}^n \cong \mathbb{R}^{2n}$. This Grassmannian coincides with the complex projective space $\mathbb{CP}^{k-1}$ and is endowed with the volume form $\eta = \omega_{\mathbb{CP}^{k-1}}^{k-1}/(k-1)!$, where $\omega_{\mathbb{CP}^{k-1}}$ is the standard Kähler form on $\mathbb{CP}^{k-1}$.

If $z : \mathbb{R}/2\pi\mathbb{Z} \to \mathbb{R}^{2n}$ is a smooth loop, we denote the Dirichlet energy of z and the symplectic area of any disk bounded by z respectively by

$$E(z) := \frac{1}{2} \int_0^{2\pi} |z'(\theta)|^2 \, d\theta, \quad A(z) := \int_{\mathbb{R}/2\pi\mathbb{Z}} z^* \Big(\sum_{j=1}^n p_j dq_j \Big).$$

Assume, for sake of simplicity, that the smooth path of symplectomorphisms φ_t starts at the identity and let H_t be the time-dependent generating Hamiltonian, that is,

$$\frac{\partial}{\partial t} \varphi_t(x) = X_{H_t}\big(\varphi_t(x)\big),$$

where $X_{H_t} = i\nabla H_t$ denotes the Hamiltonian vector field which is associated to H_t. Then we have the following second order expansion for the $2k$-volume of the $2k$-dimensional shadow of $\varphi_t(B_1)$:

Theorem 4. *There holds*

$$\mathrm{vol}_{2k}\big(P_k \varphi_t(B_1)\big) = \omega_{2k} + C(H_0) t^2 + O(t^3), \qquad \text{for } t \to 0,$$

where

$$C(H_0) := \int_{\mathrm{Gr}_1(\mathbb{C}^k)} \big(E(z_L) - A(z_L)\big) \, \eta(L),$$

and, for every L in $\mathrm{Gr}_1(\mathbb{C}^k)$, $z_L : \mathbb{R}/2\pi\mathbb{Z} \to \mathbb{R}^{2n}$ is the loop

$$z_L(\theta) = (I - P_k) X_{H_0}\big(e^{i\theta} \zeta_L\big),$$

with any vector in ζ_L in $L \cap \partial B_1$.

The proof is based on an elaborate computation which uses the Lie-Cartan formalism and fiberwise integration.

By the two-dimensional isoperimetric inequality, $E(z) \geq A(z)$ for every loop z, and the identity holds if an only if z is the constant speed counterclockwise parametrization of a circle in a complex line. Therefore, the constant $C(H_0)$ is always non-negative and vanishes only if the vector field X_{H_0} is very special.

Both Theorems 3 and 4 say that the local middle-dimensional non-squeezing property holds in the generic case.

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Aspherical groups and manifolds with extreme properties.

Mark Sapir

I prove that every finitely generated group with recursive aspherical presentation (i.e. recursive aspherical 2-dimensional $K(.,1)$) embeds into a finitely presented group with a finite 2-dimensional aspherical $K(.,1)$. Using Davis' reflection trick [1] one can further embed the group in the fundamental group of a closed aspherical 4-dimensional Riemannian manifold. Starting with a Olshanskii's Tarski monster [3]) or Gromov's random monster [2] one can then embed such groups into the fundamental group of a closed aspherical manifold. Therefore there exist closed aspherical manifolds whose universal covers have infinite asymptotic dimension, do not coarsely embed into Hilbert spaces, do not satisfy the Baum-Connes conjecture with coefficients, etc. The proof [4] is a version of the celebrated Higman embedding theorem. In the talk, I briefly described Davis' construction, sketched a proof of the Higman embedding theorem, explained why the previous versions of the proof do not produce aspherical presentations and gave some details of the new construction.

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Positive topological entropy for multi-bump magnetic fields

K. F. Siburg
(joint work with A. Knauf and F. Schulz)

Consider a magnetic field in $\mathbb{R}^3$ whose field lines are perpendicular to the plane $\mathbb{R}^2 \times \{0\} \cong \mathbb{R}^2$. Then the motion of a particle of unit mass and unit charge in that plane is modelled by Newton's Second Law

$$\ddot{q} = B(q)J\dot{q} \tag{1}$$

where $B : \mathbb{R}^2 \to \mathbb{R}$ describes the field strength and the term on the right hand side is the Lorentz force corresponding to the magnetic field, with J being the symplectic matrix $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. The differential equation (1) can be written as the Hamiltonian system generated by the Hamiltonian $H : T^*\mathbb{R}^2 \to \mathbb{R}, H(q,p) = \frac{1}{2}\|p\|^2$ on $(T^*\mathbb{R}^2, \omega)$ with the twisted symplectic form

$$\omega = \omega_0 + B(q)dq_1 \wedge dq_2$$

where $\omega_0 = d\lambda = dp_1 \wedge dq_1 + dp_2 \wedge dq_2$ stands for the standard symplectic form on $T^*\mathbb{R}^2$.

In this talk, we study the dynamics of a particle when the magnetic field consists of $n \geq 2$ disjoint bumps, i.e., when the support of B consists of n connected components $\text{supp}B_k$. Assuming that each component is a disc where the magnetic field is rotationally symmetric, we show that this dynamical systems exhibits chaotic behavior in the sense that it possesses an invariant set on which the Poincaré map induced by its flow is semi-conjugated to the full shift in n symbols. This implies that there are solutions visiting the different components $\text{supp}B_k$ in any prescribed order. Moreover, we can conclude that our system has positive topological entropy $h_{\text{top}} \geq \log n$.

Cosmic censorship of smooth structures

Stefan Nemirovski
(joint work with Vladimir Chernov)

One form of the 'strong cosmic censorship hypothesis' proposed by Roger Penrose asserts that physically relevant spacetimes should be globally hyperbolic (see [2]). Using several deep results, including Perelman's proof of the Poincaré conjecture, we observe that global hyperbolicity imposes strong restrictions on the possible smooth structures of the spacetime.

Theorem 1. Suppose that (X, g) is a globally hyperbolic spacetime such that the manifold X is contractible. Then X is diffeomorphic to the standard $\mathbb{R}^{\dim X}$.

In particular, exotic $\mathbb{R}^4$'s (that is to say, 4-manifolds homeomorphic but not diffeomorphic to $\mathbb{R}^4$) do not carry globally hyperbolic Lorentz metrics.

More generally, it seems plausible that the smooth structure of a globally hyperbolic 4-dimensional spacetime is always determined by its topology. We prove this in the case when the Cauchy surface of the spacetime is a closed orientable 3-manifold and state the following

Conjecture. If two (orientable) globally hyperbolic 4-dimensional spacetimes are homeomorphic, then they are diffeomorphic.

Examples show that this conjecture is not true in higher dimensions (e. g., for spacetimes homeomorphic to $S^7 \times \mathbb{R}$).

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Area growth and rigidity of surfaces without conjugate points

V. Bangert

(joint work with P. Emmerich)

In 1942 M. Morse and G. A. Hedlund [7] conjectured that every Riemannian 2-torus without conjugate points is flat. This was proved by E. Hopf in 1943, see [5]. The natural question, if Riemannian tori without conjugate points and of arbitrary dimension are flat, was answered affirmatively by D. Burago and S. Ivanov [3], by a completely new method.

Here, we apply E. Hopf's original method to the study of complete Riemannian planes and cylinders without conjugate points. In these cases one needs additional assumptions to prove flatness since the plane and the cylinder admit complete Riemannian metrics with negative Gaussian curvature (and, hence, without conjugate points). In this situation, conditions on the area growth are particularly natural. For the case of the plane we prove the following optimal result.

Theorem 1. Let g be a complete Riemannian metric without conjugate points on the plane $\mathbb{R}^2$. If p is a point on $\mathbb{R}^2$ then the area $A_p(r)$ of the metric ball with center p and radius r satisfies

$$\liminf_{r\to\infty} \frac{A_p(r)}{\pi r^2} \geq 1$$

with equality if and only if g is flat.

Note that, for every $\epsilon > 0$, one can easily find complete planes with non-positive Gaussian curvature and conical end such that $\lim_{r\to\infty} A_p(r)/\pi r^2 = 1 + \epsilon$. These examples show that the estimate in Theorem 1 is optimal.

To state our rigidity result for cylinders we first define what it means that an end of a cylinder has subquadratic area growth. As usual, we denote by d the distance induced by the Riemannian metric, and by $B(p, r)$ the metric ball with center p and radius r.

Definition 1. Let S be a complete, connected Riemannian surface. An end $\mathcal{E}$ of S has *subquadratic area growth* if there exists a neighborhood $U \subseteq S$ of $\mathcal{E}$ such that

$$\liminf_{r\to\infty} \frac{A(U \cap B(p,r))}{r^2} = 0$$

for one (and hence every) point $p \in S$.

Theorem 2. Let g be a complete Riemannian metric without conjugate points on the cylinder $S^1 \times \mathbb{R}$. If both ends of the cylinder have subquadratic area growth then g is flat.

There is an alternative version of Theorem 1 that involves an assumption on the growth of the lengths of shortest non-contractible loops.

Definition 2. Let $C = S^1 \times \mathbb{R}$ be a complete Riemannian cylinder and, for $p \in C$, let $l(p)$ denote the length of a shortest non-contractible loop based at p. We say that an end $\mathcal{E}$ of C *opens less than linearly* if there exists a sequence (p_i) in C converging to $\mathcal{E}$ such that

$$\lim_{i\to\infty} \frac{l(p_i)}{d(p_i, p_0)} = 0.$$

Theorem 3. Let g be a complete Riemannian metric without conjugate points on the cylinder $S^1 \times \mathbb{R}$. If both ends of the cylinder open less than linearly then g is flat.

Again, simple examples of cylinders of revolution with non-positive Gausian curvature and conical ends show that the conditions in Theorem 2 and 3 are optimal.

Rigidity results of the type of Theorem 3 have been proved by K. Burns and G. Knieper [2], H. Koehler [6], and by the present authors [1]. All of these involve stronger conditions on the growth of l and additional conditions on the Gaussian curvature. So they are far from being optimal. The basic idea, however, is the same in all these papers: E. Hopf's method is applied to an appropriate exhaustion by compact sets. This introduces boundary terms that have to be controlled in the limit and that do not appear in the case of the 2-torus treated by E. Hopf. Here the essential difficulty is that the geometric quantities that influence these boundary terms might oscillate dramatically in the non-compact situation. Any naive attempt to control them induces unwanted additional assumptions, as present in the previous results. To our surprise, a delicate analysis of the differential inequality that results from E. Hopf's method finally leads to the optimal results presented here.

In [2] the same method is applied to complete planes without conjugate points. The rigidity result proved in [2] assumes a strong "parallel axiom". In connection with this and Theorem 1 we mention the following interesting open problem mentioned in [4].

Openproblem. Suppose a complete Riemannian plane P satisfies the parallel axiom, i.e. for every geodesic c on P and every point $p \in P$ not on c there exists a unique geodesic through p that does not intersect c. Does this imply that P is isometric to the Euclidean plane?

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RAAGs in Ham

Michael Kapovich

For a graph Γ let $V(\Gamma), E(\Gamma)$ denote the vertex and edge sets of Γ. Let Γ be a graph with no loops and bigons, i.e., a simplicial complex of dimension ≤ 1. Define the Right Angled Artin group (RAAG) G_Γ with the *Artin graph* Γ by the presentation

$$\langle g_v, v \in V(\Gamma) | \quad [g_v, g_w] = 1, [vw] \notin E(\Gamma) \,.$$

We note that our definition is opposite to the usual one in the theory of RAAGs, where one imposes the relators $[g_v, g_w] = 1$ for every $[vw] \in E(\Gamma)$. However, our convention is in line with the notation in the theory of finite Coxeter groups and Dynkin diagrams. We adopted this notation because it is most suitable for our purposes, while the usual definition leads to heavy notation.

Given a symplectic manifold (M, ω) we let $Ham(M, \omega)$ denote the group of Hamiltonian symplectomorphisms of (M, ω). Since, by Moser's theorem, for a closed surface M its symplectic structure is unique up to scaling, we will abbreviate $Ham(M, \omega)$ to $Ham(M)$ if M is a closed surface.

Our main result is:

Theorem 1. For every finite Γ the group G_Γ embeds in $Ham(S^2)$. Moreover, under this embedding the group G_Γ fixes a closed disk in S^2 pointwise.

As corollaries of the proof of this theorem we establish the following:

Corollary 1. For every finite Γ and every symplectic manifold (M, ω), the group G_Γ embeds in $Ham(M, \omega)$.

Corollary 2. Let $\Lambda \subset O(n, 1)$ be an arithmetic lattice of the simplest type, $n \geq 2$. Then a finite index subgroup in Λ embeds in $Ham(M, \omega)$ for every symplectic manifold (M, ω).

Proof. According to the result of Bergeron, Haglund and Wise [HW], a finite index subgroup in Λ embeds in some RAAG G_Γ. Now, the result follows from Corollary 1. □

In contrast, suppose that M is a closed oriented surface of genus ≥ 1 with area form ω. Then it was proven first by L. Polterovich [P] and, later, by Franks and Handel [FH] using different methods, that every irreducible nonuniform arithmetic group Λ of rank ≥ 2 does not embed in $Ham(M, \omega)$. Furthermore, Franks and Handel [FH] extended this result to certain nonuniform rank 1 lattices, e.g., lattices in $PU(2, 1)$.

Outline of the proof. Theorem 1 is proven in three steps.

Step 1. Let M be a closed connected oriented surface to which Γ embeds. For technical reasons, it will be convenient to assume that M is not the torus. We first prove

Theorem 2. The group G_Γ embeds in $Ham(M)$. Moreover, each Artin generator g_v of G_Γ acts on M as an "twice-iterated Double Dehn twist" $\Psi(g_v)$ supported in a homotopically trivial annulus in M.

The key to verifying injectivity of $\Psi : G_\Gamma \to Ham(M)$ is that the action $G_\Gamma \curvearrowright M$ preserves a certain finite subset $P \subset M$, so that the restriction $G_\Gamma \curvearrowright M' = M \setminus P$ projects to a faithful representation to the mapping class group of M', $G_\Gamma \to Map(M')$. Faithfulness of this representation follows from a special case of a theorem of L. Funar [Fu] (similar results are established in the papers by T. Koberda [K] and by M. Clay, C. Leininger and J. Mangahas [CLM]). This part of our paper is similar to the arguments by J. Crisp and B. Wiest [CW].

Step 2 (Lifting). If Γ were planar, Theorem 2 would imply Theorem 1. In general, of course, Γ need not be planar (or even admit a finite planar orbi-cover), however, it has a planar universal cover (e.g., the disjoint union of simplicial trees). Suppose, therefore, that M has genus ≥ 2. Then we lift the action $\Psi : G_\Gamma \curvearrowright M$ to the universal cover $\tilde{M}$ of M, which we identify with the hyperbolic plane, i.e., the unit disk D in $S^2 = \mathbf{C} \cup \{\infty\}$. We let ω_0 be the Euclidean area form on an open disk containing D; extend ω_0 smoothly to an area form ω_0 on S^2.

Let $D' = D - P'$ denote the punctured disk where P' is the preimage of P in D. Let $Ham(M, P)$ denote the subgroup of $Ham(M)$ fixing P pointwise. We have an (injective) homomorphism

$$\iota : Ham(M,P) \to Ham(D,P')$$

obtained by choosing an appropriate lifting of Hamiltonian diffeomorphisms. We thus obtain the lift $\tilde{\Psi} = \iota \circ \Psi$ of the homomorphism Ψ. Then we show that $\tilde{\Psi}$ projects injectively to the mapping class group $Map(D')$.

Each generator g_v of G_Γ acts (via $\tilde{\Psi}$) on D as a product of infinitely many commuting twice-iterated Double Dehn twists preserving the hyperbolic area form. However, $\tilde{\Psi}(G_\Gamma)$, of course, does not preserve ω_0. Then we modify each of the Double Dehn twists in the product decomposition of $\tilde{\Psi}(g_v)$ to obtain a new diffeomorphism $\rho_0(g_v)$ which is isotopic to $\tilde{\Psi}(g_v)$ on the punctured disk D' and is the time-2-map for the appropriately chosen function $H_v : D \to \mathbf{R}$ with respect to ω_0. It then follows that the resulting representation

$$\rho_0 : G_\Gamma \to Ham(D, \omega_0)$$

is again faithful. We will see that for each v, H_v extends by zero to a $C^{1,1}$-function on S^2 and $\rho_0(g_v)$ extends Lipschitz-continuously (by the identity) to the entire sphere, so we can think of it as a Lipschitz Hamiltonian symplectomorphism. However, the function H_v need not be C^2-smooth and $\rho_0(g_v)$ need not even be differentiable.

Step 3 (Approximation). The last step of the proof is an approximation argument: We approximate $H_v : S^2 \to \mathbf{R}$ by a mollifier, a smooth function $\eta_\epsilon H_v$ which depends analytically on $\epsilon > 0$ and converges to H_v uniformly on compacts in the open disk D as $\epsilon \to 0$. Each function $\eta_\epsilon H_v$ determines its own time-2-map $\rho_\epsilon(g_v)$ and we obtain an analytic family of representations $\rho_\epsilon : G_\Gamma \to Ham(S^2)$, $\epsilon > 0$, which converge to ρ_0 as $\epsilon \to 0$. Then (since ρ_0 is injective) we establish that the representations ρ_ϵ are injective for all but countably many $\epsilon > 0$, thereby proving Theorem 1.

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Towards categorification of Lagrangian Topology

Octav Cornea
(joint work with Paul Biran (ETH))

Cobordism has played fundamental role in the modern development of algebraic and differential topology. In symplectic topology, Lagrangian cobordism has been introduced by Arnold at the beginnings of the field and it has been studied by Eliashberg and Audin who showed that without any additional restrictions it is a very flexible notion. Later on in the middle of the '90's Chekanov remarked that if additional restrictions are imposed - for instance monotonicity - then some rigidity is present. In the paper [1] - which was the main reference for the talk - we consider Lagrangian cobordism from a more categorical point of view: we first notice that it is possible to define a Lagrangian cobordism category whose objects are the Lagrangian submanifolds of a given symplectic manifold (M, ω). The morphisms between two such objects L, L' are horizontal isotopy classes of Lagrangian submanifolds $V \subset (\mathbb{C} \times M, \omega_0 + \omega)$ so that V is non-compact and has one positive end that is identified with $[0,1) \times \{1\} \times L$ as well as some negative ends identified with $(-\infty, 0] \times \{1\} \times L_1, \ldots, (-\infty, 0] \times \{k\} \times L_k, (-\infty, 0] \times \{k+1\} \times L'$ for some $k \geq 0$. It is not difficult to show that this does indeed give rise to a category that we denote by $\mathcal{C}ob_{pre}(M)$.

Remark. Another category of Lagrangian cobordisms has been introduced by Nadler and Tanaka also in an October 2011 preprint.

From now on restrict to the subcategory $\mathcal{C}ob^d_{pre}(M)$ of all Lagrangians that are uniformly monotone in the sense that the Maslov morphism and the symplectic area are proportional with the same constant, the minimal Maslov number is at least 2 and additionally the number of J-holomorphic disks through a point is the same for all Lagrangians (+ a condition having to do with the appropriate Novikov ring). The morphisms in this subcategory also satisfy the same conditions. Denote by $DFuk^d(M)$ the derived Fukaya category with the same objects as those of $\mathcal{C}ob^d_{pre}(M)$. The main result is that there exists a functor:

$$\mathcal{F} : \mathcal{C}ob^d_{pre}(M) \to DFuk^d(M)$$

that is the identity on objects and that fits, in an appropriate sense with the triangulated structure of the target. For instance, given a cobordism V as above this compatibility implies that, in $DFuk^d(M)$, L belongs to the subcategory generated by $L_1, L_2, \ldots L_k, L'$. In fact, the construction provides exact triangles in $DFuk^d(M)$: $L_2 \to L_1 \to M_2, \ldots, L_{i+1} \to M_i \to M_{i+1}$ (with $L' = L_{k+1}$) and $M_{k+1} \simeq L$.

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Ziggurats and rotation numbers

Danny Calegari
(joint work with Alden Walker)

A reference for this material is [1].

Let F be a free group of rank 2, with generators a, b. Let $\text{Homeo}^+(S^1)^\sim$ denote the universal cover of the group of orientation-preserving homeomorphisms of the circle. There is a homogeneous quasimorphism

$$\text{rot}^\sim : \text{Homeo}^+(S^1)^\sim \to \mathbb{R}$$

called rotation number.

For $w \in F$ and r, s in $\mathbb{R}$, let $X(w, r, s)$ denote the set of values of $\text{rot}^\sim(\rho(w))$ under all homomorphisms $\rho : F \to \text{Homeo}^+(S^1)^\sim$ satisfying $\text{rot}^\sim(\rho(a)) = r$ and $\text{rot}^\sim(\rho(b)) = s$, and let $R(w, r, s)$ denote the supremum of $X(w, r, s)$. It is elementary that $X(w, r, s)$ is a compact interval, with minimum equal to $-R(w, -r, -s)$, so knowledge of $R(w, \cdot, \cdot)$ completely determines knowledge of $X(w, \cdot, \cdot)$. Our investigation centers on the following:

Question: Is it true that $R(w, r, s) \in \mathbb{Q}$ for all $w \in F$ and all r, s in $\mathbb{R}$?

Problem: Give a practical method to compute or approximate $R(w, r, s)$.

The question of computing $X(w, r, s)$ for the special case $w = ab$ was considered in [2], with partial results.

1. positive words. A word w is *positive* if it is in the semigroup generated by a and b. For such a word w we give a straightforward combinatorial algorithm to compute $R(w, r, s)$ whenever r and s are rational, and show in this case that $R(w, r, s) \in \mathbb{Q}$, and has denominator no greater than the minimum of the denominators of r and s. Moreover, $R(w, \cdot, \cdot)$ is constant on some region $[r, r+\epsilon) \times [s, s+\epsilon)$ where ϵ has order of magnitude equal to the reciprocal of this denominator.

The function $R(w, \cdot, \cdot)$ is therefore locally constant and takes values in $\mathbb{Q}$ on an open dense subset of the parameter plane. Our method allows us to reduce the problem of computing $R(w, r, s)$ for $r, s \in \mathbb{Q}$ to a finite combinatorial question. This gives a very short new proof of Naimi's theorem [3] (i.e. the conjecture of Jankins–Neumann [2]) which was the last step in the classification of taut foliations of Seifert fibered spaces. See [1] for details.

2. slippery points. For w positive, define $R(w, r-, s-)$ to be the supremum of $R(w, r', s')$ over all $r' < r$, $s' < s$. It turns out that $R(w, r-, s-)$ is equal to the supremum of $\text{rot}^\sim(\rho(w))$ under all representations ρ for which $\rho(a)$ and $\rho(b)$ are conjugate to *rotations* R_r and R_s respectively. A point (r, s) is *slippery* for w if there is a strict inequality $R(w, r', s') < R(w, r-, s-)$ for all $r' < r$ and $s' < s$.

Slippery Conjecture: *If (r, s) is slippery for w then $R(w, r-, s-) = h_a(w)r + h_b(w)s$ where h_a and h_b count the number of copies of a and b in w.*

Note that $R(w, r-, s-) \geq h_a(w)r + h_b(w)s$ by definition. The Slippery Conjecture implies that $R(w, r-, s-) \in \mathbb{Q}$ whenever $r, s \in \mathbb{Q}$. It is implied by the

Refined Slippery Conjecture: *Suppose* $w = a^{\alpha_1}b^{\beta_1}\cdots b^{\beta_m}$ *is positive. If* $R(w,r,s) = p/q$ *then there is an inequality*

$$R(w,r,s) - h_a(w)r - h_b(w)s \leq m/q$$

3. arbitrary words. For w arbitrary, it still makes sense to define $R(w,r-,s-)$ to be the supremum of $\text{rot}^\sim(\rho(w))$ over all representations ρ for which a and b are conjugate to R_r and R_s. For $r,s \in \mathbb{Q}$ there is a positive word $w'(w,r,s)$ and an integer $n'(w,r,s)$ so that

$$R(w,r-,s-) = R(w',r-,s-) - n'(w,r,s)$$

The word w' and the integer n' are easily computable in terms of w,r,s. Therefore this number may be computed quickly. The slippery conjecture would imply that it was always rational, and the refined slippery conjecture would make fast approximation easy.

4. the interval game. Suppose we are given a finite collection of homeomorphisms $\phi, \varphi_i \in \text{Homeo}^+(S^1)$ for $1 \leq i \leq m$. The *interval game* asks whether there is a closed interval $I \subset S^1$ and a positive integer n so that $\phi^n(I^+) \in I$ (where I^+ denotes the rightmost point of I), but $\phi^j(I^+)$ is not in $\varphi_i(I)$ for any i and any $0 \leq j \leq n$.

Suppose ϕ is a rigid irrational rotation (this might be thought of as a "generic" case) and suppose that its rotation number satisfies a certain Diophantine condition (which is generic). If all the φ_i are C^1, and there is a point p such that $\varphi_i'(p) \neq 1$ for all i, then we show there is an interval I satisfying the conditions above. This gives strong evidence for the conjecture that $R(w,r,s)$ is rational for arbitrary w,r,s.

5. two open questions. One can restrict the analytic quality of a representation; this leads to new phenomena and new questions. Two of the most striking phenomena are as follows.

First, consider representations ρ of F into $\text{PSL}(2,\mathbb{R})$. It is a fact that for all such representations one has

$$\text{rot}^\sim(\rho(ababaabbaBBAABABAbababaabbABBAABABAB)) = 0$$

This can be proved using trace identities and numerical methods.

Question: Is there a nontrivial word $w \in F$ so that $\text{rot}^\sim(\rho(w)) = 0$ for *every* $\rho : F \to \text{Homeo}^+(S^1)^\sim$?

Second, one can consider real analytic representations. It is straightforward to show that for every x in the interior of $X(w,r,s)$ there is a real analytic representation ρ such that $\text{rot}^\sim(\rho(w)) = x$. Moreover, if w is positive, the same result holds even for $x = R(w,r,s)$.

Question: Let $w \in F$ be arbitrary. Is there a C^ω representation ρ so that $\text{rot}^\sim(\rho(w)) = R(w,r,s)$?

A positive answer, together with the proposition about the interval game alluded to above, would go a long way toward proving that $R(w, r, s) \in \mathbb{Q}$ for every w and every r, s.

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Quasi-cocycles on free groups

K. Fujiwara

(joint work with M. Bestvina, K. Bromberg)

We prove a theorem about second bounded cohomology using our earlier construction in [1] of group actions on quasi-trees. For many groups Γ and arbitrary unitary representation ρ of Γ in a (nonzero) uniformly convex Banach space, the vector space $H_b^2(\Gamma; \rho)$ is infinite dimensional. Examples include free groups; torsion-free and non-elementary hyperbolic groups; and non-virtually-abelian mapping class groups of surfaces.

Definitions are in order. Quasi-trees are graphs which are quasi-isometric to simplicial trees. For a unitary representation $\rho : \Gamma \to O(E)$, where E is a normed vector space and $O(E)$ denotes the group of norm-preserving automorphisms of E, $F : \Gamma \to E$ is a quasi-cocycle if there is a constant C such that for any $g, h \in \Gamma$,

$$|F(gh) - F(g) - \rho(g)F(h)| \leq C.$$

If ρ is trivial and $E = \mathbb{R}$, this is a quasi-homomorphism.

Following the construction by Brooks of quasi-homomorphisms on free groups using their actions on Cayley trees, we construct quasi-cocycles on free groups. This construction generalizes if a group acts on a quasi-tree in a certain way, [1]. We use that E is uniformly convex to argue that some of those cocycles are non-trivial in $H_b^2(\Gamma; \rho)$.

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Kähler groups, real hyperbolic spaces and the Cremona group

P. Py
(joint work with T. Delzant)

The purpose of this talk was to explain the results of the article [4].

We recall that a *Kähler group* is by definition the fundamental group of a compact Kähler manifold. It is a classical theorem of Carlson and Toledo [2] that a lattice in the isometry group of the real hyperbolic space $\mathbb{H}^n_{\mathbb{R}}$ is never isomorphic to a Kähler group as soon as $n \geq 3$. In [4], we generalize this result and give a complete description of actions of Kähler groups on finite and infinite dimensional real hyperbolic spaces (for the definition of the infinite dimensional real hyperbolic space, see [1]). In the case of actions of Kähler groups on the infinite dimensional real hyperbolic space $\mathbb{H}^\infty_{\mathbb{R}}$, our result takes the following form. In the following, $\mathrm{Isom}(\mathbb{H}^\infty_{\mathbb{R}})$ denotes the isometry group of the space $\mathbb{H}^\infty_{\mathbb{R}}$.

Theorem 1. Let Γ be a Kähler group. Let $\rho : \Gamma \to \mathrm{Isom}(\mathbb{H}^\infty_{\mathbb{R}})$ be a non-elementary action of Γ on $\mathbb{H}^\infty_{\mathbb{R}}$. Assume that ρ is minimal, i.e. that $\mathbb{H}^\infty_{\mathbb{R}}$ contains no nontrivial closed ρ-invariant totally geodesic subspace. Then, one of the following two cases happens.

(1) The representation ρ factors through a fibration onto a hyperbolic 2-orbifold.
(2) The representation ρ can be written as $\rho = \Psi \circ \theta$, where θ is a homomorphism from Γ to $\mathrm{PSL}_2(\mathbb{R})$ with dense image and $\Psi : \mathrm{PSL}_2(\mathbb{R}) \to \mathrm{Isom}(\mathbb{H}^\infty_{\mathbb{R}})$ is a continuous homomorphism.

We also prove that some irreducible actions of $\mathrm{PSL}_2(\mathbb{R})$ on $\mathbb{H}^\infty_{\mathbb{R}}$ exist: we describe a 1-parameter family of such actions coming from representation theory and give some of their properties (see also [5] for a more detailed study and a classification of these actions). Finally, an application is given to the study of the Cremona group $\mathrm{Bir}(\mathbb{P}^2)$, i.e. the group of all birational maps from the complex projective plane $\mathbb{P}^2$ to itself. Using ideas of Manin and Zariski, Cantat [3] has proved that the Cremona group admits a faithful action on an infinite dimensional real hyperbolic space $\mathbb{H}_{\mathbb{P}^2}$, called the Picard-Manin space. Using this action, he essentially described homomorphisms from Γ to $\mathrm{Bir}(\mathbb{P}^2)$ when Γ is a lattice in a simple Lie group with property T. The problem of describing all faithful homomorphisms from lattices in the groups $\mathrm{SO}(n,1)$ and $\mathrm{SU}(n,1)$ to the Cremona group is open. Concerning lattices in the group $\mathrm{SU}(n,1)$, we give the following partial answer:

Theorem 2. Let Γ_1 be a cocompact lattice in the group $\mathrm{SU}(n,1)$ with $n \geq 2$. If

$$\rho : \Gamma_1 \to \mathrm{Bir}(\mathbb{P}^2)$$

is an injective homomorphism, then one of the following two possibilities holds:

(1) The group $\rho(\Gamma_1)$ fixes a point in the Picard-Manin space $\mathbb{H}_{\mathbb{P}^2}$.
(2) The group $\rho(\Gamma_1)$ fixes a unique point in the boundary of the Picard-Manin space $\mathbb{H}_{\mathbb{P}^2}$.

This result is a particular case of a more general one which applies to all Kähler groups.

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Flexibility and rigidity of symplectic embeddings

Felix Schlenk

(joint work with Janko Latschev and Dusa McDuff)

For $a > 0$ consider the open 4-dimensional ball $B^4(a)$ of capacity $a = \pi r^2$ (where r is the radius), endowed with the standard symplectic form $dx_1 \wedge dy_1 + dx_2 \wedge dy_2$. Given a symplectic 4-manifold (M, ω) of finite volume $\operatorname{Vol}(M, \omega) := \frac{1}{2} \int_M \omega \wedge \omega$ we can ask how much of the volume of (M, ω) can be filled by a symplectically embedded ball. In other words, we study the *packing number*

$$p(M, \omega) \,:=\, \sup \left\{ \frac{\operatorname{Vol}(B^4(a))}{\operatorname{Vol}(M, \omega)} \;\Big|\; B^4(a) \text{ symplectically embeds into } (M, \omega) \right\}.$$

It follows from Darboux's theorem that always $p(M, \omega) > 0$. If $p(M, \omega) = 1$, one says that (M, ω) admits a *full packing* by one ball, and if $p(M, \omega) < 1$, one says that there is a *packing obstruction.* Notice that if we would consider volume preserving embeddings instead, then the packing number would always be 1 (as follows from Moser's method). Packing obstructions thus are one way to measure the difference between symplectic and volume preserving mappings.

Examples 1. Let (M, ω) be the complex projective plane CP^2 endowed with the usual Kähler form ω_{SF}, normalized such that $\int_{\mathrm{CP}^1} \omega_{\mathrm{SF}} = \pi$. Then

$$p(\mathrm{CP}^2, \omega_{\mathrm{SF}}) \,=\, 1.$$

Indeed, $B^4(\pi) \to \mathrm{CP}^2$, $z = (z_1, z_2) \mapsto [\sqrt{1 - |z|^2} : z_1 : z_2]$ is a symplectic embedding.

2. Denote by $S^2(b)$ the two-sphere endowed with an area form of total area b. (Any two such area forms are diffeomorphic by Moser's method.) Then Gromov's Non-squeezing theorem (whose proof we sketched!) shows that $p\big(S^2(100) \times S^2(1)\big) \leq$

$\frac{1}{200}$. On the other hand, the obvious symplectic embedding $B^4(1) \subset B^2(1) \times B^2(1) \hookrightarrow S^2(100) \times S^2(1)$ shows the reverse inequality, so that

$$p\big(S^2(100) \times S^2(1)\big) \;=\; \tfrac{1}{200}.$$

Instead of looking at symplectic packings by one ball, one may, more generally, look at symplectic packings by k equal balls, and study the corresponding *k'th packing number* $p_k(M, \omega)$. For the 4-ball B^4 these numbers were fully determined by Gromov [3], McDuff–Polterovich [6] and Biran [1]:

k	1	2	3	4	5	6	7	8	≥ 9
p_k	1	$\frac{1}{2}$	$\frac{3}{4}$	1	$\frac{20}{25}$	$\frac{24}{25}$	$\frac{63}{64}$	$\frac{288}{289}$	1

In order to better understand these numbers, we look at a problem that interpolates the above problem of packing by k equal balls: For $0 < a_1 < a_2$ consider the open ellipsoid

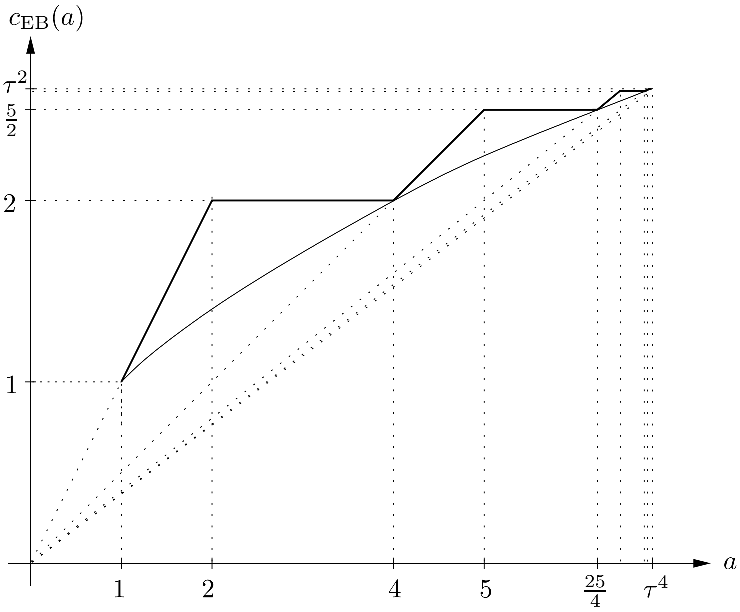
$$E(a_1, a_2) \;:=\; \left\{ (z_1, z_2) \in \mathrm{C}^2 = \mathrm{R}^4 \;\middle|\; \frac{|z_1|^2}{a_1} + \frac{|z_2|^2}{a_2} < 1 \right\}$$

and look for the smallest ball $B^4(A)$ into which $E(a_1, a_2)$ symplectically embeds. We can assume that $a_1 = 1$. We therefore would like to understand the function

$$c_{\mathrm{EB}}(a) \;:=\; \inf \big\{ A \mid E(1, a) \text{ symplectically embeds into } B^4(A) \big\} \,.$$

Since symplectic embeddings are measure preserving, an obvious lower bound for $c_{\mathrm{EB}}(a)$ is $\sqrt{a}$. It is easy to see that $\coprod_{i=1}^k B^4(1)$ symplectically embeds into $E(1, k)$. Therefore, $\coprod_{i=1}^k B^4(1)$ symplectically embeds into $B^4(A)$ whenever $E(1, k)$ symplectically embeds into $B^4(A)$. In [5], Dusa McDuff has shown that the converse is also true. Our ellipsoid embedding problem therefore indeed interpolates the problem of packing by k equal balls. In [7] we have completely determined the function c_{EB}. Let $\tau = \frac{1+\sqrt{5}}{2}$ be the golden ratio. Then the graph of $c_{\mathrm{EB}}(a)$ on $[1, \tau^4]$ is an infinite ladder determined by ratios of consecutive odd Fibonacci numbers (see the figure below). For $a \geq 8\frac{1}{36}$ we have $c_{\mathrm{EB}}(a) = \sqrt{a}$.
A similar result has been recently obtained by David Frenkel and Dorothee Müller, [2], for the embedding problem $E(1, a) \to B^2(A) \times B^2(A)$.

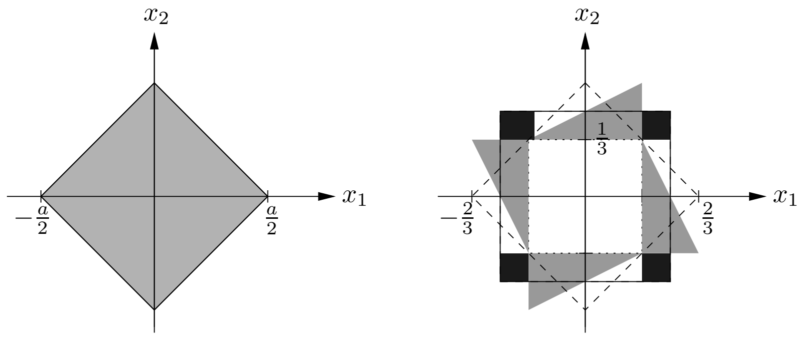
In all the above embedding problems the obstructions to full packings come from holomorphic spheres in (M, ω). It turns out that such spheres not only provide all the obstructions, but can also be used to construct "maximal" embeddings. If there are no holomorphic spheres in (M, ω), one may suspect that there are no packing obstructions. A test case for this is the the 4-torus $T^4 = \mathrm{R}^4/\mathrm{Z}^4$. We have shown in [4] that this torus (and, more generally, any 4-torus with a linear symplectic form) can be fully packed by any given collection of balls (or ellipsoids). In the sequel, we focus on explicitely constructing a full packing of T^4 by one ball. (All other cases can be reduced to known packing results.) The symplectic embedding



$B^4(1) \subset B^2(1) \times B^2(1) \to (0,1)^2 \times (0,1)^2 \to T^4$ shows that $p(T^4) \geq \frac{1}{2}$. It was known to some algebraic geometers that $p(T^4) \geq \frac{8}{9}$. The starting point for a full packing is to notice that $B^4(a)$ is symplectomorphic to the Lagrangian(!) product $\Diamond(a) \times \Box$, where $\Diamond(a) \subset \mathrm{R}^2(x)$ is the "diamond" of size a,

$$\Diamond(a) \;:=\; \left\{(x_1, x_2) \in \mathrm{R}^2 \mid |x_1| + |x_2| < \tfrac{a}{2}\right\} \;\subset\; \mathrm{R}^2(x),$$

see Figure (I), and $\Box \subset \mathrm{R}^2(y)$ is the square $\left\{(y_1, y_2) \in \mathrm{R}^2 \mid 0 < y_1, y_2 < 1\right\}$.



(I) (II)

Let $f \colon \mathrm{R} \to \mathrm{R}$ be a smooth function, and consider the x_1-shear $\varphi(x_1, x_2) = (x_1 + f(x_2),\, x_2)$ of R^2. Then the diffeomorphism

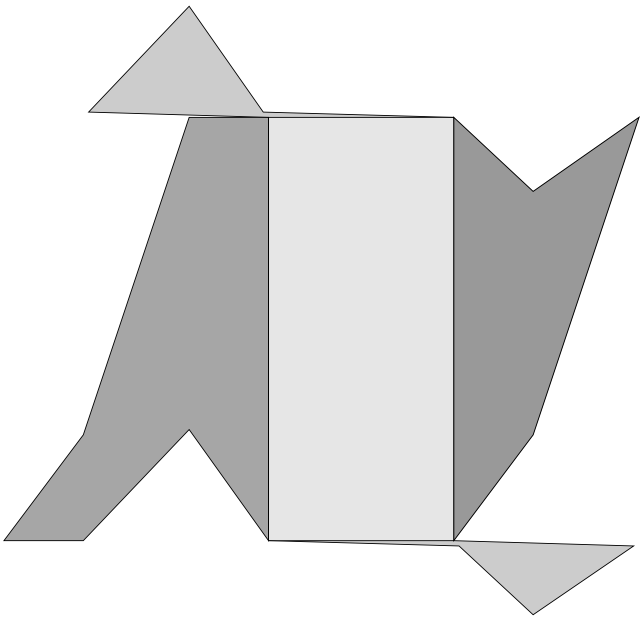
$$\widehat{\varphi}(x_1, x_2, y_1, y_2) \;=\; \big(x_1 + f(x_2),\, x_2,\, y_1,\, y_2 - f'(x_2)\, y_1\big)$$

is a symplectomorphism of $\mathbb{R}^4$. Indeed, this is just the "cotangent map"

$$(x_1, x_2, y_1, y_2) \mapsto \Big(\varphi(x_1, x_2), \big(d\varphi(x_1, x_2))^T\big)^{-1}(y_1, y_2)\Big)$$

of the shear φ. Now observe that if φ is such that $\varphi(\Diamond(a))$ projects injectively to $\mathbb{R}^2(x)/\mathbb{Z}^2$, then $\widehat{\varphi}(\Diamond(a) \times \Box)$ projects injectively to $\mathbb{R}^4/\mathbb{Z}^4$. Indeed, the image $\widehat{\varphi}(\Diamond(a) \times \Box)$ fibers over $\varphi(\Diamond(a))$, with fiber over $\varphi(x_1, x_2)$ the sheared square $\{(y_1, y_2 - f'(x_2)y_1) \mid (y_1, y_2) \in \Box\}$. For instance, taking f "piecewise-linear", we can x_1-shear the "top and bottom triangle" and x_2-shear the "left and right triangle" of $\Diamond(\frac{4}{3})$ as in Figure (II), proving that $p(T^4) \geq \frac{8}{9}$.

To obtain a full filling of T^4 by one ball $B^4(\sqrt{2}) \cong \Diamond(\sqrt{2}) \times \Box$, we start from a "distorted diamond", and shear it to the shape shown in the figure below.



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Poincaré inequalities under lower Ricci curvature bound

Gilles Courtois
(joint work with Gérard Besson and Sa'ar Hersonsky)

The following is a report on a joint work in progress with Gérard Besson and Sa'ar Hersonsky.

A lower bound on Ricci curvature has many implications on the global geometry and geometric analysis of a complete Riemannian manifold. For example non negative Ricci curvature implies a "Poincaré inequality" which was a key tool in the resolution by T. Colding and Minicozzi of a conjecture of S.T.Yau stating that the space of harmonic functions of polynomial growth of degre d is finite dimensional, [2].

A metric measured space (X, d, μ) is said to satisfy a p-Poincaré inequality if there exists positive constants C and C' such that for any Lipchitz function $u : X \to \mathbb{R}$, any $x \in X$ and any $R > 0$, we have

$$\int_{B(x,R)} |u - u_{B(x,R)}|^p d\mu \leq CR^p \int_{B(x,C'R)} |\nabla u|^p d\mu, \tag{1}$$

where $B(x, R)$ is the ball of center x and radius R, and $u_{B(x,R)}$ is the mean value of u on the ball $B(x, R)$. When there is no Lipschitz structure on X, the gradient $|\nabla u|$ should be replaced by an "upper gradient" of u that is, any non negative function $\rho : X \to \mathbb{R}_+$ such that $|u(x) - u(y)| \leq \int_{\gamma_{x,y}} \rho(\gamma_{x,y}(s))ds$ for any pair of points $x\,, y$ and any rectifiable curve $\gamma_{x,y}$ joining x and y. Note that if (X, d, μ) has no rectifiable curves, then there is no Poincaré inequality.

Examples: $\mathbb{R}^n$, $\mathbb{Z}^n$, nilpotent Lie groups, Cayley graph of finitely generated nilpotent groups, complete Riemannian manifolds of non negative Ricci curvature.

Non examples: Non abelian free groups, $\mathbb{H}^n$.

In [1] B. Kleiner proves a general Poincaré inequality for the Cayley graph of a finitely generated group:

$$\int_{B(R)} |u - u_{B(R)}|^2 d\mu \leq CR^2 \frac{V(2R)}{V(R)} \int_{B(C'R)} |\nabla u|^2 d\mu, \tag{2}$$

where $V(R)$ is the volume of a ball of radius R. This Poincaré inequality was an essential step toward the B.Kleiner's new proof of the Gromov's theorem saying that finitely generated group of polynomial growth are virtually nilpotent, [5]. This inequality together with a polynomial growth assumption leads to a 2-Poincaré inequality arguing that the ratios $\frac{V(2R)}{V(R)}$ are bounded. Note that in the proof of the above Kleiner's inequality, the homogeneity of the space X (coming from the transitive action of the group on its Cayley graph) is fundamental. These two properties (homogeneity and polynomial growth) are in a way fullfilled on complete

Riemannian manifolds of non negative Ricci curvature. This raises the question of the existence of p-Poincaré inequalities on complete Riemannian manifolds with Ricci curvature bounded from below and polynomial growth.

Consider an n-dimensional Riemannian manifold X with Ricci curvature bounded below $Ric \geq -\kappa^2$ and with α-polynomial growth $\mathrm{vol}B(x,R) \leq CR^\alpha$.

Theorem 1. Let (X^n, g) be a complete Riemannian manifold with Ricci curvature bounded below and α-polynomial growth. Then there exists a constant $C = C(n, \kappa)$ such that for any $r_0 > 0$ and any $p \geq 1$, there exists a constant $K = K(n, p, r_0)$ such that for any $u \in C^1(M^n)$, any $R \geq r_0$ and any ball $B(m, R) \subset M^n$, we have

$$\int_{B(m,R)} |u - u_{B(m,R)}|^p dx \leq KR^{\alpha+p-1} \int_{B(m,CR)} |\nabla u|^p dx. \tag{3}$$

Moreover this inequality is sharp.

The authors learnt from L. Saloff-Coste that although not stated in the litterature, the theorem follows from his work with T. Coulhon cf. [3], [4].

The sharpness in this Theorem answers by the negative to the above question and is a consequence of the following construction. Consider the planar graph G with quadratic growth $\alpha = 2$ whose vertex set is defined by $V = \{(m,n) \,/\, m\,, n \in \mathbb{Z}\}$ and whose edges are either vertical segments joining $(0,n)$ and $(0,n+1)$ or horizontal segments joining (m,n) and $(m+1,n)$, m, $n \in \mathbb{Z}$. We then embed the graph G in $\mathbb{R}^3$ and for $\epsilon > 0$ small enough the set X of points in $\mathbb{R}^3$ at distance from G equal to ϵ is a complete Riemannian manifold of quadratic growth and bounded curvature. It is easy to see that the term $R^{\alpha+p-1} = R^{p+1}$ in the above theorem cannot be promoted to R^p and therefore no p-Poincaré inequality holds on X.

This theorem raises a second question: *under which additional assumption does the Riemannian manifold M satisfy a p-Poincaré inequality?*

One example we are particularly interested in is the following. Consider a $(n+1)$-dimensional closed negatively curved manifold M. The universal cover $\tilde{M}$ of M is diffeomorphic to $\mathbb{R}^{n+1}$ and its boundary $\partial\tilde{M}$, namely the set of equivalence classes of geodesic rays staying at bounded distance of each other is homeomorphic to the standard sphere S^n. A horosphere of $\tilde{M}$ is the limit of a sequence of spheres of radius R tending to infinity and whose centers converge to a boundary point of $\partial\tilde{M}$. Each horosphere is a smooth hypersurface X of $\tilde{M}$ which is diffeomorphic to $\mathbb{R}^n$. Endowed with the *intrinsec metric* each horosphere X of $\tilde{M}$ is a complete Riemannian manifold which satisfies: 1) X has polynomial growth, 2) X has bounded curvature, in particular X has Ricci curvature bounded from below. A motivation for studying horospheres is that they are a kind of Riemannian counterpart of the boundary $\partial\tilde{M}$ endowed with the "Gromov distance" d_G. The Gromov distance d_G on $\partial\tilde{M}$ depends on the negatively curved Riemannian metric

g lying on M. In particular, when M has dimension 3, the measured metric space $(\partial\tilde{M}, d_G, \mu)$, where μ is the Hausdorff measure of d_G, has no rectifiable curves and therefore does not satisfy the 2-Poincaré inequality unless the Riemannian metric g on M is hyperbolic, cf. [1].

The horospheres of $\tilde{M}$ are identified with the strong stable foliation of the geodesic flow of the compact manifold M and therefore inherits some homogeneity. It then sounds reasonable to ask: *do horospheres X of $\tilde{M}$ endowed with the induced intrinsec Riemannian metric satisfy p-Poincaré inequalities?*. And if this is the case, *are there uniform p-Poincaré inequalities on the family of all horospheres?*

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On bi-invariant word metrics on groups

Jarek Kędrak

(joint work with M. Brandenbursky, Ś. Gal)

Let G be a group generated by a set $S \subset G$.

The associated word norm is defined by

$$\|g\| := \min\{k \in \mathbf{N} \mid g = s_1 \dots s_k, \text{ where } s_i \in S\}.$$

The corresponding metric is defined by $\mathbf{d}(g,h) := \|gh^{-1}\|$. It is right-invariant and if the set S is invariant under conjugations then it is bi-invariant. A group is called normally generated by a set S if it is generated by the set $\overline{S}$ consisting of elements from S and their conjugates. It is not difficult to show that if a group G is finitely normally generated then the boundedness of the associated bi-invariant word metric implies boundedness of any bi-invariant metric on G. In my talk I presented two theorems, one about word metrics on some finitely generated groups and another about autonomous metric on the group of area preserving diffeomorphisms of the two dimensional disc.

Finitely generated Chevalley groups.

Let $\mathcal{O}_{\mathsf{V}} \subset \mathfrak{K}$ be a ring of V-integers in a number field $\mathfrak{K}$, where V is a set of valuations containing all Archimedean ones. Let $G_\pi(\Phi, \mathcal{O}_{\mathsf{V}})$ be the Chevalley group associated with a faithful representation $\pi\colon \mathfrak{g} \to \mathfrak{gl}(V)$ of a simple complex Lie algebra of rank at least two.

Theorem 1. (joint with Ś. Gal [1]**)**
Let Γ *be a finite extension or a supergroup of finite index of the Chevalley group* $G_\pi(\Phi, \mathcal{O}_V)$. *Then any bi-invariant metric on* Γ *is bounded.*

Area preserving diffeomorphisms of the disc.
It is a well known fact that every smooth compactly supported and area-preserving diffeomorphism of the open unit disc $\mathbf{D}^2$ is a composition of finitely many autonomous diffeomorphisms. We define the *autonomous metric* on the group $\mathrm{Diff}(\mathbf{D}^2, \mathrm{area})$ of smooth compactly supported area-preserving diffeomorphisms of the disc to be the word metric associated with the set of autonomous diffeomorphisms.

Theorem 2. (joint with M. Brandenbursky [2]**)**
For every natural number $k \in \mathbf{N}$ *there exists an injective homomorphism* $\mathbf{Z}^k \to \mathrm{Diff}(\mathbf{D}^2, \mathrm{area})$ *which is bi-Lipschitz with respect to the word metric on* $\mathbf{Z}^k$ *and the autonomous metric on* $\mathrm{Diff}(\mathbf{D}^2, \mathrm{area})$.

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Free subgroups of arithmetic 3-manifold groups

M. Belolipetsky

A group Γ is called *k-free* if any subgroup of Γ generated by k elements is free. We denote the maximal k for which Γ is k-free by $\mathcal{N}_{fr}(\Gamma)$. In [Gr87, Section 5.3.A], Gromov stated that $\mathcal{N}_{fr}(\Gamma)$ of a δ-hyperbolic group Γ is bounded below by an exponential function of the systole (or injectivity radius) of the associated quotient space M. The details of the proof were not given in [Gr87], they can be found in a later paper [Gr09, Section 2.4] where it is pointed out that the argument gives only a bound of the form $\epsilon r/\log(r)$, $r = \mathrm{sys}_1(M)$. Two other proofs of the growth of $\mathcal{N}_{fr}(\Gamma)$ when $\mathrm{sys}_1(M) \to \infty$ appear in [Ar06] and [KW03], but the quantitative bounds for $\mathcal{N}_{fr}(\Gamma)$ which can be deduced from these papers are weaker than the one above: Arzhantseva gives a bound of the form $cr^{1/3}$, Kapovich and Weidmann do not present an explicit estimate but the method of their paper would not allow to produce a considerably better bound. Thus so far Gromov's estimate appears to be the best available general quantitative result about $\mathcal{N}_{fr}(\Gamma)$. Although the difference between sub-linear and exponential growth is very large, in [Gr09, p. 763] Gromov conjectured that the true bound should be exponential. He pointed out that this is not known even for the fundamental groups of hyperbolic 3-manifolds.

The main purpose of this talk is to discuss a result confirming Gromov's conjecture in this important special case.

Theorem 1. Let M be an arithmetic hyperbolic 3-manifold defined by a quadratic form and $M_i \to M$ be a sequence of its congruence covers. Then

$$\log \mathcal{N}_{fr}(\pi_1(M_i)) \gtrsim \frac{1}{3}\mathrm{sys}_1(M_i), \text{ as } i \to \infty.$$

Remark. It was pointed out to me by Ilya Kapovich that the results of [KW03] imply that the free subgroups provided by Theorem 1 are quasiconvex in $\pi_1(M_i)$.

The principal ingredient of the proof of the theorem is the following result of an independent interest. Let $\mathrm{sysg}(M)$ denote the minimal genus of a surface subgroup of $\pi_1(M)$, which we call the *systolic genus* of M.

Theorem 2. Let M be a closed hyperbolic 3-manifold. For any $\epsilon > 0$, assuming that the systole $\mathrm{sys}_1(M)$ is sufficiently large, we have

$$\mathrm{sysg}(M) \geq e^{(\frac{1}{2}-\epsilon)\mathrm{sys}_1(M)}.$$

In particular, given a sequence of closed hyperbolic 3-manifolds M with $\mathrm{sys}_1(M) \to \infty$, we have

$$\log \mathrm{sysg}(M) \gtrsim \frac{1}{2}\mathrm{sys}_1(M).$$

The proof of Theorem 2 is based on Thurston's inequality bounding the area of a π_1-injective surface in a hyperbolic 3-manifold through its genus and an important Gromov's systolic inequality for surfaces of high genus, in which we use a numerical value of the constant obtained by Katz–Sabourau. The application of these ingredients is supported by the results from the theory of minimal surfaces of Schoen–Yau and Sacks–Uhlenbeck. We refer to [Be12, Section 2] for the details of the proof and precise references.

In order to relate Theorem 2 and $\mathcal{N}_{fr}(M)$, we recall the following result of Baumslag and Shalen [BaSh89]:

Theorem 3. Let M be an irreducible, closed orientable 3-manifold, and let k be a positive integer. Suppose that $\pi_1(M)$ has no subgroup isomorphic to $\pi_1(S_g)$ for any g with $0 < g < k$, and that $\beta_1(M) > k$. Then $\pi_1(M)$ is k-free.

By the work of Xue [Xue92], if M is an arithmetic hyperbolic 3-manifold whose group is defined by a quadratic form and $M_i \to M$ is a sequence of its congruence covers, then $\log \beta_1(M_i) \gtrsim \frac{1}{3}\log \mathrm{vol}(M_i)$. The analogues results about Betti numbers are known also for some other families of arithmetic hyperbolic 3-manifolds and are conjectured to be true in general (see [Be12] for more details). In order to finish the proof of Theorem 1, it remains to recall that the systole of the congruence covers of an arithmetic hyperbolic 3-manifold grows at least as fast as $\frac{2}{3}\log \mathrm{vol}(M_i)$ [KSV07] and to combine all these facts together with Theorem 2.

Most of the above mentioned results hold for higher dimensional hyperbolic manifolds and their congruence covers, and we expect Theorem 1 to be true in higher dimensions as well. The principal problem in obtaining such a generalisation is with Theorem 3 whose proof in [BaSh89] is essentially 3-dimensional.

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Coarse geometry of Hofer's metrics on Hamiltonian diffeomorphism groups and on spaces of submanifolds

Michael Usher

For any symplectic manifold (M,ω), the group $Ham(M,\omega)$ of time-one maps of the Hamiltonian vector fields of compactly supported smooth functions $H : [0,1]\times M \to \mathbb{R}$ admits a remarkable bi-invariant metric discovered by Hofer [Ho]. Where for any such function H we denote by $\phi_H^1 : M \to M$ the corresponding time-one map, this metric is defined by setting $d(\phi,\psi) = \|\phi^{-1}\circ\psi\|$ where in general

$$\|\phi\| = \inf\left\{ \int_0^1 \left(\max_M H(t,\cdot) - \min_M H(t,\cdot)\right) dt \,\middle|\, \phi_H^1 = \phi \right\}.$$

It is a rather deep result (proven for general symplectic manifolds by Lalonde and McDuff [LM]) that this metric is nondegenerate, and to this day we have only a rather limited understanding of its large-scale properties. For instance, it is currently unknown whether the metric is unbounded for all closed symplectic manifolds, though it follows from work of Ostrover based on Floer-theoretic spectral invariants that the associated pseudometric on the universal cover of $Ham(M,\omega)$ is always unbounded. Consideration of the behavior of these invariants under monodromy sometimes allows one to use Ostrover's argument to deduce unboundedness for the original group, as has been exploited by Schwarz [S], Entov–Polterovich [EP], and McDuff [M] on manifolds including complex projective spaces and tori.

However McDuff's work also made clear that this strategy would not apply to some manifolds, such as blowups of other symplectic manifolds.

This talk discussed an application of a newer Floer-theoretic invariant developed by the speaker in [U11a], called the boundary depth, which gives a different method of providing lower bounds for the Hofer metric. Since the boundary depth is defined on $Ham(M,\omega)$ and not just on its universal cover, problems with monodromy do not arise. One application in [U11b] asserts that if a closed symplectic manifold admits a nontrivial Hamiltonian vector field all of whose contractible periodic orbits are constant, then the Hofer metric is unbounded, and in fact there are quasi-isometric monomorphisms $\Phi : V \to Ham(M,\omega)$ of infinite-dimensional normed vector spaces into $Ham(M,\omega)$ with respect to the Hofer metric. A bit more specifically, if $H : M \to \mathbb{R}$ is a function whose Hamiltonian vector field has the property indicated above, then such a monomorphism $\Phi : V \to Ham(M,\omega)$ is obtained by setting $\Phi(f) = \phi^1_{f\circ H}$, where V is a suitable vector space of smooth functions $f : \mathbb{R} \to \mathbb{R}$ equipped with the uniform norm.

The Hofer metric on $Ham(M,\omega)$ induces an invariant pseudometric δ on the orbit $\mathcal{L}(N)$ of any closed submanifold N under the group, by setting $\delta(N_1,N_2) = \inf\{\|\phi\| | \phi(N_1) = N_2\}$. It is easy to see that this pseudometric vanishes if N is a point, and a rather more subtle result of the speaker in [U12] shows that the pseudometric vanishes for a wide class of N, including images of generic closed embeddings of codimension at least two. On the other hand, Chekanov showed [C] that if N is a compact Lagrangian submanifold of a tame symplectic manifold then the pseudometric is nondegenerate. In contrast to the metric of $Ham(M,\omega)$, which is unbounded in all known cases, the metric on $\mathcal{L}(N)$ for Lagrangian N can be bounded, as observed in [U11b] in the case when N is a circle and M is the plane. At the other extreme, it was shown in [U11b] that if (M,ω) is a closed symplectic manifold admitting a Hamiltonian vector field as in the previous paragraph, then where $N \subset (M\times M, -\omega\oplus\omega)$ is the diagonal, $\mathcal{L}(N)$ admits quasi-isometrically embedded infinite-dimensional normed vector spaces (consisting of graphs of the diffeomorphisms that make up the quasi-isometrically embedded infinite-dimensional normed vector spaces that were constructed inside $Ham(M,\omega)$).

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Telescopic actions

D. Panov
(joint work with A. Petrunin)

This talk is based on paper [3] and is motivated by the following question of Gromov [2]:

Question. *Let M^n be an arbitrary compact PL manifold without boundary. Is there a co-compact lattice Γ in $O(n,1)$ such that the quotient $\mathbb{H}^n/\Gamma$ is PL isomorphic to M^n?*

Remark. It is important that Γ can have torsion, otherwise the quotient $\mathbb{H}^n/\Gamma$ is a compact hyperbolic manifold and so can never be a sphere. On the other hand, if Γ is allowed to have torsion, then one can get sphere in dimensions $2, 3, ..., 8$. Indeed in these dimensions there exist compact Coxeter hyperbolic polytopes and a sphere is obtained by doubling such a polytope along its boundary.

In our work we consider a variation of Gromov's question. We call an isometric co-compact properly discontinuous group action H on X *telescopic* if for any finitely presented group G, there exists a subgroup H' of finite index in H such that G is isomorphic to the fundamental group of X/H'. The following theorem is our main result.

Theorem 1. *The exist telescopic actions on* 3 *and* 4 *-dimensional hyperbolic spaces.*

A direct application of this theorem is the following statement:

Aitchison's statement. *Every finitely presented group G can appear as the fundamental group of M/J, where M is a closed 3-manifold and J is an involution which has only isolated fixed points.*

Another application proven together with Joel Fine [1] is the following:

Theorem 2. *For any finitely presented group G there exists a compact symlectic six-manifold with $c_1 = 0$ and with fundamental group equal G.*

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Conjugate points and radial noninjectivity of the exponential map

K. Burns
(joint work with B. Schmidt)

Two points p and q in a Riemannian manifold M are *conjugate* if there is a vector $v \in T_pM$ such that $\exp_p(v) = q$ and the derivative of $\exp_p$ is singular at v. It has long been known that if $D\exp_p$ is singular at v, then $\exp_p$ is *locally noninjective at* v. This means that any neighbourhood of v in T_pM contains vectors v' and v'' such that $\exp_p(v') = \exp_p(v'')$. Proofs were given by Littauer and Morse [LM32] in the real analytic case and Savage [Sa43] in the smooth case. Warner [Wa65] extended the results to a general class of maps that includes Riemannian and Finslerian exponential maps.

Schmidt and I have sharpened this classical result by showing that the noninjectivity of $\exp_p$ in the neighbourhood of a singularity at v can be observed along the ray $\mathbb{R}^+v$. Thus if $D\exp_p$ is singular at v, then any neighbourhood of v in T_pM contains vectors v' and v'' such that $\exp_p(v') = \exp_p(v'')$ *and* v' *is a multiple of* v. We call this property *radial noninjectivity.*

We have so far made two applications of radial noninjectivity. The first is to the study of a conjectured characterization of the compact rank one symmetric spaces (CROSSes). Lafont and Schmidt [LaSc07] showed that all CROSSes have the property that if p and q are distinct points whose distance is less than the diameter, then it is possible to find two points b_1 and b_2 (distinct from p and q) with the property that every geodesic from p to q passes through b_1 or b_2. They conjectured in the same paper that this property holds only for CROSSes. Schmidt and I cannot prove this conjecture, but we have reduced it to the well known Blaschke conjecture that the CROSSes are the only manifolds in which the diameter is equal to the injectivity radius.

Our second application of radial noninjectivity is to show that a null homotopic closed geodesic in a compact Riemannian manifold must have a *proper chord.* A proper chord is a geodesic segment that joins two distinct points on the closed geodesic and passes through points not on the closed geodesic. As a corollary to this result, we are able to show that the projective plane with constant curvature is the only compact Riemannian surface containing a closed geodesic with no proper chords. It is conjectured that the analogous result holds in higher dimensions.

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The discriminant metric on the contactomorphism group

S. Sandon
(joint work with V. Colin)

Let $\big(M, \xi = \ker(\alpha)\big)$ be a contact manifold, and consider a contactomorphism ϕ. A point q of M is called a *translated point* of ϕ (with respect to the contact form α) if $\phi(q)$ and q belong to the same Reeb orbit and if moreover the contact form α is preserved at q. A translated point of ϕ which is also a fixed point is called a *discriminant point* (in contrast to the notion of translated point, this notion does not depend on the choice of the contact form α). As it is discussed in [S11b], translated points seem to satisfy an analogue of the Arnold conjecture: at least in the case of a $\mathcal{C}^0$-small contact isotopy $\{\phi_t\}$ (but possibly also in the general case) every ϕ_t always has translated points, at least as many as the minimal number of critical points of a function on M. On the other hand, discriminant points do not necessarily exist in general. Given a contact is otopy $\{\phi_t\}_{t\in[0,1]}$ it is thus possible, after perturbing it in the same homotopy class with fixed endpoints, to find an integer N and a subdivision $0 = t_0 < t_1 < \cdots < t_{N-1} < t_N = 1$ such that for all $i = 0, \cdots, N-1$ and all $t \in [t_i, t_{i+1}]$ the contactomorphism ${\phi_{t_i}}^{-1} \circ \phi_t$ has no discriminant points. The minimal N for which we have such a subdivision is called the *discriminant length* of the element of the universal cover $\widetilde{\mathrm{Cont}_0}(M, \xi)$ of the identity component of the contactomorphism group which is represented by the contact isotopy $\{\phi_t\}_{t\in[0,1]}$. For any contact manifold (M, ξ), this notion gives rise to a (non-degenerate) bi-invariant metric on $\widetilde{\mathrm{Cont}_0}(M, \xi)$.

The problem of understanding for which contact manifolds the discriminant metric is unbounded seems to be related to other contact rigidity phenomena such as contact non-squeezing, orderability of contact manifolds and the existence of quasimorphisms on the contactomorphism group. Our results are that the discriminant metric is bounded for the standard contact structures on the Euclidean space $\mathbb{R}^{2n+1}$ and on the sphere S^{2n+1}, but unbounded for the induced contact structures on the quotients $\mathbb{R}^{2n} \times S^1$ and $\mathbb{R}P^{2n+1}$. Unboundedness for $\mathbb{R}^{2n} \times S^1$ is proved using the spectral invariants defined in [S11a], and relies crucially on the 1-periodicity of the Reeb flow. This might suggest that the discriminant metric should always be unbounded whenever there is a 1-periodic Reeb flow, but the case of S^{2n+1}, where the discriminant metric is bounded, shows that this is not true in general. Unboundedness on $\mathbb{R}P^{2n+1}$ is proved using generating functions in the setting developed by Givental [Giv90], and relies on the properties of the cohomological length of subsets of projective spaces.

Another interesting question is to understand the relation between the discriminant metric and the notion of orderability, that was introduced by Eliashberg and Polterovich in [EP00]. Recall that a contact isotopy is called positive if it moves every point in a direction which is positively transverse to the contact distribution.

This notion gives rise to a relation $\leq$ on $\widetilde{\mathrm{Cont}_0}(M,\xi)$, which is always reflexive and transitive but not necessarily anti-symmetric. If $\leq$ is also anti-symmetric, and thus defines a partial order on $\widetilde{\mathrm{Cont}_0}(M,\xi)$, then the contact manifold (M,ξ) is called *orderable*. Assuming that (M,ξ) is orderable, it is not clear to us whether the discriminant metric is compatible with the partial order $\leq$. Motivated by this question we define a second bi-invariant metric on $\widetilde{\mathrm{Cont}_0}(M,\xi)$, that we call the *oscillation metric*. We first notice that every element of $\widetilde{\mathrm{Cont}_0}(M,\xi)$ can be represented by a contact isotopy $\{\phi_t\}_{t\in[0,1]}$ which is the concatenation of a finite number of pieces $\{\phi_t\}_{t\in[t_i,t_{i+1}]}$, $i=0,\cdots,N-1$, such that each piece is either positive or negative and moreover for each $t\in[t_i,t_{i+1}]$ the contactomorphism $\phi_{t_i}^{-1}\circ\phi_t$ has no discriminant points. We then define the oscillation length of $[\{\phi_t\}]$ to be the minimal number of positive pieces plus the minimal number of negative pieces. This notion gives rise to a bi-invariant metric on $\widetilde{\mathrm{Cont}_0}(M,\xi)$, which is non-degenerate if and only if (M,ξ) is orderable, and in this case is compatible with the partial order $\leq$.

It would be important to understand the relation between the discriminant and oscillation metrics and the other (integer-valued) bi-invariant metrics on $\widetilde{\mathrm{Cont}_0}(M,\xi)$ that has been recently defined in [S10] for $\mathbb{R}^{2n}\times S^1$, by Zapolsky [Zap12] for $T^*B\times S^1$ with B closed, and by Fraser, Polterovich and Rosen [FPR] and Albers and Merry [AM] for more general classes of contact manifolds with 1-periodic Reeb flow.

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How "mixing" can an area preserving disk map be when entropy is zero?

Barney Bramham

Consider a C^∞-smooth diffeomorphism of the closed 2-disk $D = \{(x, y) \in \mathbb{R}^2 | x^2 + y^2 \leq 1\}$ with the following properties:

$$(1) \qquad \begin{cases} f \in \mathrm{Diff}^\infty(D) \\ f \text{ is area preserving} \\ f \text{ has zero topological entropy} \end{cases}$$

Thus f defines a Hamiltonian dynamical system on the disk with orbit $\{x, f(x), f^{(2)}(x), \ldots\}$ for each initial condition $x \in D$. The zero entropy condition puts strong restrictions on the "complexity" of the global orbit structure. Given this restriction how "complex" can such a system be? For example, can f have a dense orbit? Even without the zero entropy assumption this is not an easy question to answer. Back in 1930 Shnirelman constructed a (non-area-preserving) homeomorphism of the disk with a dense orbit [10].

In the area preserving case generic conditions on an elliptic periodic point will ensure that there exist invariant circles about this point (KAM theory). This is potentially a delicate obstruction to orbit travel.

Nevertheless, as we will see in a moment, the answer to the question is yes. That is, there exists a transformation f satisfying conditions (1), that has a dense orbit on the disk. This result is due to Anosov and Katok. Infact they proved a much stronger statement. Recall that for a transformation that is ergodic with respect to Lebesgue measure almost every point x is the intial condition for a dense orbit. Still stronger notions in this direction are "weak-mixing" and "mixing" transformations (still with respect to area). We can summarize this hierarchy as follows:

$$\left\{ \begin{matrix} \text{ergodic} \\ \text{transformations} \end{matrix} \right\} \subset \left\{ \begin{matrix} \text{weak-mixing} \\ \text{transformations} \end{matrix} \right\} \subset \left\{ \begin{matrix} \text{mixing} \\ \text{transformations} \end{matrix} \right\}.$$

The foundational result is the following.

Theorem 1. [Anosov-Katok, 1970, [1]] There exists an ergodic, and even weak-mixing, C^∞-smooth area preserving diffeomorphism of the closed 2-disk $D = \{(x, y) \in \mathbb{R}^2 | x^2 + y^2 \leq 1\}$ with zero topological entropy.

No mixing examples were found and their possible existence remains an open question. For surfaces with genus mixing can occur, see [4] and references within. The following is stated as Problem 3.1 in [4]:

Question 1. Does there exist a *mixing* area preserving C^∞-diffeomorphism of the closed 2-disk with zero topological entropy?

In [3] we prove a result in the direction of non-mixing for disk maps. An interesting novelty is that we use methods from symplectic geometry.

It is helpful to first recall the developments since the work of Anosov and Katok.

If $f : D \to D$ is a transformation satisfying the conditions in (1) and has a dense orbit then it turns out that f must be conjugate (via a C^∞-area preserving transformation) to a so called irrational pseudo-rotation[1]. We will therefore restrict our attention to these maps:

Definition 1. A *pseudo-rotation* with rotation number $\alpha \in \mathbb{R}/\mathbb{Z}$ is a C^∞-smooth diffeomorphism $f : D \to D$ sharing the following properties with the rigid rotation $z \mapsto e^{2\pi i\alpha} z$:

(1) f preserves area,
(2) $f(0) = 0$,
(3) For each $x \in D\backslash\{0\}$ the average rotation number of the orbit $\{x, f(x), f^{(2)}(x), \dots\}$ about the origin is α. [2]

If α is irrational then f is said to be an *irrational pseudo-rotation.*

For a precise definition of average rotation number of an orbit see [2] or the discussion in Le Calvez' survey [9].

Remark. Irrational pseudo-rotations always have zero topological entropy. This follows from Katok's formula in [8] which bounds the entropy from above by the exponential growth rate of periodic orbits.

Let $\mathcal{A}_\alpha$ denote the set of all pseudo-rotations with rotation number $\alpha \in \mathbb{R}/\mathbb{Z}$. Whether $\mathcal{A}_\alpha$ contains a transformation with "mixing" properties turns out to be closely related to the arithmetic properties of α, in particular whether α is Liouvillean. Recall that an irrational number $\alpha \in \mathbb{R}/\mathbb{Z}$ is said to be a Liouville number if for all $k \in \mathbb{N}$ there exists $(p, q) \in \mathbb{Z} \times \mathbb{N}$ relatively prime for which

$$\left|\alpha - \frac{p}{q}\right| < \frac{1}{q^k}. \tag{2}$$

Let $\mathcal{L} \subset \mathbb{R}/\mathbb{Z}$ be the set of all Liouville numbers.

The Anosov-Katok construction actually showed that the set $\mathcal{A}_\alpha$ contains a weak-mixing transformation for each α in a dense subset of $\mathcal{L}$. Later Herman proved that if $\alpha \notin \mathcal{L}$, in other words if α is Diophantine, then each element of $\mathcal{A}_\alpha$ has invariant circles near the boundary of the disk and therefore cannot even have a dense orbit. This was unpublished but follows also from [5]. In 2005 Fayad and Saprykina [6] generalized the Anosov-Katok theorem and established that for *all* $\alpha \in \mathcal{L}$ there exists a weak-mixing transformation in $\mathcal{A}_\alpha$, thus completely answering the question of when $\mathcal{A}_\alpha$ contains a weak-mixing element for α irrational.

[1]This statement does not seem to appear in the literature. Infact there are remarks in [4], see pages 1485-1486, that suggest at least the possibility of other examples, besides irrational pseudo-rotations, of zero entropy disk maps that are ergodic with respect to area. However recent strong results in [7], namely Corollary (1.6), would rule out this possibility.

[2]In some places in the literature when discussing pseudo-rotations in the C^0-category, it is stated that the rotation number of the orbit from x may only be well defined for x almost everywhere. However for smooth maps it is defined at every point in $D\backslash\{0\}$; this is proven in [7] and indeed also follows easily from the techniques used to prove theorem (3) below.

In [3] we show the following.

Theorem 2.[3] There exists a dense subset of the Liouville numbers $\mathcal{L}_* \subset \mathcal{L}$ with the following property. If $\alpha \in \mathcal{L}_*$ then there is no mixing transformation in $\mathcal{A}_\alpha$.

Regarding Question 1 this reduces the search for smooth mixing Hamiltonian disk maps with zero topological entropy to irrational pseudo-rotations in $\mathcal{A}_\alpha$ for $\alpha \in \mathcal{L}\backslash\mathcal{L}_*$. The set $\mathcal{L}_*$ is explicitely defined in [3].

The proof of Theorem 3 uses techniques from symplectic geometry. Roughly speaking pseudoholomorphic curves are used to find a sequence of periodic disk maps $f_n : D \to D$ which closely approximate any given element $f \in \mathcal{A}_\alpha$. This can be done for any irrational $\alpha \in \mathbb{R}/\mathbb{Z}$. Each periodic transformation f_n of course cannot be mixing. But the period of f_n diverges to infinity as $n \to +\infty$ and so apriori the limit map f could have fairly "wild" behavior. However, if additionally α satisfies a Liouville condition then the orbits of f_n approximate those of f uniformly on sufficiently growing time scales that one can conclude that f is at least non-mixing.

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Problem Session

General Audience

The session was moderated by Danny Calegari and the problems were edited by Strom Borman.

1. I. Kapovich

Two elements $g, g' \in G$ in a group are $SL_n(\mathbb{C})$*-character equivalent* if and only if $\text{trace}(\rho(g)) = \text{trace}(\rho(g'))$ for all homomorphisms $\rho : G \to SL_n(\mathbb{C})$. For the rank 2 free group F_2, it is known that two elements g and g' are conjugate if and only if g and g' are $SL_n(\mathbb{C})$-character equivalent for all n. This motivates the following:

Question 0.1. *Do there exist elements $g, g' \in F_2$ that are $SL_3(\mathbb{C})$-character equivalent but not conjugate?*

Remark 0.1 (D. Calegari)**.** *Since the trace is a coefficient of the characteristic polynomial, one could ask similar questions for the whole polynomial.*

2. F. Le Roux

The following is with F. Beguin and S. Crovisier.

Conjecture 0.1. *There does not exist a dense conjugacy class in Homeo$_0(S^2$, area).*

This conjecture is true for $\text{Homeo}_0(S^1)$ but is false for $\text{Homeo}_0(S^2)$. Proving Conjecture 0.1 reduces to proving the following conjecture, which is true for the group $\text{Ham}(\mathbb{T}^2) \subset \text{Homeo}_0(\mathbb{T}^2, \text{area})$. For $f \in \text{Homeo}_0(S^2, \text{area})$, let

$$A(f) := \sup\{\text{area}(D) : D \subset S^2 \text{ is a topological disk and } f(D) \cap D = \emptyset\}.$$

Conjecture 0.2. *If $f_n \to 1$ in Homeo$_0(S^2$, area), then $A(f_n) \to 0$.*

The following is with V. Humiliere and M. Khanevsky. Let $B \subset (M, \omega)$ be a ball in a symplectic manifold and let $\gamma \in \pi_1(M, x_0)$ where $x_0 \in B$, then define

$$D_B(\gamma) := \{(\varphi_t) \in \widetilde{\text{Ham}}(M, \omega) : \varphi_1|_B = 1_B \text{ and } [\varphi_t(x_0)] = \gamma\}.$$

Assume that $D_B(\gamma) \neq \emptyset$ for all $\gamma \in \pi_1(M, x_0)$, which is true in dimension 2, then

$$\|\gamma\|_B = \inf\{\|(\varphi_t)\|_{\text{Hofer}} : (\varphi_t) \in D_B(\gamma)\}$$

is a conjugation invariant norm on $\pi_1(M, x_0)$.

Question 0.2. *What can be said about this norm? Is it bounded? Is it equivalent to the word length?*

On $M = \mathbb{T}^2$ there is the bound $\|\gamma\|_B \leq 2\max\{\|(1,0)\|_B, \|(0,1)\|_B\}$, as is true for all $SL_2(\mathbb{Z})$ invariant norms on $\mathbb{Z}^2$. For $M = S^1 \times [0,1]$, if $\text{area}(B) > \frac{1}{2}\text{area}(M)$, then $\|\cdot\|_B$ is unbounded (this is a theorem by M. Khanevsky).

3. G. Courtois

Let (X^{n+1}, g) be a closed manifold with negative sectional curvature, then every horospheres H in the universal cover $(\tilde{X}^{n+1}, \tilde{g})$ is diffeomorphic to $\mathbb{R}^n$.

Question 0.3. *Are all the horospheres $(H, \tilde{g}|_H)$ in $(\tilde{X}^{n+1}, \tilde{g})$ quasi-isometric to each other? In the case this would be true, are they uniformly quasi-isometric?*

All horospheres are known to be isometric if (X, g) is a rank 1 symmetric space. The assumption of negative sectional curvature is needed since there are non-quasi-isometric horospheres in $\mathbb{H} \times \mathbb{H}$.

Question 0.4. *In dimension 3, are the horospheres in $(\tilde{X}, \tilde{g}_0)$ and $(\tilde{X}, \tilde{g})$ quasi-isometric, where g_0 is the hyperbolic metric? More generally, what does a quasi-isometry between $(\tilde{X}, \tilde{g}_0)$ and $(\tilde{X}, \tilde{g})$ say about their horospheres?*

It is known that horospheres $(H, \tilde{g}|_H)$ have polynomial volume growth:

$$\tfrac{1}{c} R^\alpha \leq \mathrm{vol}(B(R)) \leq c R^\beta$$

Question 0.5. *Under what conditions does $\alpha = \beta$?*

4. P. Py (with N. Monod)

Let G be a noncompact simple Lie group and let $X = G/K$ be its associated symmetric space. All actions of G mentioned below are continuous and by isometries. Consider the following two rigidity-type theorems.

Theorem 0.1 (Karpelevich, Mostow)**.** *If G acts on another symmetric space $X_1 = G_1/K_1$, then there exists an equivariant totally geodesic embedding $X \hookrightarrow X_1$.*

Theorem 0.2 (Caprace–Monod)**.** *If G acts cocompactly on a noncompact geodesically complete* CAT(0) *space Y, then Y is isometric to the symmetric space of G (possibly after rescaling the metric).*

In contrast, without the assumption of geodesic completeness, we have the following result in the case where G is the isometry group of the n-dimensional real hyperbolic space $\mathbb{H}^n$:

Theorem 0.3 (Monod–Py)**.** *Let $G = Isom(\mathbb{H}^n)$. There exists a family $(C_t)_{t \in (0,1]}$ of* CAT(−1) *spaces such that C_1 is isometric to $\mathbb{H}^n$, $G \simeq Isom(C_t)$ and G acts cocompactly on C_t for all t. The spaces C_t are pairwise non-isometric, even up to scaling.*

The spaces C_t can be taken to be minimal, i.e. without proper closed subsets that are invariant and convex, and most likely have infinite topological dimension.

Question 0.6. *Can one classify the* CAT(0) *or* CAT(−1) *spaces on which $Isom(\mathbb{H}^n)$ acts cocompactly?*

5. M. Kapovich

Theorem 0.4 (Otal, Croke). *Let M_i be compact surfaces with negative curvature, then a homotopy equivalence $f : M_1 \to M_2$ is homotopic to a isometry if f identifies the marked length spectrums, i.e. for any closed geodesic γ in M_1*

$$length_{M_1}(\gamma) = length_{M_2}(f(\gamma)^*)$$

where $f(\gamma)^*$ is the unique closed geodesic in M_2 homotopic to $f(\gamma)$.

Theorem 0.5 (Thurston). *Let M_i be compact hyperbolic surfaces, then a homotopy equivalence $f : M_1 \to M_2$ is homotopic to a L-Lipschitz homeomorphism if*

$$\sup_\gamma \frac{length_{M_2}(f(\gamma)^*)}{length_{M_1}(\gamma)} \leq L$$

where the supremum is taken over all closed geodesics γ in M_1.

Question 0.7. *Does Theorem 0.5 hold if M_i are only compact surfaces with negative curvature? This is open even if one assumes that one of the surfaces is hyperbolic.*

Theorem 0.4 has two proofs, a geometric one and a symplectic one, and they are both very different from the proof of Theorem 0.5. It is not clear whether one of them could be generalized in order to answer the question affirmatively.

6. J. Kędra

Let (M, ξ) be a contact manifold and let α be a contact form, i.e. $\xi = \ker(\alpha)$. There are two associated diffeomorphism groups, $\mathrm{Cont}_0(M, \alpha)$ the identity component of diffeomorphisms preserving the contact form α, and $\mathrm{Cont}_0(M, \xi)$ the identity component of diffeomorphisms preserving the contact distribution.

Question 0.8. *What can be said about the topology of the natural inclusion*

$$Cont_0(M, \alpha) \overset{j}{\hookrightarrow} Cont_0(M, \xi)$$

Some examples:

- If (M^{2n+1}, α) is a prequantization space of a symplectic manifold (X^{2n}, ω), i.e. a principle S^1-bundle $\pi : M \to X$ such that $\pi^*\omega = d\alpha$, then
$$1 \to S^1 \to \mathrm{Cont}_0(M, \alpha) \to \mathrm{Ham}(X, \omega) \to 1$$
is an exact sequence of groups.
- For $(S^1, d\theta)$, then $\mathrm{Cont}_0(S^1, d\theta) = S^1$ and $\mathrm{Cont}_0(S^1, \xi) = \mathrm{Diff}_0(S^1)$.
- If $M = S^3$ with the standard contact structure, then from the sequence of embeddings
$$\mathrm{Cont}_0(S^3, \alpha) \hookrightarrow \mathrm{Cont}_0(S^3, \xi) \hookrightarrow \mathrm{Diff}_0(S^3)$$
and the fact that $\mathrm{Cont}_0(S^3, \xi) \simeq U(2)$ and $\mathrm{Diff}_0(S^3) \simeq SO(4)$, it follows that j is a homotopy equivalence.

Question 0.9 (L. Polterovich)**.** *What can be said about the geometry of* j*? In the case of prequantization spaces there are quasi-morphisms on* $Cont_0(M, \alpha)$ *coming from* $Ham(X, \omega)$*, can such quasi-morphisms be extended to* $Cont_0(M, \xi)$*?*

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