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Classical Algebraic Geometry

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Abstract. Progress in algebraic geometry often comes through the introduction of new tools and ideas to tackle the classical problems the development of the field. Examples include new invariants that capture some aspect of geometry in a novel way, such as the derived category, and the extension of the class of geometric objects considered to allow constructions not previously possible, such as the transition from varieties to schemes or from schemes to stacks. Many famous old problems and outstanding conjectures have been resolved in this way over the last 50 years. While the new theories are sometimes studied for their own sake, they are in the end best understood in the context of the classical questions they illuminate. The goal of the workshop was to study new developments in algebraic geometry, with a view toward their application to the classical problems.

Mathematics Subject Classification (2000): 14xxx. IMU Classification 6.

Introduction by the Organisers

The workshop *Classical Algebraic Geometry* held June 18–22, 2012 at the "Mathematisches Forschungsinstitut Oberwolfach" was organized by Olivier Debarre (ENS), David Eisenbud (Berkeley), Frank-Olaf Schreyer (Saarbrücken), and Ravi Vakil (Stanford). There were 19 one hour talks with a maximum of four talks a day, and an evening session of short presentations allowing young participants to introduce their current work (and themselves). The schedule deliberately left plenty of room for informal discussion and work in smaller groups.

The extended abstracts give a detailed account of the broad variety of topics of the meeting, many of them classical questions in algebraic geometry approached with modern methods. We focus on a representative sample here:

- Farkas reported on joint work with Chiodo, Eisenbud, and Schreyer on a new use of vanishing results for Koszul cohomology to get at the birational structure of the higher genus generalization of modular curves, yielding analogues of the classical "general type" results pioneered by Eisenbud, Mumford, and Harris. The work also yields new conjectures, which turn out to be unexpectedly subtle (see the discussion of Conjectures A and B in Farkas' extended abstract). The use of computers to make progress in this area was eye-opening to much of the audience.
- Totaro gave one of his signature elegant talks. The Hodge conjecture as originally stated, over the integers, is false, and for this reason it is almost always discussed over $\mathbb{Q}$. However, the truth (or falsity) of the integral Hodge conjecture is hoped to shed light even on the Hodge conjecture over $\mathbb{Q}$. The integral Hodge conjecture is true in codimension 1, by the Lefschetz theorem $(1, 1)$, so the first case of interest is that of curves on a threefold. Counterexamples were given in this case by Kollár, Höring and Voisin, and Hassett and Tschinkel (whose examples were even varieties over $\overline{\mathbb{Q}}$). Totaro gave an extremely transparent counterexample in real time. He then argued that we should hope that the integral Hodge conjecture should hold for smooth complex projective threefolds of Kodaira dimension κ at most zero. Voisin had dispatched the case of $\kappa = -\infty$ (the "uniruled" case), as well as some other cases. Totaro gave an argument showing that the integral Hodge conjecture holds for all smooth projective threefolds with trivial canonical bundle — in particular, abelian threefolds.
- Alper reported on ongoing joint work with Fedorchuk, Smyth, and van der Wyck, a dramatic new advance in the Hassett-Keel program on understanding the stages in the minimal program applied to the moduli space of curves as moduli spaces. Only the first few steps had been successfully completed before; Alper et al. complete the first 9 steps. The new twist they introduced is that the moduli spaces in question cease to be Deligne-Mumford stacks; they are Artin stacks, and they turn up in the Hassett-Keel program through their "good moduli spaces", a notion introduced two years ago by Alper at the previous workshop. The use of the minimal model program and Artin stacks to understand the classical notion of the moduli space of curves is typical of the interactions which took place at the workshop.
- Finally, our youngest speaker was François Charles. (His selection is a perfect example of why we wait until late to fill out our invitation list.) He discussed his proof of the full Tate conjecture for K3 surfaces over finite fields of characteristic at least 5. The work built on earlier important work of Maulik, which in turn used work of Borcherds. One key idea was to consider arithmetic analogs of the classical Noether-Lefschetz divisors.

The young participants' presentations by Angela Ortega, Maksym Fedorchuk, Melanie Wood, John Ottem, Olivier Benoist, Margherita Lelli-Chiesa, Jack Hall, Claudiu Raicu, Zhi Jiang, Florian Geiss, Jonathan Wise and David Rydh covered a similarly wide spread of topics, from the structural and formal to the specific and geometric.

Workshop: Classical Algebraic Geometry

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Abstracts

Chow groups of K3 surfaces and stable maps

Daniel Huybrechts

According to the Bloch-Beilinson conjectures, an algebraic correspondence $[Z] \in \mathrm{CH}^2(X \times X)$ on a K3 surface X acts trivially on $\mathrm{CH}^2(X)$ if and only if the induced map on $H^{2,0}(X)$ is trivial. If $[Z] = [\Gamma_f] - [\Delta]$ for an automorphism f, the latter condition is saying that $f^*\sigma = \sigma$ for the unique (up to scaling) regular two-form σ, i.e. f is symplectic.

In my talk, I discussed some aspects of the proof of the following result [3]:

Theorem *Let X be a complex projective K3 surface and let $f : X \to X$ be a symplectic automorphism of finite order. Then the induced action $f^* : \mathrm{CH}^2(X) \to \mathrm{CH}^2(X)$ is the identity.*

Due to classical results of Nikulin, the order of f is bounded by eight. Symplectic automorphisms of infinite order exist, but the available techniques do not seem to be applicable in this case. (There are even examples of Picard number two.)

The proof combines [5] for $|f| = 2$ with [4] for the higher order case. The case $|f| = 2$ had been proved earlier in [2] for one third of all cases. I explained the idea of [2]: Deformation theory for stable maps is applied to prove the existence of a family of elliptic curves avoiding the fixed points of f. Eventually, the non-existence of torsion in $\mathrm{CH}^2(X)$ (Roitman) is used to conclude.

The proof for symplectic automorphisms of order three, five or seven relies on the classification of the generic Picard lattice [1] and classical results due to Kneser that show that the orthogonal group of the extended Picard lattice is generated by reflections. As explained in [4], one then can apply derived techniques to conclude.

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Syzygies of torsion bundles and the geometry of the level ℓ modular variety over $\overline{\mathcal{M}}_g$

Gavril Farkas

(joint work with A. Chiodo, D. Eisenbud and F.-O. Schreyer)

This is a report on joint work with A. Chiodo, D. Eisenbud and F.-O. Schreyer on the geometry of the level ℓ modular varieties over the moduli space of curves. The aim of this work is to show how vanishing statements for Koszul cohomology of certain line bundles on curves can be used to describe the birational nature of the higher genus generalization $\mathcal{R}_{g,\ell}$ of the modular curve $X_1(\ell)$. For integers $g, \ell \geq 2$, we denote by $\mathcal{R}_{g,\ell}$ the moduli space classifying pairs $[C, \eta]$, where C is a smooth curve of genus g and $\eta \in \mathrm{Pic}^0(C)$ is a line bundle of order ℓ. The space $\mathcal{R}_{g,\ell}$ admits a compactification $\overline{\mathcal{R}}_{g,\ell}$, which is the coarse moduli space associated to the smooth proper Deligne-Mumford stack $\overline{\mathsf{R}}_{g,\ell}$ of *level twisted curves*. One of the main results of this paper is the following:

Theorem 1. *The level three moduli space $\overline{\mathcal{R}}_{g,3}$ is a variety of general type for $g \geq 12$. Furthermore, the Kodaira dimension of $\overline{\mathcal{R}}_{11,3}$ is at least* 19*.*

In order to prove that for a given g, the moduli space $\overline{\mathcal{R}}_{g,3}$ is of general type, it suffices to express $K_{\overline{\mathcal{R}}_{g,3}} \in \mathrm{Pic}(\overline{\mathcal{R}}_{g,3})$ as a positive combination of the Hodge class $\lambda \in \mathrm{Pic}(\overline{\mathcal{R}}_{g,3})$, the class $[\overline{\mathfrak{D}}]$ of the closure of a certain effective divisor $\mathfrak{D}$ on $\mathcal{R}_{g,3}$ and boundary divisor classes corresponding to singular level curves. The key step in carrying out this program is a judicious choice of the divisor $\mathfrak{D}$. We choose $\mathfrak{D}$ to be a jumping locus for Koszul cohomology groups of paracanonically embedded curves. We now describe the general set-up for constructing such divisors.

For a curve C, a line bundle $L \in \mathrm{Pic}(C)$ and a sheaf $\mathbb{F}$ on C, we denote by $K_{p,q}(C; \mathbb{F}, L)$ the Koszul cohomology groups describing the pieces of the minimal resolution of the graded module

$$R(\mathbb{F}, L) := \bigoplus_{q \geq 0} H^0(C, \mathbb{F} \otimes L^{\otimes q})$$

as a Sym $H^0(C, L)$-module. Green's Conjecture asserts that the resolution of the coordinate ring of a general canonical curve $C \xrightarrow{|K_C|} \mathbf{P}^{g-1}$ is pure. This is equivalent to the vanishing statement

$$K_{\frac{g-1}{2},1}(C, K_C) = 0. \qquad (1)$$

Let us now consider a level curve $[C, \eta] \in \mathcal{R}_{g,\ell}$ and set $L := K_C \otimes \eta \in \mathrm{Pic}^{2g-2}(C)$ to be the *paracanonical* line bundle. Whenever $g \geq 5$, the induced map $\phi_L : C \to \mathbf{P}^{g-2}$ is an embedding for a general choice of $[C, \eta] \in \mathcal{R}_{g,\ell}$. We propose three statements concerning the purity of certain resolutions associated to $[C, \eta]$. In each of the cases, the corresponding non-vanishing locus is a *virtual* divisor on $\mathcal{R}_{g,\ell}$, that is, the degeneracy locus of a morphism between vector bundles of the same rank over the stack $\mathsf{R}_{g,\ell}$. Proving the corresponding syzygy conjecture amounts to showing that the respective degeneracy locus is a *genuine* divisor on $\mathcal{R}_{g,\ell}$.

A. Prym-Green Conjecture. *Let* $[C,\eta] \in \mathcal{R}_{g,\ell}$ *be a general level* ℓ *curve of* even *genus* $g \geq 6$*. Then the homogeneous coordinate ring of the paracanonically embedded curve* $\phi_L : C \hookrightarrow \boldsymbol{P}^{g-2}$ *has a pure resolution. Equivalently, the paracanonical curve satisfies property* $(N_{\frac{g}{2}-3})$*, that is,*

$$K_{\frac{g}{2}-2,1}\big(C, K_C \otimes \eta\big) = 0. \tag{2}$$

The name of the conjecture is justified by the analogy with Green's Conjecture. We verify the Prym-Green Conjecture for all even values of $g \leq 16$ and small ℓ, with the exception of 2-torsion in genus $g = 8$ and $g = 16$. Our findings suggest that perhaps, for 2-torsion and g a multiple of 8 (or perhaps a power of 2), the Prym-Green Conjecture might actually be false. (An experiment with $g = 24$ is unfortunately out of reach computationally.) Our verification of Conjecture A is computational via the use of *Macaulay2*. We verify condition (2) for g-nodal rational curves over a finite field (see Section 4 for details). For $g := 2i + 6$, we denote by

$$\mathcal{Z}_{g,\ell} := \Big\{[C,\eta] \in \mathcal{R}_{g,\ell} : K_{i+1,1}(C, K_C \otimes \eta) \neq 0\Big\},$$

the failure locus of Conjecture A.

The second conjecture we address concerns the resolution of torsion bundles.

B. Torsion Bundle Conjecture. *Let* $[C,\eta] \in \mathcal{R}_{g,\ell}$ *be a general level* ℓ *curve of* even *genus* $g \geq 4$*. For each* $1 \leq k \leq \ell - 2$*, the resolution of* $R(\eta^{\otimes k}, L)$ *as a* Sym $H^0(C,L)$*-module is pure, unless*

$$\eta^{\otimes(2k+1)} = \mathcal{O}_C \ \text{ and } g \equiv 2 \mod 4 \text{ and } \binom{g-3}{\frac{g}{2}-1} \equiv 1 \text{ mod } 2.$$

Equivalently, one has the vanishing statement

$$K_{\frac{g}{2}-1,1}(C; \eta^{\otimes k}, K_C \otimes \eta) = 0. \tag{3}$$

In the exceptional cases there is precisely one extra syzygy.

The first exceptional genera in the Conjecture B are $g = 6, 10, 18, 34$ or 66. For levels $\ell \geq 3$, we set $g := 2i + 2$ and denote the corresponding virtual divisor by

$$\mathcal{D}_{g,\ell} := \Big\{[C,\eta] \in \mathcal{R}_{g,\ell} : K_{i,1}(C; \eta^{\otimes(\ell-2)}, K_C \otimes \eta) \neq 0\Big\}.$$

Conjecture B can be reformulated in the spirit of the *Minimal Resolution Conjecture* for points on paracanonical curves as follows. We view a divisor $\Gamma \in |K_C \otimes \eta^{\otimes(1-k)}|$ as a 0-dimensional subscheme of $\phi_L(C) \subset \mathbf{P}^{g-2}$. Conjecture B turns out to be equivalent to the statement $b_{\frac{g}{2},1}(\Gamma) = b_{\frac{g}{2}-1,2}(\Gamma) = 0$, which amounts to the minimality of the number of syzygies of $\Gamma \subset \mathbf{P}^{g-2}$.

The exceptions to Conjecture B can be explained by (surprising) symmetries in the Koszul differentials. We verify Conjecture B for genus $g \leq 16$ and small level. In any of these cases $\mathcal{D}_{g,\ell}$ is a divisor on $\mathcal{R}_{g,\ell}$.

A similar prediction about the purity of the resolution of the module $R(\eta, K_C)$ can be made for paracanonically embedded curves of odd genus. This time we expect no exceptions and this turns out to be the case:

Theorem 2. *Let $[C, \eta] \in \mathcal{R}_{g,\ell}$ be a general level ℓ curve of* odd *genus $g \geq 5$. Then the resolution of $R(\eta, K_C)$ as a* Sym $H^0(C, K_C)$ *module is pure, that is,*

$$K_{\frac{g-1}{2},1}\big(C; \eta, K_C\big) = 0. \tag{4}$$

On connected automorphism groups of algebraic varieties

Michel Brion

Consider a smooth complete algebraic variety X over the field of complex numbers. By [6], the connected automorphism group $G := \mathrm{Aut}^o(X)$ (the connected component of $\mathrm{Aut}(X)$ containing the identity) is a complex algebraic group. In general, G is neither linear nor an abelian variety, as shown by the following examples due to Maruyama (see [7]):

Example 1. Let C be an elliptic curve, V a vector bundle of rank 2 on C, and $\pi : X := \mathbb{P}(V) \to C$ the associated ruled surface. Assume that V is a nonsplit extension of the trivial line bundle $\mathcal{O}_C$ by $\mathcal{O}_C$. Then G sits in an exact sequence of algebraic groups

$$1 \longrightarrow \mathbb{G}_a \longrightarrow G \longrightarrow C \longrightarrow 1,$$

where $\mathbb{G}_a$ denotes the additive group. Moreover, G acts on X with two orbits: the obvious section of π, a closed orbit isomorphic to $G/\mathbb{G}_a$, and its complement, an open orbit isomorphic to G. The class of the above extension in $\mathrm{Ext}^1(C, \mathbb{G}_a) \cong H^1(C, \mathcal{O}_C) \cong \mathrm{Ext}^1(\mathcal{O}_C, \mathcal{O}_C)$ is identified to the class of the extension $0 \to \mathcal{O}_C \to V \to \mathcal{O}_C \to 0$.

Example 2. Let C be as above and take $V := L \oplus \mathcal{O}_C$, where L is a line bundle of degree 0 on C. Then for the associated ruled surface $\pi : X \to C$, we have an exact sequence of algebraic groups

$$1 \longrightarrow \mathbb{G}_m \longrightarrow G \longrightarrow C \longrightarrow 1,$$

where $\mathbb{G}_m$ denotes the multiplicative group. Here G acts on X with three orbits: the two obvious sections of π, which are closed orbits isomorphic to $G/\mathbb{G}_m$, and their complement, an open orbit isomorphic to G. The class of the above extension in $\mathrm{Ext}^1(C, \mathbb{G}_m) \cong \mathrm{Pic}^o(C)$ is identified to the class of L.

For an arbitrary X, there is a unique exact sequence of algebraic groups

$$1 \longrightarrow G_{\mathrm{lin}} \longrightarrow G \longrightarrow A \longrightarrow 1,$$

where G_{lin} is connected and linear, and A an abelian variety (Chevalley's structure theorem; see [4, 3] for modern proofs).

We now determine the isogeny class of A in terms of the geometry of X. For this, we need the following definition: a morphism $f : X \to B$ is a *homogeneous fibration* if B is an abelian variety, the fibers of f are connected, and every translation of B lifts to an automorphism of X.

Proposition 3. (i) *For any homogeneous fibration $f : X \to B$, the G-action on X descends to a transitive action of A on B by translations.*
(ii) *There exists a homogeneous fibration $f : X \to B$ such that the resulting homomorphism $A \to B$ is an isogeny.*

The first assertion follows readily from a variant of a result of Blanchard (see [1]): let $\varphi : Y \to Z$ be a proper morphism of varieties such that $\varphi_*(\mathcal{O}_Y) = \mathcal{O}_Z$ and let H be a connected algebraic group acting on Y. Then this action descends to an action on Z such that φ is equivariant. The second assertion is a consequence of a theorem of Matsumura (see [5]): G acts on the Albanese variety of X via a finite quotient of A.

Note that there generally exists no homogeneous fibration $f : X \to A$, as a consequence of the following:

Example 4. Let again C be an elliptic curve. Denote by $C[n]$ the n-torsion subgroup of C, where n is a positive integer. Then $C[n] \cong (\mathbb{Z}/n\mathbb{Z})^2$ and there is a faithful irreducible projective representation $\rho : C[n] \to \mathrm{PGL}_n$. Consider the associated projective bundle $f : X := C \times^{C[n]} \mathbb{P}^{n-1} \to C/C[n]$, where $C[n]$ acts on $\mathbb{P}^{n-1}$ via ρ. Then $G \cong C$ and f is the Albanese morphism of X, a homogeneous fibration over C (where $C/C[n] \cong C$ via the multiplication n_C). The corrresponding isogeny $A = C \to C$ is just n_C.

Next, we obtain an interpretation of the Lie algebra of G_{lin} in terms of zeroes of vector fields. Recall that $\mathrm{Lie}(G)$ is the Lie algebra of vector fields, $H^0(X, T_X)$.

Proposition 5. *Let V be a subspace of $H^0(X, T_X)$. Then the following assertions are equivalent:*
(i) *Any nonzero vector field in V has a zero.*
(ii) $V \cap \mathrm{Lie}(G_{\mathrm{lin}}) = \{0\}$.
(iii) *$\mathcal{O}_X \otimes V$ is a direct factor of the tangent bundle T_X.*

This result follows easily from Proposition 3(ii); it extends to all complete normal varieties. As a direct consequence, the maximal trivial direct factors of T_X are exactly the $\mathcal{O}_X \otimes V$, where V is a complement to $\mathrm{Lie}(G_{\mathrm{lin}})$ in $\mathrm{Lie}(G)$. We may find such complements which are abelian Lie subalgebras of $\mathrm{Lie}(G)$, since we have $G = Z(G)G_{\mathrm{lin}}$, where $Z(G)$ denotes the center of G. But in general, there exists no closed subgroup $H \subset G$ such that $V = \mathrm{Lie}(H)$, as shown by Examples 1 and 2.

These examples also show that there generally exist vector fields $\xi \in H^0(X, T_X)$ without zero, such that ξ does not sit in the Lie algebra of any abelian variety acting on X, nor on any finite étale cover of X. Indeed, every such cover is the ruled surface $\pi' : X' = \mathbb{P}(V') \to C'$ associated to an isogeny $\varphi : C' \to C$ of elliptic curves, and to $V' := \varphi^*(V)$. Moreover, X' contains no elliptic curve if the extension in Example 1 is nontrivial, resp. if the line bundle in Example 2 has infinite order.

Finally, we obtain a bound for the dimension of the largest anti-affine subgroup of G in terms of the dimension of X. Here an algebraic group H is called *anti-affine* if any global regular function on H is constant. It is known that G has a largest anti-affine subgroup, G_{ant}, which is also the smallest closed subgroup H of G that maps onto A; moreover, G_{ant} is connected and contained in $Z(G)$. (For instance, the group G of Example 1 is anti-affine if and only if the extension $0 \to \mathcal{O}_C \to V \to \mathcal{O}_C \to 0$ is nontrivial; in Example 2, G is anti-affine iff L has infinite order).

Theorem 6. *With the above notation, we have* $\dim(G_{\mathrm{ant}}) \leq \max(n, 2n-4)$ *and this bound is optimal.*

To obtain examples where $\dim(G_{\mathrm{ant}}) = n$, consider an abelian variety A and an extension of algebraic groups

$$1 \longrightarrow T \longrightarrow G \longrightarrow A \longrightarrow 1,$$

where $T \cong \mathbb{G}_m^r$ is a torus. Then the *semi-abelian variety* G is classified by a group homomorphism $c : \mathrm{Hom}(T, \mathbb{G}_m) \to \mathrm{Pic}^o(A)$; moreover, G is anti-affine if and only if c is injective. We may choose a smooth complete toric variety Y with torus T. Then the associated fibered space $X := G \times^T Y \to G/T = A$ is a homogeneous fibration; moreover, G is the connected automorphism group of X, and acts there with an open orbit $G \times^T T \cong G$.

Examples where $\dim(G_{\mathrm{ant}}) = 2n-4$ are constructed from the *universal vector extension* of A. This is a connected commutative algebraic group, obtained as an extension

$$0 \longrightarrow V \longrightarrow E(A) \longrightarrow A \longrightarrow 0,$$

where V is a vector space, and universal for this property; then $V = H^1(A, \mathcal{O}_A)^*$ has dimension $g := \dim(A)$. Let $\mathbb{F}_{g-1}$ be the rational ruled surface of index $g-1$; then V acts on $\mathbb{F}_{g-1}$ by translations. The associated fibered space $X := E(A) \times^V \mathbb{F}_{g-1} \to E(A)/V = A$ is again a homogeneous fibration; moreover, G is the connected automorphism group of X, and $\dim(G) = 2g$ while $\dim(X) = g+2$.

The main ingredient in the proof of Theorem 6 is the structure of anti-affine groups, obtained in [2, 8] over an arbitrary field. In positive characteristics, every such group is a semi-abelian variety; it follows that the above theorem holds with the simpler (optimal) bound n.

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Some further geometry of Prym moduli spaces in low genus

Alessandro Verra
(joint work with Gavril Farkas)

By $\mathcal{R}_g$ we denote the Prym moduli space of pairs (C, η) such that C is a complex projective curve, smooth and connected, and $\eta \in Pic^0 C$ is a non trivial 2-torsion element. Recently there has been several results on the birational geometry of $\mathcal{R}_g$ and on its usual compactification $\overline{\mathcal{R}}_g$ by allowable double covers. Due to the results of Farkas and Ludwig on the structure of the singularities of $\overline{\mathcal{R}}_g$, and their consequent investigation on the cone of effective divisors, it is known that $\overline{\mathcal{R}}_g$ is of general type for $g \geq 13$, [FL]. On the other hand the Kodaira dimension of $\overline{\mathcal{R}}_g$ seems to be a completely open problem for $g \in [9, 12]$.

The results summarized here are part of a joint project with Gavril Farkas. They are concerned with the cases of genus 7 and 8, which appear to be linked by the methods of proof. We want to outline the proof of the following

Theorem 1. *$\mathcal{R}_7$ is unirational and $\mathcal{R}_8$ is rationally connected.*

The proof of the unirationality of $\mathcal{R}_7$ we sketch relies on Nikulin surfaces and on their strong interplay with Prym curves, see [FV]. Here a *Nikulin surface of genus $g \geq 2$* is a smooth K3 surface S endowed with a lattice of divisorial classes

$$\mathbb{Z}[C] \oplus \mathbb{Z}[F_1] \oplus \cdots \oplus \mathbb{Z}[F_8] \subset Pic\ S,$$

such that: (1) C is smooth of genus g, (2) $F_i = \mathbf{P}^1$, $i = 1 \dots 8$, (3) $F_1 + \cdots + F_8 \sim 2E$ $E \in Pic\ S$, (4) $CE_i = 0$ and $E_i E_j = -2\delta_{ij}$.

Assume that (C, η) defines a general point of $\mathcal{R}_7$ and consider the Prym canonical model $C \subset \mathbf{P}^5$ of (C, η). One can show that then $C \subset S \subset \mathbf{P}^5$, where S is a complete intersection of three quadrics and a Nikulin surface. In particular S is uniquely defined by (C, η). Let $\mathcal{N}_g$ be the moduli space of Nikulin surfaces of genus g. It turns out that the assigement $(C, \eta) \to (S, \mathcal{O}_S(C))$ defines a map

$$p : \mathcal{R}_7 \to \mathcal{N}_7$$

which is birationally equivalent to the projection $\mathbf{P}^7 \times \mathcal{N}_7 \to \mathcal{N}_7$. The unirationality of $\mathcal{R}_7$ then follows if one can prove the unirationality of $\mathcal{N}_7$. To this purpose we actually show that there exists a dominant rational map of moduli spaces

$$\phi : \mathcal{M}_{0,14} \to \mathcal{N}_7.$$

Given indeed $x = (x_1, \dots, x_{14}) \in (\mathbf{P}^1)^{14}$ we can consider the embedding of $\mathbf{P}^1$ as a rational normal quintic $R \subset \mathbf{P}^5$. One can show that there exists a unique

Nikulin surface S as above containing the curve $R \cup F_1 \cup \cdots \cup F_7$, where F_i is the line joining the points $x_i, x_{7+i} \in R$, $i = 1 \dots 7$. The construction of S defines the map ϕ.

Let $\mathcal{T}$ be the moduli space of triples (C, η, x, y), where (C, η) is a Prym curve of genus 7 and $x, y \in C$ is an unordered pair of points of C. By an additional effort one can show that $\mathcal{T}$ is unirational. In a similar way one can also deduce that some boundary divisors of $\overline{\mathcal{R}}_8$ are unirational: in particular this is true for the ramification divisor ${\delta_0}^{ram}$ of the forgetful map $f : \overline{\mathcal{R}}_8 \to \overline{\mathcal{M}}_8$. To prove the rational connectedness of $\overline{\mathcal{R}}_8$ one uses the latter property and the next, independent, result.

Let $\Delta \subset \mathbf{P}^5$ be the cubic defined by $det\ A = 0$, where A is a general symmetric 3×3 matrix of linear forms. In other words $\Delta = Sec\ V$, V a Veronese surface.

Theorem 2. *Let (C, η) be a general Prym curve of genus 8. Then there exist embeddings $C \subset X \subset \mathbf{P}^5$, where X is a transversal complete intersection of Δ and two quadrics and moreover $\mathcal{O}_C(1) \cong \omega_C$.*

It is easy to see that X is endowed with a quasi étale double cover $\pi : \tilde{X} \to X$ branched exactly at the sixteen nodes of $Sing\ X$. Furthermore $P := |C|$ is a pencil of curves $D \subset X$ endowed with an étale double covering $\pi/\tilde{D} : \tilde{D} \to D$, $\tilde{D} = \pi^{-1}(D)$. In particular the moduli map $m : P \to \overline{\mathcal{R}}_8$ is non constant and intersects the ramification divisor ${\delta_0}^{ram}$ in sixteen points.

Assume $o_1, o_2 \in \mathcal{R}_8$ are general and that they are the isomorphism classes of (C_1, η_1), (C_2, η_2) respectively. The corresponding pencil $P_i = |C_i|$, defines a rational curve $R_i = m_i(P_i)$ passing through o_i, $i = 1, 2$.

Let $p_i \in {\delta_0}^{ram} \cap R_i$, under the previous generality assumptions one can also show that p_i is a smooth point of $\overline{\mathcal{R}}_8$. Since ${\delta_0}^{ram}$ is unirational, there exists a rational curve $R_0 \subset {\delta_0}^{ram}$ containing p_1 and p_2. But then o_1, o_2 are connected by a chain of three rational curves: $R_1 \cup R_0 \cup R_2$. This implies the rational connectedness of $\mathcal{R}_8$, (cfr. [BV] for the use of an analogous argument in the case of $\mathcal{M}_{15}$).

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The integral Hodge conjecture for 3-folds

Burt Totaro

Let X be a smooth complex projective variety. The Hodge conjecture asserts that any element of $H^{2r}(X, \mathbf{Q})$ whose image in $H^{2r}(X, \mathbf{C})$ lies in the middle piece of the Hodge decomposition is the class of an algebraic cycle, meaning a $\mathbf{Q}$-linear combination of complex subvarieties of codimension r in X. The integral Hodge conjecture for X is the stronger statement that every Hodge class in $H^{2r}(X, \mathbf{Z})$ is the class of an algebraic cycle with $\mathbf{Z}$ coefficients. The integral Hodge conjecture is false in general, but it is interesting to see when it is true.

The main evidence for the Hodge conjecture is the Lefschetz $(1,1)$ theorem, which asserts that the integral Hodge conjecture is true for codimension-1 cycles. By the hard Lefschetz theorem, it follows that the Hodge conjecture (with rational coefficients) is true for 1-cycles. Not much is known beyond these cases. The Hodge conjecture for 2-cycles on a 4-fold is not known even for abelian 4-folds. It is known for smooth hypersurfaces in $\mathbf{P}^5$ of degrees at most 5 (the hypersurfaces which are Fano), but not for hypersurfaces of higher degree.

The Hodge conjecture is true for most smooth hypersurfaces of any given degree, because for most hypersurfaces, the group of Hodge classes has rank 1 (Noether-Lefschetz). But it would be much more significant to prove the Hodge conjecture for all smooth hypersurfaces of a given degree, since that would say that the "interesting" hypersurfaces (those with extra algebraic cycles beyond the obvious ones) are picked out by the Hodge structure of the hypersurface.

Atiyah and Hirzebruch disproved the integral Hodge conjecture (IHC) using topological K-theory. Their argument worked only in high dimensions, with their smallest counterexample being for 2-cycles on a 7-fold [1]. Kollár disproved the integral Hodge conjecture for 1-cycles on a 3-fold, which is the lowest possible dimension in view of the Lefschetz $(1,1)$ theorem [4]. Kollár's 3-folds are of general type. More recently, Höring and Voisin observed that IHC can also fail for 3-folds of Kodaira dimension 1 or 2 [3, after Theorem 1.2]. Finally, Hassett and Tschinkel showed how to produce counterexamples to IHC for 3-folds over $\overline{\mathbf{Q}}$, not just over $\mathbf{C}$ [2, Remarque 5.10].

Despite these counterexamples, we can hope that the integral Hodge conjecture will hold for smooth complex projective 3-folds of Kodaira dimension at most zero. Indeed, Voisin proved IHC for all 3-folds of Kodaira dimension $-\infty$ (or equivalently, for all uniruled 3-folds), and also for some 3-folds of Kodaira dimension zero: the smooth projective 3-folds with trivial canonical bundle and first Betti number zero [5].

We show that the integral Hodge conjecture is in fact true for all smooth projective 3-folds with trivial canonical bundle. In particular, we prove the integral Hodge conjecture for abelian 3-folds.

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Singularities of Cox Rings of Fano Varieties

Morgan Brown

The Cox ring, or total coordinate ring, of an algebraic variety is a generalization of the homogeneous coordinate ring of projective variety. Assume X is a normal projective variety with finitely generated class group. Then $\mathrm{Cox}(X) = \bigoplus_{D\in \mathrm{Cl}(X)} H^0(X, \mathcal{O}_X(D))$. When $X = \mathbb{P}^n$, the class group is $\mathbb{Z}$, and the Cox ring is the polynomial ring $k[x_0 \dots x_n]$ graded by degree. In general, the Cox ring of a variety X is graded by the class group of X. Cox [Cox95] studied the case when X is a toric variety, and showed that the Cox ring of a toric variety is always a polynomial ring.

The homogeneous coordinate ring of a projective variety corresponds to the affine cone over the embedded variety, and we can recover the variety by taking a $\mathbb{G}_m$ quotient. Similarly, we can recover X from its Cox ring, given some extra data. Assume X is smooth, and that $\mathrm{Pic}(X) \cong \mathbb{Z}^\rho$. Then, once we have the extra data of a linearization, we can recover X as a GIT quotient of $\mathrm{Cox}(X)$ by $\mathbb{G}_m^\rho$. Using this fact, Hu and Keel [HK00] were able to show the converse to Cox's result; X is a toric variety if and only if $\mathrm{Cox}(X)$ is a polynomial ring.

Cox rings play an important role in birational geometry. Every birational map from X to projective space is given by some linear series of divisors on X, and every such linear series is naturally a homogeneous subvector space of $\mathrm{Cox}(X)$. Thus the Cox ring algebraically encodes information about all the birational modifications of X. For this reason it is not surprising that the Cox ring often fails to be finitely generated.

If X is a smooth variety with finitely generated Picard group and finitely generated Cox ring, then X is called a Mori Dream Space. The term 'Mori Dream Space' was coined by Hu and Keel [HK00] and reflects the fact that finite generation of the Cox ring eliminates all of the very difficult technical obstacles to running the Mori program on X. Another nice consequence of finite generation of $\mathrm{Cox}(X)$ is that the effective cone of X breaks into finitely many rational polyhedral chambers corresponding to different birational modifications of X [HK00]. Figure illustrates this decomposition for $\mathbb{P}^2$ blown up at two points.

Toric varieties give a first set of examples of Mori Dream Spaces. There are many other examples. By the Lefschetz hyperplane theorem, a smooth hypersurface of any degree in $\mathbb{P}^n$ for $n \geq 4$ has Picard rank 1, so it is a Mori Dream Space. Another class of examples arises as a consequence of work done by Birkar, Cascini, Hacon and McKernan [BCHM10]. They showed that if the pair (X, Δ) is log Fano, then the Cox ring of X is finitely generated. A pair (X, Δ) is said to be log Fano if Δ is an effective $\mathbb{Q}$-divisor, (X, Δ) is klt (this is a condition on the singularities of the pair), and the divisor $-(K_X + \Delta)$ is ample. In particular, if X is a Fano variety, then the pair $(X, 0)$ is log Fano.

Our main theorem concerns the singularities of the Cox ring of a log Fano variety:

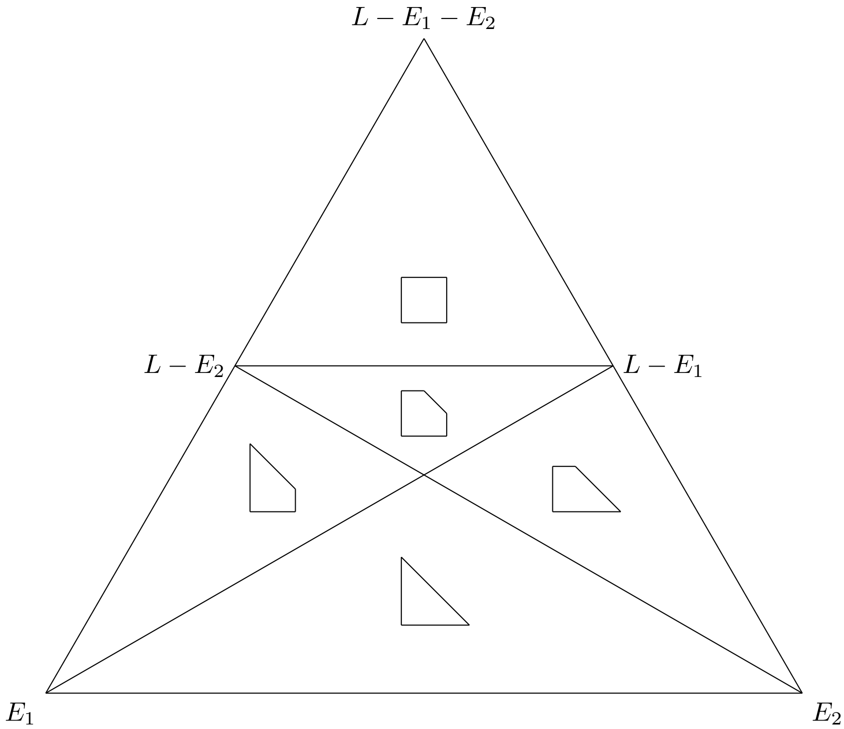

FIGURE 1. A slice of the chamber decomposition of the effective cone of $X = \mathrm{Bl}_2\mathbb{P}^2$. L denotes the pullback of the hyperplane class on $\mathbb{P}^2$, and E_1 and E_2 are the exceptional divisors. Each chamber corresponds to a birational model of X, and is labeled with the polytope corresponding to that toric variety. The ample cone is in the center.

Theorem 1. *Let (X, Δ) be a log Fano pair over $\mathbb{C}$. Then the spectrum of the Cox ring of X has log terminal singularities.*

Cox rings always have unique factorization [EKW04], and log terminal singularities are Cohen Macaulay, so an immediate corollary is that the Cox ring of a Fano variety is Gorenstein. The author [B11] first showed Theorem 1 for Fano varieties, the result in the log Fano case was proven independently by the author and Gongyo, Okawa, Sannai, and Takagi [GOST12], who also showed conversely that if X is a Mori Dream space such that $\mathrm{Cox}(X)$ has log terminal singularities, there exists Δ such that (X, Δ) is a log Fano pair. Theorem 1, along with its converse, gives a natural generalization of the fact that X is a toric variety if and only if its Cox ring is a polynomial ring.

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Iteration of rational self-maps with fixed points

Ekaterina Amerik

(joint work with Fedor Bogomolov and Marat Rovinsky)

The main motivation for this work is the problem of potential density of rational points. Recall that an algebraic variety X defined over a field K is said to be potentially dense over K if for some finite extension L of K, the set $X(L)$ of L-points is Zariski-dense in X. Lang's conjecture affirms that a variety of general type cannot be potentially dense over a number field, whereas several known conjectures agree on the feeling that the varieties with trivial canonical class should be potentially dense. This is well-known for abelian varieties, but the simply-connected case remains mysterious: for instance it is not known whether a general K3 surface is potentially dense over a number field.

When X is equipped with a rational self-map $f : X \dashrightarrow X$, one can hope to obtain many L-points on X by iterating a single one. Such examples exist for certain "closest higher-dimensional relatives" of K3 surfaces:

Example (C. Voisin): Let $V \subset \mathbb{P}^5$ be a smooth cubic and let $X = \mathcal{F}(V)$ be the variety of lines on V. Then X is a smooth irreducible holomorphic symplectic fourfold. If l is a general line on V, there is a unique plane P tangent to V along l, therefore X admits a rational self-map sending l to its residual (in $P \cap V$) line l'. This map acts as multiplication by -2 on $H^{2,0}(V) = \mathbb{C}$ and thus has degree 16.

One can show ([2]) that a "very general" point of X has Zariski-dense f-orbit, however, as "very general" means "outside of a countable union of proper subvarieties", this does not give any information over a countable field. In fact we still do not know whether X has a $\bar{\mathbb{Q}}$-point with dense f-orbit. The question of potential density of X was settled in [3] under certain genericity conditions on X, which have been shown to hold for many X defined over a number field. However this proof is quite subtle and the genericity conditions are quite complicated, so that it seems impossible to verify explicitely whether these hold for any given X. Thus our primary motivation was to find a more simple and direct approach to this problem.

Let X be an algebraic variety with a rational self-map f with a non-degenerate fixed point q. Assume that everything is defined over a sufficiently large number field K, with O_K denoting its ring of integers.

Key observation: *For almost all primes $\mathfrak{p}$ in O_K, the point q has an invariant $\mathfrak{p}$-adic neighbourhood $O_{\mathfrak{p},q,s}$ ("points with the same reduction as q modulo $\mathfrak{p}^s$"). It is naturally identified with the n-th cartesian power of $\mathfrak{p}^s$, and the map f is given by power series with coefficients in $O_{\mathfrak{p}}$ (and without constant term).*

It follows from some classical and some more modern theory that under suitable conditions on the eigenvalues $\lambda_1, \lambda_2, \dots \lambda_n$ of Df_q, the map f is linearized in $O_{\mathfrak{p},q,s}$ for some $s >> 0$. Namely, the linearization holds in the absence of "resonances" – that is, relations of the type $\lambda_i = \Pi_{j=1}^n \lambda_j^{m_j}$ with $m_j \in \mathbb{Z}_{\geq 0}$ and $m = \sum_j m_j \geq 2$. If the fixed point q is not isolated, then by definition there is always a resonance; however, one can prove that if q is a smooth point of the fixed locus in its r-dimensional component and Df_q is semisimple with eigenvalues $1, \dots, 1, \lambda_{r+1}, \dots, \lambda_n$, where λ_i, $r < i \leq n$ are constant in a neighbourhood of q and without resonances between them, the linearization is still possible.

One obtains the following consequences of linearization:

Corollary 1 *Assume that λ_i are multiplicatively independent. Then any point of $O_{\mathfrak{p},q,s}$ outside the union of coordinate hyperplanes in the linearizing coordinates has Zariski-dense open orbit.*

Corollary 2 *In the "generalized non-resonant" situation as described above (i.e. with q not necessarily isolated), if the λ_i generate a torsion-free subgroup in $O_{\mathfrak{p}}^{\times}$, then for an irreducible subvariety $Y \subset X$ passing sufficiently close to q, the union of its iterates $\bigcup_{i\geq 0} f^i(Y)$ has irreducible Zariski closure.*

The self-map f in Voisin's example has a surface of fixed points, and the eigenvalues of Df at a general one are $1, 1, -2, -2$. In particular Corollary 1 does not apply, but Corollary 2 does and together with some (not very complicated) geometric arguments gives potential density of X as soon as the Picard number of X is one. One can construct explicit examples of such X using computer verifications (as in the works by van Luijk and others).

Related to this is a conjecture by S.-W. Zhang, stating that a regular and polarized self-map f of a smooth projective variety S over a number field must have algebraic points $x \in S(\bar{\mathbb{Q}})$ with Zariski-dense orbit. "Polarized" means that there is an ample line bundle L such that $f^*L = L^{\otimes k}$ for some $k > 1$. It is possible that this condition is only needed to assure that no power of f preserves a fibration.

Consider the case $dim(S) = 2$, then by general results in holomorphic dynamics S has a lot of repulsive periodic points (in particular, those are not contained in a proper subvariety of S). Take $q \in S$ which is one of them and replace f

by a power so that q is fixed. Let λ_1, λ_2 be the eigenvalues of Df_q. If those are multiplicatively independent, then by Corollary 1 Zhang's conjecture holds. Unfortunately not every f as above admits a periodic point with multiplicatively independent eigenvalues: for instance, $f : \mathbb{P}^2 \to \mathbb{P}^2$ taking all coordinates to k-th power, does not.

Question: *Can one classify such exceptions?*

If there is a resonance between the eigenvalues, it can only be of the form $\lambda_1 = \lambda_2^a$, where $a \geq 2$, because q is repulsive. Blowing-up q, we replace it with two fixed points q_1 with eigenvalues $\lambda_1, \lambda_2/\lambda_1$ and q_2 with eigenvalues $\lambda_1/\lambda_2, \lambda_2$. So we shall eventually get rid of the resonance producing a fixed point with linearization and $\lambda_1 = \lambda_2$. If the linear part of Df is a Jordan cell, Zariski density of most orbits follows by direct calculation. If not, then in some analytic neighbourhood of q, f is multiplication by a scalar λ, so all lines through the origin are invariant. Some lines give invariant curves but most of them do not (are Zariski-dense in S). Zhang's conjecture is thus equivalent to a statement that there are "not too many" (in a sense to be made precise) invariant curves arizing in this way. It is clear that their number does not have to be finite: take the example of the k-th power map, with $q = (1 : 1 : 1)$, then for each $\alpha \in \mathbb{Q}$, $x = y^\alpha$ is an invariant algebraic curve. One would hope that the answer to the following

Question: *Can such invariant curves be naturally parameterized by the elements of $\bar{\mathbb{Q}}$ rather than $\mathbb{Q}$?*

is negative, but I still do not know how to approach it.

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Operational K-theory and localization on singular varieties

Sam Payne

(joint work with Dave Anderson)

Grothendieck groups of perfect complexes are not $\mathbf{A}^1$-homotopy invariant, and are often uncountably generated, even on $\mathbf{Q}$-factorial projective toric threefolds [Gu, CHWW]. Grothendieck groups of vector bundles are potentially even more complicated to work with; although they agree with Grothendieck groups of perfect complexes on varieties that are quasiprojective or smooth or divisorial (or, more generally, those that satisfy the resolution property, they are not known to satisfy Mayer-Vietoris when the resolution property fails, and there are singular, complete, nonprojective toric threefolds that are not known to have any nontrivial vector bundles at all [Pa2]. Some of these have Grothendieck groups of perfect complexes that are uncountably generated [GK]. It is worth noting, also, that there are no known examples of noetherian separated schemes for which the resolution property fails, so perhaps Grothendieck groups of perfect complexes and vector bundles agree on all varieties.

In joint work with Dave Anderson, we develop the basic properties of "operational K-theory," an alternative to Grothendieck groups of vector bundles and perfect complexes, and study how it localizes on spaces with torus actions, such as toric varieties. This theory is constructed following Fulton and MacPherson [FM]; to each reasonable algebraic homology theory, there is an associated operational cohomology theory that acts on the given homology theory to produce a bivariant theory. The prototypical example is the operational Chow cohomology associated to the Chow homology theory of cycles modulo rational equivalence. This operational K-theory is the cohomology theory associated to the algebraic homology theory given by Grothendieck groups of coherent sheaves.

By construction, the operational K-theory $\mathrm{op}K^\circ$ is a contravariant, ring valued functor that acts on Grothendieck groups of coherent sheaves, compatibly with proper push forward and Gysin maps for flat morphisms and regular embeddings, and there is a natural transformation from the algebraic K-theory of vector bundles to operational K-theory. We prove that it also has the following geometric properties.

(1) *Operational K-theory agrees with algebraic K-theory on smooth varieties.* If X is smooth then the natural map from $K^\circ(X)$ to $\mathrm{op}K^\circ(X)$ is an isomorphism.
(2) *Operational K-theory is* A^1*-homotopy invariant.* For any X, the pullback map from $\mathrm{op}K^\circ(X)$ to $\mathrm{op}K^\circ(X \times \mathrm{A}^1)$ is an isomorphism.
(3) *Operational K theory satisfies Kronecker duality for linear varieties.* If X is complete and linear then proper push forward to a point gives an isomorphism from $\mathrm{op}K^\circ(X)$ to $\mathrm{Hom}(K_\circ(X), \mathbf{Z})$.
(4) *Operational K theory is a sheaf in the envelope topology.* If $X' \to X$ is an envelope, then the induced complex

$$0 \to \mathrm{op}K^\circ(X) \to \mathrm{op}K^\circ(X') \to \mathrm{op}K^\circ(X' \times_X X')$$

is exact.

The last two properties are analogues of theorems for operational Chow cohomology, due to Totaro [To] and Kimura [Ki], respectively. Similar properties hold for equivariant operational K-theory, the operational cohomology theory associated to Grothendieck groups of equivariant coherent sheaves on spaces with a torus action. We apply these basic results to prove that operational equivariant K-theory satisfies localization with integer coefficients on arbitrary toric varieties.

Theorem. Let X be the toric variety associated to a fan Δ. Restriction to torus orbits gives a natural isomorphism from $\mathrm{op}K_T^\circ(X)$ to the ring of continuous, piecewise exponential functions on $|\Delta|$, with integer coefficients.

This theorem generalizes earlier localization results for equivariant algebraic K-theory of smooth toric varieties [BV, VV], and is closely related to a similar localization result in operational Chow cohomology [Pa1].

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Birational models of $\overline{M}_g$

Jarod Alper

(joint work with Maksym Fedorchuk, David Smyth, Fred van der Wyck)

In the early 2000s, Hassett and Keel asked the following question:

Question. Can the minimal model program be run on $\overline{M}_g$, the moduli space of Deligne-Mumford stable curves?

The reason that the above question is so compelling is due to the classical result of Eisenbud, Harris and Mumford asserting that $\overline{M}_g$ is of general type for $g \geq 24$, i.e., the canonical line bundle K lies in the interior of the effective cone. Therefore, the canonical model $\overline{M}_g^{\mathrm{can}} = \mathrm{Proj} \oplus_d \Gamma(\mathcal{M}_g, K^{\otimes d})$ is a birational model. On the other hand, the minimal model program has been a central theme in algebraic geometry in the last half century with a recent explosion of interest due to powerful new results. The moduli space of curves $\overline{M}_g$ is a very interesting higher dimension general type variety and therefore is a natural test candidate for the general machinery of the minimal model program.

By the results of Cornalba-Harris and Xiao, the line bundle $K + \alpha\delta$ on $\overline{\mathcal{M}}_g$ is ample for $1 \geq \alpha > \frac{9}{11}$. It is natural to introduce the projective varieties

$$\overline{M}_g(\alpha) = \mathrm{Proj} \oplus_d \Gamma(\mathcal{M}_g, (K + \alpha\delta)^{\otimes d})$$

which are birational models for $\overline{M}_g$. Note that $\overline{M}_g(1) = \overline{M}_g$ and $\overline{M}_g(0) = \overline{M}_g^{\text{can}}$. The theory of birational geometry dictates that by scaling α from 1 to 0, one expects divisorial contractions and flips which give rise to the rational morphism $\overline{M}_g \dashrightarrow \overline{M}_g^{\text{can}}$. Moreover, Hassett and Keel asked the striking question:

Question. Does $\overline{M}_g(\alpha)$ inherit a moduli description?

That is, is $\overline{M}_g(\alpha)$ a moduli space of singular curves and can the divisorial contractions and flips arising in the minimal model program be understood via the moduli interpretation? While perhaps there is no good reason to expect such a description, the striking and beautiful results of Hassett and Hyeon show that indeed for $\alpha > \frac{7}{10} - \epsilon$, the answer is "yes"! The log canonical model $\overline{M}_g(\frac{9}{11})$ parameterizes curves with cusps (i.e., an A_2: $y^2 = x^3$ singularity) but not containing an elliptic tail (i.e., an arithmetic genus 1 component meeting the rest of the curve in a node) and the morphism $\overline{M}_g \to \overline{M}_g(\frac{9}{11})$ contracts the divisor of elliptic tails [HH09]. Moreover, $\overline{M}_g(\frac{7}{10})$ and $\overline{M}_g(\frac{7}{10} - \epsilon)$ inherit a moduli description given by Chow bicanonically semistable and Hilbert bicanonically semistable curves with at worst tacnodes (i.e., an A_3: $y^2 = x^4$ singularity), and there are morphisms $\overline{M}_g(\frac{7}{10} + \epsilon) \to \overline{M}_g(\frac{7}{10}) \leftarrow \overline{M}_g(\frac{7}{10} - \epsilon)$ which flip the locus of curves containing an elliptic bridge (i.e., an arithmetic genus 1 curve meeting the rest of the curve in 2 nodes) to the locus of curves with a tacnode.

It is important to observe two issues that arise in the results of Hassett and Hyeon. First, it is necessary to parameterize tacnodal curves that have an infinite automorphism groups; that is, the moduli spaces parameterizing $\overline{M}_g(\frac{7}{10})$ and $\overline{M}_g(\frac{7}{10} - \epsilon)$ are, in fact, algebraic stacks (in the sense of Artin) which are not Deligne-Mumford. Second, the methods of Hassett and Hyeon rely on geometric invariant theory which appear difficult to extend to further steps.

This talk introduced a new approach to the minimal model program on $\overline{M}_g$. The general strategy is:

(1) Using heuristics arising from intersection theory, character theory and formal variation of GIT, define an algebraic stack $\overline{\mathcal{M}}_g(\alpha)$ of singular curves.
(2) Construct a *good moduli space* $\phi\colon \overline{\mathcal{M}}_g(\alpha) \to Y$; that is, show that there is a morphism $\phi\colon \overline{\mathcal{M}}_g(\alpha) \to Y$ to an algebraic space Y such that two points $[C_1], [C_2] \in \overline{\mathcal{M}}_g(\alpha)$ are identified in Y if and only if there is another curve C_0 in $\overline{\mathcal{M}}_g(\alpha)$ such that both C_1 and C_2 isotrivially specialize to C_0. See [Alp12a] for the precise definition of a good moduli space.
(3) Show that the line bundle $K + \alpha\delta$ on $\overline{\mathcal{M}}_g(\alpha)$ descends to an ample line bundle on Y; that is, show that $Y = \overline{M}_g(\alpha)$.

This procedure is reminiscent of the now standard approach to construct projective moduli spaces of objects with *finite* automorphism groups where the strategy is: (1) show that the moduli space $\mathcal{X}$ is a proper Deligne-Mumford stack, (2) apply the Keel-Mori theorem to obtain a coarse moduli space Y [KM97], and (3) use positivity results on tautological line bundles to show that Y is projective [Kol90].

Let us quickly elaborate on each of the above steps. For step (1), there are certain loci in $\overline{M}_g$ such as elliptic tails, elliptic bridges and their generalizations that fall in the stable base locus of $K + \alpha\delta$ beginning at certain critical values α; past these critical values, these curves must be removed in the moduli description. Second, as hinted to earlier, it is imperative to allow singular curves C with infinite automorphism group. If $\mathbb{C}^* \subseteq \mathrm{Aut}(C)$ is a subgroup, then the line bundles K and δ (which under mild hypotheses extend over $[C]$ in the stack of all curves) induce characters. Moreover, in the event that there is a good moduli space $\overline{\mathcal{M}}_g(\alpha) \to \overline{M}_g(\alpha)$, then the natural $\mathcal{O}(1)$ pulls back to a multiple of $K + \alpha\delta$ and therefore $\mathbb{C}^*$ acts trivially on the fiber of $K + \alpha\delta$ at C. It turns out that one can explicitly compute the characters of K and δ for almost any singular curve with a $\mathbb{C}^*$-action and therefore compute the critical value for which such a singular curve appears (if it appears) in the stack $\overline{\mathcal{M}}_g(\alpha)$. Finally, a powerful heuristic in determining the moduli stacks which also plays a crucial role in the proofs of steps (2) and (3) is the following: if C is a curve which is closed in $\overline{\mathcal{M}}_g(\alpha)$, then $\mathrm{Aut}(C)$ acts on the first order deformation space $\mathbb{T}^1(C)$ and the line bundle $K+\alpha\delta$ induces a character of $\mathrm{Aut}(C)$ or, in other words, a linearization of the structure sheaf on $\mathbb{T}^1(C)$. Correspondingly, there are variation of GIT chambers $V^+, V^- \subseteq \mathbb{T}^1(C)$ which is identified formally locally with the closed substacks $\overline{\mathcal{M}}_g(\alpha) \setminus \overline{\mathcal{M}}_g(\alpha+\epsilon)$ and $\overline{\mathcal{M}}_g(\alpha) \setminus \overline{\mathcal{M}}_g(\alpha-\epsilon)$. These three heuristics are each quite explicit thereby providing a useful strategy for determining which singularities should be added at α and what locus of curves should be removed at $\alpha - \epsilon$. Using these techniques, one can make the following predictions. See [AFS10] for details and notation.

α-value	Singularity added at α	Locus removed at $\alpha - \epsilon$
9/11	A_2	elliptic tails attached nodally
7/10	A_3	elliptic bridges/chains attached nodally
2/3	A_4	$g = 2$ tails attached nodally at a Weierstrass point
19/29	$A_5^{\{1\}}$	$g = 2$ tails attached nodally
	$A_{3/4}$	$g = 2$ tails attached tacnodally at a Weierstrass point
17/28	A_5	$g = 2$ bridges/chains attached nodally at conjugate points
49/83	A_6	hyperelliptic $g = 3$ tails attached nodally at a Weierstrass point
32/55	$A_7^{\{1\}}$	hyperelliptic $g = 3$ tails attached nodally
42/73	$A_{3/6}$	hyperelliptic $g = 3$ tails attached tacnodally at a Weierstrass point

Table 1. Predictions for the minimal model program for $\alpha > 5/9$.

In regard to step (2) of the above strategy, there is a serious technical hurdle on the account for the fact that the moduli stacks $\overline{\mathcal{M}}_g(\alpha)$ are not seperated nor Deligne-Mumford. Therefore, one requires a generalization of the Keel-Mori theorem [KM97]. This is a very difficult and subtle question. Nevertheless, we have

Theorem A. *Let $\mathcal{X}$ be an algebraic stack of finite type over k. Suppose that:*

(1) *For every closed point $x \in \mathcal{X}$, there exists an affine, étale neighborhood $f\colon [\mathrm{Spec(A)/G_x}] \to \mathcal{X}$ of x such that f is stabilizer preserving at closed points of $[\mathrm{Spec(A)/G_x}]$ and f sends closed points to closed points.*

(2) *For any $x \in \mathcal{X}(k)$, the closed substack $\overline{\{x\}}$ admits a good moduli space.*

Then $\mathcal{X}$ admits a good moduli space.

This theorem should only be viewed as a modest generalization of the Keel-Mori theorem as the hypotheses are much stronger. However, it is not clear that there is a better characterization of algebraic stacks admitting a good moduli space as there are compelling counterexamples to the existence of a good moduli space if any of the above hypotheses is dropped.

What makes Theorem A an interesting and powerful general tool is that for many moduli problems the above hypotheses can be in fact explicitly verified. In particular, at least for $\alpha > \frac{2}{3} - \epsilon$, one can prove the existence of a good moduli space. Finally, for the step (3) in the strategy, ampleness of $K + \alpha\delta$ can be established by standard, but by no means straightforward, intersection theory computations. In the talk, we announced a successful completion of the above strategy for the case of $\alpha > \frac{2}{3} - \epsilon$ but we expect our methods to work for the next several flips.

Theorem B. *For $\alpha > \frac{2}{3} - \epsilon$, there are algebraic stacks $\overline{\mathcal{M}}_g(\alpha)$ parameterizing certain curves with at worst ramphoid cusps (i.e., $A_4\colon y^2 = x^5$ singularities) that have as their good moduli spaces the log canonical models $\overline{M}_g(\alpha)$.*

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Algebraic cycles on singular varieties

Vasudevan Srinivas

In this talk, I discussed some aspects of algebraic cycles on singular varieties, from the perspective of the possible relations with the Grothendieck group of vector bundles. An overall reference for this talk is [9]. For simplicity I restricted attention to studying projective varieties over an algebraically closed field. I discussed 4 related themes.

1) *What is the definition of the Chow ring?*

We would like to associate to a projective variety X over an algebraically closed field k a Chow ring $CH^*(X)$ which has the "expected properties", including contravariant functoriality, and good theories of cycle classes and Chern classes, giving a Chern character isomorphism between the Grothendieck group of vector bundles and the Chow ring, both tensored with $\mathbb{Q}$ (so we do have a good definition with rational coefficients).

Such a construction of a Chow ring has not yet been accomplished in the published literature. There was a definition of Fulton ([4], §3),

$$CH^*(X) = \lim_{f:X\to Y} CH^*(Y),$$

that is, the Chow ring of X is taken to be the direct limit of Chow rings of the nonsingular varieties Y to which X maps. Fulton established some of the desired properties, but the cycle class property is not clear with his definition. M. Levine (unpublished) gave another (?) definition for $CH^*(X)$ satisfying some properties, including a cycle class property, but for which contravariant functoriality is not established. So this is an interesting open problem.

2) *Codimension of support*

One might expect that if $\mathcal{F}$ is a coherent sheaf of finite homological dimension on a projective variety X, and $\operatorname{supp}\mathcal{F}$ has codimension p, then there is an associated "cycle" $[\mathcal{F}]$ in $CH^p(X)$, such that if $\{\mathcal{F}\} \in K_0(X)$ is the associated class (which certainly exists, since $\mathcal{F}$ has finite homological dimension), then in $CH^*(X) \otimes \mathbb{Q}$ (which is defined unambigously) we would have the relations

(i) $c_i(\mathcal{F}) = 0$ for $i < p$
(ii) $c_p(\{\mathcal{F}\}) = a[\mathcal{F}]$ in $CH^p(X) \otimes \mathbb{Q}$ (with $a = (-1)^{p-1}(p-1)!$)

However, this is *wrong*. The first example (expressed in different language) is due to Dutta, Hochster and McLaughlin: they showed that if $X \subset \mathbb{P}^4$ is the projective cone over $\mathbb{P}^1 \times \mathbb{P}^1 \subset \mathbb{P}^3$, then there exists a skyscraper sheaf $\mathcal{F}$ of finite length and finite homological dimension, supported at the unique singular point of X, such that $c_2(\mathcal{F})$ is non-trivial, and in fact has non-zero intersection number with the class of a line, which is a ruling contained in $\mathbb{P}^1 \times \mathbb{P}^1$, regarded as the hyperplane at infinity in X.

This was explained and refined in work of Levine, and then Paul Roberts and myself (see [8], and further references given there), by relating this phenomenon to Thomason's localization theorem in algebraic K-theory.

3) *Hodge Conjecture*

Here, we first pointed out (with appropriate explanations) that there is a simple counterexample (see [1]) to the "naive" statement of the Lefschetz (1,1) theorem, given by the quartic surface $X \subset \mathbb{P}^3_{\mathbb{C}}$ defined by

$$w(x^3 - y^2 z) + (x^4 + y^4 + z^4) = 0.$$

We then stated the singular Lefschetz (1,1) theorem (see [2], [3]): let X be a complex projective seminormal variety; then

$$\text{image}\, c_1(\text{Pic}\,(X)) \cong \text{Hom}_{MHS}(\mathbb{Z}(-1), L^1 H^2(X, \mathbb{Z})),$$

where $L^1 H^2(X, \mathbb{Z}) \subset H^2(X, \mathbb{Z})$ consists of the subgroup of Zariski locally trivial classes. Here, a class is called *Zariski locally trivial* if it lies in the kernel of the restriction map $H^2(X, \mathbb{Z}) \to \oplus H^2(U_i, \mathbb{Z})$, for some Zariski open covering $\{U_i\}$ of X). The motivation for this is discussed in [1].

This suggests an analogue of the Hodge conjecture (see [2]). Let $\pi^X : X \to X_{Zar}$ be the identity map, considered as a continuous map where X has the complex topology, and X_{Zar} the Zariski topology. There is an associated Leray spectral sequence

$$E_2^{p,q} = H^p(X_{Zar}, R^q \pi_*^X \mathbb{Z}) \implies H^{p+q}(X, \mathbb{Z})$$

and a corresponding *Leray filtration* $L^{\cdot} H^n(X, \mathbb{Z})$ on each (Betti) cohomology group. One may ask:

(i) is $\{L^{\cdot} H^n(X, \mathbb{Z})\}$ a filtration by sub mixed Hodge structures?
(ii) is the "Hodge conjecture" true for $L^p H^{2p}(X, \mathbb{Q})$, i.e., is the image in $H^{2p}(X, \mathbb{Q})$ of the p-th Chern class map on $K_0(X) \otimes \mathbb{Q}$ isomorphic with $\text{Hom}_{MHS}(\mathbb{Z}(-p), L^p H^{2p}(X, \mathbb{Z})) \otimes \mathbb{Q}$?

Both properties hold in the nonsingular case, from Bloch-Ogus theory. This gives a sort of "cohomological Hodge conjecture" in the singular case. It is unknown in general even for curves on 3-folds.

Bloch had proposed a counterexample to a possible cohomological Hodge conjecture (see [5], Appendix A). It is unclear (to me) if this is also a counterexample to (ii); however, Bloch's example, as given, is not defined over a number field, and (conjecturally – part of the "Bloch-Beilinson conjectures") such a construction is not possible over a number field. So one may hope that (ii) always holds for varieties X which can be defined over an algebraic number field.

4) *Extra Structure*

We discussed concrete statements about vanishing of the Chow group 0-cycles of affine varieties resulting from the Bloch-Beilinson conjectures, over $\mathbb{C}$ and over $\overline{\mathbb{Q}}$: on the one hand, vanishing over $\mathbb{C}$ involves a cohomological condition on a compactification, while vanishing over $\overline{\mathbb{Q}}$ is expected to always hold, in dimensions > 1.

Thus, conjecturally, we expect that if $A = \frac{\overline{\mathbb{Q}}[x,y,z]}{(x^4+y^4+z^4-1)}$, then we must have (i) $CH^2(\text{Spec}\, A) = 0$, while (ii) $CH^2(\text{Spec}\,(A \otimes \mathbb{C})$ is nontrivial.

In fact (ii) is a consequence of Mumford's famous result on the infinite dimensionality of the Chow group of 0-cycles on surfaces with $p_g > 0$. The property in (i) is an open question, and an instance of the Bloch-Beilinson conjectures.

We closed with a brief discussion of a joint work with A. Krishna (see [6], [7]). We proved:
let $Y \subset \mathbb{P}^n_{\overline{\mathbb{Q}}}$ be a smooth projective curve over the field of algebraic numbers, with homogeneous coordinate ring A, and affine cone $X = \mathrm{Spec}\,(A)$; then

(i) $CH^2(X) = 0$
(ii) if $X_{\mathbb{C}}$ is the correponding complex surface, then

$$CH^2(X_{\mathbb{C}}) = 0 \Leftrightarrow H^1(Y, \mathcal{O}_Y(1)) = 0$$

where $\mathcal{O}_Y(1)$ is the very ample line bundle giving the above projective embedding.

A concrete consequence (of "Bloch-Beilinson type") of the above result with Krishna is the following. Let $(a, b, c) \in \mathbb{A}^3_{\mathbb{C}} \setminus \{0\}$ be a point different from the origin, satisfying (say) $a^4 + b^4 + c^4 = 0$. Then:
the maximal ideal $(x - a, y - b, z - c) \subset \mathbb{C}[x, y, z]$ can be expressed as

$$(x - a, y - b, z - c) = (x^4 + y^4 + z^4, g(x, y, z), h(x, y, z))$$

for *some* polynomials $g, h \in \mathbb{C}[x, y, z] \Leftrightarrow$
the point $[a : b : c] \in \mathbb{P}^2$ is a $\overline{\mathbb{Q}}$-rational point.

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Logarithmic Gromov–Witten invariants

Dan Abramovich
(joint work with Qile Chen, Danny Gillam, Mark Gross, Steffen Marcus, Bernd Siebert and Jonathan Wise)

The definition of standard Gromov–Witten invariants of a complex projective variety $\underline{X}$ requires a moduli space $M := \overline{M}_{g,n}(\underline{X}, \beta)$ of n-pointed stable maps $\underline{f} : \underline{C} \to \underline{X}$, admitting evaluation maps $e_i : M \to \underline{X}$, $i = 1, \ldots n$, and possessing a well-behaved virtual fundamental class $[M]^{vir}$. Given these, and given n cohomology classes $\gamma_i \in H^*(X, \mathbb{Z})$ one defines the invariants as follows:

$$\langle \gamma_1 \cdots \gamma_n \rangle^X_{g,\beta} = \int_{[M]^{vir}} e_1^* \gamma_1 \cdots e_n^* \gamma_n.$$

Such moduli spaces and their evaluation maps exist in great generality. Virtual fundamental classes usually require $\underline{X}$ to be smooth. It is often advantageous to define Gromov–Witten invariants of singular varieties, for instance when $\underline{X}$ is a degeneration, which might have simple geometry, of a smooth variety X', which might have complicated geometry.

Logarithmic Gromov–Witten theory, envisioned by Siebert in 2001 [8], allows us to do that for *logarithmically smooth* varieties X, where the underlying variety $\underline{X}$ often has singularities of toric nature. The main results are

(1) the existence of moduli spaces $M := \overline{M}_{g,n}(X, \beta)$ of n-pointed stable *logarithmic* maps $f : C \to X$ of genus g and class β ([5, 4, 1]);
(2) the existence of evaluation maps $e_i : M \to \underline{\wedge}X$, $i = 1, \ldots n$ ([5, 2]); and
(3) the existence of a well-behaved virtual fundamental class $[M]^{vir}$ (automatic when X is logarithmically smooth).

The talk introduced logarithmic structures ([7]), the moduli of log smooth curves ([6]), indicated the meaning of the results above, comparison with earlier work ([3]) and a computation, an example of ongoing joint work with Chen, Gross and Siebert.

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Universal polynomials for singular sections of vector bundles

Yu-jong Tzeng
(joint work with Jun Li)

It is a classical problem to find the number of r-nodal curves of some fixed degree d on the complex projective plane $\mathbb{P}^2$. One can also ask a similar question for general smooth projective surface S and a line bundle L, i.e. the number of r-nodal curves in a generic r-dimensional linear subsystem of $|L|$. Göttsche conjectured that for any $r \geq 0$, there exists a universal polynomial T_r of degree r, such that $T_r(L^2, LK_S, c_1(S)^2, c_2(S))$ equals the number of r-nodal curves in a general linear subsystem, provided that L is $(5r-1)$-very ample. Moreover, the generating series of T_r has a multiplicative structure and satisfies the Göttsche-Yau-Zaslow formula ([Gö], [Tz]). There are now two proofs of Göttsche's universality conjecture by the author [Tz] using degeneration methods, and by Kool-Shende-Thomas [KST] using BPS calculus and computation of tautological integrals on Hilbert schemes (see also the approach of Liu [Liu1] [Liu2]). In this report, we answer the question whether a similar phenomenon is true for curves with higher singularities, and for subvarieties in higher dimensional spaces?

Consider a collection of isolated singularity type $\underline{\alpha} = (\alpha_1, \alpha_2, \ldots, \alpha_{l(\underline{\alpha})})$. We say a variety Y has singularity type $\underline{\alpha}$ if there exists $l(\underline{\alpha})$ points y_1, y_2,$\ldots y_{l(\underline{\alpha})}$, such that the singularity type of Y at y_i are exactly α_i and Y has no more singular points. The following two theorem give affirmative answers to the previous question:

Theorem 1. For every collection of isolated analytic or topological singularity type $\underline{\alpha}$, there exists a universal polynomial $T_{\underline{\alpha}}(x, y, z, t)$ of degree $l(\underline{\alpha})$ with the following property: given a smooth projective surface S and an $(N(\underline{\alpha}) + 2)$-very ample line bundle L on S, a general codim($\underline{\alpha}$)-dimensional sublinear system of $|L|$ contains exactly $T_{\underline{\alpha}}(L^2, LK, c_1(S)^2, c_2(S))$ curves with singularity type $\underline{\alpha}$.

Theorem 2. Suppose X a smooth projective variety and E is an $(N(\underline{\alpha}) + 2)$-very ample line bundle vector bundle on X, consider subvarieties given by sections of $|E|$ in a general codim($\underline{\alpha}$)-dimensional sublinear system. For every collection of isolated analytic singularity type $\underline{\alpha}$, there exists a universal polynomial $T_{\underline{\alpha}}$ of degree $l(\underline{\alpha})$ in the Chern numbers of X and E that gives the number of such subvarties with singularity type $\underline{\alpha}$.

For instance, there is a universal polynomial which counts curves with one triple point, two E_8 singularity, and one 5-fold point analytic equivalent to $x^5 - y^5 = 0$ on surfaces.

The main difference of two theorems is we are only able to deal with topological singularities for curves, not for higher dimensional varieties. This is because higher dimensional singularities may have wild behavior even if they are topological the same. For example it is unknown that whether hypersurfaces with same topological singularity have the same multiplities at the singular points.

To characterize the conditions of given singularity type, we study the locus of zero-dimensional closed subschemes of a special shape. This is possible because our subvarieties are zero locus of sections of vector bundles so all singularities are isolated complete intersection singularities. Also, it is well known that analytic isolated complete intersection singularities are finitely determined (i.e. only depends on the quotient of the local ring of the singular point by some power of the maximal ideal). These facts allow us to associate every singularity type α with the all isomorphism classes of $\operatorname{Spec} \mathcal{O}_{Y,y}/\mathfrak{m}^{k+1}$ (called type α) for some Y has singularity α at y and α is k-determined. It is easy to check if a variety has singularity α then it must contain a zero-dimensional closed subscheme of type α. Moreover, the converse is an open condition for subvarties in linear systems of vector bundles.

The numbers $N(\underline{\alpha})$, $\operatorname{codim}(\underline{\alpha})$ in the theorems are defined as follows: $N(\underline{\alpha})$ is the sum of the lengths of zero-dimensional schemes of type α_i, $\operatorname{codim}(\underline{\alpha})$ is expected codimension of subvarties to have singularities $\underline{\alpha}$.

Consequently, the enumeration of subvarieties with give singularity types can be achieved by counting subvarieties containing zero dimensional schemes of corresponding types. By the Thom–Porteous formula, the latter can be expressed again as intersection numbers on Hilbert schemes of points on S, if the vector bundle is sufficiently ample. Then we apply a degeneration argument developed in [Tz] to show the existence of universal polynomials using algebraic cobordism.

The version of algebraic cobordism theory we use is developed by Levine and Pandharipande [LP]. They formulated the algebraic cobordism theory by identifying schemes which satisfy the double point relation. Since our goal is to count subvarieties given by zero sections of a vector bundle E on X, we need the algebraic cobordism theory of vector bundles and varieties constructed by [LeeP].

Let X_i be smooth projective surfaces and L_i be line bundles on X_i. We call

$$[X_0, E_0] - [X_1, E_1] - [X_2, E_2] + [X_3, E_3]$$

a *double point relation* if there exists a flat family of smooth projective surfaces $\pi : \mathcal{X} \to \mathbb{P}^1$ and a line bundle $\mathcal{E}$ on $\mathcal{X}$ satisfying the following properties: $\mathcal{X}$ is smooth and X_0 is the fiber over $0 \in \mathbb{P}^1$ and it is smooth. Secondly, the fiber over $\infty \in \mathbb{P}^1$ is the union of two smooth components X_1 and X_2 intersecting transversally along $D = X_1 \cap X_2$. Thirdly, E_i is the restriction of $\mathcal{E}$ onX_i for $i = 0, 1, 2$. Lastly, $X_3 = \mathbb{P}(1_D \oplus N_{X_1/D})$ and $E_3 = \eta^*(\mathcal{E}|_D)$ where $\eta : X_3 \to D$ is the projection.

Define the algebraic cobordism group of varieties of dimension n and vector bundles of rank r $\omega_{n,r}$ to be the vector space over $\mathbb{Q}$ spanned by all pairs $[X, E]$ modulo double point relations. Lee and Pandharipande [LeeP] proved $\omega_{n,r}$ is a vector space of dimension equal to the number of Chern numbers of X and E. Also they can write down a set of basis for every n and r.

Recall that the number of subvarties with singularity type $\underline{\alpha}$ can be counted by the number of subvarties containing subschemes of type $\alpha_1, \ldots, \alpha_{l(\underline{\alpha})}$, which is evaluated by a intersection number on Hilbert schemes of X. Call this intersection number to be $d_{\underline{\alpha}}(X, E)$, and define

$$\phi(X, E) = \sum_{\underline{\alpha}}^{\infty} d_{\underline{\alpha}}(X, E) x_{\alpha_1} x_{\alpha_2} \cdots x_{\alpha_{l(\underline{\alpha})}}.$$

For a family of surfaces $\pi : \mathcal{X} \to \mathbb{P}^1$ which defines a double point relation, Li and Wu [LW] constructed a family of Hilbert schemes of n points. By studying the limit of $\phi(X_t, E_t)$ we obtain

Proposition. If $[X_0, E_0] - [X_1, E_1] - [X_2, E_2] + [X_3, E_3]$ is a double point relation, then

$$\phi(X_0, E_0)\phi(X_1, E_1)^{-1}\phi(X_2, E_2)^{-1}\phi(X_3, E_3) = 1.$$

Consequently, $[X, E] \to \phi(X, E)$ is a group homomorphism from $\omega_{n,r}$ to the group of power series with multiplication.

Corollary. For every smooth projective variety X and vector bundle E on X,

$$\phi(X, E) = \prod A_I^{C_I(X,E)}$$

indexed over all Chern numbers of X and E. Therefore $d_{\underline{\alpha}}(X, E)$ is the universal polynomial that counts subvarieties with singularity $\underline{\alpha}$.

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Enriques surfaces with small/large automorphism groups

Shigeru Mukai

The automorphism group $\operatorname{Aut} S$ of an Enriques surface S, being always discrete, is infinite if S is (very) general, and shrinks when S becomes special and has smooth rational curves on it. As the most extremal case, Nikulin[12] and Kondo[5] classified the Enriques surfaces with only finitely many automorphisms into 7 types I,–, VII. In this talk I explained my recent result (and its proof) on the next extremal case.

A group is called *almost abelian* if it contains a finitely generated abelian group as normal subgroup of finite index.

Theorem A. *The automorphism group* $\operatorname{Aut} S$ *of an Enriques surface* S *is almost abelian if and only if either* $\operatorname{Aut} S$ *is finite or* S *is of type* E_8*, that is, the twisted Picard group* $\operatorname{Pic}^{\omega} S$ *contains the (negative definite) root lattice of type* E_8 *as sublattice.*

An Enriques surface of type E_8 is studied by Horikawa[4], Barth-Peters[1], Dolgachev[3], etc. It is characterized by the presence of a cohomologically trivial involution and isomorphic to the quotient $X_{a:b:c}/\varepsilon$, where $X_{a:b:c}$ is the minimal model of the function field of two variables

$$\mathbb{C}\left(x, y, \sqrt{a(x+\frac{1}{x}) + b(y+\frac{1}{y}) + 2c}\right)$$

and the involution ε interchanges $(x, y, \sqrt{})$ and $(1/x, 1/y, -\sqrt{})$, where $a, b \in \mathbb{C}^{\times}$ and $c \in \mathbb{C}$ are constants with $a \pm b \pm c \neq 0$ (*cf.* [10] and [7, §4]).

By virtue of the Torelli type theorem, the moduli space of Enriques surfaces S is the quotient D^{10}/Γ of the 10-dimensional bounded symmetric domain D^{10} of type IV by a certain arithmetic group Γ. The stabilizer group $\operatorname{Stab} S$ of Γ at the period of S, which is a point of D^{10}, is always finite, trivial if S is general and gets larger along closed subvarieties in D^{10}/Γ. (For a primitively polarized K3 surface (X, h) of degree $(h^2) \geq 4$, the automorphism group $\operatorname{Aut}(X, h)$ coincides with the stabilizer group.)

I also reported the following joint result, which is an Enriques version of [6] (*cf.* [8], [9] and [11]).

Theorem B. (with H. Ohashi) *A finite group* G *has an (effective) Mathieu semi-symplectic action on an Enriques surface if and only if* G *is a subgroup of* $\mathfrak{A}_6, \mathfrak{S}_5, (C_3)^2.D_8$*,* $C_2 \times \mathfrak{A}_4$ *or* $C_2 \times C_4$*.*

Here $\mathfrak{A}_k$ (resp. $\mathfrak{S}_k$) is the alternating (resp. symmetric) group of degree k, and C_n (resp. D_n) is the cyclic (resp. dihedral) group of order n. A finite Mathieu semi-symplectic automorphism group G injects to $\operatorname{Stab} S$.

To prove both Theorem A and B, the lattice invariant in the twisted cohomology group, which is a refinement of the root invariants of Nikulin[12], plays an essential role. For example the following is used for that purpose.

Proposition C. *If an Enriques surface is of type A_9 or $A_4 + A_5$, then it has a Mathieu semi-symplectic action of the alternating group $\mathfrak{A}_5$.*

Enriques surfaces of type A_9 (resp. $A_4 + A_5$) form a 1-dimensional family, which contains the Enriques surface of Kondo's type VII (resp. type VI) as special member.

1. Twisted cohomology

The kernel of the Gysin map $\pi_* : H^2(X, \mathbb{Z}) \to H^2(S, \mathbb{Z})$ is a free $\mathbb{Z}$-module of rank 12, which we call the *twisted cohomology group* of an Enriques surface S and denote by $H^\omega(S, \mathbb{Z})$, where $\pi : X \to S$ is the canonical covering. $H^\omega(S, \mathbb{Z})$ is isomorphic to the second cohomology group $H^2(S, \mathbb{Z}_S^\omega)$ of S with coefficients in the non-trivial local system $\mathbb{Z}_S^\omega$. Hence, the natural pairing $\mathbb{Z}_S^\omega \times \mathbb{Z}_S^\omega \to \mathbb{Z}_S$ induces a $\mathbb{Z}$-valued bilinear form on $H^\omega(S, \mathbb{Z})$, for which the following hold (see [2]):

- $H^\omega(S, \mathbb{Z})$ is an odd unimodular lattice $I_{2,10}$ of signaure $(2, 10)$.
- $H^\omega(S, \mathbb{Z})$ carries a polarized Hodge structure $H^\omega(S)$ of weight 2 with Hodge number $(1, 10, 1)$.
- The $(1, 1)$-part of $H^\omega(S)$ is the kernel of the pushforward map $\pi_* : \operatorname{Pic} X \to \operatorname{Pic} S$. We call it the *twisted Picard lattice* and denote by $\operatorname{Pic}^\omega S$.

The twisted Picard lattice $\operatorname{Pic}^\omega S$ is negative definite and does not contain a (-1)-element by Riemann-Roch. Let L be such a lattice, that is, negative definite and $\not\ni (-1)$-element.

Definition D. An Enriques surface S is *of lattice type* L (resp. L_w) if the twisted Picard group $\operatorname{Pic}^\omega S$ contains L as primitive sublattice and if the orthogonal complement of $L \hookrightarrow H^\omega(S, \mathbb{Z})$ is odd (resp. even).

The number of moduli of Enriques surfaces of type L is equal to $10 - \operatorname{rank} L$.

Example E. An Enriques surface is of Lieberman type if it is isomorphic to the quotient of a Kummer surface $\operatorname{Km}(E_1 \times E_2)$ of product type by ε_+, where E_i, $i = 1, 2$, is an elliptic curve and ε_+ is the composite of $(-1_E, 1_E)$ and the translation by a 2-torsion (a_1, a_2) with $0 \neq a_i \in (E_i)_{(2)}$. An Enriques surface S is of Lieberman type if and only if it is of type $D_{8,w}$, that is, $\operatorname{Pic}^\omega S$ contains D_8 primitively and the orthogonal complement of $D_8 \hookrightarrow H^\omega(S, \mathbb{Z})$ is isomorphic to $U + U(2)$, where U (resp. $U(2)$) denotes the rank 2 lattice given by the symmetric matrix $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ (resp. $\begin{pmatrix} 0 & 2 \\ 2 & 0 \end{pmatrix}$). (If S is of type D_8, then the orthogonal complement is $\langle 1 \rangle + \langle -1 \rangle + U(2)$.) The Enriques surface of Kondo's type III is of Lieberman type with $E_1 = E_2 = \mathbb{C}/(\mathbb{Z} + \mathbb{Z}\sqrt{-1})$.

If C is a smooth rational curve on an Enriques surface, then its pullback splits into the disjoint union of two curves C_0 and C_1. Their difference $[C_0] - [C_1]$ defines a (-2)-element in $\operatorname{Pic}^\omega S$ up to sign. In particular, a tree of smooth rational curves on S of ADE-type defines a negative definite sublattice of the same type in $\operatorname{Pic}^\omega S$.

2. Outline of proof of Theorem A

An Enriques surface S has always an elliptic fibration $\Phi : S \to \mathbb{P}^1$. The rank of the Mordell-Weil group of its Jacobian fibration is called the *MW-rank* of Φ for short.

Assume that $\operatorname{Aut} S$ is not finite but almost abelian. Then S has an elliptic fibration $\Phi_0 : S \to \mathbb{P}^1$ of positive MW-rank and other elliptic fibrations $\Phi \neq \Phi_0$ have MW-rank zero. In particular S has one and only one elliptic fibration of positive MW-rank modulo $\operatorname{Aut} S$. By a computation of $\operatorname{Pic}^\omega S$ similar to [12], S is of type either $E_8, A_9, E_7 + A_2, (A_5 + A_5)^+$ or $(D_6 + A_3 + A_1)^+_w$, where L^+ denotes an odd lattice which contains L as sublattice of index 2. Except for E_8, S has more than one elliptic fibrations of positive MW-rank. For example, in the case of type A_9, it is deduced from the action of $\mathfrak{A}_5$ given by Proposition C. In the last case, it is deduced from the fact that S is the normalization of the diagonal Enriques sextic

$$\bar{S} : (x_0^2 + x_1^2 + x_2^2 + x_3^2) + \sqrt{-1}\left(\frac{1}{x_0^2} + \frac{1}{x_1^2} + \frac{1}{x_2^2} + \frac{1}{x_3^2}\right) x_0 x_1 x_2 x_3 = 0$$

in $\mathbb{P}^3$. (This equation was found in [11] as the octahedral Enriques sextic.)

In the case of type E_8, the automorphism group of $S = X_{a:b:c}/\varepsilon$ is almost abelian by [1].

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Prime Fano fourfolds of degree ten

Laurent Manivel
(joint work with Olivier Debarre and Atanas Iliev)

A general complex prime Fano fourfold of index two and degree ten is the intersection of the Grassmannian $G(2,5) \subset \mathbb{P}^9$ with a hyperplane and a quadric. Such fourfolds behave in many respects very much like cubic fourfolds:

(1) They are unirational.
(2) Some of them are known to be rational, but the general one is expected to be non rational.
(3) Their Hodge number $h^{3,1} = 1$, and their Hodge structure in middle dimension is *of K3 type*.
(4) One can construct from them some holomorphic symplectic manifolds, and even a locally complete family of such manifolds.
(5) The period map defined by their special Hodge structure, or that of the associated symplectic manifolds, behave nicely.
(6) *Special* fourfolds, characterized by the existence of extra Hodge classes, are parametrized by a countable family of divisors in the period domain.

Extending the Beauville-Donagi construction. When $Y \subset \mathbb{P}^5$ is a smooth cubic fourfold, the Fano variety $F(Y)$ parametrizing lines contained in Y is a holomorphic symplectic fourfold, and there is an isometry of Hodge structures between the primitive cohomology lattices $H^4(Y,\mathbb{Z})_0$ and $H^2(F(Y),\mathbb{Z})_0$ [1].

If $X = G(2,5) \cap H \cap Q$ is a generic prime Fano fourfold of degree ten, the variety $F_c(X)$ parametrizing conics contained in X is a smooth projective variety of dimension five. Moreover, a suitable rational quotient is a holomorphic symplectic fourfold, denoted dEPW_X (see [3] for a precise statement). In fact these symplectic fourfolds were constructed before in a different way by O'Grady, as double covers $\mathrm{dEPW}_X \to \mathrm{EPW}_X \subset \mathbb{P}^5$ of very special singular sextic hypersurfaces in $\mathbb{P}^5$, called Eisenbud-Popescu-Walter sextics [4]. Again there is an isometry of Hodge structures between the primitive cohomology lattices $H^4(X,\mathbb{Z})_0$ and $H^2(\mathrm{dEPW}_X,\mathbb{Z})_0$.

Period maps. The lattice $H^4(X,\mathbb{Z})_0$ is isomorphic to $\Lambda = 2E_8 \oplus 2U \oplus 2A_1$. There is an associated local period domain Ω, a bounded symmetric domain of type IV, with an action of the *stable orthogonal group* Γ, a finite index subgroup of $O(\Lambda)$. Then we have natural period maps from the moduli space $\mathcal{EPW}$ of double EPW sextics, and from the moduli stack $\mathcal{X}$ of prime Fano fourfolds of degree ten, to the global period domain $\mathcal{D} = \Omega/\Gamma$. They fit into a commutative diagram

$$\begin{array}{ccc} \mathcal{X} & \longrightarrow & \mathcal{EPW} \\ & {\scriptstyle p}\searrow \quad \swarrow{\scriptstyle p_{EPW}} & \\ & \mathcal{D} & \end{array}$$

Note that $\mathcal{X}$ has dimension 24 while $\mathcal{EPW}$ and $\mathcal{D}$ both have dimension 20. The map p_{EPW} is known to be an open immersion.

Theorem. *The period map p is everywhere submersive.*

In particular its fibers are smooth and four dimensional. We expect that:

(1) The fibers of p are connected and parametrize birationally equivalent prime Fano fourfolds.
(2) The moduli stack $\mathcal{X}$ is (birationally) equivalent to the universal family over $\mathcal{EPW}$.

Special loci. A prime Fano fourfold X of degree ten is *special* if its cohomology contains Hodge classes not coming from the Grassmannian. (Special cubic fourfolds were studied in [2].) The Hodge conjecture being true at least over $\mathbb{Q}$, this means that X contains some special algebraic cycles. In terms of lattices, a special Hodge class will generate, with those classes coming from the Grassmannian, a rank three positive definite lattice K with $2A_1 \subset K \subset \Lambda$, and the condition of being orthogonal to K defines a hyperplane in the local period domain Ω. The discriminant d of K is always equal to $0, 2$ or 4 mod 8.

Proposition. *When $d = 0$ mod 4, by projection to $\mathcal{D}$ one obtains an irreducible divisor $\mathcal{C}_d$, depending only on d. When $d = 2$ mod 8 one obtains two irreducible divisors $\mathcal{C}'_d$ and $\mathcal{C}''_d$.*

Examples.

(1) Suppose that X contains a plane P, parametrizing lines in $\mathbb{P}^4$ contained in a fixed plane. Then X defines a point in $\mathcal{C}_{12}$. The projection from P maps X birationally to a cubic Y in $\mathbb{P}^5$, containing a cubic scroll. The variety of conics $F_c(X)$ is birationally equivalent to the Fano variety of lines $F(Y)$.
(2) Suppose that X contains a plane P, parametrizing lines in $\mathbb{P}^4$ having a common point. Then X defines a point in $\mathcal{C}'_{10}$. The projection from P maps X birationally to a quadric Q in $\mathbb{P}^5$. In particular X is rational.
(3) Suppose that X contains a quadric surface Σ, parametrizing lines in $\mathbb{P}^4$ incident to two given skew lines. Then X defines a point in $\mathcal{C}''_{10}$. The projection from Σ maps X birationally to $\mathbb{P}^4$. In particular X is again rational.

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Geometric methods for descent on K3 surfaces

Brendan Hassett
(joint work with Anthony Várilly-Alvarado)

Let X be a K3 surface defined over a number field K, and $X(K)$ the rational points on X. Let $K_v \supset K$ denote the completion of K at some place v. Clearly

$$X(K) \subset \prod_v X(K_v)$$

so if $X(K_v) = \emptyset$ for some v then $X(K) = \emptyset$. When the converse holds we say X satisfies the *Hasse principle*; when $X(K)$ is dense in the product then X satisfies *weak approximation*. We do not have a general algorithm for determining whether a K3 surface admits rational points over a given field, and if so, whether there are infinitely many such points over that field. However, the Hasse and weak approximation principles give such algorithms, when they hold true.

The Hasse principle and weak approximation sometimes fail due to the Brauer-Manin obstruction. Each rational point $x \in X(K)$ induces a homomorphism $x^* : \mathrm{Br}(K) \to \mathrm{Br}(X)$ left-inverse to the homomorphism $\mathrm{Br}(K) \to \mathrm{Br}(X)$ induced by the structure map. Using the fundamental exact sequence of class field theory

$$0 \to \mathrm{Br}(K) \to \oplus_v \mathrm{Br}(K_v) \to \mathbb{Q}/\mathbb{Z} \to 0,$$

an element $\alpha \in \mathrm{Br}(X)$ may obstruct an element of $\prod_v X(K_v)$ from arising from a rational point. A recent result of Skorobogatov-Zarhin [4] implies $\mathrm{Br}(X)/\mathrm{Br}(K)$ is finite, so it is natural to seek to enumerate these obstructions.

By definition, a K3 surface X is general if $\mathrm{Pic}(X_{\mathbb{C}}) \simeq \mathbb{Z}$. While general K3 surfaces have straightforward geometry, their defining equations tend to be complicated; K3 surfaces with simple equations tend to have complicated geometry and *vice versa*. Indeed, before 2011 no examples of general K3 surfaces violating weak approximation or the Hasse principle were known. (See [1, 2] for detailed references.)

Theorem 1. [1] *Let X be the K3 surface of degree two over $\mathbb{Q}$ given by $w^2 = \det M$, where M is the matrix:*

$$\begin{pmatrix} 2(2x+3y+z) & 3x+3y & 3x+4y & 3y^2+2z^2 \\ 3x+3y & 2z & 3z & 4y^2 \\ 3x+4y & 3z & 2(x+3z) & 4x^2+5xy+5y^2 \\ 3y^2+2z^2 & 4y^2 & 4x^2+5xy+5y^2 & 2(2x^3+3x^2z+3xz^2+3z^3) \end{pmatrix}$$

Then X is general and there is a (transcendental) Brauer-Manin obstruction to weak approximation on X.

Theorem 2. [2] *Let X be a K3 surface of degree two over $\mathbb{Q}$ given by*

$$w^2 = -\frac{1}{2} \cdot \det \begin{pmatrix} 2A & B & C \\ B & 2D & E \\ C & E & 2F \end{pmatrix},$$

where

$$
\begin{aligned}
A &:= -7x^2 - 16xy + 16xz - 24y^2 + 8yz - 16z^2\\
B &:= 3x^2 + 2xz + 2y^2 - 4yz + 4z^2\\
C &:= 10x^2 + 4xy + 4xz + 4y^2 - 2yz + z^2\\
D &:= -16x^2 + 8xy - 23y^2 + 8yz - 40z^2\\
E &:= 4x^2 - 4xz + 11y^2 - 4yz + 6z^2\\
F &:= -40x^2 + 32xy - 40y^2 - 8yz - 23z^2.
\end{aligned}
$$

Then X is general and there is a (transcendental) Brauer-Manin obstruction to the Hasse principle on X.

We shall illustrate how these examples arise from geometric considerations. In each case, the obstruction arises from a two-torsion element $\alpha \in \mathrm{Br}(X)$. The challenge is to represent such elements explicitly and canonically via étale projective bundles:

$$
\begin{array}{ccc}
\mathbb{P}^n & \to & F \\
 & & \downarrow \\
 & & X
\end{array} \tag{1}
$$

The geometric construction behind the first theorem involves cubic fourfolds: Let $P \subset Y \subset \mathbb{P}^5$ denote a cubic hypersurface containing a plane P. Projection from P induces a quadric surface bundle

$$q : \mathrm{Bl}_P(Y) \to \mathbb{P}^2$$

with a plane sextic curve B as discriminant. Consider the relative variety of lines $F_1(q) \to \mathbb{P}^2$, with Stein factorization

$$F_1(q) \to X \to \mathbb{P}^2.$$

Generically, the first morphism is an étale $\mathbb{P}^1$-bundle and the second is a double cover branched over B; thus X is a degree two K3 surface equipped with a $\mathbb{P}^1$-bundle. The class $\alpha \in \mathrm{Br}(X)$ classifies this bundle.

The construction giving the second theorem involves a different class of Fano fourfolds: Let $W \subset \mathbb{P}^2 \times \mathbb{P}^2$ denote a divisor of bidegree $(2, 2)$ and $Y \to \mathbb{P}^2 \times \mathbb{P}^2$ the double cover branched over W. The projections induce quadric surface fibrations

$$q_1, q_2 : Y \to \mathbb{P}^2$$

with discriminant curves $B_1, B_2 \subset \mathbb{P}^2$ of degree six. As before, we obtain degree two K3 surfaces X_1 and X_2, each equipped with an étale $\mathbb{P}^1$-bundle.

These constructions may be placed in a broader context: van Geemen [5] classified orbits of $H^2(X, \mathbb{Z}/2\mathbb{Z})$ under the action of the monodromy group, where X is a degree two K3 surface. Altogether, there are three orbits giving rise to two-torsion Brauer classes on the general such K3 surface; we have exploited two of these orbits. The third has been studied geometrically, e.g., by Mukai [3], but we are not aware of applications to rational points.

This raises some broader questions:

- Are there geometric interpretations for the moduli spaces of K3 surfaces with level-N structure on their middle cohomology?
- Do these give canonical geometric realizations of étale projective bundles (1) classified by the corresponding Brauer classes?
- Are these mediated via the geometry of higher-dimensional holomorphic symplectic manifolds?

For instance, in the first example above we have an embedding

$$F_1(q) \hookrightarrow F_1(Y)$$

into the variety of lines on the cubic fourfold Y, which is holomorphic symplectic.

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The Tate conjecture for $K3$ surfaces over finite fields

François Charles

Let k be a finite field, and let X be a smooth projective variety over k. Given an integer i, let $CH^i(X)$ be the group of algebraic cycles of codimension i in X up to rational equivalence. If ℓ is any prime number invertible in k, general properties of étale cohomology provide us with a cycle class map

$$CH^i(X_{\overline{k}}) \otimes \mathbb{Q}_\ell \to H^{2i}(X_{\overline{k}}, \mathbb{Q}_\ell(i)).$$

The Tate conjecture, as stated in [6], deals with the image of the cycle class map. Recall that the absolute Galois group G of k acts on the étale cohomology group on the right.

Conjecture. The image of $CH^i(X_{\overline{k}}) \otimes \mathbb{Q}_\ell$ in $H^{2i}(X_{\overline{k}}, \mathbb{Q}_\ell(i))$ is the space of elements with a finite orbit under G.

The Tate conjecture is known in significantly fewer cases than the Hodge conjecture. In particular it is not proven for $i = 1$ in full generality. However, it is a theorem of Tate for abelian varieties and $i = 1$.

The case of $K3$ surfaces over finite fields has been addressed in [4] and [5], where the conjecture is proved for $i = 1$ in case the characteristic of k is at least 5 and the $K3$ surface is not supersingular, where by definition a supersingular $K3$ surface is such that the Galois action on the second étale cohomology group is finite.

In the case of supersingular $K3$ surfaces, the conjecture takes a particularly simple form also known as Artin's conjecture.

Conjecture Let X be a supersingular $K3$ surface. Then the second étale cohomology group of X is spanned by cohomology classes of divisors.

In the paper [2], we prove Artin's conjecture over fields of characteristic at least 5, hence completing the proof of the Tate conjecture for $K3$ surfaces over finite fields of characteristic at least 5.

Our proof builds on the important idea, introduced by Maulik in [3], that Borcherds' results [1] constructing automorphic forms related to orthogonal Shimura varieties are relevant in this arithmetic context.

The proof proceeds by studying arithmetic analogs of the classical Noether-Lefschetz divisors, using the Kuga-Satake correspondence to relate them to algebraic cycles on abelian varieties, and by relying on Borcherds' results to get ampleness properties for these divisors.

This method also proves the Tate conjecture for codimension 2 cycles on cubic fourfolds and for divisors on certain reductions of holomorphic symplectic varieties.

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Varieties with vanishing first Chern class

Stefan Kebekus

(joint work with Daniel Greb, Thomas Peternell)

The minimal model program aims to reduce the birational study of projective manifolds with Kodaira dimension zero to the study of associated *minimal models*, that is, normal varieties X with terminal singularities whose canonical divisor is numerically trivial. The ideal case, where the minimal variety X is smooth, is described in the fundamental Decomposition Theorem of Beauville and Bogomolov.

Theorem 1 (Beauville-Bogomolov Decomposition, [Bea83] and references there)**.** *Let X be a compact Kähler manifold whose canonical divisor is numerically trivial. Then there exists a finite étale cover, $X' \to X$ such that X' decomposes as a product*

$$X' = T \times \prod_{\nu} X_{\nu} \tag{1}$$

where T is a compact complex torus, and where the X_ν are irreducible and simply-connected Calabi-Yau– or holomorphic-symplectic manifolds.

Remark 2. The decomposition (1) induces a decomposition of the tangent bundle $T_{X'}$ into a direct sum whose summands have vanishing Chern class, and are integrable in the sense of Frobenius' theorem. Those summands that correspond to the X_ν are stable with respect to any polarisation.

In view of recent progress in minimal model theory, an analogue of Theorem 1 for minimal models would clearly be a substantial step towards a complete structure theory for varieties of Kodaira dimension zero. However, since the only known proof of Theorem 1 heavily uses Kähler-Einstein metrics and the solution of the Calabi conjecture, a full generalisation of Beauville-Bogomolov Decomposition Theorem 1 to the singular setting is difficult.

Main result. The main result presented in this talk is the following Decomposition Theorem for the tangent sheaf of minimal varieties with vanishing first Chern class. Presenting the tangent sheaf as a direct sum of integrable subsheaves which are stable with respect to any polarisation, Theorem 3 can be seen as an infinitesimal analogue of the Beauville-Bogomolov Decomposition 1 in the singular setting.

Theorem 3 (Decomposition of the tangent sheaf, [GKP11, Thm. 1.3])**.** *Let X be a normal projective variety with at worst canonical singularities, defined over the complex numbers. Assume that the canonical divisor of X is numerically trivial: $K_X \equiv 0$. Then there exists an Abelian variety A as well as a projective variety $\widetilde{X}$ with at worst canonical singularities, a finite cover $f : A \times \widetilde{X} \to X$, étale in codimension one, and a decomposition*

$$\mathscr{T}_{\widetilde{X}} \cong \bigoplus \mathscr{E}_i$$

such that the following holds.

(1) *The $\mathscr{E}_i$ are integrable saturated subsheaves of $\mathscr{T}_{\widetilde{X}}$, with trivial determinants.*

Further, if $g : \widehat{X} \to \widetilde{X}$ is any finite cover, étale in codimension one, then the following properties hold in addition.

(2) *The sheaves $(g^*\mathscr{E}_i)^{**}$ are stable with respect to any ample polarisation on $\widehat{X}$.*
(3) *The irregularity of $\widehat{X}$ is zero, $h^1\big(\widehat{X}, \mathscr{O}_{\widehat{X}}\big) = 0$.*

The decomposition found in Theorem 3 is characterised by additional uniqueness properties, [GKP11, Rem. 7.5]. Taking g to be the identity, we see that the irregularity of $\widetilde{X}$ is zero, and that the summands $\mathscr{E}_i$ are stable with respect to any polarisation.

Other results. In the course of the proof, we show the following two additional results, pertaining to stability of the tangent bundle, and to wedge products of differentials forms that are defined on the smooth locus of a minimal model. We feel that these results might be of independent interest.

Proposition 4 (Stability of $\mathscr{T}_X$ does not depend on polarisation, [GKP11, Prop. 5.7])**.** *Let X be a normal projective variety having at worst canonical singularities. Assume that K_X is numerically equivalent to zero. If the tangent sheaf $\mathscr{T}_X$ is stable with respect to one polarisation, then it is also stable with respect to any other polarisation.* □

Proposition 5 (Non-degeneracy of the wedge product, [GKP11, Prop. 6.1])**.** *Let X be a normal n-dimensional projective variety X having at worst canonical singularities. Denote the smooth locus of X by X_{reg}. Suppose that the canonical divisor is trivial. If $0 \le p \le n$ is any number, then the natural pairing given by the wedge product on X_{reg},*

$$\bigwedge : H^0\big(X_{\text{reg}}, \Omega^p_{X_{\text{reg}}}\big) \times H^0\big(X_{\text{reg}}, \Omega^{n-p}_{X_{\text{reg}}}\big) \longrightarrow H^0\big(X_{\text{reg}}, \omega_{X_{\text{reg}}}\big) \cong H^0\big(X, \omega_X\big) \cong \mathbb{C},$$

is non-degenerate. □

Singular analogues of Calabi-Yau and irreducible symplectic manifolds. Based on the Decomposition Theorem 3, we argue that the natural building blocks for any structure theory of projective manifolds with Kodaira dimension zero might be canonical varieties with *strongly stable* tangent sheaf. Strong stability is a formalisation of condition (3.2) that appears in the Decomposition Theorem 3.

We remark that in dimension no more than five, canonical varieties with strongly stable tangent sheaf fall into two classes, which naturally generalise the notions of irreducible Calabi-Yau– and irreducible holomorphic-symplectic manifolds, respectively. There is evidence to suggest that this dichotomy should hold in arbitrary dimension.

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Divisors on the Hilbert scheme of points in $\mathbb{P}^2$ (work of Jack Huizenga)

Joe Harris

Let $\mathcal{H}_n$ be the Hilbert scheme of subschemes of $\mathbb{P}^2$ of dimension 0 and degree n. The group $\text{Pic}(\mathcal{H}_n)$ is freely generated by two classes, the class of the divisor

$$H = \{\Gamma \subset \mathbb{P}^2 \mid \Gamma \cap L \neq \emptyset\}$$

of subschemes containing a point on a fixed line $L \subset \mathbb{P}^2$, and the divisor class $\Delta/2$, where

$$\Delta = \{\Gamma \subset \mathbb{P}^2 \mid \Gamma \text{ is nonreduced}\}$$

is the "diagonal."

Since $\mathcal{H}_n$ is log Fano (the divisor class $K_{\mathcal{H}_n} + \epsilon\Delta = -(3H - \epsilon\Delta)$ is antiample), it is a Mori dream space. Thus the effective cone is closed, and our goal is to find for each n an extremal divisors D_n, together with a moving curve C_n having intersection number 0 with them.

For special values of n this can be done by elementary constructions. For example, if $n = \binom{d+2}{2}$ for some d, we can take D_n to be the locus of subschemes contained in a curve of degree d. In general, Huizenga adopts a generalization of this approach: he considers vector bundles E on $\mathbb{P}^2$ such that

$$h^0(E) = n \cdot \mathrm{rank}(E)$$

and defines

$$D_n = \{\Gamma \subset \mathbb{P}^2 \mid h^0(\mathcal{I}_\Gamma \otimes E) \neq 0\},$$

assuming this is indeed a proper subscheme of $\mathcal{H}_n$. The class of the divisor D_n is then given by

$$[D_n] = c_1(E) \cdot H - \mathrm{rank}(E)\frac{\Delta}{2}.$$

To start, Huizenga considers vector bundles E defined by the sequence

$$0 \to \mathcal{O}_{\mathbb{P}^2}(r-2)^s \to \mathcal{O}_{\mathbb{P}^2}(r-1)^{r+s} \to E \to 0$$

where we write

$$n = \binom{r+1}{2} + s, \quad \text{with} \quad 0 \leq s \leq r,$$

and the map $\mathcal{O}_{\mathbb{P}^2}(r-2)^s \to \mathcal{O}_{\mathbb{P}^2}(r-1)^{r+s}$ is given by a general matrix of linear forms. The slope of the corresponding divisor $D_n \subset \mathcal{H}_n$—again, assuming it exists—is then

$$\sigma_n = \frac{r^2 - r + s}{r},$$

which we can see is extremal by considering a moving curve $C_n \subset \mathcal{H}_n$ consisting of a pencil on a curve $B \subset \mathbb{P}^2$ of degree $r+1$.

The issue is then whether D_n exists, that is, whether n general points in $\mathbb{P}^2$ impose independent conditions on $H^0(E)$. (We say in this case that E *has the interpolation property.*) The central theorem of Huizenga's work is the

Theorem The following are equivalent:

1. E has the interpolation property
2. E is semistable
3. If $f : \mathbb{P}^1 \to \mathbb{P}^2$ is a general map of degree r, then f^*E is balanced
4. For a general 3-dimensional subspace $V \subset H^0(\mathcal{O}_{\mathbb{P}^2}(r))$ and any subspace of $H^0(\mathcal{O}_{\mathbb{P}^2}(s-1))$, we have

$$\frac{\dim(VW)}{r+s} \geq \frac{\dim(W)}{s}$$

(5) The ratio $s/r \in \Phi$, where

$$\Phi = (\phi^{-1}, 1] \cup \{0, \frac{1}{2}, \frac{3}{5}, \frac{8}{13}, \dots\}$$

with ϕ the golden mean $(1+\sqrt{5})/2$.

This settles the question of the effective cone of $\mathcal{H}_n$ for roughtly 38% of all cases. Using variants of this method (for example, bundles with quadratic, rather than linear, resolutions), Huizenga succeeds in answering the question for approximately 96% of all values of n, and in particular all $n < 142$.

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