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Mathematical Aspects of General Relativity

Organised by
Mihalis Dafermos, Cambridge UK
Jim Isenberg, Eugene
Hans Ringström, Stockholm

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Abstract. Mathematical general relativity, the subject of this workshop, is a remarkable confluence of different areas of mathematics. Einstein's equation, the focus of mathematical relativity, is one of the most fruitful nonlinear hyperbolic PDE systems under study. As well, some of the most challenging geometric analysis problems in Reimannian geometry and elliptic PDE theory arise from the study of the initial data for Einstein's equations. In addition, these studies play a crucial role in modeling the physics of astrophysical and cosmological systems. This workshop reflected the rapid progress seen in the field in recent years, and highlighted some of the most interesting questions under study in mathematical relativity.

Mathematics Subject Classification (2000): 83Cxx.

Introduction by the Organisers

Black holes and their stability: The black hole stability problem, a topic which has generated intense work in the past ten years by researchers spanning the areas of harmonic analysis, spectral theory, and partial differential equations all the way to theoretical physics, remained one of the main areas of activity in this year's workshop. The proof of quantitative boundedness and decay estimates for solutions of the wave equation on Kerr backgrounds in the entire sub-extremal range $|a| < M$, reported on in the previous 2009 Mathematical Aspects of Relativity workshop, was complemented in this year's workshop by the elucidation of the remaining extremal case, in which $|a| = M$. Stability results for this case were discussed, but even more surprisingly, *instability* results were also presented, in the talk of Stefanos Aretakis. Another new direction for black hole stability has been that of black holes with negative cosmological constant, an area marked by intense interest on the part of the string theory community. New results concerning Schwarzschild-AdS and Kerr-AdS spacetimes were reported in the talk of Jacques Smulevici, which included in particular a proof of logarithmic decay for solutions of the wave and Klein-Gordon equation, and, moreover, a proof that such rates are in fact sharp. The latter might suggest that *all asymptotically AdS space times are in fact non-linearly unstable*, a statement with important implications for high energy physics. This would extend the conjecture, made already at a 2007 Oberwolfach workshop on Analysis and Geometric Singularities, that pure-AdS space-time is unstable; this conjecture is now substantially supported by the recent numerical work which was presented in the talk of Piotr Bizon. (This numerical work features the very interesting phenomenon that the instability appears to be driven by a mechanism similar to fluid turbulence, in which energy is transported from low to high frequencies.) Yet another direction for the black hole stability problem has been to consider what happens in higher dimensions, where the zoology of stationary black holes is much richer. Harvey Reall reported on a very pretty application of the Penrose inequality to infer *non-linear instability* for certain higher dimensional black holes. This latter work was the original motivation for the result presented in Bob Wald's talk, which related a *linear stability*-type statement directly to the validity of a local Penrose inequality.

The above multifaceted progress on linear aspects of the stability problem makes a proof of the non-linear stability of the Kerr family (as solutions of the vacuum Einstein equations) an exciting prospect for the years to come. A new and different step in that direction was presented in the talk of Gustav Holzegel, which discussed a proof of the existence of a large class of vacuum spacetimes without symmetries which dynamically approach Schwarzschild or Kerr. These are constructed by imposing "scattering type data" on the horizon and on null infinity, and solving the Einstein equations "backwards".

Several other important issues involving black holes and other asymptotically flat space times were discussed at the meeting. Lydia Bieri presented extensions of Christodoulou's celebrated non-linear memory effect to the Einstein-Maxwell case, and showed in particular that non-linear memory has also a non-trivial contribution arising from the total electromagnetic radiation. Håkan Andréasson reported on a construction of spherically symmetric black hole solutions with collisionless matter arising from a complete regular past. Carsten Gundlach presented numerical evidence for a new type of critical behaviour at the threshold of immediate merger in binary black hole systems.

Cosmology:. Turning to mathematical problems arising in cosmology, Qian Wang spoke about a new local existence result in the spatially harmonic constant mean curvature (CMCSH) gauge. In particular, she described how to obtain local existence in H^s for $s > 2$. Needless to say, improved local existence results lead to improved breakdown criteria, and are therefore of interest in the context of proving global results.

Jared Speck presented joint work with Igor Rodnianski concerning big bang singularities in the scalar field and stiff fluid case. One particular consequence of the result is that perturbing initial data for a spatially flat FLRW solution leads to solutions that are similar to FLRW towards the past singularity. This constitutes the first known result which gives a detailed description of the singularity (including curvature blow up) for an open set of initial data. Juan Antonio Valiente Kroon presented joint work with C. Lübbe concerning future global non-linear stability of FLRW spacetimes containing a de Sitter-like cosmological constant and a radiation fluid.

Several important questions concerning oscillatory singularities in the spatially homogeneous setting remain unanswered. These concern the generic behaviour as well as the causal structure. In his talk, Alan Rendall presented joint work with S. Liebscher and S. Tchapnda on this topic. In particular, a construction of an unstable manifold of solutions converging to a heteroclinic cycle was described in the class of magnetic Bianchi VI_0 solutions.

Regularity: Issues related to causality theory for metrics of low regularity appear in many different contexts. In his talk, Piotr Chruściel discussed this topic, and presented a proof of the existence of a maximal globally hyperbolic development for sets of initial data with low regularity. Philippe LeFloch presented a general theory (developed together with several authors) for treating low regularity spacetimes with symmetries.

Another promising direction for future research is that which has been opened up by the study of interacting, impulsive gravitational wave-solutions of the vacuum equations, without symmetries, as discussed in the talk of Jonathan Luk. These singular solutions lie below the well-posedness threshold of curvature in L^2, and their existence is yet another manifestation of the remarkable structure which is present in the Einstein equations. The analysis of these solutions gives hope that spacetimes which have been conjectured to have even more severe singularities propagating on null cones, like those of generic vacuum black hole interiors, will soon be understood in complete generality.

Initial Data sets and their properties: The study of initial data sets which satisfy the Einstein constraint equations has been a major feature of all five of the Oberwolfach meetings on mathematical relativity. Many important issues involving this topic remain unresolved, and are currently very active areas of research. One of the long-standing issues is determining the extent to which the conformal method can be used to parametrize and construct the range of solutions of the Einstein constraint equations. For the constant mean curvature and near constant mean curvature solutions, this is well understood, but for others it is not. Romain Gicquaud discussed his recent results (with M. Dahl and E. Humbert) which provide a means for showing that, for certain sets of conformal data with non constant mean curvature, the conformal method does produce solutions. Another issue of long-standing interest is the positivity of the ADM mass in asymptotically flat initial data sets. While such positivity for $3 + 1$ dimensional spacetimes has been known for a number of years, it is known for higher dimensional spacetimes only for spin manifolds. Lan-Hsuan Huang, reporting on research done with M. Eichmair, D. Lee, and R. Schoen, showed that positivity holds for spacetime dimensions of $7+1$ dimension or less.

A key tool used in the work reported by Huang is the "marginally outer trapped surface", or "MOTS". These geometric structures play an increasingly important role in the study of initial data sets and their development. In Greg Galloway's talk, MOTS play a major role in the formulation and proof (with M. Eichmair and D. Pollack) of a new topological censorship theorem which uses criteria involving initial data only to restrict the allowed asymptotic topology of such data sets. The talk of Marc Mars (with M. Reiris) also discussed MOTS, focussing on the relationship between MOTS (locally defined) and black holes (globally defined) in stationary and static spacetimes.

Helmut Friedrich's talk focussed on issues related to asymptotic properties of asymptotically flat initial data sets. Working with time symmetric vacuum data, he studies the relationship between those data sets which are asymptotically conformal to static data and those which are asymptotically static. The talk of Martín Reiris dealt with an interesting phenomenon related to sequences of axisymmetric initial data sets for which an inequality relating areas and angular momenta is saturated. He shows (with S. Dain) that for such sequences, the limiting data set exhibits the phenomenon of an "extreme Kerr throat". The talk of Christine Sormani addressed the mathematics of convergence of Riemannian manifolds. She introduced the notion of "intrinsic flat convergence", and discussed ways in which this notion could be useful in studying initial data sets.

Workshop: Mathematical Aspects of General Relativity

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Abstracts

Topological censorship from the initial data point of view

Gregory J. Galloway
(joint work with Michael Eichmair and Dan Pollack)

Topological censorship is a basic principle of spacetime physics. It is a set of results, beginning with the topological censorship theorem of Friedman, Schleich, and Witt [3], that establishes the topological simplicity at the fundamental group level of the domain of outer communications (the region outside all black holes and white holes) under a variety of physically natural circumstances. An important precursor to the principle of topological censorship, which serves to motivate it, is the Gannon-Lee singularity theorem [5, 6]. All of these results are *spacetime* results, i.e., they involve conditions that are essentially global in time. From the evolutionary point of view, there is the difficult question of determining whether a given initial data set will give rise to a spacetime satisfying these conditions. In order to separate out the principle of topological censorship from these difficult questions of global evolution, it would be useful to have a pure initial data version of topological censorship. We report here on some recent results along these lines; cf., [2].

We first consider an initial data version of the Gannon-Lee singularity theorem [5, 6] for 3-dimensional initial data sets. In this theorem, one considers a globally hyperbolic spacetime (M, g) which satisfies the null energy condition (NEC), $\mathrm{Ric}(X, X) \geq 0$ for all null vectors X, and which contains an asymptotically flat Cauchy surface V. It is shown that if V is nonsimply connected, $\pi_1(V) \neq 0$, then M is future null geodesically complete. This theorem captures the idea that nontrivial topology tends to induce gravitational collapse.

To obtain an initial data version of the Gannon-Lee singularity theorem, we first need to decide on what should be meant by an initial data singularity theorem. Let (V^3, h, K) be a 3-dimensional initial data set in four dimensional spacetime (M^4, g), by which we mean V is a smooth spacelike hypersurface in M with induced metric h and second fundamental form K. Certainly, in view of the Penrose singularity theorem, conditions on an initial data set that imply the existence of a trapped surface should be view as an initial data singularity theorem.

Let Σ be a closed two-sided surface in V. Since it is two-sided it admits two globally defined future directed null normal vector fields l_+ ('outward pointing') and l_- ('inward pointing'). Let θ_+ and θ_- be the null expansions associated to l_+ and l_-, respectively. Then recall, Σ is a trapped surface if both null expansions are negative, $\theta_+ < 0$ and $\theta_- < 0$. Focusing on the outward null normal, if $\theta_+ < 0$ we say that Σ is outer trapped. If θ_+ vanishes identically, $\theta_+ = 0$, we say that Σ is a marginally outer trapped surface, or MOTS for short. In the time-symmetric case ($K = 0$), a MOTS is just a minimal surface.

Should conditions on an initial data set that imply the existence of a MOTS be viewed as an initial data singularity result? From the point of view taken here, the answer is yes. As shown in [2], there is a Penrose-type singularity theorem that applies to a noncompact Cauchy surface V that contains a MOTS Σ. From this result we conclude that conditions on an initial data set that imply the existence of a MOTS should be viewed as an initial data singularity theorem.

We must take our point of view one step further to accommodate other basic examples. There is a more general type of object in an initial data set that implies a Penrose-type singularity theorem, which we refer to as an *immersed* MOTS. Given an initial data set (V, h, K), we say that a subset $\Sigma \subset V$ is an *immersed MOTS* if there exists a finite cover $\tilde{V}$ of V with covering map $p : \tilde{V} \to V$ and a MOTS $\tilde{\Sigma}$ in $(\tilde{V}, p^*h, p^*K)$ such that $p(\tilde{\Sigma}) = \Sigma$.

The best known example of an immersed MOTS (that is not a MOTS) occurs in the so-called $\mathbb{RP}^3$ geon. The $\mathbb{RP}^3$ geon is a globally hyperbolic spacetime which is double covered by the extended Schwarzschild spacetime. Its Cauchy surfaces have topology $\mathbb{RP}^3$ minus a point. The Cauchy surface V covered by the $t = 0$ slice in extended Schwarzschild spacetime contains a projective plane Σ which is covered by the unique minimal sphere in $t = 0$. Since it is not two-sided, Σ is an immersed MOTS, but not a MOTS.

It is easily shown that the Penrose type singularity theorem for MOTSs extends to immersed MOTSs. We conclude that *any result that implies the existence of an immersed MOTS in an initial data set should be viewed as an initial data singularity theorem.*

We are now ready to state our initial data version of the Gannon-Lee result.

Theorem 1. *Let (V^3, h, K) be a 3-dimensional AF initial data set. If V^3 is not diffeomorphic to $\mathbb{R}^3$ then V^3 contains an immersed MOTS.*

Thus, if V^3 is not $\mathbb{R}^3$, spacetime is singular, from the initial data point of view. We note that the dominant energy condition is not required; if one assumes it the conclusion can be refined slightly (see [2]). Theorem 1 may be viewed as a non-time-symmetric version of work of Meeks, Simon and Yau [7], which implies, without any curvature assumptions, that an asymptotically flat 3-manifold that is not diffeomorphic to $\mathbb{R}^3$ contains an embedded stable minimal sphere or projective plane. However, while their work did not rely on the positive resolution of the Poincaré conjecture, our proof of Theorem 1 makes use of the geometrization of closed orientable 3-manifolds. Specifically it uses the fact that the fundamental group of every closed orientable 3-manifold is *residually finite.* This fact is used in the proof in conjunction with recent existence results for MOTSs; see [1] for an excellent exposition of these existence results.

Now we turn to a brief discussion of topological censorship. The notion of topological censorship may be described as follows: As the Gannon-Lee theorem suggests, nontrivial topology tends to induce gravitational collapse. By the weak cosmic censorship conjecture, the process of gravitational collapse leads to the formation of an event horizon which shields the singularities from view. As a result, nontrivial topology should become hidden behind the event horizon, and the domain of outer communications should have simple topology.

This notion was formalized by the topological censorship theorem of Friedman, Schleich and Witt [3], which applies to globally hyperbolic spacetimes that satisfy the null energy condition and that are asymptotically flat, i.e. that admit a regular *null infinity* $\mathscr{I} = \mathscr{I}^+ \cup \mathscr{I}^-$. Their theorem then asserts, roughly, that any causal curve from $\mathscr{I}^-$ to $\mathscr{I}^+$ can be deformed, with fixed end points, to $\mathscr{I}$. In physical terms, observers traveling from $\mathscr{I}^-$ to $\mathscr{I}^+$ are unable to probe any nontrivial topology. In [4] using their result, it was shown under similar conditions that the domain of outer communications (DOC) $D = I^-(\mathscr{I}^+) \cap I^+(\mathscr{I}^-)$ is simply connected.

In the context of proving an initial data version of topological censorship, one should think of the initial data manifold V as representing an asymptotically flat spacelike slice in the DOC, whose boundary Ṿ corresponds to a cross section of the event horizon. At the initial data level, we represent this cross section by a MOTS. By extending known results, it can be shown, assuming the null energy condition, that there can be no immersed MOTS in the DOC. As such, for the initial data version, we assume that there are no immersed MOTS in $V \setminus$ Ṿ. The following initial data version of topological censorship shows that under these circumstances, the topology of V is essentially as simple as possible.

Theorem 2. *Let (V, h, K) be a 3-dimensional asymptotically flat initial data set such that V is a manifold-with-boundary, whose boundary ∂V is a compact MOTS. If all components of ∂V are spherical and if there are no immersed MOTS in $V \setminus \partial V$, then V is diffeomorphic to $\mathbb{R}^3$ minus a finite number of open balls.*

The proof is similar to the proof of the Theorem 1, but some added care is needed in dealing with the MOTS boundary. See [2] for details and further results.

References

[1] L. Andersson, M. Eichmair, and J. Metzger, *Jang's equation and its applications to marginally trapped surfaces*, in: Complex Analysis and Dynamical Systems IV: Part 2. General Relativity, Geometry, and PDE, Contemporary Mathematics, vol. 554, (AMS and Bar-Ilan), 2011.

[2] M. Eichmair, G. J. Galloway, and D. Polack *Topological censorship from the initial data point of view*, 2012, arXiv:1204.0278.

[3] J. L. Friedman, K. Schleich, and D. M. Witt, *Topological censorship*, Phys. Rev. Lett. **71** (1993), no. 10, 1486–1489.

[4] G. J. Galloway, *On the topology of the domain of outer communication*, Classical Quantum Gravity **12** (1995), no. 10, L99–L101.

[5] D. Gannon, *Singularities in nonsimply connected space-times*, J. Mathematical Phys. **16** (1975), no. 12, 2364–2367.

[6] C. W. Lee, *A restriction on the topology of Cauchy surfaces in general relativity*, Comm. Math. Phys. **51** (1976), no. 2, 157–162.

[7] W. Meeks, III, L. Simon, and S.-T. Yau, *Embedded minimal surfaces, exotic spheres, and manifolds with positive Ricci curvature*, Ann. of Math. (2) **116** (1982), no. 3, 621–659.

Construction of dynamical vacuum black holes

Gustav Holzegel
(joint work with Mihalis Dafermos and Igor Rodnianski)

In this talk, we construct a class of solutions to the vacuum Einstein equations

$$R_{\mu\nu} = 0\,,$$

which are non-stationary and asymptotically converge (at an exponential rate) to a Kerr spacetime $(\mathcal{M}, g_{M,a})$ with $|a| \leq M$. These dynamical spacetimes are constructed as solutions of a characteristic scattering problem from infinity with scattering data prescribed on the event horizon and on null-infinity. More precisely, we prove the following

Theorem 1. Given suitable smooth scattering data on the horizon $\mathcal{H}^+$ and future null infinity $\mathcal{I}^+$ asymptoting to the induced Kerr geometry of parameters $|a| \leq M$, then there exists a corresponding smooth vacuum black hole spacetime $(\mathcal{M}, g)$ asymptotically approaching in its exterior region the Kerr solution with parameters $|a| \leq M$.

For simplicity (mainly the advantage of working with explicit expressions for the curvature components), we will prove the above theorem for Schwarzschild and explain carefully below why the proof carries through for Kerr as well.

Theorem 1 should be compared with the Kerr stability conjecture, which asserts that sufficiently small perturbations of a Kerr spacetime $(\mathcal{M}, g_{M,a})$ also form black holes, which moreover eventually settle down to another member of the Kerr family on their exterior. In this context, the above theorem promises that at least there *exists* a "large" class of data (without any symmetry assumptions!) which exhibit dynamical convergence to a stationary black hole spacetime. Such behavior was previously known only in symmetry classes and for the special class of Robinson-Trautman metrics, which converge to Schwarzschild at an exponential rate [5] but – by construction – do not radiate through the horizon and are moreover of limited regularity (C^k for $k < 123$, see [6]). While the spacetimes of Theorem 1 are suitably parametrized by their scattering data and hence admit the full functional degrees of freedom, it should be kept in mind that generic perturbations of Kerr are expected to exhibit *polynomial* tails. Therefore, at the level of "initial data", the subset of data leading to the solutions of Theorem 1 (i.e. exponential convergence) is "small".

Let us make some remarks about the proof. By interpolating the scattering data on the horizon and null-infinity with a spacelike slice (which eventually gets pushed to infinity) the problem is turned into a mixed characteristic-Cauchy problem, which is well-posed in the smooth category, cf. [4]. The main problem, therefore, is to propagate (uniformly) the exponential decay estimates. Here the reason one needs to impose *exponential* decay becomes apparent: It is rooted in the famous red-shift effect [7, 8] near the event horizon which – when solving backwards as above – turns into a blue-shift leading to exponential growth of the solution. With this in mind it also becomes clear why Theorem 1 includes the case of extremal Kerr, $a = M$: While for the forward problem the degenerate redshift of extremal black holes leads to an instability of solutions along the horizon [1, 2], for the scattering problem this degeneration is a benefit rather than a difficulty, which may even allow one to establish a version of Theorem 1 for polynomially decaying scattering data in the extemal case.

At the technical level, we work in a double-null foliation with the equations phrased in terms of hyperbolic equations for the curvature components (Bianchi) coupled with transport- (as well as elliptic) equations for the Ricci-coefficients (structure equations), similar to [3, 11]. Of course, all these equations need to be suitably renormalized in order to capture the decay to Schwarzschild (or Kerr). In this formulation, the important null-condition is manifest at the level of the Bianchi equations. The latter is crucial in the context of the r-weighted energy estimates (cf. [10]), which we employ. Remarkably, these weighted estimates are the only estimates required from the Bianchi equations. In particular, we neither exploit the almost conservation law arising from ∂_t at this stage nor do we derive a Morawetz estimate (hence an understanding of trapping is not required). It is mainly for these two reasons that the argument applies to Schwarzschild as well as Kerr. Finally, there is a remarkable (decay)-hierarchy at the level of the transport equations which allows one to propagate the exponential decay also for the Ricci-coefficients.

Details can be found in [9].

References

[1] S. Aretakis, *Stability and Instability of Extreme Reissner-Nordström Black Hole Spacetimes for Linear Scalar Perturbations I*, Commun. Math. Phys. **307** (2011) 17 , arXiv:1110.2007 [gr-qc].

[2] S. Aretakis, *Horizon Instability of Extremal Black Holes*, arXiv:1206.6598 [gr-qc].

[3] D. Christodoulou, *The Formation of Black Holes in General Relativity*, EMS Monographs in Mathematics, arXiv:0805.3880 [gr-qc].

[4] A. D. Rendall, Reduction of the characteristic initial value problem to the Cauchy problem and its applications for the Einstein equation, Proc. Roy. Soc. Lond. **427** (1992) 221-239

[5] P. T. Chruściel, *On the global structure of Robinson-Trautman space-times*, Proc. Roy. Soc. Lond. **436** (1992) 299.

[6] P. T. Chruściel and D. B. Singleton, *Nonsmoothness of event horizons of Robinson-Trautman black holes*, Commun. Math. Phys. **147** (1992) 137.

[7] M. Dafermos and I. Rodnianski, *The Red-shift effect and radiation decay on black hole spacetimes*, Commun. Pure Appl. Math. **62** (2009) 859.

[8] M. Dafermos and I. Rodnianski, *Lectures on black holes and linear waves*, Institut Mittag Leffler, Report **14**, arXiv:0811.0354 [gr-qc].

[9] M. Dafermos, G. Holzegel and I. Rodnianski, *Construction of Dynamical Vacuum Black Holes*, preprint (2012)

[10] G. Holzegel, *Ultimately Schwarzschildean Spacetimes and the Black Hole Stability Problem*, arXiv:1010.3216 [gr-qc].

[11] S. Klainerman and I. Rodnianski, *On the formation of trapped surfaces*, arXiv:0912.5097 [gr-qc].

On the isoperimetric structure of initial data sets

Michael Eichmair

Let $m > 0$. The complete Riemannian manifold

$$\left(\mathbb{R}^n \setminus \{(0,\dots,0)\}, \left(1+\frac{m}{2|x|^{n-2}}\right)^{\frac{4}{n-2}} \sum_{i=1}^n (dx^i)^2\right)$$

is time-symmetric initial data for the $(n+1)$-dimensional Schwarzschild spacetime of mass m. We will refer to these Riemannian manifolds as "Schwarzschild of mass m" for short. These Riemannian manifolds are conformally flat and scalar flat. We will denote the centered coordinate spheres and balls by S_r and B_r respectively. The coordinate sphere $S_{(\frac{m}{2})^{1/(n-2)}}$ is called the horizon. The map $x \mapsto \frac{x}{|x|^2}(\frac{m}{2})^{2/(n-2)}$ is a reflection symmetry of Schwarzschild of mass m about the horizon; it flips the two asymptotically flat ends.

We say that a complete asymptotically flat n-dimensional initial data set (M,g), possibly with compact minimal boundary, is $\mathcal{C}^k$-asymptotic to Schwarzschild of mass $m > 0$ if it has one asymptotically flat end and if there exist asymptotic coordinates for that end such that

$$g_{ij} = \left(1+\frac{m}{2|x|^{n-2}}\right)^{\frac{4}{n-2}} \delta_{ij} + O(|x|^{1-n}) \text{ as } |x| \to \infty$$

holds up to and including derivatives of order k. Such initial data sets, in particular those of dimension three, have been subject of extensive investigation. A landmark result on the structure of 3-dimensional initial data sets that are $\mathcal{C}^4$-asymptotic to Schwarzschild of mass $m > 0$ due to G. Huisken and S.-T. Yau [9] says that the complement of a certain compact region in every such initial data set is foliated by stable constant mean curvature spheres $\{\Sigma_H\}_{0<H<H_0}$. These spheres are parametrized by their (small) mean curvature H and they diverge to infinity as H tends to zero, becoming rounder and rounder in the limit. The leaf Σ_H is the unique stable constant mean curvature sphere of mean curvature H that contains the centered coordinate ball $B_{H^{-q}}$, where $q \in (\frac{1}{2}, 1]$, provided H is sufficiently small (depending on q). The centers of mass of Σ_H computed with respect to the asymptotic coordinate system chosen above converge to the "Huisken-Yau geometric center of mass" as $H \searrow 0$. J. Qing and G. Tian [12] improved the uniqueness result of Huisken-Yau by showing that the leaves are the unique stable constant mean curvature spheres of their respective mean curvature containing a certain large but fixed compact set in (M,g). L.-H. Huang has shown that the Huisken-Yau center of mass coincides with other notions of the center of mass (introduced and studied by Regge-Teitelboim, Beig-Ó Murchadha, and Corvino-Schoen). She has also proven an extension of the result of Huisken-Yau to initial data sets with more general asymptotics. An extension of the results of Qing-Tian to more general asymptotics has been established by S. Ma in [10, 11].

The study of stable constant mean curvature spheres in 3-dimensional initial data sets was initiated by work of D. Christodoulou and S.-T. Yau [4], who showed that the Hawking mass of such a surface is non-negative provided the initial data set has non-negative scalar curvature. Note that stable constant mean curvature surfaces are exactly the stable critical points for the classical isoperimetric problem of finding the least area surface that encloses a given amount of volume. H. Bray in his thesis [1] established a special case of the Penrose inequality using a Hawking-type mass defined in terms of the isoperimetric data of asymptotically flat three manifolds. In the process, he characterized the isoperimetric surfaces of Schwarzschild of mass $m > 0$ where volume is measured relative to the horizon: they are exactly the centered coordinate spheres. A natural and long-standing conjecture of Bray's is whether the leaves of the foliation through stable constant mean curvature spheres of Huisken-Yau are in fact isoperimetric. In joint work with J. Metzger [5] we answer this conjecture in the affirmative. Moreover, we show that, as isoperimetric surfaces enclosing a given volume, the leaves of the foliation are *globally* unique. The proof of this result connects with the uniqueness result of Huisken-Yau only at the very end; the existence of isoperimetric regions that are close to centered coordinate spheres is independent and relies on a delicate quantitative refinement of Bray's characterization of the isoperimetric regions in exact Schwarzschild initial data. In further joint work with J. Metzger [6], under the additional hypothesis that the initial data set have positive scalar curvature, we strengthen the uniqueness result of Huisken-Yau and Qing-Tian to assert that the leaves of the foliation $\{\Sigma_H\}_{0<H<H_0}$ are the unique connected large constant mean curvature surfaces in the initial data set that enclose a given point. The mechanism we use to obtain this result is that of the Schoen-Yau proof of the positive energy theorem [13].

The techniques in the work of [9, 12, 6] are tied to 3-dimensional initial data sets in many ways, except for the existence of the foliation, cf. the work of R. Ye [14]. In [7], we show in collaboration with J. Metzger that the existence of foliations through isoperimetric surfaces, and the uniqueness of the leaves of this foliation as isoperimetric surfaces enclosing a given volume, holds in all dimensions. The list of geometries where isoperimetric regions have been characterized explicitly is short and had been limited to highly symmetric geometries or to the case of isoperimetric regions of very small volume; cf. Appendix H of [7] . The recent characterization of closed embedded constant mean curvature surfaces in exact Schwarzschild that are disjoint from the horizon due to S. Brendle [2] is an important ingredient in the proof of the main result in [7].

In joint work with S. Brendle, we characterize the null-homologous isoperimetric regions in Schwarzschild of mass $m > 0$. These results complement the work of H. Bray [1]. If the enclosed volume is large, these regions are connected and rotationally symmetric. If the dimension n is less than eight and if the regions are not rotationally symmetric, then these regions are connected and their boundaries intersect the horizon.

References

[1] H. Bray, *The Penrose inequality in general relativity and volume comparison theorems involving scalar curvature (thesis)*, arXiv:0902.3241
[2] S. Brendle, *Hypersurfaces of constant mean curvature in deSitter-Schwarzschild space*, arXiv:1105.4273
[3] S. Brendle, M. Eichmair, *Isoperimetric and Weingarten surfaces in the Schwarzschild manifold*, arXiv:1208.3988
[4] D. Christodoulou, S.-T. Yau, *Some remarks on the quasi-local mass*, Mathematics and general relativity (Santa Cruz, CA, 1986), 9–14, Contemp. Math., 71, Amer. Math. Soc., Providence, RI, 1988
[5] M. Eichmair, J. Metzger, *Large isoperimetric surfaces in initial data sets*, arXiv:1102.2999
[6] M. Eichmair, J. Metzger, *On large volume preserving stable constant mean curvature surfaces in asymptotically flat manifolds*, arXiv:1102.3001, to appear in J. Differential Geom.
[7] M. Eichmair, J. Metzger, *Unique isoperimetric foliations of asymptotically flat manifolds in all dimensions*, arXiv:1204.6065
[8] L.-H. Huang, *On the center of mass in general relativity*, Fifth International Congress of Chinese Mathematicians. Part 1, 2, 575–591
[9] G. Huisken, S.-T. Yau, *Definition of center of mass for isolated physical systems and unique foliations by stable spheres with constant mean curvature*, Invent. Math. 124 (1996), no. 1-3, 281–311
[10] S. Ma, *Uniqueness of the Foliation of Constant Mean Curvature Spheres in Asymptotically Flat* 3*-Manifolds*, arXiv:1012.3231
[11] S. Ma, *A new result on the uniqueness of the CMC foliation in asymptotically flat manifold*, arXiv:1207.0281
[12] J. Qing, G. Tian, *On the uniqueness of the foliation of spheres of constant mean curvature in asymptotically flat 3-manifolds*, J. Amer. Math. Soc. 20 (2007), no. 4, 1091–1110
[13] R. Schoen, S.-T. Yau, *On the proof of the positive mass conjecture in general relativity*, Comm. Math. Phys. 65 (1979), no. 1, 45–76
[14] R. Ye, *Foliation by constant mean curvature spheres on asymptotically flat manifolds*, Geometric analysis and the calculus of variations, 369–383, Int. Press, Cambridge, MA, 1996

Impulsive gravitational waves

Jonathan Luk
(joint work with Igor Rodnianski)

Explicit solutions of impulsive gravitational waves were first found by Penrose [8], building on earlier works of [1] and [3]. These are spacetimes satisfying the vacuum Einstein equations such that the Riemann curvature tensor has a delta singularity across a null hypersurface. However, the Penrose explicit solution was constructed in plane symmetry. The impulsive gravitational wave thus has plane wavefront and can only be thought of as an idealization that the source of the gravitational wave is at an infinite distance. Moreover, plane symmetry also assumes that the gravitational wave has infinite extent and automatically imposes the assumption of non-asymptotic flatness.

The first study of general spacetimes satisfying the Einstein equations and admitting possible 3-surface delta singularities was first undertaken by Taub [10], who derived a system of consistency relations linking the metric, curvature tensor and the geometry of the singular hypersurface.

We study the dynamical problem for general impulsive gravitational waves by solving the characteristic initial value problem without symmetry assumptions. Characteristic initial data is given on a truncated outgoing cone H_0 and a truncated incoming cone $\underline{H}_0$ intersecting at a two sphere $S_{0,0}$.

To study the propagation of impulsive gravitational waves, the initial data on the outgoing hypersurface H_0 is prescribed such that the null second fundamental form has a jump discontinuity across an embedded two sphere $S_{0,\underline{u}_s}$ but is smooth otherwise. The curvature tensor for the initial data thus has a delta singularity across $S_{0,\underline{u}_s}$. On the initial incoming hypersurface, the data is smooth but otherwise does not satisfy any smallness assumption. For this class of data, we prove existence and uniqueness of local solutions, as well as a result on the propagation of singularity:

Theorem 1 (L.-Rodnianski [4])**.** *Given the characteristic initial data as above, there exists a unique local solution to the vacuum Einstein equations $R_{\mu\nu} = 0$. Moreover, the curvature has a delta singularity across the null hypersurface emanating from the initial singularity prescribed on $S_{0,\underline{u}_s}$. The spacetime is smooth away from this null hypersurface.*

The theorem gives a precise description of how the singularity propagates. This can be thought of as an analog in general relativity of the work of Majda on the propagation of shocks in compressible fluids [6], [7].

In view of the examples of colliding impulsive gravitational waves found by Khan-Penrose [2] and Szekeres [9], we also studied the collision of these impulsive gravitational waves. More specifically, we prescribe a jump discontinuity across an embedded two sphere in the null second fundamental forms both on the outgoing and the incoming initial hypersurfaces. Locally, by Theorem 1, a unique solution exists and the curvature has a delta singularity across each of the null hypersurfaces emanating from the initial singularity. We show that we can understand the spacetime after the interaction of the two gravitational impulsive waves, which is represented geometrically by the intersection of these two null hypersurfaces. In particular, while the two gravitational impulsive waves interact nonlinearly, the resulting spacetime is smooth except on the union of the two null hypersurfaces emanating from the initial singularities even beyond the interaction:

Theorem 2 (L.-Rodnianski [5])**.** *Suppose on the initial outgoing hypersurface H_0, the null second fundamental form has a jump discontinuity across the two sphere $S_{0,\underline{u}_s}$ but is smooth otherwise; on the initial incoming hypersurface $\underline{H}_0$, the null second fundamental form has a jump discontinuity across the two sphere $S_{u_s,0}$ but is also smooth otherwise. Then there exists a unique local solution to the vacuum Einstein equations $R_{\mu\nu} = 0$. Moreover, the spacetime is smooth away from the union of incoming null hypersurface H_{u_s} emanating from $S_{u_s,0}$ and the outgoing null hypersurface $H_{\underline{u}_s}$ emanating from $S_{0,\underline{u}_s}$.*

The main difficulty in studying this class of spacetimes is that the Riemann curvature tensor is not in L^2. In this case, the standard energy estimates based on the Bel Robinson tensor do not apply. In the proof, we introduced a new type of energy estimates, which is based on the L^2 norm of only some (renormalized) components of the Riemann curvature tensor. Moreoever, we show that the space-time geometry can be controlled only with the knowledge of these components of the curvature tensor. In fact this allows us to prove existence and uniqueness of solutions to the vacuum Einstein equations for a more general class of initial data.

References

[1] M. W. Brinkmann, M. W., *On Riemann spaces conformal to Euclidean space*, Proc. Natl. Acad. Sci. U.S.A. 9 (1923), 1-3.

[2] K. A. Khan and R. Penrose, *Scattering of two impulsive gravitational plane waves*, Nature 229 (1971), 185-186.

[3] V. K. Lanczos, *Über eine stationäre Kosmologie im Sinne der Einsteinischen Gravitationstheorie*, Z. Phys. 21.2 (1924), 73-110.

[4] J. Luk and I. Rodnianski, *Local propagation of impulsive gravitational waves*, Preprint.

[5] J. Luk and I. Rodnianski, *Nonlinear interactions of impulsive gravitational waves for the vacuum Einstein equations*, Preprint.

[6] A. Majda, *The existence of multidimensional shock fronts*, Mem. Amer. Math. Soc. 43 (1983).

[7] A. Majda, *The stability of multidimensional shock fronts*, Mem. Amer. Math. Soc. 41 (1983).

[8] R. Penrose, *The geometry of impulsive gravitational waves*, General relativity (papers in honour of J. L. Synge) (1972), 101–115.

[9] P. Szekeres, *Colliding gravitational waves*, Nature 228 (1970), 1183-1184.

[10] A. H. Taub, *Space-times with distribution-valued curvature tensors*, J. Math. Phys. 21.6 (1980), 1423-1431.

On the formation of Kerr-throats in (vacuum) axisymmetry.

Martín Reiris

The Kerr family of black holes displays a remarkable property: fixed the angular momentum J, and decreasing the mass from $m^2 > |J|$ to $m^2 = |J|$ a long neck forms around the bifurcating sphere, ending up in an exact cylindrical state called the extreme Kerr-throat. Moreover the area of the bifurcating sphere decreases to the limit value of $8\pi|J|$. In this talk we show that this phenomenon is not specific only to the Kerr family of black holes but rather it is a general phenomenon of axisymmetry (at least). More precisely we show that

A. In any maximal, vacuum and axisymmetric initial data, the area of any surface is always greater than eight times its angular momentum, namely

$$A(S) > 8\pi|J(S)|, \tag{1}$$

B. For any convergent sequence of initial data having a sequence of surfaces saturating asymptotically the inequality (1), a long neck forms converging to the extreme Kerr-throat.

Property B is achieved by proving the rigidity of the Kerr-throat an then applying A. All these results are natural developments of the joint work with Sergio Dain [1].

References

[1] Sergio, Dain; Martin, Reiris. Area-Angular-Momentum Inequality for Axisymmetric Black Holes. Physical Review Letters, vol. 107, Issue 5, id. 051101

Black hole formation from a complete regular past for collisionless matter

Håkan Andréasson

An important question in the study of gravitational collapse is to identify physically admissible initial data, and it is natural to require that the past evolution of the data is regular. However, most of the existing mathematical results which ensure a regular past also ensure a regular future, cf. [11, 8], which rules out the study of the formation of black holes. The exceptions being the classical result for dust [10], and the recent result [9] for a scalar field. In the latter work, which in part rests on the studies [6, 7], initial data whose past evolution is regular and whose future evolution forms a black hole is constructed. Now, neither dust nor a scalar field are realistic matter models in the sense that they are used by astrophysicists. Dust is a perfect fluid where the pressure is assumed to be zero, and a scalar field is merely a toy model. Thus, there is so far no example of a solution to the Einstein-matter system for a realistic matter model possessing a regular past and a singular future.

Here we consider collisionless matter governed by the Vlasov equation. Although this is a simple matter model, it has rich dynamics and many features that are desirable of a realistic matter model, cf. [1]. Indeed, it allows for anisotropic pressure, there is a large number of stable and unstable spherically symmetric and axially symmetric stationary solutions, there is numerical support that time periodic solutions exist, it behaves as Type I matter in critical collapse, and it is used by astrophysicists, cf. [5]. The following theorem is the main result.

Theorem 1. *There exists a class of initial data J for the spherically symmetric Einstein-Vlasov system with the property that black holes form in the future time direction and in the past time direction spacetime is causally geodesically complete.*

A consequence of this result is that for any $\epsilon > 0$, initial data can be constructed with the property that the ratio $\mathring{m}/r$ of the initial Hawking mass $\mathring{m} = \mathring{m}(r)$, and the area radius r, is less than ϵ everywhere, such that a black hole forms in the evolution. We formulate this as a corollary.

Corollary 1. *Given $\epsilon > 0$, there exists a class J_r of initial data for the spherically symmetric Einstein-Vlasov system which satisfy*

$$\sup_r \frac{\mathring{m}(r)}{r} \leq \epsilon,$$

for which black holes form in the evolution.

This result improves the main result of [4] and is analogous to the result [7] in the case of a scalar field where conditions on the data are given which ensure the formation of black holes. These conditions give no lower bound on $2\mathring{m}/r$ but involve other restrictions. Another consequence of our result is the following corollary.

Corollary 2. *There exists a class J_s of black hole initial data for the spherically symmetric Einstein-Vlasov system such that the corresponding solutions exist for all Schwarzschild time $t \in (-\infty, \infty)$.*

In the future time direction this corollary was shown in [4], the improvement here is that the solutions exist on the entire real line.

The present result relies in part on the previous studies [2], [3] and [4], which now will be reviewed. In [3] global existence in a maximal time gauge is shown for a particular class of initial data where the particles are moving rapidly outwards. One of the restrictions imposed on the initial data is that

$$\sup_r \frac{2\mathring{m}(r)}{r} < k_0, \tag{1}$$

where the constant k_0 is roughly $1/10$. The situation considered in [4] is in a sense the reverse since the initial data is such that the particles move rapidly inwards and the quantity $\sup_r 2\mathring{m}/r$ is required to be close to one. The main result of [4] is that data of this kind guarantee the formation of black holes in the evolution. The analysis in [4] is carried out in Schwarzschild coordinates, i.e., in a polar time gauge. Now, particles that move inward in the future time direction move outward in the past time direction. It is thus natural to try to combine these two results with the goal of constructing solutions with a regular past and a singular future. The conditions on the ratio $2\mathring{m}/r$ are clearly very different in [3] compared to [4], and moreover, the Cauchy hypersurfaces are different since a maximal time gauge and a polar time gauge are imposed in the respective cases. The main reason why a maximal time gauge is used in [3] is due to the difficulties related to the so called pointwise terms in the characteristic equations in Schwarzschild coordinates. In [2] the problem of global existence for general initial data is investigated under conditional assumptions on the solutions. The analysis along characteristics is applied to a modified quantity for which the problems with the pointwise terms in Schwarzschild coordinates do not appear.

Here we combine the strategies in [2] and [3] and show global existence for rapidly outgoing particles in Schwarzschild coordinates. In particular the result in [3] is improved by showing that the restriction (1) can be relaxed, and for sufficiently fast moving particles $2\mathring{m}/r$ is allowed to be arbitrarily close to one. By combining this result with the result in [4] we are able to construct data whose past is regular and whose future contains a black hole.

References

[1] H. Andréasson, *The Einstein-Vlasov System/Kinetic Theory*, Living Rev. Relativity **14** (2011), 4.

[2] H. Andréasson, *Regularity results for the spherically symmetric Einstein-Vlasov system*, Ann. Henri Poincaré **11** (2010), 781–803.

[3] H. Andréasson, M. Kunze, G. Rein, *Global existence for the spherically symmetric Einstein-Vlasov system with outgoing matter*, Comm. Partial Differential Eqns. **33** (2008), 656–668.

[4] H. Andréasson, M. Kunze, G. Rein, *The formation of black holes in spherically symmetric gravitational collapse*, Math. Ann. **350** (2011), 683–705.

[5] J. Binney, S. Tremaine, *Galactic Dynamics*, Princeton University Press 1987.

[6] D. Christodoulou, *Bounded variation solutions of the spherically symmetric Einstein-scalar field equations*, Comm. Pure Appl. Math. **46** (1993), 1131–1220.

[7] D. Christodoulou, *The formation of black holes and singularities in spherically symmetric gravitational collapse*, Comm. Pure Appl. Math. **44** (1991), 339-373.

[8] D. Christodoulou, S. Klainerman, *The global nonlinear stability of Minskowski space*, Princeton Mathematical Series **41**. Princeton University Press 1993.

[9] M. Dafermos, *Black hole formation from a complete regular past*, Commun. Math. Phys. **289** (2009), 579–596.

[10] J. R. Oppenheimer, H. Snyder, *On continued gravitational contraction*, Phys. Rev. **56** (1939), 455–459.

[11] G. Rein, A.D. Rendall, *Global existence of solutions of the spherically symmetric Vlasov-Einstein system with small initial data*, Commun. Math. Phys. **150** (1992), 561–583. Erratum: Commun. Math. Phys. **176** (1996), 475–478.

Conformal structures of static vacuum data

Helmut Friedrich

In the Cauchy problem for asymptotically flat vacuum data the solution-jets along the cylinder at space-like infinity develop in general logarithmic singularities at the *critical sets* at which the cylinder touches future/past null infinity [1]. The tendency of these singularities to spread along the null generators of null infinity obstructs the development of a smooth conformal structure at null infinity. We consider time reflection symmetric vacuum data which are given by an asymptotically Euclidean metric h on $\mathbb{R}^3$ that admits a smooth conformal 1-point compactification at space-like infinity. For the solution-jets arising from these data to extend smoothly to the critical sets it is *necessary* that the Cotton tensor of h satisfies a certain conformally invariant condition $(*)$ at space-like infinity [1], it is *sufficient* that the initial three-metric h be asymptotically static at space-like infinity [2]. Data which are asymptotically conformal to static data at space-like infinity must necessarily be asymptotically static for the solution-jets to extend smoothly [3]. The purpose of this article is to characterize the gap between these conditions. We show that with the class of metrics which satisfy condition $(*)$ on the Cotton tensor and a certain non-degeneracy requirement is associated a one-form κ with conformally invariant differential $d\kappa$. The conditions that this one-form be closed and one of it integrals satisfy a certain equation provide a criterion under which h is asymptotically conformal to static data at space-like infinity.

References

[1] H. Friedrich, *Gravitational fields near space-like and null infinity*, J. Geom. Phys. **24** (1998), 83–163.

[2] H. Friedrich, *Smoothness at null infinity and the structure of initial data*, in: P. T. Chruściel, H. Friedrich (eds.), *The Einstein Equations and the Large Scale Behaviour of Gravitational Fields.* Birkhäuser, Basel, 2004.

[3] J. A. Valiente Kroon, *Asymptotic simplicity and static data*, Ann. Henri Poincaré **13** (2012) 363-397.

Solutions of the constraint equations with non constant mean curvature

Romain Gicquaud

(joint work with M. Dahl and E. Humbert)

Initial data for the Cauchy problem in general relativity are given by a triple $(M, \widetilde{g}, \widetilde{K})$ where $(M, \widetilde{g})$ is a Riemannian manifold and $\widetilde{K}$ is a symmetric 2-tensor on M. M is to be interpreted as a Cauchy surface in the space-time solving the Einstein equations, $\widetilde{g}$ corresponds to the induced metric and $\widetilde{K}$ to the second fundamental form. As a consequence of the Einstein equations together with the Gauss and Codazzi equations, the metric $\widetilde{g}$ and the tensor $\widetilde{K}$ cannot be chosen independently one from the other. They are linked by the so-called constraint equations:

$$\begin{cases} \mathrm{Scal}^{\widetilde{g}} + \left(\mathrm{tr}_{\widetilde{g}}\, \widetilde{K}\right)^2 - \left|\widetilde{K}\right|^2_{\widetilde{g}} = 0 \\ \mathrm{div}^{\widetilde{g}} \widetilde{K} - d\, \mathrm{tr}_{\widetilde{g}}\, \widetilde{K} = 0. \end{cases}$$

Constructing and classifying triples (M, g, K) solving these equations is an important problem in general relativity. One of the main methods to address this question is the conformal method introduced by Y. Choquet-Bruhat, A. Lichnerowicz and J. York. It consists in prescribing part of the initial data and adjusting the other part to satisfy the constraint equations.

More precisely, we fix:

- (M, g) a Riemannian manifold (e.g. closed) of dimension $n \geq 3$,
- $\tau : M \to \mathbb{R}$ a function,
- σ a symmetric traceless ($g^{ij}\sigma_{ij} = 0$) and divergence-free ($\nabla^i \sigma_{ij} = 0$) 2-tensor.

The unknowns are:

- $\phi : M \to \mathbb{R}_+^*$ a conformal factor,
- W a 1-form.

We define the metric $\widetilde{g}$ and the tensor $\widetilde{K}$ by:

$$\begin{cases} \widetilde{g} = \phi^{N-2} g, \\ \widetilde{K} = \tau \widetilde{g} + \phi^{-2}(\sigma + LW), \end{cases}$$

where $N = \frac{2n}{n-2}$ and L is the conformal Killing operator defined by

$$(LW)_{ij} = \nabla_i W_j + \nabla_j W_i - \frac{2}{n} \nabla^k W_k g_{ij}.$$

The triple $(M, \widetilde{g}, \widetilde{K})$ solves the constraint equations if and only if ϕ and W solve the following system:

$$-\frac{4(n-1)}{n-2}\Delta\phi + \mathrm{Scal}\ \phi + n(n-1)\tau^2\phi^{N-1} = |LW+\sigma|_g^2\,\phi^{-N-1}, \tag{1a}$$

$$-\frac{1}{2}L^*LW = (n-1)\phi^N d\tau. \tag{1b}$$

Equation (1a) is called the *Lichnerowicz equation* while Equation (1b) is called the *vector equation*. L^* is the formal L^2-adjoint of L.

Remark that $\tau = \frac{1}{n}\widetilde{g}^{ij}\widetilde{K}_{ij}$ is, up to a constant, the mean curvature of M in the space-time solving the Einstein equations with initial data $(M, \widetilde{g}, \widetilde{K})$. In the particular case of constant τ, Equation (1b) reduces to $L^*LW = 0$ which implies that $W = 0$ unless (M, g) admits conformal Killing vector fields. Thus the study of the system boils down to the study of the Lichnerowicz equation (1a). This equation appears as a generalization of the Yamabe problem and is now well understood. We refer the reader to [1] for more details.

The major difficulty of the system (1) with $d\tau \not\equiv 0$ can be explained as follows. If we imagine that we can define some sort of order of magnitude λ of the function ϕ, then from Equation (1b), W has order of magnitude λ^N. Looking at Equation (1a), the two dominant terms are $n(n-1)\tau^2\phi^N$ and $|LW|^2\phi^{-N-1}$ which both are of order λ^{N+1} while all other terms have lower order of magnitude. As a consequence, it is hard to obtain a priori estimates for the solutions of this system.

The case of small $d\tau$ can be treated by the implicit function theorem or by a Banach fixed point argument. Recently, two different methods were introduced to handle the case when $d\tau$ is not small.

The first one was introduced by M. Holst, G. Nagy and G. Tsogtgerel in [4, 5] and refined by D. Maxwell in [7]. The naive idea is to remark that we implictly assumed in our reasoning that the order of magnitude λ was large. But if we assume that λ is small then other terms in the Lichnerowicz equation (1a) can dominate and hence yield an a priori upper bound for the solution. The following theorem is taken from [7]:

Assume that (M, g) has $Y(g) > 0$ and that $\sigma \not\equiv 0$ but is small then there exists at least one solution (ϕ, W) to the equations of the conformal method.

Here $Y(g)$ is the Yamabe invariant of the metric g. The assumptions on σ may look a bit strange at first glance. However smallness is required to make the scalar curvature term dominate alone the Lichnerowicz equation while $\sigma \not\equiv 0$ is an essential ingredient to provide a positive lower bound for ϕ. Indeed, if $\sigma \equiv 0$ and $d\tau$ is small, it can be proven that there is no solution to the system, see e.g. [2].

One the other hand, the second method addresses the case of potentially large order of magnitude λ. It was introduced by M. Dahl, E. Humbert and the speaker in [2]. Heuristically, if λ becomes very large then all subdominant terms in the

Lichnerowicz equation should become negligible. Hence, we are left with the following relation between ϕ and LW:

$$n(n-1)\tau^2\phi^{N-1} = |LW|_g^2\,\phi^{-N-1}.$$

In particular, inserting this relation in the vector equation (1b), we get a non trivial solution to the so-called *limit equation*:

$$-\frac{1}{2}L^*LW = \sqrt{\frac{n-1}{n}}\,|LW|\,\frac{d\tau}{\tau}.$$

The exact theorem we prove is actually the following:

Assume that $\sigma \not\equiv 0$ if $Y(g) \geq 0$, that $\tau > 0$ and that the equation

$$-\frac{1}{2}L^*LW = \alpha\sqrt{\frac{n-1}{n}}\,|LW|\,\frac{d\tau}{\tau}$$

admits no non zero solution for any $\alpha \in (0,1]$. Then the set of solutions (ϕ, W) of the equations of the conformal method is non empty and compact.

As a consequence of a Bochner formula for the operator L^*L, this equation can be shown to have no non zero solution for any given function $\tau > 0$ provided that the Ricci tensor of the metric is very negative. By a result of J. Lohkamp [6], these metrics form a dense subset for the C^0-norm.

This method has been adapted to the asymptotically hyperbolic context by A. Sakovich and the speaker in [3].

References

[1] R. Bartnik and J. Isenberg, *The constraint equations*, The Einstein equations and the large scale behavior of gravitational fields, Birkhäuser, Basel, 2004, pp. 1–38.

[2] M. Dahl, R. Gicquaud, and E. Humbert, *A limit equation associated to the solvability of the vacuum Einstein constraint equations using the conformal method*, (to appear in Duke Math. J.), 2010,

[3] R. Gicquaud and A. Sakovich, *A large class of non constant mean curvature solutions of the Einstein constraint equations on an asymptotically hyperbolic manifold*, Comm. Math. Phys. **310** (2012), no. 3, 705–763.

[4] M. J. Holst, G. Nagy, and G. Tsogtgerel, *Far-from-constant mean curvature solutions of Einstein's constraint equations with positive Yamabe metrics*, Phys. Rev. Lett. **100** (2008), no. 16, 161101, 4.

[5] ———, *Rough solutions of the Einstein constraints on closed manifolds without near-CMC conditions*, Comm. Math. Phys. **288** (2009), no. 2, 547–613.

[6] J. Lohkamp, *Curvature h-principles*, Ann. of Math. **142** (1995), no. 3, 457–498.

[7] D. Maxwell, *A class of solutions of the vacuum Einstein constraint equations with freely specified mean curvature*, Math. Res. Lett. **16** (2009), no. 4, 627–645.

Stability of heteroclinic cycles and construction of oscillatory singularities

Alan D. Rendall
(joint work with Stefan Liebscher, Sophonie Blaise Tchapnda)

In the theory of dynamical systems a heteroclinic orbit is a solution which converges to one stationary solution as $t \to -\infty$ and to another stationary solution as $t \to +\infty$. If $\{x_i\},\ i = 1, 2, \dots n$, is a finite sequence of stationary solutions of the dynamical system $\dot{x} = f(x)$ with the property that for each $i < n$ there is a heteroclinic orbit starting at x_i and ending at x_{i+1} then the resulting configuration is called a heteroclinic chain. In what follows the points x_i will be referred to as the vertices of the chain. If, in addition, $x_n = x_1$ the chain is called a heteroclinic cycle. It is of interest to investigate the stability properties of objects of this kind. In other words, if a solution starts close to a heteroclinic cycle, does it converge to it as $t \to \infty$? If a solution has this property then, because the function f vanishes at the vertices, this solution spends most of its time near the vertices. It is therefore plausible that the stability is controlled by the behaviour of the flow near the vertices. In favourable circumstances this is in turn controlled by the eigenvalues of the linearization of the system about the vertices. 'Favourable circumstances' usually includes the requirement that the stationary points at the vertices should be hyperbolic, i.e. that there are no eigenvalues with vanishing real part. In the examples of interest in this report the condition of hyperbolicity is not satisfied. The difficulty can nevertheless be overcome in this case and this issue is not discussed further here. The solutions in these examples represent solutions of the Einstein equations.

The behaviour of solutions of the Einstein equations, the fundamental equations of general relativity, is hard to analyse mathematically. For this reason it is common to make the simplifying assumptions provided by symmetry conditions. Even in the case of spatially homogeneous solutions of the Einstein equations, where the PDE reduce to ODE (essentially everything depends only on time), complicated dynamical properties and challenging mathematical problems remain. In what follows one large class of spatially homogeneous solutions, the solutions of Bianchi Class A, are considered. In order to have a complete system of equations a choice of the matter content must also be made. The choices we concentrate on are vacuum (no matter at all) and a magnetic field. The vacuum models can be described by a four-dimensional dynamical system, the Wainwright-Hsu (WH) system. There are invariant submanifolds defined by different symmetries of the spacetime metric, called Bianchi types I, II, VI_0, VII_0, VIII and IX. The most general of these types are VIII and IX and they are represented by open subsets. There is a similar four-dimensional system describing spacetimes of types I, II and VI_0 with magnetic field. For convenience call it the magnetic system.

Physically the spatially homogeneous solutions represent cosmological models and they have an initial big bang singularity. The issue of the structure of this singularity has been a key question in general relativity for more than forty years.

A heuristic picture of this structure was developed by Belinskii, Khalatnikov and Lifshitz (BKL). The validity of this picture is still largely unclear although some positive results are available. This picture can be applied to the spatially homogeneous case where it makes contact with ideas developed independently by Misner at about the same time. In general it is not even clear how to translate the BKL picture into clear mathematical conjectures. In the spatially homogeneous case it can be formulated as saying that generic solutions converge to heteroclinic chains of a certain type as the initial singularity is approached. The BKL picture also includes the idea that generic inhomogeneous solutions should be approximated by homogeneous solutions near the singularity, but this aspect will not be pursued here. The natural conjecture in the Bianchi Class A case is that generic solutions of Bianchi types VIII and IX are approximated in the past by heteroclinic chains. Here a relation with the discussion of general dynamical systems above can be seen. Note, however, that the direction of time has been reversed - here we consider the dynamics in the limit $t \to -\infty$ which represents the singularity in the WH system. The convention is that the physical time coordinate should increase towards the future.

The variables in the WH system are dimensionless. This has the consequence that self-similar spacetimes correspond to stationary solutions of the WH system. The Bianchi type I subset is a circle consisting of stationary solutions called the Kasner circle. Each Bianchi type II solution is a heteroclinic orbit joining two points on the Kasner circle. Its projection on two of the variables is a straight line. The Bianchi II solutions define many heteroclinic chains including many heteroclinic cycles. For purposes of exposition we will concentrate on the simplest case, a heteroclinic cycle with three vertices. It will be called 'the triangle' in what follows since under the projection already mentioned it maps onto a triangle. Until very recently little was known about the extent to which Bianchi Class A solutions can be approximated by heteroclinic cycles near their singularities. It was a big advance when in was shown in [1] that there is a large class of solutions which are approximated in this way. This was done by starting with particular heteroclinic chains and proving the existence of corresponding solutions of Bianchi types VIII and IX. An informal statement of the main theorem in the case of the triangle is as follows.

Theorem 1 The triangle in the Wainwright-Hsu system has an unstable manifold which is of codimension one.

In other words, among solutions starting close to the triangle those which converge to it as $t \to -\infty$ form a set of codimension one.

The magnetic system for Bianchi type VI_0 solutions was first introduced as a useful model for studying cosmological singularities in [3]. It contains a heteroclinic cycle which can be related in a natural way to the triangle in the vacuum case. Superficially it looks like it should have similar stability properties. This was investigated in [2] where the following result was obtained.

Theorem 2 The triangle in the magnetic system for Bianchi type VI_0 has an unstable manifold which is of codimension one.

The final results for the vacuum case and the case with magnetic field are very similar and this is not surprising. What is surprising is that the proof of Theorem 2 is much more difficult than that of Theorem 1 and requires several essentially new ideas. The difficulty arises through a difference in the eigenvalues of the linearization at the vertices in the two cases. In both cases there is a zero eigenvalue which, although important for the proofs, will be ignored here. It arises from the fact that the stationary solution lies on the Kasner circle which is a manifold of stationary solutions. There is a negative eigenvalue which will be denoted by $-\mu$ and two positive eigenvalues which will be denoted by λ_1 and λ_2. In the vacuum case $\mu < \min\{\lambda_1, \lambda_2\}$. The inequality satisfied by the eigenvalues is crucial for the proof of Theorem 1. The eigenvalues λ_1 and λ_2 are distinct and the notation will be chosen so that $\lambda_1 < \lambda_2$. The eigendirection corresponding to the eigenvalue λ_2 is tangent to one of the heteroclinic orbits belonging to the triangle. This latter fact is not important for the proof of Theorem 1 but it is also true in the magnetic case and it is important for the proof of Theorem 2. In that case it is still true that $\mu < \lambda_2$.

If a heteroclinic cycle in a general dynamical system has the eigenvalue properties of the triangle in the magnetic system this is not enough to prove the analogue of Theorem 2 for that cycle, even when the dominant outgoing direction belongs to the cycle. An important additional property, which follows from the geometric properties of the underlying application to the Einstein equations, is that there exist certain invariant manifolds which cannot be expected to exist on the basis of the eigenvalue structure alone. The existence of the unstable manifold in the vacuum case is based on the graph transform approach to the stable manifold theorem which makes use of a contraction property and the Banach fixed point theorem. In the proof of Theorem 1 the norm in which the contraction is obtained is built using the Euclidean metric defined by the coordinates in the Wainwright-Hsu system. In the magnetic case the corresponding mapping is not a contraction in that norm. This problem is overcome in [2] by using a different norm which is associated to a metric which becomes singular on the boundary. This metric is of the form $\frac{y^2+z^2}{y^2}dy^2 + \frac{y^2+z^2}{z^2}dz^2 + dw^2$ in suitable coordinates. It should be thought of as a metric on a section transverse to one of the heteroclinic orbits belonging to the cycle. The intersection of the section with the orbit itself lies at the origin of the coordinates (y, z, w) with the Kasner circle being represented by $y = z = 0$.

References

[1] S. Liebscher, J. Härterich, K. Webster and M. Georgi, *Ancient dynamics in Bianchi models: approach to periodic cycles*, Commun. Math. Phys. **305** (2011), 59–83.

[2] S. Liebscher, A. D. Rendall and S. B. Tchapnda, *Oscillatory singularities in Bianchi models with magnetic fields*, Preprint arXiv:1207.2655.

[3] M. Weaver, *Dynamics of magnetic Bianchi type VI_0 cosmologies*, Class. Quantum Grav. **17** (2000), 421–434.

Stationary and static initial data sets with MOTS

Marc Mars
(joint work with Martín Reiris)

Mars, Marc

Given a spacetime, a marginally outer trapped surface (MOTS) is a codimension-two, spacelike, oriented and compact embedded surface such that one of its two null expansions vanishes identically. These surfaces signal the presence of a strong gravitational field and, from a physics perspective, they are widely believed to be closely related to black holes. Investigating this relationship is an important problem in mathematical gravitation. Since the concept of black hole is global in nature, while MOTS are defined locally, studying this relationship is a hard problem, which, in fact, is closely connected to the weak cosmic censorship hypothesis. It becomes natural to study the simplest cases first.

Stationary spacetimes are an important class because they represent equilibrium configurations and hence, constitute the class of end-states of many dynamical processes. They are also simple enough so that the problem above becomes tractable. The expectation is that stationary, asymptotically flat and globally hyperbolic spacetimes containing a MOTS should be a black hole and that, moreover, the MOTS should be a section of the corresponding event horizon. Indeed, if MOTS are to be dynamical, quasi-local replacements of black holes, then in an equilibrium situation, where there is no evolution at all, they should simply coincide. The aim of this talk is to address this problem.

The objective being to find global-in-time properties of spacetimes, the natural setup is to make assumptions only at the initial data level. In order to ensure that the corresponding maximal Cauchy development is stationary, it is necessary (and sufficient) to assume that the initial data is a so-called Killing initial data, namely a set $\mathcal{D} := (\Sigma^m, g, K; \rho, \mathcal{J}, \mathcal{T}; N, Y)$ consisting of a standard initial data set $(\Sigma^m, g, K; \rho, \mathcal{J})$ together with a symmetric covariant two-tensor $\mathcal{T}$, a scalar N and a vector Y satisfying the so-called KID equations (see e.g. [5]).

In this definition, N, Y represent the lapse and shift of the Killing vector that exists in the spacetime generated by the data. When the KID is asymptotically flat we assume that $\lambda := N^2 - g(Y, Y)$ tends to a positive constant at infinity.

A particular case of KIDs are **static KID**, where additional equations (see [2]) are imposed to ensure that the Killing vector of the development is integrable.

There are at least two natural approaches to analyze whether the maximal Cauchy development of an asymptotically flat KID containing a MOTS is a black hole. The first one (A) consists in trying to determine sufficient information on its maximal Cauchy development so as to make sure that it is a black hole spacetime and, afterwards, study the relative position of the given MOTS and the intersection of the black event horizon with the given initial data. The second approach (B) uses a detour via the black hole uniqueness theorems. The idea is to extend the black hole uniqueness theorems to stationary (or static), asymptotically flat initial data sets containing a MOTS boundary. As a consequence, the spacetime generated by the data is automatically a black hole (in fact, within the corresponding uniqueness class) and it only remains to show that the MOTS lies in the black hole horizon.

The first time approach (B) was implemented is due to P. Miao [1] who proved that an asymptotically flat, vacuum, three-dimensional, time-symmetric (i.e. with $K = 0$) and static Killing initial data set such that $\partial\Sigma$ is the outermost minimal surface is necessarily isometric to a so-called Schwarzschild half-space.

The proof of this result relies heavily on the vacuum field equations. In [2, 3, 4] an alternative approach based solely on energy inequalities was found that allowed for much more general matter models and for general static KID (i.e. not necessarily time-symmetric). Note that allowing for non time-symmetric static KID is a relevant generalization because, given any such data, it is not true a priori that its maximal Cauchy development admits a time-symmetric Cauchy hypersurface. Dealing with non time-symmetric KID introduces an important complication, namely the potential existence of so-called *Killing prehorizons*. These are injectively immersed, but not necessarily embedded, hypersurfaces of Σ where λ vanishes identically (this corresponds to points where the Killing vector becomes null in the spacetime development). The presence of these objects prevents from applying the standard black hole uniqueness proofs. In the papers above, technical conditions where added in order to deal with this extra difficulty.

In recent work in collaboration with Martín Reiris [5] we have followed approach (A). The asymptotically flat KID is assumed to have a compact outer trapped boundary, i.e. such that the mean curvature H of $\partial\Sigma$ with respect to the unit normal towards Σ satisfies $H + \mathrm{tr}_{\partial\Sigma} K < 0$. This is a condition slightly stronger than $\partial\Sigma$ being a MOTS. Our main result establishes that the maximal Cauchy development of such data is a black hole spacetime.

For a more precise statement, let $(\mathcal{M}^+, \hat{g})$ denote the future of Σ in the maximal Cauchy development $(\mathcal{M}, \hat{g})$ of the data. Denote by S_{r_0} a coordinate sphere $\{r = r_0\}$ in the asymptotically flat end and by $\Sigma^E(r)$ the exterior of S_r. Let also $T(\partial\Sigma, d) := \{p \in \Sigma / \mathrm{dist}_g(p, \partial\Sigma) \leq d\}$. An important object for the theorem is the so-called future Killing development of the data. This is a spacetime generated by dragging the initial data information with the Killing vector. It only exists on points where the Killing vector is transverse (i.e. where $N \neq 0$) and, in particular, on any connected, open subset Ω of the exterior region Σ^T defined as the connected component of $\{\lambda > 0\}$ containing the asymptotically flat end. The future Killing development of Ω is the spacetime $\mathcal{K}(\Omega) := \big(\Omega \times [0, \infty), g_D = -\lambda dt^2 + Y \otimes dt + dt \otimes Y + g\big)$. Our main result reads [5]

Theorem 1. Let $\mathcal{D}$ be an m-dimensional ($m \geq 3$) asymptotically flat, stationary initial data set with well-posed matter model satisfying the null energy condition. Suppose that $\partial\Sigma$ (if non-empty) is future outer trapped. Then the maximal Cauchy development $(\mathcal{M}, \hat{g})$ of $\mathcal{D}$ satisfies

(1) There is $r > 0$ such that $\mathcal{K}(\Sigma^E(r))$ can be isometricaly embedded in $\mathcal{M}^+$.
(2) There is $d > 0$, such that $T(\partial\Sigma, d) \cap J^-(\mathcal{K}(\Sigma^E(r))) = \emptyset$ where $J^-(V)$ is the causal past of $V \subset \mathcal{M}$.

Moreover, $\overline{\Sigma^T} \cap \partial\Sigma = \emptyset$.

Since the future Killing development $\mathcal{K}(\Sigma^{(E)}(r))$ is asymptotically flat in a spacetime sense and, according to (2), its causal past does no cover the whole spacetime, this theorem states that the spacetime $(\mathcal{M}, \hat{g})$ is indeed a black hole.

As mentioned above, the presence of Killing prehorizons is an obstruction to proving black hole uniqueness theorems (at least, with the present proofs). So, the problem of studying whether Killing prehorizons are possible or not is interesting on its own. Under suitable global hypotheses (both to the future and past), Killing prehorizons have been excluded in the domain of outer communications [6]. Since the theorem above only provides information to the future, this global result is not directly applicable at the initial data level. Using an argument that only requires initial data information, the following result on non-existence of Killing prehorizons can be proved [5].

Theorem 2. Let $\mathcal{D}$ be an asymptotically flat static Killing initial data set satisfying the null energy condition. Suppose $\overline{\Sigma^T} \cap \partial\Sigma = \emptyset$. Then, all Killing prehorizons are embedded.

Combining these results a fully satisfactory static uniqueness theorem follows.

Theorem 3. Let $\mathcal{D}$ be an m-dimensional ($m \geq 3$), static, asymptotically flat, Killing initial data satisfying the following assumptions:

A1. The matter model is well-posed and satisfies the null energy condition.
A2. The matter model satisfies the static black hole uniqueness theorem.
A3. Σ has outer trapped boundary.

Then (Σ^T, g, K) can be isometrically embedded in a spacetime within the black hole uniqueness class.

The notion of a matter model satisfying the static black hole uniqueness theorem is defined in [5] and simply makes precise the requirement that black holes of this matter model can be classified.

The strict inequality $H + \mathrm{tr}_{\partial\Sigma} K < 0$ is required for technical reasons. It would be interesting to relax this to $H + \mathrm{tr}_{\partial\Sigma} K = 0$, i.e. to the case of MOTS boundary.

Another interesting problem is to study whether uniqueness extends to axially symmetric, stationary KID when $m = 3$ (and e.g. vacuum).

References

[1] P. Miao, *A remark on boundary effects in static vacuum initial data sets*, Classical and Quantum Gravity **22** (2005), L53-L59.

[2] A. Carrasco, M. Mars, *On marginally outer trapped surfaces in stationary and static spacetimes*, Classical and Quantum Gravity **25** (2008), 055011.

[3] A. Carrasco, M. Mars, *Uniqueness theorem for static spacetimes containing marginally outer trapped surfaces*, Classical and Quantum Gravity **28** (2011), 175018.

[4] A. Carrasco, *Trapped surfaces in spacetimes with symmetries and applications to uniqueness theorems*, Ph.D. Thesis (2011).

[5] M. Mars, M. Reiris, *Global and uniqueness properties of stationary and static spacetimes with outer trapped surfaces*, submitted.

[6] P.T. Chruściel, G.J. Galloway, *Uniqueness of static black-holes without analyticity*, Classical and Quantum Gravity **27** (2010), 152001.

The wave equation on extremal black holes

Stefanos Aretakis

One problem that attracted a lot of interest during the past decade is the understanding of the evolution of solutions to the wave equation

$$\Box_g \psi = 0$$

on black holes spacetimes. This problem is also closely related to the stability of black holes, the latter being solutions to the Einstein equations.

Previous works by Dafermos and Rodnianski [7], Andersson and Blue [1] and Tataru and Tohaneanu [10] have independently provided quantitative decay rates for solutions to the wave equation in the exterior region of slowly rotating Kerr black holes. In fact, Dafermos and Rodnianski [8] presented analogous rates for the general **subextremal** Kerr family. These decay rates hold for solutions and all their derivatives up to and including the event horizon.

This talk concerns the case previously not covered, i.e. the extremal case. Extremal Kerr is characterized by the vanishing of the surface gravity on the event horizon. Geometrically, this means that if V is Killing and null normal on the horizon then

$$\nabla_V V = 0.$$

The above condition manifestly implies that there is no bifurcate sphere in the extremal case. Note that from a classical physics point of view, the extremality of the horizon implies that the redshift effect degenerates there. Furthermore, from the quantum physics point of view, the same condition implies that the temperature of the horizon vanishes and thus extremal black holes do not radiate.

For the problem at hand, the fundamentally new aspect (compared to the subextremal case) is the degeneracy of the redshift effect on the horizon.

Instead of starting with extremal Kerr, one could first consider the simpler extremal Reissner–Nordström black holes; the simplicity of the latter black holes being manifested in their spherical symmetry. In this case, if Y is a translation-invariant transversal to the horizon $\mathcal{H}$ vector field and ψ is a *spherically symmetric* solution to the wave equation then we obtain that the quantity

$$H[\psi] = Y\psi + \frac{1}{M}\psi$$

is **conserved** along $\mathcal{H}$, where $M > 0$ is the mass parameter. Since for generic initial data $H[\psi] \neq 0$, we have that for generic ψ either ψ or $Y\psi$ **cannot decay**! Note that this result is in stark contrast with the subextremal case for which decay is known for ψ and all its derivatives.

In view of the fact that extremal Kerr is not spherically symmetric, the above argument cannot be readily adapted to extremal Kerr. Nonetheless, we can prove the following (S.A. 2012):

Let $(\mathcal{M}, g)$ be a 4-dimensional Lorentzian manifold containing an extremal axisymmetric horizon $\mathcal{H}$. Let also V denote the Killing field null and normal to $\mathcal{H}$

and Φ denote the axial Killing tangential to $\mathcal{H}$ and such that $[V, \Phi] = 0$. If the distribution of the planes orthogonal to the planes spanned by V and Φ is integrable, then we have a conservation law on the horizon $\mathcal{H}$.

The conservation law holds for the spherical mean of an expression of ψ and first order derivatives of ψ. Specifically, for extremal Kerr, we obtain that the quantity

$$H^{\mathrm{Kerr}}[\psi](\tau) = \int_{S_\tau} \Big(M \sin^2\theta \, (T\psi) + 4M \, (Y\psi) + 2\psi \Big)$$

is conserved along the event horizon $\mathcal{H}$.

We note that similar conservation laws hold for Majumdar–Papapetrou multi black holes. Note further that the integrability condition of the above theorem is consistent with general relativity in view of Papapetrou theorem. We also mention that Lucietti and Reall [9] have recently extended the conservation laws to the Teukolsky equation governing linearized gravitational (or electromagnetic) perturbations of (extremal) Kerr.

The above conservation laws are completely determined by the local properties of extremal horizons (namely, by the induced metric and the Christoffel symbols Γ on $\mathcal{H}$) and hence do not depend on global aspects of the spacetime. Hence, although they imply that not all quantities decay in the extremal case, they do not provide a satisfactory picture of the evolution of solutions to the wave equation in the domain of outer communications. Nonetheless, in the case of extremal Reissner–Nordström and extremal Kerr we were able to show the following **instability results** (S.A. 2010-2012):

For generic solutions ψ to the wave equation on extremal Reissner–Nordström or extremal Kerr backgrounds we have:

Non-Decay:

The translation-invariant transversal to $\mathcal{H}$ derivative $Y\psi$ does not decay along $\mathcal{H}$.

Pointwise Blow-up:

$$\left|Y^k\psi\right| \to +\infty,$$

along $\mathcal{H}$ as advanced time tends to infinity $k \geq 2$.

Energy Blow-up:

$$\left\|Y^k\psi\right\|_{L^2} \to +\infty$$

as advanced time tends to infinity $k \geq 2$ for all $k \geq 2$ (the L^2 norm is taken over spacelike hypersurface crossing $\mathcal{H}$).

The above instability results concern the transversal to $\mathcal{H}$ derivatives of ψ. For the solutions ψ themselves, we can in fact prove the following **stability result** (S.A. 2010-2011):

For all solutions ψ (with sufficiently regular initial data on Σ_0) to the wave equation on extremal Reissner–Nordström and all axisymmetric solutions on extremal Kerr we have

$$\textbf{\textit{Pointwise Decay:}}\ |\psi(\tau, \cdot)| \leq C \cdot D \cdot \frac{1}{\tau^{\frac{3}{5}}}$$

up to and including the event horizon $\mathcal{H}$.

Proving the above decay rate requires overcoming the degeneracy of the redshift and circumventing the use of quantities which do not decay. This was made possible by the construction of a novel current that captures the degenerate redshift and the use of new Hardy-like inequalities. The so-called *trapping* was overcome by constructing microlocal currents for axisymmetric solutions which completely decouple from the redshift effect (and are in fact localized in phase space near the region where trapping takes place). Removing the axisymmetry assumption for the stability result remains an open problem.

It is our belief that extremality has opened a new area in the stability problem and we are certain that more fascinating results are to be found.

References for the talk include: [2, 3, 4, 5, 6].

References

[1] L. Andersson and P. Blue, *Hidden symmetries and decay for the wave equation on the Kerr spacetime*, arXiv:0908.2265

[2] S. Aretakis, *The wave equation on extreme Reissner–Nordström black hole spacetimes: stability and instability results*, arXiv:1006.0283

[3] S. Aretakis, *Stability and instability of extreme Reissner–Nordström black hole spacetimes for linear scalar perturbations I*, Comm. Math. Phys. **307** (2011), 17–63

[4] S. Aretakis, *Stability and instability of extreme Reissner–Nordström black hole spacetimes for linear scalar perturbations II*, Ann. Henri Poincaré **8** (2011), 1491–1538

[5] S. Aretakis, *Decay of axisymmetric solutions to the wave equation on extreme Kerr backgrounds*, 2011, to appear in Journal of Functional Analysis

[6] S. Aretakis, *Horizon instability of extremal black holes*, arXiv:1206.659

[7] M. Dafermos and I. Rodnianski, *A proof of the uniform boundedness of solutions to the wave equation on slowly rotating Kerr backgrounds*, Invent. Math. **185** (2011), 467–559, arXiv:0805.4309

[8] M. Dafermos and I. Rodnianski, *The black hole stability problem for linear scalar perturbations*, to appear in Proceedings of the 12 Marcel Grossmann Meeting, arXiv:1010.5137

[9] J. Lucietti and H. Reall, *Gravitational instability of an extreme Kerr black hole*, arXiv:1208.1437

[10] D. Tataru and M. Tohaneanu, *Local energy estimate on Kerr black hole backgrounds*, Int. Math. Res. Not. **2011** (2008), 248–292

Critical phenomena at the threshold of immediate merger in binary black hole systems: the extreme mass ratio case

Carsten Gundlach

There has been some interest in the formation of black holes from particle collisions in accelerators. Pretorius and Khurana [1] have simulated this numerically, using two equal mass Schwarzschild black holes, boosted relative to each other, as the particles. For the sake of argument, keep the initial separation (large enough to be effectively infinite) and relative boost constant and vary only the impact parameter in the initial data. Clearly, for large impact parameter the "particles" will shatter, and for small impact parameter they will merge into a single black hole.

Clearly, there is more generally a "threshold of immediate merger" in the abstract space of initial data, and it can be explored by varying any one parameter p

of the initial data, such as the impact parameter. It is this threshold Pretorius and Khurana were interested in, because of the "critical phenomena in gravitational collapse" that Choptuik found around 1992 at the "threshold of gravitational collapse" for spherically symmetric scalar field.

Choptuik's initial work was generalised to other systems that have a threshold between collapse and dispersion (as does the two black hole system considered by Pretorius and Khurana). Most of these are spherically symmetric for simplicity, but axisymmetric scalar fields and axisymmetric vacuum gravity have also been looked at. One distinguishes type I and type II critical phenomena at the threshold of collapse. Type I is perhaps more familiar: In this scenario there exists a stationary solution with precisely one growing (unstable) perturbation mode. An elementary calculation (which I went through in the talk) then shows that near the threshold the time evolution hovers near this "critical solution" for a time T that scales as $T \sim -\gamma \ln |p - p_*|$, where p is any parameter of the initial data, p_* its value at the collapse threshold, and γ is a universal "critical exponent", calculated to be the inverse of the growth rate of the one unstable mode.

Type II is not really relevant for my talk but is what Choptuik discovered and is more surprising, and so merits a paragraph aside. Here the critical solution is self-similar (homothetic) and it is not the survival time T of the metastable star that scales with the initial data, but the mass M of the black hole that is eventually made: $M \sim |p - p_*|^\gamma$, with γ once again related to the growth rate (now per logarithm of time) of the one unstable mode of the self-similar critical solution.

Back to the binary black hole case. Pretorius and Khurana found that the number of orbits scales as $n \simeq -\gamma \ln |p - p_*|$ along any one-parameter family of initial data such that the threshold is at $p = p_*$. Hence they conjectured that in ultrarelavistic collisions almost all the kinetic energy can be converted into gravitational waves if the impact parameter is fine-tuned to the threshold. As a toy model for the binary, they considered the geodesic motion of a test particle in a Kerr black hole spacetime, where the unstable circular geodesics play the role of critical solutions (of type I), and calculate the critical exponent γ.

With Leor Barack, Sarp Akcay and Alessandro Nagar, I have incorporated radiation reaction into this model using the self-force approximation. That is, we turn the toy model into a physical model of a finite mass ratio binary in the limit where the mass ratio is much smaller than one.

The critical solution now evolves adiabatically along a sequence of unstable circular geodesic orbits under the effect of the self-force. We confirm that almost all the initial energy and angular momentum are radiated on the critical solution. Our calculation suggests that, even for infinite initial energy, this happens over a finite number of orbits given by $n_\infty \simeq 0.41/\eta$, where η is the (small) mass ratio. We derive expressions for the time spent on the critical solution, number of orbits and radiated energy as functions of the initial energy and impact parameter. I won't give them here.

At the end, I speculated a bit about what this may mean for the equal (or comparable) mass ratio case. Say the masses are equal and the two black holes are nonspinning for definiteness. Then we believe that there is a unique critical solution starting at infinite energy and and which ends with a merger only after all the kinetic energy has been radiated to infinity (as far as compatible with the area theorem). Any finite energy collision fine-tuned close to the critical impact parameter (say) joins the critical solution at the appropriate energy, and leaves it at a (much) lower energy depending on how good the fine-tuning was. Our extreme mass ratio critical solution spiralled out from $r = 3M$ (the light ring) to $6M$ (the point particle ISCO), but the equal mass critical solution must be spiralling in to avoid immediate merger by the hoop conjecture. But the two black holes will be moving essentially at the speed of light all along, held together by the gravity of their kinetic energy... It is quite unfamiliar, and worth exploring more. At (relatively) low orbital energy, one may be able to use an adiabatic approximation, similar to what we used for the extreme mass ratio case - here this would mean a helical Killing vector approximation. But at high to infinite energy I have no idea what this solution looks like. Our extreme mass ratio results suggest that it only makes a finite number of orbits, perhaps very few. So perhaps one could use a colliding impulsive waves approximation?

References

[1] F. Pretorius and D. Khurana, *Black hole mergers and unstable circular orbits*, Class. Quant. Grav. **24**, S83 (2007).

Black hole instabilities and local Penrose inequalities

Harvey S. Reall
(joint work with Pau Figueras, Keiju Murata)

In this talk I shall describe work done in Ref. [1] and a new paper to appear.

Vacuum black holes in four spacetime dimensions are believed to be stable against gravitational perturbations. A qualitatively new feature that emerges in higher dimensions is the possibility of black objects with unstable horizons. The first example to be discovered was the Gregory-Laflamme instability of a black string [2]. Later, heuristic arguments were presented which suggest that "ultra-spinning" Myers-Perry [3] black holes should suffer from a similar kind of instability [4]. The existence of this instability has been confirmed by studies of linearized perturbations [5, 6, 7, 8].

The equations governing linearized perturbations of higher-dimensional rotating black holes are very complicated. It would be nice if there were a simpler method of demonstrating black hole instabilities. In this talk, we will explain how the existence of certain types of instability can be demonstrated using inequalities analogous to the Penrose inequality.

The usual Penrose inequality is supposed to apply to any asymptotically flat initial data containing an apparent horizon. We introduce the notion of a *local*

Penrose inequality, in which one restricts attention to initial data describing a small perturbation of a black hole. We use local Penrose inequalities to demonstrate instability of certain stationary black hole solutions. If such a black hole is stable, then initial data describing a small perturbation of the black hole must satisfy a local Penrose inequality. Therefore, if we can find initial data that violates the inequality then the black hole must be unstable. Constructing initial data usually is much easier than solving the linearized Einstein equation so this is an easy method of demonstrating instabilities.

We explain how the method can be used to demonstrate existence of the Gregory-Laflamme instability and the instability of ultraspinning Myers-Perry black holes. Next we consider the stability of the black ring solutions of Ref. [9, 10]. So far, investigations of black ring stability have been heuristic. For a given mass, there is a finite range of angular momenta for which there exist two distinct ring solutions, referred to as "thin" and "fat" because of the shape of the horizon. Heuristic arguments suggest that "fat" rings should be unstable. We find that this is indeed the case.

Finally, we consider the case of "non-uniform" black string solutions, which we construct numerically. Such solutions exhibit an interesting change in their stability around the critical spacetime dimension $d = 13$. Weakly non-uniform solutions are expected to be unstable for $d \leq 13$ and stable for $d > 13$, which is what we find. However, our results can be extended to finite non-uniformity, with the surprising result that strongly non-uniform black strings appear to be stable for $d = 12, 13$.

References

[1] P. Figueras, K. Murata and H. S. Reall, Class. Quant. Grav. **28**, 225030 (2011) [arXiv:1107.5785 [gr-qc]].
[2] R. Gregory and R. Laflamme, Phys. Rev. Lett. **70**, 2837 (1993) [arXiv:hep-th/9301052].
[3] R. C. Myers and M. J. Perry, Annals Phys. **172**, 304 (1986).
[4] R. Emparan and R. C. Myers, JHEP **0309**, 025 (2003) [arXiv:hep-th/0308056].
[5] O. J. C. Dias, P. Figueras, R. Monteiro, J. E. Santos and R. Emparan, Phys. Rev. D **80**, 111701 (2009) [arXiv:0907.2248 [hep-th]].
[6] O. J. C. Dias, P. Figueras, R. Monteiro, H. S. Reall and J. E. Santos, JHEP **1005**, 076 (2010) [arXiv:1001.4527 [hep-th]].
[7] O. J. C. Dias, P. Figueras, R. Monteiro and J. E. Santos, Phys. Rev. D **82**, 104025 (2010) [arXiv:1006.1904 [hep-th]].
[8] O. J. C. Dias, R. Monteiro and J. E. Santos, arXiv:1106.4554 [hep-th].
[9] R. Emparan and H. S. Reall, Phys. Rev. Lett. **88**, 101101 (2002) [arXiv:hep-th/0110260].
[10] A. A. Pomeransky and R. A. Sen'kov, arXiv:hep-th/0612005.

Stability of Black Holes and Black Branes

Robert M. Wald

I describe recent work with Stefan Hollands, discussed in much more detail in [1].

It is of considerable interest to determine the linear stablity of black holes in D-dimensional general relativity. It is also of interest to determine the linear stability of the corresponding black branes in $(D+p)$-dimensions, i.e., spacetimes $M \times \mathbb{T}^p$ with metric of the form

$$d\tilde{s}^2_{D+p} = ds^2_D + \sum_{i=1}^{p} dz_i^2 \ . \tag{1}$$

where ds^2_D is a black hole metric. One can analyze this issue by writing out the linearized Einstein equation off of the black hole or black brane background spacetime. One way of establishing linear stability is to find a suitable positive definite conserved norm for perturbations. Linear instability can be established by finding a solution for which some gauge independent quantity has unbounded growth in time. However, even in the very simplest cases—such as the Schwarzschild black hole and the Schwarzschild black string—it is highly nontrivial to carry out the decoupling of equations and the fixing of gauge needed to determine stability or instability directly from the equations of motion. It is particularly difficult to analyze stability when the background is stationary but not static. Thus, it would be extremely useful to have a stability criterion for black holes and black branes that does not require one to perform a complete analysis of the linearized perturbation equations.

For the case of black branes, a simple criterion for stability was proposed by Gubser and Mitra [2, 3], based on the analogy between laws of black hole mechanics and the laws of thermodynamics. Another relatively simple stability criterion, which has been proposed to be applicable to black holes, is the "local Penrose inequality," discussed in [4].

In our work, we establish a new criterion for the dynamical stability of black holes in $D \geq 4$ spacetime dimensions in general relativity with respect to axisymmetric perturbations: Dynamical stability is equivalent to the positivity of the canonical energy, $\mathcal{E}$, on a subspace, $\mathcal{T}$, of linearized solutions that have vanishing linearized ADM mass, momentum, and angular momentum at infinity and satisfy certain gauge conditions at the horizon. This is shown by proving that—apart from pure gauge perturbations and perturbations towards other stationary black holes—$\mathcal{E}$ is nondegenerate on $\mathcal{T}$ and that, for axisymmetric perturbations, $\mathcal{E}$ has positive flux properties at both infinity and the horizon. We further show that $\mathcal{E}$ is related to the second order variations of mass, angular momentum, and horizon area by $\mathcal{E} = \delta^2 M - \sum_A \Omega_A \delta^2 J_A - \frac{\kappa}{8\pi}\delta^2 A$, thereby establishing a close connection between dynamical stability and thermodynamic stability. Thermodynamic instability of a family of black holes need not imply dynamical instability because the perturbations towards other members of the family will not, in general, have vanishing linearized ADM mass and/or angular momentum. However, we prove that for any black brane corresponding to a thermodynamically unstable black hole, sufficiently long wavelength perturbations can be found with $\mathcal{E} < 0$ and vanishing linearized ADM quantities. Thus, all black branes corresponding to thermodynmically unstable black holes are dynamically unstable, as conjectured by Gubser and Mitra. We also prove that positivity of $\mathcal{E}$ on $\mathcal{T}$ is equivalent to the satisfaction of a "local Penrose inequality," thus showing that satisfaction of this local Penrose inequality is necessary and sufficient for dynamical stability. Although we restrict our considerations to vacuum general relativity, most of our results are derived using general Lagrangian and Hamiltonian methods and therefore can be straightforwardly generalized to allow for the presence of matter fields and/or to the case of an arbitrary diffeomorphism covariant gravitational action.

References

[1] S. Hollands and R.M. Wald, *Stability of Black Holes and Black Branes*, arXiv:1201.0463.
[2] S. S. Gubser and I. Mitra, *The Evolution of unstable black holes in anti-de Sitter space*, JHEP **0108**, 018 (2001) [hep-th/0011127].
[3] S. S. Gubser and I. Mitra, *Instability of charged black holes in Anti-de Sitter space*, hep-th/0009126.
[4] P. Figueras, K. Murata and H. S. Reall, *Black hole instabilities and local Penrose inequalities*, Class. Quant. Grav. **28**, 225030 (2011) [arXiv:1107.5785 [gr-qc]].

Waves, modes and quasimodes on asymptotically Anti-de-Sitter black hole spacetimes

Jacques Smulevici
(joint work with Gustav Holzegel)

In this talk, we presented results concerning the behaviour of waves propagating in asymptotically Anti-de-Sitter (AdS) black hole spacetimes. The presence of an asymptotically AdS end strongly modifies the decay properties of solutions compared to the asymptotically flat or de-Sitter case. Indeed, for asymptotically AdS spacetimes, null infinity can be seen as a timelike boundary and, in the case of Dirichlet boundary conditions, no radiation can escape through it. In the pure AdS case, this leads to the existence of mode solutions, with finite energy, to the scalar wave or the Klein-Gordon equation. These solutions are in particular time periodic and therefore do not decay.

One may hope to bypass this obstruction to decay by considering black hole spacetimes. To fix the notation, let us consider, for an asymptotically Anti-de-Sitter spacetime $(\mathcal{M}, g)$ with cosmological constant $\Lambda = -\frac{3}{l^2} < 0$, the equation

$$\Box_g \psi + \frac{\alpha}{l^2}\psi = 0, \qquad (1)$$

where α is a constant.

In [9], we proved

Theorem 1. *Let $(\mathcal{M}, g)$ be a Kerr-AdS spacetime with parameters (M, a, l) for which the boundedness theorem of [7] holds*[1] *and ψ be a CH^2_{AdS} solution of (1) with $\alpha < \frac{9}{4}$ on this background. Assume moreover than one of the following three conditions on the parameters hold:*

1. $\alpha \leq 1$,
2. $1 < \alpha \leq 2$ *and* $|a| < \frac{l}{2}$,
3. *a is sufficiently small depending only on M, l and α.*

Then, we have the decay estimate

$$\|\psi\|^2_{H^1_{AdS}(\Sigma_{t^\star})} \lesssim (\log\ t^\star)^{-1} \left[\|\psi\|^2_{H^2_{AdS}(\Sigma_0)} \right] \tag{2}$$

for any $t^\star \geq 2$, where $\lesssim$ allows a constant depending only on the parameters M, a, l of the background spacetime and on α.

In the above theorem, the $\Sigma_{t^\star}$ foliation is a standard foliation, intersecting the event horizon, by slices of constant $t^\star$, where $t^\star$ is the standard time function in Kerr-AdS-star coordinates (see [9]). The norms $||.||_{H^1_{AdS}}$ and $||.||_{H^2_{AdS}}$ are radially weighted Sobolev type norms controlling respectively one and two derivatives. The set CH^2_{AdS} is the space of functions which are continuous in time with values in H^2_{AdS}. In particular, the Dirichlet conditions at the asymptotically AdS end are incorporated in the condition that the solutions lie in the space CH^2_{AdS}.

Remarks. 1. The condition $\alpha < \frac{9}{4}$, known as the Breiterlohner-Freedman bound, is related to the issue of well-posedness for (1). See [11, 1, 12, 6] as well as the recent [13] for related work on this issue.

2. One can improve the $(\log\ t^\star)^{-1}$ weight in the decay estimate to $(\log\ t^\star)^{-k}$, for any $k \geq 1$, if one is willing to commute more, i.e. the replace the H^2_{AdS} norm by a higher norm H^{1+k}_{AdS} on the right-hand side of (2).

This logarithmic decay rate is much weaker than what can be proven in the asymptotically flat case[2]. Hence, one may think that this estimate is far from being sharp. However, distributions of quasinormal modes have been computed numerically which approach the real axis exponentially (see [5]). Such convergence typically leads to logarithmic decay rates. Moreover, in [9], we unravelled a new trapping phenomenon and conjectured that it was responsible for this slow decay. Indeed, this trapping phenomenon, which essentially consists in light rays unable to cross the photon sphere (or its analogue in Kerr-AdS) and being reflected at null infinity, is easily seen to be stable. We have recently managed to validate these heuristics and proved

Theorem 2 ([10])**.** *Let $(\mathcal{M}, g)$ denote a Schwarzschild-AdS spacetime, with mass M and cosmological constant $\Lambda = -\frac{3}{l^2}$. Let $\alpha \leq 2$. Let $t^*_0 > 0$ be fixed. Let*

[1] in particular both the no-superradiance condition $r^2_+ > |a|l$, with r_+ the area-radius at the event horizon, and the no-naked singularities condition $|a| < l$ hold.

[2] For the reader interested in the asymptotically flat case, as well as for many more references, we refer to the standard lecture notes on the subject, [3], for a detailed study of decay of linear waves in black hole spacetimes.

SCH^2_{AdS} denotes the set of CH^2_{AdS} solutions to (1)*. Then there exist a universal (depending only on $M>0$) constant $C>0$ such that*

$$\lim_{t^\star\to+\infty}\sup_{\psi\in SCH^2_{AdS}}\log(2+t^\star)\frac{||\psi_t||_{L^2(\Sigma_{t^\star})}+||\psi||_{H^1(\Sigma_{t^\star})}}{||\psi_t||_{H^1(\Sigma_{t^\star_0})}+||\psi||_{H^2(\Sigma_{t^\star_0})}}>C>0.$$

In particular, this theorem shows that one cannot prove a uniform decay rate faster than logarithmic[3]. The proof of this theorem relies on the construction of the following approximate solutions or *quasimodes*, which we state only for the conformal case for simplicity in the exposition,

Theorem 3 ([10])**.** *Let $(\mathcal{M},g)$ denote a Schwarzschild-AdS spacetime, with mass M and cosmological constant $\Lambda=-\frac{3}{l^2}$. Let $\widetilde{\Box}_g$ denote the conformal wave operator $\Box_g+\frac{2}{l^2}$. Let (t,r,θ,φ) be standard Schwarzschildian coordinates on the black hole exterior of $\mathcal{M}$. Then, there exists a family of functions $(\psi_\ell\in H^2_{AdS})_{\ell\geq 0}$ such that*

- *$\psi_\ell=e^{i\omega_\ell t}\phi_\ell(r,\theta,\varphi)$,*
- *for all $k\geq 0$, $||\widetilde{\Box}_g\psi_\ell||_{H^k_{Ads}}\leq Ce^{-C\ell}||\psi_\ell||_{H^k_{AdS}}$, for some $C>0$ independent of ℓ,*
- *The support of $F_\ell:=\widetilde{\Box}_g\psi_\ell$ is contained in $[3M-\delta,3M+\delta]$,*
- *$\phi_\ell(r,\theta,\varphi)=0$, for all $r\leq 3M-\delta$.*

Since the ψ_ℓ have constant in time H^k_{AdS} norms, they exhibits no decay. Moreover, a standard application of Duhamel's formula shows that they are good approximations to the solutions of $\widetilde{\Box}_g\psi=0$ with the data induced by ψ_ℓ up to a time $t\sim e^{C\ell}$, hence the relation to Theorem 2.

All these results hold for solutions in the CH^2_{AdS} class, which, as mentioned earlier, corresponds to imposing Dirichlet boundary conditions. Other boundary conditions are however possible, depending on the value of α. In particular, local well-posedness for asymptotically Anti-de-Sitter spacetimes has been proven in [13] and boundedness of solutions will be studied in [8].

As far as nonlinear evolution is concerned, one naturally expects that this slow decay will lead to instability. Thus, we conjecture that Schwarzschild-AdS and Kerr-AdS are dynamically unstable. In this regard, note also that AdS itself has been conjectured, independently by M.T Anderson and M. Dafermos, to be unstable and that there is growing numerical and heuristic arguments supporting this conjecture (see [2, 4]).

References

[1] A. Bachelot, *The Dirac System on the Anti-de Sitter Universe*, Comm. Math. Phys. **283** 2008, 127–167.

[2] P. Bizoń and A. Rostworowski, *Weakly turbulent instability of anti-de Sitter space*, arXiv:1104.3702.

[3] A careful reader would have noticed that the powers of the log differ in Theorems 1 and 2, leaving a small room for improvement in Theorem 1. We plan to address this small gap in future work.

[3] M. Dafermos and I. Rodnianski, *Lectures on black holes and linear waves*, Institut Mittag-Leffler Report no. 14, 2008/2009, arXiv:0811.0354.
[4] O. Dias, G. Horowitz and J.E. Santos, *Gravitational Turbulent Instability of Anti-de Sitter Space*, arXiv:1109.1825.
[5] G. Festuccia and H. Liu, *A Bohr-Sommerfeld quantization formula for quasinormal frequencies of AdS black holes*, arXiv:0811.1033.
[6] G. Holzegel, *Well-posedness for the massive wave equation on asymptotically anti-de-Sitter spacetimes*, J. Hyperbolic Differ. Equ. **9** 2012, 239.
[7] G. Holzegel, *On the massive wave equation on slowly rotating Kerr-AdS spacetimes*, Comm. Math. Phys. **294** 2010, 169–197, arXiv:0902.0973.
[8] G. Holzegel and C. M. Warnick, *Boundedness of the wave equation for asymptotically Anti-de-Siter black holes*, preprint.
[9] G. Holzegel and J. Smulevici, *Decay properties of Klein-Gordon fields on Kerr-AdS spacetimes*, arXiv:1110.6794.
[10] G. Holzegel and J. Smulevici, *Quasimodes and lower bound on uniform energy decay rate on Schwarzschild-AdS spacetimes*, preprint.
[11] A. Ishibashi and R. M. Wald, *Dynamics in non-globally-hyperbolic static spacetimes. III. Anti-de Sitter spacetime*, Classical Quantum Gravity **21** 2004, 2981–3013.
[12] A. Vasy, *The wave equation on asymptotically Anti-de Sitter spaces*, Analysis and PDE **5** 2012, 81–144.
[13] C. M. Warnick, *The massive wave equation on asymptotically AdS spacetimes*, preprint arXiv:1202.3445.

Causality with continuous metrics

Piotr T. Chrusciel
(joint work with James Grant)

After a brief review of some (non-)existence and (non-)uniqueness for linear wave equations on manifolds, and for the Einstein equations, I will describe some ideas which enter in the proof of the theorem of existence of maximal globally hyperbolic vacuum developments of general relativistic Cauchy data with $H^3 \times H^2$ initial data. I will concentrate on the difficulties that arise when trying to carry out the required causality arguments for metrics which are merely continuous. The talk is based on my paper [1] and on a joint paper with James Grant [2].

References

[1] P.T. Chrusciel, *On maximal globally hyperbolic vacuum space-times*, arXiv:1112.5779v1.
[2] P.T. Chrusciel and J.E. Grant, *On Lorentzian causality with continuous metrics*, arXiv:1111.0400v3.

On Big Bang Spacetimes

Jared Speck
(joint work with Igor Rodnianski)

The purpose of this report is to summarize a new past-global stability result for the Einstein equations. The result concerns the well-known Friedmann-Lemaître-Robertson-Walker (FLRW) solution on the manifold $(0, \infty) \times \mathbb{T}^3$ to the Einstein-stiff fluid system. Relative to an arbitrary coordinate system, the equations are

$$\mathbf{Ric}_{\mu\nu} - \frac{1}{2}\mathbf{R}\mathbf{g}_{\mu\nu} = \mathbf{T}_{\mu\nu}, \qquad (\mu, \nu = 0, 1, 2, 3), \tag{1a}$$

$$\mathbf{D}_{\mu}\mathbf{T}^{\mu\nu} = 0, \qquad (\nu = 0, 1, 2, 3), \tag{1b}$$

where $\mathbf{Ric}_{\mu\nu}$ denotes the Ricci tensor of $\mathbf{g}_{\mu\nu}$, $\mathbf{R}$ denotes the scalar curvature of $\mathbf{g}$, $\mathbf{D}_{\mu}$ denotes the Levi-Civita connection of $\mathbf{g}_{\mu\nu}$, and $\mathbf{T}_{\mu\nu}$ denotes the energy-momentum tensor of a perfect fluid:

$$\mathbf{T}^{\mu\nu} = (\rho + p)\mathbf{u}^{\mu}\mathbf{u}^{\nu} + p(\mathbf{g}^{-1})^{\mu\nu}. \tag{2}$$

Above, ρ is the fluid's proper energy density, p is the pressure, and $\mathbf{u}$ is the (future-directed) four-velocity, which is normalized by

$$\mathbf{g}_{\alpha\beta}\mathbf{u}^{\alpha}\mathbf{u}^{\beta} = -1. \tag{3}$$

In order to close the equations, we assume the equation of state of a *stiff fluid*:

$$p = \rho. \tag{4}$$

The aforementioned FLRW solution can be expressed as

(5)
$$\widetilde{\mathbf{g}} = -dt^2 + t^{2/3}\sum_{j=1}^{3}(dx^j)^2, \qquad \widetilde{p} = \frac{1}{3}t^{-2}, \qquad \widetilde{\mathbf{u}}^{\mu} = \delta_0^{\mu}, \qquad (t, x) \in (0, \infty) \times \mathbb{T}^3.$$

The solution has a Big Bang singularity at $\{t = 0\}$ at which the Kretschmann scalar $|\widetilde{\mathbf{Riem}}|^2_{\widetilde{\mathbf{g}}}$ blows up like t^{-4}.

Our main goal in our upcoming article is to study small perturbations of the FLRW solution from the point of view of the initial value problem. More precisely, we view the solution (5) as having been launched by its Cauchy data on the spacelike hypersurface $\Sigma'_1 := \{t = 1\}$. We would like to answer the following question: what happens if we perturb the Cauchy data of the FLRW solution? In particular, does the perturbed solution also collapse into a Big Bang singularity at some past time? The famous singularity theorems of Penrose and Hawking (see e.g. [Pen65], [Haw67]) ensure that under certain assumptions verified by perturbed data, the resulting perturbed spacetimes are necessarily past-timelike incomplete. However, these theorems don't provide any information about the nature of the incompleteness. There are at least two possible scenarios behind the incompleteness, and they are quite distinct. The first scenario is that the incompleteness is caused by the blow-up of some curvature invariant, as is the case at the FLRW solution's Big Bang singularity. Another possibility is that the incompleteness is caused by the development of a Cauchy horizon, as is the case in e.g. the Taub family of solutions. Our main results confirm that for solutions launched by near-FLRW data, the first scenario always occurs.

Before stating our main results in more detail, we first describe some important prior work. In [AR01], Andersson and Rendall constructed a family of spatially analytic solutions to the Einstein-scalar field and the Einstein-stiff fluid systems. The family has the same number of degrees of freedom as do the Einstein data, and each solution has a Big Bang singularity. A neighborhood of the singularity is foliated by a Gaussian coordinate system that synchronizes the singularity at $\{t = 0\}$, and near the singularity, the spatial derivatives become negligible. We remark that previous heuristic arguments had been given suggesting that a scalar field or a stiff fluid could have such a mollifying effect on solutions near the singularity (see e.g. [BK72], [Bar78]). The solutions in [AR01] were constructed by first solving a simpler system known as the velocity term dominated (VTD) system. The VTD system is obtained by throwing away the spatial derivatives from the original equations. The VTD solutions can be viewed as a family of ODE solutions parameterized by the spatial coordinates. Andersson and Rendall then used Fuchsian PDE techniques to construct a solution to the original equations that converges to the VTD solution. This strategy is sometimes known as the "backward" approach because it requires that one prescribe the asymptotics near the singularity *before* constructing the solution. In the conclusions of [AR01], Andersson and Rendall stated that it would be desirable to prove an analogous result from the point of view of the initial value problem. More precisely, they stated that it would be desirable to show that an open set of near-FLRW Cauchy data (in particular without the assumption of spatial analyticity) launches a solution with a Big Bang singularity. Our main theorem confirms that such an open set exists. This strategy is known as the "forward" approach (even though our choice of time coordinate in our main theorem is such that we solve the Cauchy problem "backward" in time). We also note that in [Rin01], Ringström proved (among many other things) an analogous version of our main results under the assumption that the solution is a member of the symmetry class Bianchi class A. These solutions depend only on a time coordinate t. We remark that unlike in our main results, Ringström did not assume that the data are near-FLRW.

We now state our main results. A full proof will be given in an upcoming article.

Main Results. The spatially homogeneous, isotropic FLRW spacetime (5) is a past-globally stable singular solution to (1a) - (4). More precisely, if initial data for the system are given on the spacelike hypersurface Σ'_1 and the data are near-FLRW as measured by a Sobolev norm, then the maximal globally hyperbolic development of the perturbed data contains a hypersurface Σ_1 of constant mean curvature (CMC) $-1/3$ that is near Σ'_1. Furthermore, the past of Σ_1 is foliated by a family of spacelike hypersurfaces Σ_t corresponding to a time function $t \in (0,1]$ upon which the CMC condition $k^a{}_a(t,x) = -t^{-1}$ holds. Here, $k^i{}_j$ is the (mixed) second fundamental form of Σ_t. Relative to the time coordinate t, the perturbed solution exists on the manifold-with-boundary $(0,1] \times \mathbb{T}^3$ and remains close in a Sobolev sense to the FLRW solution.

Furthermore, certain time-rescaled versions of the volume form of Σ_t, the mixed second fundamental form, and the pressure have a non-zero, finite limit as $t \downarrow 0$. The limiting field components are functions on $\mathbb{T}^3$ that are close to the corresponding time-rescaled FLRW field components, which are globally constant. The coordinate frame relative to which these estimates hold is such the FLRW solution has the form (5) in this frame.

In addition, the spacetime $\big((0,1] \times \mathbb{T}^3, \mathbf{g}\big)$ is past-timelike geodesically incomplete and inextendible beyond $t = 0$. As $t \downarrow 0$, the 3−volume of the constant-time hypersurfaces Σ_t collapses to 0, the pressure p blows-up like t^{-2}, and the spacetime Kretschmann scalar $|\mathbf{Riem}|^2_{\mathbf{g}}$ blows-up like t^{-4}. Furthermore, the ratio $\left|\frac{|\mathbf{W}|^2_{\mathbf{g}}}{|\mathbf{Riem}|^2_{\mathbf{g}}}\right|$ remains small for all $t \in (0,1]$, where $\mathbf{W}$ denotes the spacetime Weyl curvature tensor. This shows that the curvature singularity is dominated by the $\mathbf{Ric}$ components of $\mathbf{Riem}$.

The main strategy of the proof is to study the Cauchy problem relative to a CMC-transported spatial coordinate system. In order for this approach to be viable, we must show that near-FLRW spacetimes contain a constant mean curvature hypersurface. This is especially important in view of the fact that Bartnik showed [Bar88] that some spacetimes do not contain any such hypersurfaces. Our argument for their existence in near-FLRW spacetimes is based on a result of Bartnik [Bar84], which is itself an extension/modification of earlier work by Gerhardt [Ger83].

In CMC-transported spatial coordinates (t,x), where $t \in \mathbb{R}$ and $x \in \mathbb{T}^3$, the spacetime metric can be decomposed as

$$\mathbf{g} = -n^2 dt^2 + g_{ab} dx^a dx^b, \tag{6}$$

where n is the lapse. The lapse verifies an elliptic PDE that enforces the CMC condition, and the Einstein-stiff fluid equations become a mixed elliptic-hyperbolic system. Our analysis of solutions is based on a combination of L^2 energy estimates, elliptic estimates, and sup norm estimates. We identified a hierarchy of special structures that are visible in the aforementioned coordinates. One particularly important ingredient is our identification of a new energy almost-monotonicity inequality for the solutions under consideration. Roughly speaking, our derivation of this inequality is based on using the lapse equation, the Hamiltonian constraint, and the CMC condition in order to show that a certain quadratic spatial integral has a favorable sign. In fact, the integral provides dynamic control over the lapse and plays a key role in closing our stability proof. The integral arises during our derivation of energy estimates for the fluid variables. Another important structure is that, roughly speaking, the lower order derivatives of the solution can be treated as if they were ODEs with small sources. This allows us to derive sup norm estimates for the lower order derivatives that are much stronger than the the bounds implied by our energy estimates. These strong sup norm estimates are essential for closing our main bootstrap argument. The assumption that the matter is a stiff fluid is heavily used in the derivation of the sup norm estimates.

References

[AR01] Lars Andersson and Alan D. Rendall, *Quiescent cosmological singularities*, Comm. Math. Phys. **218** (2001), no. 3, 479–511.

[Bar78] John D. Barrow, *Quiescent cosmology*, Nature **272** (1978), 211–215.

[Bar84] Robert Bartnik, *Existence of maximal surfaces in asymptotically flat spacetimes*, Comm. Math. Phys. **94** (1984), no. 2, 155–175.

[Bar88] ———, *Remarks on cosmological spacetimes and constant mean curvature surfaces*, Comm. Math. Phys. **117** (1988), no. 4, 615–624.

[BK72] V. A. Belinskiĭ and I. M. Khalatnikov, *Effect of scalar and vector fields on the nature of the cosmological singularity*, Ž. Èksper. Teoret. Fiz. **63** (1972), 1121–1134.

[Ger83] Claus Gerhardt, *H-surfaces in Lorentzian manifolds*, Comm. Math. Phys. **89** (1983), no. 4, 523–553.

[Haw67] S. W. Hawking, *The occurrence of singularities in cosmology. iii. causality and singularities*, Proceedings of the Royal Society of London. Series A, Mathematical and Physical Sciences **300** (1967), no. 1461, 187–201 (English).

[Pen65] Roger Penrose, *Gravitational collapse and space-time singularities*, Phys. Rev. Lett. **14** (1965), 57–59.

[Rin01] Hans Ringström, *The Bianchi IX attractor*, Ann. Henri Poincaré **2** (2001), no. 3, 405–500.

Weakly regular spacetimes with symmetry

Philippe G. LeFloch

In the past five years, the author established new existence results concerning the initial value problem for Einstein's field equations, when *weak regularity,* only, is assumed on the initial data set and, therefore, on the spacetime itself. In this theory, the essential metric coefficients belong to the Sobolev space H^1, while other metric coefficients are solely in the Lebesgue space L^∞ or even in L^1, while the spacetime curvature is understood *in the sense of distributions* [5]. Under the assumption of T^2 symmetry, Gowdy symmetry, or plane symmetry, several techniques were developed in order to construct future Cauchy developments with weak regularity and to analyze their global geometry.

In particular, motivated by the work of Christodoulou [1] on the formation of trapped surfaces along the evolution determined by the Einstein equations, the author and Stewart [9] studied the global causal geometry of plane-symmetric colliding spacetimes with (possibly impulsive) interacting gravitational waves, and established the strong version of Penrose conjecture *in the weak regularity class.*

When T^2 symmetry is assumed, the Einstein vacuum system can be expressed in the so-calles areal coordinates When the time function is chosen to coincide with the area R of the surfaces of symmetry, the metric takes the form

$$g = e^{2(\eta-U)}\big(- dR^2 + a^{-2}\, d\theta^2\big) \quad + e^{2U}\big(dx + A\, dy + \big(G + AH\big)\, d\theta\big)^2 \\ + e^{-2U} R^2\big(dy + H\, d\theta\big)^2,$$

in which the coefficients U, A, η, a, G, H are functions of the time variable R and the spatial variable θ. Here, R belongs to a real interval, while x, y, θ belong to the one-dimensional torus S^1. The weak version of the field equations can be defined fully geometrically by taking into account certain *cancellation properties* and specifically, in areal coordinates, reads as follows in the sense of distributions:

(1) Four evolution equations for the metric coefficients U, A, η, a:

$$\begin{aligned} &(R\, a^{-1}U_R)_R - (R\, a\, U_\theta)_\theta = 2R\, \Omega^U, \\ &(R^{-1}\, a^{-1}A_R)_R - (R^{-1}\, a\, A_\theta)_\theta = e^{-2U}\Omega^A, \\ &(a^{-1}\eta_R)_R - (a\, \eta_\theta)_\theta = \Omega^\eta - R^{-3/2}\big(R^{3/2}\big(a^{-1}\big)_R\big)_R, \\ &(2\ln a)_R = -R^{-3}K^2\, e^{2\eta}, \end{aligned}$$

where $K \geq 0$ is a constant and the right-hand sides are defined by

$$\begin{aligned} \Omega^U &:= (2R)^{-2}e^{4U}\, \big(a^{-1}\, A_R^2 - a\, A_\theta^2\big), \\ \Omega^A &:= 4R^{-1}e^{2U}\, \big(- a^{-1}\, U_R A_R + a\, U_\theta A_\theta\big), \\ \Omega^\eta &:= (-a^{-1}U_R^2 + a\, U_\theta^2) + (2R)^{-2}e^{4U}(a^{-1}A_R^2 - a\, A_\theta^2). \end{aligned}$$

(2) Two constraint equations for the metric coefficient η:

$$\eta_R + \frac{1}{4}\, R^{-3}\, e^{2\eta}K^2 = a\, RE, \qquad \eta_\theta = R\, F,$$

where

$$\begin{aligned} E &:= \big(a^{-1}\, U_R^2 + a\, U_\theta^2\big) + (2R)^{-2}e^{4U}\big(a^{-1}\, A_R^2 + a\, A_\theta^2\big), \\ F &:= 2U_R U_\theta + 2R^{-2}e^{2U}A_R A_\theta. \end{aligned}$$

(3) Four auxiliary equations for the twists:

$$\begin{aligned} &\big(R\, e^{4U-2\eta}a\, (G_R + AH_R)\big)_\theta = 0, \qquad &&\big(R^3\, e^{-2\eta}a\, H_R\big)_\theta = 0, \\ &\big(R\, e^{4U-2\eta}\, a\, (G_R + AH_R)\big)_R = 0, \qquad &&\big(R^3\, e^{-2\eta}\, aH_R\big)_R = 0. \end{aligned}$$

(4) Two equations for the metric coefficients G, H:

$$G_R = -AKe^{2\eta}\, a^{-1}R^{-3}, \qquad H_R = Ke^{2\eta}\, a^{-1}R^{-3}.$$

In [8], the author and Smulevici propose a fully geometric, weak formulation of the Einstein equations for T^2 symmetric spacetimes, and solve the initial value problem for the above system when the essential metric coefficients U and A belong to H^1 in the variable $\theta \in S^1$, while the other metric coefficients have even weaker regularity.

In [6], the author and Rendall investigate the coupling of the Einstein equations and the Euler equations of compressible fluids, when weak regularity only is assumed, and contruct a global areal foliation of Einstein-Euler spacetimes with

Gowdy symmetry. We also refer to the papers [3] and [4] for a review of these techniques and to [2] for further results.

References

[1] D. Christodoulou, *The formation of black holes in general relativity,* European Math. Soc., 2009.

[2] N. Grubic and P.G. LeFloch, Weakly regular Einstein–Euler spacetimes with Gowdy symmetry. The global areal foliation, Arch. Rational Mech. Anal. (2013).

[3] P.G. LeFloch, Einstein spacetimes with weak regularity, in "Advances in Lorentzian Geometry", Ed. M. Plaue, A.D. Rendall, and M. Scherfner, AMS/IP series, Vol. **49,** 2011, pp. 81–96.

[4] P.G. LeFloch, Hyperbolic conservation laws on spacetimes, in "Nonlinear conservation laws and applications", IMA Vol. Math. Appl. **153**, Springer, New York, 2011, pp. 379–391.

[5] P.G. LeFloch and C. Mardare, Definition and weak stability of spacetimes with distributional curvature, Portugal Math. **64** (2007), 535–573.

[6] P.G. LeFloch and A.D. Rendall, A global foliation of Einstein-Euler spacetimes with Gowdy-symmetry on T^3, Arch. Rational Mech. Anal. **201** (2011), 841–870.

[7] P.G. LeFloch and J. Smulevici, Global geometry of T^2–symmetric spacetimes with weak regularity, Note C.R. Acad. Sc. **348** (2010), 1231–1233.

[8] P.G. LeFloch and J. Smulevici, Weakly regular T^2–symmetric spacetimes. The global geometry of future developments, Preprint ArXiv 1012.5931 (2010).

[9] P.G. LeFloch and J.M. Stewart, The characteristic initial value problem for plane–symmetric spacetimes with weak regularity, Class. Quantum Grav. **28** (2011), 145019–145035.

Spacetime positive mass theorem in dimensions less than eight

Lan-Hsuan Huang

(joint work with M. Eichmair, D. Lee, and R. Schoen)

We prove the spacetime positive mass theorem in dimensions less than eight. This theorem states that for any asymptotically flat initial data set satisfying the dominant energy condition, the inequality $E \geq |P|$ holds, where (E, P) is the ADM energy-momentum vector of the initial data set. Previously, this theorem was proved only for spin manifolds [4]. Our proof is a modification of the minimal hypersurface technique that was used by the last author and S.-T. Yau to establish the time-symmetric case [3, 2]. Instead of minimal hypersurfaces, we use marginally outer trapped hypersurfaces (MOTS) whose existence is guaranteed by earlier work of the first author [1]. Since MOTS do not arise from a variational principle, an important part of our proof to introduce an appropriate substitute for the area functional used in the time-symmetric case. As part of our proof, we establish a density theorem of independent interest that allows us to reduce the general case of our theorem to the case of initial data that has harmonic asymptotics and satisfies the strict dominant energy condition. A refinement of the density argument allows us to approximate any given data set by one which is identical with a Kerr slice outside a compact set preserving the dominant energy condition. This enables us to give an alternative proof of the main theorem by reducing it to the positive energy theorem.

References

[1] Michael Eichmair, *The Plateau problem for marginally outer trapped surfaces*, J. Differential Geom. 83 (2009), no. 3, 551–583. MR 2581357

[2] Richard Schoen, *Variational theory for the total scalar curvature functional for Riemannian metrics and related topics, Topics in calculus of variations* (Montecatini Terme, 1987), Lecture Notes in Math., vol. 1365, Springer, Berlin, 1989, pp. 120-154. MR 994021 (90g:58023)

[3] Richard Schoen and Shing Tung Yau, *On the proof of the positive mass conjecture in general relativity*, Comm. Math. Phys. 65 (1979), no. 1, 45-76. MR 526976 (80j:83024)

[4] Edward Witten, *A new proof of the positive energy theorem*, Comm. Math. Phys. 80 (1981), no. 3, 381-402. MR 626707 (83e:83035)

A conformal approach for the analysis of the non-linear stability of radiation cosmologies

Juan Antonio Valiente Kroon

(joint work with Christian Lübbe)

The conformal Einstein field equations have proven to be a powerful tool to analyze the stability and the global properties of vacuum, electro-vacuum and Yang-Mills-electro-vacuum spacetimes —see e.g. [4, 5, 6, 9, 10, 11]. By contrast, to the best of our knowledge, there has been no attempt to make use of conformal methods to analyze similar issues in spacetimes whose matter content is given by a perfect fluid. The work reported in this talk constitutes a first step in this direction. We discuss the stability and the global properties of a class of cosmological spacetimes having as a source a perfect fluid with trace-free energy-momentum tensor. The solutions we construct are non-linear perturbations of a Friedman-Lemaître-Robertson-Walker (FLRW) reference spacetime. We have restricted our attention to the simplest case from the point of view of conformal methods: *perturbations of a trace-free prefect fluid cosmological model with compact spatial sections of positive constant curvature.*

The problem of the non-linear stability of the irrotational Euler-Einstein system for de Sitter-like spacetimes has been analyzed in [12]. This analysis shows that FLRW background solutions with pressure $\tilde{p}$ and density $\tilde{\rho}$ related by a barotropic equations of state of the form $\tilde{p} = (\gamma - 1)\tilde{\rho}$ with $1 < \gamma < \frac{4}{3}$ are future asymptotically stable under small irrotational perturbations. An extension of this analysis to the case of fluids with non-zero vorticity has been given in [13]. It is notable that the case of a pure radiation perfect fluid cannot be covered by the analysis of [12, 13]. By contrast, from the point of view of conformal methods, the pure radiation perfect fluid case turns out to be one of the simplest scenarios to be considered. It should be mentioned that conformal methods have been used to pose an initial value problem for the Einstein-Euler system at the Big Bang for a class of cosmological models with isotropic singularities —see [1]. The methods used in that work do not allow, however, to obtain global existence assertions towards the future. Our main result can be stated as follows:

Theorem. *Suppose one is given Cauchy initial data for the Einstein-Euler system with a de Sitter-like cosmological constant and equation of state for pure radiation. If the initial data is sufficiently close to data for a FLRW cosmological model with the same equation of state, value of the cosmological constant and spatial curvature $k = 1$, then the development exists globally towards the future, is future geodesically complete and remains close to the FLRW solution.*

Similar future global existence and stability results can be obtained using the methods of this article for a FLRW background solution with pure radiation equation of state, de Sitter-like or vanishing cosmological constant, λ, and $k = 0, -1$. These models expand indefinitely towards the future, and remarkably, their scale factor can be computed explicitly —see [2]. In the cases with $\lambda = 0$, minor technical modifications need to be introduced to account for a null conformal boundary. The stability of these models will be discussed elsewhere by means of different (conformal) methods.

The restriction of our analysis to the case of perfect fluids with trace-free energy-momentum tensor is a technical one. In the trace-free case the equation $\tilde{\nabla}^\mu \tilde{T}_{\mu\nu} = 0$ transforms homogeneously under conformal rescalings. As a result, the majority of conformal field equations for a radiation fluid can be treated with the same methods used in the analysis of the Einstein-Maxwell case [5, 11]; only the equations for the fluid variables need to be analysed in more detail, to ensure that they fit in correctly with the remainder of the system of partial differential equations (PDEs). In the case of an energy-momentum tensor with non-vanishing trace, it is an open question whether a suitable choice of conformal variables exist which lead to a regular set of conformal field equations with matter.

In this article, we follow the conformal approach developed by H. Friedrich in a series of articles [4, 5, 6, 7]. The key idea of our analysis is that by conformally rescaling a solution $(\tilde{\mathcal{M}}, \tilde{g}_{\mu\nu})$ to the Einstein field equations with matter, one is able to carry out a global stability analysis to the future in terms of a local analysis near the conformal boundary of the conformally extension $(\mathcal{M}, g_{\mu\nu})$ where $g_{\mu\nu} = \Theta^2 \tilde{g}_{\mu\nu}$ and Θ is a suitable conformal factor. The unphysical manifold $(\mathcal{M}, g_{\mu\nu})$ does not satisfy the Einstein field equations with matter. Instead, the conformally rescaled metric $g_{\mu\nu}$ and derived fields satisfy the conformal Einstein field equations with trace-free matter. In these equations the matter content, which is not specified at this stage, only appears in the form of source terms. The system needs to be supplemented with suitable evolution equations for the matter variables once a specific model has been chosen.

The system of equations given by the conformal field equations has more variables than, say, the so-called ADM evolution equations This feature is necessary in order to obtain a closed system of first order symmetric hyperbolic PDEs which is regular at points for which the conformal factor Θ vanishes. In particular, note that the principal parts do not contain the conformal factor Θ and no negative powers of Θ appear in the lower order terms of the equations. This is achieved by the use of suitable variables which absorb formally singular terms involving the conformal conformal factor. As already mentioned, in this work we have analyzed trace-free perfect fluids. The matter variables are the rescaled density, the spatial components of the conformal fluid velocity and their first derivatives.

From the conformally rescaled fluid equations it is possible to deduce a symmetric hyperbolic system. In view of the presence in the geometric subsystem of terms involving derivatives of the matter fields, it is necessary to extend the matter subsystem by considering further unknowns representing these derivatives. These new fields can be shown to satisfy suitable hyperbolic equations. Following the approach of [3, 4], in order to obtain a closed system of evolution equations one has to supplement the geometric part of the conformal field equations with appropriate *gauge source functions* As a result of the reduction procedure one obtains a symmetric hyperbolic evolution system.

The reduction procedure leading to the evolution system discards certain components of the geometric subsystem. These equations are, in turn, regarded as constraints which need to be shown to hold at later times if their are satisfied at some initial hypersurface —this is the so-called *propagation of the constraints*. The propagation of constraints is shown by studying the evolution of the zero quantities. The equations satisfied by the geometric zero quantities is obtained following a procedure described in [5]. For the matter subsystem there are no constraints for which propagation needs to be shown. Instead, it is necessary to show that the fluid evolution equations imply the original equations from which they are derived.

Given the overall evolution system for the geometric and matter fields, existence, uniqueness and stability can be asserted using results for symmetric hyperbolic systems which are local in time like the ones presented in [8]. The stability analysis requires the knowledge of a reference spacetime. Hence, the reference FLRW spacetime under consideration needs to be casted in the form of a solution to the conformal field equations.

References

[1] K. Anguige & K. P. Tod, *Isotropic Cosmological Singularities I. Polytropic Perfect Fluid Spacetimes*, Ann. Phys. **276**, 257 (1999).

[2] J. B. Griffiths & J. Podolský, *Exact space-times in Einstein's General Relativity*, Cambridge University Press, 2009.

[3] H. Friedrich, *On the hyperbolicity of Einstein's and other gauge field equations*, Comm. Math. Phys. **100**, 525 (1985).

[4] H. Friedrich, *On the existence of n-geodesically complete or future complete solutions of Einstein's field equations with smooth asymptotic structure*, Comm. Math. Phys. **107**, 587 (1986).

[5] H. Friedrich, *On the global existence and the asymptotic behaviour of solutions to the Einstein-Maxwell-Yang-Mills equations*, J. Diff. geom. **34**, 275 (1991).

[6] H. Friedrich, *Einstein equations and conformal structure: existence of anti-de Sitter-type space-times*, J. Geom. Phys. **17**, 125 (1995).

[7] H. Friedrich, *Conformal Einstein evolution*, in *The conformal structure of spacetime: Geometry, Analysis, Numerics*, edited by J. Frauendiener & H. Friedrich, Lecture Notes in Physics, page 1, Springer, 2002.

[8] T. Kato, *The Cauchy problem for quasi-linear symmetric hyperbolic systems*, Arch. Ration. Mech. Anal. **58**, 181 (1975).

[9] C. Lübbe & J. A. Valiente Kroon, *On de Sitter-like and Minkowski-like spacetimes*, Class. Quantum Grav. **26**, 145012 (2009).
[10] C. Lübbe & J. A. Valiente Kroon, *A stability result for purely radiative spacetimes*, J. Hyp. Diff. Eqns. **7**, 545 (2010).
[11] C. Lübbe & J. A. Valiente Kroon, *The extended Conformal Einstein field equations with matter: the Einstein-Maxwell system*, J. Geom. Phys. **62**, 1548 (2012).
[12] I. Rodnianski & J. Speck, *The Stability of the Irrotational Euler-Einstein System with a Positive Cosmological Constant*, In `arXiv:0911.5501`, 2009.
[13] J. Speck, *The Nonlinear Future-Stability of the FLRW Family of Solutions to the Euler-Einstein System with a Positive Cosmological Constant*, In `arXiv:1102.1501`, 2011.

Intrinsic flat convergence as a gauge invariant means of defining weak convergence of manifolds

Christina Sormani

In a variety of questions arising in general relativity, one needs a weak notion of convergence of manifolds in order to understand how two different models are close to one another. Recall that Schoen-Yau [11] proved that an asymptotically flat Riemannian manifold with nonnegative scalar curvature whose ADM mass is 0 must be Euclidean space. If the ADM mass is close to 0, one may ask whether the manifold close to Euclidean space. Ishibashi-Wald and Green-Wald ask how close an inhomogeneous model of the universe is to the homogeneous description used in the Friedmann model of cosmology [6] [4]. In these and in many other settings one can see that the perturbations are not smooth and that one can have thin deep gravity wells. Here we propose that the intrinsic flat convergence of Riemannian manifolds provides a natural weak notion of convergence that can be applied to provide an appropriate measure of closeness. It is possible that combining this notion with the cosmological time function of Andersson-Galloway-Howard [2], one can define a weak notion of convergence for those Lorentzian manifolds which have such time functions.

The intrinsic flat distance between compact oriented Riemannian manifolds with boundaries was defined by the author and Stefan Wenger in [15] building upon work of Ambrosio-Kirchheim [1] and Gromov [5]. It extends Federer-Fleming's notion of a flat distance between submanifolds of Euclidean space that was applied to solve the Plateau Problem [3]. Like the classical flat distance, a sequence of manifolds $M_j \xrightarrow{\mathcal{F}} M_\infty$, can have increasingly thin wells which disappear in the limit. Thus examples, like Ilmanen's spheres of increasingly many splines, have intrinsic flat limits even when no subsequence converges in the Gromov-Hausdorff or smooth sense.

Like the Gromov-Hausdorff distance, the intrinsic flat distance is gauge invariant (i.e. it does not depend on the coordinate charts) [15]. In fact, the intrinsic flat distance between two Riemannian manifolds is 0, $d_{\mathcal{F}}(M_1, M_2) = 0$, iff there is an orientation preserving isometry between them [15].

A sequence of pointed oriented complete noncompact Riemannian manifolds, (M_j, p_j), are said to converge to (M_∞, p_∞) in the pointed intrinsic flat sense if for any $R > 0$, the balls converge: $d_{\mathcal{F}}(B_{p_j}(R), B_{p_\infty}(R)) \to 0$. By Wenger's Compactness Theorem, providing a uniform upper bound on the volumes of the balls and the areas of their boundaries guarantees the existence of a subsequence which converges in this sense [16]. In general, the limit space is no longer a Riemannian manifold. It is an oriented weighted countably $\mathcal{H}^m$ rectifiable metric space called an integral current space. Such spaces have biLipschitz coordinate charts covering almost every point in the space [15]. See also the article of Urs Lang and Stefan Wenger [8].

In joint work with Dan Lee [9][10], the intrinsic flat convergence has been applied to study manifolds with small ADM mass and manifolds which almost achieve the Poincare Inequality. We prove the rotationally symmetric case and propose more general conjectures. These papers include methods for estimating the intrinsic flat distances between Riemannian manifolds which can be embedded into a higher dimensional Euclidean space. The rotational symmetry is used to provide that embedding. The proofs are constructive.

In joint work with Sajjad Lakzian [7], new methods for measuring the intrinsic flat distance between manifolds which have diffeomorphic subregions are provided. This allows us to understand when a sequence of manifolds which converges smoothly away from a singular set also converges in the intrinsic flat sense. Essentially one must control the volumes of the bad regions and the areas of the boundaries of the bad regions and ensure that the good regions are close in a Lipschitz sense. Explicit bounds on the intrinsic flat distance are given. Sajjad Lakzian is conducting further work in this direction in his doctoral dissertation.

In [13], the author studies the stability of the Friedmann model of cosmology using the Gromov-Hausdorff distance. That is a Riemannian manifold which is almost locally isotropic, allowing for both weak gravitational lensing and strong gravitational lensing, is GH-close to a homogeneous space as long as one has the uniform bounds needed to apply Gromov's Compactness Theorem. Applying Wenger's Compactness Theorem instead, the author is working to prove such an inhomogeneous model is close in the intrinsic flat sense to a homogeneous space. One must show that almost isometries in the sequence converge to isometries of the limit space.

One of the advantages of intrinsic flat convergence is that certain regions in spaces will disappear in the limit. This may happen due to collapse, where the volume of the region approaches 0, or due to cancellation, where two regions come close together with opposite orientation. In joint work with Stefan Wenger [14], it is shown that a sequence of compact oriented manifolds, M_j^3, created by connecting two standard spheres with increasingly tiny increasingly dense Gromov-Lawson tunnels disappears completely in the limit: $d_{\mathcal{F}}(M_j, 0) \to 0$. We conjecture that sequences of manifolds with nonnegative scalar curvature and no closed interior minimal surfaces (other than their boundaries) do not disappear in the limit unless their volumes converge to 0. We prove that assuming $Ricci \geq 0$ or a uniform contractibility of the spaces, one avoids such cancellation.

In [12], the author examines which points in a sequence of manifolds disappear and which converge under intrinsic flat convergence. If $p_j \to p_\infty$, then $\liminf_{j\to\infty} Vol(B_{p_j}(r)) \geq \mu(B_{p_\infty}(r))$ where μ is the mass measure of the limit space. In contrast, the filling volumes of spheres are continuous,

$$\lim_{j\to\infty} FillVol(\partial B_{p_j}(r)) = FillVol(\partial B_{p_\infty}(r)).$$

A sequence of points can be prevented from disappearing in the limit if the filling volume or the sliced filling volume of the ball around it can be bounded from below by Cr^m. If the balls have a uniform tetrahedral property or integral tetrahedral property, their centers will not disappear [12].

The author is then able to prove an Arzela-Ascoli theorem for sequences of functions, $f_j : M_j \to \mathbb{R}$, when $M_j \xrightarrow{\mathcal{F}} M_\infty$. She proves level sets of certain Lipschitz distance functions, $f_j^{-1}(t) \subset M_j$, converge in the intrinsic flat sense to level sets of their limits, $f_\infty^{-1}(t) \subset M_\infty$, for almost every $t \in \mathbb{R}$ [12]. In ongoing discussions with Lars Andersson and Ralph Howard and their students, the author is exploring the possibility that one may use this property of convergence of level sets applied to the cosmological time function of [2] to define an intrinsic flat convergence of Lorentzian manifolds which have such time functions.

References

[1] Luigi Ambrosio and Bernd Kirchheim. Currents in metric spaces. *Acta Math.*, 185(1):1–80, 2000.

[2] Lars Andersson, Gregory J. Galloway, and Ralph Howard. The cosmological time function. *Classical Quantum Gravity*, 15(2):309–322, 1998.

[3] Herbert Federer and Wendell H. Fleming. Normal and integral currents. *Ann. of Math. (2)*, 72:458–520, 1960.

[4] Stephen R. Green and Robert M. Wald. New framework for analyzing the effects of small scale inhomogeneities in cosmology. *Phys. Rev. D*, 83:084020, Apr 2011.

[5] Misha Gromov. *Metric structures for Riemannian and non-Riemannian spaces*, volume 152 of *Progress in Mathematics*. Birkhäuser Boston Inc., Boston, MA, 1999.

[6] Akihiro Ishibashi and Robert M. Wald. Can the acceleration of our universe be explained by the effects of inhomogeneities? *Classical Quantum Gravity*, 23(1):235–250, 2006.

[7] Sajjad Lakzian and Christina Sormani. Smooth convergence away from singular sets. *arxiv preprint*, 2012.

[8] Urs Lang and Stefan Wenger. The pointed flat compactness theorem for locally integral currents. *Comm. Anal. Geom.*, 19(1):159–189, 2011.

[9] Dan A. Lee and Christina Sormani. Stability of the positive mass theorem for rotationally symmetric riemannian manifolds. *preprint on arxiv*, 2011.

[10] Dan A. Lee and Christina Sormani. Almost inequality in the Penrose equality for rotationally symmetric manifolds. *Annales Henri Poincare*, to appear.

[11] Richard Schoen and Shing Tung Yau. On the proof of the positive mass conjecture in general relativity. *Comm. Math. Phys.*, 65(1):45–76, 1979.

[12] Christina Sormani. Properties of the intrinsic flat distance. *http://comet.lehman.cuny.edu/sormani.*

[13] Christina Sormani. Friedmann cosmology and almost isotropy. *Geom. Funct. Anal.*, 14(4):853–912, 2004.

[14] Christina Sormani and Stefan Wenger. Weak convergence and cancellation, appendix by Raanan Schul and Stefan Wenger. *Calculus of Variations and Partial Differential Equations*, 38(1-2), 2010.
[15] Christina Sormani and Stefan Wenger. Intrinsic flat convergence of manifolds and other integral current spaces. *Journal of Differential Geometry*, 87, 2011.
[16] Stefan Wenger. Compactness for manifolds and integral currents with bounded diameter and volume. *Calc. Var. Partial Differential Equations*, 40(3-4):423–448, 2011.

Instability of AdS – one year later

Piotr Bizoń
(joint work with Andrzej Rostworowski, Joanna Jałmużna, Maciej Maliborski)

Anti-de Sitter (AdS) space is the maximally symmetric solution of the vacuum Einstein equations $R_{\alpha\beta} - \frac{1}{2}g_{\alpha\beta}R + \Lambda g_{\alpha\beta} = 0$ with negative cosmological constant Λ. In $d+1$ dimensions the AdS metric takes the form

$$ds^2 = \frac{\ell^2}{\cos^2 x}\left(-dt^2 + dx^2 + \sin^2 x\, d\Omega^2_{S^{d-1}}\right), \tag{1}$$

where $\ell^2 = -d(d-1)/2\Lambda$, $0 \le x < \pi/2$, $-\infty < t < \infty$, and $d\Omega^2_{S^{d-1}}$ is the round metric on S^{d-1}. My talk was concerned with global-in-time evolution of small perturbations of AdS space. In the first part, I discussed our recent work [1, 2] on spherically symmetric massless scalar field minimally coupled to gravity. In this model, we gave numerical evidence that, for $d \ge 3$, arbitrarily small generic perturbations of AdS evolve into a black hole after a time of order $\mathcal{O}(\varepsilon^{-2})$, where ε measures the size of a perturbation. On the basis of nonlinear stability analysis, we conjectured that the instability is due to the resonant mode mixing that transfers energy from low to high frequencies, or equivalently, from coarse to fine spatial scales, until eventually an apparent horizon forms. This mechanism is reminiscent of the turbulent energy cascade in fluids and may have implications in studies of the gauge/gravity duality. To emphasize this relationship, I showed a preliminary result indicating that the energy spectrum develops a universal power-law scaling at late times.

In the second part, some work in progress generalizing the above result was presented. First, the critical case $d = 2$ was discussed [3]. In this case, small perturbations of AdS cannot lead to collapse because there is a mass gap between AdS and the lightest black hole. Nonetheless, the turbulent instability is present and is conjectured to lead to equipartition of energy for $t \to \infty$. No tendency for naked singularity formation was observed which suggests that weak cosmic censorship holds in this model.

Second, partly motivated by the AdS/CFT correspondence, we considered the Bianchi IX cohomogeneity-two vacuum model in $4+1$ dimensions with negative cosmological constant [4]. The results are qualitatively similar to the scalar field case which indicates that the mechanism of turbulent instability of AdS is model-independent.

Finally, I mentioned very recent result of Maliborski on instability of Minkowski space enclosed in a cavity [5]. His result demonstrates that the turbulent behavior is not an exclusive domain of asymptotically AdS spacetimes but probably a typical feature of solutions of Einstein's equations with compact (or effectively compact) spatial slices.

References

[1] P. Bizoń, A. Rostworowski, *Weakly Turbulent Instability of Anti–de Sitter Spacetime*, Phys. Rev. Lett. **107** (2011), 031102.

[2] J. Jałmużna, A. Rostworowski, P. Bizoń, *AdS collapse of a scalar field in higher dimensions*, Phys. Rev. D **84** (2011), 085021.

[3] J. Jałmużna, P. Bizoń, *Turbulent instability of* AdS_3, in preparation

[4] P. Bizoń, R. Janik, A. Rostworowski, *Instability of* AdS_5 *under gravitational perturbtions*, in preparation

[5] M. Maliborski, *Instability of Minkowski spacetime enclosed in a cavity*, arXiv:1208.2934

Rough Solution of Einstein vacuum equation in CMCSH gauge

Qian Wang

We consider very rough solutions to Cauchy problem for the Einstein vacuum equations (**EV**) in CMC spacial harmonic gauge, and obtain the local well-posedness result in $H^s, s > 2$. The novelty of our approach lies in that, without resorting to the standard paradifferential regularization over the rough, Einstein metric $\mathbf{g}$, we manage to implement the vector field approach to prove Strichartz estimate for geometric wave equation $\Box_{\mathbf{g}}\phi = 0$ directly.

Relative to the wave coordinates, Einstein vacuum equation takes the form of a strictly hyperbolic, quasilinear equations,

$$\mathbf{g}^{\alpha\beta}\partial_\alpha\partial_\beta\phi = N(\phi, \partial\phi) \tag{1}$$

where $\phi = (\mathbf{g}_{\mu\nu})$ and the function $N_{\mu\nu}(\phi, \partial\phi)$ is smooth in its arguments and is quadratic in $\partial\phi$. The study of local well-posedness of Einstein equation was pioneered by Y.C. Bruhat [2] who proved a local in time existence result under the assumption that $\|(g, k)\|_{H^s\times H^{s-1}}$ with $s \geq 4$. By classic energy method and Sobolev embedding, it was improved in [4] to $s > 5/2$. The most significant progresses toward the optimal regularity of initial data were the improvement to $s > 2$ for Einstein vacuum equation under wave coordinates, which was achieved in [7]-[9], and the work [11] for the general class of quasi-linear wave equation.

We consider vacuum Einstein equation under constant mean curvature and spatial harmonic coordinate (CMCSH) gauge, which was raised by Andersson and Moncrief in [1] to produce solutions in a more dynamic way.

Let $\hat{g}$ be a fixed smooth Riemannian metric on Σ with Levi-Civita connection $\widehat{\nabla}$ and Christoffel symbol $\hat{\Gamma}^k_{ij}$. The solution of (**EV**) constructed in [1] is to find a Lorentzian metric

$$\mathbf{g} = -n^2 dt \otimes dt + g_{ij}(dx^i + Y^i dt) \otimes (dx^j + Y^j dt)$$

via a pair (g, k) that satisfies the vacuum Einstein evolution equations, the constraint equations and the CMCSH condition

$$\text{Tr}k := g^{ij} k_{ij} = t \quad \text{and} \quad U^j = 0 \tag{2}$$

where $U = U^k \partial_k$ denotes the vector field with

$$U^k := g^{ij} U^k_{ij} \ \text{ with } \ U^k_{ij} = \Gamma^k_{ij} - \hat{\Gamma}^k_{ij},$$

and Γ^k_{ij} denotes the Christoffel symbol with respect to g.

This leads to an elliptic-hyperbolic system

$$\begin{cases} \partial_t g_{ij} = -2nk_{ij} + \mathcal{L}_Y g_{ij} \\ \partial_t k_{ij} = -\nabla_i \nabla_j n + n(R_{ij} + \text{Tr}k k_{ij} - 2k_{im} k^m_j) + \mathcal{L}_Y k_{ij} \\ -\Delta n + |k|^2 n = 1 \\ \Delta Y^i + R^i_j Y^j = \left(-2nk^{jl} + 2\nabla^j Y^l\right) U^i_{jl} + 2\nabla^j n k^i_j - \nabla^i n k^j_j, \end{cases} \tag{3}$$

where the hyperbolic part is the Einstein evolution equations, and where the elliptic part consists of the equations for the lapse and shift. It is natural to ask under what minimal regularity on the initial data the CMCSH Cauchy problem is locally well-posed. We proved in [13] the following result which shows the well-posedness of the problem when the initial data is in $H^s \times H^{s-1}$ with $s > 2$.

Theorem For any $s > 2$, $t_0 < 0$ and $M_0 > 0$, there exist positive constants T_*, M_1 and M_2 such that the following properties hold true:

(i) For any initial data set (g^0, k^0) satisfying constraint equations with $t_0 := \text{Tr}k^0 < 0$ and $\|g^0\|_{H^s(\Sigma_{t_0})} + \|k^0\|_{H^{s-1}(\Sigma_{t_0})} \leq M_0$, there exists a unique solution $(g, k) \in C(I_*, H^s \times H^{s-1}) \times C^1(I_*, H^{s-1} \times H^{s-2})$ to (2);

(ii) There holds

$$\|\widehat{\nabla} g, k\|_{L^2_{I_*} L^\infty_x} + \|\widehat{\nabla} g, k\|_{L^\infty_{I_*} H^{s-1}} \leq M_1;$$

(iii) For $2 < r \leq s$, and for each $\tau \in I_*$ the linear equation

$$\begin{cases} \Box_{\mathbf{g}} \psi = 0, & (t, x) \in I_* \times \Sigma \\ \psi(\tau, \cdot) = \psi_0 \in H^r(\Sigma), & \partial_t \psi(\tau, \cdot) = \psi_1 \in H^{r-1}(\Sigma) \end{cases}$$

admits a unique solution $\psi \in C(I_*, H^r) \times C^1(I_*, H^{r-1})$ satisfying the estimates

$$\|\psi\|_{L^\infty_t H^r} + \|\partial_t \psi\|_{L^\infty_t H^{r-1}} \leq M_2 \|(\psi_0, \psi_1)\|_{H^r \times H^{r-1}}$$

and

$$\|\mathbf{D}\psi\|_{L^2_t L^\infty_x} \leq M_2 \|(\psi_0, \psi_1)\|_{H^r \times H^{r-1}};$$

where $I_* := [t_0 - T_*, t_0 + T_*]$.

To improve the classical result on the local well-posedness of quasilinear wave equation, the key step is to derive the Strichartz estimate for the wave operator $\Box_{\mathbf{g}(\phi)}$ with rough coefficients $\mathbf{g}^{ij}(\phi)$. A series of progresses was achieved by Smith [10], Bahouri-Chemin [3] and Tataru [12] using parametrix constructions. One interesting progress was made by Klainerman in [5] where a vector field approach was developed to establish the Strichartz estimate. Blended with the paradifferential localization procedure, this approach was further developed by Klainerman-Rodnianski in [6] where they successfully improved the local well-posedness of (1) in $\mathbb{R}^{3+1}$ to the Sobolev space H^s with $s > 2+\frac{2-\sqrt{3}}{2}$. The core progress that enables the improvement from $s > 2 + \frac{2-\sqrt{3}}{2}$ to $s > 2$ for (**EV**) under wave coordinate was made in [9] by showing that $\mathbf{Ric}(\mathbf{h})$ relative to the frequency-truncated metric $\mathbf{h} = \mathbf{g}_{\le\lambda}$ does not deviate from 0 to a harmful level. However, similar estimates for $\mathbf{Ric}(\mathbf{h})$ can hardly be obtained for general quasilinear wave equation of type (1). The sharp local well-posedness for type (1) in H^s with $s > 2$ was achieved by Smith and Tataru in [11] based on the wave packet parametrix construction.

A reduction to consider $\Box_{\mathbf{g}_{\le\lambda^a}}\psi = 0$, with $0 < a \le 1$, appeared in all the above mentioned work, where $\mathbf{g}_{\le\lambda^a} := S_{\lambda^a}(\mathbf{g}(S_{\lambda^a}(\phi)))$ is the truncation of $\mathbf{g}(\phi)$ at the frequency level λ^a. Here $S_\lambda := \sum_{\mu\le\lambda} P_\mu$ and P_λ is the Littlewood-Paley projector with frequency λ being a dyadic number. This regularization on metric is used to phase-localize the solution, and in most of the works, to balance the differentiability on coefficients required either by parametrix construction or by energy method. Such a regularization on metric, nevertheless, poses major technical baggage, in particular, to carry out the vector field approach in Einstein vacuum spacetime, since $\mathbf{Ric}(\mathbf{g}_{\le\lambda})$ no longer vanishes. The analysis in [9] on the defected Ricci tensor and its derivatives is a very delicate procedure, which relies crucially on full force of $\partial\mathbf{h}$, hence, on their non-smoothed counter part $\partial\mathbf{g}$ as well. One particular issue tied to CMCSH gauge itself arises due to the lack of control on $\mathbf{D_T}Y$, the time derivative of the shift vector field. Although $\mathbf{D_T}Y$ satisfies an elliptic equation, that equation is not good enough to provide a valid control on $\mathbf{D_T}Y$ even in terms of L^2-norm. The loss of control over some components of $\partial\mathbf{g}$ becomes a serious hurdle in obtaining the control of $\mathbf{Ric}(\mathbf{h})$ and its derivatives. The potential issue on Ricci defect forces us to abandon the frequency truncation on metric.

The important aspect of our analysis is to implement the vector field approach directly relative to the non-smoothed Einstein metric $\mathbf{g}$ to establish the Strichartz estimate with an arbitrarily small loss for the linearized problem $\Box_{\mathbf{g}}\psi = 0$. This confirms that, due to the better behavior of $\mathbf{Ric}$, Einstein metric is in nature "smooth" enough to implement the vector field approach without truncation on $\mathbf{g}$ in Fourier space, and leads to the H^s well-posedness result with $s > 2$ for Einstein equation and a simplification over the the methodology in [7]-[9]. Note that in [6], [7]-[9], deriving the bounded Morawetz type energy of derivatives of ψ, with $\Box_{\mathbf{g}_{\le\lambda^a}}\psi = 0$, is the main building block to obtain the dispersive estimate for $P\partial_t\psi$ required by Strichartz estimate. This procedure relies on $H^\sigma, \sigma > \frac{1}{2}$ norm of curvature, which is impossible to be obtained relative to the rough and non-smoothed metric. Our strategy is to derive the dispersive estimate merely using Morawetz type energy for ψ itself. The analysis to control such energy is mainly focused on the Ricci coefficients relative to Einstein metric. Although part of such analysis benefits from $\mathbf{Ric} = \mathbf{0}$, the crucial estimates such as strichartz type norm $\|\hat{\chi}\|_{L_t^2 L_x^\infty}$ require the bound of $H^\sigma, \sigma > 1/2$ for $\widehat{\nabla} k$ and $\widehat{\nabla}^2 g$ if only the classic L_x^∞ Calderson-Zygmund inequality is employed. We solve this problem by modifing

Calderson Zygmund inequality followed by taking advantage of the extra differentialbity for $\widehat{\nabla} g, k$ that can be obtained by Strichartz estimates. The difficulty coming from $\mathbf{D_T} Y$ still penetrates in key steps in the vector fields approach, where all components of $\partial \mathbf{g}$ were typically involved. We exclude such term by modifying the standard treatments including modifying energy momentum tensor, refining $\mathcal{T}\mathcal{T}^*$ argument and curvature decomposition into more invariant fashion.

References

[1] Andersson, L. and Moncrief, V *Elliptic-hyperbolic systems and the Einstein equations.* Ann. Henri Poincare 4 (2003), no. 1, 1–34.
[2] Choquet Bruhat, Y. *Theoreme d'existence pour certains systemes d'equations aux derivees partielles non lineaires* Acta Math. 88 (1952), 141-225
[3] Bahouri H. and Chemin J.Y. *Eq̀uations dòndes quasilineaires et estimation de Strichartz*, Amer. J. Math. 121 (1999), 1337–1377.
[4] Hughes, T., Kato, T. and Marsden, J. *Well-posed quasi-linear second-order hyperbolic systems with applications to nonlinear elastodynamics and general relativity,* Arch. Rational Mech. Anal. 63 (1976), 273-294
[5] Klainerman, S. *A commuting vectorfield approach to Strichartz type inequalities and applications to quasilinear wave equations.* IMRN, 2001, No 5, 221–274.
[6] Klainerman, S. and Rodnianski, I. *Improved local well-posedness for quasilinear wave equations in dimension three*, Duke Mathematical Journal, Vol.117, No.1, 2003.
[7] Klainerman, S. and Rodnianski, I. *Rough solutions to the Einstein vacuum equations*, Annals of Mathematics., 161 (2005), 1143-1193.
[8] Klainerman, S. and Rodnianski, I., *The Causal Structure of Microlocalized Rough Einstein Metrics*, Annals of Mathematics, 161(2005), 1195-1243.
[9] Klainerman, S. and Rodnianski, I. *Ricci defects of microlocalized Einstein metrics.* J. Hyperbolic Differ. Equ. 1 (2004), no. 1, 85–113.
[10] Smith, H. F., *A parametrix construction for wave equations with $C^{1,1}$ coefficients*, Ann. Inst. Fourier (Grenoble) 48 (1998), 797–835.
[11] Smith, H. F. and Tataru, D. *Sharp local well-posedness results for the nonliear wave equation* . Annals of Mathematics, 162 (2005), 291–366
[12] Tataru, D, Strichartz estimates for second order hyperbolic operators with nonsmooth coefficients, III, J. Amer. Math. Soc. 15 (2002), 419–442.
[13] Wang, Q. *Rough solution of Einstein vacuum equation in CMCSH gauge.* arXiv:1201.0049

Null Asymptotic Analysis of Spacetimes and Memory

Lydia Bieri
(joint work with P. Chen, S.T. Yau)

To study astrophysical objects like galaxies, black holes and neutron stars we have to investigate asymptotically flat solutions of the corresponding Einstein equations. When performing an astrophysical experiment, we can think of ourselves as 'sitting at null infinity' of the spacetime in question. Thus besides stability issues, it is crucial to understand the null asymptotic analysis and geometry which describe the physics. Precise geometric-analytic description of null infinity for the Einstein vacuum (EV) equations are derived by D. Christodoulou and S. Klainerman in [8]. I generalize these results in [1], [2]. The Cauchy problem is solved with more general, asymptotically flat initial data, relaxing decay and regularity. In contrast to the original result, I establish the borderline case from the point of view of decay of the data by relaxing the decay of the initial data by one power of r at spatial infinity, indicating that this decay is sharp. My latest results in [4] yield a precise description of null infinity for spacetimes including a non-isotropic mass term. For further studies of the radiation field see my paper [3]. This geometric-analytic approach turns out to be the perfect way to reach another major goal of mathematical General Relativity (GR) and astrophysics: to precisely describe and finally observe gravitational radiation, one of the predictions of GR. In order to do so, one has to study the null asymptotical limits of the spacetimes for typical sources such as mergers of binary neutron stars and binary black holes. In these processes typically mass and momenta are radiated away in form of gravitational waves. Whereas in the afore-mentioned stability proofs there are assumptions on the smallness of the data, the results at null infinity hold for all data, including large data such as for binary black hole or binary neutron star mergers.

D. Christodoulou showed that every gravitational-wave burst has a nonlinear memory [7] displacing test masses permanently. See also [7] for further references on earlier papers within a linearized theory. Together with my co-authors P. Chen and S.-T. Yau I establish in [5] the nonlinear electromagnetic Christodoulou memory effect of gravitational waves. We show that in the Einstein-Maxwell (EM) equations there is a contribution from the electromagnetic field to the nonlinear memory effect. We use N. Zipser's studies of the Einstein-Maxwell equations [9], [10] to further investigate null infinity. In [6] we apply the geometric-analytic result to astrophysical data.

In this talk, we discuss the null asymptotics for spacetimes solving the EV and EM equations. Further, we consider the nonlinear memory effect in the EM case and compare our results to the EV situation. We answer mainly two questions for the gravitational wave experiment. First, the electromagnetic field enters the formula for the instantaneous displacement only at lower order. Second, the electromagnetic field contributes at highest order to the Christodoulou effect, i.e. to the permanent displacement. We rely on the methods introduced in the works of Christodoulou and Klainerman [8], Bieri [1], [2], [4] and Zipser [9], [10].

The nonlinear memory effect is governed by

$$\Sigma^{+} \ - \ \Sigma^{-} \ = \ - \int_{-\infty}^{\infty} \Xi(u) \ du$$

(See [7] for a derivation in the EV case.) Here, Σ denotes the asymptotic shear of outgoing null hypersurfaces C_u and Σ^+ and Σ^- are limits of Σ as u tends to $+\infty$ and $-\infty$, respectively. And Ξ is the (weighted) limit of the trace-free part of the conjugate null second fundamental form of a closed spacelike surface S in spacetime.

We find in the EM case [5], [6] that the total energy $\frac{\mathbf{F}}{\mathbf{8\pi}}$ radiated to infinity in a given direction per unit solid angle is given by

$$F = \int_{-\infty}^{+\infty} \left(\mid \Xi \mid^2 + \frac{1}{2} \mid A_F \mid^2 \right) du$$

with A_F denoting a component of the electromagnetic field. In the EV case [8], [7] the A_F term is not there.

Christodoulou [7] relates $\Sigma^+ - \Sigma^-$ for the EV equations to a permanent displacement of test masses. We [5], [6] investigate the corresponding question for $\Sigma^+ - \Sigma^-$ in the EM case.

Theorem 1. **[Bieri-Chen-Yau** [5]**]** *Let* $\Sigma^+(\cdot) = \lim_{u\to\infty} \Sigma(u,\cdot)$ *and* $\Sigma^-(\cdot) = \lim_{u\to-\infty} \Sigma(u,\cdot)$. *Let*

$$F(\cdot) = \int_{-\infty}^{\infty} \big(\mid \Xi(u,\cdot) \mid^2 + \frac{1}{2} \mid A_F(u,\cdot) \mid^2 \big) du \quad . \tag{1}$$

Moreover, let Φ *be the solution with* $\bar{\Phi} = 0$ *on* S^2 *of the equation*

$$\overset{\circ}{\not{\triangle}} \ \Phi = F - \bar{F} \quad .$$

Then $\Sigma^+ - \Sigma^-$ *is given by the following equation on* S^2*:*

$$\overset{\circ}{\not{div}} \ (\Sigma^+ - \Sigma^-) = \overset{\circ}{\not{\nabla}} \ \Phi \quad . \tag{2}$$

We also show for the EM case in [5], [6] how $\Sigma^+ - \Sigma^-$ is equivalent to an overall displacement $\triangle x$ of the test masses in a laser interferometer gravitational wave detector. In particular, we derive a formula of the form

$$\triangle x = -\frac{d_0}{r}(\Sigma^+ - \Sigma^-) \quad .$$

To derive the solution at the observation point, denote the direction of observation by $\xi \in S^2 \subset \mathbb{R}^3$. Let X, Y be arbitrary vectors lying in the tangent plane at ξ, i.e. in $T_\xi S^2$. Let Π be the projection to the plane through the origin orthogonal to ξ.

$< \ , \ >$ denotes the inner product. The solution at the observation point ξ is expressed as an integral over S^2 of a contribution from each $\xi' \in S^2$:

$$(\Sigma^+ \ - \ \Sigma^-) \ (X, \ Y) =$$

$$-\frac{1}{2\pi} \int_{\xi' \in S^2} (F - F_{[1]})(\xi') \frac{< X, \xi' >< Y, \xi' > - \frac{1}{2} < X, Y > \mid \Pi\xi' \mid^2}{1 - < \xi, \xi' >} d\mu_{\overset{\circ}{\gamma}}(\xi')$$

Subscript [1] denotes the projection onto the sum of the 0^{th} $(l = 0)$ and 1^{st} $(l = 1)$ eigenspaces of $\overset{\circ}{\not{\triangle}}$. Multiplicity of the lth eigenspace $2l + 1$, eigenvalue $l(l + 1)$. For derivation and discussion of this formula see [7]. For proof of our theorem 1 see [5] and for further discussions and applications of our result we refer to our [6].

Thus we prove that the electromagnetic field contributes to the nonlinear Christodoulou memory effect of gravitational waves. A typical source where large magnetic fields are generated and radiated away is the coalescence of a neutron star binary. Our geometric-analytic investigations of the spacetimes yield exact solutions, which hold for all data. In particular, they hold for large data describing the astrophysical phenomena mentioned above.

References

[1] L. Bieri. *An Extension of the Stability Theorem of the Minkowski Space in General Relativity.* ETH Zurich, Ph.D. thesis. **17178**. Zurich. (2007).
[2] L. Bieri. *Extensions of the Stability Theorem of the Minkowski Space in General Relativity. Solutions of the Einstein Vacuum Equations.* AMS-IP. **45**. Studies in Advanced Mathematics. Cambridge. MA. (2009).
[3] L. Bieri. *Convergence of Hawking Mass to Bondi Mass in Asymptotically Flat Spacetimes in General Relativity.* Preprint. (2012).
[4] L. Bieri. *Spacetimes with Non-Isotropic Mass in General Relativity.* Preprint. (2012).
[5] L. Bieri, P. Chen, S.-T. Yau. *Null Asymptotics of Solutions of the Einstein-Maxwell Equations in General Relativity and Gravitational Radiation.* Advances in Theoretical and Mathematical Physics, 15, 4, (2011).
[6] L. Bieri, P. Chen, S.-T. Yau. *The Electromagnetic Christodoulou Memory Effect and its Application to Neutron Star Binary Mergers.* Submitted. (2012).
[7] D. Christodoulou. *Nonlinear Nature of Gravitation and Gravitational-Wave Experiments.* Phys.Rev.Letters. **67**. (1991). no.12. 1486-1489.
[8] D. Christodoulou, S. Klainerman. *The global nonlinear stability of the Minkowski space.* Princeton Math.Series **41**. Princeton University Press. Princeton. NJ. (1993).
[9] N. Zipser. *The Global Nonlinear Stability of the Trivial Solution of the Einstein-Maxwell Equations.* Ph.D. thesis. Harvard Univ. Cambridge MA. (2000).
[10] N. Zipser. *Extensions of the Stability Theorem of the Minkowski Space in General Relativity. - Solutions of the Einstein-Maxwell Equations.* AMS-IP. **45**. Studies in Advanced Mathematics. Cambridge. MA. (2009).

The Geometry of Static Spacetimes: Geometrostatics

Carla Cederbaum

Geometrostatics is an important sub-domain of Einstein's General Relativity. It describes the mathematical and physical properties of static isolated relativistic systems such as stars, galaxies, or black holes. For example, geometrostatic systems have a well-defined ADM-mass [1, 2]. If this mass is non-zero, they also have a well-defined center of mass [3, 4, 5] induced by a CMC-foliation at infinity.

Geometrostatic systems are described by an asymptotically flat Riemannian 3-manifold (M, g) and a lapse function $N : M \to \mathbb{R}^+$ describing the lapse of time in the corresponding Lorentzian space-time $(\mathbb{R} \times M, ds^2 = -N^2c^2dt^2 + g)$, where c denotes the speed of light. In the vacuum region outside the matter, these variables satisfy the so-called *vacuum static metric equations*

$$\begin{aligned} N\,\mathrm{Ric} &= \nabla^2 N \\ \triangle N &= 0, \end{aligned}$$

where Ric is the Ricci curvature tensor of g, $\nabla^2 N$ denotes the Hessian, and $\triangle N$ denotes the Laplacian of N.

Using these equations and a covariant reformulation of them, I have obtained several geometric, analytic, and physical results for geometrostatic systems [6]. For example, the lapse function N can be proven to be unique for a given metric g that is known to be static. Complementarily, the metric g is unique given the lapse function and a set of asymptotically flat wave-harmonic coordinates

(coordinates harmonic with respect to ds^2). This can be reinterpreted as saying that geometrostatic systems have four degrees of freedom – which is relevant for Bartnik's conjecture on minimal mass extensions [7].

Furthermore, proving asymptotic estimates, I obtained information on the fall-off of geometrostatic systems linking these variables to the center of mass (generalizing [8]). Applying the conformal transformation $\gamma := N^2 g$, $U := c^2 \ln N$ into *pseudo-Newtonian variables*, these asymptotic expansions allowed for a localization of both ADM-mass m_{ADM} and CMC-center of mass $\vec{z}_{CMC}$ via

$$m_{ADM} = \int_\Sigma \frac{\partial U}{\partial \nu}\, d\sigma, \quad \text{and} \quad \vec{z}_{CMC} = \int_\Sigma \left(\frac{\partial U}{\partial \nu} \vec{x} - U \frac{\partial \vec{x}}{\partial \nu} \right) d\sigma,$$

where Σ is any surface enclosing the support of the matter, ν and $d\sigma$ are the outer unit normal to and area measure of Σ with respect to γ, and $\vec{x}$ is the vector of γ-harmonic coordinates.

Finally, I used Ehlers' frame theory [9] to prove that the Newtonian limit ($c \to \infty$) of ADM-mass and CMC-center of mass coincide with the Newtonian mass and center of mass of the Newtonian limit – along any family of geometrostatic space-times that possesses a static Newtonian limit. This might turn out useful for the study of the static n-body problem [10].

References

[1] R. Bartnik, *The Mass of an Asymptotically Flat Manifold*, Communications on Pure and Applied Mathematics **39** (1986), 661–693.

[2] P. Chrúsciel, *On the invariant mass conjecture in general relativity*, Communications in Mathematical Physics **120** (1988), 233–248.

[3] G. Huisken and S-T. Yau, *Definition of Center of Mass for Isolated Physical Systems and Unique Foliations by Stable Spheres with Constant Mean Curvature*, Inventiones Mathematicae **124** (1996), 281–311.

[4] J. Metzger, *Foliations of asymptotically flat 3-manifolds by 2-surfaces of prescribed mean curvature*, Journal of Differential Geometry **77** (2007), 201–236.

[5] L-H. Huang, *Foliations by Stable Spheres with Constant Mean Curvature for Isolated Systems with general Asymptotics*, Communications in Mathematical Physics **300** (2010), 331–373.

[6] C. Cederbaum, *The Newtonian Limit of Geometrostatics*, PhD thesis (2011), arXiv:1201.5433

[7] R. Bartnik, *New definition of quasilocal mass*, Physical Review Letters **62** (1989), 2346–2348.

[8] D. Kennefick and N. O´Murchadha, *Weakly decaying asymptotically flat static and stationary solutions to the Einstein equations*, Classical and Quantum Gravity **12** (1995), 149–158.

[9] J. Ehlers, *Examples of Newtonian limits of relativistic spacetimes*, Classical and Quantum Gravity **14** (1997), A119–A126.

[10] R. Beig and R. Schoen, *On static n-body configurations in General Relativity*, Classical and Quantum Gravity **26** (2009), 075014.

Reporter: Volker Schlue

Participants

Prof. Dr. Paul Tyler Allen
Lewis & Clark College
Dept. of Mathematical Sciences
MSC 110
0615 S.W. Palatine Hill Road
PORTLAND OR 97219
UNITED STATES

Prof. Dr. Michael T. Anderson
Department of Mathematics
Stony Brook University
Math. Tower
STONY BROOK, NY 11794-3651
UNITED STATES

Prof. Dr. Lars Andersson
MPI für Gravitationsphysik
Albert-Einstein-Institut
Am Mühlenberg 1
14476 Golm

Dr. Hakan Andreasson
Department of Mathematics
Chalmers University of Technology
412 96 GÖTEBORG
SWEDEN

Dr. Stefanos Aretakis
Department of Mathematics
Princeton University
Fine Hall
Washington Road
PRINCETON, NJ 08544-1000
UNITED STATES

Prof. Dr. Robert Bartnik
School of Mathematical Sciences
Monash University
CLAYTON, Victoria 3800
AUSTRALIA

Prof. Dr. Robert Beig
Institut für Theoretische Physik
Universität Wien
Währinger Straße 17
1090 WIEN
AUSTRIA

Dr. Lydia Bieri
Department of Mathematics
University of Michigan
530 Church Street
ANN ARBOR, MI 48109-1043
UNITED STATES

Prof. Dr. Piotr Bizon
Institute of Physics
Jagiellonian University
ul. Reymonta 4
30-059 KRAKOW
POLAND

Dr. Pieter Blue
School of Mathematics
University of Edinburgh
James Clerk Maxwell Bldg.
King's Buildings, Mayfield Road
EDINBURGH EH9 3JZ
UNITED KINGDOM

Dr. Carla Cederbaum
Max Planck Institute for Gravitational Physics
Am Mühlenberg 1
14476 Golm

Prof. Dr. Piotr T. Chrusciel
Fakultät für Physik
Universität Wien
Boltzmanngasse 5
1090 WIEN
AUSTRIA

Prof. Dr. Justin Corvino
Department of Mathematics
Lafayette College
EASTON, PA 18042-1781
UNITED STATES

Prof. Dr. Mihalis Dafermos
Dept. of Pure Mathematics and
Mathematical Statistics
University of Cambridge
Wilberforce Road
CAMBRIDGE CB3 0WB
UNITED KINGDOM

Prof. Dr. Michael Eichmair
Departement Mathematik
ETH-Zentrum
Rämistr. 101
8092 ZÜRICH
SWITZERLAND

Dr. David Fajman
MPI für Gravitationsphysik
Albert-Einstein-Institut
Am Mühlenberg 1
14476 Golm

Prof. Dr. Helmut Friedrich
MPI für Gravitationsphysik
Albert-Einstein-Institut
Am Mühlenberg 1
14476 Golm

Prof. Dr. Gregory Galloway
Dept. of Mathematics and Computer
Science
University of Miami
P.O. Box 248011
CORAL GABLES, FL 33124
UNITED STATES

Dr. Romain Gicquaud
Departement de Mathematiques
Faculte des Sciences
Universite de Tours
Parc de Grandmont
37200 TOURS
FRANCE

Prof. Dr. Carsten Gundlach
School of Mathematics
University of Southampton
Highfield Campus
SOUTHAMPTON SO17 1BJ
UNITED KINGDOM

Dr. Gustav Holzegel
Department of Mathematics
Princeton University
402 Fine Hall
Washington Road
PRINCETON, NJ 08544
UNITED STATES

Dr. Lan-Hsuan Huang
Department of Mathematics
Columbia University
2990 Broadway
NEW YORK, NY 10027
UNITED STATES

Prof. Dr. Gerhard Huisken
MPI für Gravitationsphysik
Albert-Einstein-Institut
Am Mühlenberg 1
14476 Golm

Prof. Dr. James Isenberg
Department of Mathematics
University of Oregon
EUGENE, OR 97403-5203
UNITED STATES

Jonathan Kommemi
Dept. of Applied Mathematics and
Theoretical Physics
University of Cambridge
Silver Street
CAMBRIDGE, CB3 9EW
UNITED KINGDOM

Prof. Dr. Philippe G. LeFloch
Laboratoire Jacques-Louis Lions
Universite Paris 6
4, Place Jussieu
75252 PARIS Cedex 05
FRANCE

Dr. Jonathan Luk
Department of Mathematics
Princeton University
Fine Hall
Washington Road
PRINCETON, NJ 08544-1000
UNITED STATES

Prof. Dr. Marc Mars
Facultad de Ciencias
Universidad de Salamanca
Plaza de la Merced s/n
37008 SALAMANCA
SPAIN

Prof. Dr. David Maxwell
Department of Mathematical Sciences
University of Alaska Fairbanks
PO Box 756660
FAIRBANKS, AK 99775-6660
UNITED STATES

Dr. Luc Nguyen
Department of Mathematics
Princeton University
Fine Hall
Washington Road
PRINCETON, NJ 08544-1000
UNITED STATES

Tim Paetz
Institut für Theoretische Physik
Universität Wien
Währinger Straße 17
1090 WIEN
AUSTRIA

Prof. Dr. Daniel Pollack
Department of Mathematics
University of Washington
Padelford Hall
Box 354350
SEATTLE, WA 98195-4350
UNITED STATES

Prof. Dr. Harvey Reall
Department of Applied Mathematics &
Theoretical Physics (DAMTP)
Centre for Mathematical Sciences
Wilberforce Road
CAMBRIDGE CB3 OWA
UNITED KINGDOM

Dr. Martin Reiris
MPI für Gravitationsphysik
Albert-Einstein-Institut
Am Mühlenberg 1
14476 Golm

Prof. Dr. Alan Rendall
MPI für Gravitationsphysik
Albert-Einstein-Institut
Am Mühlenberg 1
14476 Golm

Prof. Dr. Hans Ringstrm
Department of Mathematics
Royal Institute of Technology
Lindstedtsvägen 25
100 44 STOCKHOLM
SWEDEN

Dr. Volker Schlue
Department of Mathematics
University of Toronto
40 St George Street
TORONTO, Ont. M5S 2E4
CANADA

Dr. Chung-Tse Shao
Department of Mathematics
University of Toronto
40 St George Street
TORONTO, Ont. M5S 2E4
CANADA

Dr. Walter Simon
Fakultät für Physik
Universität Wien
Boltzmanngasse 5
1090 WIEN
AUSTRIA

Dr. Jacques Smulevici
Laboratoire de Mathematiques
Universite Paris Sud (Paris XI)
Batiment 425
91405 ORSAY Cedex
FRANCE

Prof. Dr. Christina Sormani
Dept. of Mathematics & Computer Science
Lehman College & CUNY Graduate Center
365 Fith Avenue
NEW YORK CITY NY 10016
UNITED STATES

Prof. Dr. Jared Speck
Department of Mathematics
Massachusetts Institute of Technology
77 Massachusetts Avenue
CAMBRIDGE, MA 02139-4307
UNITED STATES

Prof. Dr. Iva Stavrov
Lewis & Clark College
Dept. of Mathematical Sciences
MSC 110
0615 S.W. Palatine Hill Road
PORTLAND OR 97219
UNITED STATES

Prof. Dr. Paul Tod
Mathematical Institute
Oxford University
24-29 St. Giles
OXFORD OX1 3LB
UNITED KINGDOM

Dr. Juan Antonio Valiente Kroon
School of Mathematical Sciences
Queen Mary
University of London
Mile End Road
LONDON E1 4NS
UNITED KINGDOM

Prof. Dr. Robert Wald
Enrico Fermi Institute and
Department of Physics
University of Chicago
5640 S. Ellis Ave.
CHICAGO, IL 60637
UNITED STATES

Prof. Dr. Mu-Tao Wang
Department of Mathematics
Columbia University
2990 Broadway, Math. Building 509
MC 4406
NEW YORK NY 10027
UNITED STATES

Dr. Qian Wang
MPI für Gravitationsphysik
Albert-Einstein-Institut
Am Mühlenberg 1
14476 Golm

Dr. Willie Wong
EPFL SB MA PDE
Ecole Polytechnique Federale de
Lausanne
1015 LAUSANNE
SWITZERLAND