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Optimal and Near Optimal Configurations on Lattices and Manifolds

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Abstract. Optimal configurations of points arise in many contexts, for example classical ground states for interacting particle systems, Euclidean packings of convex bodies, as well as minimal discrete and continuous energy problems for general kernels. Relevant questions in this area include the understanding of asymptotic optimal configurations, of lattice and periodic configurations, the development of algorithmic constructions of near optimal configurations, and the application of methods in convex optimization such as linear and semidefinite programming.

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Introduction by the Organisers

The workshop *Optimal and Near Optimal Configurations on Lattices and Manifolds*, organised by Christine Bachoc (Bordeaux), Peter Grabner (Graz), Ed Saff (Nashville) and Achill Schürmann (Rostock), was held 19 August –25 August 2012. This meeting has gathered 27 participants from various areas of mathematics, mostly Approximation Theory, Optimization, Number Theory and Discrete Geometry. Bringing together mathematicians with a variety of expertise on optimal configurations on lattices and manifolds was one of the aims of this workshop and was indeed fully achieved. A broad range of topics have been covered during the talks, especially minimal energy configurations, best packings in Euclidean space or on other spaces, spherical designs, Euclidean lattices and linear and semidefinite programming methods. Theoretical as well as computational issues have been addressed.

Eight survey talks were given by four participants at the beginning of the workshop, at the request of the organisers, in order to provide the younger participants with a better perspective on fruitful research directions. In addition, nineteen 40min talks have addressed more specialised topics. Also one *Problems Session* took place one evening, and one session gathered our group with the participants of the parallel workshop *Rough Paths and PDE's*.

Many recent and exciting results were presented during this week, including (but not limited to): the proof that for each $N \geq c_d t^d$ there exists a spherical t-design on S^d consisting of N points, where c_d is a constant depending only on d, the computation of semidefinite programming bounds for binary packings, the construction of kissing configurations in dimension 25 and larger that are better than the previously known, the discovery of an extremal even unimodular lattice in dimension 72, the solution of Tammes problem for thirteen points, and low complexity methods for computing near minimal energy points on general manifolds.

The diversity of participants and the broad range of topics covered during the presentations, as well as the warm atmosphere provided by the Mathematisches Forschungsinstitut Oberwolfach have stimulated fruitfull discussions and new collaborations among the participants. A detailed program of the week and slides for many of the talks are available from `https://sites.google.com/site/optimalconfigurations2012/`. In the following the abstracts of the talks are included in alphabetical order of their authors.

Workshop: Optimal and Near Optimal Configurations on Lattices and Manifolds

Table of Contents

Abstracts

Euclidean representation of strongly regular graphs

Andriy Bondarenko
(joint work with Danylo Radchenko)

Definition 1. A strongly regular graph $\Gamma = (V, E)$ with parameters (v, k, λ, μ) is an undirected regular graph on v vertices of valency k such that each pair of adjacent vertices has λ common neighbors, and each pair of nonadjacent vertices has μ common neighbors.
The main question is:

For which parameter set (v, k, λ, μ) there exists a strongly regular graph?

It is easy to show that incidence matrix of a strongly regular graph A has 3 eigenvalues: k of multiplicity 1, one positive eigenvalue r of multiplicity f, and one negative eigenvalue s of multiplicity g. The list of all suitable parameters together with known existence results for $v \leq 1300$ could be found at [3] (except trivial case when Γ is a disjoint union of complete graphs mK_n or its complement).

A strongly regular graph implies 2-distance sets both in S^{f-1}, and in S^{g-1} which are spherical 2-designs. Conversely, if there is a 2-distance set on S^n then it is naturally implies a strongly regular graph. This simple fact is useful for constructing of many well-known strongly regular graphs. Strongly regular graph which are spherical 3-designs provide universally optimal configurations [6].

The main tool we will use for description of strongly regular graphs is the fact that each subset of vectors $\{x_i : i \in U\}$ for Euclidean representation in S^{g-1}, where $U \subset V$, has a positive definite Gram matrix $\{(x_i, x_j)\}_{i,j \in U}$ of rank at most g.

We will investigate strongly regular graphs with $\lambda = 1$ and negative eigenspaces of dimension $g = k$, see [1]. One can deduce that such a graph is either K_3 or belongs to the family

$$((n^2 + 3n - 1)^2, n^2(n + 3), 1, n(n + 1)),$$

where $n \in \mathbb{N}$ is the positive eigenvalue.

This family includes the lattice graph $L_{3,3}$ with parameters $(9, 4, 1, 2)$, the Brouwer-Haemers graph with parameters $(81, 20, 1, 6)$, which is also known to be unique [4], and the Games graph with parameters $(729, 112, 1, 20)$, for which the uniqueness question was open. We will show that these are the only graphs in the family.

Theorem 1. *Suppose that there exists a strongly regular graph with parameters* $((n^2 + 3n - 1)^2, n^2(n + 3), 1, n(n + 1))$. *Then* $n \in \{1, 2, 4\}$.

The proof consists of two parts. First, using Euclidean representation we will show that each graph in the family exhibits certain symmetries (in particular, its group of automorphisms is vertex-transitive). Then, we will use different properties of these symmetries to prove that the set of vertices can be given a vector space structure over the finite field F_3. In particular, the number of vertices in the graph is a power of 3.

Finally, the resulting diophantine equation has the only three mentioned solutions by virtue of [2, Theorem B]. This equation has appeared during the studying of ternary linear codes with exactly two nonzero weights and with the minimal weight of the dual code at least 4. It was actually shown in [5] that each of such codes of dimension $2m$ implies a strongly regular graph from the family with $v = 3^{2m}$ vertices.

The following result completes description of the family.

Theorem 2. *The strongly regular graph with parameters* $(729, 112, 1, 20)$ *is unique up to isomorphism.*

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Asymptotically optimal *d*-energy configurations on *d*-dimensional sets

Sergiy Borodachov

(joint work with Doug Hardin and Edward Saff)

We find a class of asymptotically optimal sequences of N-point configurations for the Riesz d-energy minimization problem on Jordan measurable sets in $\mathbb{R}^d$, $d \in \mathbb{N}$. We also provide sufficient conditions for the asymptotic optimality of sequences of N-point configurations on a certain class of d-dimensional manifolds (for the d-energy problem).

Let $\|\cdot\|$ be an arbitrary norm in $\mathbb{R}^d$ and η be the metric generated by this norm. Given a finite set $\omega = \{x_1, \ldots, x_N\} \subset \mathbb{R}^d$, let

$$E_d^\eta(\omega) = \sum_{1 \le i \ne j \le N} \frac{1}{\|x_i - x_j\|^d}.$$

The *minimal N-point d-energy* of a compact set $K \subset \mathbb{R}^d$ is defined as

$$\mathcal{E}_d^\eta(K,N) = \inf_{\substack{\omega\subset K\\ \#\omega=N}} E_d^\eta(\omega).$$

Here $\#X$ stands for the cardinality of a set X.

Denote by $\mathcal{L}_d$ the d-dimensional Lebesgue measure and let β_d^η be the $\mathcal{L}_d$-measure of the unit ball in $\mathbb{R}^d$ with respect to the norm $\|\cdot\|$.

The following asymptotic result is known when η is the Euclidean distance (see [1, Theorem 2.1]). In this case we omit the index η in the notation of the minimal N-point d-energy and use the notation β_d instead of β_d^η.

Theorem A. *Suppose that K is a compact subset of $\mathbb{R}^d$ endowed with the Euclidean norm. Then*

$$\lim_{N\to\infty}\frac{\mathcal{E}_d(K,N)}{N^2\ln N} = \frac{\beta_d}{\mathcal{L}_d(K)}.$$

When the boundary ∂K of K has Lebesgue measure zero; i.e., K is Jordan measurable, we are able to obtain asymptotically d-energy minimizing configurations.

Let Y be an infinite point configuration in $\mathbb{R}^d$ such that

$$\delta_\eta(Y) = \inf_{\substack{x,y\in Y\\ x\neq y}} \|x-y\| > 0. \tag{1}$$

Given a point $x=(x_1,\ldots,x_d)\in\mathbb{R}^d$, denote

$$C[x,R] := [x_1-R, x_1+R)\times\ldots\times[x_d-R,x_d+R)\,.$$

Assume that the limit defining the density of Y,

$$\Delta(Y) := \lim_{R\to\infty}\frac{\#(Y\cap C[\mathbf{0},R])}{(2R)^d}$$

exists as a finite and positive number. Denote

$$J(Y,R) := \sup_{x\in\mathbb{R}^d}\left|\frac{\#(Y\cap C[x,R])}{(2R)^d} - \Delta(Y)\right|.$$

Given a compact subset $K\subset\mathbb{R}^d$ such that $\mathcal{L}_d(K)>0$ and $\mathcal{L}_d(\partial K)=0$, define the following sequence of point configurations:

$$X_N := (\sigma_N Y)\cap K,\quad N\in\mathbb{N},\quad \text{where}\quad \sigma_N = \sqrt[d]{\frac{\Delta(Y)\mathcal{L}_d(K)}{N}}. \tag{2}$$

Theorem 1. *Let K be a compact subset of $\mathbb{R}^d$ such that $\mathcal{L}_d(K)>0$ and $\mathcal{L}_d(\partial K)=0$. If a point set $Y\subset\mathbb{R}^d$ of a positive density $\Delta(Y)$ satisfies the conditions $\delta_\eta(Y)>0$ and*

$$\lim_{R\to\infty} J(Y,R) = 0,$$

then for the sequence of configurations $\{X_N\}_{N=1}^\infty$ defined by (2) one has

$$\lim_{N\to\infty}\frac{E_d^\eta(X_N)}{N^2\ln N} = \frac{\beta_d^\eta}{\mathcal{L}_d(K)}. \tag{3}$$

Remark 1. Theorem 1, in particular, holds true when Y is any full-rank lattice in $\mathbb{R}^d$ or a periodic set; i.e., a union of finitely many shifts of a given full-rank lattice.

Remark 2. One can prove that $\#X_N = N(1 + o(1))$, $N \to \infty$. Hence, in the case of the Euclidean distance η in $\mathbb{R}^d$, Theorem A and relation (3) imply that the sequence of configurations $\{X_N\}$ is asymptotically d-energy minimizing on K. In particular, we show the asymptotic optimality when $d = 2$ of the sequence $\{X_N\}$ obtained by intersecting the regular hexagonal lattice with a Jordan measurable region in the Euclidean plane.

Remark 3. Theorem 1 is a consequence of a more general result that we obtain in this work. This result gives the asymptotic behavior of the d-energy of well-separated N-point configurations with a sufficiently low discrepancy from a given continuous density on the set K, which belongs to a certain class of d-dimensional manifolds embedded in $\mathbb{R}^p$, $p > d$.

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Spherical Nets, Sequences and Lattice rules — construction of low-discrepancy sequences on spheres

Johann S. Brauchart

(joint work with Christoph Aistleitner, Josef Dick, Edward B. Saff, Ian H. Sloan, Robert S. Womersley)

How does one explicitly construct sequences of N-point configurations on the unit sphere $\mathbb{S}^d$ in the Euclidean space $\mathbb{R}^{d+1}$ that will provide "good" integration node sets for equal-weight numerical integration, i.e. Quasi Monte Carlo (QMC), rules, in particular, for integrating functions from Sobolev spaces over $\mathbb{S}^d$? In this talk we approach this question by looking at ways to utilize the fact that low-discrepancy sequences in the unit cube in $\mathbb{R}^{d+1}$ can be efficiently computed (see, e.g., [8]).

An essential tool for measuring the quality of a low-discrepancy sequence (X_N) is a (closed form) expression for the worst-case error in terms of a reproducing kernel for the Sobolev space $\mathbb{H}^s(\mathbb{S}^d)$ with $s > d/2$. The space $\mathbb{H}^{(d+1)/2}(\mathbb{S}^d)$ provided with the distance kernel $K_{\mathrm{dist}}(\mathbf{x}, \mathbf{y}) = 2V_{-1}(\mathbb{S}^d) - |\mathbf{x} - \mathbf{y}|$ is special as the worst-case error has a geometrical interpretation in terms of the spherical cap $\mathbb{L}_2$-discrepancy:

$$\left[\mathrm{wce}(Q[X_N]; \mathbb{H}^{(d+1)/2}(\mathbb{S}^d))\right]^2 = V_{-1}(\mathbb{S}^d) - \frac{1}{N^2} \sum_{j=1}^{N} \sum_{k=1}^{N} |\mathbf{x}_j - \mathbf{x}_k| = \frac{1}{C_d} \left[D_{\mathbb{L}_2}^{C}(X_N)\right]^2,$$

where $X_N = \{\mathbf{x}_1, \ldots, \mathbf{x}_N\}$, cf. [5, 7] and [3] for a generalization to other distance kernels. The second equation is also known as *Stolarsky's invariance principle* [13].

A sequence (X_N) of N-point node sets X_N on $\mathbb{S}^d$ is said to be a low-discrepancy sequence on $\mathbb{S}^d$ if the spherical cap discrepancy

$$D^C_{\mathbb{L}_\infty}(X_N) := \sup\left\{ \left| \frac{|X_N \cap \mathcal{C}|}{N} - \sigma_d(\mathcal{C}) \right| : \mathcal{C} \text{ spherical cap in } \mathbb{S}^d \right\}$$

satisfies (for some some positive constant c independent of N) for all X_N

$$D^C_{\mathbb{L}_\infty}(X_N) \leq c \frac{\sqrt{\log N}}{N^{[(d+1)/2]/d}}. \tag{1}$$

Such sequences are almost QMC design sequences for the Sobolev space $\mathbb{H}^s(\mathbb{S}^d)$ with $s = (d+1)/2$ in that the worst case error of the QMC rules $Q[X_N]$ have up to a logarithmic term optimal order; i.e. for every $d/2 < s \leq (d+1)/2$,

$$\frac{\beta_1(s,d)}{N^{s/d}} \leq \mathrm{wce}(Q[X_N]; \mathbb{H}^s(\mathbb{S}^d)) \leq \beta_3(s,d)\,\frac{(\log N)^{s/(d+1)}}{N^{s/d}}.$$

(*QMC design sequences for* $\mathbb{H}^s(\mathbb{S}^d)$ were introduced in the talk of Ian Sloan, [6].)

We discuss theoretical and numerical results concerning sequences on $\mathbb{S}^d$ that are obtained from "lifting" to the sphere well-known low-discrepancy sequences on the unit square using the *Lambert cylindrical equal-area projection*,

$$\mathbf{\Phi}(\alpha,\tau) = \left(2\sqrt{\tau-\tau^2}\,\cos(2\pi\alpha), 2\sqrt{\tau-\tau^2}\,\sin(2\pi\alpha), 1-2\tau\right), \qquad \alpha,\tau \in [0,1],$$

in case of $\mathbb{S}^2$ and its area-preserving higher-dimensional analogue in case of $\mathbb{S}^d$. Under such an area-preserving map a discrepancy notion in the unit cube $[0,1)^d$ with respect to some collection $\mathcal{R}^{(*)}$ of test sets immediately translates into a corresponding spherical version,

$$D_N^{(*)}(\mathbb{S}^d, \Omega^{(*)}; Z_N) = D_N^{(*)}([0,1)^d, \mathcal{R}^{(*)}; \mathcal{Z}_N),$$

where $\Omega^{(*)}$ and Z_N are the images of $\mathcal{R}^{(*)}$ and $\mathcal{Z}_N$ under the Lambert map.

Digital (t,m,d)*-nets* and (t,d)*-sequences*[1] as introduced by H. Niederreiter [10] provide a very efficient method to generate low-discrepancy sequences of point sets in the d-dimensional unit cube $[0,1)^d$. A $(t,m,2)$*-net in base* b with $N = b^m$ points is characterized by the requirement that every elementary interval $\left[\frac{a_1}{b^{d_1}}, \frac{a_1+1}{b^{d_1}}\right) \times \left[\frac{a_2}{b^{d_2}}, \frac{a_2+1}{b^{d_2}}\right)$ with $0 \leq a_1 \leq b^{d_1}$, $0 \leq a_2 \leq b^{d_2}$, $d_1 + d_2 = m - t$, and $0 \leq d_1, d_2 \in \mathbb{Z}$ contains exactly b^t points. The star-discrepancy $D_N^*(X_N)$ of such a net X_N satisfies $D_N^*(X_N) \leq C(m-t)/b^{m-t}$. In [4] it is shown that the *spherical rectangle star discrepancy* (derived from the star discrepancy on the unit square which uses rectangles anchored at $\mathbf{0}$ as test sets) of a $(0,m,2)$-net lifted to $\mathbb{S}^2$ obeys the bound

$$D^*_{b^m}(\mathbb{S}^2, \Omega^*, \mathcal{Z}_N) \leq \frac{b^2/4}{b+1}\frac{m}{b^m} + \frac{1}{b^m}\left(\frac{9}{4} + \frac{1}{b}\right) + \frac{1}{b^{2m}}\left(\frac{b}{2} - \frac{1}{4} - \frac{4b+3}{4(b+1)^2}\right).$$

This bound is essentially best possible as can be seen from Roth's lower bound [12]

$$D^*_{b^m}(\Omega^*, \mathcal{Z}_N) \geq (\lfloor \log_2 N \rfloor + 3)/(2^8 N).$$

[1]Because of a clash of notation we shall not use here the traditional notation (t,m,s)-net.

It is worth to note that the minimal spherical cap discrepancy of N-point sets on $\mathbb{S}^d$ also satisfies the bound in (1), hence, is of order $\mathcal{O}(\sqrt{\log N}/N^{3/4})$ if $d = 2$. However, the best estimates so far for the spherical cap $\mathbb{L}_\infty$-discrepancy of spherical $(0, m, 2)$-nets in base b are obtained in [1] by using as an intermediate step the spherical analogue (via Lambert transformation) of the *isotropic discrepancy* $J_N(\mathcal{Z}_N)$ which uses convex subsets of the unit square as test sets:

$$D^C_{\mathbb{L}_\infty}(\mathbf{\Phi}(\mathcal{Z}_N)) \leq 11\, J_N(\mathcal{Z}_N) \leq 11 \frac{4\sqrt{2b}}{\sqrt{N}}, \qquad \mathcal{Z}_N \text{ a } (0,m,2)\text{-net in base } b.$$

This result implies an improvement upon the bounds on the spherical cap $\mathbb{L}_2$ and $\mathbb{L}_\infty$-discrepancy of the construction given in [9]. (In contrast it is also shown in [1] that the expected value of the spherical cap $\mathbb{L}_\infty$-discrepancy is of order $N^{-1/2}$.) Numerical experiments suggests the conjecture that a sequence of so-called Sobol' points in $[0, 1)^d$ $(d \geq 2)$ forming a $(0, d)$-sequence lifted to the d-sphere $\mathbb{S}^d$ achieves optimal convergence rate of the sum of all pairwise distances which is related to the worst-case error and the spherical cap $\mathbb{L}_2$-discrepancy in the aforementioned sense (see first displayed formula).

Fibonacci lattice rules are the optimal lattice rules for 2-dimensional numerical integration (cf. [11]). Let F_n denote the nth Fibonacci number. Using the Lambert map the *spherical Fibonacci lattice points* with $0 \leq k \leq F_n - 1$ can be defined as

$$\mathbf{z}_k = \left(\alpha_{k,n} \cos\left(2\pi k \frac{F_{n-1}}{F_n}\right), \alpha_{k,n} \sin\left(2\pi k \frac{F_{n-1}}{F_n}\right), 1 - \frac{2k}{F_n}\right)$$

where

$$\alpha_{k,n} := 2\sqrt{\frac{k}{F_n}\left(1 - \frac{k}{F_n}\right)}.$$

The F_n-point set $Z_{F_n} = \{\mathbf{z}_1, \dots, \mathbf{z}_{F_n}\}$ is the *spherical Fibonacci lattice point set.* The smallest pairwise distance $\delta(X_N)$ between points of a node set X_N affects the quality of the QMC rule $Q[X_N]$. For a Fibonacci point set $\mathcal{Z}_{F_n}$ in $[0, 1)^2$ this distance has an explicit formula and we conjecture that $|\mathbf{z}_0 - \mathbf{z}_1| = 2/\sqrt{F_n}$ is the least pairwise distance in a spherical Fibonacci lattice F_n-point set [2]. It is shown in [1] that the spherical cap $\mathbb{L}_\infty$-discrepancy of Z_{F_n} is of order $\mathcal{O}(1/\sqrt{F_n})$. Numerical results suggest $\mathcal{O}((\log F_n)^c/F_n^{3/4})$ for some $1/2 \leq c \leq 1$.

A numerics section concludes the talk: Notable is that spherical nets (based on Sobol' points) and spherical Fibonacci lattice points seem to form QMC design sequences for $\mathbb{H}^{3/2}(\mathbb{S}^2)$. Shifting the spherical Fibonacci points (to avoid the poles of $\mathbb{S}^2$) improves the rate of convergence by a factor $\sqrt{N}$ compared to the unshifted version when worst-case error formulas for $\mathbb{H}^s(\mathbb{S}^2)$ with $s = 3.5$ and 4.5 are used. Integration of the smooth Franke test function gives a similar result.

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Examples and open problems I and II

Henry Cohn

These expository talks covered three of the topics I find most puzzling:

(1) Asymptotic lower bounds for the sphere packing density on $\mathbb{R}^n$, including recent work of Vance and Venkatesh.
(2) Maxwell's double tangent construction and its implications for the Gaussian core model in $\mathbb{R}^3$.
(3) Configurations of n^2 equiangular lines in $\mathbb{C}^n$, i.e. Zanner's conjecture.

An overall theme is the role of symmetry in geometric optimization problems.

On Rankin constants and slope filtration of Euclidean lattices

Renaud Coulangeon

Rankin constants are a variant of the classical Hermite constant in the theory of Euclidean lattices. Whereas the Hermite constant is concerned with the shortest non-zero vectors in a lattice, and thus measures the density of the associated sphere packing, Rankin constant account for the sublattices (of fixed dimension) with minimal covolume. More precisely, if L is a Euclidean lattice, and k is a fixed integer between 1 and $\dim L$, one defines, following Rankin ([8]):

$$\delta_k(L) = \inf_{M \in L(k)} \det M$$

where $L(k)$ stands for the set of k-dimensional sublattices of L, and

$$\gamma_{n,k} = \sup_{\det L=1} \delta_k(L),$$

the supremum over lattices of fixed covolume. When $k = 1$, one recovers the Hermite constant. The geometric interpretation of $\gamma_{n,k}$ for general k is less transparent, although this quantity shows up naturally in a variety of contexts.

In the first part of the talk we briefly explain why, from an algorithmic point of view, the computation of $\gamma_{n,k}$ is essentially more difficult when $k > 1$, because of the lack of a complete *Voronoi theory*. Indeed, when $k = 1$, the computation of the classical Hermite constant can be achieved, at least theoretically, by computing the vertices of a certain polyhedron (the *Ryshkov polyhedron*) in $\operatorname{Sym} \mathbf{R}^n$, as a byproduct of the so-called Voronoi algorithm. For $k > 1$, the same approach is possible, but unfortunately does not yield a polyhedron, but rather a convex set (*Ryshkov generalised cone*) bounded by finitely many algebraic hypersurfaces (see *e.g.* [4]). Nevertheless, we show that it is possible to approximate this Ryshkov cone with an arbitrary precision by polyhedra for which a complete Voronoi theory is available.

The second part of the talk is concerned with the *slope filtration* of Euclidean lattices, first described by Stuhler [9, 10] and Grayson [5]. This construction, inspired by the classical *Harder-Narasinham filtration* of vector bundles over projective curves, seems not to have received the interest it deserves among lattice-theorists. The slope filtration of a Euclidean lattice L can be defined as follows : for every primitive sublattice M of L, one plots in $\mathbf{R}^2$ the point $\ell(M) = (\dim M, \log\det M)$ – by convention, $\ell(0) = (0, 0)$ – and take the convex hull of the resulting set of points. The *profile* of L is the polygonal boundary of this convex hull. It turns out that each vertex of the profile of L corresponds to a unique sublattice of L. Any set of sublattices of L corresponding to vertices of the profile of L form a flag which we call the *canonical* or *slope filtration* of L (see [9, 10, 5] for the justification of these properties, or the very nicely written introductory paper [3]). Obviously, the Rankin invariants of L have a geometric interpretation in terms of the profile of L. In particular, if we set

$$\mu(L) = \min_{1 \le k \le n} (\delta_k L)^{1/k}$$

then the *minimal slope* of the profile is given by

$$\min_{M \subset L} \frac{\log\det M}{\dim M} = \log \min_k (\delta_k L)^{1/k} = \log \mu(L).$$

About 15 years ago, J.-B. Bost proposed a conjecture predicting a very rigid behaviour of this invariant, namely that the equality

$$\mu(L \otimes M) = \mu(L)\mu(M) \tag{1}$$

should hold for any Euclidean lattices L and M. As strange as it may sound –metric invariants of Euclidean lattices are usually *not* well-behaved under tensor product – this conjecture is nevertheless supported by the fact that the corresponding statement for vector bundles on projective curves in characteristic 0 is actually a theorem ([7]). Note that (1) is trivially satisfied if L and M are *unimodular*.

In the talk, we report on recent results about this conjecture ([1, 2, 6]), and put emphasis on the case of *isodual* lattices, which is a natural class to investigate after that of *unimodular* lattices.

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On a characterization theorem for stationary logarithmic configurations

Peter D. Dragnev

For every $\mathbf{x}, \mathbf{y} \in \mathbb{R}^d$ let $\mathbf{x} \cdot \mathbf{y} = x_1 y_1 + \cdots + x_d y_d$ be the inner product and $|\mathbf{x}| = (\mathbf{x} \cdot \mathbf{x})^{1/2}$ the Euclidean distance. Denote the unit sphere with $\mathbb{S}^{d-1} := \{\mathbf{x} \in \mathbb{R}^d : |\mathbf{x}| = 1\}$. For any N-point configuration $\omega_N = \{\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_N\} \subset \mathbb{S}^{d-1}$ the points $\{\mathbf{x}_i\}$ and the segments $\{\mathbf{x}_i\mathbf{x}_j\}_{i \neq j}$ will be called respectively *vertices* and *edges* of the configuration. Throughout $d_{i,j} := |\mathbf{x}_i - \mathbf{x}_j|^2$ will denote the square of the length of the corresponding edge. Here we are interested in configurations of points ω_N^* such that

$$(1) \qquad P(\omega_N^*) = \mathcal{P}(N,d) := \max_{\omega_N \subset \mathbb{S}^{d-1}} P(\omega_N), \quad P(\omega_N) = \prod_{1 \le i < j \le N} d_{i,j}$$

These configurations minimize the logarithmic energy

(2) $$E_{\log}(\omega_N) := \sum_{1\le i<j\le N} \log \frac{1}{|\mathbf{x}_i - \mathbf{x}_j|} = -2\log P(\omega_N),$$

and hence are called *logarithmic configurations*. More generally, the *optimal s-energy configurations* minimize (maximize when $s<0$) the *s-energy*

(3) $$E_s(\omega_N) := \sum_{i<j} \frac{1}{|\mathbf{x}_i - \mathbf{x}_j|^{\alpha}}.$$

The logarithmic configurations are the limiting case of the optimal s-energyl configurations as $s\to 0$. As $s\to\infty$ we arrive at *best packing configurations* (centers of N identical spherical caps with maximal radius packed on the unit sphere). For further reference on various optimal configurations see [3], [5], [7], [8], [9].

The questions of finding logarithmic configurations was posed by Whyte in 1952 [10] (for $d=3$), yet only very few are known. If $d=2$ this is a well studied problem of Fekete points on the circle, and the solution is the regular N-gon. If $d=3$ the solution is known for $N=1-6$, and 12. For $N=1-4$ the solution is trivial (regular simplex of dimension $0-3$). For $N=12$ Andreev [1] showed that the regular icosahedron is an optimal configuration. He used the fact that the configuration is a spherical 5-design, a technique that follows closely the results of [6], where the authors provided a lower bound for $\mathcal{P}(N,d)$, which they showed is attained for the regular d-simplex when $N=d+1$, and the generalized octahedron when $N=2d$. The analysis is based on the fact that these special configurations are suitable spherical designs. However, when the optimal configuration is not a design of sufficiently high degree, then the method fails, as noted by the authors for the particular case when $N=5$ and $d=3$. The solution in this case, given in [4], is a triangular bipyramid, i.e. two points at the Poles and three points forming an equilateral triangle on the Equator.

Our goal here is to characterize the stationary configurations (or "ground states") of the discrete logarithmic energy (2). The following vector conditions are essentially found in [2] for $d=3$.

Proposition *Let* $\omega_N = \{\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_N\}$ *be a stationary logarithmic configuration on the unit sphere* S^{d-1} *in* $\mathbb{R}^d$*. Then the following force conditions hold:*

(4) $$\sum_{j\ne i} \frac{\mathbf{x}_i - \mathbf{x}_j}{d_{i,j}} = \frac{N-1}{2}\mathbf{x}_i \quad i=1,\ldots,N.$$

Moreover, the center of mass of the configuration coincides with the center of the sphere $\mathbf{0}$ *and* $\sum_{j\ne i} d_{i,j} = 2N \quad i=1,\ldots,N$.

In general, for stationary s-energy configurations similar vector equations hold, but the coefficients on the right-hand side of (4) vary with i, which adds significant difficulty.

Definition: Let $\omega_N = \{\mathbf{x}_1, \ldots, \mathbf{x}_N\} \subset \mathbb{S}^{d-1}$. Two vertices $\mathbf{x}_i$ and $\mathbf{x}_j$ are called *mirror related* (we write $\mathbf{x}_i \sim \mathbf{x}_j$), if $d_{i,k} = d_{j,k}$, for every $k\ne i,j$. A configuration is called *degenerate* if the points of the configuration do *not* span the whole $\mathbb{R}^d$.

Note that the mirror relation property defined above is an equivalence relation and that an equivalence class of k vertices forms a regular k-simplex inscribed in a $(k-1)$-dimensional sphere of radius less or equal to 1.

Theorem 1 *Let $N = d + 2$ and let the configuration ω_N be stationary. Then at least one of the following three possibilities occurs:*

(a) *The configuration ω_N is degenerate;*
(b) *There exists a vertex with all edges stemming out being equal;*
(c) *Every vertex is mirror related to another vertex.*

The following strict monotonicity property of $\mathcal{P}(N, d)$ shows that degenerate stationary logarithmic configurations are not optimal when $N \geq d + 1$.

Theorem 2 *For fixed N, the sequence $\mathcal{P}(N, d)$ is strictly increasing for $d < N$ and $\mathcal{P}(N, d) = \mathcal{P}(N, N-1)$ for $d \geq N$.*

We illustrate Theorem 1 with the following classifications of the stationary configurations of $d + 2$ points for $d = 2$, 3, and 4.

Example 1 Let $d = 2$ $(N = 4)$. Then (a) and (b) are impossible,and the only stationary configurations satisfy (c). There could be only two equivallence classes of two points each, which are easily seen to be the diagonals of a square.

Example 2 Let $d = 3$ $(N = 5)$. Now all, (a), (b) and (c) are possible. The only degenerate stationary configuration ω_5^r is the regular pentagon. The only stationary configuration ω_5^p satisfying (b) is the square pyramid with vertex at the North pole and a square base in a horizontal plane of altitude $-1/4$.

If (c) holds, there could be only two equivallence classes, one with two points, a segment, and the other with three points, an equilateral triangle, which we orient horizontally. The two points from the segment have to be equidistant to the vertices of the equilateral triangle, so clearly they are the North and South Poles. The center of mass shows that the equilateral triangle lies on the equator. Comparing the energies we observe that the bipyramid configuration ω_5^b minimizes the energy (2), which is another proof of the result in [4].

We note that numerical evidence shows that the triangular bypyramid configuration ω_5^b is minimizing the s-energy for $s < 15.023...$, while the square pyramid $\omega_5^{p,s}$ is optimal (the base altitude is adjusted with s).

Example 3 Let $d = 4$ $(N = 6)$. Again all, (a), (b) and (c) are possible. The degenerate configuration with minimal logarithmic energy of dimension two is the regular hexagon and dimension three the octahedron (see [6]).

The situation when (c) holds is richer. The various equivallence classes under the mirror relation give rise to the following configurations:

(i) Two orthogonal simplexes, a diameter and regular tetrahedron, with 2 and 4 points respectively;

$$\omega_{\{2,4\}} = \{(0,0,0,\pm 1)\} \cup \left\{ (1,0,0,0), (-\frac{1}{3}, \frac{2\sqrt{2}}{3} \cos \frac{2k\pi}{3}, \frac{2\sqrt{2}}{3} \sin \frac{2k\pi}{3}, 0) \right\}_{k=0}^{2}$$

(ii) Two orthogonal simplexes with 3 points each (equilateral triangles);

$$\omega_{\{3,3\}} = \left\{(\cos\frac{2k\pi}{3}, \sin\frac{2k\pi}{3}, 0, 0)\right\}_{k=0}^{2} \cup \left\{(0, 0, \cos\frac{2k\pi}{3}, \sin\frac{2k\pi}{3})\right\}_{k=0}^{2}.$$

(iii) Three orthogonal simplexes

$$\{(u, \pm\sqrt{1-u^2}, 0, 0)\} \cup \{(v, 0, \pm\sqrt{1-v^2}, 0)\} \cup \{(w, 0, 0, \pm\sqrt{1-w^2})\}, \quad u+v+w=0.$$

Comparing the enrgies we conclude that the optimal logarithmic configuration of six points on $\mathbb{S}^3$ is $\omega_{\{3,3\}}$. This result is new in the literature.

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Signal reconstruction from the magnitude of subspace components

Martin Ehler

(joint work with Christine Bachoc)

Frames have become a powerful tool in signal processing that can offer more flexibility than orthogonal bases, cf. [10, 24, 23] and [12, 13, 14, 15]. Many signal processing problems in engineering such as X-ray crystallography and diffraction imaging require signal reconstruction from the magnitude of its frame coefficients, also known under the term "reconstruction without the phase", cf. [5, 7] and references therein. By making additional assumptions on the underlying frame, exact solutions are presented in [4, 5], see [26, 27] for relations to quantum measurements. Recently, reconstruction from frame coefficients without phase has been numerically addressed via semidefinite programming, see [7, 8].

Frame coefficients can be thought of as projections onto one-dimensional subspaces. In image reconstruction from averaged diffraction patterns by means of incoherent addition of k wavefields [17], the original signal must be recovered from the norms of its k-dimensional subspace components. Notably, the latter is a common problem in crystal twinning [11]. Here, we pose the following questions: Can we reconstruct the original signal from the norms of its k-dimensional subspace components by means of a closed formula? Also, can we reconstruct even if a certain number of norms of subspace components are erased?

We will provide an affirmative answer in the following sense: Given k-dimensional linear subspaces $\{V_j\}_{j=1}^n$ in $\mathbb{R}^d$, we aim to reconstruct the signal $x \in S^{d-1}$ from any subset of $\{\|P_{V_j}(x)\|\}_{j=1}^n$ that has cardinality at least $n-p$. Here, P_{V_j} denotes the orthogonal projector onto V_j. Under suitable conditions on the subspaces, we are able to compute a finite list of candidate signals, one of which is the correct one.

Finding a set of candidate signals is known as list decoding and was introduced in [16]. We determine this list in a two-step procedure, both steps are related to codes, designs, and cubatures in Grassmann spaces [1, 2]. Without loss of generality, let us suppose that the first p norms were erased. If there are positive weights $\{\omega_j\}_{j=1}^n$ such that $\{(V_j,\omega_j)\}_{j=1}^n$ forms a tight p-fusion frame as recently introduced in [3], then we are able to reconstruct the erased norms $\{\|P_{V_j}(x)\|\}_{j=1}^p$ at least up to permutations. Here, we needed to verify that a certain system of algebraic equations has only finitely many solutions. Notice that our input are not the subspace components but their norms, as opposed to signal reconstruction under the erasures discussed in [6, 9, 22, 25].

In the second step, we assume that $\{(V_j,\omega_j)\}_{j=1}^n$ yields a cubature formula of strength 4, see [3], enabling us to reconstruct the orthogonal projection onto $x\mathbb{R}$ from knowledge of $\{\|P_{V_j}(x)\|\}_{j=1}^n$. Indeed, if $x \in S^{d-1}$, then

$$P_{x\mathbb{R}} = \frac{1}{\alpha}\sum_{j=1}^n \omega_j \|P_{V_j}(x)\|^2 P_{V_j} - \frac{\beta}{\alpha}\,\mathrm{Id}, \tag{1}$$

where $\alpha = \frac{2k(d-k)}{d(d+2)(d-1)}$ and $\beta = \frac{k(kd+k-2)}{d(d+2)(d-1)}$. This extends the one-dimensional results in [4] to k-dimensional projections addressed in the present paper. Note that the authors in [4] require cubature formulas for the projective space whose weights are $\omega_j = 1/n$, i.e., so-called projective designs. In practice, however, the choice of subspaces may underlie some restrictions that prevent them from being a design, but there is still the chance to compute weights enabling cubature formulas, cf. [18, 19, 21]. Thus, our results in Step 2 are a significant improvement for one-dimensional projections already. We also extend the above concepts to the complex setting, in which some constants need adjustments.

The cardinality of a cubature formula of strength 4 scales at least quadratically with the ambient dimension d, cf. [20]. For $k=1$, it is known that semidefinite programming yields signal recovery with high probability when the one-dimensional subspaces are chosen at random [8]. The cardinality of the subspaces can then scale linearly in the ambient dimension up to a logarithmic term. Here, we shall extend the latter to general k-dimensional subspaces chosen at random.

Let $\mathcal{H}$ denote the collection of symmetric matrices in $\mathbb{R}^{d\times d}$. For $\{V_j\}_{j=1}^n \subset \mathcal{G}_{k,d}$, we define the operator

$$\mathcal{F}_n : \mathcal{H} \to \mathbb{R}^n, \quad X \mapsto \frac{d}{k}(\langle X, P_{V_j}\rangle)_{j=1}^n. \tag{2}$$

If $f := \frac{d}{k}(\|P_{V_j}(x)\|^2)_{j=1}^n = \mathcal{F}_n(P_{x\mathbb{R}}) \in \mathbb{R}^n$ and assuming $f \neq 0$, then $P_{x\mathbb{R}}$ solves

$$\min_{X\in\mathcal{H}}(\operatorname{rank}(X)), \quad \text{subject to} \quad \mathcal{F}_n(X) = f,\ X \succeq 0. \tag{3}$$

The notation $X \succeq 0$ stands for X being symmetric and positive semidefinite. Rank minimization is in general NP-hard. Therefore, (3) is commonly replaced with

$$\min_{X\in\mathcal{H}}(\operatorname{trace}(X)), \quad \text{subject to} \quad \mathcal{F}_n(X) = f,\ X \succeq 0. \tag{4}$$

If $\{V_j\}_{j=1}^n$ are chosen at random, then we shall verify that, with high probability, $P_{x\mathbb{R}}$ is the unique solution to (4). Our proof is guided by the approach in [8] but some steps are more involved and need to be taken care of with different tools in this more general setting. For instance, the case $k = 1$ relies on i.i.d. Gaussian random matrices modeling the measurements. For $k > 1$, we must deal with measurement matrices involving rank-k orthogonal projectors that clearly have dependent entries. Hence, the extension from $k = 1$ to $k > 1$ is not obvious and requires special care. Later, we also verify numerically that semidefinite programming enables us to recover a signal from the norms of its random k-dimensional subspace components.

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Lattices and Designs

Noam Elkies

This two-part exposition introduces several variations on the theta functions of Euclidean lattices that can yield precise results on the distribution of lattice vectors.

In the first lecture we introduce Poisson summation and harmonic polynomials; we then use them to define harmonic theta functions, and outline how these functions' modularity leads to asymptotic equidistribution results on the spherical shells of rational lattice translates.

In the second lecture we introduce spherical designs, and show how in special cases the asymptotic results of the first lecture improve to exact formulas with diverse applications.

Asymptotics of Minimal Energy and Maximal Polarization Configurations I

Douglas P. Hardin and Edward B. Saff

Let $A \subset \mathbb{R}^p$ be a compact set and $K(\cdot,\cdot) : A \times A \to (-\infty, \infty]$ be a symmetric lower semi-continuous kernel on A. Then with each set of N points $\omega_N = \{x_1, \ldots, x_N\} \subset A$ we can associate a K-energy; namely

$$E_K(\omega_N) := \sum_{1 \le i \ne j \le N} K(x_i, x_j) = \sum_{i=1}^{N} \sum_{\substack{j=1 \\ j \ne i}}^{N} K(x_i, x_j). \tag{1}$$

We denote by $\mathcal{E}_K(A, N)$ the minimum K-energy taken over all sets of N points of A; that is,

$$\mathcal{E}_K(A, N) := \min\{E_K(\omega_N) : \omega_N \subset A, \, |\omega_N| = N\}, \tag{2}$$

where $|X|$ denotes the cardinality of the set X. A point set ω_N that attains this minimum is denoted by ω_N^*, and is referred to as an *optimal N-point K-energy configuration.*

Of particular interest in classical potential theory is the class of *Riesz kernels*

$$K_s(x, y) := \begin{cases} |x - y|^{-s}, & \text{for } s > 0, \\ \log |x - y|^{-1}, & \text{for } s = \log, \end{cases} \tag{3}$$

where $|\cdot|$ denotes Euclidean distance and $s > 0$ (or $s = \log$) is a fixed parameter. For such kernels we write E_s in place of E_{K_s} and $\mathcal{E}_s$ in place of $\mathcal{E}_{K_s}$. Optimal and near optimal Riesz s-energy configurations are of interest for a variety of reasons. For example, in the case when $A = S^d$, the d-dimensional unit sphere in $\mathbb{R}^{d+1}$, optimal Riesz s-energy configurations are, for fixed $s > 0$, and $s = \log$, asymptotically (as $N \to \infty$) uniformly distributed with respect to surface area (more precisely d-dimensional Hausdorff measure on S^d). This fact together with properties of "well-separation" and "small fill radius" make some of these point set sequences particularly attractive for such purposes as interpolation, cubature, and CAGD.

For $s = 1$ and $A = S^2$, finding optimal configurations is the classical *Thomson problem* which has been studied for over 100 years. Other examples of physical interest include the case $s = 3$ which corresponds to a dipole interaction, appropriate for neutral colloids at the interface between two liquids. The case $s = 12$ is the repulsive part of the Lennard-Jones potential and is the important piece of the interaction for driving crystallization.

It is important to note that as $s \to \infty$, with N fixed, the s-energy $E_s(\omega_N)$ is increasingly dominated by the term(s) involving the smallest of pairwise distances and, in this sense, leads to the *best-packing problem* on A.

If the Riesz parameter s is less than the Hausdorff dimension d of the compact set A, then the limiting distribution of minimal s-energy configurations as $N \to \infty$ is characterized from classical potential theory as the equilibrium measure for the corresponding continuous energy problem for integrals. Even in this case, the support of the equilibrium measure may be a proper subset of A that is not easily determined. For the case of a surface of revolution in $\mathbb{R}^3$, e.g. a torus, the equilibrium measure for $s = \log$ will have support contained in the 'radially outer most' portion of the surface [4]. When the Riesz parameter $s \geq d$, classical potential theory does not apply; however we have shown that the asymptotic distribution of minimal energy points is uniform as described in the following theorem for d-rectifiable sets, i.e., Lipschitz images of bounded sets in $\mathbb{R}^d$.

Theorem 1 (Poppy-Seed Bagel Theorem, [3])**.** *Let $s > d$ and $p \geq d$, where d and p are integers. For every infinite compact d-rectifiable set $A \subset \mathbb{R}^p$, we have*

$$\lim_{N\to\infty} \frac{\mathcal{E}_s(A,N)}{N^{1+s/d}} = \frac{C_{s,d}}{\mathcal{H}_d(A)^{s/d}},$$

where $C_{s,d}$ is a positive and finite constant independent of A and $\mathcal{H}_d$ denotes d-dimensional Hausdorff measure on $\mathbb{R}^p$.

Moreover, if A is d-rectifiable with $\mathcal{H}_d(A) > 0$, then any sequence $\{\omega_N^*\}_{N=2}^\infty$ of s-energy minimizing configurations on A such that $\#\omega_N = N$ is asymptotically uniformly distributed on A with respect to $\mathcal{H}_d$, i.e.

$$\frac{1}{N}\sum_{x\in\omega_N^*} \delta_x \xrightarrow{\ *\ } \frac{\mathcal{H}_d(\cdot)}{\mathcal{H}_d(A)}, \quad N \to \infty. \tag{4}$$

For the purpose of generating point configurations that are not uniform, but rather approximate a given distribution, we introduce a weighted energy. Moreover, by including a cutoff function in the weight, we obtain a truncated energy expression of low computational complexity. Before stating this result, we introduce some needed notation.

For a collection of $N(\geq 2)$ distinct points $\omega_N := \{x_1, \ldots, x_N\} \subset A$, a nonnegative symmetric weight function w on $A \times A$, and $s > 0$, the *weighted Riesz s-energy of ω_N* is defined by

$$E_s^w(\omega_N) := \sum_{1\leq i\neq j\leq N} \frac{w(x_i,x_j)}{|x_i - x_j|^s} = \sum_{i=1}^N \sum_{\substack{j=1\\ j\neq i}}^N \frac{w(x_i,x_j)}{|x_i-x_j|^s}, \tag{5}$$

while the *N-point weighted Riesz s-energy of A* is defined by

$$\mathcal{E}_s^w(A,N) := \inf\{E_s^w(\omega_N) : \omega_N \subset A, |\omega_N| = N\}. \tag{6}$$

We call $w : A \times A \to [0,\infty]$ a *CPD-weight function on $A \times A$* if

(a) w is continuous (as a function on $A \times A$) at $\mathcal{H}_d$-almost every point of the diagonal $D(A) := \{(x,x) : x \in A\}$,
(b) there is some neighborhood G of $D(A)$ (relative to $A \times A$) such that $\inf_G w > 0$, and
(c) w is bounded on any closed subset $B \subset A \times A$ such that $B \cap D(A) = \emptyset$.

If A is a compact set in $\mathbb{R}^p$ and w is a CPD-weight function on $A \times A$, then for $s \geq d$ we define the *weighted Hausdorff measure* $\mathcal{H}_d^{s,w}$ on Borel sets $B \subset A$ by

$$\mathcal{H}_d^{s,w}(B) := \int_B (w(x,x))^{-d/s} \mathrm{d}\mathcal{H}_d(x). \tag{7}$$

Finally, we say that a sequence $\{\omega_N\}_{N=1}^{\infty}$ of N-point configurations in A is *asymptotically (w, s)-energy minimizing for A* if

$$E_s^w(\omega_N)/\mathcal{E}_s^w(A, N) \to 1 \text{ as } N \to \infty. \tag{8}$$

For Riesz s-energy we prove in [1] the following version of the poppy-seed bagel theorem for d-rectifiable sets (also see [2]).

Theorem 2. *Let $A \subset \mathbb{R}^p$ be as in Theorem 1 and let w be a CPD-weight function on $A \times A$. Suppose $\Phi : (0, \infty) \to [0, 1]$ is such that* (a) $\lim_{t \to 0^+} \Phi(t) = 1$ *and* (b) *$\Phi(t) = 0$, $t > 1$, and $\{r_N\}_{N \in \mathbb{N}}$ is a sequence of positive numbers such that*

$$\lim_{N \to \infty} r_N N^{1/d} = \infty. \tag{9}$$

For $N \in \mathbb{N}$, let

$$v_N(x, y) := \Phi\left(\frac{|x - y|}{r_N}\right) w(x, y), \quad x, y \in A, \quad x \neq y. \tag{10}$$

If $s > d$, then

$$\lim_{N \to \infty} \frac{\mathcal{E}_s^{v_N}(A, N)}{N^{1+s/d}} = \frac{C_{s,d}}{\mathcal{H}_d^{s,w}(A)^{s/d}}, \tag{11}$$

where the constant $C_{s,d}$ is as in Theorem 1. Furthermore, any sequence of asymptotically $(\{v_N\}, s)$-energy minimizing configurations on A is uniformly distributed with respect to $\mathcal{H}_d^{s,w}$ as $N \to \infty$.

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Asymptotics of Minimal Energy and Maximal Polarization Configurations II

Douglas P. Hardin and Edward B. Saff

1. Minimum Energy, continued

Using a truncated energy as in Theorem 2 of part I of this report with constant weight and cutoff function $\Phi(t) = (1 - t^2)^2$, $s = 3.5$, and r_N chosen proportional to $N^{-1/2}$ for $N = 30,000$, we obtain the following 'low energy' configuration on S^2. The figure shows the associated Voronoi decomposition. The majority of the Voronoi cells are nearly congruent hexagons, while the 'scars' or 'dislocations' form 'grain boundaries' composed of alternating pentagons and heptagons.

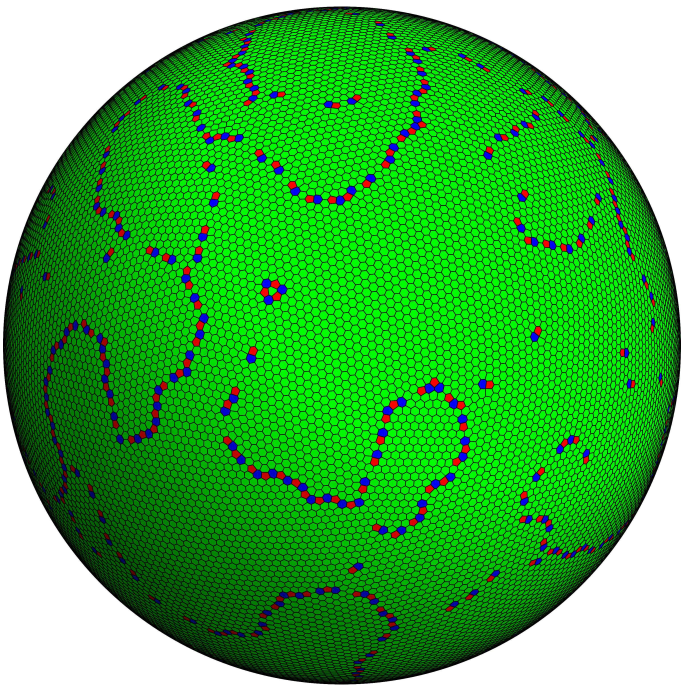

Figure 1. A near optimal configuration of 30, 000 points on the sphere S^2 for $s = 3.5$.

2. Maximum Polarization

For $N \in \mathbb{N}$, let $\omega_N = \{\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_N\}$ denote N (not necessarily distinct) points in p-dimensional Euclidean space $\mathbb{R}^p$. We define for $s > 0$ and a compact set $A \subset \mathbb{R}^p$, the *polarization quantities*

$$(1) \qquad M^s(\omega_N, A) := \min_{\mathbf{x}\in A} \sum_{j=1}^{N} \frac{1}{|\mathbf{x}-\mathbf{x}_j|^s}\,, \quad M_N^s(A) := \max_{\omega_N \subset A} M^s(\omega_N, A).$$

Such max-min quantities for potentials were first introduced by M. Ohtsuka who explored (for very general kernels) their relationship to various definitions of capacity that arise in electrostatics (see [4]). In particular, he showed that for any compact set $A \subset \mathbb{R}^p$, the following limit, called the *Chebyshev constant* of A, exists as an extended real number:

$$(2) \qquad \mathcal{M}^s(A) := \lim_{N\to\infty} \frac{M_N^s(A)}{N}.$$

Moreover, he showed that $\mathcal{M}^s(A)$ is not smaller than the Wiener constant $W_s(A)$ for A. We are primarily interested in the case when the limit (2) is infinite and the set A is the unit sphere or the unit ball.

Just as minimum energy problems are related to best-packing, maximum polarization configurations are related to best-covering problems as $s \to \infty$ as we now describe. The *fill (covering) radius* of $\omega_N \in A^N$ is given by

$$\rho(\omega_N; A) := \max_{y\in A} \min_{x\in\omega_N} |y-x|.$$

Proposition 1. *For each fixed N,*

$$\lim_{s\to\infty} M_N^s(A)^{1/s} = \frac{1}{\rho_N(A)},$$

*where $\rho_N(A)$ is the N**-point fill (covering) radius of** A :*

$$\rho_N(A) := \inf\{\rho(\omega_N; A) : \omega_N \in A^N\}.$$

Furthermore, every cluster point of optimal $N-$point polarization configurations ω_N^s is an $N-$point best covering configuration for A as $s \to \infty$.

In [1], the authors conjecture that for the unit circle $\mathbb{S}^1$ and every $s > 0$ and $N \geq 1$, the maximum polarization is attained for equally spaced points. The case $s = 2$ was first proved by Ambrus [1] and the case $s = 4$ by Erdélyi and Saff [2]. We announce that this conjecture is now proved.

Theorem 3 ([3])**.** *Let $f : [0,\pi] \to [0,\infty]$ be non-increasing and strictly convex. For a configuration $\omega_N = (z_1, ..., z_N)$ on $\mathbb{S}^1$, set*

$$M^f(\omega_N; \mathbb{S}^1) := \min_{z\in\mathbb{S}^1} \sum_{k=1}^{n} f(d(z, z_k)),$$

$$M_N^f(\mathbb{S}^1) := \max\{M^f(\omega_N; \mathbb{S}^1) : \omega_N \in (\mathbb{S}^1)^N\},$$

where $d(z,w)$ denotes geodesic distance on $\mathbb{S}^1$. Then $M^f(\omega_N; \mathbb{S}^1) = M_N^f(\mathbb{S}^1)$ if and only if ω_N consists of N distinct points equally spaced on $\mathbb{S}^1$.

The research team at Vanderbilt is in the process of establishing analogs of minimum energy results on d-rectifiable sets for maximum polarization. In particular, progress is being made on the following conjectured analog of the poppy-seed bagel theorem.

Conjecture 1. *Let $s > d$ and $p \geq d$, where d and p are integers. For every infinite compact d-rectifiable set $A \subset \mathbb{R}^p$, we have*

$$\lim_{N\to\infty} \frac{M_N^s(A)}{N^{s/d}} = \frac{\sigma_{s,d}}{\mathcal{H}_d(A)^{s/d}},$$

where $\sigma_{s,d}$ is a positive and finite constant independent of A and $\mathcal{H}_d$ denotes d-dimensional Hausdorff measure on $\mathbb{R}^p$.

Moreover, if A is d-rectifiable with $\mathcal{H}_d(A) > 0$, then any sequence $\{\omega_N^*\}_{N=2}^\infty$ of s-polarization maximizing configurations on A is asymptotically uniformly distributed on A with respect to $\mathcal{H}_d$.

In particular, Theorem 3 implies that the constant $\sigma_{s,1}$ appearing in Conjecture 1 would have to equal $2(2^s-1)\zeta(s)$, where $\zeta(s)$ denotes the classical *Riemann zeta function.*

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An Application of Weighted Theta Functions to t-core Partitions and Numerical Semigroups

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(joint work with Noam Elkies)

We first present a problem about the set of positive integers represented by a certain quadratic form subject to some additional constraints. This particular problem is motivated by an application to combinatorics, which we explain below. Let

$$Q_5(x_1,x_2,x_3,x_4) = 2\left(\sum_{j=1}^4 x_j^2\right) - \sum_{1\leq j<k\leq 4} x_j x_k.$$

This quadratic form is associated to a lattice, in this case a scaled copy of A_4^*. We do not consider integer inputs but (x_1,x_2,x_3,x_4) satisfying $x_j \in \mathbb{Z} + \frac{j}{5}$ for

$1 \leq j \leq 4$. Knowing the set of integers represented by this form subject to this condition is equivalent to knowing the possible norms of vectors in a particular translate of this lattice. Standard results in the theory of quadratic forms in four or more variables imply that the vectors of norm N in this lattice translate asymptotically become equidistributed. For our particular application these types of asymptotic results are not sufficient. We want to determine exactly the set of positive integers represented by $Q_5(x_1, x_2, x_3, x_4)$ subject to the above constraint and also satisfying $\max_{j \neq k} \frac{x_j}{x_k} \leq 2$. Our main result is the following.

Theorem 4 (Elkies, K.)**.** *For each $n \geq 1$, $n \notin \{1, 2, 3, 6\}$, there exist (x_1, x_2, x_3, x_4) satisfying $x_j \in \mathbb{Z} + \frac{j}{5}$, $\max_{j \neq k} \frac{x_j}{x_k} \leq 2$, and $Q_5(x_1, x_2, x_3, x_4) = n$.*

This result has consequences in the study of t-core partitions and numerical semigroups. A *partition* λ of n is nonincreasing list of positive integers, $\lambda_k \geq \lambda_{k-1} \geq \cdots \geq \lambda_1 \geq 1$, that sums to n. We can represent a partition by its *Young diagram*, an array of k left-justified rows of boxes with λ_k boxes in the first row, λ_{k-1} boxes in the second row, and so on, down to λ_1 boxes in the last row. Each box in the Young diagram has a *hook length*, the number of boxes in the hook attached to this box. Hook lengths play an important role in the correspondence between irreducible representations of S_n and partitions of n. The *hook set* of a partition is the set of hook lengths of boxes of the Young diagram. A *t-core partition* is a partition none of whose hook lengths is divisible by t. We recall the t-core theorem of Granville and Ono [2].

Theorem 5 (Granville-Ono, 1996)**.** *Fix $t \geq 4$. For any $n \geq 1$, there exists a t-core partition of n.*

The proof of the t-core theorem involves studying the generating function for t-cores as a modular form and carefully considering the coefficients of its q-expansion. We will use similar modular forms methods to strengthen the Granville-Ono theorem in the case $t = 5$.

A *numerical semigroup* S is an additive submonoid of $\{0, 1, 2, \ldots, \}$ with finite complement. The size of this complement is known as the *genus* of S, denoted $g(S)$, terminology which comes from the theory of Weierstrass semigroups on algebraic curves. The *weight* of S, denoted $w(S)$, is the sum of the elements of $\mathbb{N} \setminus S$ minus $g(S)(g(S)+1)/2$. The connection between numerical semigroups and t-core partitions is summarized in the following propositions, the second of which builds on work of Bras-Amorós and de Mier [1].

Proposition 2. *The hook set of a partition is the complement of a numerical semigroup.*

Proposition 3. *Given a numerical semigroup S there is a unique partition $\lambda(S)$ with hook set equal to $\mathbb{N} \setminus S$ and satisfying the additional property that for any integer j in the hook set there is a box in the first column of the Young diagram of λ with hook length equal to j.*
Suppose m is the smallest nonzero element of S. The $\lambda(S)$ is an m-core partition but not a t-core for any $t < m$ and the size of $\lambda(S)$ is $w(S) + g(S)$.

We can express $w(S)+g(S)$ in terms of a particular generating set of S known as the Apéry set. If S has smallest nonzero element n, then $w(S)+g(S)$ is equal to an inhomogeneous quadratic form in $m-1$ variables evaluated at inputs coming from the Apéry set of S. A simple change of variables gives the following quadratic form:

$$Q_m(x_1, x_2, \ldots, x_{m-1}) := \frac{m-1}{2}\left(\sum_{j=1}^{m-1} x_j^2\right) - \sum_{1\le j<k\le m-1} x_j x_k.$$

We want to determine the values represented by this form when our inputs come from the Apéry set of a semigroup. A sufficient condition is that $\max_{j\neq k} \frac{x_j}{x_k} \le 2$. Setting $m=5$ gives the problem from the beginning of the abstract. Our main result implies the following.

Corollary 1. *For each $n \ge 6$ there exists a numerical semigroup S with multiplicity 5 and $w(S)+g(S)=n$.*
For each $n \ge 6$ there exists a 5-core partition λ of n which is not a t-core partition for any $t<5$ with the additional property that for any $j \in H(\lambda)$ there exists a box in the first column of the Young diagram of λ with hook number equal to j.

We describe some of the ingredients of the proof below. In order to impose the condition that (x_1, x_2, x_3, x_4) satisfies $\max_{j\neq k} \frac{x_j}{x_k} \le 2$ we note that this vector is in this cone if and only if the absolute value of the cosine of the angle defined by this vector and $(1,1,1,1)$ is large. We treat this scaled inner product as a variable and find a polynomial in this variable which is negative outside of the cone and has positive average on the unit sphere. Therefore, if we show that this average of this polynomial taken over all vectors of norm N is positive, then there exists a vector of norm N in our cone.

We express this polynomial as a linear combination of Chebyshev polynomials of the second kind, which lead to closely related harmonic polynomials. These harmonic polynomials give weighted theta functions. In this setting, the theta function weighted by the harmonic polynomial of degree $2n$ is a modular form of level 5, weight $2+2n$, and character χ, the Legendre character mod 5.

One of these weighted theta functions is the Eisenstein series for our space and the other nonzero ones are cusp forms. We can determine the q-expansion of the Eisenstein series explicitly and note that the q^n coefficient is $\gg n^{1-\epsilon}$. We use a theorem of Deligne to bound the contribution from the cusp forms. Combining these bounds with the expansion described above shows that the average of this polynomial taken over vectors of norm N must be positive if N is squarefree and either N has 9 or more prime factors, or N has a prime factor at least 5471. We now have a large but finite list of squarefree N to consider.

We generate a list of candidates and for each one determine the relevant cusp form coefficients and the corresponding term of the overall q-expansion. This completes our analysis for squarefree N. It is not difficult to extend these ideas to general N. We show that the only N for which the average of the polynomial taken over vectors of norm N is negative are $\{1,2,3,4,6,8,10,14\}$. We can give explicit representatives of vectors of norm N in our cone for some of these values, completing the proof of the main theorem.

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On some kissing numbers in low dimensions

Abhinav Kumar

(joint work with Henry Cohn, Yang Jiao, Salvatore Torquato)

The *kissing number* problem in dimension n asks for the maximum size of a configuration of non-overlapping unit spheres in $\mathbb{R}^n$ which touch a central unit sphere. It is equivalent to the question of finding the maximum size of a spherical code of minimal angle $\pi/3$. The kissing numbers are exactly known only in dimensions 1 through 4 [18, 15], 8 and 24 [16, 14]. In other low dimensions, we have upper bounds from linear programming [9, 11] or semidefinite programming [1], and lower bounds from explicitly known kissing arrangements, often arising from dense sphere packings. Note that the kissing configurations of size 240 and 196560 in $\mathbb{R}^8$ and $\mathbb{R}^{24}$ respectively are unique [2], coming from the E_8 and Leech lattices. The kissing configuration of 24 points in $\mathbb{R}^4$ is expected to be unique, but this question is open.

In our recent work [3], we described an algorithm to test whether a spherical code is infinitesimally rigid, motivated by studies of rigidity of sphere packings [19, 10]. We say that a spherical code is *rigid* or *jammed* if the only continuous motions of its points which do not decrease its minimal distance are induced by continuous rotations (i.e. paths in $SO(n)$ starting at the identity). Similarly, there is a notion of being infinitesimally rigid, which says that any infinitesimal deformation is an infinitesimal rotation. If $\mathcal{C} = \{x_1, \ldots, x_N\} \subset S^{n-1}$ is a spherical code, and the collection of vectors $y_1, \ldots, y_N$ describes the first order change in the positions, then we must have $\langle x_i, y_i \rangle = 0$ to stay on the sphere (to first order). In addition, in order that the minimal angle of the code does not decrease to first order, one needs that for every pair (i, j) of distinct points at the smallest distance, $\langle x_i, y_j \rangle + \langle x_j, y_i \rangle \leq 0$. We say that the collection of y_i is an infinitesimal deformation if these conditions are satisfied. On the other hand, an infinitesimal rotation is one for which $y_i = \Phi x_i$ for some skew-symmetric matrix Φ (recall that the Lie algebra of $SO(n)$ is the space of skew-symmetric matrices). It follows from the work of Connelly, Roth and Whiteley [5, 17] that an infitesimally rigid spherical code is rigid. We do not know whether the converse is true or false, if the code is assumed to span the ambient space (a harmless restriction). In any case, the considerations above lead almost immediately to a linear programming algorithm to check if a spherical code is rigid, once we establish the following easy fact: for a spherical code which spans the ambient space, an infinitesimal deformation is an infinitesimal rotation if and only if no distances change to first order, i.e., $\langle x_i, y_j \rangle + \langle x_j, y_i \rangle = 0$ for *all pairs* (i, j). Since this objective function is linear in the variables (the coordinates of the y_i) and so are the constraints, one merely has to run $\binom{N}{2}$ linear programs to check if a code is infinitesimally rigid. The presence of symmetry may simplify often simplifies matters, since one may then restrict to checking one pair of vectors in each orbit of $\mathrm{Aut}(\mathcal{C})$ on $\mathcal{C} \times \mathcal{C}$.

We implemented this algorithm and ran it on the best kissing configurations known in low dimensions. We were then able to rigorously verify our results, by showing (via more conceptual arguments) that the spherical codes in question were rigid or non-rigid. Here, we briefly report on these results.

In dimension 2, the best kissing arrangement comes from the vertices of the regular hexagon, and it is obviously rigid, for instance by a simple argument with angles. In dimension 3, one may take the vertices of a rhombic dodecahedron (the kissing configuration of the face-centered cubic lattice). This is well-known to be non-rigid, and in fact there is so much space that one can perform any permutation of the twelve points through continuous deformations, without decreasing the mininal distance [8, Appendix to Chapter 1]. In dimension 4, the root system of the D_4 lattice gives an optimal kissing configuration of 24 points. We showed that it is (infinitesimally) rigid, by first using an embedding argument from the A_2 root system to see that the inner products of $\pm 1/2$ cannot change to first order, and then using a bit of algebra to see that the inner product of 0 cannot change to first order, either. The same embedding argument shows that the root systems of D_n (for $n \geq 5$) and of E_6, E_7, E_8 are also rigid.

In dimension 5, we showed that both the best known kissing configurations of 40 points, coming from the D_5 lattice and its non-lattice competitor [12, 7] are rigid. Similarly, in dimensions 6, we showed that the kissing configurations of E_6 and the three other uniform "tight" packings [13, 7] are rigid. In dimension 7, again we have four competitors for the current record, two of which are the E_6 root system and a uniform non-lattice packing [7, 4]. The other two do not occur in any of the tight packings. All these kissing configurations are rigid.

In dimension 8, the unique solution to the kissing problem is obviously rigid. In dimensions 9 through 12, the best kissing configurations known are obtained from certain constant weight binary codes through Construction A [8, Chapter 5], and are all rigid. However, the best lattice kissing configuration in $\mathbb{R}^9$ is obtained from the laminated lattice Λ_9 [6]. It consists of the E_8 kissing configuration of 240 points on the "equator", with cross-polytopes of 16 points above and below. This code contains the kissing configuration of D_9 (consisting of 144 points), which we have shown is rigid. However, the remaining 128 points are not even locally rigid, and in fact we can move half of them simultaneously into the northern hemisphere to deform the code. An adaptation of this construction is lets us improve the kissing numbers in dimensions 25 through 31, which we describe next.

For concreteness, we describe dimension 25. The previously best kissing configuration known was that of the laminated lattice Λ_{25}, consisting of $196560 + 96$

points, with a similar structure as in 9 dimensions: the Leech lattice kissing configuration $\mathcal{C}_{24}$ on the equator, and a cross-polytope above and below. To improve upon it, we first discard the 96 points, and then proceed to find a subset S of $\mathcal{C}_{24}$ of size 480 such that no pair of points of S is at the minimal distance. We then remove S from the equatorial hyperplane, and place a copy at an appropriate latitude in the northern hemisphere, as well as one in the southern hemisphere. The new kissing configuration has size $196560 + |S|$, which beats the previous record. Similar constructions, utilizing good kissing configurations in dimension 2 through 7, work in dimensions $24 + 2$ through $24 + 7$.

Finally, we mention two further interesting observations from our work. In dimension 12, the best lattice kissing configuration is that of the Coxeter-Todd lattice, with 756 points. We showed that it is not rigid, by finding an interesting family of deformations: one can decompose the code into 126 hexagons (using the Eisenstein structure, for instance) and then independently rotate each hexagon through a different angle! In dimension 16, the best kissing configuration known is that of the Barnes-Wall lattice, which is also the densest packing in that dimension. We showed that this 4320-point spherical code is rigid. However, we have not checked the large number of competitors [7], which may not be rigid and may offer a possibility of improving the kissing number in $\mathbb{R}^{16}$.

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Polydisperse spherical cap packings

David de Laat

(joint work with Fernando Mário de Oliveira Filho and Frank Vallentin)

Given a finite set $\{\alpha_1, \dots, \alpha_N\}$ of angles, we give upper bounds on the maximal density of spherical cap packings using these angles, where the spherical cap with center $x \in S^{n-1}$ and angle α is defined by $C(x, \alpha) = \{y \in S^{n-1} \colon x \cdot y \geq \cos \alpha\}$. To do this we define a graph G with vertex set $V = S^{n-1} \times \{1, \dots, N\}$ where vertices (x, i) and (y, j) are adjacent if $\cos(\alpha_i + \alpha_j) < x \cdot y$ and $(x, i) \neq (y, j)$. We define the weight of a vertex to be the normalized area of the corresponding cap. The independent sets in this graph correspond precisely to the spherical cap packings, and the weight of an independent set gives the cap packing density.

Determining the independence number of a finite graph is NP-hard, but in many cases the ϑ number, which can be computed in polynomial time, gives good upper bounds. When we generalize the weighted theta number of Grötschel, Lovász, and Schrijver [3] to infinite graphs we obtain

$$\begin{aligned} \vartheta'_w(G) = \inf M \colon\ & K - \sqrt{w} \otimes \sqrt{w} \in \mathcal{C}(V \times V)_{\succeq 0}, \\ & K(u,u) \leq M \text{ for all } u \in V, \\ & K(u,v) \leq 0 \text{ for all } \{u,v\} \notin E \text{ where } u \neq v, \end{aligned}$$

where $\mathcal{C}(V \times V)_{\succeq 0}$ denotes the cone of positive semidefinite kernels.

Any feasible solution of this program gives an upper bound on the packing density, but to find good feasible solutions by computer we reformulate the problem as a semidefinite program with finitely many variables and constraints. This is done using the symmetry of G expressed by the group action

$$O(n) \times V \to V,\ A(x,i) = (Ax, i).$$

It follows that in the above program we can replace $\mathcal{C}(V \times V)_{\succeq 0}$ by the smaller cone $\mathcal{C}(V \times V)_{\succeq 0}^{O(n)}$ of $O(n)$-invariant positive kernels. Using an explicit characterization

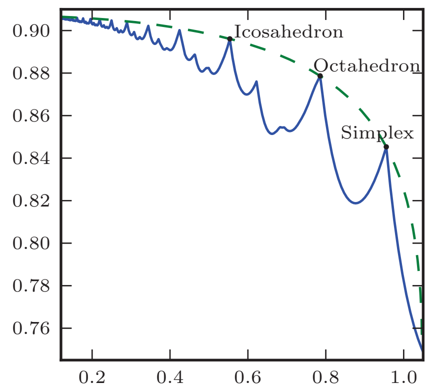

FIGURE 1. Upper bounds on the packing density for $N = 1$ and $n = 3$. The solid line is the semidefinite programming bound; the dashed line is the geometric bound.

of the kernels in this cone, we can reformulate the above program as

$$\begin{aligned} \vartheta'_w(G) = \inf M\colon & \big(f_{ij,0} - w(\alpha_i)^{1/2} w(\alpha_j)^{1/2}\big)_{i,j=1}^N \succeq 0, \\ & \big(f_{ij,k}\big)_{i,j=1}^N \succeq 0 \text{ for } k \geq 1, \\ & f_{ij}(u) \leq 0 \text{ whenever } -1 \leq u \leq \cos(\alpha_i + \alpha_j), \\ & M \geq f_{ii}(1) \text{ for } 1 \leq i \leq N, \end{aligned}$$

where

$$f_{ij}(u) = \sum_{k=0}^{\infty} f_{ij,k} P_k^n(u)$$

and where P_k^n denotes the Jacobi polynomial $P_k^{((n-3)/2,(n-3)/2)}$ of degree k, normalized so that $P_k^n(1) = 1$. By restricting the degree of the polynomials f_{ij} in this program, and by using sums of squares characterizations, we can reformulate the above as a semidefinite program with finitely many variables and constraints.

This semidefinite programming bound reduces to the Delsarte, Goethals, and Seidel [1] bound when $N = 1$. Figure 1 shows a plot of this bound for $n = 3$. The other bound in this plot is a geometric bound due to Florian [2]. We know that the bounds meet at the three given configurations, and the numerical solution suggest that the bounds in fact meet infinitely often as the angle decreases. An interesting feature of the semidefinite programming bound is that it has a similar shape between any two of these meeting points.

Figure 2 shows the bound for $N = 2$ and $n = 3$. When we compare this bound with the geometric bound we see that it depends on the specific combination of angles which bound is sharper. One specific set of angles where the semidefinite bound is sharper – and where the bound is in fact tight – is those that occur in the configuration related to the 5-prism. Here we project a 5-prism whose vertices

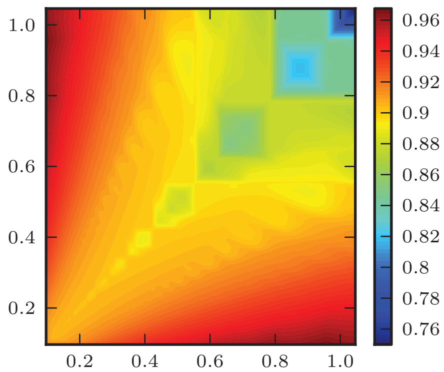

FIGURE 2. Upper bounds on the packing density for $N = 2$ and $n = 3$. The horizontal and vertical axes carry the spherical cap angle; the colors indicate the density.

lie on the sphere radially on the sphere so that we obtain a so called spherical archimedean tiling. Taking the incircles of this tiling results in a binary spherical cap packing with big caps on the north and south poles, and five smaller caps on the equator. We show how an exact solution of the semidefinite program can be obtained, proving that the bound is tight.

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Self-similar set as Martin Boundary and Induced Dirichlet forms

Ka-Sing Lau

(joint work with Sze-Man Ngai, Xiang-Yang Wang and Tim-Kam Wong)

In this report, we discuss an approach using the Markov chain theory and its associated discrete potential theory to study the induced Dirichlet forms and Laplacians on the self-similar sets K. The basic idea is to establish a homeomorphism of K with the limit set of the chain, i.e., the *Martin boundary* [3], then apply the established theory to obtain the desired harmonic structure on K. This technique was first used by Denker and Sato [2] on the Sierpinski gasket. The study here is closely connected to the current developments on the analysis on fractals.

We let K be the self-similar set generated by a contractive iterated function system (IFS) $\{S_i\}_{i=1}^N$, $N \geq 2$, which satisfies the *open set condition* (OSC), and assume the maps have equal contraction ratio. The set K admits a symbolic representation as follows: for each integer $n \geq 0$, let $\Sigma = \{1, \cdots, N\}$ be the set of alphabets, $\Sigma^* = \bigcup_{n=0}^{\infty} \Sigma^n$ the set of finite words, and Σ^∞ the set of infinite words. Then each $x \in K$ can be represented by an $u \in \Sigma^\infty$ with $\{x\} = \cap_{n=1}^{\infty} S_{u_1 \cdots u_n}(K)$. From the graph-theoretic point of view, we can consider Σ^* as a *tree* with edges connecting the descendants of each word (the root is at $\Sigma^0 := \{\vartheta\}$), and Σ^∞ is a certain *boundary* of Σ^*. It is conceivable that the tree structure on Σ^* is too rough and does not reflect the nature of K. We therefore introduce an *augmented tree* structure on Σ^* to reflect more properties of the underlying set K.

Definition 1. *Let $X := \Sigma^*$ be the symbolic space associated with $\{S_i\}_{i=1}^N$. The augmented tree (X, E) is defined to have an edge set $E = E_{\mathrm{v}} \cup E_{\mathrm{h}}$ of vertical edges and horizontal edges:*

$$\begin{aligned} (u, v) \in E_{\mathrm{v}} &\quad \Leftrightarrow \quad v = ui \text{ or } u = vi, \quad i \in \Sigma; \\ (u, v) \in E_{\mathrm{h}} &\quad \Leftrightarrow \quad |u| = |v| \text{ and } S_u(K) \cap S_v(K) \neq \emptyset. \end{aligned}$$

We write $u \sim v$, $u \sim_{\mathrm{v}} v$ and $u \sim_{\mathrm{h}} v$ for $(u, v) \in$E, E_{v} and E_{h} respectively. The augmented tree was introduced by Kaimanovich [5] on the symbolic space of the Sierpinski gasket K. The following is a useful extension.

Theorem 2. [10] *With the above assumptions on the IFS, then*
(i) the augmentd tree (X, E) is a hyperbolic graph ([14]);
(ii) the self-similar set K is homeomorphic to the hyperbolic boundary $\partial_H X$ of (X, E).

On this augmented tree, we can impose different Markov chains that walk on the whole or part of the paths. For example, the simple Markov chain considered by Denker and Sato [2] is that when it is at u, it will walk to its descendants and the descendants of the *conjugate* v (i.e., $u \sim_{\mathrm{h}} v$ and u, v have different parents) (see [4], [11] for more discussions and extensions). In the following we consider two other classes of Markov chains that have different perspectives.

We first consider a class of random walks on Σ^* so that the induced harmonic structure on K coincides with the one that is now classical in the analysis on fractals [6]. On the Sierpinski gasket, we let V_n be the three vertices in Σ^n, and define a transition probability on Σ^* by

$$p(u, v) = \begin{cases} 1/3, & \text{if } u, v \in \Sigma^n \setminus V^n, \ u \sim_{\mathrm{h}} v; \\ 1/3, & \text{if } u \in V^n, \ v = ui, \ i = 1, 2, 3; \\ 0, & \text{otherwise.} \end{cases}$$

The chain walks on the horizontal edges of the augmented tree in each level Σ^n, and goes down to the next level Σ^{n+1} when it reaches the three vertices of Σ^n. We call this a *transversal* random walk. The main result is

Theorem 3. [9] *The Martin boundary $\mathcal{M}$ is homeomorphic to the SG and the minimal boundary $\mathcal{M}_{\min} = \{\dot{1}, \dot{2}, \dot{3}\}$.*

Moreover the class of induced harmonic functions on the SG is 3-dimensional, satisfies the "1/5 − 2/5 rule", and coincides with Kigami's harmonic functions.

We point out that for the about type of Markov chains, the self-similar set is *not* always homeomorphic to the Martin boundary. For example, in [10], we prove that on the Hata tree (a non-symetric case), the Martin boundary is homeomorphic to the "trunk" of the tree, yet the harmonic structure still holds as desired. There are fruitful researches concerning the existence and properties of the Laplacian on self-similar sets, in particular, for some symmetric fractals and their ramifications [6]. A still unsolved problem is whether a Laplacian will exist on a general non-symmetric (connected) self-similar set. The above Markov chain approach is likely to be extended to larger class of self-similar sets, and offers an alternative approach to attack the problem.

Note that for the above Markov chain, the induced harmonic structure on the self-similar set corresponds to a Laplacian, which is a *local* Dirichlet form. In the following we consider another type of Markov chain which induces a *non-local* Dirichlet form. This is initiated by Kigami [7] where the underlying state space is a tree and the Martin boundary is a Cantor-type set.

On the augmented tree (Σ^*, E), we define a transition probability by

$$p(u,v) = \begin{cases} \frac{c(u,v)}{m(u)}, & u \sim v; \\ 0, & \text{otherwise}, \end{cases} \tag{1}$$

where $c(u,v) = c(v,u)$, $0 < M_1 \le c(u,v) \le M_2$ and $m(u) = \sum_{v:v\sim u} c(u,v)$. Note that unlike the previous case, this $\{Z_n\}_{n=0}^\infty$ visits its neighbor in all directions, and we call this a *reversible random walk*. Also note that if $c(u,v) = 1$, then $\{Z_n\}_{n=0}^\infty$ is the nearest neighborhood random walk. In [1], Ancona proved a deep theorem that says for a hyperbolic graph with a uniformly irreducible transition probability (plus some other minor conditions), the hyperbolic boundary $\partial_H X$, the Martin boundary $\mathcal{M}$ and the minimal boundary $\mathcal{M}_{\min}$ are all identical. Hence together with Theorem 2, we prove

Theorem 4. [13] *Let $p(\cdot,\cdot)$ be as above, then $K \approx \partial_H X \approx \mathcal{M} = \mathcal{M}_{\min}$*

Let $\mathcal{E}$ be a discrete Dirichlet form on (Σ^*, E) defined by

$$\mathcal{E}[f] = \frac{1}{2}\sum_{x\sim y} c(x,y)(f(x)-f(y))^2.$$

By using the representation theory of harmonic functions on $\mathcal{M}$, $\mathcal{E}$ induces a closed, non-negative definite, symmetric bilinear form $\mathcal{E}_K$ on K: $\mathcal{E}_K(u,v) = \mathcal{E}(Hu, Hv)$ where Hu on Σ^* is the Poisson integral of u on K.

Theorem 5. [13] *Suppose* $\dim_H K < 2$*, and the random walk in (1) satisfies the* uniform drift condition*:* $\inf_x \mathbb{P}_x\{|Z_1| = |x|+1\} > \sup_x \mathbb{P}_x\{|Z_1| = |x|-1\}$*. Then*

$\mathcal{E}_K$ is a non-local Dirichlet form on $L^2(K, \mu_\vartheta)$, and has the expression

$$\mathcal{E}_K[u] = \int_K \int_K (u(x) - u(y)^2 J(x,y) d\mu_\vartheta(x) d\mu_\vartheta(y).$$

We note that the technical conditions in the theorem are used to prove the domain $\mathcal{D}_K$ of $\mathcal{E}_K$ is dense in $L^2(K, \mu_\vartheta)$; $J(x, y)$ is the Naïm kernel and the integral expression of $\mathcal{E}_K$ follows from Silverstein [12], which implies $\mathcal{E}_K$ is a non-local Dirichlet form. There are many unanswer questions arisen from Theorem 5, for example, it is not clear what the explicit expression of the hitting distribution μ_ϑ is, and an estimation of the kernel $J(x, y)$ has not been obtained. Also it is not known whether the technical conditions in the theorem can be weakened.

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Some applications of local operators

H. N. Mhaskar
(joint work with M. Ehler and F. Filbir)

A recent idea for analyzing high dimensional, unstructured data is to let the data speak for themselves. In theoretical analysis, one assumes that the data represents a sample from some *unknown* low dimensional manifold embedded in a high dimensional ambient Euclidean space. The objective is then to understand the geometry of this manifold. Thus, statistical techniques have been devised to estimate the dimension of this manifold [4]. Laplacian eigenmaps/diffusion maps are suggested to reveal the relative neighborhoods of different data points [3, 1], as well as provide local coordinate systems for the manifold [9, 10]. See the special issue [2] for an introduction to these ideas.

In many practical applications, one needs to go beyond an understanding of the manifold and answer queries based on the data. These queries can be modeled mathematically as functions on the (unknown) manifold. This function may be known to us on few training points and we aim to accurately predict the value of the function at items which are not yet observed. The theory on the approximation of functions on data-defined manifolds has been developed in a series of papers [11, 7, 8, 12, 13]. A particularly interesting aspect of this theory is a definition of pointwise smoothness of the target function. The research has also enabled us to devise specific algorithms, extending the theory developed for the understanding of the geometry, with the property that the rate of convergence of these algorithms in neighborhoods of different points completely characterize the local smoothness properties of the target function at those points.

Let $\mathbb{X}$ be the hypothetical manifold from which all data is assumed to be sampled. Let $\{\varphi_k\}_{k=0}^\infty$ be the eigenfunctions of the Laplace-Beltrami operator Δ on $\mathbb{X}$ and $\{\ell_k^2\}_{k=0}^\infty$ the associated eigenvalues, ordered in a nondecreasing way with $\ell_0 = 0$ and $\ell_k \to \infty$ as $k \to \infty$. The space of *diffusion polynomials* up to degree N is $\Pi_N := \mathrm{span}\{\varphi_k : \ell_k < N\}$. Note that $\{\varphi_k\}_{k=0}^\infty$ was replaced by a more general orthonormal basis for $L_2(\mathbb{X}, \mu)$ in [7, 8].

The object of interest in approximation theory is the *degree of approximation* of the target function:

$$E_N(f) = \min_{P \in \Pi_N} \|f - P\|_\infty. \tag{1}$$

This leads us to define the (global) smoothness class W^s in terms of the quantities $E_N(f)$ as

$$W^s(\mathbb{X}) := \{f \in \mathcal{C}(\mathbb{X}) : E_N(f) = \mathcal{O}(N^{-s})\},$$

endowed with the norm $\|\cdot\|_{W^s} := \|\cdot\|_\infty + \|(N^s E_N(f))_N\|_\infty$. The set of all infinitely differentiable functions is denoted by $\mathcal{C}^\infty(\mathbb{X}) = \bigcap_{s>0} W^s(\mathbb{X})$. If $x_0 \in \mathbb{X}$, we will define the local smoothness of f at x_0 by the natural windowing construction. Thus, we say that $f \in W^s_{x_0}(\mathbb{X})$ if there exists a neighborhood U of x_0 such that for every $\phi \in \mathcal{C}^\infty(\mathbb{X})$, supported on U, $\phi f \in W^s(\mathbb{X})$.

If we define

$$\Phi_N(x,y) := \sum_{k=0}^{\infty} h(\frac{\ell_k}{N})\varphi_k(x)\varphi_k(y), \qquad \text{for all } x,y \in \mathbb{X}, \tag{2}$$

where $h : \mathbb{R} \to [0,1]$ satisfies $h(t) = 1$ if $|t| \leq 1/2$ and $h(t) = 0$ if $|t| \geq 1$, then formally, we have

$$f(x) = \sum_{k=0}^{\infty} \int_{\mathbb{X}} f(y)\varphi_k(y)d\mu(y)\varphi_k(x) = \int_{\mathbb{X}} f(y)\Phi(x,y)d\mu(y), \tag{3}$$

with $\Phi(x,y) = \sum_{k=0}^{\infty} \varphi_k(y)\varphi_k(x)$. To approximate f from training data, we need some technical assumptions that lead to

$$|\Phi_N(x,y)| \lesssim \frac{N^\alpha}{\max(1,(N\rho(x,y))^S}, \tag{4}$$

where α is the dimension of $\mathbb{X}$, ρ the metric on $\mathbb{X}$, and S any integer [7, 11, 13]. If the training data $\mathcal{C} = \{y_i\}_{i=1}^M$ are sufficiently dense in $\mathbb{X}$, then there are quadrature weights $\{\omega_i\}_{i=1}^M$ such that

$$\int_{\mathbb{X}} P(x)d\mu(x) = \sum_{j=1}^{M} \omega_j P(x_j), \qquad \text{for all } P \in \Pi_{aN}.$$

We now replace the right-hand side of (3) with the approximation

$$\sigma_N(f,x) := \sum_{j=1}^{M} \omega_j f(y_j)\Phi_N(x,y_j) \tag{5}$$

that is defined for all $x \in \mathbb{X}$, although f must only be known on $\{y_i\}_{i=1}^M$. We verified in [6] that, for $f \in W_{y_0}^s(\mathbb{X})$, there is $\delta > 0$, such that,

$$\sup_{x\in B_\delta(y_0)} |f(x) - \sigma_N(f,x)| \lesssim N^{-s}, \tag{6}$$

where $B_\delta(y_0) \subset \mathbb{X}$ denotes a ball of radius δ around y_0. Thus, when f is locally smooth in a neighborhood around y_0, then we can locally reconstruct f from the training data. The analogous result for functions that are globally smooth is contained in [7, 8, 11], and [13] contains local estimates using spectral data rather than the values of f at finitely many points.

Local smoothness ideas and local function approximation as outlined above are particularly useful in biomedicine as disease progression underlies natural variations, medication leads to abrupt changes in disease progression, and environmental factors can vary quickly, so that the query function might not be globally smooth. While late disease stages underlie large variations, the transition from healthy to early pathology can be smooth, leading to query functions that are locally smooth within such early disease transitions. Therefore, we decided to test our methods for classification problems in two standard biomedical datasets, available at the UCI database library site : the Cleveland heart disease data set and the Wisconsin breast cancer data set. Our methods yielded substantially better results than the commonly used support vector machines.

After this verification, we use our scheme to analyze multi-spectral retinal images of age-related macular degeneration (AMD) patients. This is a first–of–its–kind work, an announcement of which was published in [5]. The full results are published in [6]

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Packing of congruent spheres on a sphere

Oleg R. Musin

We say that X in $\mathbb{S}^{d-1}$ is a spherical φ-code if for any $x, y \in X$ with $x \neq y$, we have $\operatorname{dist}(x, y) \geq \varphi$. Denote by $A(d, \varphi)$ the maximum cardinality of a φ-code in $\mathbb{S}^{d-1}$. In other words, $A(d, \varphi)$ is the maximum cardinality of a sphere of radius $\varphi/2$ packing in $\mathbb{S}^{d-1}$.

We consider several classical and new methods for upper bounds on $A(d, \varphi)$ i.e. upper bounds on densest packing of congruent spheres on a sphere:
(1) Fejes Tóth's bound of circles packings (1943). Coxeter (1963) extended this bound for higher dimensions.

(2) Distance and irreducible graphs of circles packings [Schutte and van der Waerden, Leech, Danzer].
(3) Linear programming (Delsarte method) and SDP;
(4) Combination of (2) and (3).

(1) L. Fejes Tóth proved the following bound

$$A(3,\varphi) \leq \frac{2\pi}{\Delta(\varphi)} + 2,$$

where

$$\Delta(\varphi) = 3\arccos\left(\frac{\cos\varphi}{1+\cos\varphi}\right) - \pi,$$

i.e. $\Delta(\varphi)$ is the area of a spherical regular triangle with side length φ.

The Fejes Tóth bound is tight for $n = 3, 4, 6$ and 12. So for these n it gives a solution of the Tammes problem. This bound is also tight asymptotically.

Coxeter (1963) and Böröczky (1978) extended the Fejes Tóth bound for all dimensions:

$$A(d,\varphi) \leq 2F_{d-1}(\alpha)/F_d(\alpha),$$

where

$$\sec 2\alpha = \sec\varphi + d - 2,$$

and the function F is defined recursively by

$$F_{d+1}(\alpha) = \frac{2}{\pi}\int\limits_{arcsec(d)/2}^{\alpha} F_{d-1}(\beta)\,d\theta, \ \sec 2\beta = \sec 2\theta - 2,$$

with the initial conditions $F_0(\alpha) = F_1(\alpha) = 1$.

(2) Let X be a finite subset of $\mathbb{S}^2$. Denote

$$\psi(X) := \min_{x,y\in X}\{\text{dist}(x,y)\}, \text{ where } x \neq y.$$

Then X is a spherical $\psi(X)$-code.

Denote by d_N the largest angular separation $\psi(X)$ with $|X| = N$ that can be attained in $\mathbb{S}^2$, i.e.

$$d_N := \max_{X\subset\mathbb{S}^2}\{\psi(X)\}, \text{ where } |X| = N.$$

In other words, *how are N congruent, non-overlapping circles distributed on the sphere when the common radius of the circles has to be as large as possible?*

The problem was first asked by the Dutch botanist Tammes, who was led to this problem by examining the distribution of openings on the pollen grains of different flowers.

The Tammes problem is presently solved only for several values of N: for $N = 3, 4, 6, 12$ by L. Fejes Tóth (1943); for $N = 5, 7, 8, 9$ by Schütte and van der Waerden (1951); for $N = 10, 11$ by Danzer (1963) and for $N = 24$ by Robinson (1961).

Recently, we solved the Tammes problem for $N = 13$ [7]. It is joint work with Alexey Tarasov. Our computer-assisted proof is based on an enumeration of irreducible graphs.

(3) I. J. Schoenberg proved that a function is positive definite in the unit sphere if and only if this function is a nonnegative linear combination of Gegenbauer polynomials. This fact plays a crucial role in Delsarte's method for finding bounds for the density of sphere packings on spheres and Euclidean spaces.

One of the most excited applications of Delsarte's method is a solution of the kissing number problem in dimensions 8 and 24. However, 8 and 24 are the only dimensions in which this method gives a precise result. For other dimensions (for instance, three and four) the upper bounds exceed the lower.

Recently, were found extensions of Schoenberg's theorem for multivariate positive-definite functions. Using these extensions and semidefinite programming some upper bounds for spherical codes can be improved.

(4) We have found an extension of the Delsarte method that allows to solve the kissing number problem (as well as the one-sided kissing number problem) in dimensions three and four.

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On extremal lattices in jump dimensions

Gabriele Nebe

Let (L, Q) be an even unimodular lattice, so L is a free $\mathbb{Z}$-module of rank n, and $Q : L \to \mathbb{Z}$ a positive definite regular integral quadratic form. Then L can be embedded into Euclidean n-space $(\mathbb{R}^n, (,))$ with bilinear form defined by $(x, y) := Q(x + y) - Q(x) - Q(y)$ and L defines a lattice sphere packing, whose density measures its error correcting properties. One of the main goals in lattice theory is to find dense lattices. This is a very difficult problem, the densest lattices are known only in dimension $n \leq 8$ and in dimension 24 [3], for $n = 8$ and $n = 24$ the densest lattices are even unimodular lattices. The density of a unimodular lattice is proportional to its **minimum**, $\min(L) := \min\{Q(\ell) \mid 0 \neq \ell \in L\}$. For even unimodular lattices the theory of modular forms allows to bound this minimum $\min(L) \leq 1 + \lfloor \frac{n}{24} \rfloor$ and **extremal lattices** are those even unimodular lattices L that achieve equality. The link is the **theta series** of L,

$$\theta_L = \sum_{\ell \in L} q^{Q(\ell)} = 1 + \sum_{k=\min(L)}^{\infty} a_k q^k$$

where $a_k = |\{\ell \in L \mid Q(\ell) = k\}|$. After substituting the formal variable q by the holomorphic function $\exp(2\pi i z)$ with $z \in \mathbb{C}$, $\Im(z) > 0$, $\theta_L(z)$ becomes a modular form of weight $\frac{n}{2}$ for the full modular group $\mathrm{SL}_2(\mathbb{Z})$. So one may apply explicit transformation rules to conclude that the dimension n is always a multiple of 8 (see [4, Theorem 2.1]), which also follows from the theory of quadratic forms. The space of modular forms of weight $4k$ has dimension $m_k := \lfloor \frac{k}{3} \rfloor + 1$ and contains a unique form

$$f^{(k)} := 1 + 0q + 0q^2 + \ldots + 0q^{m_k - 1} + a(f^{(k)})q^{m_k} + b(f^{(k)})q^{m_k+1} + \ldots$$

the **extremal modular form** of weight $4k$. Already Siegel [12, end of proof of Satz 2] has shown that $a(f^{(k)}) > 0$ for all k, therefore $\min(L) \leq 1 + \lfloor \frac{n}{24} \rfloor$ for all even unimodular lattices of rank n. Lattices achieving equality are called **extremal**. Recently Jenkins and Rouse [5] have shown that the next coefficient $b(f^{(k)})$ of the extremal modular form becomes negative for all $k \geq 20408$, so there are no extremal lattices of dimension $n \geq 163,264$.

Extremal even unimodular lattices $L \leq \mathbb{R}^n$

n	8	16	**24**	32	40	**48**	**72**	80	$\geq 163,264$
min(L)	1	1	**2**	2	2	**3**	**4**	4	
number of extremal lattices	1	2	**1**	$\geq 10^7$	$\geq 10^{51}$	$\geq$ **3**	$\geq$**1**	≥ 4	0

Of particular interest are extremal even unimodular lattices L in the **jump dimensions** $24m$. Then $\theta_{L,p} = 0$ for all harmonic polynomials of degree $1 \leq \deg(p) \leq 11$ hence all non-empty layers $\{\ell \in L \mid Q(\ell) = a\}$ form spherical 11-designs. In particular the minimal vectors of L form a spherical 4-design, so all these lattices are strongly perfect [14] and their density realises a local maximum of the density function on the space of all $24m$-dimensional lattices. For $m = 1$ there is a unique extremal even unimodular lattice, the **Leech lattice**, which is the densest 24-dimensional lattice [3]. The 196560 minimal vectors of the Leech lattice form the unique tight spherical 11-design and realise the maximal kissing number in dimension 24. Also $m = 2, 3$ these lattices are the densest known lattices and realise the maximal known kissing number. There are only 5 extremal lattices known in jump dimensions. Using the classification of finite simple groups, one may show that the automorphism groups of these lattices are [11]

$\mathrm{Aut}(\Lambda_{24}) \cong 2.Co_1$	order	8315553613086720000
	$=$	$2^{22}3^95^47^2 \cdot 11 \cdot 13 \cdot 23$
$\mathrm{Aut}(P_{48p}) \cong (\mathrm{SL}_2(23) \times S_3) : 2$	order	$72864 = 2^53^211 \cdot 23$
$\mathrm{Aut}(P_{48q}) \cong \mathrm{SL}_2(47)$	order	$103776 = 2^53 \cdot 23 \cdot 47$
$\mathrm{Aut}(P_{48n}) \cong (\mathrm{SL}_2(13) \mathsf{Y}\, \mathrm{SL}_2(5)).2^2$	order	$524160 = 2^73^25 \cdot 7 \cdot 13$
$\mathrm{Aut}(\Gamma_{72}) \cong (\mathrm{SL}_2(25) \times \mathrm{PSL}_2(7)) : 2$	order	$5241600 = 2^83^25^27 \cdot 13$

A **canonical construction** of a lattice is a construction that is respected by (a big subgroup of) its automorphism group. Two of the 48-dimensional extremal lattices have a canonical construction with codes:
Let $(e_1, \ldots, e_n)$ be a p-**frame**, so $(e_i, e_j) = p\delta_{ij}$. Given $C \leq \mathbb{F}_p^n$ the **codelattice** is $\Lambda(C) := \{\frac{1}{p}\sum c_i e_i \mid (\overline{c}_1, \ldots, \overline{c}_n) \in C\}$.

Theorem [6], [7]
Let $C = C^\perp \leq \mathbb{F}_3^{48}$ with $d(C) = 15$. Then one of the two even neighbors of the codelattice $\Lambda(C)$ is an extremal even unimodular lattice. The other even neighbor has minimum 4, its minimal vectors form a 4-frame and hence this is a codelattice for some extremal code modulo 4. This is one explanation of the surprising bijection between Hadamard matrices mod 4 and mod 3 given in [7].

Having this application to extremal lattices in mind, I classified all extremal ternary codes of length 48 that have an automorphism prime order ≥ 5 in [9]. It turned out that the two known codes are the only such codes: the extended quadratic residue code Q_{48} with $\mathrm{Aut}(Q_{48}) \cong \mathrm{SL}_2(47)$ and the Pless code P_{48} with $\mathrm{Aut}(P_{48}) \cong (\mathrm{SL}_2(23) \times C_2) : 2$. These codes yield the two lattices P_{48q} and P_{48p}.

In [8] I found the third lattice P_{48n} which has a canonical construction as a tensor product of lattices over quaternions which is very similar to the construction of Γ_{72} as a Hermitian tensor product over $\mathbb{Z}[\alpha]$ where $\alpha = \frac{1+\sqrt{-7}}{2}$. For sake of brevity I will only comment on Γ_{72} and show how one may apply the theory from [1] to obtain the minimum of Γ_{72}. A $\mathbb{Z}[\alpha]$-lattice P is a free $\mathbb{Z}[\alpha]$ module of rank n together with a positive definite Hermitian form $h : P \times P \to \mathbb{Q}[\alpha]$. The **minimum** of P is $\min(P) := \min\{h(\ell, \ell) \mid 0 \neq \ell \in P\}$, the **determinant** of P is the determinant of any Gram matrix of P and the **Hermitian dual lattice** is $P^* := \{v \in V \mid h(v, \ell) \in \mathbb{Z}[\alpha]$ for all $\ell \in P\}$ We call P **Hermitian unimodular**, if $P = P^*$ One example of such a lattice is the **Barnes lattice** P_b with Hermitian Gram matrix $\begin{pmatrix} 2 & \alpha & -1 \\ \beta & 2 & \alpha \\ -1 & \beta & 2 \end{pmatrix}$ where $\beta = \overline{\alpha} = 1 - \alpha$. Then P_b is Hermitian unimodular, $\det(P_b) = 1$, $\min(P_b) = 2$ and $\mathrm{Aut}(P_b) = \pm\,\mathrm{PSL}_2(7)$.

Any Hermitian $\mathbb{Z}[\alpha]$-lattice (P, h) is also a $\mathbb{Z}$-lattice (L, Q) of dimension $2n$, where $L = P$ and $Q(x) := h(x, x) \in \mathbb{R} \cap \mathbb{Q}[\alpha] = \mathbb{Q}$. Then the polar form of Q is $(x, y) = \mathrm{Trace}_{\mathbb{Q}[\alpha]/\mathbb{Q}}(h(x, y))$ and (L, Q) is called the **trace lattice** of (P, h). We have $\min(L) = \min(P)$, $L^\# = \frac{1}{\sqrt{-7}}P^*$ and $\det(L) = 7^n \det(P)^2$.

Transfering ideas of Kitaoka, Renaud Coulangeon [1] obtained bounds on the minimum of the tensor product of Hermitian lattices: Let K be an imaginary quadratic field and (L, h_L) and (M, h_M) be Hermitian $\mathbb{Z}_K$-lattices, $n = \dim_{\mathbb{Z}_K}(L) \leq m := \dim_{\mathbb{Z}_K}(M)$. Each $v \in L \otimes M$ is the sum of at most n pure tensors $v =$

$\sum_{i=1}^r \ell_i \otimes m_i$ where r is minimal. Put $A := (h_L(\ell_i, \ell_j))$ and $B := (h_M(m_i, m_j))$, then $h(v, v) = \text{Trace}\, A\overline{B} \geq r \det(A)^{1/r} \det(B)^{1/r}$. so

$$\min(L \otimes M) \geq \min\{r d_r(L)^{1/r} d_r(M)^{1/r} \mid r = 1, \ldots, n\}$$

where $d_r(L) = \min\{\det(T) \mid T \leq L, Rg(T) = r\}$.

Theorem [2]
Let P be an Hermitian $\mathbb{Z}[\alpha]$-lattice with $\min(P) = 2$. Then $\min(P \otimes P_b) \geq 3$ and $\min(P \otimes P_b) > 3$ if and only if P has no sublattice isometric to P_b.
Proof: Clearly $d_1(P_b) = \min(P_b) = 2$, $d_3(P_b) = \det(P_b) = 1$ and $d_2(P_b) = d_1(P_b^*) = 2$. By assumption $d_1(P) = \min(P) = 2$ and so $d_2(P) \geq 2^2\frac{3}{7}$ and $d_3(P) \geq 1$, as these are the minimal determinants of the densest $\mathbb{Z}[\alpha]$-lattices of minimum 2 and dimension 2 respectively 3. So

$$r d_r(P_b)^{1/r} d_r(P)^{1/r} \begin{cases} = 4 & r = 1 \\ \geq 3.7 & r = 2 \\ \geq 3 & r = 3 \end{cases}$$

So the bound on $\min(P \otimes P_b)$ is strictly bigger than 3, if P does not represent the lattice P_b.

The nine $\mathbb{Z}[\alpha]$ structures of the Leech lattice

i	group	$\#P_b \leq P_i$
1	$\text{SL}_2(25)$	0
2	$2.A_6 \times D_8$	$2 \cdot 20,160$
3	$\text{SL}_2(13).2$	$2 \cdot 52,416$
4	$(\text{SL}_2(5) \times A_5).2$	$2 \cdot 100,800$
5	$(\text{SL}_2(5) \times A_5).2$	$2 \cdot 100,800$
6	$2^9 3^3$	$2 \cdot 177,408$
7	$\pm\text{PSL}_2(7) \times (C_7 : C_3)$	$2 \cdot 306,432$
8	$\text{PSL}_2(7) \times 2.A_7$	$2 \cdot 504,000$
9	$2.J_2.2$	$2 \cdot 1,209,600$

In particular we may apply this to the nine 12-dimensional $\mathbb{Z}[\alpha]$-lattices P_i given in the table such that $\text{Trace}(P_i) \cong \Lambda_{24}$. The representation number of P_b in P_i can be obtained by computations within the set of minimal vectors of the Leech latticed and is given in the last column of this table. It gives the number of vectors of norm 3 in $P_i \otimes P_b$. Therefore the trace lattice $\text{Trace}(P_1 \otimes P_b) =: \Gamma_{72}$ is an extremal even unimodular lattice. Two computational proofs of the extremality of Γ_{72} have been given in [10] a third proof by M. Watkins is based on the following idea.

Theorem. [13]
Let L be an even unimodular lattice of dimension 72 with $\min(L) \geq 3$. Then L is extremal, if and only if it contains at least $6,218,175,600$ vectors v with $Q(v) = 4$.
Proof: L is an even unimodular lattice of minimum ≥ 3, so its theta series is

$$\theta_L = 1 + a_3 q^3 + a_4 q^4 + \ldots = f^{(9)} + a_3 \Delta^3.$$

$$\begin{array}{rclllll} f^{(9)} & = & 1 & + & 6,218,175,600q^4 & +\dots \\ \Delta^3 & = & & q^3 & -72q^4 & +\dots \end{array}$$

So $a_4 = 6,218,175,600 - 72a_3 \geq 6,218,175,600$ if and only if $a_3 = 0$.

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Packings of bodies in Euclidean space

Fernando Mário de Oliveira Filho

(joint work with David de Laat and Frank Vallentin)

Let $K_1, \ldots, K_N \subseteq \mathbb{R}^n$ be convex bodies. A *packing* of $K_1, \ldots, K_N$ is a union of translated copies of these bodies in which any two copies have disjoint interiors. The *density* of a packing is the fraction of space it covers, and one is interested in finding lower and upper bounds for the maximum density of a packing of the bodies $K_1, \ldots, K_N$. There are different ways to formalize the definition of density, and then problems arise as to whether every packing has a density and so on. These technical details are not really important for us, as they do not change the maximum density that can be achieved by a packing; the reader interested in a brief discussion of these matters and further references may consult e.g. Appendix A of Cohn and Elkies [1].

When $N = 1$ and K_1 is a ball we are dealing with the classical sphere-packing problem. For this problem, Cohn and Elkies [1] proposed an upper bound for the maximum density based on linear programming. Their bound is the best known upper bound for dimensions 4, …, 24 and is conjectured to be tight in dimensions 2, 8, and 24.

In our work [3] we give a generalization of the Cohn-Elkies bound to packings of N convex bodies. Let $f\colon \mathbb{R}^n \to \mathbb{C}$ be an L^1 function. The *Fourier coefficient* associated with $u \in \mathbb{R}^n$ is

$$\hat{f}(u) = \int_{\mathbb{R}^n} f(x) e^{-2\pi i u \cdot x}\, dx.$$

Our theorem is the following:

Theorem. *Let $K_1, \ldots, K_N \subseteq \mathbb{R}^n$ be convex bodies and let $f\colon \mathbb{R}^n \to \mathbb{R}^{N\times N}$ be a matrix-valued function whose every component is an L^1 and continuous function. Suppose f satisfies:*

1. *the matrix $\big(\hat{f}_{ij}(0) - (\operatorname{vol} K_i)^{1/2}(\operatorname{vol} K_j)^{1/2}\big)_{i,j=1}^N$ is positive semidefinite;*
2. *the matrix $\big(\hat{f}_{ij}(u)\big)_{i,j=1}^N$ is positive semidefinite for all $u \in \mathbb{R}^n$;*
3. *$f_{ij}(x) \le 0$ whenever $K_i^\circ \cap (x + K_j^\circ) = \emptyset$.*

Then the density of any packing of translated copies of $K_1, \ldots, K_N$ is at most $\max\{\, f_{ii}(0) : i = 1, \ldots, N\,\}$.

We use the theorem to compute explicit upper bounds for the densities of binary sphere packings. A *binary sphere packing* is a packing of balls of two different radii. Recently, Hopkins, Jiao, Stillinger, and Torquato [2] provided constructive lower bounds for the densities of binary sphere packings in dimension 3. For instance, they give a packing of balls of radius 0.224744… and 1 in $\mathbb{R}^3$ with density 0.824539…, whereas the upper bound we computed is 0.861712….

Our approach to finding functions f satisfying the conditions required by the theorem is computational. Roughly speaking, we specify the function via its Fourier coefficients (so that they become our optimization variables) and use the computer to find a best function satisfying the conditions of the theorem.

More precisely, we specify a template for the Fourier transform of f. We fix an odd integer d and write

$$\hat{f}_{ij}(u) = \sum_{k=0}^{d} a_{ij,k} \|u\|^{2k} e^{-\pi\|u\|^2},$$

where the $a_{ij,k}$ would be our optimization variables, being such that $a_{ij,k} = a_{ji,k}$.

This template for the Fourier transform makes it easy to compute back the function f. Indeed we have

$$f_{ij}(x) = \sum_{k=0}^{d} a_{ij,k} k! \pi^{-k} e^{-\pi\|x\|^2} L_k^{n/2-1}(\pi\|x\|^2),$$

where L_k^α is the Laguerre polynomial of degree k and parameter α.

Both f and $\hat{f}$ are polynomials multiplied by an exponential function, which is always nonnegative. We may use this fact to rewrite the conditions of the theorem. Condition (1) can be easily expressed in our formulation. Condition (2) is equivalent to the 3-variable polynomial

$$\sigma(t, y_1, y_2) = \sum_{i,j=1}^{2} \sum_{k=0}^{d} a_{ij,k} t^{2k} y_i y_j$$

being nonnegative for all t and all y_1, y_2. In this particular case, this is equivalent to σ being a sum of squares, that is, it is equivalent to the existence of polynomials $p_1, \ldots, p_m$ such that $\sigma = p_1^2 + \cdots + p_m^2$.

Condition (3) can be similarly expressed in terms of nonnegative polynomials and sums of squares. Finally, since determining whether a polynomial is a sum of squares is the same as solving a semidefinite programming problem, we obtain at the end a semidefinite programming problem giving an upper bound for the maximum density.

The approach sketched above is in principle numerical. Since we use a semidefinite programming solver, working with floating-point numbers, it is not *a priori* clear that the numbers we obtain are really bounds. It is possible however to obtain rigorous results out of this approach by carefully checking the solution data provided by the solver.

Finally, a similar but simpler computational approach can also be used to compute upper bounds for the densities of sphere packings, which in this case coincide with the Cohn-Elkies bounds. We also show how to strengthen the Cohn-Elkies bounds for sphere packings by adding extra valid inequalities. This strengthening, together with the computational approach above, gives stronger bounds in dimensions 4, 5, 6, and 7.

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Formally dually sequences in finite abelian groups

Christian Reiher

(joint work with Henry Cohn, Abhinav Kumar, Achill Schürmann)

Let G be a finite abelian group and $\hat{G}$ its Pontryagin dual. Two sequences $v_1, \ldots, v_N \in G$ and $w_1, \ldots, w_M \in \hat{G}$ of distinct elements from these groups are said to be formally dual to each other if each $y \in \hat{G}$ satisfies

$$\Big| \sum_{i=1}^{N} (v_i, y) \Big|^2 = \frac{N^2}{M} \sharp \big\{ (j,k) \mid 1 \le j, k \le M \text{ and } y = w_j - w_k \big\}.$$

It is elementary that this implies $|G| = |\hat{G}| = MN$ and

$$\Big|\sum_{j=1}^{M}(x, w_j)\Big|^2 = \frac{M^2}{N}\sharp\big\{(i,k) \mid 1 \le i,k \le N \text{ and } x = v_i - v_k\big\}$$

for all $x \in G$. In the talk we described some examples of formally dual sequences and described our approach to the classification of all such sequences in case G has rank 1, which is still work in progress.

QMC designs – optimal order Quasi Monte Carlo Integration in Sobolev spaces on the sphere

Ian Hugh Sloan

In this joint work with Johann Brauchart, Edward Saff and Robert Womersley, we introduce a new concept, that of QMC designs. These are sequences of finite point sets on the unit sphere $S^d \subset R^{d+1}$, which if used for equal weight (or Quasi Monte Carlo, or QMC) numerical integration, give optimal order of convergence for functions in a Sobolev space $H^s(S^d)$ on the unit sphere $S^d \in R^{d+1}$, where $s > d/2$ is the smoothness parameter of the Sobolev space. Thus if (X_N) is a sequence of N-point QMC designs for the Sobolev space $H^s(S^d)$ then the worst case error in $H^s(S^d)$ of equal weight numerical integration based on X_N (the worst case error being the supremum of the quadrature error for all functions in the unit ball of $H^s(S^d)$) has order of convergence $O(N^{-s/d})$, with an implied constant that depends on the $H^s(S^d)$-norm, but is independent of N. Here $N = |X_N|$ is the cardinality of X_N .

Spherical t-designs with a suitably small number of points are prime examples of QMC designs. A spherical t-design, a concept introduced in a ground breaking paper by Delsarte, Goethals and Seidel, is a finite subset $X_N \subset S^d$ with the characterizing property that an equal weight integration rule with nodes from X_N integrates exactly all spherical polynomials of degree $\le t$, where a spherical polynomial of degree $\le t$ is the restriction to S^d of a polynomial of degree $\le t$ on R^{d+1}. From earlier results of Hesse and Sloan, spherical designs have a known fast convergence property in Sobolev spaces, namely that the worst case error in $H^s(S^d)$ for an N-point design is of order $O(t^s)$. Thus if a sequence of N-point spherical t-designs has N of exact order t^d then it is a sequence of QMC designs for all $s > d/2$. That spherical designs with this number of points do exist was proved recently by Bondarenko, Radchenko and Viazowska, and is the subject of a different presentation at this workshop.

We show that if a sequence (X_N) is a sequence of QMC designs for $H^s(S^d)$ then it is also a sequence of QMC designs for s' satisfying $d/2 < s' < s$. This implies that for every sequence of QMC designs on S^d there is a supremum s^* of the values of s for which the QMC design property holds. We call s^* the QMC index of the sequence of designs. A sequence of spherical t-designs with $t \to \infty$ has $s^* = \infty$.

An essential tool for our analysis is an expression for the worst-case error in terms of a reproducing kernel for the space $H^s(S^d)$ with $s > d/2$. As a consequence of this and the recent result of Bondarenko et al., we show that minimizers of the N-point energy for this kernel form a sequence of QMC designs for $H^s(S^d)$. We also show, without appealing to the Bondarenko et al. result, that point sets that maximize the sum of suitable powers of the Euclidean distance between pairs of points form a sequence of QMC designs for $H^s(S^d)$ for s in the interval $(d/2, d/2+1)$. For such spaces there exist reproducing kernels with simple closed forms that are useful for numerical testing of optimal order Quasi Monte Carlo integration.

Numerical experiments suggest that many familiar sequences of point sets on the sphere (equal area points, spiral points, minimal [Coulomb or logarithmic] energy points, and Fekete points) are QMC designs for s up to some number s^*. We present estimated values of s^* for each such sequence. For comparison purposes we show that configurations of random points that are independently and uniformly distributed on the sphere do not constitute QMC designs for any $s > d/2$.

Linear and semidefinite programming bounds

Frank Vallentin

To show that a given point configuration on a manifold is optimal or near optimal the use of linear programming methods has become one of the most successful approaches over the last 40 years. This development started in 1973 with Delsarte's fundamental work on linear programming bounds for finite Hamming and Johnson spaces. In 1977 it was extended by Delsarte, Goethals and Seidel to the unit sphere and in 1978 by Kabatiansky and Levenshtein to general compact 2-point homogeneous manifolds. Cohn and Elkies extended this to Euclidean space in 2003. Using semidefinite programming instead of linear programming improvements were established by Schrijver in 2005 for Hamming and Johnson spaces. In 2008 Bachoc and I extended Schrijver's approach to the unit sphere.

Linear and semidefinite programming

Linear programming (LP) is maximizing a linear function over a polyhedron, the intersection of the nonnegative orthant $\mathbb{R}^n_{\geq 0}$ with an affine subspace. Semidefinite programming (SDP) — a rich generalization of linear programming — is maximizing a linear function over a spectrahedron, the intersection of the cone of positive semidefinite matrices $\mathcal{S}^n_{\succeq 0}$ with an affine subspace. Both, LP and SDP, are conic optimization problems for which efficient (in theory and practice) algorithms are available. Furthermore duality theory gives a systematic tool for rigorously proving that a solution is (globally) optimal or close to optimal.

For solving LPs and SDPs two types of algorithms are available: The ellipsoid method focuses on the existence/non-existence of polynomial time algorithm but no practical implementation is available. In contrast to this there are many very good implementations of the interior point method. However, we currently do not know if interior point methods run in polynomial time (in the Turing machine bit model!) for non-linear convex programs. **Question:** Are interior point methods the right choice for performing *rigorous* computations in computer proofs?

Three applications of SDP

For the design and implementation of LP and SDP bounds three applications of SDP are important: eigenvalue optimization, approximating $\mathcal{NP}$-hard graph parameters, and polynomial optimization:

1. Let $X \in \mathcal{S}^n$ be a symmetric matrix with (real) eigenvalues $\lambda_1(X) \geq \lambda_2(X) \geq \ldots \geq \lambda_n(X)$. Finding the sum of the largest k eigenvalues is an SDP

$$\lambda_1(X) + \cdots + \lambda_k(X) = \max_{Y \in \mathcal{E}_k} \langle X, Y \rangle, \quad \mathcal{E}_k = \{Y \in \mathcal{S}^n : I_n \succeq Y \succeq 0, \langle I_n, Y \rangle = k\}.$$

This gadget can be used to show that optimizing convex functions which only depend on the eigenvalues is in many cases SDP representable.

2. Let $G = (V, E)$ be a finite graph. A subset $I \subseteq V$ is called independent if $\{x, y\} \notin E$ or all $x, y \in I$. The independence number $\alpha(G)$ is the largest cardinality of an independent set. This graph parameter is $\mathcal{NP}$-hard to compute. Lovász' ϑ is another graph parameter which upper bounds α and which is efficiently computable by solving an SDP. Let J denotes the all-one matrix.

$$\begin{array}{lll} \vartheta'(G) = \min & M & \\ & K - J & \text{is positive semidefinite,} \\ & K(x,x) \leq M & \text{for all } x \in V, \\ & K(x,y) \leq 0 & \text{for all } \{x,y\} \notin E \text{ where } x \neq y, \\ & M \in \mathbb{R},\, K \in \mathcal{S}^V. & \end{array}$$

3. Polynomial optimization amounts to finding the solution p_{min} of the following minimization problem

$$\begin{array}{ll} \text{minimize} & p(x) \\ & x \in K = \{x \in \mathbb{R}^n : g_1(x) \geq 0, \ldots, g_m(x) \geq 0\}, \end{array}$$

where $p, g_1, \ldots, g_m \in \mathbb{R}[x]$ are given polynomials. Equivalently, one finds the largest t so that $p - t \in \mathcal{P}(K)$ where $\mathcal{P}(K)$ is the cone of polynomials which are nonnegative on K. Again, this is in general an $\mathcal{NP}$-hard problem which one can relax to an SDP by using sum of squares

$$p_{sos} = \sup\{t : p - t \in \Sigma + g_1\Sigma + \cdots + g_m\Sigma\},$$

where

$$\Sigma = \{h_1^2 + \cdots + h_r^2 : r \in \mathbb{N}, h_i \in \mathbb{R}[x]\}$$

is the cone of sum of squares polynomials. Detecting that a polynomial $p \in \mathbb{R}[x]_{\leq d}$ of degree at most d is a sum of squares is equivalent to deciding whether there is a positive semidefinite matrix Q with

$$p = [x]_d^{\mathsf{T}} Q [x]_d, \quad \text{i.e.} \quad \sum_{\substack{\beta,\gamma \in \mathbb{N}^n_d \\ \beta + \gamma = \alpha}} Q_{\beta,\gamma} = p_\alpha \quad \forall \alpha \in \mathbb{N}^n_{2d},$$

where $[x]_d$ denotes the vector of all monomials of degree up to d. Clearly, $p_{sos} \leq p_{min}$ and generally $p_{min} \neq p_{sos}$. Putinar's theorem (1993) guarantees equality in many important cases: If there is a natural number N such that $N - \sum_{i=1}^{n} x_i^2 \in \Sigma + g_1\Sigma + \cdots + g_m\Sigma$, then

$$\forall x \in K : p(x) > 0 \Longrightarrow p \in \Sigma + g_1\Sigma + \cdots + g_m\Sigma.$$

Geometric packing problems

Many, often notoriously difficult, problems in geometry can be modeled as packing and coloring problems of continuous graphs $G = (V, E)$ where the vertex set V is a manifold. Packing problems correspond to finding the independence number $\alpha(G)$. Examples:

- Kissing numbers:

  $$V = S^{n-1} = \{x \in \mathbb{R}^n : x \cdot x = 1\}, \quad x \sim y \Longleftrightarrow 0 < \angle(x, y) < \pi/3.$$

- Translational body packings: Let $\mathcal{K}_1, \ldots, \mathcal{K}_N \subseteq \mathbb{R}^n$ be convex, compact bodies:

  $$V = \mathbb{R}^n \times \{1, \ldots, N\}, \quad (x, i) \sim (y, j) \Longleftrightarrow x + \mathcal{K}_i^o \cap y + \mathcal{K}_j^o \neq \emptyset.$$

A combination of SDP and harmonic analysis often gives the best known upper bounds for these packing problems.

Symmetry reduction and harmonic analysis

Now I illustrate how to apply harmonic analysis in order to be able to perform the calculations of $\vartheta'(G)$ for continuous graphs. To simplify the notation I consider the case when (V, μ) is a compact measure space. Then, we replace symmetric matrices $\mathcal{S}^V$ by continuous symmetric kernels $\mathcal{C}(V \times V)$. Suppose the graph G has Γ as its symmetry group. If $K \in \mathcal{C}(V \times V)_{\succeq 0}$ is feasible for ϑ', then also its *group average* $\overline{K}$ is:

$$\overline{K}(x, y) = \int_\Gamma K(\gamma x, \gamma y) d\gamma.$$

So it suffices to consider only the Γ-invariant cone

$$\mathcal{C}(V \times V)_{\succeq 0}^\Gamma = \{K : \forall \gamma \in \Gamma : K(\gamma x, \gamma y) = K(x, y)\}$$

and a theorem of Bochner (1941) gives an explicit parametrization of $\mathcal{C}(V \times V)_{\succeq 0}^\Gamma$.

We state Bochner's theorem now which requires a bit of technical vocabulary. The group Γ acts on $\mathcal{C}(V)$ by $(\gamma f)(x) = f(\gamma^{-1} x)$. So one can speak about Γ-invariant and Γ-irreducible subspaces of $\mathcal{C}(V)$. The Peter-Weyl theorem (1927) says that one can decompose $\mathcal{C}(V)$ orthogonally (using the inner product from $L^2(V)$)

$$\mathcal{C}(V) = (H_{0,1} \perp \ldots \perp H_{0,m_0}) \perp (H_{1,1} \perp \ldots \perp H_{1,m_1}) \perp \ldots,$$

where $H_{k,l}$ is Γ-irreducible and $\dim H_{k,l} < \infty$ and where $H_{k,l} \sim H_{k',l'}$ iff $k = k'$. We fix an orthonormal basis $e_{k,1,1}, \ldots, e_{k,1,\dim H_{k,1}}$ and Γ-isomorphisms $\varphi_{k,l}$:

$H_{k,1} \to H_{k,l}$ and set $e_{k,l,1} = \varphi_{k,l}(e_{k,1,1}), \ldots, e_{k,l,\dim H_{k,1}} = \varphi_{k,l}(e_{k,1,\dim H_{k,1}})$. Bochner's theorem:

$$\mathcal{C}(V \times V)^{\Gamma}_{\succeq 0} = \Big\{ K(x,y) = \sum_{k=0}^{\infty} \Big\langle F_k, \Big(\sum_{i=1}^{\dim H_{k,1}} e_{k,l,i}(x)\overline{e_{k,l',i}(y)} \Big)_{l,l'=1,\ldots,m_k} \Big\rangle : F_k \in \mathcal{S}^{m_k}_{\succeq 0} \Big\}.$$

This means that instead of optimizing over the cone $\mathcal{C}(V \times V)_{\succeq 0}$ we can optimize over the direct product of semidefinite cones $S^{m_0}_{\succeq 0} \times S^{m_1}_{\succeq 0} \times \ldots$ Since for finding upper bounds for $\alpha(G)$ we are interested only in feasible solutions we can set $F_k = 0$ for large enough k. In the case when $\sum_i e_{k,l,i}(x)\overline{e_{k,l',i}(y)}$ is a polynomial one can use polynomial optimization to find feasible solutions for $\vartheta'(G)$ by a finite-dimensional SDP.

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On optimal asymptotic bounds for spherical designs

Maryna Viazovska
(joint work with Andriy Bondarenko, Danylo Radchenko)

Let S^d be the unit sphere in $\mathbb{R}^{d+1}$ with Lebesgue measure $d\mu_d$ normalized by $\mu_d(S^d) = 1$.

A set of points $x_1, \ldots, x_N \in S^d$ is called a *spherical t-design* if

$$\int_{S^d} P(x)\, d\mu_d(x) = \frac{1}{N} \sum_{i=1}^{N} P(x_i)$$

for all algebraic polynomials in $d+1$ variables and of total degree at most t. The concept of a spherical design was introduced by Delsarte, Goethals, and Seidel [6]. For each $t, d \in \mathbb{N}$ denote by $N(d,t)$ the minimal number of points in a spherical t-design on S^d. The following lower bounds,

$$N(d,t) \geq \begin{cases} \binom{d+k}{d} + \binom{d+k-1}{d} & \text{if } t = 2k, \\ 2\binom{d+k}{d} & \text{if } t = 2k+1, \end{cases} \tag{1}$$

are also proved in [6].

Spherical t-designs attaining these bounds are called tight. Exactly eight tight spherical designs are known for $d \geq 2$ and $t \geq 4$. All such configurations of points are highly symmetrical and possess other extreme properties. Several of these configurations are described in the book of and Conway and Sloane [5].

Let us give a short history of asymptotic upper bounds on $N(d,t)$ for fixed d and $t \to \infty$. First, Seymour and Zaslavsky [11] have proved that spherical designs exist for all d, $t \in \mathbb{N}$. Then, Wagner [12] and Bajnok [2] independently proved that $N(d,t) \le c_d t^{Cd^4}$ and $N(d,t) \le c_d t^{Cd^3}$, respectively. Korevaar and Meyers [8] have improved these inequalities by showing that $N(d,t) \le c_d t^{(d^2+d)/2}$. They have also conjectured that $N(d,t) \le c_d t^d$. Note that (1) implies $N(d,t) \ge C_d t^d$.

The conjecture of Korevaar and Meyers was attacked by many mathematicians. For instance, Kuijlaars and Saff [10] emphasized the importance of this conjecture and revealed its relation to the energy problems. Then, Mhaskar, Narcowich, and Ward [9] have constructed positive quadrature formulas on S^d with $c_d t^d$ points having *almost* equal weights. Very recently, An, Chen, Sloan, and Womersley, see, e.g. [1], [4], have proved the existence of spherical t-designs on S^2 having $(t+1)^2$ points, for $t \le 100$. In order to prove their result they extensively used computer calculations.

For $d = 2$ there exists even stronger conjecture by Hardin and Sloane [7], that $N(2,t) = \frac{1}{2}t^2 + o(t^2)$ as $t \to \infty$. They also provided a numerical evidence for the conjecture.

In the paper [3] we combine the ideas of Brouwer degree theory and the notion of area regular partitions in order to find the optimal asymptotic lower bounds for spherical designs. Firstly, starting from a partition of the sphere into regions of equal area and small diameter we find a configuration of points on S^d which is "almost" a t-design. Then, using Brouwer degree theory we show that one can slightly move these points so that they become a t-design. Thus, we prove the following.

Theorem 6. *For each $N \ge c_d t^d$ there exists a spherical t-design on S^d consisting of N points, where c_d is a constant depending only on d.*

This proves the conjecture of Korevaar and Meyers.

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Efficient spherical designs with good geometric properties

Robert S. Womersley

Spherical t-designs, introduced in Delsarte, Goethals and Seidel [5], are sets $\mathcal{X}_N$ of N points on the unit sphere $\mathbb{S}^d \in \mathbb{R}^{d+1}$ which are equal weight numerical integration rules for $\mathbb{S}^d$ that are exact for all spherical polynomials of degree at most t. Spherical t-designs are known to exist for all N sufficiently large. A lower bound $N^*(d,t)$ on the number of points N was given in [5], but point sets which achieve this lower bound, known as tight spherical t-designs, only exist in a few special cases (see, for example, the survey [1]). The lower bounds were improved by Yudin [11]. Recently Bondarenko, Radchenko and Viazovska [2] established that spherical t-designs exist for all $N \geq c_d t^d$ for some constant c_d. A spherical design is efficient if N is less than $2N^*(d,t)$.

We are interested in sequences of efficient spherical t-designs $\mathcal{X}_N = \{\mathbf{x}_1, \ldots, \mathbf{x}_N\}$ that also have good geometric properties. The geometric properties used are the mesh norm $h_{\mathcal{X}_N} = \max_{\mathbf{x}\in\mathbb{S}^d} \operatorname{dist}(\mathbf{x}, \mathcal{X}_N)$ (the radius for covering the sphere by congruent spherical caps with centres from $\mathcal{X}_n$) and the separation $\delta_{\mathcal{X}_N} = \min_{i\neq j} \operatorname{dist}(\mathbf{x}_i, \mathbf{x}_j)$ (twice the packing radius). The mesh ratio is

$$\rho_{\mathcal{X}_N} = \frac{2h_{\mathcal{X}_N}}{\delta_{\mathcal{X}_N}} \geq 1.$$

A common requirement for computational methods using the nodes in $\mathcal{X}_N$ is that $\mathcal{X}_N$ is quasi-uniform, that is the mesh ratio is uniformly bounded.

A number of variational characterizations of spherical designs have been proposed (see for example Grabner and Tichy [6], Cohn and Kumar [4] and Sloan and Womersley [8]). These all rely on the property that they can be expressed as weighted (with strictly positive weights) sums of the squares of the functions

$$r_{\ell,k}(\mathcal{X}_N) := \sum_{j=1}^{N} Y_{\ell,k}(\mathbf{x}_j), \quad \ell = 1, \ldots, t, \quad k = 1, \ldots, Z(d,\ell).$$

Here $Z(d,\ell)$ is the dimension of the space of homogeneous harmonic polynomials on $\mathbb{S}^d$ of exact degree ℓ and $\{Y_{\ell,k}, k = 1, \ldots, Z(d,\ell)\}$ are an orthonormal set of spherical harmonics of degree ℓ. The dimension of the space $\mathbb{P}_t(\mathbb{S}^d)$ of spherical polynomials of degree at most t is $Z(d+1,t)$. A set $\mathcal{X}_N \subset \mathbb{S}^d$ is a spherical t-design if and only if the $m = Z(d+1,t)-1$ conditions $r_{\ell,k}(\mathcal{X}_N) = 0$ for $k = 1, \ldots, Z(d,\ell)$

and $\ell = 1, \ldots, t$ are satisfied. (The term $\ell = 0$ corresponding to the constant polynomial is not included).

A spherical t-design $\mathcal{X}_N$ is invariant under rotations of the whole point set. Thus, on $\mathbb{S}^2$ a point set can be rotated so that the first point is at the north pole and the second point is on the prime meridian, giving $n = 2N - 3$ remaining parameters. The simple argument of matching the number of parameters n with the number of equations $m = Z(3,t) - 1 = (t+1)^2 - 1$, suggests that a solution to the system of equations may be found when

$$N \geq \widehat{N}(2,t) := \lceil (t+1)^2/2 \rceil + 1.$$

The idea of using symmetries to ensure that certain polynomials are integrated exactly and hence reducing the number of conditions that must be satisfied goes back to at least the work of Sobolev [9]. A simple example of that is that an antipodal (or symmetric) point set such that $\mathbf{x} \in \mathcal{X}_N \Leftrightarrow -\mathbf{x} \in \mathcal{X}_N$, automatically integrates all odd degree spherical polynomials exactly. Counting constraints from the even degree polynomials and imposing the symmetry condition gives

$$N \geq \overline{N}(2,t) := 2 \left\lceil \frac{(t-1)(t+2)+6}{4} \right\rceil.$$

A Levenberg-Marquardt based optimization procedure for solving the system of nonlinear equations numerically found spherical t-designs with $\widehat{N}(2,t)$ points for $t = 1, \ldots, 180$ and symmetric t-designs with $\overline{N}(2,t)$ points for $t = 1, \ldots, 231$. A key factor in the performance of the algorithm was the choice of starting point, with different starting points producing different spherical designs. A version of the generalized spiral points was very effective as they are close to spherical designs (see [3]). Although the results are only numerical, the confidence that the calculated points are close to true spherical t-designs is increased by observing a rapid rate of convergence of the method and the condition number of the Jacobian of the system of nonlinear equations. These results agree with Hardin and Sloane [7] who conjectured that spherical t-designs exist for $\mathbb{S}^2$ with $N = \frac{t^2}{2}(1 + o(1))$.

Yudin [10] has has proved results for the mesh norm (covering radius) of spherical t-designs, which imply that $h_{\mathcal{X}_N}$ decays as $cN^{-1/d}$ for spherical designs with $N = ct^d$ points. Thus spherical t-designs with $\widehat{N}(2,t)$ or $\overline{N}(2,t)$ points must have good mesh norm.

It is well know that the combination of two spherical t-designs with N points, produces a spherical t-design with $2N$ points. Moreover, as a rotation does not affect the conditions for a spherical design, the separation of the combined set of $2N$ points can be made arbitrarily small. However as

$$2N^*(2,t) - \widehat{N}(2,t) = t, \qquad 2N^*(2,t) - \overline{N}(2,t) = \frac{3}{2}t - \zeta_t,$$

(where $\zeta_t = \frac{3}{2}$ if $\mod (t,4) = 1$ and $\zeta_t = \frac{1}{2}$ if $\mod (t,4) = 3$) this construction of spherical t-designs with $\widehat{N}(2,t)$ or $\overline{N}(2,t)$ points and small separation is not possible. The calculated spherical t-designs with $\widehat{N}(2,t)$ points had mesh ratio

$\rho_{\mathcal{X}_{\widehat{N}}} \leq 1.85$ for all $t = 1, \ldots, 180$, while the symmetric point sets had mesh ratio less than 2.2 for all $t = 1, \ldots, 231$.

The calculated spherical designs for $\mathbb{S}^2$ have $N \approx \frac{t^2}{2}$ points and are efficient in that they have less than twice the lower bounds on the number of points. They also have good geometric properties with bounded mesh ratio, which in turn implies that the point sets are well separated (although this has not yet been proved).

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Gauss variational problem for infinite dimensional vector measures and Riesz kernels

Natalia Zorii

The interest in the Gauss variational problem (i.e., minimal energy problems with external fields), which goes back to the work by Gauss, reappeared in the 1980's when Gonchar and Rakhmanov, Mhaskar and Saff efficiently applied logarithmic potentials with external fields in the investigation of orthogonal polynomials and rational approximations to analytic functions. E.g., the vector setting of the problem, suggested earlier by Ohtsuka [4], nowadays became particularly interesting in connection with Hermite–Padé rational approximations. However, the potential–theoretical methods, applied in these studies, were mainly based on the *vague* (=*weak**) topology, which made it possible to prove the existence of a solution only for vector measures of finite dimensions and compact support [4].

In order to treat the Gauss variational problem for vector measures $\boldsymbol{\mu}$ of infinite dimensions and/or noncompact support, in [8, 9] we have suggested an approach based on the introducing a metric structure on the class of all $\boldsymbol{\mu}$ with finite energy and also on the establishing an infinite dimensional version of a completeness theorem. This enabled us to obtain *simultaneously necessary and sufficient conditions* for the solvability of the problem [8, 9]. Although these results have been obtained therein for a general positive definite kernel on a locally compact space, satisfying Fuglede's condition of perfectness [1], for the sake of simplicity we shall concentrate on the Riesz kernels $\kappa_\alpha(x, y) = |x - y|^{\alpha - n}$, $0 < \alpha < n$, in $\mathbb{R}^n$, $n \geqslant 2$.

Let $\mathfrak{M}$ be the linear space of all Borel measures on $\mathbb{R}^n$ equipped with the vague topology, and let $\mathcal{E} = \mathcal{E}_\alpha$ consist of all $\nu \in \mathfrak{M}$ with finite (Riesz) energy $E(\nu) := \int \kappa_\alpha \, d(\nu \otimes \nu)$. Then, by Cartan, $\mathcal{E}$ forms an (incomplete) pre-Hilbert space with the scalar product $E(\nu, \nu_1) := \int \kappa_\alpha \, d(\nu \otimes \nu_1)$ and the norm $\|\nu\| := \sqrt{E(\nu)}$, while by Deny, it can be completed by making use of slowly increasing distributions with finite energy [3]. Let $C_\alpha(\cdot)$ denote the *capacity* of a Borel set relative to κ_α.

We consider a countable collection $\boldsymbol{A} = (A_i)_{i \in I}$ of pairwise disjoint, nonempty, closed sets $A_i \subset \mathbb{R}^n$ with the sign $+1$ or -1 prescribed, and let $\mathfrak{M}^+(\boldsymbol{A})$ consist of all vector measures $\boldsymbol{\mu} = (\mu^i)_{i \in I}$ with $\mu^i \in \mathfrak{M}^+(A_i) := \{\nu \geqslant 0 : \ \operatorname{supp} \nu \subset A_i\}$. That is, $\mathfrak{M}^+(\boldsymbol{A}) = \prod_{i \in I} \mathfrak{M}^+(A_i)$. The product space topology on $\mathfrak{M}^+(\boldsymbol{A})$, where each $\mathfrak{M}^+(A_i)$ is endowed with the induced vague topology, is likewise called *vague*.

Treating $\boldsymbol{A}$ as a condenser, we assume that the interaction between the charges on the conductors A_i, $i \in I$, is characterized by the matrix $(s_i s_j)_{i,j \in I}$, where $s_i := \operatorname{sign} A_i$. Then the *energy* of $\boldsymbol{\mu} \in \mathfrak{M}^+(\boldsymbol{A})$ is given by $E(\boldsymbol{\mu}) := \sum_{i,j \in I} s_i s_j E(\mu^i, \mu^j)$. Let $\mathcal{E}^+(\boldsymbol{A})$ denote the class of all $\boldsymbol{\mu} \in \mathfrak{M}^+(\boldsymbol{A})$ with $-\infty < E(\boldsymbol{\mu}) < \infty$.[1]

Fix a vector-valued external field $\boldsymbol{f} = (f_i)_{i \in I}$; then the $\boldsymbol{f}$*-weighted energy* of $\boldsymbol{\mu} \in \mathcal{E}^+(\boldsymbol{A})$ is defined by $G_{\boldsymbol{f}}(\boldsymbol{\mu}) := E(\boldsymbol{\mu}) + 2\langle \boldsymbol{f}, \boldsymbol{\mu} \rangle$, where $\langle \boldsymbol{f}, \boldsymbol{\mu} \rangle := \sum_{i \in I} \int f_i \, d\mu^i$. Let $\mathcal{E}^+_{\boldsymbol{f}}(\boldsymbol{A})$ consist of all $\boldsymbol{\mu} \in \mathcal{E}^+(\boldsymbol{A})$ with $-\infty < G_{\boldsymbol{f}}(\boldsymbol{\mu}) < \infty$. In what follows, we suppose that either all the f_i are $\geqslant 0$ and lower semicontinuous on $\mathbb{R}^n$ (*Case* 1), or $f_i = s_i \int \kappa_\alpha(\cdot, y) \, d\sigma(y)$ for all $i \in I$, where $\sigma \in \mathcal{E}$ is given (*Case* 2).

Also fix $\boldsymbol{a} = (a_i)_{i \in I}$ with $a_i \in \mathbb{R}_+$ for all $i \in I$ and $g \in C(\mathbb{R}^n)$ satisfying the assumptions $\sum_{i \in I} a_i < \infty$ and $0 < c_1 < g(x) < c_2 < \infty$ for all $x \in \mathbb{R}^n$. Write

$$\mathcal{E}^+_{\boldsymbol{f}}(\boldsymbol{A}, \boldsymbol{a}, g) := \{\boldsymbol{\mu} \in \mathcal{E}^+_{\boldsymbol{f}}(\boldsymbol{A}) : \ \langle g, \mu^i \rangle = a_i \quad \text{for all} \quad i \in I\}$$

and suppose that[2]

$$G_{\boldsymbol{f}}(\boldsymbol{A}, \boldsymbol{a}, g) := \inf_{\boldsymbol{\mu} \in \mathcal{E}^+_{\boldsymbol{f}}(\boldsymbol{A}, \boldsymbol{a}, g)} G_{\boldsymbol{f}}(\boldsymbol{\mu}) < \infty.$$

Problem 1. *Does there exist $\boldsymbol{\lambda} = \boldsymbol{\lambda}_{\boldsymbol{A}} \in \mathcal{E}^+_{\boldsymbol{f}}(\boldsymbol{A}, \boldsymbol{a}, g)$ with $G_{\boldsymbol{f}}(\boldsymbol{\lambda}) = G_{\boldsymbol{f}}(\boldsymbol{A}, \boldsymbol{a}, g)$?*

If $\boldsymbol{A}$ is finite and compact and Case 1 holds, then the existence of $\boldsymbol{\lambda}_{\boldsymbol{A}}$ can be established by exploiting the vague topology only, since then $\mathcal{E}^+_{\boldsymbol{f}}(\boldsymbol{A}, \boldsymbol{a}, g)$ is vaguely compact, while $G_{\boldsymbol{f}}(\cdot)$ is vaguely lower semicontinuous (see [4]). However, these arguments break down if any of the above-mentioned hypotheses is dropped, and then Problem 1 becomes rather nontrivial. E.g., $\mathcal{E}^+_{\boldsymbol{f}}(\boldsymbol{A}, \boldsymbol{a}, g)$ is no longer vaguely

[1] For $\boldsymbol{\mu}$ to have finite energy, it is sufficient that $\sum_{i \in I} \|\mu^i\| < \infty$ (see [8]).

[2] See [8] for necessary and sufficient conditions for this to hold.

compact if any of the A_i is noncompact. Another difficulty is that, in Case 2, $G_{\boldsymbol{f}}(\cdot)$ might not be vaguely lower semicontinuous. These difficulties have been overcome in [8] in the frame of an approach[3] based on the following crucial arguments.

Given $\boldsymbol{\mu} \in \mathcal{E}^+(\boldsymbol{A})$ and a Borel set $B \subset \mathbb{R}^n$, write $R\boldsymbol{\mu}(B) := \sum_{i\in I} s_i\mu^i(B)$.

Theorem 7. *$\mathcal{E}^+(\boldsymbol{A})$ forms a metric space with the metric*

$$\|\boldsymbol{\mu}_1 - \boldsymbol{\mu}_2\|_{\mathcal{E}^+(\boldsymbol{A})} := \Big[\sum_{i,j\in I} s_is_jE(\mu_1^i - \mu_2^i, \mu_1^j - \mu_2^j)\Big]^{1/2}.$$

Moreover, $\|\boldsymbol{\mu}_1 - \boldsymbol{\mu}_2\|_{\mathcal{E}^+(\boldsymbol{A})} = \|R\boldsymbol{\mu}_1 - R\boldsymbol{\mu}_2\|_{\mathcal{E}}$, so that $\mathcal{E}^+(\boldsymbol{A})$ becomes isometric to its R-image, the latter being regarded as a metric subspace of $\mathcal{E}$.

Theorem 8. *The metric space $\mathcal{E}^+(\boldsymbol{A})$ is complete.*[4]

All these enabled us to establish the following (see [8]):

Theorem 9. *Let $\varrho_{\boldsymbol{A}} > 0$, where $\varrho_{\boldsymbol{A}}$ is the Euclidean distance between the oppositely signed plates of $\boldsymbol{A}$. Then, for the Gauss variational problem to be solvable for any $\boldsymbol{a}$, g and $\boldsymbol{f}$, it is sufficient that $C_\alpha(A_i) < \infty$ for all $i \in I$.*

However, if $C_\alpha(A_\ell) = \infty$ for some ℓ, then in general there exists a vector $\boldsymbol{a}$ such that the problem admits *no* solution [8]. *Then, for given $\boldsymbol{A}$, g and $\boldsymbol{f}$, what is a description of the set $\mathcal{S}_{\boldsymbol{f}}(\boldsymbol{A}, g)$ of all $\boldsymbol{a}$ for which the problem is nevertheless solvable?* We shall formulate an answer in the case where A_ℓ is the only plate with $C_\alpha(A_\ell) = \infty$; then the characterization obtained has the most complete form [9].

Suppose that Case 2 holds; then, in fact, $G_{\mathbf{f}}(\boldsymbol{\mu}) = -\|\sigma\|^2 + \|\sigma + R\boldsymbol{\mu}\|^2$. Also assume that σ is compactly supported in $\mathbb{R}^n \setminus \bigcup_{i\in I} A_i$, while $\varrho_{\boldsymbol{A}} > 0$.

Theorem 10. *Then $\mathcal{S}_{\boldsymbol{f}}(\boldsymbol{A}, g)$ consists of all $\boldsymbol{a} = (a_i)_{i\in I}$ with $a_\ell \leqslant \langle g, \widetilde{\lambda}^\ell\rangle$, where $\widetilde{\lambda}^\ell$ is the ℓ-component of the solution $\widetilde{\boldsymbol{\lambda}}$ (it exists) to the auxiliary problem* inf $G_{\boldsymbol{f}}(\boldsymbol{\mu})$*, the infimum being taken over all $\boldsymbol{\mu} \in \mathcal{E}^+(\boldsymbol{A})$ such that $\langle g, \mu^i\rangle = a_i$ for all $i \neq \ell$. Actually,*

$$\langle g, \widetilde{\lambda}^\ell\rangle = \Big\langle g, \mathcal{P}_{A_\ell}\Big(\sigma + \sum_{i\neq\ell} s_i\widetilde{\lambda}^i\Big)\Big\rangle,$$

where $\mathcal{P}_{A_\ell}$ is the operator of orthogonal projection in $\mathcal{E}$ onto $\mathcal{E}^+(A_\ell)$.

Assume, in addition, $\alpha \in (0, 2]$ and $\alpha < n$. Then $\mathcal{P}_{A_\ell}$ is, in fact, the operator of Riesz balayage onto A_ℓ. On account of [6, Theorem 4], we thus led to the following:

Corollary 2. *Let $I = \{1, 2\}$, $s_1 = 1$, $s_2 = -1$, $C_\alpha(A_1) < \infty$, $g = 1$, $\sigma \geqslant 0$, $a_2 = a_1 + \sigma(\mathbb{R}^n)$, and let $\mathbb{R}^n \setminus A_2$ be connected. Then the Gauss variational problem admits no solution if and only if $C_\alpha(A_2) = \infty$ though A_2 is α-thin at $\infty_{\mathbb{R}^n}$.*[5]

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[3] For a background of this approach, see Fuglede's pioneering work [1] (where $I = \{1\}$, $g = 1$, and $f = 0$) and the author's study [7] (where I is finite and $\boldsymbol{f} = \mathbf{0}$).

[4] The proof of Theorem 8 is based on the isometric between $\mathcal{E}^+(\mathbf{A})$ and its R-image, Deny's result mentioned above, and the fact that a positive distribution is a measure. Thus, if compared with Cartan's counterexample on the incompleteness of $\mathcal{E}$, a crucial assumption in Theorem 8 is that the supports of $R\boldsymbol{\mu}^+$ and $R\boldsymbol{\mu}^-$, where $\boldsymbol{\mu}$ ranges over $\mathcal{E}^+(\mathbf{A})$, are uniformly disjoint.

[5] A closed set F is *α-thin at $\infty_{\mathbb{R}^n}$* if F^*, the inverse of F relative to the unit sphere, is α-thin at $x = 0$; or equivalently [3], if either F is bounded or $x = 0$ is an α-irregular point for F^*.

Example 1. *Under the assumptions of Corollary 2, let* $n = 3$ *and let* A_2 *be a rotational body consisting of all* $x = (x_1, x_2, x_3) \in \mathbb{R}^3$ *with* $q \leqslant x_1 < \infty$*, where* $q \in \mathbb{R}$*, and* $0 \leqslant x_2^2 + x_3^2 \leqslant \rho(x_1)$*. Consider the following three cases:*

$$\text{(1)} \qquad \alpha \in (0,2], \qquad \rho(r) = r^{-t}, \qquad t \in [0,\infty),$$

$$\text{(2)} \qquad \alpha \in (0,2], \qquad \rho(r) = \exp(-r^t), \qquad t > 1,$$

$$\text{(3)} \qquad \alpha = 2, \qquad \rho(r) = \exp(-r^t), \qquad t \in (0,1].$$

Then A_2 *is not* α*-thin at* $\infty_{\mathbb{R}^3}$ *in case (1), has finite (Riesz) capacity in case (2), and it is* 2*-thin at* $\infty_{\mathbb{R}^3}$ *though* $C_2(A_2) = \infty$ *in case (3) (see* [5]*). Hence, by Corollary 2, the Gauss variational problem is solvable in both cases (1) and (2), while it is nonsolvable in case (3). See also* [2] *for some related numerical experiments.*

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Open Problems Session

Mixed packings of balls (Oleg Musin): Assume that we pack balls of two sizes, r_1 and r_2, in the Euclidean space. Assume also that the sizes are distributed (in terms of the limit cardinalities of the sets of balls touching a large ball B_R when $R \to +\infty$) in a prescribed proportion $\alpha : (1-\alpha)$. Denote by k_{ij} the mean kissing (contact) number of balls of radius r_j around balls of radius r_i and consider the packing density defined in terms of the limit ratio of the covered volume. Some experimental data were obtained in [1] raising many questions to be rigorously considered. For example:

Problem. *What is the maximal packing density and kissing numbers* k_{11}*,* k_{12}*,* k_{21}*, and* k_{22} *in such packings? Is it true that in the case when* r_1 *and* r_2 *are close to each other, the packing density coincides with the packing density for balls of same size, that is different balls do not mix?*

Minimizing the sum of squares (Oleg Musin): An olympiad problem (communicated by Fedor Petrov) asserts that given a finite point set P in the unit square I^2 it is possible to connect its points by a Hamiltonian cycle with the sum of squares of edge lengths at most 4. Evgeniy Shchepin noted that the cycles minimizing the sum of squares generate certain interesting Peano curves in the unit square.

A bunch of similar questions were discussed by Bern and Eppstein in [2]. For example, it is true that for any set P of an even number of points in the d-dimensional cube I^d it is possible to find a full matching (a partition into pairs) of these points such that the sum of d-th powers of the lengths of matching segments is bounded by a constant C_d.

Returning to the planar case and sums of squares one may ask:

Problem. *Let B be the disk of radius one in the plane. Is it true that any finite $P \subset B$ can be spanned by a Hamiltonian cycle with the sum of squares at most 8? Is it true that any finite P with even number of points has a full matching with sum of squares at most 4? What can be done for other convex figures C in place of B or I^2?*

It seems like the case $|P| = 4$ can be done by hand, the extremal configurations being inscribed quadrangles with perpendicular diagonals and inscribed triangles together with their orthocenters, as well as the limiting cases of them.

Subcodes of $S(\Lambda_{24})$ (Henry Cohn): This problem is closely related to Abhinav Kumar talk.

Let $S(\Lambda_{24})$ denote the set of minimal vectors in the leech lattice Λ_{24}. The inner product of two distinct elements in $S(\Lambda_{24})$ belongs to $\{-4, -2, -1, 0, 1, 2\}$. The problem is to find a large subset $C \subset S(\Lambda_{24})$ such that $(x, y) \leq 1$ for all $x \neq y \in C$. A set of 480 element was found by computer search. A set of size 554 would improve the best known kissing number in dimension 32.

The s-polarization of N points (Ed Saff): For $w_N = (x_1, \dots, x_N) \in (S^2)^N$, where S^2 is the unit sphere of $\mathbb{R}^3$, let

$$M^s(w_N, S^2) = \min_{x \in S^2} \sum_{j=1}^{N} \frac{1}{|x - x_j|^s}$$

be the s-polarization of w_N with respect to S^2. According to numerical experiments, the square-based pyramid is optimal for the 5-point s-polarization:

$$M^s(S^2) = \max_{w_5} M^s(w_5, S^2)$$

for s up to a certain value $s_0 \approx 2.69$. Prove it.

The optimal configuration of 7 points for the log energy minimization problem (Peter Dragnev): For $w_N = (x_1, \dots, x_N)$ a set of N points on the unit sphere of $\mathbb{R}^3$, let

$$E_{\log}(w_N) = \sum_{1 \leq i \neq j \leq N} \log \frac{1}{\|x_i - x_j\|}.$$

Prove that the configuration of 7 points given by a regular pentagon on the equator, together with the north and south poles, minimizes $E_{\log}(w_7)$. This is Rakhmanov's problem.

The maximal measure of a subset of S^2 that avoids $\pi/2$ (Frank Vallentin): Let A be a measurable subset of S^2 such that, for $x, y \in A$, $(x, y) \neq 0$. What is the largest possible A ? If A is the union of two antipodal spherical caps of angular radius $\pi/4$, and if μ denotes the Lebesgue measure on S^2, we have $\mu(A)/\mu(S^2) = 1 - 1/\sqrt{2} \approx 0.29$. It is conjectured that this set is optimal. The best known upper bound for $\mu(A)/\mu(S^2)$ is 1/3, easily obtained from the set $\{(1,0,0),(0,1,0),(0,0,1)\} \subset S^2$. Can one prove a better upper bound ?

This problem first appeared in [3].

The maximal density of a 1-avoiding set in $\mathbb{R}^2$ (Fernando Mario de Oliveira Filho): This problem is closely related to the chromatic number of the plane $\chi(\mathbb{R}^2)$, for which we only know that $4 \leq \chi(\mathbb{R}^2) \leq 7$. Let $A \subset \mathbb{R}^2$ be a measurable set, such that $\|x - y\| \neq 1$ for all $x, y \in A$. Let $\delta(A)$ denotes its density, and let $m_1(\mathbb{R}^2)$ denotes the supremum of the densities of all such subsets. Erdös has conjectured that $m_1(\mathbb{R}^2) < 1/4$. Currently the best known upper bound is by 0.26.., and the best known lower bound is 0.22...

If G is a finite subgraph of the unit distance graph, if V is its vertex set, and $\alpha(G)$ its independence number, we have $m_1(\mathbb{R}^2) \leq \alpha(G)/|V|$. The Moser graph gives $2/7 = 0.28$.. Problem: find a finite graph with $\alpha(G)/|V| < 2/7$.

The number of local minima for the Coulomb potential on S^2 (Henry Cohn): Let L_n be the number of local minima for the Coulomb potential $1/\|x-y\|$ taken over n points of S^2.

Q1 Is it true that $L_n > 1$ for n large enough ?
Q2 Does L_n grow exponentialy with n ?
Q3 What is the energy gap from the global minimum ?
Q4 What it the size of the basins of attraction in the gradiant descent method ?

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