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Rough Paths and PDEs

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Abstract. The purpose of the Oberwolfach workshop "Rough Paths and PDEs" was to bring together these researchers, both young and senior, with the aim to promote progress in rough path theory, the connections with partial differential equations and its applications to numerical methods.

Mathematics Subject Classification (2000): 34A99, 35R99, 60H10, 60H15, 65C99.

Introduction by the Organisers

The rough path theory, initiated be T. Lyons (workshop participant) in the nineties has had a profound influence on stochastic analysis; its single most important results is that solutions to stochastic differential equations can be solved pathwise and that the solution map is continuous (even locally Lipschitz) in rough path metric. This continuity property has since become the key in many striking applications, ranging from the Stroock-Varadhan support theorem in its as-of-yet strongest form to a new understanding of Hörmander's theory without Markovian structure. Much of this has been summarized in a recent monograph of Friz (workshop organizer). By applying and extending rough paths ideas to (stochastic) partial differential equations, a fruitful connection was established between the stability of (stochastic) flows in rough path sense and the stability properties of viscosity solutions to PDEs. In particular, large classes of SPDEs are reduced to (deterministic) partial differential equations driven by rough signals. This is closely related to the (essentially pathwise) Lions-Souganidis theory of stochastic viscosity solution. Souganidis was a participant at the workshop. A related set of new ideas is to introduce rough path stability in the context of backward (doubly) stochastic differential equations (BSDEs); in a sense this amounts to non-linear Feynman-Kac formulae for rough partial differential equations. BSDEs have been introduce in the eighties by another workshop participant, Shige Peng. Another important application: stochastic filtering is concerned with the estimation of the conditional law of a Markov process, given observations of some function of it. Using the tools provided by rough paths one can show that is essential to measure not just the observation process but also its associated area process. In other words, filtering has now become an outlet for rough paths developments. The following workshop participants are active in this area Diehl, Oberhauser, Friz and Crisan. Lastly, rough paths theory has had an importance influence in the area of numerical approximations of solutions of PDEs deterministic as well as stochastic. Litterrer, Lyons and Crisan work on this topic.

The Mathematisches Forschungsinstitut Oberwolfach offered the ideal environment to enhance the synergy between the participant experts working in these related areas.

Workshop: Rough Paths and PDEs

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Abstracts

Geometric Structure of the Reachability set

Youness Boutaib
(joint work with Terry Lyons)

In the seventies, mathematicians working on Control Theory (e.g. [1], [2]) studied the geometric structure of the reachability set (the formal definition of which we give later) associated to different families of controlled differential equations in order to get sufficient conditions for (bang-bang) controllability. More specifically, given an initial condition $\xi \in E$, where E is a (finite-dimensional) vector space, and a set of controls Ω, one looks at the problem of controllabity associated to the differential equations:

$$\begin{cases} \mathrm{d}y_t &= A(y_t, u_t, x_t)\mathrm{d}t \quad ,\forall t \in [0,T], u \in \Omega \\ y_0 &= \xi \end{cases}$$

We use the same idea (studying the geometric structure of the reachability set) in the case of the following rough differential equation:

$$\begin{cases} \mathrm{d}y_t &= \sum_{i=1}^{d} A^i(y_t)\mathrm{d}x_t^i \quad ,\forall t \in [0,T] \\ y_0 &= \xi \end{cases} \tag{1}$$

where d is a positive integer, $T > 0$, $A_1, \ldots, A_d$ are γ-Lipschitz vector fields on E (with $\gamma > 1$) and $x := (x_1, \ldots, x_d)$ is a geometric p-rough path in $\mathbb{R}^d$ (with $p < \gamma$ so that (1) makes sense). Before developping more on the problems we seek to solve, let us first give a formal definition of the reachability set:

Definition 1 (Reachability set)**.** *Let G be a family of geometric p-rough paths. We call the reachability set associated with the family of rough differential equations (1) defined by the vector fields $A := (A^1, \ldots, A^d)$, the initial condition ξ and the set of controls G the set:*

$$\mathcal{R}(\xi, A, G) = \{y_T(x) | x \in G\}$$

With the example of the signature of paths and Chow-Rashevskii's theorem in mind (see for example [3] and [4] (chapter 2)), we ask ourselves the question whether the reachability set defined by all p-rough paths is the same as the one defined by all lattice paths. The idea is to endow the latter with "enough" smooth structure for (1) to make sense.

In [5], the authors develop a theory of Lipschitz manifolds on which rough paths and rough differential equations make sense and which is consistent with the classical theory when the manifold in question is a finite-dimensional vector space. Now smooth manifolds look locally like Lipschitz manifolds and one could locally make sense (and solve) rough differential equations on them. It would be then enough to use the existing work (e.g. [1]) to state that under suitable conditions on the vector fields $A_1, \ldots, A_d$, the reachability set defined by lattice paths is a smooth manifold on which (1) locally makes sense and is therefore the same

as the reachability set defined by all geometric p-rough paths. If one thinks of the truncated signature as a solution to a particular rough differential equation, it is known then that the reachability set defined by all rough paths is the free nilopotent group (which is a Lie group and therefore a smooth manifold) and the terminal value of the truncated signature of a rough path, by Chow-Rashevskii's theorem, corresponds to the terminal value of the truncated signature of a lattice path.
A more interesting question to ask though is to quantify the p-variation of the lattice path which gives the same terminal value of the solution to (1) driven by a given geometric p-rough paths. It is for this purpose that we seek to put more structure on the reachability set defined by lattice paths. This structure turns out to be exactly a structure of a Lipschitz manifold given that the vector fields $A_1, \dots, A_d$ satisify a "non-degeneracy" condition in addition to the UFG condition (see for example [6]) or the locally of finite type condition (see [1]) necessary to obtain the smooth structure discussed above. This opens the door to more interesting results like a quantitative mean value theorem in the case of rough paths.

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Smoothness of the density for solutions to Gaussian RDEs

Thomas Cass
(joint work with Martin Hairer, Chrisitan Litterer, Samy Tindel)

This abstract is based on the introduction to the recent preprint [7].

Over the past decade our understanding of stochastic differential equations (SDEs) driven by Gaussian processes has evolved considerably. As a natural counterpart to this development, there is now much interest in investigating the probabilistic properties of solutions to these equations. Consider an SDE of the form

$$dY_t = V(Y_t)dX_t + V_0\,(Y_t)\,dt, \qquad Y\,(0) = y_0 \in \mathbb{R}^e, \tag{1}$$

driven by an $\mathbb{R}^d$-valued continuous Gaussian process X along C_b^∞-vector fields V_0 and $V = (V_1, \dots, V_d)$ on $\mathbb{R}^e$. Once the existence and uniqueness of Y has been settled, it is natural to ask about the existence of a smooth density of Y_t for

$t > 0$. In the context of diffusion processes, the theory is classical and goes back to Hörmander [23] for an analytical approach, and Malliavin [28] for a probabilistic one.

For the case where X is fractional Brownian motion, this question was first addressed by Nualart and Hu [24], where the authors show the existence and smoothness of the density when the vector fields are elliptic, and the driving Gaussian noise is fractional Brownian motion (fBM) for $H > 1/2$. Further progress was achieved in [1] where, again for the regime $H > 1/2$, the density was shown to be smooth under Hörmander's celebrated bracket condition. Rougher noises are not directly amenable to the analysis put forward in these two papers. Additional ingredients have since gradually become available with the development of a broader theory of (Gaussian) rough paths (see [26], [9], [13]). The papers [6] and [5] used this technology to establish the existence of a density under fairly general assumptions on the Gaussian driving noises. These papers however fall short of proving the smoothness of the density, because the proof demands far more quantitative estimates than were available at the time.

More recently, decisive progress was made on two aspects which obstructed the extension of this earlier work. First, the paper [8] established sharp tail estimates on the Jacobian of the flow $J^{\mathbf{X}}_{t\leftarrow 0}(y_0)$ driven by a wide class of (rough) Gaussian processes. The tail turns out to decay quickly enough to allow to conclude the finiteness of all moments for $J^{\mathbf{X}}_{t\leftarrow 0}(y_0)$. Second, [22] obtained a general, deterministic version of the key Norris lemma (see also [25] for some recent work in the context of fractional Brownian motion). The lemma of Norris first appeared in [30] and has been interpreted as a quantitative version of the Doob-Meyer decomposition. Roughly speaking, it ensures that there cannot be too many cancellations between martingale and bounded variation parts of the decomposition. The work [22] however shows that the same phenomenon arises in a purely deterministic setting, provided that the one-dimensional projections of the driving process are sufficiently and uniformly rough. This intuition is made precise through the notion of the "modulus of Hölder roughness". Together with an analysis of the higher order Malliavin derivatives of the flow of (1), also carried out in [22], these two results yield a Hörmander-type theorem for fractional Brownian motion if $H > 1/3$.

In this paper we aim to realise the broader potential of these developments by generalising the analysis to a wide class of Gaussian processes. This class includes fractional Brownian motion with Hurst parameter $H \in (\frac{1}{4}, \frac{1}{2}]$, the Ornstein-Uhlenbeck process, and the Brownian bridge. Instead of focusing on particular examples of processes, our approach aims to develop a general set of conditions on X under which Malliavin-Hörmander theory still works.

The probabilistic proof of Hörmander's theorem is intricate, and hard to summarise in a few lines, see [19] for a relatively short exposition. However, let us highlight some basic features of the method in order to see where our main contributions lie:

(i) At the centre of the proof of Hörmander's theorem is a quantitative estimate on the non-degeneracy of the Malliavin covariance matrix $C_T(\omega)$.

Our effort in this direction consists in a direct and instructive approach, which reveals an additional structure of the problem. In particular, the conditional variance of the process plays an important role, which does not appear to have been noticed so far. More specifically, following [6] we study the Malliavin covariance matrix as a 2D Young integral against the covariance function $R(s,t)$. This provides the convenient representation:

$$v^T C_t(\omega) v = \int_{[0,t]\times[0,t]} f_s(v;\omega) f_r(v;\omega)\, dR(s,r),$$

for some γ-Hölder continuous $f(v;\omega)$, which avoids any detours via the fractional calculus that are specific to fBM. Compared to the setting of [5] we have to impose some additional assumptions on $R(s,t)$, but our more quantitative approach allows us in return to relax the zero-one law condition required in this paper.

(ii) An essential step in the proof is achieved when one obtains some lower bounds on $v^T C_t v$ in terms of $|f|_{\infty;[0,t]}$. Towards this aim we prove a novel interpolation inequality, which lies at the heart of this paper. It is explicit and also sharp in the sense that it collapses to a well-known inequality for the space $L^2([0,T])$ in the case of Brownian motion. Furthermore, this result should be important in other applications in the area, for example in establishing bounds on the density function (see [2] for a first step in this direction) or studying small-time asymptotics.

(iii) Hörmander's theorem also relies on an accurate analysis and control of the higher order Malliavin derivatives of the flow $J^{\mathbf{X}}_{t\leftarrow 0}(y_0)$. This turns out the be notationally cumbersome, but structurally quite similar to the technology already developed for fBm. For this step we therefore rely as much as possible on the analysis performed in [22]. The integrability results in [8] then play the first of two important roles in showing that the flow belongs to the Shigekawa-Sobolev space $\mathbb{D}^\infty(\mathbb{R}^e)$.

(iv) Finally, an induction argument that allows to transfer the bounds from the interpolation inequality to the higher order Lie brackets of the vector fields has to be set up. This induction requires another integrability estimate for $J^{\mathbf{X}}_{t\leftarrow 0}(y_0)$, plus a Norris type lemma allowing to bound a generic integrand A in terms of the resulting noisy integral $\int A\,dX$ in the rough path context. This is the content of our second main contribution, which can be seen as a generalisation of the Norris Lemma from [22] to a much wider range of regularities and Gaussian structures for the driving process X. Namely, we extend the result of [22] from p-rough paths with $p < 3$ to general p under the same "modulus of Hölder roughness" assumption. It is interesting to note that the argument still only requires information about the roughness of the path itself and not its lift.

Let us further comment on the Gaussian assumptions allowing the derivation of the interpolation inequality briefly described in Step (ii) above. First, we need a standing assumption that regards the regularity of $R(s,t)$ (expressed in terms of its so called 2D ρ-variation, see [13]) and complementary Young regularity of X and its Cameron-Martin space. This is a standard assumption in the theory of Gaussian rough paths. The first part of the condition guarantees the existence of a natural lift of the process to a rough path. The complementary Young regularity in turn is necessary to perform Malliavin calculus, and allows us to obtain the integrability estimates for $J^{\mathbf{X}}_{t\leftarrow 0}(y_0)$ in [8].

In order to understand the assumptions on which our central interpolation inequality hinges, let us mention that it emerges from the need to prove lower bounds of the type:

$$\int_{[0,T]\times[0,T]} f_s f_t \, dR(s,t) \geq C \, |f|^a_{\gamma;[0,T]} \, |f|^{2-a}_{\infty;[0,T]}, \tag{2}$$

for some exponents γ and a, and all γ-Hölder continuous functions f. After viewing the integral in (2) along a sequence of discrete-time approximations to the integral, relation (2) relies on solving a sequence of finite dimensional partially constrained quadratic programming (QP) problems. These (QP) problem involve some matrices Q whose generic element can be written as $Q^{ij} = E[X^1_{t_i,t_{i+1}} X^1_{t_j,t_{j+1}}]$, where $X^1_{t_i,t_{i+1}}$ designates the increment $X^1_{t_{i+1}} - X^1_{t_i}$ of the first component of X. Interestingly enough, some positivity properties of Schur complements computed within the matrix Q play a prominent role in the resolution of the aforementioned (QP) problems. In order to guarantee these positivity properties, we shall make two non-degeneracy type assumptions on the conditional variance and covariance structure of our underlying process X^1. This is obviously quite natural, since Schur complements are classically related to conditional variances in elementary Gaussian analysis. We also believe that our conditions essentially characterise the class of processes for which we can quantify the non-degeneracy of $C_T(\omega)$ in terms of the conditional variance of the process X.

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Solving semilinear partial differential equations using the cubature method

Dan Crisan
(joint work with J-F Chassaneux and K. Manolarakis)

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a complete probability space endowed with a filtration $\{\mathcal{F}_t\}_{t\geq 0}$ satisfying the usual conditions and $W = \{W_t,\, t \geq 0\}$ be an $\mathcal{F}_t$-adapted Brownian motion defined on $(\Omega, \mathcal{F}, \mathbb{P})$. Let T be a fixed time horizon which we fix henceforth and consider the triplet $(X, Y, Z) = \{(X_t, Y_t, Z_t), t \in [0, T]\}$ of $\mathcal{F}_t$-adapted stochastic processes satisfying the following system of equations

$$\begin{cases} dX_t &= V_0(X_t)dt + \sum_{j=1}^d V_j(X_t) \circ dW_t^j \\ -dY_t &= f(t, X_t, Y_t, Z_t)dt - Z_t dW_t \end{cases} . \tag{1}$$

The system (1) is called a forward-backward stochastic differential equation (FB-SDE). The process X, called the forward component of the FBSDE, is a d-dimensional diffusion satisfying an SDE with coefficients $V_i : \mathbb{R}^d \to \mathbb{R}^d$, $i = 0, 1, ..., d$ with all entries belonging to $C_b^\infty(\mathbb{R}^d)$, the space of bounded infinitely differentiable functions with all partial derivatives bounded. The notation "$\circ$" indicates that the stochastic term in the equation satisfied by X is a Stratonovitch integral. The process Y, called the backward component of the SDE is a one-dimensional stochastic process with final condition $Y_T = \Phi(X_T)$, where $\Phi : \mathbb{R}^d \to \mathbb{R}$ is a continuous function and the function $f : [0, T] \times \mathbb{R}^d \times \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}$ referred to as " the driver", is Lipschitz.

In [1] and [2], it is shown that these processes provide a Feynman-Kac representation for solutions of semilinear partial differential equations (PDEs) which appear in many applications in the field of Mathematical Finance. In particular, let u be the solution of the final value Cauchy problem

$$\begin{cases} \partial_t u = Lu + f(t, x, u, V_1 u, ..., V_d u), \quad t \in (0, T], x \in \mathbb{R}^m \\ u(0, x) = \Phi(x), \quad x \in \mathbb{R}^m \end{cases} , \tag{2}$$

where L is the second order differential operator

$$L\varphi = V_0\varphi + \frac{1}{2}\sum_{i=1}^d V_i^2 \varphi. \tag{3}$$

In particular, if $d = m$ and $V_i = \partial_{x_i}$, $i = 1, ..., d$ and $V_0 = 0$, then (2) becomes

$$\partial_t u = \frac{1}{2}\Delta u + f(t, x, u, \nabla u).$$

Then the solution of the PDE (2) admits the Feynman-Kac representation

$$u(t, x) = Y_{T-t}^{T-t,x} = \mathbb{E}\left[\Phi(X^{T-t,x}(T)) + \int_{T-t}^T f(s, X_s^{t,x}, Y_s^{t,x}, Z_s^{t,x})ds\right], \tag{4}$$

where $(X^{r,x}, Y^{r,x}, Z^{r,x})$ is the 'stochastic flow' associated to the FBSDE. We deduce from the Feynman-Kac representation (4) that there exists $\Lambda'_{t,x} : C_{\mathbb{R}^m}[0, T] \to$

$\mathbb{R}$ such that

$$u(t,x) = \mathbb{E}\left[\Lambda_{t,x}\left(W\right)\right] = \int_{\omega \in C([0,\infty),\mathbb{R}^d)} \Lambda_{t,x}(\omega) dP_W(\omega).$$

We exploit this property to construct a numerical approximation of the solution of the semilinear PDE that involves the following three procedures:

- replacing P_W with $P_{\tilde{W}} = \frac{1}{n}\sum_{i=1}^{n} \delta_{\omega_i}$ - $\tilde{W}$ approximates the signature of W.
- approximating $\Lambda_{t,x}$ with an explicit/simple version $\tilde{\Lambda}_{t,x}$.
- controlling the computational effort (using the Tree Based Branching Algorithm).

Let us define

$$\tilde{u}^{\delta}(0,x) = \int_{\omega \in C([0,\infty),\mathbb{R}^d)} \tilde{\Lambda}_{t,x}(\omega) dP_{\tilde{W}}(\omega) = \tilde{\Lambda}_{t,x}(\omega_i).$$

We prove (under additional assumptions on the coefficients of the PDE) that

$$\sup_{x\in\mathbb{R}^d} \mathbb{E}[|\tilde{u}^{\delta}(0,x) - u(0,x)|] \leq C\left(\delta^k + \delta^{\frac{m-1}{2}} + \frac{1}{\sqrt{N}}\right),$$

where δ is the mesh of the discretizing partition of $\Lambda_{t,x}$, k is discretization order, m is the level of approximation of the signature of W $(m = 3, 5, ...)$ and N is the size of the computational effort.

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Convolutional rough paths

Aurélien Deya

The talk will consist in a brief survey on the method introduced by Gubinelli and Tindel ([1]) to study evolution equations with perturbation driven by non-differentiable paths. We will review some of the basic principles of this approach, as well as a few stochastic applications derived from it.

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Controlled rough differential equations and applications to stochastic control

Joscha Diehl
(joint work with P. Friz, P. Gassiat)

We study controlled rough differential equations (controlled RDEs) of the form

$$dX_t^{s,x,\nu} = b(X_t^{s,x,\nu}, \nu_t)dt + \sigma(X_t^{s,x,\nu})d\eta_t, \qquad X_s^{s,x,\nu} = x, \tag{1}$$

for some geometric rough path $\eta \in C^{0,p-var}([0,T], G^{[p]}(\mathbb{R}^d)$. Here $\nu \in \mathcal{M}$, the class of all measurable functions on $[0,T]$ taking values in some control set U. Existence, uniqueness and stability follow easily from the general theory (which is exposed for example in [7]), by defining the vector-field valued path of bounded variation $B_t := \int_0^t b(\cdot, \nu_r)dr$ and solving

$$dX = XdB + \sigma(X)d\eta.$$

We study the related (finite time horizon) optimal control problem and define the value function

$$v(t,x) := \sup_\nu \left\{ \int_t^T f(r, X_r^{t,x,\nu}, \nu_r)dr + g(X_T^{t,x,\nu}) \right\}.$$

It turns out, that v is the solution to the rough HJB equation.

$$-dv(t,x) - H(x, Dv)dt - \langle \sigma(x), Dv(t,x) \rangle d\eta_t. \tag{2}$$

There are several approaches to make sense of equations of this type ([2, 1, 3]). We follow the viscosity solution-setting of [8] and say that v solves (2) if it is the limit of viscosity solutions v^n to (2) whith η beeing replaced by smooth approximations (in rough path metric).

At this points it is still an open question whether solutions to such equations can be intrinsically characterised; say analogously to classical viscosity theory by using "touching test functions" (but see [1] for results in this directions).

As described just now, the approach of dynamic programming (which corresponds to the HJB equation in the infinitesimal) works for controlled RDEs as it does for classical controlled ODEs. We currently investigate whether a maximum principle does also hold.

Consider now the stochastic optimal control problem

$$V(t,x) := \sup_u \mathbb{E}[\int_t^T f(r, X_r^{t,x,u}, u_r)dr + g(X_T^{t,x,u})],$$

where the supremum is taken over all progressively measurable controls u and X solves the stochastic differential equation

$$dX_t^{s,x,u} = b(X_t^{s,x,u}, u_t)dt + \sigma(X_t^{s,x,u})dB_t, \qquad X_t^{s,x,u} = x,$$

driven by some d-dimensional Brownian motion B.

In the literature there exist several ideas to use pathwise optimization (e.g. fixing a realization ω and *then* performing optimization) in order to obtain duality formulas, see e.g. [4, 5, 6].

Using rough path theory one avoids technical difficulties in the continuous time setting, and we obtain immediately the following result.

Theorem 1. *Let $p \in (2,3)$. Let $\eta \in C^{0,p-var}([0,T],\mathbb{R}^d)$ be a rough path. Let $\gamma > p$. Let $b : \mathbb{R}^e \times U \to \mathbb{R}^e$ be continuous and let $b(\cdot,u) \in \mathrm{Lip}^1(\mathbb{R}^e)$ uniformly in $u \in U$. Let $\sigma_1,\dots,\sigma_d \in \mathrm{Lip}^\gamma(\mathbb{R}^e)$. Let $g \in BUC(\mathbb{R}^e)$. Let $f : [0,T]\times\mathbb{R}^e\times U \to \mathbb{R}$ be bounded, continuous and locally uniformly continuous in t,x, uniformly in u.*

Let $\mathcal{Z}_\mathcal{F}$ be the class of all mappings $z : C^{0,p-var}([0,T],\mathbb{R}^d)\times\mathcal{M} \to \mathbb{R}^d$ such that

- *z is measurable*
- *$\mathbb{E}[z(\mathbf{B},u)] \geq 0$, if u is adapted*

Let $\mathbf{B}$ be the rough-path lift of the Brownian motion B.
Then

$$V(t,x) = \inf_{z\in\mathcal{Z}_\mathcal{F}} \mathbb{E}[\sup_{\nu\in\mathcal{M}} \left\{ \int_t^T f(r, X_r^{t,x,\nu,\eta}, u_r)dr + g(X_T^{t,x,\nu,\eta}) + z(\eta,\nu) \right\}\Bigg|_{\eta=\mathbf{B}(\omega)}].$$

This general statement can be specialized to obtain the duality result in [5] as well as the (continuous time analogue) of a a result in [4]. We are currently looking for other variants, especially ones that could be used for competitive numerical methods.

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Holomorphic functions and subelliptic heat kernels over Lie groups

Bruce Driver
(joint work with Len Gross, Laurent Sallof-Coste)

A Hermitian form q on the dual space, $\mathfrak{g}^*$, of a Lie algebra, $\mathfrak{g}$, of a Lie group, G, determines a Laplacian, Δ, on G. Assuming Hörmander's condition for hypoellipticity, the subelliptic heat semigroup, $e^{t\Delta/4}$, is given by convolution by a C^∞ probability density ρ_t. Analogous to earlier work in the strongly elliptic case, we are able to show that if G is complex, connected, and simply connected then the Taylor expansion defines a unitary map from the space of holomorphic functions in $L^2(G, \rho_t)$ onto (a subspace of) the dual of the universal enveloping algebra in the norm induced by q. This work is related to an extension of the bosonic Fock space to the noncommutative Lie group setting.

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Examples from physics and economics where rough paths matter

Peter K. Friz

1. Phyiscal Brownian motion in a magnetic field as rough path, joint with P. Gassiat

Newton's second law for a particle in $\mathbb{R}^3$ with mass m, and position $z = z(t)$, (for simplicity: constant) frictions $\alpha_1, \alpha_2, \alpha_3 > 0$ in the coordinate axis, subject to a (3-dimensional) white noise $\xi = \xi(t)$ reads

$$mz'' = -Az' + \xi \tag{1}$$

where $A = diag(\alpha_1, \alpha_2, \alpha_3)$. Orthonormal change of coordinates implies that the "correct" assumption for A is to be symmetric with strictly positive spectrum,

$$\sigma(A) \subset (0, \infty).$$

$\sigma(A)$. The process $z(t)$ describes what is known as *physical Brownian motion.* Let us now assume that our particle (with position z, velocity z') carries an electric charge $q \neq 0$ and moves in a (for simplicity, constant) magnetic field $\mathbb{B}$. Recall that such a particle experiences a sideways force ("Lorentz force") that is proportional to the strength of the magnetic field, the component of the velocity that is perpendicular to the magnetic field and the charge of the particle, $F_{Lorentz} = qz' \times \mathbb{B}$..

When $\mathbb{B}$ is constant, which we assume for simplicity, the Lorentz force experienced by the particle (at time t) can be written as as linear function of $z' = z'(t)$, namely qBz' for some anti-symmetric matrix B. In other words,

$$\begin{aligned} mz'' &= -Az' + qBz' + \xi \\ &\equiv -Mz' + \xi. \end{aligned}$$

Observe that for zero mass ($m = 0$), and $M = A - qB$, the process Mz is a bona fide 3-dimensional Brownian motion. We want to study the limit $m \to 0$.

Set $m = \varepsilon^2$ and rewrite the above differential equations as evolution in phase space, introducing moment $p := \varepsilon y := mz'$. Since $mz'' = \varepsilon y'$ anbd $z' = \varepsilon y/m = y/\varepsilon$, it follows that $\varepsilon y' = -M\varepsilon^{-1}y + \xi$ and hence we are led to the 6-dimensional SDE, a special case of which was studied in [4] with multiscale methods,

$$\begin{aligned} dY^\varepsilon &= -\varepsilon^{-2}MY^\varepsilon dt + \varepsilon^{-1}dW \\ dZ^\varepsilon &= \varepsilon^{-1}Y^\varepsilon dt. \end{aligned}$$

We can then show the following result. As $\varepsilon \to 0$, $MZ^\varepsilon \in C^1$ with its canonical area converges to a Brownian rough path, where the area (over time $[0,t]$) is equal to Lévy's area if and only if $qB = 0$, that is in the case of absence of a magnetic field or in the case that the particle carries no charge. Extensions to (friction resp. magnetic) vector fields non constant in space, and also situations with fractional noise, are currently under investigation.

2. A multi-dimensional asset model under presence of infinitesimally delayed market reaction

Consider an IID sequence (ξ) of standard d-dimensional Gaussians, $\xi_i \sim N\left(0, I^d\right)$. A simple discrete asset price model (under the market measure; here we are not interested in hedging, risk-neutral pricing, completeness of the market etc.) is

$$X_{i+1} - X_i = \alpha X_i \xi_{i+1} + \delta X_i$$

where $\alpha = (\alpha_j), \delta \in \mathbb{R}^{n \times n}$, such as to model n assets, $X = \left(X^1, \ldots, X^n\right)$. One may interprete this equation in saying that exogenous randomness ("information"), modelled by (ξ), trigger market moves modelled by α. (Of course, there are many ways to enrich this model but the simple, linear model given above already has all the structure which leads to the phenomena described below.)

It is rather obvious that not all market participants react to new information at same speed; although most do try to act quickly. We can incorporate this in the discrete model via

$$X_{i+1} - X_i = \{\alpha X_i \xi_{i+1} + \beta X_i \xi_i + \gamma X_i \xi_{i-1} + ...\} + \delta X_i$$

where (in some sense) $\alpha >> \beta >> \gamma >>$ .... Here, β models the (bulk) behaviour of market participants - let us call them β agents - which react one time unit later than α agents. One may regard α agents as "fast investors" with immediate reaction modelled by α. In this spirit, β agents may be regarded as "not-so-fast investors" (but far from lazy); in fact, hegde funds which deliberatly try to take advantage of market overreactions caused by the α agents (and their algorithmic trading systems) may - at least partially - act as β-agents. And so on. To keep things simple, we take $\gamma = 0$ and all further dots above to be zero (somewhat ignoring effects due to medium - very lazy investors; extensions are possible of course) and consider

$$X_{i+1} - X_i = \{\alpha X_i \xi_{i+1} + \beta X_i \xi_i\} + \delta X_i.$$

Since new information arrives in continuous time, as are the reactions of the market participants, there is every reason to switch to continuous time and we shall consider in the same spirit

$$dX(t) = \alpha X dW(t) + \beta X dW(t-\varepsilon) + \delta X dt$$

in the $\varepsilon \to 0$ regime. (This is not a simple scaling limit of the discrete model, but at least with $\varepsilon = 1$, the standard Euler-scheme with step-size 1 will bring us back to the discrete equation.) One may be tempted to believe that, in this limit, the effective behaviour simply reduces to

$$\begin{aligned} dX(t) &= \alpha X dW(t) + \beta X dW(t) + \delta X dt \\ &= (\alpha + \beta) X dW(t) + \delta X dt; \end{aligned}$$

which amounts to a superposition principle for the behaviour of α- and β-agents. Curiously enough this is false, and the asset prices (in the small delay limit) is affected by a non-linear interplay between α and β. In essence, the reason is that Brownian motion and its delay (in the small delay limit) produce a non-trivial rough path; as was first understood in [2].

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Spatial rough path lifts of stochastic convolutions

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(joint work with Peter Friz, Archil Gulisashvili, Sebastian Riedel)

The lack of spatial regularity of solutions to SPDE often causes serious obstacles concerning well-posedness and stability. As a basic example of such effects one may consider vector-valued stochastic Burgers type equations of the form

$$dX_t^i = \Delta X_t^i + \sum_{j=1}^{n} g_j^i(X_t)\partial_x X_t^j dt + dW_t^i, \quad i = 1, ..., d, \tag{1}$$

where $g^i_j : \mathbb{R}^d \to \mathbb{R}$ are smooth functions and W_t is space-time white noise. Even when $g^i_j \equiv 0$, the solutions to such equations are known to be spatially only β-Hölder continuous for every $\beta < \frac{1}{2}$. Therefore, the term $\partial_x X^j_t$ is not rigorously defined and a weak formulation has to be used instead. However, as long as g^i_j are not of gradient type one cannot just rely on partial integration in order to pass to a weak formulation. Recently, an alternative approach to (1) based on the theory of rough paths has been developed and successfully applied in order to prove well-posedness and stability (cf. [1, 8]). Closely related, instability of spatial discretizations of (1) and the occurrence of correction terms has been observed in [2, 4, 7]. The crucial step in the formulation of a weak notion of solution to (1) is the construction of geometric rough paths lifting strictly stationary solutions to the stochastic heat equation in their space variable. I.e., considering

$$d\Psi^i_t = (\Delta - 1)\Psi^i_t dt + dW^i_t, \quad i = 1, ..., d,$$

one needs to construct rough paths $x \mapsto \mathbf{\Psi}(t,x)$ lifting $x \mapsto \Psi(t,x)$. In the special case of stochastic heat equations with space-time white noise on the one dimensional torus, the existence of a corresponding rough path has been shown in [1]. However, the reasoning strongly relied on the simple structure of the equation and on explicit calculations that break down for fractional stochastic heat equations or colored noise. Similar constructions are also fundamental for the recent progress on the KPZ equation [3], again on the one dimensional torus.

We provide a general sufficient condition for the existence of a rough path lift of centered, continuous Gaussian processes with stationary increments and convex or concave variance function. As applied to fractional stochastic heat equations, i.e.

$$d\Psi^i_t = (-(-\Delta)^\alpha - 1)\Psi^i_t dt + dW^i_t, \quad \alpha \leq 1, \; i = 1, ..., d, \tag{2}$$

with possibly colored noise this proves the existence of a geometric rough path, lifting $x \mapsto \Psi(t,x)$ for all $t \geq 0$ under suitable assumptions on the diffusion coefficients.

The study of vector-valued Burgers equations of the form (1) is motivated by path sampling problems. More precisely, let Z be the solution to a linear SDE

$$dZ_u = AZ_u du + CdB_u, \quad \text{in } \mathbb{R}^d, \tag{3}$$

on $[0, 2\pi]$, where B denotes standard Brownian motion in $\mathbb{R}^d$. In the simplest case $A \equiv 0$, $C \equiv \mathrm{Id}$, $Z(0) = 0$, the covariance of Z is given by $R_Z(u,v) = (u \wedge v)$ Id, which is the fundamental solution to $(-\Delta, \mathcal{D}(-\Delta))$ with $\mathcal{D}(-\Delta) := \{f \in H^2([0,2\pi];\mathbb{R}^d) |\; f(0) = 0, \frac{d}{du}f(2\pi) = 0\}$. In other words, the covariance operator of the Gaussian measure $\mathcal{L}(Z)$ on $L^2([0,2\pi];\mathbb{R}^d)$ is given by $(-\Delta)^{-1}$. On the other hand, the invariant measure μ corresponding to the SPDE

$$dX_t = \Delta X_t dt + dW_t, \quad \text{on } [0, 2\pi],$$

with W being space-time white noise and $\mathcal{D}(-\Delta)$ as before, is known to be Gaussian with covariance operator $(-\Delta)^{-1}$. Hence, $\mu = \mathcal{L}(Z)$.

In the general case, in [6] the covariance operator of the solution Z to (3) has been identified as $R_Z = \big(-(\partial_u + A^*)(CC^*)^{-1}(\partial_u - A)\big)^{-1}$ with analogous boundary conditions.

One then aims to solve the corresponding bridge sampling problem, i.e. to sample from the distribution μ of Z conditioned on

$$Z_0 = z_0, \quad Z_1 = z_1.$$

In [6] it has been shown, that μ equals the unique ergodic invariant measure of the SPDE

$$\begin{aligned} d\Psi_t &= -R_Z^{-1}\Psi dt + dW_t, \quad \forall (t,u) \in \mathbb{R}_+ \times (0, 2\pi) \\ \Psi(t,0) &= z_0, \quad \Psi(t,2\pi) = z_1, \quad \forall t \in \mathbb{R}_+, \end{aligned} \tag{4}$$

where R_Z is given as before endowed with inhomogeneous Dirichlet boundary conditions. Since the speed of convergence of solutions Ψ_t to (4) to the invariant measure μ is exponential, efficient sampling algorithms for μ may be based on solving (4).

The case of linear SDE (3) has subsequently been extended in [5, 1] to non-linear SDE of the form

$$dZ_u = AZ_u du + f(Z_u)du + CdB_u, \quad \text{in } \mathbb{R}^d,$$

with non-linear $f : \mathbb{R}^d \to \mathbb{R}^d$. At least informally, this leads to an SPDE of the form (1) if we set $A \equiv 0$, $C \equiv \mathrm{Id}$ and drop lower order terms for simplicity. In the gradient case (i.e. $f = \nabla F$), this has been rigorously worked out in [5], based on standard SPDE methods. The non-gradient case presented a substantial difficulty as the non-linearity in the SPDE could not be made meaningful with classical methods, as it has been outlined above. In [1], Hairer resolved this problem by constructing a suitable spatial rough path to the linear stochastic heat equation, so that the nonlinearity $g^i_j(X_t)\partial_x X^j_t$ makes sense after pairing with a Schwartz test function φ as a rough integral $\int_0^{2\pi} \varphi(u) g^i_j(X_t(u)) dX^j_t(u)$.

It is tempting to try a similar approach in the fractional case; that is, to sample the law of

$$dZ_u = AZ_u du + f(Z_u)du + CdB^H_u, \quad \text{in } \mathbb{R}^d, \tag{5}$$

conditional on its endpoints, via the stationary solution of a suitable fractional SPDE. However, combining the heuristics found in [5], notably the relation to Onsager-Machlup functionals, and the known form of these functionals in the fractional case [9], suggests an SPDE of the form (2) with appropriate boundary conditions and with an additional non-local, nonlinear term[1]. For the linear case we give a heuristic derivation below. It is not difficult to make this case rigorous, but since many questions remain open in the non-linear case we shall return to this in its own right. We believe, that the solution of the resulting SPDE will rely on the construction of a suitable (spatial) rough path associated to the linear problem much as in Hairer's work.

[1]Here B^H is Fourier fractional Brownian motion with Hurst parameter $2H = \alpha$.

We will now give a heuristic derivation of the SPDE associated to the sampling problem for (5) at least in the linear case, i.e. for A, $f \equiv 0$, $B \equiv \mathrm{Id}$. Let B^H is Fourier fractional Brownian motion (cf. [10]) with Hurst parameter $H \in (0,1)$, that is B^H is a continuous, centered Gaussian process starting at 0 with covariance

$$R^H(s,t) = \sum_{k\in\mathbb{N}} \frac{\sin((k+\frac{1}{2})s)\sin((k+\frac{1}{2})t)}{(k+\frac{1}{2})^{4H}}.$$

Note that for $H = \frac{1}{2}$ we recover standard Brownian motion.

If we consider Δ on $L^2([0,2\pi];\mathbb{R}^d)$ with domain

$$\mathcal{D}(\Delta) := \{f \in H^2([0,2\pi];\mathbb{R}^d)|\ f(0) = 0,\ \frac{d}{du}f(2\pi) = 0\},$$

then an orthonormal basis of $-\Delta$ is given by $\{e_k(\cdot) := \sin((k+\frac{1}{2})\cdot)\}_{k\in\mathbb{N}}$ with eigenvalues $\lambda_k^2 := (k+\frac{1}{2})^2$. Hence, the corresponding fractional Laplace operator $(-\Delta)^\alpha$ is given by

$$(-\Delta)^\alpha f(t) = \sum_{k\in\mathbb{N}} \left(k+\frac{1}{2}\right)^{2\alpha} f_k \sin\left((k+\frac{1}{2})t\right),$$

with $f_k := \int_0^{2\pi} f(s)\sin((k+\frac{1}{2})s)ds$ and $\alpha \in (0,1]$. The covariance operator of B^H on $L^2([0,2\pi];\mathbb{R}^d)$ is given by

$$\begin{aligned} R^H f(t) &= \int_0^{2\pi} R^H(s,t) f(s) ds \\ &= \sum_{k\in\mathbb{N}} \int_0^{2\pi} \frac{\sin((k+\frac{1}{2})s)\sin((k+\frac{1}{2})t)}{(k+\frac{1}{2})^{4H}} f(s) ds \\ &= (-\Delta)^{-\alpha} f, \end{aligned}$$

with $\alpha = 2H$ and domain of definition $\mathcal{D}(\Delta)$ given as above. Suppose $\alpha > \frac{1}{2}$. As in the case $\alpha = 1$, the invariant measure μ of the SPDE

$$\begin{aligned} &d\Psi_t = -(-\Delta)^\alpha \Psi dt + dW_t, \quad \forall (t,x) \in \mathbb{R}_+ \times (0,2\pi) \\ &\Psi(t,0) = 0, \quad \frac{d}{dx}\Psi(t,2\pi) = 0, \quad \forall t \in \mathbb{R}_+, \end{aligned} \tag{6}$$

is a Gaussian measure with covariance operator $(-\Delta)^{-\alpha}$ on $L^2([0,2\pi];\mathbb{R}^d)$. Hence, $\mu = \mathcal{L}(B^H)$ and the path sampling problem for B^H may be approached by considering Ψ_t for large values of t.

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A theory of controlled distributions

Massimiliano Gubinelli
(joint work with P. Imkeller, N. Perkowski)

If $(X_t)_{t\geq 0}$ and $(Y_t)_{t\geq 0}$ are two independent d-dimensional Brownian motions it is well known that it is, in general, not possible to give a meaning analytically to the point-wise product $Y_t \partial_t X_t$ of the continuous path Y_t with the distribution $\partial_t X_t$. Itô theory of integration exploits probabilistic independence to give sense to the Riemmann-like integral $Z_t = \int_0^t Y_s dX_s$ as a substitute to the point wise quantity $Y_t \partial_t X_t$: formally $\partial_t Z_t = Y_t \partial_t X_t$. In Lyons' theory of rough paths [5, 6, 7, 3] an appropriate candidate for the integral Z (the cross integral of Y and X) allows to define integrals of the form $\int_0^t F_s dX_s$ for a large class of paths F which "locally" behave like Y (it is *controlled* by Y, see [3]), in the sense that $F_t - F_s \simeq F'_s(Y_t - Y_s)$ modulo a small remainder. Euristically, the local information contained in Z is enough to reconstruct the product $F_s \partial_s X_s$ at least as a distribution. The probabilistic structure takes part only in the definition of Z but the construction of the integral $\int_0^t F_s dX_s$ from the data (X, Y, Z, F') is analytic and does not rely on any special probabilistic structure. Inspired by these results we sketched in [1] a general approach to the analysis of products of distributions which are a-priori not well defined. Essentially we take the controlled path idea and apply it to the multi scale development of the distributions, instead of their parameter dependence. In this way we can easily generalize rough path theory to the multi-parameter setting. (en passant: this idea seems also fruitful for stochastic integration theory, which in the many parameters setting, for example w.r.t. the Brownian sheet, looses its simplicity). Let us explain our construction in the particular case of giving a meaning to the quantity $F_t \partial_t X_t$ for a large class of functions F. Consider standard Littlewood-Paley projectors $\{\Delta_i\}_{i\geq -1}$ on $\mathcal{S}'(\mathbb{R})$ (the space of Schwartz distributions on $\mathbb{R}$). A distribution f belongs to the Hölder-Besov space $C^\gamma = B^\gamma_{\infty,\infty}$ iff $\|\Delta_i f\|_{L^\infty(\mathbb{R})} \lesssim 2^{-i\gamma}$ for all $i \geq -1$ (see e.g. [2] for details on Besov spaces and the L-P decomposition). It is easy to show that (upon suitable localization) $X \in C^\gamma$ for any $\gamma < 1/2$ and so that $\partial_t X \in C^{\gamma-1}$. If $F \in C^\rho$ then the product $F \partial_t X$ is well defined only if $\gamma + \rho - 1 > 0$ which is the standard Young condition in this setting. If we want to allow a larger range of ρ (for example for SDEs we would need $\rho = \gamma$) we proceed by decomposing the product over all the relevant scales and partitioning the sum into three terms:

$$F\partial_t X = \sum_{i<j-1} \Delta_i F \partial_t \Delta_j X + \sum_{i>j+1} \Delta_i F \partial_t \Delta_j X + \sum_{|i-j|\le 1} \Delta_i F \partial_t \Delta_j X$$

The first two terms in this decomposition are well defined whatever the value of ρ is. In particular

$$\pi_<(F, \partial_t X) = \sum_{i<j-1} \Delta_i F \partial_t \Delta_j X \in C^{\gamma-1}$$

and

$$\pi_>(F, \partial_t X) = \sum_{i>j+1} \Delta_i F \partial_t \Delta_j X \in C^{\gamma+\rho-1}$$

The diagonal term in the double sum is the origin of the difficulties: only oscillations on almost the same scale give problems when trying to point-wise multiply distributions. At this point the key observation come from defining a suitable class of controlled functions, so we say that F is controlled by Y off

$$F = \pi_<(F', Y) + F^\sharp$$

where $F' \in C^\delta$ and $F^\sharp \in C^{\rho+\delta}$ with $\delta > 0$. This definition implies for example that $\Delta_i F \simeq F' \Delta_i Y$ modulo smoother correction terms. As we see we just transposed the controlled path definition on the multiscale expansion. With this assumption we can show that

$$\sum_{|i-j|\le 1} \Delta_i F \partial_t \Delta_j X \simeq F' \sum_{|i-j|\le 1} \Delta_i Y \partial_t \Delta_j X$$

modulo a term belonging to $C^{\gamma+\delta+\rho-1}$. In other terms we reduced the problem of the definition of the product $F\partial X$ to that of the product $Y\partial X$ for all F controlled by Y. As in rough path theory, this last piece of data can be synthesized using probabilistic arguments: almost surely there exists a version of $\sum_{|i-j|\le 1} \Delta_i Y \partial_t \Delta_j X$ living in $C^{\gamma+\rho-1}$. This completes the construction.

Suitable commutator estimates and results on paralinearization of maps of Besov functions allow to show the continuity of this product under a suitable "controlled" topology and set up fixed-point arguments to solve rough differential equations and more general problems.

To exemplify the applicability of our ideas, in [1] we consider two SPDEs for which previously it was not known how to describe solutions:

(1) A Burgers type SPDE driven by time-space white noise on the d-dimensional torus $\mathbb{T}^d = [-\pi, \pi]^d$ with periodic boundary conditions:

$$\partial_t u(t,x) = -Au(t,x) + g(u(t,x))Du(t,x) + \xi(t,x),$$

where $u : \mathbb{R}_+ \times \mathbb{T}^d \to \mathbb{R}^n$ is a vector valued function, $-A = -(-\Delta)^\sigma$ is the fractional Laplacian with $\sigma > 1/2$, ξ is a space-time white noise taking values in $\mathbb{R}^n$ and D denotes the spatial derivative. Moreover g :

$\mathbb{R}^n \to \mathcal{L}(\mathbb{R}^n; \mathbb{R}^n)$ is a smooth field of linear transformations of $\mathbb{R}^n$. Here our results complement the results of [4].

(2) A non-linear heat equation with rough space dependence:

$$\partial_t u(t,x) = \Delta u(t,x) + f(u(t,x))\xi(x)$$

where $x \in \mathbb{T}^n$, $n = 2, 3$, ξ is a space white-noise which does not depend on time and $f : \mathbb{R} \to \mathbb{R}$ is a regular function.

In both cases we are able to exibith a space of controlled distribution where the equations are well-posed (in a suitable sense) and admit a local solution.

During the workshop we become aware of a different but related approach developed by M. Hairer to treat non-linear operation on distributions. In his approach, instead of imposing some control on the multi-scale decomposition, he prescribe the local behavior of the distributions and then reconstruct the global object using a generalization of the one-dimensional sewing map considered in [3].

All these developments hint to a new territory which become suddenly more amenable to exploration along rough paths.

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Modelled distributions and the KPZ equation

Martin Hairer

In [1], we introduce a novel robust concept of solution to the KPZ equation which is shown to extend the classical Cole-Hopf solution. Similarly to what can be shown in the context of the "rough paths" approach to the solution to controlled ODEs, this new notion provides a solution map which is jointly locally Lipschitz continuous as a function of the initial condition and the driving noise. The price to pay is that the space containing the driving noise is not a classical Banach space, but rather a genuinely nonlinear metric space $\mathcal{X}$.

The Cole-Hopf solution then factorises into a "universal" measurable map from the probability space into $\mathcal{X}$, composed with the new solution map. The advantage of such a formulation is that it essentially provides a pathwise notion of a solution, together with a very detailed approximation theory. In particular, our construction completely bypasses the Cole-Hopf transform, thus laying the groundwork for proving that the KPZ equation describes the fluctuations of systems in the KPZ universality class.

As a corollary of our construction, we obtain very detailed new regularity results about the solution, as well as its derivative with respect to the initial condition. Other byproducts of the proof include an explicit approximation to the stationary solution of the KPZ equation, a well-posedness result for the Fokker-Planck equation associated to a particle diffusing in a rough space-time dependent potential, and a new periodic homogenisation result for the heat equation with a space-time periodic potential. One ingredient in our construction is an example of a non-Gaussian rough path such that the area process of its natural approximations needs to be renormalised by a diverging term for the approximations to converge.

At a technical level, our construction extends and sharpens the tools developed in [4, 3] in the context of the analysis of a class of "Burgers-type" equations. However, it can also be interpreted as an instance of a much more general theory of "modelled distributions" [2]. The idea of this theory is essentially to describe a function (or distribution) f by a kind of "local Taylor expansion" or "germ" $F(x)$ at every (space-time) point x. Here, the function F takes values in a vector space T that encodes the coefficients of the expansion. The twist is that, unlike in the case of the classical Taylor expansion, we do *not* in general assume that the basis functions of the expansion are given by polynomials. In particular, our basis functions are allowed to contain irregular functions and / or distributions. The main additional structure required in the theory is the action of a Lie group G onto T, which "translates" the coefficients of an expansion around a given point into the coefficients of the expansion around a different point. In other words, if $\Pi_x \colon T \to \mathcal{D}$, where $\mathcal{D}$ is some space of distributions, is the linear map that, to a given set of coefficients a, associates the corresponding expansion around x (think of it as giving the polynomial with coefficients a based at x), then there exists a function $(x, y) \mapsto \Gamma_{xy} \in G$ such that

$$\Gamma_{xy}\Gamma_{yz} = \Gamma_{xz} \;, \qquad \Pi_y = \Pi_x \Gamma_{xy} \;.$$

Under natural analytic conditions on Π and Γ, we can then construct a "reconstruction map" $\mathcal{R}$ which, to a given "modelled distribution" $F \colon x \mapsto F(x)$ associates a unique Schwartz distribution $\mathcal{R}F$ such that, for every x, $\mathcal{R}F \approx \Pi_x F(x)$ near x. Furthermore, the reconstruction map $\mathcal{R}$ is continuous both as a function of F and as a function of the model (Γ, Π). The natural condition for this to be the case is that

$$\|\Gamma_{xy}F(y) - F(x)\|_\alpha \le C|x-y|^{\gamma-\alpha} \;, \tag{1}$$

for some $\gamma > 0$, where $\|\cdot\|_\alpha$ denotes the norm of the component in T_α. If we define an abstract product $\star$ on T, one can then define the product between two modelled distributions F and $\bar{F}$ by $\big(F \star \bar{F}\big)(x) = F(x) \star \bar{F}(x)$. We can give explicit and natural conditions on $\star$ which ensure that a bound of the type (1) is again satisfied for $F \star \bar{F}$, thus allowing to define a product between classes of distributions that one could not classically multiply. In particular, the solution to the KPZ equation can be described as such a modelled distribution, and its nonlinearity is given by a particular instance of the structure just described.

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The signature of a path of bounded variation

Ben Hambly
(joint work with Terry Lyons)

We consider the signature $\mathbf{X}_{[0,T]}$ of a bounded variation path $\{X_t; 0 \leq t \leq T\}$, taking values in $\mathbb{R}^d$, $d > 1$, that is the element of the tensor algebra over $\mathbb{R}^d$ consisting of the sequence of all the iterated integrals of the path over a fixed time T. This can be viewed as a non-commutative transform of the path and we address the question of whether or not the signature uniquely determines the path. That is we seek an analogue of the result that the Fourier coefficients determine integrable functions on the circle up to Lebesgue null sets. The key idea is to identify those paths which have a null signature and for this we introduce the idea of a tree-like path.

Definition 2. *$\{X_t,\ t \in [0,T]\}$ is a tree-like path in $\mathbb{R}^d$ if there exists a positive real valued continuous function h defined on $[0,T]$ such that $h(0) = h(T) = 0$ and such that*

$$\|X_t - X_s\| \leq h(s) + h(t) - 2 \inf_{u \in [s,t]} h(u).$$

The function h is called a height function for X.

With this notion our main theorem, published in [1], is

Theorem 1. *A path of bounded variation has $\mathbf{X}_{[0,T]} = \mathbf{0} = (1,0,0,\dots)$ if and only if the path $\{X_t; 0 \leq t \leq T\}$ is tree-like.*

As a result we see that the signatures of paths are unique up to tree-like pieces and we can use the signature to define an equivalence relation on bounded variation paths. Within each equivalence class there is a path of minimal length which contains no tree-like pieces. We call this the tree reduced path associated with the signature $\mathbf{X}_{[0,T]}$ and such tree reduced paths form a group under concatenation.

Two open problems are:

(1) to extend this result to the case of paths with finite p-variation for $p > 1$. That is are such paths determined up to tree-like pieces by their signatures. The result is known for Brownian paths almost surely from work of Le Jan and Qian [2].
(2) to reconstruct the path given its signature. There is work on this problem which is solved efficiently in the case of lattice paths by Lyons and Xu [3].

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Large deviation principle of Freidlin-Wentzell type for pinned diffusion measures

Yuzuru Inahama

Summary: Since T. Lyons invented rough path theory, one of its most successful applications is a new proof of Freidlin-Wentzell's large deviation principle for diffusion processes. In this talk we extend this method to the case of pinned diffusion processes under a mild ellipticity assumption. Besides rough path theory, our main tool is quasi-sure analysis, which is one of the deepest theories in Malliavin calculus. (A preprint of this work can be found on ArXiv math).

For the canonical realization of d-dimensional Brownian motion $(w_t)_{0\le t\le 1}$ and the vector fields $V_i : \mathbf{R}^n \to \mathbf{R}^n$ $(1 \le i \le d)$ with sufficient regularity, let us consider the following Stratonovich-type stochastic differential equation (SDE):

$$dy_t = \sum_{i=1}^{d} V_i(y_t) \circ dw_t^i \qquad \text{with} \quad y_0 = a \in \mathbf{R}^n.$$

For simplicity of explanation, no drift term is added, but modification is easy. The correspondence $w \mapsto y$ is called the Itô map and denoted by $y = \Phi(w)$. It is well-known that the Itô map is not continuous as a map from the Wiener space. Now, introduce a small positive parameter $\varepsilon \in (0, 1]$ and consider

$$dy_t^\varepsilon = \sum_{i=1}^{d} V_i(y_t^\varepsilon) \circ \varepsilon dw_t^i \qquad \text{with} \quad y_0^\varepsilon = a \in \mathbf{R}^n.$$

Formally, $y^\varepsilon = \Phi(\varepsilon w)$. The process $(y_t^\varepsilon)_{0\le t\le 1}$ takes its values in $\mathbf{R}^n$ and its law is a diffusion measure associated with the starting point a and the generator $L^\varepsilon = (\varepsilon^2/2)\sum_{i=1}^{d} V_i^2$.

A classic result of Freidlin and Wentzell states the laws of $(y_t^\varepsilon)_{0\le t\le 1}$ satisfies a large deviation principle as $\varepsilon \searrow 0$. The proof was not so easy. If Φ were continuous, we could use contraction principle and the proof would be immediate from Schilder's large deviation principle for the laws of $(\varepsilon w)_{0\le t\le 1}$. However, it cannot be made continuous in the framework of the usual stochastic analysis.

Ten years ago, Ledoux, Qian, and Zhang [3] gave a new proof by means of rough path theory. Roughly speaking, a rough path is a couple of a path itself and its iterated integrals. Lyons established a theory of line integrals along rough paths and ordinary differential equation (ODE) driven by rough paths. The Itô map in the rough path sense is deterministic and is sometimes called the Lyons-Itô map. The most important result in the rough path theory could be Lyons's continuity theorem (also known as the universal limit theorem), which states that the Lyons-Itô map is continuous in the rough path setting. Brownian motion (w_t) admits a natural lift to a random rough path W, which is called Brownian rough path. If we put W or εW into the Lyons-Itô map, then we obtain the solution of Storatonovich SDE (y_t) or (y^ε_t), respectively. Ledoux, Qian, and Zhang proved that the laws of εW satisfy a large deviation principle of Schilder type with respect to the topology of the rough path space. Large deviation principle of Freidlin-Wentzell type for the laws of (y^ε_t) is immediate from this, since the contraction principle can be used in this framework. Since then many works on large deviation principle on rough path space have been published.

There arises a natural question; can one obtain a similar result for pinned diffusion processes with this method, too? More precisely, does the family of measures $\{\mathbb{Q}^\varepsilon_{a,a'}\}_{\varepsilon>0}$ satisfy a large deviation principle as $\varepsilon \searrow 0$? Here, $\mathbb{Q}^\varepsilon_{a,a'}$ is the pinned diffusion measure associated with L^ε, which starts at a at time $t=0$ and ends at a' at time $t=1$. Heuristically, $\mathbb{Q}^\varepsilon_{a,a'}$ is the law of y^ε_1 under the conditional probability measure $\mathbb{P}(\,\cdot\,|y^\varepsilon_1 = a')$, where $\mathbb{P}$ stands for the Wiener measure.

The aim of this talk is to answer this question affirmatively under a certain mild ellipticity assumption for the coefficient vector fields. Besides rough path theory, our main tool is quasi-sure analysis, which is a sub-field of Malliavin calculus. It deals with objects such as Watanabe distributions (i.e., generalized Wiener functionals) and capacities associated with Gaussian Sobolev spaces. Recall that motivation for developing this theory was to analyse the pullbacks of pinned diffusion measures on the Wiener space.

In 1993, Takanobu and Watanabe [4] presented this kind of large deviation principle under a hypoellipticity assumption for coefficient vector fields. This result seems very general and nice, but they gave no proof. Their tool are Malliavin calculus, and in particular, quasi-sure analysis. Recall that rough path theory did not exist, then. Presumably, they computed Besov norm of the solution of SDE, but details are unknown.

Since we use rough path theory, we will compute, not the output, but the input of the (Lyons-)Itô map. Here, the input means (w_t) itself and its iterated Storatonovich stochastic integrals. So, we believe that our proof via rough paths is probably simpler. Extending our method to the hypoelliptic case is an interesting and important future task.

Now we give a precise setting and state our main result. Let $(w_t)_{0\le t\le 1}$ be the canonical realization of d-dimensional Brownian motion. We consider the following $\mathbf{R}^n$-valued Stratonovich-type SDE;

$$dy_t^\varepsilon = \sum_{i=1}^d V_i(y_t^\varepsilon)\circ \varepsilon dw_t^i + V_0(\varepsilon, y_t^\varepsilon)dt \qquad \text{with} \quad y_0^\varepsilon = a \in \mathbf{R}^n.$$

Here, $\varepsilon \in [0,1]$ is a small parameter and $V_i \in C_b^\infty(\mathbf{R}^n,\mathbf{R}^n)$ for $1 \le i \le d$ and $V_0 \in C_b^\infty([0,1]\times\mathbf{R}^n,\mathbf{R}^n)$. (A function is said to be of class C_b^∞ if it is a bounded, smooth function with bounded derivatives of all order.) For each ε, (y_t^ε) is a diffusion process with its generator $L^\varepsilon = (\varepsilon^2/2)\sum_{i=1}^d V_i^2 + V_0(\varepsilon,\,\cdot\,)$.

We assume everywhere ellipticity:

(A1): For all $a \in \mathbf{R}^n$, the set of vectors $\{V_1(a),\dots,V_d(a)\}$ linearly spans $\mathbf{R}^n$.

Under this assumption, the pinned diffusion measure $\mathbb{Q}_{a,a'}^\varepsilon$ associated with L^ε exists for any $\varepsilon > 0$, the starting point a and the terminal point a'. This measure sits on

$$C_{a,a'}^\alpha([0,1],\mathbf{R}^n) = \{x \in C([0,1],\mathbf{R}^n) \mid \alpha\text{-Hölder conti. and } x_0 = a,\ x_1 = a'.\ \}$$

for any $\alpha \in (1/3, 1/2)$.

Let H be Cameron-Martin space for (w_t). For $h \in H$, we denote by $\phi = \phi(h)$ be a unique solution of the following ODE;

$$d\phi_t = \sum_{i=1}^d V_i(\phi_t)dh_t^i + V_0(0,\phi_t)dt \qquad \text{with} \quad \phi_0 = a.$$

We set $K^{a,a'} = \{h \in H \mid \phi(h)_1 = a'\}$, which is not empty under **(A1)**.

Define a good rate function $I : C_{a,a'}^{\alpha-H}([0,1],\mathbf{R}^n) \to [0,\infty]$ by

$$I(y) = \inf\{\frac{\|h\|_H^2}{2} \mid h \in K^{a,a'} \text{ with } y = \phi(h)\} - \min\{\frac{\|h\|_H^2}{2} \mid h \in K^{a,a'}\}$$

if $y = \phi(h)$ for some $h \in K^{a,a'}$ and define $I(y) = \infty$ if no such $h \in K^{a,a'}$ exists.

Now we state our main result in this paper.

Theorem 2. *Let $1/3 < \alpha < 1/2$ and assume* **(A1)**. *The family $\{\mathbb{Q}_{a,a'}^\varepsilon\}_{\varepsilon>0}$ of probability measures on $C_{a,a'}^\alpha([0,1],\mathbf{R}^d)$ satisfies a large deviation principle as $\varepsilon \searrow 0$ with a good rate function I.*

A rough sketch of our proof is as follows. (1) Brownian motion (w_t) admits a lift, not only almost surely, but aslo quasi-surely. See [1, 2] for instance.

(2) By Sugita's theorem, a positive Watanabe distribution $\delta_{a'}(y_1^\varepsilon) = \delta_{a'}(y^\varepsilon(1,a))$ is actually a finite Borel measure on the Wiener space. We can think of its push-forward measure $\mu_{a,a'}^\varepsilon$ of $\delta_{a'}(y_1^\varepsilon)$ by the lift map. Notice that the pushforward measure of $\mu_{a,a'}^\varepsilon$ by the Lyons-Itô map is the pinned diffusion measure in question.

(3) We prove large deviation for $\{\mu_{a,a'}^\varepsilon\}$ as $\varepsilon \searrow 0$ on the geometric rough path space. (In fact, we need to assume ellipticity only at the starting point a.) Three

key facts in this part are as follows; (i) large deviation estimate for capacities, not for measures, on geometric rough path space, (ii) integration by parts formula in the sense of Malliavin calculus for Watanabe distributions, (iii) uniform non-degeneracy of Malliavin covariance matrix for solutions of the shifted scaled SDE.

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Perturbation of linear rough differential equations and applications

Antoine Lejay

(joint work with Laure Coutin)

Although linear Rough Differential Equations may be considered as a particular case of Rough Differential Equations (RDE), they could also be studied as objects with their own properties.

A linear RDE could be defined from a *resolvent* $(A_{s,t})_{0\leq s\leq t\leq T}$ where the $A_{s,t}$ are linear applications on a Banach space V satisfying

$$\|A_{s,t} - \mathrm{Id}\| \leq C\omega(s,t)^{1/p} \text{ for } 0 \leq s \leq t \leq T, \tag{1}$$

$$A_{s,r} = A_{r,t}A_{s,r} \text{ for } 0 \leq s \leq r \leq t \leq T, \tag{2}$$

for some $p \geq 1$, $C \geq 0$ and a super-additive function $\omega : \{0 \leq s \leq t \leq T\} \to \mathbb{R}_+$ which is continuous on its diagonal.

Such a family may be constructed from a family $(B_{s,t})_{0\leq s\leq t\leq T}$ satisfying (1) but where (2) is replaced by

$$\|B_{s,t} - B_{r,t}B_{s,r}\| \leq K\omega(s,t)^{\theta},\ K \geq 0,\ \theta > 1.$$

This construction is strongly inspired by the work of D. Feyel et al. [5]. The construction of $(A_{s,t})_{0\leq s\leq t\leq T}$ from $(B_{s,t})_{0\leq s\leq t\leq T}$ is similar to the one which allows one to pass from a Chen series living in a truncated tensor algebra to a Chen series in the full tensor algebra $\mathbb{R} \oplus \mathbb{R}^d \oplus (\mathbb{R}^d)^{\otimes 2} \oplus \cdots$ (See *e.g.* [7]). Indeed, Chen series are solutions to linear differential equations [1].

If $1 \leq p < 2$ and $\mathcal{A} : [0,T] \to \mathrm{L}(\mathrm{V},\mathrm{V})$ is a path of finite p-variation, then a resolvent $(A_{s,t})_{0\leq s\leq t\leq T}$ may be constructed from the family $B_{s,t} = \mathrm{Id} + \mathcal{A}_t - \mathcal{A}_s$ or $B_{s,t} = \exp(\mathcal{A}_t - \mathcal{A}_s)$ for $0 \leq s \leq t \leq T$. This is why we write a linear RDE as

$$\mathrm{d}Y_t = \mathrm{d}\mathcal{A}_t Y_t. \tag{3}$$

In this case, $Y_t = A_{s,t}Y_s$ for any $0 \le s \le t \le T$. Of course, when $p \ge 2$, knowing $\mathcal{A}$ is no longer sufficient to define (3) and a *lift* of $\mathcal{A}$ is needed.

In [2], we give several results on linear RDE which are extension of properties known in smooth case: convergence of Magnus/Chen-Strichartz series in small time, development in Dyson series, ...

We study also perturbations of linear RDE for $1 \le p < 3$. By this, we mean solutions to

$$\mathrm{d}Y_t = \mathrm{d}\mathcal{A}_t Y_t + \mathrm{d}b_t$$

for a path b of finite p-variation with values in V. For $1 \le p < 2$, this could be understood for example as

$$|Y_t - Y_s - A_{s,t}Y_s - (b_t - b_s)| \le D\omega(s,t)^\theta \text{ for any } 0 \le s \le t \le T$$

for some constant $D \ge 0$ and $\theta > 1$. When $2 \le p < 3$, the path b needs to be *lifted*, in a way which depends on $(A_{s,t})_{0 \le s \le t \le T}$. With proper extensions of the notion of integral, Duhamel/Variation of constant principle could be given.

Finally, in [3], perturbed linear RDE are used to study differentiability and flow properties of the Itô map in its full generalities, while such results are generally proved using geometric rough paths and approximation by smooth paths (See [6, 8] for example) without relying to $(p, p/2)$-rough paths. Let us consider

$$\mathfrak{I} : a \in \mathrm{V} \mapsto y \text{ with } y_t = a + \int_0^t f(y_s)\,\mathrm{d}x_s$$

for a rough path x of finite p-variation, $2 \le p < 3$ and a smooth enough vector field f, so that y is well defined and $\mathfrak{I}$ is continuous. Then $\mathfrak{I}$ is differentiable in the sense of Fréchet. Besides, for some $0 < \beta \le 1$ and $C \ge 0$,

$$\|\mathfrak{I}(a+\epsilon) - \mathfrak{I}(a) - \nabla\mathfrak{I}(a)\cdot\epsilon\|_p \le C(a)|\epsilon|^{1+\beta},$$

for $a, \epsilon \in \mathrm{V}$. In addition, $\nabla\mathfrak{I}$ is itself Hölder continuous. The main idea is to study the difference between $\mathfrak{I}(a+\epsilon) - \mathfrak{I}(a)$ and $Y_t \cdot \epsilon$ where Y is a family of linear operators solution to $Y_t = \mathrm{Id} + \int_0^t \nabla f(y_s) Y_s\,\mathrm{d}x_s$ which is indeed equal to $\nabla\mathfrak{I}(a)_t$. This difference may then be written as a perturbed linear RDE for which estimates are provided. With our construction and the notion of solution in the sense of A.M. Davie [4], we are not bound in using the iterated integrals of Y.

Similar results hold for other perturbations, such as perturbation of the vector field and perturbation of a driving rough path x by a path of finite q-variation with $1/p + 1/q > 1$.

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Integrability and Tail Estimates for Gaussian Rough Differential Equations

Christian Litterer
(joint work with Thomas Cass and Terry Lyons)

We study stochastic differential equations of the form

$$dY_t = V(Y_t)dX_t, \quad Y(0) = y_0, \tag{1}$$

driven by a Gaussian process X. Over the past decade extensive progress has been made understanding the behaviour of solutions to such equations. In particular, for the case of fBm with Hurst parameter $H > 1/4$ the work of Cass and Friz [2] shows the existence of the density for (1) under Hörmander's condition; Hairer et al. [1], [7] have shown the smoothness of this density and established ergodicity under the regime $H > 1/2$.

If we consider the flow $U^{\mathbf{X}}_{t\leftarrow 0}(y_0) \equiv Y_t$ of the RDE (1) under sufficient regularity on V, the map $U^{\mathbf{X}}_{t\leftarrow 0}(\cdot)$ is a differentiable function and its derivative ("the Jacobian"):

$$J^{\mathbf{X}}_{t\leftarrow 0}(y_0) \equiv DU^{\mathbf{X}}_{t\leftarrow 0}(\cdot)\,|_{\cdot=y_0}$$

satisfies path-by-path an RDE of linear growth driven by $\mathbf{X}$.

In the diverse applications in [1], [7] a surprisingly generic common obstacle to the extensions of such results to the rough path regime emerges in the need for sharp estimates on the integrability of the Jacobian of the flow $J^{\mathbf{X}}_{t\leftarrow 0}(y_0)$ of an RDE. Cass, Lyons [4] and Inahama [9] establish such integrability for the Brownian rough path but only by using the independence of the increments; for more general Gaussian processes a more careful analysis is needed.

Our results allow us to deduce the existence of moments of all orders for $J^{\mathbf{x}}_{t\leftarrow 0}(y_0)$ for RDEs driven by a class of Gaussian processes (including, but not restricted to, fBm with Hurst index $H > 1/4$). In fact, we show that the logarithm of the Jacobian has a tail that decays faster than an exponential: More precisely,

$$P\left(\log\left[\left|J^{\mathbf{X}}_{\cdot\leftarrow 0}(y_0)\right|_{p-\mathrm{var};[0,T]}\right] > x\right) \lesssim \exp\left(-x^r\right), \tag{2}$$

for any $r < r_0 \in (1,2]$, where the constant r_0 is described in terms of the regularity properties of the Gaussian path.

Our results are relevant to a number of important problems. First, they are a necessary ingredient if one wants to extend the work of [6] and [7] on the ergodicity of non-Markovian systems. Second, they allow one to achieve an analogue of Hörmander's Theorem on the smoothness of the density for Gaussian RDEs in conjunction with a suitable version of Norris's Lemma (see [3] and [8]).

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A conjecture on the optimal approximation of the fractional Lévy area

Andreas Neuenkirch

Let $(B_t)_{t\geq 0} = (B_t^{(1)}, B_t^{(2)})_{t\geq 0}$ be a two-dimensional fractional Brownian motion with Hurst parameter $H \in (1/4, 1)$. The stochastic differential equation

$$\begin{aligned} dY_t^{(1)} &= dB_t^{(1)}, & t \geq 0, \qquad & Y_0^{(1)} = 0, \\ dY_t^{(2)} &= Y_t^{(1)}\, dB_t^{(2)}, & t \geq 0, \qquad & Y_0^{(2)} = 0, \end{aligned}$$

is the prototype example of a stochastic differential equation driven by fractional Brownian motion with non-commutative noise. Here, the first component of the solution is simply $(B_t^{(1)})_{t\geq 0}$, while the second component of the solution is given by

$$Y_t^{(2)} = \int_0^t B_s^{(1)}\, dB_s^{(2)}, \quad t \geq 0.$$

The process $(Y_t^{(2)})_{t\geq 0}$ is usually denoted as fractional Lévy area (in a slight abuse of notation, see e.g. [2] for the precise notion of a Lévy area).

For the case of Brownian motion, i.e. $H = 1/2$, a well known result of Cameron and Clark ([1]) states that the best possible mean square approximation of $X_T = Y_T^{(2)}$ given $B_{T/n}, B_{2T/n}, \ldots, B_T$, i.e.

$$\overline{X}_T^n = \mathbf{E}(\, X_T \,|\, B_{T/n}, B_{2T/n}, \ldots, B_T\,),$$

satisfies

$$(\mathbf{E}|X_T - \overline{X}_T^n)|^2)^{1/2} = \frac{T}{2} \cdot n^{-1/2}.$$

Here, $\overline{X}_T^n$ coincides in fact with the trapezoidal rule

$$\widetilde{X}_T^n = \frac{1}{2} \sum_{i=0}^{n-1} (B^{(1)}_{iT/n} + B^{(1)}_{(i+1)T/n})(B^{(2)}_{(i+1)T/n} - B^{(2)}_{iT/n}).$$

In the general case, i.e. $H \neq 1/2$, the conditional expectation of X_T given $B_{T/n}, B_{2T/n}, \ldots, B_T$ does not coincide with the trapezoidal rule, in particular since fractional Brownian motion does not have independent increments for $H \neq 1/2$. It has been shown in [3] that

$$(\mathbf{E}|X_T - \overline{X}_T^n)|^2)^{1/2} \leq (\mathbf{E}|X_T - \widetilde{X}_T^n)|^2)^{1/2} \leq C_H \cdot T^{2H} \cdot n^{-2H+1/2}.$$

We strongly suppose that the trapezoidal scheme and the conditional expectation have exact root mean square convergence rate $n^{-2H+1/2}$. In other words, we conjecture that there exists a constant $0 < c_H < C_H$ such that

$$(\mathbf{E}|X_T - \overline{X}_T^n)|^2)^{1/2} \geq c_H \cdot T^{2H} \cdot n^{-2H+1/2}.$$

This is supported by the fact that no construction of the Lévy area for $H = 1/4$ based on the interpolation of point evaluations of B seems to be possible, which corresponds to the zero convergence rate in the above conjecture for $H \to 1/4$. (For constructions of a fractional Lévy area for $H \leq 1/4$, see e.g. [5, 4].)

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Expected signature of stochastic processes

Hao Ni

(joint work with Prof. Terry Lyons)

The signature of the path provides a top down description of a path in terms of its effects as a control (see [1]). It is a group-like element in the tensor algebra and is an essential object in rough path theory. When the path is random, the linear independence of the signatures of different paths leads one to expect, and it has been proved in simple cases, that the expected signature would capture the complete law of this random variable. It becomes of great interest to be able to compute examples of expected signatures. In this presentation, we explain how to compute the expected signature of various stochastic processes by solving one PDE system, which fully characterise the expected signature. We consider the case for an Itô diffusion process up to a fixed time, and the case for the Brownian motion up to the first exit time from a domain(see [2]). We manage to derive the PDE of the expected signature for both cases, and find that this PDE system could be solved recursively. Some specific examples are discussed in this talk as well, e.g. Ornstein-Uhlenbeck (OU) processes, Brownian motion, and Brownian motion coupled with its Lévy area.

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A new semi-closed form solutions to some financial problems: a note on Bayer-Friz-Loeffen's work

Syoiti Ninomiya

(joint work with Yusuke Kubo, Tokyo Institute of Technology)

Recently some algorithms that solve the higher-order weak approximation of SDEs following the theory of Kusuoka [Kus01] and the theory of Cubature on Wiener space by Lyons and Victoir [LV04] are found in [NV08] and [NN09] and they have been generalized to various directions [OTV12][Fuj06][JS09] etc. Their practical efficiencies also have been demonstrated there.

When one weakly approximates an SDE following those algorithms, he draws a set of ODEs per one simulation. If all these drawn ODEs have closed form solutions, the algorithm is called semi-closed form solution to the SDE [BFL10]. In [BFL10], Bayer, Friz and Loeffen construct semi-closed form solutions to derivative pricing problems under generalized SABR models by transforming the higher-order algorithm presented by Ninomiya and Victor[NV08].

Inspired by Bayer-Friz-Loeffen's work, the authors obtain new semi-closed form solutions to some finance problems. In this presentation, the authors show new semi-closed form solutions to the problems of derivative pricing under Heston and SABR models. The new solutions are obtained by generalization of Bayer-Friz-Loeffen's technique and the other higher-order algorithm presented by Ninomiya and Ninomiya [NN 2009]. The newly obtained semi-closed form solution to the Heston model also gives a new positive simulation scheme for CIR type processes [Alf10].

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Rough path robustness in nonlinear filtering

Harald Oberhauser

(joint work with Dan Crisan, Joscha Diehl, Peter Friz)

Nonlinear filtering is concerned with the estimation of a Markov process based on some observation of it; e.g. consider the classic case when the Markov process (X,Y) takes values in $\mathbb{R}^{d_X+d_Y}$ and

$$\begin{aligned} dX_t &= \mu\left(X_t\right)dt + V\left(X_t\right)dB_t + \sigma\left(X_t\right)d\tilde{B}_t \\ dY_t &= h\left(X_t\right)dt + d\tilde{B}_t \end{aligned} \tag{1}$$

with B and $\tilde{B}$ independent, multidimensional Brownian motions. The goal is to compute for a given real-valued function φ

$$\pi_t\left(\varphi\right) = \mathbb{E}\left[\varphi\left(X_t\right)|\sigma\left(Y_r, 0 \le r \le t\right)\right].$$

From basic measure theory it follows that there exists a measurable map

$$\phi_t^{\varphi} : C\left(\left[0,T\right],\mathbb{R}^{d_Y}\right) \to \mathbb{R}$$

such that

$$\phi_t^{\varphi}\left(Y|_{[0,t]}\right) = \pi_t\left(\varphi\right)\ \ \mathbb{P}-a.s. \tag{2}$$

As was first pointed out by Clark [3], this classic formulation is not justified in practice since only discrete observations of Y are available and the functional ϕ_t^{φ}

is only defined up to nullsets on pathspace (which includes the observed, bounded variation path); further the model choosen for the observation process might only be close in law to the "real-world" observation process. Clark showed ([3]; a complete proof was given in [4]) that in the uncorrelated noise case ($\sigma \equiv 0$ in (1)) there exists a unique $\overline{\phi}_t^{\varphi} : C([0,T], \mathbb{R}^n) \to \mathbb{R}$ which is continuous in uniform norm and fulfills (2), thus providing a "robust" version of the conditional expectation $\pi_t(\varphi)$ which resolves above mentioned problems. Unfortunately in the correlated noise case this is no longer true (it is easy to construct counterexamples)! Nevertheless, we show that one can also for the correlated case construct a version of $\pi_t(\varphi)$ which is a robust functional of the observation process seen as a rough path — combined with well-known approximation results of rough path lifts of semimartingales this solves above mentioned problems; further it suggests that it is natural to include the Levy area in the observation process.

0.1. **Robustness via the Kallianpur–Striebel functional** [5]**.** Applying Girsanov to transform Y into a Brownian motion under a measure $\mathbb{P}_0$ in combination with the conditional Bayes formula leads to the classic Kallianpur–Striebel representation

$$\pi_t(\varphi) = \frac{\rho_t(\varphi)}{\rho_t(1)}, \;\; \rho_t(\varphi) = \mathbb{E}_{\mathbb{P}_0}\left[\varphi(X_t) W_t | \sigma(Y_r : 0 \le r \le t)\right]$$

where $W_t = \exp\left(\int_0^t h(X) \cdot dY - \frac{1}{2}\int_0^t h^2(X)\, dr\right)$. Such a representation suggests to find functionals

$$\overline{\rho}_1, \overline{\rho}_f : C^{0,p-var}\left([0,T], G^2_{d_Y}\right) \to \mathbb{R}$$

(as usual $C^{0,p-var}\left([0,T], G^2_{d_Y}\right)$ denotes the set of geometric p-rough paths, $p \in (2,3)$) s.t.

$$\pi_t(\varphi) = \frac{\overline{\rho_f}\left(\boldsymbol{Y}|_{[0,t]}\right)}{\overline{\rho_1}\left(\boldsymbol{Y}|_{[0,t]}\right)}$$

This was carried out in [5]: under the right assumptions on the vector fields appearing in (1) (see [5] for details) we have

Theorem 2. *The map*

$$C^{0,p-var}\left([0,T], G^2_{d_Y}\right) \ni \bullet \mapsto \frac{\overline{\rho_f}(\bullet)}{\overline{\rho_1}(\bullet)} \in \mathbb{R}$$

is locally uniformly continuous (under stronger assumptions on the coefficients in (1) even locally Lipschitz). Further,

$$\pi_t(\varphi) = \frac{\overline{\rho_f}\left(\boldsymbol{Y}|_{[0,t]}\right)}{\overline{\rho_1}\left(\boldsymbol{Y}|_{[0,t]}\right)} \;\; \mathbb{P} - a.s.$$

where $\boldsymbol{Y}$ denotes the canonical rough path lift of the semimartingale Y.

0.2. **Robustness via the Zakai–SPDE** [6]**.** Under well-known conditions [1], π_t has a density. Using this density in unnormalized form leads to the representation

$$\pi_t(\varphi) = \int_{\mathbb{R}^d X} \varphi(x) \frac{u_t(x)}{\int u_t(\tilde{x})\, d\tilde{x}} dx \tag{3}$$

where $u_t \in L^1(\mathbb{R}^n)$ a.s. and (u_t) is the L^2-solution of the so-called (dual) Zakai SPDE

$$du_t = \left(G^\star + \frac{1}{2} \sum_k N_k N_k \right) u_t dt + \sum_k N_k u_t \circ dY_t^k$$

with G denoting the generator of the diffusion X, Y a Brownian motion under a measure change and

$$(N_k u)(t,x) = \sigma_k^i(t,x)\, \partial_i u_t(x) + h(t,x)\, .u_t(x)\, . \tag{4}$$

Hence, Clark's robustness problem is related to the robustness of linear, parabolic SPDEs. In [2] an approach to parabolic SPDE based on ideas of rough path theory and work of Lions–Souganidis on viscosity solutions of SPDEs [7] was started. In more recent work [6] linear SPDEs with affine linear rough noise were studied. In the Theorem below, L denotes a (semi-)linear, (possibly degenerate) elliptic operator of the form

$$L(t,x,r,p,X) = -\mathrm{Tr}\left[A(t,x) \cdot X\right] + b(t,x) \cdot p + c(t,x,r)$$

and Λ a collection of first order different operators $\Lambda_k = \Lambda_k(t,x,r,p)$ which are affine linear in r, p, that is,

$$\Lambda_k(t,x,r,p) = p \cdot \sigma_k(t,x) + r\, \nu_k(t,x) + g_k(t,x)\,, \quad k = 1, \ldots, d. \tag{5}$$

Under the right regularity assumptions (see [6]) the following result holds:

Theorem 3. *Let $p \geq 1$, $u_0 \in BUC(\mathbb{R}^n)$ and $\mathbf{z}$ be a geometric p-rough path. Then there exists a unique $u = u^{\mathbf{z}} \in BUC([0,T] \times \mathbb{R}^n)$ such that for any sequence $(z^\epsilon)_\epsilon \subset C^1([0,T], \mathbb{R}^d)$ such that $z^\varepsilon \to \mathbf{z}$ in p-rough path sense, the viscosity solutions $(u^\varepsilon) \subset BUC([0,T] \times \mathbb{R}^n)$ of*

$$\dot{u}^\varepsilon + L\left(t,x,u^\varepsilon, Du^\varepsilon, D^2 u^\varepsilon\right) = \sum_{k=1}^d \Lambda_k(t,x,u^\varepsilon, Du^\varepsilon)\, \dot{z}_t^{k;\varepsilon}, \;\; u^\epsilon(0,\cdot) = u_0(\cdot)\,,$$

converge locally uniformly to $u^{\mathbf{z}}$. We write formally,

$$du + L\left(t,x,u,Du,D^2 u\right) dt = \Lambda(t,x,u,Du)\, d\mathbf{z}_t, \;\; u(0,\cdot) = u_0(\cdot)\,.$$

Moreover, we have the contraction property

$$\sup_{(t,x) \in \mathbb{R}^n \times [0,T]} \left| u^{\mathbf{z}}(t,x) - \hat{u}^{\mathbf{z}}(t,x) \right| \leq e^{CT} \sup_{x \in \mathbb{R}^n} \left| u_0(x) - \hat{u}_0(x) \right|$$

and continuity of the solution map $(\mathbf{z}, u_0) \mapsto u^{\mathbf{z}}$

$$C^{0,p\text{-}var}\left([0,T], G^{[p]}\left(\mathbb{R}^d\right)\right) \times \mathrm{BUC}(\mathbb{R}^n) \to \mathrm{BUC}([0,T] \times \mathbb{R}^n)\,.$$

When applied to Brownian motion enhanced with Levy area one can show thata above solution coincides with the classic variational solution (resp. the dual Zakai SPDE in the filtering context) due to Pardoux, Krylov, Rozovoski et. al. To apply it to derive a robust representation of (3) it only remains to show that $u_t^{\boldsymbol{z}}(.) \in L^1(\mathbb{R}^n)$ for every $\boldsymbol{z} \in C^{0,p-var}\left([0,T], G_d^2\right)$ – an easy way is to use the results in [5], another way is to directly work with the RPDE. We finish by noting that the gradient term in the noise $N_k u$ as in (4) explains rather intuitively why — in the general, correlated noise case of Clark's robustness problem — rough path metrics are required: as toy example consider $L \equiv 0$ and $h \equiv 0$; then solving (0.2) for the case of $\sigma \equiv 0$ reduces via the method of characteristics to solving an SDE with commuting vector fields which is well-known to be robust under approximations of the driving signal (i.e. the observation Y) in uniform norm, hence explains why in the uncorrelated case robustness in uniform norm holds but why in the general case rough path metrics are necessary.

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Likelihood construction for discretely observed rough differential equations

Anastasia Papavasiliou

Let Y be the solution of

$$dY_t = a(Y_t;\theta)dZ_t + b(Y_t;\theta)dX_t, \quad Y_0 = y_0, \tag{1}$$

where (X, Z) is a geometric rough path.

We develop a methodology for constructing the likelihood and consequently performing statistical inference for differential equation (1) driven by any geometric rough path (Z, X), assuming that we know Z and the distribution of X. In particular, we are interested in estimating parameter θ in (1), given that we discretely observe a realization of the solution $Y(\omega)$. Our methodology can be summarized as follows:

(i) first, we solve the **inverse problem**, i.e. given $(Y,Z)(\omega)$ and a $\theta \in \Theta$, we construct $\hat{X}(\theta)(\omega)$ such that $(Y,Z)(\omega)$ solves

$$dY(\omega)_t = a(Y(\omega)_t;\theta)dZ(\omega)_t + b(Y(\omega)_t;\theta)d\hat{X}(\theta)(\omega)_t, \quad Y_0 = y_0. \tag{2}$$

(ii) then, instead of writing down the likelihood of observing a given realization of $Y(\omega)$, we write down the likelihood of the corresponding $\hat{X}(\theta)$ being a realization of X.

Note that in addition to the standard assumptions of the Universal Limit Theorem needed to make sense of (1), we make the following assumptions:

(a) Y and X are both n-dimensional;
(b) for each $\theta \in \Theta$, $b^{-1}(\cdot,\theta)$ exists and is also $\mathrm{Lip}(\gamma)$;

It is straight forward to see that the solution of the inverse problem will be given by

$$\hat{X}(\theta)_{s,t} = \int_s^t b^{-1}(Y_u;\theta)dY_u - \int_s^t (b^{-1}a)(Y_u;\theta)dZ_u. \tag{3}$$

Constructing $\hat{X}(\theta)$ requires integration with respect to the rough path (Y,Z). This cannot be done exactly since we only observe Y discretely but we can approximate the integrals using appropriate Taylor expansions of the integrands, assumed to be $\mathrm{Lip}(\gamma)$ and approximations to the higher iterated integrals appearing in the approximation. Let us denote by $\tilde{X}(\theta)$ our approximation to $\hat{X}_\theta$.

Then, we define the *approximate* likelihood by

$$\tilde{L}_{\mathcal{D}}(\theta) = \frac{dP^{\mathcal{D}}}{d\mu}(\tilde{X}(\theta)^1_{\mathcal{D}}). \tag{4}$$

where $\mathcal{D}$ is a partition of $[0,T]$ corresponding to observation times and $P^{\mathcal{D}}$ is the distribution of the increments of X on the partition $\mathcal{D}$, which is assumed to be absolutely continuous with respect to some measure μ. Finally, $\tilde{X}(\theta)^1_{\mathcal{D}}$ are the increments of $\tilde{X}(\theta)$ on the partition.

This problem has only be considered for specific rough paths X, such as Brownian motion fractional Brownian motion [1].

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An upper bound for the distance between the signatures of two Gaussian processes and applications

Sebastian Riedel

(joint work with Christian Bayer, Peter Friz, John Schoenmakers, Weijun Xu)

Let $X = (X^1, \dots, X^d) \colon [0,T] \to \mathbb{R}^d$ be a centered, continuous Gaussian process with independent components. Using the results of Friz and Victoir [6], we can lift the sample paths $X(\omega)$ to geometric rough paths $\mathbf{X}(\omega)$ in a natural way, provided the covariance function $R_X \colon [0,T]^2 \to \mathbb{R}^{d\times d}$ fulfills a certain regularity condition; namely has finite two-dimensional ρ-variation for some $\rho < 2$. This covers, for instance, fractional Brownian motion with Hurst parameter $H > 1/4$. Natural in this context means that for suitable approximations X^ε of the process X we have convergence of the respective rough paths lifts, i.e.

$$|\varrho_{p-\mathrm{var}}(\mathbf{X}, \mathbf{X}^\varepsilon)|_{L^q(\mathbb{P})} \to 0 \tag{1}$$

for $\varepsilon \to 0$ where $\varrho_{p-\mathrm{var}}$ denotes the inhomogeneous rough paths metric and $p > 2\rho$. Our main result (cf. [5]) gives a quantitative upper bound for the distance between two Gaussian lifts $\mathbf{X}$ and $\mathbf{Y}$: If (X,Y) is jointly Gaussian and $|R_{(X,Y)}|_{\rho-\mathrm{var};[0,T]^2} \le K$, then for every $\delta > 0$ there is a $p = p_\delta > 2\rho$ and a constant $C_{K,\delta}$ such that

$$|\varrho_{p-\mathrm{var}}(\mathbf{X}, \mathbf{Y})|_{L^q(\mathbb{P})} \le C_{q,K,\delta} \sup_{t\in[0,T]} \mathbb{E}\left[|X_t - Y_t|^2\right]^{1-\frac{\rho}{2}-\delta} \tag{2}$$

for all $q > 0$. The choice of p can be made explicit and increases for $\delta \searrow 0$. This result has various implications which we list below:

1. **Almost sure Wong-Zakai convergene rates.** Assume that $t \mapsto X^\varepsilon_t$ has bounded variation for $\varepsilon > 0$ and consider the Riemann-Stieltjes ODEs

$$dY^\varepsilon_t = \sum_{i=1}^d V_i(Y^\varepsilon_t)\, dX^{i;\varepsilon}_t \equiv V(Y^\varepsilon_t)\, dX^\varepsilon_t; \quad Y^\varepsilon_0 = Y_0 \in \mathbb{R}^m. \tag{3}$$

For simplicity, assume that $X(\omega)^{\varepsilon_n}$ is the piecewise-linear approximation of $X(\omega)$ at the time points $\{0 < \frac{1}{n} < \ldots < \frac{\lfloor Tn\rfloor}{n} \le T\}$. Using a Borell-Cantelli argument and the Lipschitz property of the Itō-Lyons map, (2) implies that for V sufficiently smooth

$$|Y - Y^{\varepsilon_n}|_{\infty;[0,T]} \le C\left(\frac{1}{n}\right)^{\frac{1}{\rho}-\frac{1}{2}-\delta}$$

a.s. for all $\delta > 0$ and $n \in \mathbb{N}$ where Y solves the random rough differential equation

$$dY = V(Y)\, d\mathbf{X}; \quad Y_0 \in \mathbb{R}^m. \tag{4}$$

In particular, the Wong-Zakai approximations for fractional Brownian motion converge with a rate of (almost) $2H - 1/2$ and we find the well-known rate of (almost)

1/2 in the Brownian case.

2. **Convergence rates for implementable Milstein schemes.** Milstein schemes (of higher order) and rough paths are closely connected, see e.g. [3]. Replacing the (hard-to-simulate) iterated integrals in this scheme by an (easy-to-simulate) product of increments of the driving Gaussian process leads to a numerical scheme which is much easier to implement (see [4] for a first result in the context of fractional Brownian motion). The convergence rates of this scheme coincide with the Wong-Zakai convergence rates. This answers a conjecture stated in [4].

3. **L^q-rates.** In general, the Lipschitz constant from the Itō-Lyons map enjoys very bad integrability properties which is a serious obstacle when passing from almost sure to L^q convergence rates for the schemes above. However, improving the estimate slightly and using the results of [2] this can be done, see the forthcoming article [1]. As a corollary we obtain strong convergence rates in the classical sense which opens the door to a multilevel Monte Carlo approach for evaluating the quantity $\mathbb{E}[f(Y_T)]$, Y being the solution of (4), which can reduce the computational complexity significantly; see [7].

4. **Optimal time regularity for rough SPDEs.** Hairer realized that it can be useful to consider the solution of certain stochastic partial differential equations as an evolution in a rough paths space, see e.g. [8]. It turns out that our estimate (2) can be used to derive the optimal time regularity of the solution, see [9] for details in this direction.

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Exponential bounds for solutions to rough differential equations driven by fractional Brownian motion

Samy Tindel
(joint work with F. Baudoin, M. Besalú, A. Kohatsu, E. Nualart, C. Ouyang)

Let $B = (B^1, \dots, B^d)$ be a d dimensional fractional Brownian motion with Hurst parameter $H > 1/4$, defined on a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Recall that it means that the components B^j are i.i.d, satisfy the relation $\mathbb{E}[(B_t^j - B_s^j)^2] = (t-s)^{2H}$, and that each B^j admits a representation of Volterra type, namely

$$B_t^j = \int_0^t K(t,u)\, dW_u^j, \quad j = 1, \dots, d, \tag{1}$$

for a d-dimensional Wiener process W and a kernel K such that $K(t,\cdot) \in L^2([0,1])$ for any $t \in [0,1]$.

We are concerned here with the following class of equations driven by B:

$$X_t^x = x + \int_0^t V_0(X_s^x) ds + \sum_{i=1}^d \int_0^t V_i(X_s^x) dB_s^i, \tag{2}$$

where x is a generic initial condition and $\{V_i;\, 0 \le i \le d\}$ is a collection of smooth and bounded vector fields of $\mathbb{R}^m$. The unique solution to equation (2) is understood thanks to the rough paths theory [6, 7, 10].

Once equations like (2) are solved, it is natural to wonder how the density of the random variable X_t behaves for an arbitrary strictly positive t. Theorems concerning existence and smoothness of those densities can be found in [1, 8] for $H > 1/2$ and in [3, 4, 5] for the rough case $1/4 < H < 1/2$. However, results concerning Gaussian type estimates for those densities are scarce, and the only effort we are aware of in this direction is contained in [2], in a special skew-symmetric case.

Our report thus focuses on upper and lower Gaussian bounds for solutions to (2), under two types of elliptic hypothesis: the first one is considered as the standard elliptic assumptions, and states that

$$V(z)\, V^*(z) \ge \epsilon\, \mathrm{id}_n, \quad \text{for all} \quad z \in \mathbb{R}^n, \tag{3}$$

where V stands for the matrix $(V^1, \dots, V^d)$. The second assumption is more clumsy, and can be stated as follows:

$$\frac{1}{2} \left[V(z_1)\, V^*(z_2) + V(z_2)\, V^*(z_1) \right] \ge \epsilon\, \mathrm{id}_n, \quad \text{for all} \quad z_1, z_2 \in \mathbb{R}^n. \tag{4}$$

Under those hypothesis, we are able to prove the following lower bound:

Theorem 3. *Let B be a d-dimensional fBm, X the solution to* (2) *and V a smooth and bounded coefficient satisfying relation* (3)*. Then if $H > 1/2$ and $t \in (0,1]$ the* *density $p_t(z)$ of y_t satisfies*

$$p_t(z) \geq \frac{c_1}{t^{nH}} \exp\left(-\frac{c_2\,|z-a|^2}{t^{2H}}\right),$$

for two strictly positive constants c_1, c_2 depending on n, d, V, H.

The upper bound we have obtained is the following:

Theorem 4. *Let B be a d-dimensional fBm, X the solution to (2) and V a smooth and bounded coefficient satisfying relation (4). Then if $H > 1/4$ and $t \in (0,1]$ the density $p_t(z)$ of y_t satisfies*

$$p_t(z) \leq \frac{c_3}{t^{nH}} \exp\left(-\frac{c_4\,|z-a|^{(2H+1)\wedge 2}}{t^{2H}}\right)$$

for two strictly positive constants c_3, c_4 depending on n, d, V, H.

Let us make a few remarks on the results:

- Up to constants and for $H > 1/2$, our bounds seem to be optimal in the sense that they mimic the Gaussian behavior of the underlying fBm itself. This is not true anymore for $H < 1/2$, where an exponential decay $|z-a|^{2H+1}$ shows up in the upper bound.
- Our methods of proof all rely on Gaussian analysis combined with rough paths techniques. Specifically for the lower bound we rely on the techniques introduced in [9], for which we have to express the solution to equation (2) in terms of the underlying Wiener process W appearing in (1).
- Relation (4) is needed only to retrieve the term t^{nH} in front of the exponential term in our upper bound. If we do not wish to achieve optimality for this coefficient, we can work under relation (3) as well.
- Generalizations of our lower bound to the case $H < 1/2$ do not seem to be out of reach, but would certainly require a tremendous additional technical effort.

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Approximating rough stochastic PDEs

Jan Maas and Hendrik Weber
(joint work with M. Hairer)

We study a class of vector-valued equations of Burgers type driven by a multiplicative space-time white noise. These equations are of the form

$$\partial_t u = \nu\, \partial_x^2 u + F(u) + G(u)\partial_x u + \theta(u)\, \xi \ , \tag{1}$$

where the function $u = u(t, x; \omega) \in \mathbb{R}^n$ is vector-valued. We assume that the functions $F\colon \mathbb{R}^n \to \mathbb{R}^n$ and $G, \theta\colon \mathbb{R}^n \to \mathbb{R}^{n\times n}$ are smooth and the products in the terms $G(u)\partial_x u$ as well as in $\theta(u)\xi$ are to be interpreted as matrix vector multiplication. The noise term ξ denotes an $\mathbb{R}^n$-valued space-time white noise and the multiplication should be interpreted in the sense of Itô integration against an L^2-cylindrical Wiener process.

In the case where G is the gradient of a function $\mathcal{G}$ the equation (1) is classically well-posed. The definition of weak solutions and their construction uses the conservation law structure of (1): The nonlinearity is rewritten as

$$G(u)\partial_x u = \partial_x \mathcal{G}(u),$$

and the derivative can be treated by integration by parts. However, several seemingly natural approximation schemes fail to produce solutions of (1), but converge to different limit equations in which extra terms may appear.

In the case where G is not a total derivative it is not even clear how to make sense of (1). The solution does not have the regularity required to make sense of the nonlinearity. We use rough path theory to resolve this issue. Weak solutions can be defined by testing against a smooth test function φ and defining the term

$$\int_{-\pi}^{\pi} \varphi(x) G(u(t, x))\partial_x u(t, x) dx$$

as a rough integral.

We study approximations to (1) of the form

$$du_\varepsilon = \Big(\nu \Delta_\varepsilon u_\varepsilon + F(u_\varepsilon) + G(u_\varepsilon) D_\varepsilon u_\varepsilon\Big) dt + \theta(u_\varepsilon)\, H_\varepsilon dW,$$

for a large class of regularisations $\Delta_\varepsilon, D_\varepsilon$, and H_ε. We show that the u_ε converge to a process $\bar{u}$ that solves an equation similar to (1) with an extra term

$$-\Lambda\, \theta(u) \nabla G(u)\, \theta^T(u).$$

This term is the local spatial cross variation of u and $G(u)$ and can be interpreted as a spatial Itô-Stratonovich correction. The constant Λ depends on the specific choice of the approximations and can be calculated explicitly. We obtain a rate of convergence of $\varepsilon^{1/6}$.

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