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Mini-Workshop: Nichols Algebras and Weyl Groupoids

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Abstract. Nichols algebras are graded braided Hopf algebras satisfying a universal property. Many structural results of a Nichols algebra can be obtained by studying its Weyl groupoid and its homology. In the mini-workshop, introductions to and recent developments on these structures were presented and open problems were discussed.

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Introduction by the Organisers

Since its introduction in 1998, the Lifting Method grew to be one of the most powerful and fruitful methods to study Hopf algebras. Over the years, the method showed strong relationship with other areas of mathematics such as quantum groups, non-commutative differential geometry, knot theory, combinatorics of root systems and Weyl groups, Lyndon words, cohomology of flag varieties, projective representations, conformal field theory. The influence of the method was corroborated by the "New Hot Paper" award of Essential Science Indicators$^{\mathrm{SM}}$, Thomson Reuters, May 2011, for

N. Andruskiewitsch and H.-J. Schneider, *On the classification of finite-dimensional pointed Hopf algebras*, Ann. Math. **171** (2010), 375–417.

The heart of the Lifting Method is formed by the structure theory of Nichols algebras. Nichols algebras are connected graded braided Hopf algebras generated by primitive elements, all primitive elements having degree one. They were first studied by Nichols and were re-discovered by Lusztig, Woronowicz and others. A major problem, which has been open since the introduction of the Lifting Method, is the classification of finite-dimensional Nichols algebras over groups. This problem was completely solved for finite abelian groups when the base field is algebraically closed of characteristic 0; the solution uses Lie theoretic structures, in particular the very flexible notion of the Weyl groupoid. A generalization of this theory to arbitrary groups is possible and opens new research directions. On the other hand, Weyl groupoids can be and are investigated for their own sake, for example in relation with Coxeter groups, simplicial arrangements, cluster algebras and toric varieties.

The purpose of this meeting was to bring together experts in Nichols algebras and Weyl groupoids, to analyze the present state-of-the-art and elaborate new strategies to further deepen our knowledge of these mathematical objects with a view towards applications in the classification program and the areas evoked above, with emphasis in the cohomological aspects. At this place we would like to thank MFO for providing us with an NSF grant with which we could invite Julia Pevtsova to our mini-workshop. At the mini-workshop, talks were given by N. Andruskiewitsch, I. Angiono, M. Cuntz, F. Fantino, J. Pevtsova, V. Kharchenko, A. Lochmann, M. Rosso, H.-J. Schneider, Ø. Solberg, L. Vendramin, V. Welker, S. Witherspoon, and H. Yamane.

On the first day of the mini-workshop there were two three-hour introductory talks on Nichols algebras (by H.-J. Schneider and I. Angiono) and on Weyl groupoids (by M. Cuntz), respectively. With these talks we tried to compensate the large differences in the backgrounds and main research interests of the participants. The introductory talks were attributed by all participants to be very helpful and professional to create a common starting point.

Starting with Tuesday we tried to keep the number of talks per day on a low level to allow intensive discussions between and after the talks. So we had four talks on Tuesday and Thursday, two on Wednesday and three on Friday. On Wednesday afternoon we made a trip to St. Roman which we enjoyed very much.

We planned as a highlight the talk of H.-J. Schneider, which brought together the two topics in the title of the mini-workshop. However, we indeed had many excellent talks, so at the end of the meeting it was hard to say that one particular talk should be called the highlight. In the talks the speakers often mentioned questions and conjectures; in a problem session at the end of the meeting we were discussing possible ways to attack these problems and directions of further research.

Concluding the above, all organizers and all participants agreed that the mini-workshop was an extremely effective and enjoyable meeting which helped to get an overview on the recent developments of the field, to initiate discussions between experts of various fields, to motivate each other to think sometimes differently, and to discuss possible perspectives.

We strongly believe that the subject of our mini-workshop also has the potential for a successful half-size workshop which we would like to organize in the future.

Mini-Workshop: Nichols Algebras and Weyl Groupoids

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Abstracts

Introduction to Nichols algebras

Hans-Jürgen Schneider

Let H be a Hopf algebra with comultiplication $\Delta : H \to H \otimes H$ and bijective antipode $S : H \to H$. A left Yetter-Drinfeld module V over H (or a YD-module) is a vector space V which is a left H-module and a left H-comodule with comodule structure map $\delta : V \to H \otimes V$ such that

$$\delta(hv) = h_{(1)}v_{(-1)}S(h_{(3)}) \otimes h_{(2)}v_{(0)}, \qquad \text{for all } h \in H, v \in V.$$

Here the symbolic notation $\delta(v) = v_{(-1)} \otimes v_{(0)}$ and $(\Delta \otimes id_H)\Delta(h) = h_{(1)} \otimes h_{(2)} \otimes h_{(3)}$ is used. The YD-modules form a category where morphisms are left H-linear and H-colinear maps. This category is monoidal: if V, W are YD-modules, then so is their tensor product $V \otimes W$ over the ground field with diagonal action and coaction of H. And it is braided: the braiding is given by

$$c_{V,W} : V \otimes W \to W \otimes V,\ v \otimes w \mapsto v_{(-1)}w \otimes v_{(0)}.$$

Let V be a YD-module. Then the tensor algebra $T(V)$ is a Yetter-Drinfeld module, and $T(V)$ becomes a Hopf algebra in the braided monoidal category of YD-modules, where the comultiplication is the algebra map with

$$\Delta_{T(V)} : T(V) \to T(V) \otimes T(V),\ v \mapsto v \otimes 1 + 1 \otimes v.$$

Here the algebra structure on $T(V) \otimes T(V)$ is the braided, that is

$$(a \otimes b)(c \otimes d) = a(b_{(-1)} \cdot c) \otimes b_{(0)}d, \qquad a, b, c, d \in T(V).$$

The Nichols algebra $\mathcal{B}(V)$ is defined by $\mathcal{B}(V) = T(V)/I(V)$, where $I(V)$ is the largest coideal of $T(V)$ contained in $\bigoplus_{n\geq 2} T^n(V)$. The Nichols algebra is a graded quotient Hopf algebra of $T(V)$ whose primitive elements are exactly the elements in V. A special case of a Nichols algebra is $U_q^+(\mathfrak{g})$, $\mathfrak{g}$ a semisimple Lie algebra, see [2]; the YD-modules in this case is defined over the group algebra $H = k\Gamma$ of a free abelian group of finite rank Γ. The relations of the Nichols algebra can be described by the braided symmetrizer maps $S_n : T^n(V) \to T^n(V)$, $n \geq 2$. Thus $\mathcal{B}(V)$ can be defined for any vector space with a braiding $c : V \otimes V \to V \otimes V$. Important early papers on the subject are [3], [5]. There is a dual description as a subalgebra of the braided shuffle algebra [4]. Nichols algebras arise naturally in the classification of pointed Hopf algebras as subalgebras of the graded Hopf algebra associated to the coradical filtration of a pointed Hopf algebra [1].

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Nichols algebras of diagonal type

Iván Angiono

A PBW basis of an algebra A is understood as a basis whose elements are products of elements of a totally ordered subset of A. The Poicaré-Birkhoff-Witt Theorem establishes the existence of such basis for the enveloping algebra of a Lie algebra, where the ordered set is any ordered basis of the Lie algebra

The existence of PBW bases for more general algebras was a frequent topic of study. Several combinatorics methods were discovered to show that some families of algebras have this property, see for example [1] as a pioneer reference. One of these methods involve the Lyndon words and it is applied to show the existence of such PBW bases for braided connected Hopf algebras obtained as quotients of a tensor algebra, with an extra element, a function controlling the *height* of the generators. One sufficient condition about the braiding is that it should be of diagonal type [3, 4], or more generally of triangular type [5]. This result is an important tool to show the existence of the Weyl groupoid associated to a Nichols algebra of diagonal type [2].

In this talk we recall the definition of Lyndon words, their properties and the hyperwords related with the braiding. We state the main results from [3, 4, 5] describing the following elements:

- the braided bracket and the comultiplication of hyperwords on the tensor algebra, and the existence of a family of coideal subalgebras,
- the definition of some PBW generators for a given quotient by a braided Hopf ideal and the height function,
- a proof that they give a PBW basis of such quotient, and an necessary condition of an element to have finite height.

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Introduction to Weyl groupoids

Michael Cuntz

Reflections appear in many areas of mathematics. For instance, certain groups generated by involutions may be investigated by representing them as reflection groups. In particular, the Weyl groups belong to this class. They appear naturally inside semisimple algebraic groups and are fundamental for their classification. In their reflection representation, the Weyl groups are in fact subgroups of $\mathrm{GL}(\mathbb{Z}^r)$ for some r. This integrality is a very strong and important restriction; reflection groups with this property are also called *crystallographic*.

Closely related to the algebraic group is another important structure, the Lie algebra. Lie algebras arise in nature as vector spaces of linear transformations, for example differential operators. It turns out that finite dimensional semisimple complex Lie algebras decompose into a direct sum labeled by *roots* and a Cartan subalgebra. These roots are (up to signs) the normal vectors defining the reflection hyperplanes of a Weyl group. Again, we have an integrality property for the roots. Let $\mathcal{A}$ be the real hyperplane arrangement given by the orthogonal complements of the roots. Then this is a simplicial arrangement and for each chamber K, the roots labeling the walls of K form a simple system Δ, and in particular all other roots are integer linear combinations of the roots in Δ.

So apparently the combinatorics of root systems and Weyl groups play an important role in mathematics and moreover, a certain integrality is an essential feature of these structures. Recent results on Nichols algebras have led to a new symmetry structure, the *Weyl groupoid*. Again one has vectors called "roots", but this time the object acting on the roots is in general a groupoid and not a group anymore. A remarkable fact is that even in this much more general setting, the above integrality still plays a crucial role.

If the *real roots* of a Weyl groupoid form a finite root system, then we will say that the Weyl groupoid is finite. As for root systems from Coxeter groups, a finite Weyl groupoid $\mathcal{W}$ defines a hyperplane arrangement: Fix an object a and its set of positive roots R^a_+. The arrangement associated to $\mathcal{W}$ and a is the set of orthogonal complements of the elements of R^a_+ in $\mathbb{R}^r$. It turns out that these arrangements are simplicial (see [10]). Moreover, it turns out that in terms of simplicial arrangements the axioms of a finite Weyl groupoid reduce to one single integrality property (see [7]). We call simplicial arrangements satisfying this axiom *crystallographic arrangements*.

Among other things, this explains why the class of arrangements obtained from Weyl groupoids is so large. In fact this class is so large that for example in rank three, 53 of the 67 known sporadic simplicial arrangements (see [9] and [8]) over $\mathbb{Q}$ are crystallographic. Like reflection arrangements, all crystallographic arrangements are free [1]. They could provide examples or counterexamples in geometry or topology, especially since they may also be viewed as compact smooth toric varieties (see [6]).

Our lecture on finite Weyl groupoids was divided into three talks. We first gave the definitions of Cartan schemes, Weyl groupoids, and crystallographic arrangements. Then we explained the relations between these notions, and finally we sketched various results (see [3], [2], [5], [4]) needed for the classification of finite Weyl groupoids.

The following open problems were formulated during the lecture.

(1) Are all finite Weyl groupoids invariants of Nichols algebras?
(2) Classify simplicial arrangements.
(3) Extend the connection between reflection groupoids and cluster algebras.
(4) Elaborate the connection to toric varieties.
(5) Understand the structure of the sporadic finite Weyl groupoids.
(6) Deduce the classification of finite dimensional Nichols algebras of diagonal type from the classification of finite Weyl groupoids.

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Some problems on Nichols algebras

Nicolás Andruskiewitsch

Most of the open problems about Nichols algebras fit into the general question:

Main Problem. *Classify all the braided vector spaces (V, c) (in a class $\mathcal{C}$) such that the corresponding Nichols algebras $\mathcal{B}(V) = T(V)/\mathcal{I}(V)$ have finite dimension, or finite GK dimension, or finite growth. For the resulting (V, c), give an optimal set of defining relations of $\mathcal{I}(V)$.*

Although the Nichols algebra $\mathcal{B}(V)$ depends only on the braiding c, we are interested in realizations of (V, c) as a Yetter-Drinfeld module over a Hopf algebra H. In the next Table we summarize some problems related with some kind of

Hopf algebras, their finite-dimensional Yetter-Drinfeld modules. Here Γ denotes a group, and *Applications* means *Applications to the classification of Hopf algebras...*

Table 1. Some problems on Nichols algebras

Shape H	Shape c	Problem	Applications
$H = \mathbf{k}\Gamma$, Γ finite abelian	Diagonal type	$\dim \mathcal{B}(V) < \infty$	pointed over abelian groups, finite dim.
$H = \mathbf{k}\Gamma$, Γ abelian	Triangular type	$\mathrm{GKdim}\mathcal{B}(V) < \infty$	pointed over abelian groups, finite GK dimension
$H = \mathbf{k}\Gamma$, $\mathbf{k}^\Gamma$ Γ finite	Rack type	$\dim \mathcal{B}(V) < \infty$	pointed and copointed, finite dimension
H semisimple	Unknown	$\dim < \infty$	Chevalley type, finite dim.
H finite dim., generated by the coradical	Unknown	$\dim < \infty$	all finite dimensional

Now we describe advances and open questions for some particular problems.

1. Diagonal type, finite dimension

Let us briefly comment the main results in this setting.

◦ Each Nichols algebra $\mathcal{B}(V)$ has a PBW basis [10, 12] formed by some hyperletters called roots.

◦ The definition of the Weyl groupoid of $\mathcal{B}(V)$ with finite set of roots [8].

◦ The classification of all matrices (q_{ij}) with a finite set of roots [9].

◦ The complete set of relations for each (q_{ij}) in Heckenberger's list [6].

As we see, there are many important results about the Nichols algebras of this class of braidings; but there are still some open questions. To state them, let us divide the list of indecomposable matrices in [9] in three families:

◦ *Standard braidings:* the Cartan matrices of all the braidings with the same Weyl groupoid coincide. This Cartan matrix should be of finite type. It includes properly the family of braidings of *Cartan type* [4]

◦ *Braidings of super type:* the Weyl groupoid behaves as the one for some contragredient Lie superalgebra. It intersects the family of standard braidings.

◦ *Unidentified braidings:* a finite family with braidings of rank ≤ 7.

Question 1. *Deduce the classification in* [9] *from the classification of Cartan schemes* [7].

There are some objects in the list in [7] that do not appear in the list in [9]. Are they related to Lie algebras in positive characteristic?

Question 2. *Characterize the braidings of super type.*

Question 3. *Identify the unidentified braidings.*

2. Triangular type, finite GK-dimension

We recall that a braiding is *triangular* if there exists a basis $x_1, \dots, x_\theta$, non-zero scalars q_{ij}, $1 \le i, j \le \theta$, and vectors $v_{ij}^k \in V$, $1 \le i, j, k \le \theta$, $j < k$ such that

$$c(x_i \otimes x_j) = q_{ij} x_j \otimes x_i + \sum_{k>j} x_k \otimes v_{ij}^k, \qquad 1 \le i, j \le \theta$$

An example of a braiding of triangular type is the *Jordanian quantum plane*. Any diagonal braiding is of course triangular with $v_{ij}^k = 0$ always. About the problem of Nichols algebras with finite GK-dimension in this context, we know:

∘ The Nichols algebra (V, c) has a PBW basis [13].

∘ If the braiding is of diagonal type, q_{ii} is not a root of unity, $\mathcal{B}(V)$ is a domain with finite GK dimension, then the braiding is of finite Cartan type [4, 1].

3. Rack type, finite dimension

Given a (finite) rack X and a cocycle $q : X \times X \to GL(n, k)$, there is a braiding in the vector space $kX \otimes k^n$; the corresponding Nichols algebra is denoted $\mathcal{B}(X, q)$. One seeks to classify all pairs (X, q) such that $\dim \mathcal{B}(X, q) < \infty$. There are two different kind of contributions to this problem:

(1) Determination of families of racks that collapse (see below).
(2) Computation by hand, or by computer, some specific examples.

As the computation of all cocycles for a given rack X is hard, it is nice to have tools that guarantee that $\dim \mathcal{B}(X, q) = \infty$ for all cocycles q. For shortness, we say that X *collapses* if this happens. The main features are:

∘ A rack is *of type D* if there exists a subrack Y of X, which admits a decomposition of subracks $Y = R \amalg S$, an elements $r \in R$, $s \in S$ such that $r \triangleright (s \triangleright (r \triangleright s))) \neq s$.

∘ A rack of type D collapses.

∘ The list of all finite simple racks is known [3, 11].

∘ If X is of type D and we have a surjective map of racks $W \to X$, then W is of type D. Also, any rack has a simple quotient. Therefore it is natural to ask:

Question 4. *Classify all simple racks of type D.*

For instance, (twisted) conjugacy classes of finite simple groups are simple racks and most of them are of type D. See [2] for details and references for the alternating and sporadic groups (for groups of Lie type, this is work in progress). An exception:

Question 5. *Determine* $\dim \mathcal{B}(\mathcal{O}_2^n, -1)$, $n \ge 6$. *Is this Nichols algebra quadratic?*

Here $\mathcal{O}_x^\Gamma$ is the conjugacy class of x in Γ, with abbreviation $\mathcal{O}_{1^{n_1}, 2^{n_2}, \dots}^n$ when $\Gamma = \mathbb{S}_n$ and x is of type $(1^{n_1}, 2^{n_2}, \dots)$. As for the list of all pairs (X, q) with $\dim \mathcal{B}(X, q) < \infty$ (in char 0), see [2, Table 6]. However the real reason for the finite-dimensionality of these Nichols algebras is yet to be understood. It would also be interesting to compute a few more examples of Nichols algebras $\dim \mathcal{B}(X,q)$, e. g. when (X,q) is one of the following (where ω is a primitive cubic root of 1):

$$(\mathcal{O}_3^{A_4}, \omega), \qquad (\mathcal{O}_2^3, q),\ q \neq -1, \qquad (\mathcal{O}_{2,2}^5, -1).$$

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Fomin-Kirillov algebras

Leandro Vendramín

For an integer $n \geq 3$ denote by $\mathcal{E}_n$ the algebra (of type A_{n-1}) with generators $x_{(ij)}$, where $1 \leq i < j \leq n$, and relations

$$\begin{aligned} &x_{(ij)}^2 = 0, && \text{for } 1 \leq i < j \leq n,\\ &x_{(ij)}x_{(jk)} = x_{(jk)}x_{(ik)} + x_{(ik)}x_{(ij)}, && \text{for } 1 \leq i < j < k \leq n,\\ &x_{(jk)}x_{(ij)} = x_{(ik)}x_{(jk)} + x_{(ij)}x_{(ik)}, && \text{for } 1 \leq i < j < k \leq n,\\ &x_{(ij)}x_{(kl)} = x_{(kl)}x_{(ij)}, && \text{for any distinct } i,j,k,l. \end{aligned}$$

The algebras $\mathcal{E}_n$ are graded by $\deg(x_{(ij)}) = 1$. Of course, it is natural to ask if $\mathcal{E}_n$ is finite-dimensional. It is known that $\mathcal{E}_n$ is finite-dimensional if $n \leq 5$. It has been conjectured that $\dim \mathcal{E}_n = \infty$ for $n \geq 6$.

Example 1. *The algebra $\mathcal{E}_3$ has dimension 12. The Hilbert series $\mathcal{H}_3(t)$ of $\mathcal{E}_3$ is a polynomial of degree 4: $\mathcal{H}_3(t) = (2)_t^2(3)_t$, where $(k)_t = 1 + t + \cdots + t^{k-1}$.*

Example 2. *Computer calculations yield $\dim \mathcal{E}_4 = 576$. The Hilbert series $\mathcal{H}_4(t)$ of $\mathcal{E}_4$ is a polynomial of degree 12: $\mathcal{H}_4(t) = (2)_t^2(3)_t^2(4)_t^2$.*

Example 3. *Computer calculations yield* $\dim \mathcal{E}_5 = 8294400$*. The Hilbert series* $\mathcal{H}_5(t)$ *of* $\mathcal{E}_5$ *is a polynomial of degree* 40*:* $\mathcal{H}_5(t) = (4)_t^4(5)_t^2(6)_t^4$.

Example 4. *The Hilbert series* $\mathcal{H}_6(t)$ *of* $\mathcal{E}_6$ *cannot be written as a product of* t*-numbers. Further,*

$$\mathcal{H}_6(t) = 1 + 15t + 125t^2 + 765t^3 + 3831t^4 + 16605t^5 + 64432t^6 + 228855t^7 + \cdots$$

In [3], Fomin and Kirillov introduced the algebras $\mathcal{E}_n$ as a new model for the Schubert calculus of a flag manifold. They proved that $\mathcal{E}_n$ contains a commutative subalgebra isomorphic the cohomology ring of the flag manifold. In [1], Bazlov proved that Nichols algebras provide the correct setting for this model of Schubert calculus. But what is the relation between the algebras $\mathcal{E}_n$ and Nichols algebras?

Let V_n be the vector space with basis $\{v_{(ij)} \mid 1 \leq i < j \leq n\}$ and consider the map $c \in \mathbf{GL}(V_n \otimes V_n)$ defined by

$$c(v_\sigma \otimes v_\tau) = \chi(\sigma,\tau) v_{\sigma\tau\sigma^{-1}} \otimes v_\sigma, \qquad \chi(\sigma,\tau) = \begin{cases} 1 & \text{if } \sigma(i) < \sigma(j), \\ -1 & \text{otherwise,} \end{cases}$$

where σ and τ are transpositions, and $\tau = (i\ j)$ with $i < j$. Since (V_n, c) is a braided vector space, it is possible to consider the Nichols algebra $\mathfrak{B}(V_n)$. Bazlov proved that $\mathfrak{B}(V_n)$ contains a commutative subalgebra isomorphic to the cohomology ring of the flag manifold.

It is known that $\mathfrak{B}(V_n) = \mathcal{E}_n$ if $3 \leq n \leq 5$; this was proved by Milinski and Schneider for $n \leq 4$, and by Graña for $n = 5$. It has been conjectured that $\mathfrak{B}(V_n)$ is quadratic and $\mathfrak{B}(V_n) = \mathcal{E}_n$; see for example [5] and [1].

There are many others interesting conjectures about Fomin–Kirillov algebras. In [5], Majid wrote that it might be possible to find a relation between Fomin–Kirillov algebras and the representation theory of preprojective algebras.

Let Λ be the preprojective algebra of a quiver of type A_{n-1}. It is known that the number of indecomposable modules over Λ is 4 if $n = 3$, 12 if $n = 4$, and 40 if $n = 5$. Further, Λ is of infinite representation type if $n \geq 6$. Majid noticed that the number of indecomposable modules over Λ is equal to the degree of the Hilbert series of $\mathcal{E}_n$, at least for $3 \leq n \leq 5$. Majid's conjecture does not have a precise formulation, but it states that this numerology is not an accident.

To conclude, we restate Majid's observation in terms of cluster algebras. Let $n \geq 2$, $G = \mathbf{SL}_n$, and N be the subgroup of upper triangular matrices with ones in the diagonal. In [2], Berenstein, Fomin and Zelevinski proved that $\mathbb{C}[N]$, the coordinate ring of N, has a cluster algebra structure. Furthermore, the number of clusters of $\mathbb{C}[N]$ is given by the following table:

Lie type of G	Number of clusters
A_2	4
A_3	12
A_4	40
others	∞

Geiss, Leclerc and Schröer established a relation between the number of clusters of $\mathbb{C}[N]$ and the number of indecomposable modules over the preprojective algebra

Λ, see for example [4]. This implies that Majid's observation can be translated into the combinatorial language of cluster algebras.

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On racks of type D

Fernando Fantino

In the context of the Lifting Method, the study of the classification of finite-dimensional pointed Hopf algebras with non-abelian group leads to the study of finite-dimensional Nichols algebras $\mathfrak{B}(X, q)$ associated with pairs (X, q), where X is a rack an q a 2-cocycle of racks.

Since the computation of the dimension of $\mathfrak{B}(X, q)$ is a hard task, it is useful to determine those racks X such that $\dim \mathfrak{B}(X, q) = \infty$ for all q. A family of racks with this property is the class of racks of type D. We say that a rack X is *of type D* if it contains a decomposable subrack $Y = R \coprod S$ and elements $r \in R$, $s \in S$ such that $r \triangleright (s \triangleright (r \triangleright s)) \neq s$.

Another property of the racks of type D is the following: if $Z \to X$ is an epimorphism of finite racks and X is of type D, then Z is of type D. On the other hand, any rack has a projection onto a simple rack and the classification of finite simple racks is known, see [1] and [6]. For that reasons, it is an important:

Problem: to classify all finite simple racks of type D.

By [1], a finite simple rack belongs to one of the following classes:

(a) simple affine racks,
(b) non-trivial conjugacy classes of non-abelian finite simple groups,
(c) twisted conjugacy classes of non-abelian finite simple groups,
(d) simple twisted homogeneous racks.

In this talk I will show the present state of the classification of racks of type D in the list above. For the families of racks (b), (c) and (d) our approach uses the classification of finite simple non-abelian groups. For the class of racks in (b), the problem if finished for the alternating groups [2, 4] and the sporadic simple groups except 19 conjugacy classes of the Monster group [3]. For the class of racks in (c), the problem if finished for the alternating groups [2] and the sporadic simple groups [5]. There are works in progress considering the finite groups of Lie type.

Finally, I will mention how the results obtained in the previous paragraph are used to obtain the classification of finite-dimensional pointed Hopf algebras over some families of non-abelian (simple) groups.

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Harish-Chandra type theorem of Drinfeld doubles

Hiroyuki Yamane

(joint work with Punita Batra)

Let $\mathbb{K}$ be an algebraically closed field, and $\mathbb{K}^\times$ be $\mathbb{K}\setminus\{0\}$. Let I be a non-empty finite set. Let $q_{ij}\in\mathbb{K}^\times$ for $i,j\in I$. Let $\widetilde{U}$ be an associative unital $\mathbb{K}$-algebra defined by the generators $\widetilde{K}_i$, $\widetilde{K}_i^{-1}$, $\widetilde{L}_i$, $\widetilde{L}_i^{-1}$, $\widetilde{E}_i^{+}$, $\widetilde{E}_i^{-}$ $(i\in I)$, and the relations

$$\widetilde{X}\widetilde{Y}=\widetilde{Y}\widetilde{X},\ \widetilde{X},\widetilde{Y}\in\{\widetilde{K}_i^{\pm1},\widetilde{L}_i^{\pm1}|i\in I\},\qquad \widetilde{K}_i\widetilde{K}_i^{-1}=\widetilde{L}_i\widetilde{L}_i^{-1}=1,$$
$$\widetilde{K}_i\widetilde{E}_j^{\pm}=q_{ij}^{\pm1}\widetilde{E}_j^{\pm}\widetilde{K}_i,\qquad \widetilde{L}_i\widetilde{E}_j^{\pm}=q_{ji}^{\mp1}\widetilde{E}_j^{\pm}\widetilde{L}_i,$$
$$\widetilde{E}_i^{+}\widetilde{E}_j^{-}-\widetilde{E}_j^{-}\widetilde{E}_i^{+}=\delta_{ij}(-\widetilde{K}_i+\widetilde{L}_i).$$

Let $\mathcal{A}$ be a free $\mathbb{Z}$-module with a basis $\{\alpha_i|i\in I\}$, so $\mathrm{rank}(\mathcal{A})=\mathrm{Card}(I)$. Let $\mathcal{A}_+:=\{\sum_{i\in I}m_i\alpha_i|m_i\in\mathbb{Z}_{\geq0}(i\in I)\}$. Regard $\widetilde{U}$ as an $\mathcal{A}$-graded $\mathbb{K}$-algebra with $\mathrm{deg}\widetilde{K}_i^{\pm1}=\mathrm{deg}\widetilde{L}_i^{\pm1}=0$, and $\mathrm{deg}\widetilde{E}_i^{\pm}=\pm\alpha_i$. Let $\widetilde{U}^{\pm}$ be the unital $\mathbb{K}$-subalgebras of $\widetilde{U}$ generated by $\widetilde{E}_i^{\pm}$'s. Let $\widetilde{U}^0$ be the unital $\mathbb{K}$-subalgebra of $\widetilde{U}$ generated by $\widetilde{K}_i^{\pm1}$'s and $\widetilde{L}_i^{\pm1}$'s. Let $\widetilde{I}^{\pm}$ be the $\mathbb{K}$-subspaces of $\widetilde{U}^{\pm}$ spanned by the homogeneous elements $\widetilde{X}^{\pm}$ with $\mathrm{deg}\widetilde{X}^{\pm}\notin\{0\}\cup\{\pm\alpha_i|i\in I\}$ such that for every i,

$$\widetilde{X}^{\pm}\widetilde{E}_i^{\mp}-\widetilde{E}_i^{\mp}\widetilde{X}^{\pm}=\widetilde{X}^{\pm,\prime}\widetilde{K}_i+\widetilde{X}^{\pm,\prime\prime}\widetilde{L}_i$$

for some $\widetilde{X}^{\pm,\prime}$, $\widetilde{X}^{\pm,\prime\prime}\in\widetilde{I}^{\pm}$. Let $\widetilde{J}:=\mathrm{Span}_{\mathbb{K}}(\widetilde{I}^{-}\widetilde{U}^0\widetilde{U}^{+}+\widetilde{U}^{-}\widetilde{U}^0\widetilde{I}^{-})$. Then $\widetilde{J}$ is an ideal of $\widetilde{U}$. Let $U:=\widetilde{U}/\widetilde{J}$, and $\pi:\widetilde{U}\to U$ be the canonical map. Let $U^{\pm}:=\pi(\widetilde{U}^{\pm})$, and $U^0:=\pi(\widetilde{U}^0)$.

For $\lambda=\sum_{i\in I}m_i\alpha_i$, $\mu=\sum_{i\in I}l_i\alpha_i\in\mathcal{A}$ with m_i, $l_i\in\mathbb{Z}$, let $\chi(\lambda,\mu):=\prod_{ij}q_{ij}^{m_il_j}$, and $\mathrm{ord}(\lambda):=\mathrm{Card}(\{\,\chi(k\lambda,\lambda)\in\mathbb{K}^\times\mid k\in\mathbb{Z}\,\})$. Assume that there exist a positive integer n and an injection $\varphi:\{1,\ldots,n\}\to\mathcal{A}_+$ satisfying the conditions:

(1) $\mathrm{ord}(\varphi(k))\geq2$ for all $1\leq k\leq n$.

(2) There exist homogeneous elements $\bar{E}_1^{\pm}, \ldots, \bar{E}_n^{\pm} \in U^{\pm}$, $\deg \bar{E}_k^{\pm} = \pm\varphi(k)$, such that $(\bar{E}_1^{\pm})^{r_1} \cdots (\bar{E}_n^{\pm})^{r_n}$ $(0 \leq r_k < \mathrm{ord}(\varphi(k)))$ form $\mathbb{K}$-bases of $U^{\pm}$.

Let $\{K_\lambda L_\mu | \lambda, \mu \in \mathcal{A}\}$ be the $\mathbb{K}$-basis of U^0 such that $K_\lambda L_\mu \cdot K_{\lambda'} L_{\mu'} = K_{\lambda+\lambda'} L_{\mu+\mu'}$, $K_{\pm\alpha_i} L_0 = \pi(\widetilde{K}_i^{\pm 1})$, $K_0 L_{\pm\alpha_i} = \pi(\widetilde{L}_i^{\pm 1})$. We have $U \cong U^- \otimes U^0 \otimes U^+$ as a $\mathbb{K}$-linear space. Let $R^+ := \{\beta_k \in \mathcal{A} | 1 \leq k \leq n\}$. By the Kharchenko's PBW theorem, R^+ is unique. We call $R := R^+ \cup -R^+$ *the root system of* U.

Let $E_i^{\pm} := \pi(\widetilde{E}_i^{\pm})$. Define the $\mathbb{K}$-linear map $\mathrm{Sh} : U \to U^0$ by $\cup_{i \in I}(\mathrm{Sh}(UE_i^+) \cup \mathrm{Sh}(E_i^- U)) = \{0\}$ and $\mathrm{Sh}_{|U^0} = \mathrm{id}_{U^0}$. We call Sh *the Shapovalov map of* U. Let U_0 be the $\mathbb{K}$-subalgebra of U of all homogeneous zero-degree elements. Then $U^0 \subset U_0$, $U^0 \neq U_0$. Let $\Omega : \mathcal{A} \to \mathbb{K}^\times$ be a multiplicative character. Let $\mathcal{Z}_\Omega(U) := \{ X \in U_0 \,|\, E_i^{\pm} X = \Omega(\alpha_i)^{\pm 1} X E_i^{\pm} \, (i \in I) \}$. We call $\mathcal{Z}_\Omega(U)$ *the* Ω*-skew graded center of* U. We call $\mathrm{HC}_\Omega := \mathrm{Sh}_{|\mathcal{Z}_\Omega(U)}$ *the* Ω*-Harish-Chandra map of* U. We can prove that HC_Ω is injective. Define the multiplicative character $\hat{\rho} : \mathcal{A} \to \mathbb{K}^\times$ by $\hat{\rho}(\alpha_i) := q_{ii}$.

Theorem. *In the following, let* $h := \mathrm{ord}(\beta)$ *and* $\omega := \Omega(\beta) \cdot \frac{\chi(\beta,\mu)}{\chi(\lambda,\beta)}$. *Let* $Z = \sum_{\lambda,\mu \in \mathcal{A}} a_{\lambda,\mu} K_\lambda L_\mu \in U^0$, *where* $a_{\lambda,\mu} \in \mathbb{K}$. *Then* $Z \in \mathrm{ImHC}_\Omega$ *if and only if* Z *satisfies the following conditions for all* $\beta \in R^+$ *and all* $(\lambda, \mu) \in \mathcal{A} \times \mathcal{A}$.

(e1) *If* $h = \infty$ *and* $\omega = \chi(\beta,\beta)^t$ *for some* $t \in \mathbb{Z} \setminus \{0\}$, *then*

$$a_{\lambda+t\beta,\mu-t\beta} = \hat{\rho}(\beta)^t \cdot a_{\lambda,\mu}.$$

(e2) *If* $h = \infty$ *and* $\omega \neq \chi(\beta,\beta)^t$ *for all* $t \in \mathbb{Z} \setminus \{0\}$, *then* $a_{\lambda,\mu} = 0$.

(e3) *If* $h < \infty$ *and* $\omega = \chi(\beta,\beta)^t$ *for some* $t \in \{1, \ldots, h-1\}$, *then*

$$\sum_{k=-\infty}^{\infty} a_{\lambda+(hk+t)\beta,\mu-(hk+t)\beta} \cdot \hat{\rho}(\beta)^{-(hk+t)} = \sum_{k=-\infty}^{\infty} a_{\lambda+hk\beta,\mu-hk\beta} \cdot \hat{\rho}(\beta)^{-hk}.$$

(e4) *If* $h < \infty$ *and* $\omega \neq \chi(\beta,\beta)^s$ *for all* $s \in \mathbb{Z}$, *then for all* $t \in \{1, \ldots, h-1\}$,

$$\sum_{k=-\infty}^{\infty} a_{\lambda+(hk+t)\beta,\mu-(hk+t)\beta} \cdot \hat{\rho}(\beta)^{-(hk+t)} = \sum_{k=-\infty}^{\infty} a_{\lambda+hk\beta,\mu-hk\beta} \cdot \hat{\rho}(\beta)^{-hk}.$$

Cohomology of Hopf algebras and Nichols algebras

Sarah Witherspoon

About 50 years ago, Golod [6], Venkov [11], and Evens [3] proved that the cohomology ring of a finite group is finitely generated, thus opening the door to a study of its representations using geometric methods. Within the past two decades, analogous results for various classes of finite dimensional Hopf algebras have been proven. For example, Friedlander and Suslin [4] generalized the result to finite dimensional cocommutative Hopf algebras. Ginzburg and Kumar [5] proved that the (noncocommutative) small quantum groups have finitely generated cohomology, using techniques that Friedlander and Parshall had used for restricted Lie algebras. Similarly Gordon [7] proved that cohomology of finite quantum function algebras is finitely generated.

Etingof and Ostrik [2] conjectured much more generally that the cohomology of any finite tensor category is finitely generated, which would imply in particular that the cohomology of any finite dimensional Hopf algebra is finitely generated. With Mastnak, Pevtsova, and Schauenburg [8], we proved the conjecture for pointed Hopf algebras with abelian groups of group-like elements, under some conditions. The proof used the recent classification result of Andruskiewitsch and Schneider [1], in particular the structure of such Hopf algebras and their Nichols algebras.

Specifically, let A be any augmented algebra, that is an algebra over a field k with algebra homomorphism $\varepsilon : A \to k$, for example, A could be a Hopf algebra or a Nichols algebra. The *cohomology* of A is

$$\mathrm{H}^*(A) := \mathrm{Ext}^*_A(k,k),$$

defined for example by choosing a projective resolution

$$\ldots \to P_n \to P_{n-1} \to \cdots \to \cdots P_1 \to P_0 \to k \to 0$$

of k as an A-module, applying $\mathrm{Hom}_A(\cdot, k)$, and taking homology. The resulting graded vector space $\mathrm{H}^*(A)$ becomes a k-algebra under multiplication given by Yoneda composition.

If A is a Hopf algebra, then its cohomology $\mathrm{H}^*(A)$ is graded commutative (see e.g. [10]), while if A is a Nichols algebra, then $\mathrm{H}^*(A)$ is braided graded commutative [8]. If $A = R\#H$ is the bosonization of a Nichols algebra R with a semisimple Hopf algebra H, then there is an action of H on the cohomology $\mathrm{H}^*(R)$ of R for which the cohomology $\mathrm{H}^*(A)$ of A is isomorphic to $(\mathrm{H}^*(R))^H$, the subalgebra of H-invariants of $\mathrm{H}^*(R)$. More generally, if H is not necessarily semisimple, a spectral sequence yields a relationship between the cohomology of A and the cohomology of R. In either case, in order to understand the cohomology of the Hopf algebra A, it is necessary to understand that of the Nichols algebra R.

In the case that A is a pointed Hopf algebra with abelian group of group-like elements, under the conditions in the classification result of Andruskiewitch and Schneider [1], in [8] we adapted the techniques of Friedlander and Parshall, and of Ginzburg and Kumar, to prove finite generation of cohomology. Knowing the structure of the Nichols algebras involved is crucial in the proof. An outline of this approach is: (1) A is filtered with associated graded algebra $R\#kG$ for some Nichols algebra R. A spectral sequence for a filtered algebra is used to reduce the question of finite generation to that for $\mathrm{H}^*(R\#kG) \cong \mathrm{H}^*(R)^G$. (2) The Nichols algebra R is filtered, has a PBW basis, and its associated graded algebra $\mathrm{gr}R$ is a Nichols algebra of type $A_1 \times \cdots \times A_1$. A spectral sequence for a filtered algebra is used to reduce the question of finite generation to that for $\mathrm{H}^*(\mathrm{gr}R)$. (3) The cohomology $\mathrm{H}^*(\mathrm{gr}R)$ is computed explicitly, using a resolution adapted from Benson and Green; the cohomology is finitely generated.

We remark that each proof of finite generation of cohomology of a class of Hopf algebras uses the structure of the specific type of Hopf algebra in crucial ways.

The problem remains open: Prove or find a counterexample to the conjecture of Etingof and Ostrik. More specifically, do this first for some known classes of

Hopf algebras, such as the following. (1) The remaining pointed Hopf algebras with abelian groups of group-like elements in characteristic 0, or more generally the bosonizations of Nichols algebras of diagonal type. Existence of PBW bases for some classes may mean that existing techniques can be adapted. In particular, the general result of Shroff [9] on quotients of PBW algebras may apply to some of the classes of examples. (2) Pointed Hopf algebras with nonabelian groups of group-like elements, and pointed Hopf algebras in positive characteristic (with any group of group-like elements having nonsemisimple group algebra). There are no such examples of $\mathrm{H}^*(A)$ known at this time.

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Computational aspects of projective resolutions

Oyvind Solberg

The main focus behind this talk was to understand the 12-dimensional Nichols algebra

$$R = k\langle a, b, c\rangle/(a^2, b^2, c^2, ab + bc + ca, ac + ba + cb)$$

with respect to the cohomology ring $H^*(R) = \oplus_{i\geq 0}\mathrm{Ext}^n_R(k, k)$, where k is a field and k also denotes the trivial R-module. The cohomology ring is important, among other things, since it gives rise to support varieties of modules given certain finite generation conditions. See the talk of Sarah Witherspoon for further details and background.

One way of trying to understand the cohomology ring $H^*(R)$ is to compute a projective resolution of the trivial module k and then compute the Yoneda algebra

$H^*(R)$. One projective resolution of an R-module is the so called Bongartz-Butler-Gruenberg resolution: Let M be a finitely generated R-module, where $R = kQ/I$ for a field k, a quiver Q and an admissible ideal I in the path algebra kQ. Given a projective presentation

$$\eta\colon 0 \to P_1 \hookrightarrow P_0 \to M \to 0$$

of M as a kQ-module, it is known that the sequence of submodules of P_0,

$$\cdots \subseteq P_1I^4 \subseteq P_0I^4 \subseteq P_1I^3 \subseteq P_0I^3 \subseteq P_1I^2 \subseteq P_0I^2 \subseteq P_1I \subseteq P_0I \subseteq P_1 \subseteq P_0,$$

modulo the ideal I induces a projective resolution of M as a R-module. This resolution can be analyzed using right Gröbner basis theory, and in [3] it is shown that knowing a finite set of equations this projective resolution and the Yoneda algebra $\mathrm{Ext}^*_R(M, M)$ can be computed for any finitely generated R-module. In general this resolution is far from minimal. For the above example one can show that the number of indecomposable projective summands in n-th projective in the resolution is $25^{\frac{n}{2}}$ and $3 \cdot 25^{\frac{n-1}{2}}$, when n is even and odd, respectively (and given a certain Gröbner basis). The minimal projective resolution has a linear growth.

A more efficient projective resolution is described in [2, 1]. It takes the Bongartz-Butler-Gruenberg resolution as a starting point, but makes adjustments along the way. For instance, it can be shown that

$$0 \to P_0I \hookrightarrow P_1 \to \Omega^1_R(M) \to 0$$

is a kQ-projective presentation of a first syzygy of M over R. The projective module P_1 can be written as $\amalg_{i=1}^t w_ikQ$ for some w_i in P_0. If w_i is in P_0I, then it will be mapped to zero in $\Omega^1_R(M)$. Hence, consider only those w_i which is not in P_0I. Denote this set by $T_1 \subseteq \{w_1, w_2, \ldots, w_t\}$. Then

$$0 \to P_0I \cap \amalg_{w\in T_1} wkQ \hookrightarrow \amalg_{w\in T_1} wkQ \to \Omega^1_R(R) \to 0$$

is an exact sequence. We now use this sequence as η was used above. This gives rise to the projective resolutions described in [2, 1], where [1] explains in some detail the algorithm for constructing the resolution.

Using the software package QPA (see `http://sourceforge.net/projects/quiverspathalg/`), the structure of $H^*(R)$ has been computed up to degree 40. The dimension of $H^n(R)$ for $n = 0, 1, 2, \ldots, 40$ is the following:

1, 3, 5, 6, 7, 9, 11, 12, 13, 15, 17, 18, 19, 21, 23, 24, 25, ...

which is all positive integers congruent to $\{0, 1, 3, 5\}$ according to the *On-Line Encyclopedia of Integer Sequences* (`oeis.org`). If the dimension of $H^n(R)$ is given by this sequence, the Hilbert series for $H^*(R)$ is $\frac{(1+t)(1+t+t^2)}{(1-t)(1-t^4)}$. This indicates that $H^*(R)$ has generators in degrees 1 and 4 as an algebra over $H^0(R)$. Again using the software package QPA, one can calculate that it has three generators in degree 1 and one generator in degree 4, as an algebra over the degree zero part. We hope to unravel the structure of the cohomology ring through an explicit knowledge of a projective resolution of k.

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Weyl groupoid and root system of Nichols algebras

Hans-Jürgen Schneider

(joint work with István Heckenberger)

Let H be a Hopf algebra with bijective antipode, $\theta \geq 2$, and $M = (M_1, \dots, M_\theta)$ a tuple of finite-dimensional irreducible Yetter-Drinfeld modules M_i over H. A fundamental problem for Nichols algebras is to understand the Nichols algebra

$$\mathcal{B}(M) = \mathcal{B}(M_1 \oplus \cdots \oplus M_\theta).$$

In the case of $\mathcal{B}(M) = U_q^+(\mathfrak{g})$, the M_i are all one-dimensional, and their Nichols algebra is simply a polynomial algebra in one variable, but $\mathcal{B}(M)$ is given by the complicated braided Serre relations. In general the computation of $\mathcal{B}(M)$ is very difficult or out of reach. The Weyl groupoid and the generalized root system of $\mathcal{B}(M)$ are important combinatorial invariants of the Nichols algebra. These invariants are used in all the deeper results on Nichols algebras. In the diagonal case the Weyl groupoid was introduced by Heckenberger [2]. He used Kharchenko's PBW-basis of the Nichols algebra to define the root system. In the general case of a semisimple Yetter-Drinfeld module, the Weyl groupoid was defined in [1]. The existence of root systems of $\mathcal{B}(M)$ was established in [3], and in [4] with a new proof in the case of a finite Weyl groupoid. In our new work we describe the i-th reflection operator defining the Weyl groupoid in a completely new way by a more general and natural approach. The basic result is a braided monoidal equivalence between the categories of left Yetter-Drinfeld modules over $\mathcal{B}(M_i)\#H$ and of $\mathcal{B}({M_i}^*)\#H$, where M_i^* is the dual Yetter-Drinfeld module over H.

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Primitively generated braided Hopf algebras

Vladislav Kharchenko

The braided Hopf algebras appeared firstly in the famous paper by Milnor and Moore [6] as graded Hopf algebras, and then as universal enveloping algebras of colored super-algebras [7]. A more general concept of "generalized Lie algebra" related to an involutive braiding (a symmetry) has been introduced by Gurevich [3] and appeared later in the categorical context in the paper by Manin [5]. The universal enveloping algebra construction then provided a new class of braided Hopf algebras. More generally, braided Hopf algebras are Hopf algebras in braided tensor categories. A standard way to obtain a braided tensor category is to consider all modules over a quasitriangular Hopf algebra or all comodules over coquasitriangular Hopf algebra.

We discuss possible generalizations of the Cartier—Kostant theorem for braided Hopf algebras. By definition a connected braided coalgebra C is *cosymmetric* if the image of the *linearization map* defined by M. Sweedler is contained in the Nichols algebra defined by the braided space of primitive elements. We show that a connected braided Hopf algebra H is cosymmetric if and only if it is strictly generated by the primitive elements: $H_n = H_1^n$, [4]. Additionally all Hopf subalgebras of H and all homomorphic images of H in the related tensor category are cosymmetric, while all biideals are generated by the primitive elements, provided that the braiding is diagonal or the category is the category of left comodules over coquasitriangular cosemisimple bialgebra. The latter statement somehow defines a category equivalence to some algebraic structure on the space of primitive elements, which is naturally to consider as a "quantum Lie algebra". From this point of view, we discuss approaches of S. L. Woronowicz [8] and Ardizzoni [1], and consider in more detail the case of involutive braiding.

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Nichols algebras of diagonal type and convex orders

Iván Angiono

An important and difficult question about Nichols algebras is to obtain a set of defining relations for $I(V)$, where $\mathcal{B}(V) = T(V)/I(V)$ denotes the Nichols algebra associated to a braided vector space (V, c). It is divided in two parts: first, to obtain some relations; second, to prove that our set generates $I(V)$. For the second part, a main tool is the existence of PBW bases in the diagonal case [7]. But a problem about them is that we do not know neither the PBW generators explicitly nor the relations between them.

Fix a braided vector space of diagonal type with matrix $(q_{ij})_{1\leq i,j\leq\theta}$ and χ the bicharacter on $\mathbb{Z}^\theta$ given by $\chi(\alpha_i, \alpha_j) = q_{ij}$, (α_i) the canonical basis on $\mathbb{Z}^\theta$. Assume that the associated set of roots Δ^χ is finite. Using the Weyl groupoid [3] we can know the $\mathbb{Z}^\theta$-degrees of the generators and that all of them are different, because all the roots are real. Therefore the order on the PBW generators induces a total order on the positive roots. An interesting property of these PBW bases is that they generate a chain of coideal subalgebras when we admit ordered products up to each generator. Therefore we can relate them with the classification of $\mathbb{N}^\theta$-graded coideal subalgebras of $\mathcal{B}(V)$ [6]. We deduce that the previous order on Δ^χ_+ is *convex*: if $\alpha < \beta \in \Delta^\chi_+$ are such that $\alpha + \beta \in \Delta^\chi_+$, then $\alpha < \alpha + \beta < \beta$. This notion generalizes the previous one for classical root systems in [8].

From this point we obtain recursively the set of Lyndon words giving the PBW generators, and prove that the PBW basis is orthogonal for the non-degenerate bilinear form existing when the matrix (q_{ij}) is symmetric. The orthogonality on the PBW basis gives a family of relations generating the ideal $I(V)$ in every case, i. e. not only for symmetric matrices, see [1].

We consider now the classification of diagonal braidings with a finite root system [4] and the Lusztig isomorphisms [5], in order to obtain a minimal set of relations for each braidings from the previous one [2]. It contains the classical power root vectors, quantum Serre relations and generalizations of this last family. Using this minimal representation we can answer positively in the diagonal case a conjecture made by Andruskiewitsch and Schneider:

Theorem. [2] *Let $R = \oplus_{n\geq 0} R_n$ be a finite-dimensional connected graded braided Hopf algebra such that $R_1 = V$ is a braided vector space of diagonal type generating R as an algebra. Then $R = \mathcal{B}(V)$.*

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Geometric combinatorics of Weyl groupoids

Volkmar Welker
(joint work with István Heckenberger)

In [1] we extend properties of the weak order on finite Coxeter groups to Weyl groupoids admitting a finite root system. We do so by adapting the usual definition of the weak order to the set of morphisms with a fixed target object in the category underlying the Weyl groupoid using the length function for morphisms in Weyl groupoids. We exhibit examples that show that the isomorphism type of the weak order can depend on the chosen object. Combinatorially, the weak orders of Weyl groupoids are graded posets that have the structure of ortho-complemented lattices. In addition, we exhibit examples of Weyl groupoids where the posets occurring for this Weyl groupoid do not resemble all of the nice structural properties shared by the weak orders of finite Coxeter groups. In particular, the rank generating function of the weak order of a Weyl groupoid does not factor into factors of type $1 + t + \cdots + t^l$.

The main geometric result for weak orders of Weyl groupoids states.

Theorem. *Let a be an object in a Weyl groupoid and let u, v be two morphisms with target a such that u is smaller than v in the weak order. Then either the order complex of the open interval (u, v) is contractible or homotopy equivalent to a sphere.*

Using the definition of a descent set for morphisms in Weyl groupoids we can provide conditions on which the two cases occurs and the dimension of the sphere.

1. Coxeter complex

We show that to each object in a Weyl groupoid there is an associated simplicial arrangement of hyperplanes defined by the root system in that object. The arrangements for different objects are linearly isomorphic and induce a triangulation of the sphere that is as an abstract simplicial complex isomorphic to the simplicial complex of cosets of parabolic subgroups. This is the usual way to introduce the Coxeter complex for an abstract Coxeter group. Thus we can speak of the Coxeter complex of a Weyl groupoid. Combinatorially the Coxeter complexes of Weyl groupoids are shown to contain elements that do not occur for Coxeter groups.

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Classifying Nichols algebras by their Hilbert series

Andreas Lochmann

(joint work with István Heckenberger, Leandro Vendramín)

The classification of finite dimensional Nichols algebras has been solved in the abelian case by Heckenberger. In the non-abelian case, a description of Nichols algebras in terms of racks was developed, see e.g. [1], but their classification is still an open problem and of large interest to the study of pointed Hopf algebreas. In [2], Graña, Heckenberger and Vendramin classified finite dimensional Nichols algebras which satisfy a certain factorization property for their Hilbert series. We present a continuation to this work, which allows for more general Hilbert series, but restricts on the type of racks to be used:

Theorem. [3] *Let $\mathcal{B}$ be a Nichols $\mathbf{k}$-algebra over an indecomposable faithful braided quandle X. Assume the Hilbert series of $\mathcal{B}$ is of the form*

$$H_{\mathcal{B}}(t) \;=\; (\alpha_1)_t \cdots (\alpha_n)_t \cdot (\beta_1)_{t^2} \cdots (\beta_m)_{t^2}\,, \tag{1}$$

where $(n)_t := 1+t+t^2+\cdots+t^{n-1}$. Then, for each field $\mathbf{k}$, $\mathcal{B}$ is one of 10 possible Nichols algebras (11 if $\mathbf{k}$ is of characteristic 2).

Conjecture. *Let V be an absolutely irreducible Yetter-Drinfeld module over a group G. Assume that G is generated by the support of V. If the Nichols algebra $\mathcal{B}(V)$ is finite-dimensional, then it is one of the Nichols algebras in Theorem , or one of the two known Nichols algebras over an affine rack with five elements.*

As a necessary input to the theorem, we need the special factorization in equation (1). All known examples so far satisfy this factorization. Surprisingly enough, it is possible to deduce some factors of the Hilbert series of a Nichols algebra without knowing much about it. The Nichols-Zoeller-Theorem states as a corollary, that any Nichols algebra $\mathcal{B}$ is isomorphic to the tensor product $\ker \partial_t \otimes \mathbf{k}[x_t]/x_t^m$ for some $m \in \mathbb{N}$. From this follows $(m)_t \mid \mathcal{H}_{\mathcal{B}}(t)$. We demonstrate a slight improvement on this:

Theorem. *Let $\mathcal{B}$ be a finite-dimensional Nichols algebra over an indecomposable rack X and a 2-cocycle χ with diagonal elements of order m. Let X' be a non-empty proper subrack of X and $\mathcal{B}'$ its corresponding Nichols sub-algebra of $\mathcal{B}$. Then the Hilbert series $\mathcal{H}_{\mathcal{B}}(t)$ is divisible by $(m)_t \cdot \mathcal{H}_{\mathcal{B}'}(t)$.*

Assume that the degree of X divides m. Assume further that $X \setminus X'$ still generates $\operatorname{Inn} X$. Then $\#\operatorname{Inn} X \cdot \dim \mathcal{B}'$ divides $\dim \mathcal{B}$.

The proof uses a similar idea as has been used in [4], and combines it with Graña's Freeness Theorem. We conjecture that each finite dimensional Nichols algebra satisfies equation (1), and hope to further extend the above theorem in the near future.

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Beyond Nichols algebras, and back to quantum groups

Marc Rosso

(joint work with Xin Fang)

We give the general construction of all Hopf algebra structures on the cotensor algebra $T_H^C(M)$, the subalgebra generated by H and M being a natural generalization of Nichols algebra. A particular example are quantum quasi-symmetric algebras.

We show that we can reconstruct the whole quantum group $U_q(\mathfrak{g})$, as such an algebra, and one can realize all highest weight irreducible representations in this framework. It also provides a systematic way to construct simple modules over the quantum double of a quantum group.

Cohomology and support varieties

Julia Pevtsova

This talk was an introduction to the theory of support varieties for modules which complemented S. Witherspoon's talk on finite generation of cohomology at the same mini-workshop.

For a finite group G and a field k of characteristic p, dividing the order of G, the cohomology ring $H^*(G,k)$ is a graded commutative algebra over k. It had been shown by Golod, Venkov, and Evens in 1959-1961 that the cohomology algebra was finitely generated. This finite generation result was a precursor of the celebrated "stratification theorem" of Quillen [5], which gives a beautiful geometric description of the ring $H^*(G,k)$ in terms of the elementary abelian subgroups of G. Work of several prominent group theorists followed to employ Quillen's geometric techniques to study representation theory of G. The power of those techniques is one of the main driving forces behind the search for finite generation results for cohomology of other types of algebraic structures.

The central geometric invariant that appeared in the works of Alperin-Evens and Carlson in the 1980's is the support variety of a module.

Definition. *Let M be a kG-module. The support variety of M, denoted $V_G(M)$, is the closed subset of* $\operatorname{Spec} H^*_{\text{red}}(G,k)$ *defined by the ideal* $\operatorname{Ann}_{H^*_{\text{red}}(G,k)}\operatorname{Ext}^*_G(M,M)$.

Support varieties satisfy a number of good properties:

Theorem. *Let M, M_1, M_2, M_3 be kG-modules.*

(1) $V_G(M_1 \oplus N_1) = V_G(M_1) \cup V_G(N_1)$.
(2) $V_G(M) = V_G(\Omega M)$, *where ΩM is the syzygy of M.*
(3) *If* $0 \longrightarrow M_1 \longrightarrow M_2 \longrightarrow M_3 \longrightarrow 0$ *is a short exact sequence then* $V_G(M_i) \subset V_G(M_j) \cup V_G(M_\ell)$ *for any permutation* $\{i,j,\ell\}$ *of* $\{1,2,3\}$.
(4) *Dimension of $V_G(M)$ is the complexity of M (i.e., the growth of the minimal projective resolution of M).*
(5) *"Tensor product theorem"* $V_G(M_1 \otimes M_2) = V_G(M_1) \cap V_G(M_2)$.

The proof of the last property involves a different characterization of the support variety, the "rank variety", conjectured by Carlson ([2]) and proved by Avrunin-Scott [1] for elementary abelian p-groups. The theory of rank and support varieties had been subsequently developed for restricted Lie algebras [3], Frobenius kernels [6], and finite group schemes. A suitable analogue of the Avrunin-Scott theorem holds in all those cases which, in turn, allows one to prove Theorem .5 for any finite group scheme.

Support varieties have been considered for other Hopf and augmented algebras. These include (but are not limited to) small quantum groups, general self-injective algebras (via Hochschild cohomology), Lie superalgebras, reduced enveloping algebras, complete intersections, quantum elementary abelian groups and truncated polynomial algebras.

In this talk we concentrate on the case of a finite-dimensional cocommutative Hopf algebra H (equivalently, a finite group scheme), describing the theory of π-points developed in [4]. We also discuss one application: classification of thick tensor ideals in the stable module category of H.

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