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Non-Archimedean Analytic Geometry

Organised by
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Abstract. The workshop focused on recent developments in non-Archimedean analytic geometry with various applications to arithmetic and algebraic geometry. These applications include questions in Arakelov theory, p-adic differential equations, p-adic Hodge theory and the geometry of moduli spaces. Various methods were used in combination with analytic geometry, in particular perfectoid spaces, model theory, skeleta, formal geometry and tropical geometry.

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Introduction by the Organisers

The half-size workshop Non-Archimedean Analytic Geometry, organized by Vladimir Berkovich (Rehovot), Walter Gubler (Regensburg) and Annette Werner (Frankfurt) had 26 participants. Non-Archimedean analytic geometry is a central area of arithmetic geometry. The first analytic spaces over fields with a non-Archimedean absolute value were introduced by John Tate and explored by many other mathematicians. They have found numerous applications to problems in number theory and algebraic geometry. In the 1990s, Vladimir Berkovich initiated a different approach to non-Archimedean analytic geometry, providing spaces with good topological properties which behave similarly as complex analytic spaces. Independently, Roland Huber developed a similar theory of adic spaces. Recent years have seen a growing interest in such spaces since they have been used to solve several deep questions in arithmetic geometry.

We had 19 talks in this workshop reporting on recent progress in non-Archimedean analytic geometry and its applications. All talks were followed by lively discussions. Several participants explained work in progress. The workshop provided a useful platform to discuss these new developments with other experts.

During the workshop, we saw applications to complex singularity theory and to Brill–Noether theory in algebraic geometry. Progress was made in the study of Berkovich spaces over $\mathbb{Z}$, and they were used for an arithmetic Hodge index theorem with applications to the non-archimedean Calabi-Yau problem. An analog of complex differential geometry was developed on Berkovich spaces which allows us to describe non-archimedean Monge-Ampère measures as a top-dimensional wedge product of first Chern forms or currents. Two talks focused on p-adic differential equations where Berkovich spaces help to understand the behaviour of radii of convergence. Scholze's perfectoid spaces, which have led to spectacular progress regarding the monodromy weight conjecture, and their relations to p-adic Hodge theory were the topic of two other lectures. Methods from Model Theory become increasingly important in arithmetics, and we have seen two talks adressing this in connection with analytic spaces. Skeleta and tropical varieties are combinatorial pictures of Berkovich spaces, and these tools were used in several talks. In the one-dimensional case these methods lead to a better understanding of well-studied objects of algebraic geometry such as moduli spaces of curves or component groups.

Workshop: Non-Archimedean Analytic Geometry

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Abstracts

Linear series on metrized complexes of algebraic curves

Matthew Baker
(joint work with Omid Amini)

This is a report on the paper [1], written jointly with Omid Amini.

A *metric graph* Γ is the geometric realization of an edge-weighted graph G in which each edge e of G is identified with a line segment of length $\ell(e)$. We call G a *model* for Γ.

Let κ be an algebraically closed field. A *metrized complex* $\mathfrak{C}$ *of* κ*-curves* is the following data:

- A connected finite graph G with vertex set V and edge set E.
- A metric graph Γ having G as a model (or equivalently, a length function $\ell : E \to \mathbb{R}_{>0}$).
- For each vertex v of G, a complete, nonsingular, irreducible curve C_v of genus $g_v \geq 0$ over κ.
- For each vertex v of G, a bijection $e \mapsto x_v^e$ between the edges of G incident to v and a subset $\mathcal{A}_v = \{x_v^e\}_{e \ni v}$ of $C_v(\kappa)$.

A *divisor* on a metrized complex of curves $\mathfrak{C}$ is an element $\mathcal{D} = D_\Gamma \oplus \sum_v D_v$ of $\mathrm{Div}(\Gamma) \oplus (\oplus_v \mathrm{Div}(C_v))$ such that $\deg(D_v) = D_\Gamma(v)$ (the coefficient of v in D_Γ) for all v in V. The *degree* of $\mathcal{D}$ is defined to be the degree of D_Γ.

Let $\mathbb{K}$ be a complete and algebraically closed non-Archimedean field with residue field isomorphic to κ and let $X/\mathbb{K}$ be a smooth, proper, connected algebraic curve. There is a metrized complex $\mathfrak{C} = \mathfrak{C}\mathfrak{X}$ canonically associated to any semistable model $\mathfrak{X}$ of X over the valuation ring R of $\mathbb{K}$, along with a canonical *specialization map* $\tau_*^{\mathfrak{C}\mathfrak{X}} : \mathrm{Div}(X) \to \mathrm{Div}(\mathfrak{C}\mathfrak{X})$. Both $\mathfrak{C}\mathfrak{X}$ and the map $\tau_*^{\mathfrak{C}\mathfrak{X}}$ are most conveniently defined using Berkovich's theory of non-Archimedean analytic spaces.

A *rational function* $\mathfrak{f}$ on a metrized complex of curves $\mathfrak{C}$ is the data of a rational function f_Γ on Γ and nonzero rational functions f_v on C_v for each $v \in V$. One can define the divisor of $\mathfrak{f}$ in a natural way, and two divisors in $\mathrm{Div}(\mathfrak{C})$ are called *linearly equivalent* if they differ by the divisor of some rational function. A divisor $\mathcal{E} = E_\Gamma \oplus \sum_v E_v$ is called *effective* if E_Γ and the divisors E_v are all effective. The *rank* $r_\mathfrak{C}$ of a divisor $\mathcal{D} = D_\Gamma \oplus \sum_v D_v$ in $\mathrm{Div}(\mathfrak{C})$ is defined to be the largest integer k such that $\mathcal{D} - \mathcal{E}$ is linearly equivalent to an effective divisor for all effective divisors $\mathcal{E}$ of degree k on $\mathfrak{C}$. If $r_X(D)$ denotes the usual rank function $r_X(D) = \dim |D| = h^0(D) - 1$ on $\mathrm{Div}(X)$, we have:

Theorem 1 (Specialization Theorem)**.** *For all $D \in \mathrm{Div}(X)$, we have*

$$r_X(D) \leq r_{\mathfrak{C}\mathfrak{X}}(\tau_*^{\mathfrak{C}\mathfrak{X}}(D)).$$

The theory of divisors and linear equivalence on metrized complexes of curves generalizes both the classical theory for algebraic curves and the corresponding theory for metric graphs and tropical curves found in [4, 6]. The former corresponds to the case where G consists of a single vertex v and no edges and $C = C_v$ is an arbitrary smooth proper curve. The latter corresponds to the case where the curves C_v have genus zero for all $v \in V$. Since any two points on a curve of genus zero are linearly equivalent, the divisor theory (and rank function) on $\mathfrak{C}$ and Γ are essentially equivalent.

As in [2], the main utility of Theorem 1 is that the rank function $r_{\mathfrak{C}}$ on a metrized complex of curves is surprisingly well-behaved; for example, it satisfies an analogue of the Riemann-Roch formula. The *genus* of a metrized complex of curves $\mathfrak{C}$ is $g(\mathfrak{C}) = g(\Gamma) + \sum_{v\in V} g_v$, where g_v is the genus of C_v. A *canonical divisor* on $\mathfrak{C}$, denoted $\mathcal{K}$, is defined to be any divisor linearly equivalent to $K^{\#} \oplus \sum_{v\in V}(K_v + A_v)$, where $K^{\#} = \sum_v (\deg_G(v) + 2g_v - 2)\,(v)$, K_v is a canonical divisor on C_v, and A_v is the sum of the $\deg_G(v)$ points in $\mathcal{A}_v$. The following result generalizes both the classical Riemann-Roch theorem for algebraic curves and the Riemann-Roch theorem for metric graphs:

Theorem 2 (Riemann-Roch for metrized complexes of algebraic curves)**.** *Let $\mathfrak{C}$ be a metrized complex of algebraic curves over κ and $\mathcal{K}$ a divisor in the canonical class of $\mathfrak{C}$. For any divisor $\mathcal{D} \in \mathrm{Div}(\mathfrak{C})$, we have*

$$r_{\mathfrak{C}}(\mathcal{D}) - r_{\mathfrak{C}}(\mathcal{K} - \mathcal{D}) = \deg(\mathcal{D}) - g(\mathfrak{C}) + 1.$$

Our theory of linear series on metrized complexes of curves has close connections with the Eisenbud-Harris theory of limit linear series for strongly semistable curves of compact type, and suggests a way to generalize the Eisenbud-Harris theory to more general semistable curves. A proper nodal curve X_0 over κ is of *compact type* if its dual graph G is a tree. For such curves, Eisenbud and Harris define a *limit* $\mathfrak{g}^r_d$ L on X_0 to be the data of a (not necessarily complete) degree d and rank r linear series L_v on X_v for each vertex $v \in V$ such that if two components X_u and X_v of X_0 meet at a node p, then for any $0 \le i \le r$,

$$a_i^{L_v}(p) + a_{r-i}^{L_u}(p) \;\ge\; d\,,$$

where $a_i^L(p)$ denotes the i^{th} term in the vanishing sequence of a linear series L at p.

Let $\mathfrak{C}$ be a metrized complex and let $\mathcal{F} = \{F_v \;:\; v \in V\}$, where F_v is a κ-subspace of the function field $\kappa(C_v)$ for each $v \in V$. For $\mathcal{D} \in \mathrm{Div}(\mathfrak{C})$, we define the $\mathcal{F}$-rank of $\mathcal{D}$, denoted $r_{\mathfrak{C},\mathcal{F}}(D)$, to be the largest integer k such that for any effective divisor $\mathcal{E}$ of degree k on $\mathfrak{C}$, there is a rational function $\mathfrak{f} = (f_\Gamma, (f_v)_{v\in V})$ with $f_v \in F_v$ for all $v \in V$ such that $\mathcal{D} - \mathcal{E} + \mathrm{div}(\mathfrak{f}) \ge 0$. A g^r_d on a metrized complex $\mathfrak{C}$ is an equivalence class of pairs $(\mathcal{D}, \mathcal{H})$ with $r_{\mathfrak{C},\mathcal{H}}(\mathcal{D}) = r$ and $\mathcal{H} = \{H_v\}$ with H_v an $(r+1)$-dimensional subspace of $\kappa(C_v)$ for all v. The equivalence relation is $(\mathcal{D}, \mathcal{H}) \sim (\mathcal{D}', \mathcal{H}')$ iff there is a nonzero rational function $\mathfrak{f} = (f_\Gamma, \{f_v\})$ on $\mathfrak{C}$ such that $D' = D + \mathrm{div}(\mathfrak{f})$ and $H'_v = H_v \cdot f_v$ for all v.

If R is a dvr with residue field κ and $\mathfrak{X}/R$ is a regular semistable arithmetic surface whose generic fiber X is smooth, then for any divisor D on X with $r_X(D) = r$ and $\deg(D) = d$, our specialization machine gives rise in a natural way to a limit $\mathfrak{g}^r_d$ on the special fiber X_0 of $\mathfrak{X}$. Moreover:

Theorem 3. *If X_0 is of compact type, then there is a bijection between limit g^r_d's in the sense of Eisenbud-Harris and g^r_d's on the metrized complex $\mathfrak{C}\mathfrak{X}$.*

These ideas have Diophantine applications to the study of rational points on curves over number fields (specifically, to the method of Coleman-Chabauty). Eric Katz and David Zureick-Brown have recently used a special case of Clifford's theorem for metrized complexes to prove the following theorem [5] which answers affirmatively a question of M. Stoll:

Theorem 4. *Let X be a smooth projective geometrically irreducible curve of genus $g \geq 2$ over $\mathbf{Q}$ and assume that the Mordell-Weil rank r of the Jacobian of X is less than g. Fix a prime number $p > 2r + 2$ and let $\mathfrak{X}$ be a proper (not necessarily semistable) regular model for X over $\mathbf{Z}_p$. Then (letting $\bar{\mathfrak{X}}^{\mathrm{sm}}$ denote the smooth locus of $\bar{\mathfrak{X}}$)*

$$\#X(\mathbf{Q}) \leq \bar{\mathfrak{X}}^{\mathrm{sm}}(\mathbf{F}_p) + 2r.$$

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Monodromy and the Lefschetz fixed point formula

François Loeser
(joint work with Ehud Hrushovski)

Let X be a smooth complex algebraic variety and $f\colon X \to \mathbb{A}^1_{\mathbb{C}}$ be a non-constant morphism to the affine line. Let x be a singular point of $f^{-1}(0)$, that is, such that $df(x) = 0$.

Fix a metric on X. Let $0 < \eta \ll \varepsilon \ll 1$. By Milnor's local fibration Theorem the morphism f restricts to a fibration, called the Milnor fibration,

$$B(x, \varepsilon) \cap f^{-1}(B(0, \eta) \setminus \{0\}) \longrightarrow B(0, \eta) \setminus \{0\}.$$

Here $B(a, r)$ denotes the closed ball of center a and radius r.

The Milnor fiber at x,

$$F_x = f^{-1}(\eta) \cap B(x, \varepsilon),$$

has a diffeomorphism type that does not depend on η and ε and is endowed with an automorphism, defined up to homotopy, the monodromy M_x, induced by the characteristic mapping of the fibration. In particular the cohomology groups $H^q(F_x, \mathbb{C})$ are endowed with an automorphism M_x, and we can consider the Lefschetz numbers

$$\Lambda(M_x^m) = \mathrm{tr}(M_x^m; H^\bullet(F_x, \mathbb{C})) = \sum_i (-1)^i \mathrm{tr}(M_x^m; H^i(F_x, \mathbb{C})).$$

In [1], N. A'Campo proved that if x is a singular point of $f^{-1}(0)$, then $\Lambda(M_x^1) = 0$ and this was later generalized by Deligne to the statement that $\Lambda(M_x^m) = 0$ for $0 < m < \mu$, with μ the multiplicity of f at x, cf. [2].

In [6], $\Lambda(M_x^m)$ was expressed in terms of arcs in the following way. Set

$$\mathcal{X}_{m,x} = \{\varphi \in X(\mathbb{C}[t]/t^{m+1}); f(\varphi) = t^m \mod t^{m+1}, \varphi(0) = x\}.$$

Note that $\mathcal{X}_{m,x}$ can be viewed in a natural way as the set of closed points of a complex algebraic variety.

Then, according to [6], for every $m \geq 1$,

$$(*) \qquad \chi_c(\mathcal{X}_{m,x}) = \Lambda(M_x^m).$$

Here χ_c denotes the usual Euler characteristic with compact supports. Note that one recovers Deligne's statement as a corollary since $\mathcal{X}_{m,x}$ is empty for $0 < m < \mu$. The original proof in [6] proceeds as follows. One computes explicitly both sides of $(*)$ on an embedded resolution of $f = 0$ and checks both quantities are equal. The computation of the left hand side relies on the change of variable formula for motivic integration in [5] and the one on the right hand side on A'Campo's formula in [2]. In this talk we present a geometric proof of $(*)$ not using resolution of singularities which is given in our paper [9].

Our approach uses étale cohomology of non-archimedean spaces and motivic integration. More precisely we use the following ingredients:

- The analytic Milnor fiber introduced by Nicaise and Sebag in [10].
- Finiteness of the étale cohomology with compact supports (cf. [3]) of locally closed semi-algebraic subsets of Berkovich spaces associated to algebraic varieties proved by F. Martin in [7]
- A proved in [9] stating that motivic integration as developed by Hrushovski and Kazhdan in [8] is compatible with étale realization.
- The Lefschetz fixed point theorem for finite order automorphisms of algebraic varieties, cf. [4].

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Berkovich analytic spaces and tubular descent

Michael Temkin
(joint work with Oren Ben-Bassat)

1. Motivation

This report is devoted to a joint work with Oren Ben-Bassat that will be published in Advances in Mathematics (see [4] for a preprint version). The main aim of our research was to study how a coherent sheaf $\mathcal{F}$ on an algebraic variety can be reconstructed from its restrictions onto an open subvariety U and its complement $Z = X - U$. Informally speaking, this can be reformulated as a question of gluing coherent sheaves $\mathcal{F}|_U$ and $\mathcal{F}|_Z$ to a coherent $\mathcal{O}_X$-module $\mathcal{F}$. Solution of analogous descent problems were known in the literature under certain affineness assumptions (see [1], [2], and [3]) and the main achievement of our solution is that it is of global nature. On the other hand, in the earlier known situations one used another scheme $\mathcal{W}$ for the gluing; in a sense, $\mathcal{W}$ played the role of the intersection. In our approach, $\mathcal{W}$ is replaced with a Berkovich analytic space W, so the construction becomes subtler.

2. Main result

If one wants to reconstruct $\mathcal{F}$, then at the very least one should know the restrictions of $\mathcal{F}$ onto all closed subschemes whose reduction is Z, or equivalently one should know the formal completion $\widehat{\mathcal{F}}_Z$ of $\mathcal{F}$ along Z. So, we consider the formal completion $\mathfrak{X} = \widehat{X}_Z$ of X along Z and show how to reconstruct $\mathcal{F}$ from the restriction $\mathcal{F}_U$ and the formal completion $\widehat{\mathcal{F}}_Z$. In fact, we introduce an analytic space W analogous to a punctured tubular neighborhood of Z, construct "restriction" functors $\mathrm{Coh}(U) \to \mathrm{Coh}(W)$ and $\mathrm{Coh}(\mathfrak{X}) \to \mathrm{Coh}(W)$, and prove our main result that takes the following very intuitive form:

$$\mathrm{Coh}(X)\widetilde{\to}\mathrm{Coh}(U) \times_{\mathrm{Coh}(W)} \mathrm{Coh}(\mathfrak{X})$$

Naturally, we call this type of descent *tubular descent*.

3. Choice of W

Before explaining what W is, let us consider the case that $k = \mathbb{C}$. Then one often considers a tubular neighborhood T_ε of Z. It is an open neighborhood of Z in the classical analytic topology and is contractible to Z. The gluing of sheaves can then be performed along the punctured tubular neighborhood $T_\varepsilon - Z$. Having this case in mind, one can view $\mathfrak{X}$ as an (infinitesimal) algebraic version of a tubular neighborhood and wonder if a "punctured formal scheme $\mathfrak{X} - Z$" (or a "generic fiber") can be meaningfully defined.

It was discovered by Tate and Grothendieck that a generic fiber of a formal scheme can be defined in some cases as a non-archimedean analytic space. Today there are three different theories of such spaces: Tate's rigid spaces, Berkovich analytic spaces, and Huber's adic spaces. We chose to define W as a Berkovich k-analytic space, where the valuation on k is trivial, but other alternatives are possible (and would lead to the same category $\mathrm{Coh}(W)$). Our definition runs as follows: one considers the generic fiber $\mathfrak{X}_\eta$ of the k-special formal scheme $\mathfrak{X}$, and removes the generic fiber Z_η from it. Note that $\mathfrak{X}_\eta$ can be viewed as a tubular neighborhood of Z_η, and $W = \mathfrak{X}_\eta - Z_\eta$ is an analog of the punctured tubular neighborhood. If X is proper then we also prove that W depends only on U because actually $W = U_\infty = U^{\mathrm{an}} - U_\eta$ (the space U_∞ was introduced by Berkovich in a private correspondence with Drinfeld and called the "infinity" of U in that correspondence).

4. A work of A. Thuillier on the homotopical type of divisor at infinity

I would also like to note that when giving the Oberwolfach lecture I learned from Sam Payne that the space W was used in a spectacular way by Amaury Thuillier in [5]. The main result [5, Th. 4.10] states that if the ground field k is perfect, X is regular and proper, and Z is a normal crossings divisor, then the topological type of the simplicial complex $\Delta(Z)$ depends only on U and not on its compactification $U \hookrightarrow X$. In fact, Thuillier constructs a deformation retract $W \to \Delta(Z)$ and shows that W depends only on U when X is proper (as remarked above, it is U_∞).

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Analytic spaces over Z

Jérôme Poineau

In the late eighties, Vladimir G. Berkovich has defined a notion of analytic space over a general Banach ring $(A, \|.\|)$ (see [1]). In this talk, we are mainly concerned with the case where $A = \mathbf{Z}$ endowed with the usual absolute value.

Let n be an integer. The affine analytic space of dimension n over $\mathbf{Z}$, which is denoted $\mathbf{A}^{n,\mathrm{an}}_{\mathbf{Z}}$, is defined as the set of semi-norms over the polynomial ring $\mathbf{Z}[T_1, \ldots, T_n]$. (For a general Banach ring $(A, \|.\|)$, we also require that the semi-norm be bounded by $\|.\|$ over A ; this condition is always satisfied in the case of $\mathbf{Z}$.) It is endowed with the topology generated by the subsets of the form $\{r < |P| < s\}$, with $P \in \mathbf{Z}[T_1, \ldots, T_n]$ and $r, s \in \mathbf{R}$, and with the sheaf $\mathcal{O}$ of functions that are locally uniform limits of rational functions without poles.

The affine analytic space $\mathbf{A}^{n,\mathrm{an}}_{\mathbf{Z}}$ is naturally fibered over $\mathcal{M}(\mathbf{Z}) = \mathbf{A}^{0,\mathrm{an}}_{\mathbf{Z}}$. Among its fibers, there are usual analytic spaces $\mathbf{C}^n$ (quotiented by the action of the conjugation) as well as non-archimedean p-adic analytic spaces, for every prime number p.

In [2], we carried out a detailed study of the one-dimensional case. The purpose of the talk is to investigate the local theory of the space $\mathbf{A}^{n,\mathrm{an}}_{\mathbf{Z}}$, for any n (see [3]).

The basic tool is a local Weierstraß division theorem in a rather general setting. Let B be an analytic space and X be a relative line over it. Let $b \in B$ and x be the point 0 over b. If two analytic functions f and g are given on a neighbourhood of x, the Weierstraß division theorem allows to divide f by g, as soon as g does not vanish identically on the fiber over b, with a polynomial remainder. This result may be extended to the case where x is a rigid point over b, which means that the induced extension of residue fields is algebraic.

Let us mention that the case of a point x over b which is not rigid is indeed simpler: the local ring at the point x in its fiber over b is a field, hence any analytic function which vanishes at x vanishes on the whole fiber.

Using those results, we may now carry out a local study of analytic spaces by an induction process. It closely follows the strategy that is used in complex analytic geometry. Let x be a point in the analytic space $\mathbf{A}^{n,\mathrm{an}}_{\mathbf{Z}}$. We prove that the local ring $\mathcal{O}_x$ is Noetherian, regular and excellent. A direct study ensures that it is also Henselian. Pushing the methods further, we show that the structure sheaf $\mathcal{O}$ is coherent.

The results we have just mentioned actually hold for more general rings such as rings of integers of number fields or discrete valuation rings (with the additional assumption that their fraction fields have characteristic 0 as regards excellence).

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p-adic boundary values

Ehud de Shalit
(joint work with Eran Assaf)

This is a report on work in progress with Eran Assaf.

Let K be a finite extension of $\mathbb{Q}_p$ and $X \subset \mathbb{P}^d_K$ Drinfeld's p-adic symmetric domain of dimension d, regarded as a rigid analytic space over K. The module $\Omega^d(X)$ of rigid analytic d-forms on X is a rich and interesting representation of $G = GL_{d+1}(K)$. It has been studied extensively in the paper [1]. It was shown to possess a G-equivariant filtration of length $d+1$, whose graded pieces were completely classified. These graded pieces are the prototypes of the admissible representations studies by Schneider and Teitelbum, whose construction mixes smooth and algebraic representations, locally analytic induction and modules of differential equations. These graded pieces were later on shown to be irreducible in [2].

Nevertheless, the structure of $\Omega^d(X)$ as a whole remains somewhat mysterious. Let $\mathcal{F} = G/P$ be the full flag variety of G (P denoting a Borel subgroup) and $C(\mathcal{F}, K)$ the space of continuous K-valued functions on $\mathcal{F}$. Let $C(\mathcal{F}, K)_{inv}$ be the sum of the spaces $C(G/Q, K)$ where Q is a parabolic properly containing P. Thus $C(\mathcal{F}, K)/C(\mathcal{F}, K)_{inv}$ is the "continuous Steinberg representation". The starting point of [1] is a map

$$I : \Omega^d(X)' \to C(\mathcal{F}, K)/C(\mathcal{F}, K)_{inv}$$

($\Omega^d(X)'$ is the strong dual of $\Omega^d(X)$, which is a locally convex G-module of compact type), whose image consists of locally analytic vectors. The map I is constructed as follows. Fix a flag $\xi^0 = (\xi^0_0 \supset \xi^0_1 \supset \cdots \supset \xi^0_d \supset 0)$, where ξ^0_i is a subspace of codimension i in K^{d+1}. If ξ is a another flag in general position w.r.t. ξ^0 (a flag lying in the big Bruhat cell w.r.t.P, where P is the stabilizer of ξ^0), then intersecting subspaces of complementary dimensions gives a frame $\{e_i\}$ of K^d, hence a differential form

$$\omega(\xi^0, \xi) = \sum_{i=0}^{d} (-1)^i d\log e_0 \wedge \cdots \wedge \widehat{d\log e_i} \wedge \cdots \wedge d\log e_d.$$

Let $\lambda \in \Omega^d(X)'$. The map $\xi \mapsto \lambda(\omega(\xi^0, \xi))$, extended by zero outside the big cell, is continuous. It depends on ξ^0 and has singularities on the boundary of the big cell, but modulo $C(\mathcal{F}, K)_{inv}$ is independent of any choice, G-equivariant and "analytic". Call it $I(\lambda)$.

Let $\Omega^d(X)^j$ be the subspace topologically spanned by rational d-forms on $\mathbb{P}^d$, whose polar divisor is supported on the union of at most $d+1-j$ K-rational hyperplanes. Then $\Omega^d(X) = \Omega^d(X)^0$, $\Omega^d(X)^1$ is the space of exact forms, and $\Omega^d(X)^{d+1} = 0$. The filtration thus obtained is clearly G-stable. The main result of [1] is a description of $\Omega^d(X)^j/\Omega^d(X)^{j+1}$.

Motivated by the case $d = 1$, which was studied long ago by Morita, and by our approach to the cohomology of X in [3], we study a map J in the opposite direction.

More precisely, let $\pi : \mathcal{F} \to \mathbb{P}^d(K)$ be the map sending a full flag ξ as above to the hyperplane ξ_1 (here we regard $\mathbb{P}^d(K)$ as the space of hyperplanes in K^{d+1}). Let $C^S(\mathcal{F}, K)$ be the space of locally analytic K-valued functions on $\mathcal{F}$ which are locally constant in the fibers of π. Let $C^S(\mathcal{F}, K)_{inv} = C^S(\mathcal{F}, K) \cap C(\mathcal{F}, K)_{inv}$. We define a G-equivariant map

$$J : C^S(\mathcal{F}, K) \to \Omega^d(X)'$$

and prove the following.

Theorem 5. *The map J induces an isomorphism*

$$C^S(\mathcal{F}, K)/C^S(\mathcal{F}, K)_{inv} \simeq \Omega^d(X)'_2$$

(the filtration on $\Omega^d(X)'$ is orthogonal to the filtration on $\Omega^d(X)$). Restricted to the smooth Steinberg representation, J recovers the Schneider-Stuhler isomorphism

$$C^\infty(\mathcal{F}, K)/C^\infty(\mathcal{F}, K)_{inv} \simeq \Omega^d(X)'_1 = H^d_{dR}(X)'.$$

Moreover, $I \circ J$ is the identity on $C^S(\mathcal{F}, K)/C^S(\mathcal{F}, K)_{inv}$.

As a result we obtain a characterization of $I(\Omega^d(X)'_2)$ which is not supplied by knowing its two graded pieces.

The construction of J relies on some lemmas on the Bruhat-Tits building of G. Denote this building by $\mathcal{T}$. If $\sigma \in \widehat{\mathcal{T}}_d$ is an oriented d-cell in $\mathcal{T}$, then σ is a chain of lattices

$$\sigma = [L_0 \supset \cdots \supset L_d \supset \pi L_0]$$

(here π is the uniformizer of K, the inclusions are strict, and the whole chain is considered up to a common homothety). We denote by $\mathcal{F}(\sigma)$ the compact and open subset of flags $\xi \in \mathcal{F}$ which are compatible with σ in the sense that for all i,

$$\xi_i \cap L_0 + \pi L_0 = L_i.$$

These sets form a basis for the topology of $\mathcal{F}$. Furthermore, $\mathcal{F}(\tau) \subset \mathcal{F}(\sigma)$ if and only if (i) τ is parallel to σ (ii) the *elementary divisors* of τ w.r.t.σ are non-increasing $m_0 \geq m_1 \geq \cdots \geq m_d$. Let $r : X \to |\mathcal{T}|$ be the reduction of X to the real realization of $\mathcal{T}$. If $\sigma \in \mathcal{T}_d$, we let X_σ be the pre-image of the open chamber $|\sigma|$, isomorphic to the multi-annulus

$$1 > |z_1| > \cdots > |z_d| > |\pi|.$$

Given a finite partition of $\mathcal{F}$ into a union of $\mathcal{F}(\sigma)$, for $\sigma \in \Sigma$, we define a finite subcomplex Π_Σ of $\mathcal{T}$, which becomes larger and larger as the partition becomes finer and finer, and which does not contain the σ from Σ. In particular, using fine enough partitions, we can guarantee that the complement of the affinoid $X_\Sigma = r^{-1}(\Pi_\Sigma)$ is an arbitrarily small neighborhood of the complement of X in $\mathbb{P}^d$.

Let now $\phi \in C^S(\mathcal{F}, K)$. Choosing a fine enough partition as above, we may write

$$\phi = \sum_{\sigma \in \Sigma} 1_{\mathcal{F}(\sigma)} \otimes \pi^* \phi_\sigma$$

where the $\phi_\sigma \in C^{an}(\mathbb{P}^d(K), K)$ is locally analytic, and extends to a *rigid* analytic function in a polydisk in $\mathbb{P}^d$ containing X_σ, and in fact any X_τ if $\mathcal{F}(\tau) \subset \mathcal{F}(\sigma)$. We then define, for $\omega \in \Omega^d(X)$

$$J(\phi)(\omega) = \sum_{\sigma \in \Sigma} res_{X_\sigma}(\phi_\sigma \omega).$$

The residue is defined as in [3]. This is then well-defined and independent of all choices, because of a variant of "Cauchy's theorem" for residues.

To get beyond $\Omega^d(X)'_2$ we consider also ϕ as above which are locally *meromorphic,* and whose polar divisor is (locally) supported on the union of K-rational hyperplanes. We can show that the image of J then intersects every step in the filtration of $\Omega^d(X)'$ in a dense subspace. We can also compute $I \circ J$ by means of what we call "partial Mittag-Leffler decompositions". However, the results are not as nice as for locally *analytic* ϕ, and the spaces of locally meromorphic ϕ do not carry a good topology.

We have also made progress on a similar construction for k-forms, $0 \leq k \leq d$.

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Logarithmic radii of convergence and finite coverings of non-archimedean curves

Francesco Baldassarri

1. Polystable formal schemes

Let k be a non-archimedean valued field of characteristic 0. A *standard k°-formal scheme* is an affine k°-formal scheme $\mathfrak{T}$ of the form

$$\mathfrak{T} = \mathfrak{S}(m) \times \prod_{i=1}^{h} \mathfrak{T}_{d_i,a_i} \ , \tag{1}$$

where

$$\mathfrak{S}(m) = \mathrm{Spf}\, k^\circ\{X_1, \dots, X_m\} \ ,$$

while, for $i = 1, \dots, h$,

$$\mathfrak{T}_{d_i,a_i} := \mathrm{Spf}\, k^\circ\{X_{i,0}, \dots, X_{i,d_i}\}/(X_{i,0} \cdot \dots \cdot X_{i,d_i} - a_i) \ ,$$

for $a_i \in k^{\circ\circ}$; it is *non-degenerate* if $a_1 \cdot \dots \cdot a_h \neq 0$. A k°-formal scheme $\mathfrak{X}$ is *strictly polystable nondegenerate* if every point $\mathbf{x} \in \mathfrak{X}_s$ admits a Zariski open connected affine neighborhood $\mathfrak{U}$ in $\mathfrak{X}$ endowed with an étale morphism

$$\varphi : \mathfrak{U} \to \mathfrak{T} \ , \tag{2}$$

to a standard non-degenerate k°-formal scheme $\mathfrak{T}$. Then,

$$(\mathfrak{U}, \varphi = (X_1, \dots, X_m, X_{1,0}, \dots, X_{1,d_1}, \dots, X_{h,0}, \dots, X_{h,d_h})) \tag{3}$$

is a *polystable coordinate neighborhood of* $\mathbf{x} \in \mathfrak{X}$. The special fiber $\mathfrak{X}_s \subset \mathfrak{X}$ defines a canonical log-structure on $\mathfrak{X}$. From now on, $\mathfrak{X}$ will thus be regarded as a log-formal scheme, log-smooth over $\mathrm{Spf}\, k^\circ$, equipped with its own canonical log-structure. We denote by $X = \mathfrak{X}_\eta$ the generic fiber of $\mathfrak{X}$, equipped with the *specialization morphism* of G-ringed spaces

$$\mathrm{sp}_{\mathfrak{X}} : X_G \longrightarrow \mathfrak{X} .$$

For any k-analytic space Y, a *k-rational strict open polydisk in Y* (resp. a *k-rational strict open polyannulus in Y of* height $\rho \in \sqrt{|k^\times|}$) is an open analytic domain in Y, isomorphic to the *standard open unit polydisk*

$$D^n_k(0, 1^-) = \{x \in \mathbb{A}^n_k \mid |T_i(x)| < 1 \ , \ \forall i = 1, \dots, n\}$$

(resp. to the *standard open polyannulus of height* ρ

$$C^n_k(0; (\rho, 1)) = \{x \in \mathbb{A}^n_k \mid \rho < \prod_{i=1}^{n} |T_i(x)| < 1\} \ ) \quad ,$$

in the k-analytic affine n-space $\mathbb{A}^n_k$, with coordinates $(T_1, \dots, T_n)$. So, using the coordinates

$$(X_1, \dots, X_m, X_{1,1}, \dots, X_{1,d_1}, \dots, X_{h,1}, \dots, X_{h,d_h}) \ , \tag{4}$$

φ_η the isomorphism

$$\mathrm{sp}^{-1}_{\mathfrak{X}}(\mathbf{x}) =:]\mathbf{x}[_{\mathfrak{X}} \xrightarrow{\ \sim\ } P = D^m_k(0, 1^-) \times \prod_{i=1}^{h} C^{d_i}_k(0; (|a_i|, 1)) \subset \mathbb{A}^n_k \ , \tag{5}$$

for

$$n = m + d_1 + \dots + d_h \ .$$

Notice that every k-rational point $x = (x_1, \dots, x_m, x_{1,1}, \dots, x_{h,d_h}) \in P(k)$ admits a unique maximal open neighborhood which is a strict open k-rational polydisk centered at x, namely

$$D_P(x, 1^-) := \left(\prod_{u=1}^{m} D(x_u, 1^-) \right) \times \prod_{i=1}^{h} \prod_{j=1}^{j=d_i} D(x_{i,j}, |x_{i,j}|^-) \ ,$$

where $D(x, \rho^-)$ stands for the standard open disk of radius $\rho \geq 0$.

So, any $x \in X(k)$ admits a unique maximal open neighborhood in $]\mathrm{sp}_{\mathfrak{X}}(x)[_{\mathfrak{X}} \subset X$, which is a k-rational strict open polydisk centered at x. We call it the *$\mathfrak{X}$-normalized unit open polydisk centered at* x and denote it by $D_{\mathfrak{X}}(x, 1^-)$. An explicit choice of an isomorphism

$$(T_1, \dots, T_n) : D_{\mathfrak{X}}(x, 1^-) \xrightarrow{\ \sim\ } D^n_k(0, 1^-) \tag{6}$$

is given, in terms of the coordinates (4) by

$$(T_1, \dots, T_m) = (X_1 - X_1(x), \dots, X_m - X_m(x)) \ , \tag{7}$$

$$(T_{m+1},\dots,T_n) = \left(\frac{X_{1,1}-X_{1,1}(x)}{X_{1,1}(x)},\dots,\frac{X_{1,d_1}-X_{1,d_1}(x)}{X_{1,d_1}(x)},\dots,\frac{X_{h,1}-X_{h,1}(x)}{X_{h,1}(x)},\dots,\frac{X_{h,d_h}-X_{h,d_h}(x)}{X_{h,d_h}(x)}\right).$$

Locally at $\mathbf{x}$, the $\mathcal{O}_{\mathfrak{X}}$-module $\mathcal{D}er(\mathfrak{X}/k^\circ)$ of k°-linear continuous derivations of $\mathcal{O}_{\mathfrak{X}}$, which preserve the ideal sheaf of $\mathfrak{X}_s$, is freely generated by

$$(8)\quad (\partial_1^{(\mathfrak{X})},\dots,\partial_n^{(\mathfrak{X})}) := \left(\frac{\partial}{\partial X_1},\dots,\frac{\partial}{\partial X_m},X_{1,1}\frac{\partial}{\partial X_{1,1}},\dots,X_{1,d_1}\frac{\partial}{\partial X_{1,d_1}},\dots,X_{h,1}\frac{\partial}{\partial X_{h,1}},\dots,X_{h,d_h}\frac{\partial}{\partial X_{h,d_h}}\right).$$

We denote by $\mathcal{D}iff(\mathfrak{X}/k^\circ)$ the $\mathcal{O}_{\mathfrak{X}}$-module of differential operators of $\mathcal{O}_{\mathfrak{X}}$ which preserve the ideal sheaf of $\mathfrak{X}_s$ and, for any $N = 0,1,2,\dots$, by $\mathcal{D}iff^{(N)}(\mathfrak{X}/k^\circ)$ the $\mathcal{O}_{\mathfrak{X}}$-submodule of differential operators of degree $\leq N$. Free generators of $\mathcal{D}iff^{(N)}(\mathfrak{X}/k^\circ)$ are

$$(\underline{\partial}^{(\mathfrak{X})})^{[\underline{\alpha}]} := \frac{(\partial_1^{(\mathfrak{X})})^{\alpha_1}}{\alpha_1!}\cdot\dots\cdot\frac{(\partial_n^{(\mathfrak{X})})^{\alpha_n}}{\alpha_n!}\,, \tag{9}$$

of degree $\alpha_1+\dots+\alpha_n \leq N$.

This situation can easily be generalized to the case when we also have a strict normal crossings divisor $\mathfrak{Z} \subset \mathfrak{X}$, relative to $\mathrm{Spf}\, k^\circ$. We get a smooth log-scheme $(\mathfrak{X},\mathfrak{Z})$ over $\mathrm{Spf}\, k^\circ$, and the same definitions ($D_{(\mathfrak{X},\mathfrak{Z})}(x,\rho^\pm)$, $\partial_i^{(\mathfrak{X},\mathfrak{Z})}$, $\mathcal{D}iff^{(N)}((\mathfrak{X},\mathfrak{Z})/k^\circ)$, ...).

2. The convergence polygon

Definition 6. *Let $x \in X(k)$. An ($(\mathfrak{X},\mathfrak{Z})$-normalized) optimal basis of horizontal sections of $(\mathcal{E},\nabla)$ at x, is a basis of $\mathcal{E}_x^\nabla$, $(e_1,\dots,e_\mu)$, such that there exists real numbers $f_1(x) \geq f_2(x) \geq \dots \geq f_{\mathrm{rk}\,\mathcal{E}}(x) \geq 0$ such that e_i prolongs to a horizontal section of $(\mathcal{E},\nabla)$ on $D_{(\mathfrak{X},\mathfrak{Z})}(x,(e^{-f_i(x)})^-)$, and that $h_{(\mathfrak{X},\mathfrak{Z})}(x) := f_1(x)+f_2(x)+\dots+f_{\mathrm{rk}\,\mathcal{E}}(x) \geq 0$ be minimal for this property.*

An optimal basis exists at any $x \in X(k)$. We define the $(\mathfrak{X},\mathfrak{Z})$-*normalized convergence polygon of* $(\mathcal{E},\nabla)$ *at* x as in Fig. 1 [1], [2], [8], [9], [10], [11]. A general point $x \in X$, corresponds to a bounded character

$$\chi_x : B \to \mathscr{H}(x) =: K\,.$$

We define the *canonical K-rational* point $x_K \in X_K(K)$ above x by its character

$$\chi_{x_K} = \chi_x \widehat{\otimes} \mathrm{id}_K : B\widehat{\otimes}_k K \to K\,.$$

We then set

$$\mathcal{N}_{(\mathfrak{X},\mathfrak{Z})}(x;(\mathcal{E},\nabla)) := \mathcal{N}_{(\mathfrak{X},\mathfrak{Z})\widehat{\otimes}K^\circ}(x_K;(\mathcal{E},\nabla)_K)\,. \tag{10}$$

We expect that $x \mapsto \mathcal{N}_{(\mathfrak{X},\mathfrak{Z})}(x)$ be continuous and piecewise affine on X, and that it define a finite $n+1$-polyhedron above X. In the case of curves and for $\zeta = \emptyset$, these results have been announced here [10], [11]. The function $x \mapsto h_{(\mathfrak{X},\mathfrak{Z})}(x)$ is

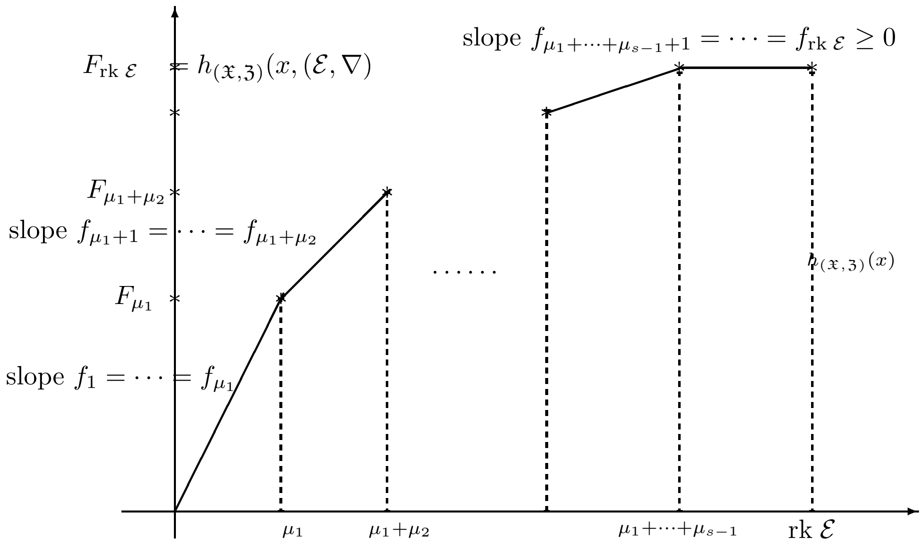

FIGURE 1. The convergence polygon $\mathcal{N}_{(\mathfrak{X},\mathfrak{Z})}(x)$

harmonic at points of type 2,3 in $X \setminus Z$, except for a finite number of points of type 2, where is still is subharmonic [8]. We interpret the laplacian of $x \mapsto h_{(\mathfrak{X},\mathfrak{Z})}(x)$ at those points as an *index. i.e.* as an Euler-Poincaré characteristic in De Rham cohomology. There is no difficulty in extending these notions to compact smooth dagger curves.

3. Finite morphisms of non-archimedean curves

Let $\varphi : Y \to X$ be a finite morphism of connected compact rig-smooth strictly k-analytic dagger curves branched over $B \subset X(k)$, and let $Z = \varphi^{-1}(B)$. Then φ may be seen as a finite log-étale morphism $(X, Z) \to (X, B)$. We assume that this extends to a finite morphism $\Phi : (\mathfrak{Y}, \mathfrak{Z}) \to (\mathfrak{X}, \mathfrak{B})$ of strictly polystable formal models [5]. The direct image connection $\varphi_*(\mathcal{O}_Y, d_Y)$ is a logarithmic connection $(\mathcal{F}, \nabla_{\mathcal{F}})$ on $(X, B)/k$, and the convergence polygon $x \mapsto \mathcal{N}_{(\mathfrak{X},\mathfrak{Z})}(x; (\mathcal{F}, \nabla_{\mathcal{F}}))$ is defined all over X. The De Rham cohomology of Y (resp. of X) is finite-dimensional [7]. We are trying to prove that, for a sufficiently refined choice of the formal models $(\mathfrak{Y}, \mathfrak{Z})$ and $(\mathfrak{X}, \mathfrak{B})$, a Riemann-Hurwitz formula exists and is formulated in terms of the Laplacian of the function $x \mapsto h_{(\mathfrak{X},\mathfrak{B})}(x; (\mathcal{F}, \nabla_{\mathcal{F}}))$.

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Real differential forms and currents on Berkovich spaces

Antoine Chambert-Loir
(joint work with Antoine Ducros)

In this talk, I described the first steps of a theory of differential forms with real coefficients and currents on an analytic space in the sense of Berkovich. The motivation for this work is the quest for an analytic non-archimedean Arakelov geometry. Two main ideas lie at the hear of the construction: a local version of tropical geometry (see for example [11]) and the definition in [12] of a bicomplex of differential superforms on a real vector space. A preliminary version of this work is available in [6].

To any hermitian line bundle $\bar{L}$ on a complex manifold X, complex geometry attaches a curvature form $c_1(\bar{L})$ which is a differentialform of type $(1,1)$ on X and represents the class of the line bundle L in the De Rham cohomology of X. If $\dim(X) = n$, its power $c_1(\bar{L})^n$ is a form of type (n,n), hence gives rise to a measure on X. Such measures appear naturally in arithmetic geometry, notably in equidistributions theorems.

Indeed, let $\mathscr{X}$ be a flat and projective $\mathbf{Z}$-scheme, let $\overline{\mathscr{L}}$ be a pair consisting of a line bundle $\mathscr{L}$ on $\mathscr{X}$ together with a hermitian metric on the holomorphic line bundle L on $X = \mathscr{X}(\mathbf{C})$. Using Arakelov geometry, [4] defines a height function $h_{\overline{\mathscr{L}}}$ on the set of cycles on $\mathscr{X}$, in particular a height function on $\mathscr{X}(\bar{\mathbf{Q}})$ which is invariant under the action of the Galois group. Moreover, if $\mathscr{L}$ is relatively semipositive, $\mathscr{L}_{\mathbf{Q}}$ is ample and the hermitian metric is nonnegative, [16] proved an equidistribution theorem for points of "small height" which generalizes the theorem of [13]. The limit measure on X is precisely the measure $c_1(\bar{L})^{\dim(X)}/\deg_L(X)$. (In fact, it is sufficient that the metric be only continuous and nonnegative, then $c_1(\bar{L})$ is a current whose product $c_1(\bar{L})^n$ exists as a measure.) In turn, such equidistribution theorems have been essential for the proof of the Bogomolov conjecture ([14, 18]) that describes points of small Néron-Tate height on subvarieties of abelian varieties over number fields.

Let us now pass to non-archimedean geometry. Let p be a prime number and let X_p be the analytic space in the sense of [2] associated to $\mathscr{X} \otimes \mathbf{Q}_p$. [5] defined a measure on X_p attached to the p-adically metrized line bundle induced by $(\mathscr{X}, \mathscr{L})$ on X_p; the construction holds for the more general admissible metrics defined by [17]. Moreover, similar equidistribution theorems hold, as shown there for curves, and by [16] in general. As observed by [8, 9, 10, 7], there is also an analogous theory for function fields which allowed [9, 15] to prove new instances of the Bogomolov conjecture over function fields.

I now describe my work [6] with Ducros.

Let k be a field with a non-archimedean absolute value, complete; let X be an analytic space over k in the sense of [2, 3].

A *smooth function* on X is a real valued function which can be written locally $\phi(\log|f_1|, \dots, \log|f_r|)$, where $f_1, \dots, f_r$ are holomorphic invertible functions on X and ϕ is a $\mathscr{C}^\infty$-function on $\mathbf{R}^r$. Smooth functions form a subsheaf $\mathscr{A}_X$ of the sheaf of real valued continuous functions on X. If X is good and paracompact, for example if X is the analytic space associated to an algebraic variety, this sheaf is fine (there are partitions of unity).

If $f_1, \dots, f_r$ are defined on an open subset U, we obtain an analytic morphism $f \colon U \to \mathbf{G}_{\mathrm{m}}^r$ from U to the r-dimensional torus $T = \mathbf{G}_{\mathrm{m}}^r$. Such a datum allows to borrow techniques from tropical geometry. Let $\mathrm{trop} \colon \mathbf{G}_{\mathrm{m}}^r \to \mathbf{R}^r$ be the tropicalization map $x \mapsto (\log|T_1(x)|, \dots, \log|T_r(x)|)$; we write $f_{\mathrm{trop}} = \mathrm{trop} \circ f$. By a theorem of Ducros which generalizes a result of Berkovich, $f_{\mathrm{trop}}(V)$ is a compact polyhedron of $\mathbf{R}^r$ for every compact analytic domain V of U. We shall hence call a triple (U, f, P), where U and f are as above, and P is a polyhedron containing $f_{\mathrm{trop}}(U)$ a *tropical chart*.

To define differential forms on X, we first work locally and consider such a tropical chart (U, f, P). On an open subset Ω of the real space $E = \mathbf{R}^r$, A. Lagerberg [12] defined a superform of type (p, q) to be an element of

$$\mathscr{A}^{p,q}(\Omega) = \mathscr{C}^\infty(\Omega) \otimes \bigwedge^p E^* \otimes \bigwedge^q E^*,$$

thus doubling the De Rham complex of $\mathbf{R}^r$. In coordinates, one may write such a form as

$$\omega = \sum_{\substack{1 \le i_1 < \dots < i_p \le r \\ 1 \le j_1 < \dots < j_q \le r}} \omega_{IJ} \, \mathrm{d}' x_{i_1} \wedge \dots \wedge \mathrm{d}' x_{i_p} \otimes \mathrm{d}'' x_{j_1} \wedge \dots \wedge \mathrm{d}'' x_{j_q}.$$

Superforms form a graded commutative algebra, possess two differentials d' and d'', respectively defined by the graded Leibniz rule and the formulas

$$\mathrm{d}' \phi = \sum_{i=1}^r \frac{\partial \phi}{\partial x_i} \, \mathrm{d}' x_i, \quad \mathrm{d}'' \phi = \sum_{j=1}^r \frac{\partial \phi}{\partial x_j} \, \mathrm{d}'' x_j.$$

Lagerberg's definition can be extended to superforms on polyhedra.

Let U be an open subset of X. The limit of all spaces $\mathscr{A}^{p,q}(P)$, where (U, f, P) ranges over all tropical charts of U, define a pre-sheaf on X; its associated sheaf is the sheaf $\mathscr{A}_X^{p,q}$ of smooth (p,q)-forms on X. Forms of type $(0,0)$ are identified with smooth functions.

As observed by Lagerberg, one may integrate forms of type (r,r) on an open subset of $\mathbf{R}^r$. However, the resulting number depends on the choice of a coordinate system. Consequently, to integrate forms of type (n,n) on a real polyhedron P of dimension n, one needs to fix Haar measures on all n-dimensional faces of P. Precisely, we introduce the notion of a calibration of P which amounts, up to the choice of a fine enough polytopal decomposition of P, of a pair (o, v) consisting of an orientation and of a n-vector for each n-dimensional face of P, and identifying (o, v) with $(-o, -v)$.

We prove that any polyhedron written as the tropicalization $f_{\mathrm{trop}}(X)$ of a compact analytic space, for some morphism $f\colon X \to T$, has a *canonical calibration.* The construction of this calibration is defined using basic properties of finite flat morphisms on analytic spaces. This allows to integrate forms of type (n,n) on an analytic space of dimension n.

The usual Stokes formula on half-spaces of a real vector space gives rise to an analog for polyhedra, where the integration on $(n-1)$-dimensional faces is done with respect to an ambient n-vector which is the sum of n-vectors on all adjacent faces. This gives rise to the "boundary integral" of a $(n-1,n)$-forms and to a Stokes formula in analytic geometry. A far reaching generalization of the balancing condition in tropical geometry states that if a $(n-1,n)$-form vanishes on the boundary $\partial(X)$ of the analytic space X (boundary in the sense of Berkovich), then its boundary integral vanishes.

At that point, one can begin to define currents by duality. Space is too short to describe the rest of the construction here. There is an analog of the Poincaré-Lelong equation $\mathrm{d}'\,\mathrm{d}'' \log|f| = \delta_{\mathrm{div}(f)}$. Line bundles with smooth metrics have a curvature $(1,1)$-form; using the Poincaré-Lelong equation, we prove that for proper analytic spaces, integrating the maximal power of curvature forms captures the degree of line bundles in intersection theory.

Taking models in formal geometry gives rise to continuous metrics for line bundles. However these metrics are not smooth in general; consequently, their curvature form is not a form, but a current. Adapting methods of [1] from pluripotential theory, we can show that these currents multiply. Finally, we prove that if $\dim(X) = n$, the nth power of such a current is equal the measure previously defined in [5].

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Perfectoid spaces and étale local systems

Kiran S. Kedlaya

We report on some recent developments providing a new geometric basis for p-adic Hodge theory[1], which appear in our papers [3, 4, 5] and independently in the work of Scholze [6, 7].

Let A be a uniform (commutative) Banach algebra over $\mathbb{Q}_p$ equipped with its spectral norm. We say A is *perfectoid* if for all $x \in A$, there exists $y \in A$ with $|x - y^p| \leq p^{-1}|x|$. For example, if A is an analytic field, then A is perfectoid if and only if A is not discretely valued and the Frobenius map on $A^\circ/(p)$ is surjective. Also, the completion of any arithmetically profinite extension of $\mathbb{Q}_p$ is perfectoid.

Theorem 7. *Let A be a perfectoid algebra with Gel'fand spectrum $\mathcal{M}(A)$.*

[1] It is likely that there exist numerous applications beyond p-adic Hodge theory. For instance, [6] includes a striking partial result on the weight-monodromy conjecture for ℓ-adic étale cohomology.

(a) *Every rational subspace of* $\mathcal{M}(A)$ *is represented (in the category of uniform Banach algebras) by a bounded homomorphism* $A \to B$ *with* B *perfectoid.*
(b) *The structure sheaf on the G-topology of special subsets (finite unions of rational subdomains) of* $\mathcal{M}(A)$ *has ring of global sections equal to* A*.*
(c) *The adic spectrum* $\mathrm{Spa}(A, A^\circ)$ *is an adic space*[2] *(by (a) and (b)).*
(d) *Let* B, C *be additional perfectoid algebras and let* $A \to C$ *and* $B \to C$ *be bounded homomorphisms. Then the completed tensor product* $A \widehat{\otimes}_C B$ *is uniform and perfectoid.*
(e) *Let* B *be a finite étale* A*-algebra. Then the Banach* A*-module norm on* B *is equivalent to a power-multiplicative norm under which* B *is again perfectoid.*

These and other basic properties of perfectoid algebras are derived using the *perfectoid correspondence*. For A perfectoid, let $R(A)^+$ denote the inverse limit of $A^\circ/(p)$ under Frobenius. As usual in p-adic Hodge theory, $R(A)^+$ admits the power-multiplicative norm $|(\cdots, x_1, x_0)| = \lim_{n\to\infty} |\tilde{x}_n|^{p^n}$ where $\tilde{x}_n \in A^\circ$ lifts $x_n \in A^\circ/(p)$ (the limit exists because the sequence stabilizes). There is a natural surjective[3] homomorphism $\theta : W(R(A)^+) \to A^\circ$ (where W denotes Witt vectors); put $R(A) = R(A)^+[\theta(p)^{-1}]$ and extend the norm multiplicatively.

Theorem 8. *Let* A *be a perfectoid algebra.*
(a) *The ring* $R(A)$ *is a perfect uniform Banach algebra over* $\mathbb{F}_p$ *and* $R(A)^\circ = R(A)^+$*.*
(b) *For* $A \to B$ *a bounded morphism representing a rational subdomain of* $\mathcal{M}(A)$*,* $R(A) \to R(B)$ *represents a rational subdomain of* $\mathcal{M}(R(A))$*, and every rational subdomain of* $\mathcal{M}(R(A))$ *arises uniquely in this way.*
(c) *For* B *a finite étale* R*-algebra,* $R(B)$ *is a finite étale* $R(A)$*-algebra, and every finite étale* $R(A)$*-algebra arises uniquely in this way. (This can be used to recover the* almost purity theorem *of Faltings.)*
(d) *The construction in (b) induces a functorial homeomorphism* $\mathcal{M}(A) \cong \mathcal{M}(R(A))$ *for the natural topology, the strictly special G-topology, and the special G-topology, and a functorial homeomorphism* $\mathrm{Spa}(A, A^\circ) \cong \mathrm{Spa}(R(A), R(A)^\circ)$ *for the natural topology and the étale topology.*

A *perfectoid space* is an adic space which is locally isomorphic to $\mathrm{Spa}(A, A^+)$ where A is a perfectoid algebra and A^+ is an open subring of A° which is integrally closed in A such that the Frobenius map on $A^+/(p)$ is surjective. Unlike the category of adic spaces, the category of perfectoid spaces admits[4] fibred products.

Problem. Let A be a uniform Banach algebra over $\mathbb{Q}_p$ equipped with its spectral norm. Suppose that $\mathrm{Spa}(A, A^\circ)$ is a perfectoid adic space. Must A be a perfectoid algebra?

[2] This gives numerous examples of adic spaces for which A is not strongly noetherian.

[3] The kernel of θ is principal; its generators are *primitive of degree* 1 in the sense of [2].

[4] In addition, if X, Y are perfectoid spaces and $X \to Z, Y \to Z$ are morphisms in the category of *uniform* adic spaces, we expect that $X \times_Z Y$ exists and is perfectoid.

Certain constructions in p-adic Hodge theory naturally give rise to sheaves on perfectoid spaces. For example, for any perfectoid algebra A, there is an analogue of the Fargues-Fontaine construction [2] giving a scheme C_A some of whose vector bundles (those satisfying a pointwise stability condition) correspond to the étale $\mathbb{Q}_p$-local systems on $\mathrm{Spa}(A, A^\circ)$. The C_A glue as adic spaces.

To study étale local systems on Berkovich spaces, we cover these spaces by perfectoids. For example, put $S_n = \mathbb{A}^n_{\mathbb{Q}_p}$ and let Y be the perfectoid space obtained from S_n by adjoining the p-power roots of 1 and of the coordinate functions. Then for any unramified morphism $\psi : X \to S_n$, the product $X \times_{\psi, S_n} Y$ is perfectoid and admits an action of a group $\Gamma = \mathbb{Z}_p^\times \ltimes \mathbb{Z}_p^n$. We can then study étale $\mathbb{Q}_p$-local systems on X using (φ, Γ)-modules over a relative Robba ring, generalizing Andreatta and Brinon [1], or Γ-equivariant vector bundles on relative Fargues-Fontaine curves.

To globalize further, it is convenient to use Scholze's *proétale topology*; this simplifies matters because étale $\mathbb{Q}_p$-local systems correspond to genuinely locally constant $\mathbb{Q}_p$-sheaves for the proétale topology. For example, Scholze uses this language to construct a de Rham-étale comparison isomorphism for analytic spaces; we plan to extend this construction to nontrivial coefficients. We also plan to consider universal local systems on (transversal subspaces of) Rapoport-Zink period domains; this should connect to the ongoing work of Scholze and Weinstein on the moduli of p-divisible groups.

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Finiteness results for vanishing cycles of formal schemes

Vladimir G. Berkovich

Let k be a non-Archimedean field with nontrivial valuation, k° its ring of integers, and $\widetilde{k}$ its residue field. A formal scheme $\mathfrak{X}$ over k° is said to be locally finitely presented if it is a locally finite union of open affine subschemes of the form $\mathrm{Spf}(A)$ with A isomorphic to a quotient of $k^\circ\{T_1, \ldots, T_m\}$ by a finitely generated ideal. If the valuation on k is discrete, a formal scheme $\mathfrak{X}$ over k° is said to be special if it is a locally finite union of open affine subschemes $\mathrm{Spf}(A)$ with A isomorphic to a quotient of $k^\circ\{T_1,\dots,T_m\}[[S_1,\dots,S_n]]$. In both cases, the generic fiber $\mathfrak{X}_\eta$ of $\mathfrak{X}$ is a paracompact strictly k-analytic space, and the closed fiber $\mathfrak{X}_s$ of $\mathfrak{X}$ is a scheme of locally finite type over $\widetilde{k}$. In [1] and [2], we defined for both classes of formal schemes, respectively, a vanishing cycles functor Ψ_η from the category of étale sheaves on $\mathfrak{X}_\eta$ to the category of étale sheaves on $\mathfrak{X}_{\overline{s}} = \mathfrak{X}_s \otimes_{\widetilde{k}} \widetilde{k}^{\mathrm{s}}$ provided with an action of the Galois group of k. The comparison theorem from [1, 5.3] (resp. [2, 3.1]) implies that if $\mathfrak{X}$ is the formal completion $\widehat{\mathcal{X}}_{/\mathcal{Y}}$ of a scheme $\mathcal{X}$ of finite type over k° along an open (resp. arbitrary) subscheme $\mathcal{Y} \subset \mathcal{X}_s$, then for any finite abelian group Λ of order prime to $\mathrm{char}(\widetilde{k})$ there is a canonical isomorphism $R\Psi_\eta(\Lambda_{\mathcal{X}_\eta})\big|_{\mathcal{Y}} \xrightarrow{\sim} R\Psi_\eta(\Lambda_{(\widehat{\mathcal{X}}_{/\mathcal{Y}})_\eta})$. In particular, the vanishing cycles sheaves $R^q\Psi_\eta(\Lambda_{\mathfrak{X}_\eta})$ of such a formal scheme $\mathfrak{X}$ are constructible.

Theorem 1. *Let $\mathfrak{X}$ be a locally finitely presented (resp. special) formal scheme over k° for an arbitrary (resp. discretely valued) k. Then for any finite abelian group Λ of order prime to $\mathrm{char}(\widetilde{k})$, the vanishing cycles sheaves $R^q\Psi_\eta(\Lambda_{\mathfrak{X}_\eta})$ are constructible.*

The statement of Theorem 1 for locally finitely presented formal schemes is deduced from the following finiteness result.

Theorem 2. *Let k be an algebraically closed non-Archimedean field, and let X be a compact k-analytic space. Then for any finite abelian group Λ of order prime to $\mathrm{char}(\widetilde{k})$, the étale cohomology groups $H^q(X,\Lambda)$ are finite.*

In [1, 5.6], the statement of Theorem 2 was proven under the assumption that X is locally isomorphic to an analytic domain in the analytification of a scheme of finite type over k. If the characteristic of k is zero, the statements of Theorem 2 as well as of Theorem 1 for locally finitely presented formal schemes (resp. the formal completions $\mathfrak{X}_{/\mathcal{Y}}$ of a locally finitely presented formal scheme $\mathfrak{X}$ along a subscheme $\mathcal{Y} \subset \mathfrak{X}_s$) follow from results of Huber ([3] and [4]). In the general case, Theorem 2 is deduced from Gabber's weak uniformization theorem [5, Exp. VII, 1.1]. Theorem 1 for special formal schemes is deduced from a version of Gabber's result.

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Galois representations and perfectoïd spaces

Jean-Marc Fontaine

This talk is an attempt to describe the link between the constructions I made with Laurent Fargues [1],[2] and the work of Peter Scholze on perfectoíd spaces [3] as well as some natural generalizations, some of them already considered by Kiran Kedlaya and Ruochuan Liu [4].

We fix a prime number p. In this talk:

– An *ultrametric field* is a field E complete with respect to a non trivial non archimedean absolute value whose residue field k_E is of characteristic p.

– An *ultrametric submultiplicative norm* on a commutative ring A is a map

$$|\cdot| : A \to \mathbb{R}_{\geq 0}$$

such that $|a| = 0 \iff a = 0$, $|a+b| \leq \max\{|a|, |b|\}$, $|1| = 1$, $|ab| \leq |a||b|$).

– A *Banach ring* is a commutative ring A equipped with an equivalence class of ultrametric submultiplicative norms (called the *admissible norms*), complete with respect to the topology they define and admitting a *pseudo-uniformizer*, that is an invertible topologically nilpotent element.

If A is a Banach ring, it is reduced. Moreover, if we set

$$A^0 = \{a \in A \mid \exists C \text{ s.t. } |a^n| \leq C\}\ ,\ A^{00} = \{a \in A \mid (a^n)_{n\in\mathbb{N}} \text{ is top. nilp.}\}\ ,$$

then A^0 is an open subring independent of $|\cdot|$ and A^{00} is an open ideal of A^0.

– A *spectral ring* is a Banach ring A such that A^0 is bounded. This is equivalent to requiring the existence of a power multiplicative admissible norm ($|a^n| = |a|^n$, if $a \in A$ and $n \in \mathbb{N}$).

– A *perfectoïd ring* is a spectral ring admitting a pseudo-uniformizer π which is a p^{th}-power and such that $A^0/\pi A^0$ is a ring of characteristic p on which the absolute Frobenius $x \mapsto x^p$ is surjective.

– A *perfectoïd field* is an ultrametric field which is a perfectoïd ring.

– If E is any ultrametric field, a *perfectoïd E-algebra* is a Banach E-algebra which is a perfectoïd ring, though a *perfectoïd k_E-algebra* is a perfectoïd ring containing k_E as a discrete subfield.

– If E is any ultrametric field, π a pseudo-uniformizer of A and R any perfect ring containing $k_E = \widetilde{E}$, the π-adic completion $W_{E^0}(R)$ of $E^0 \otimes_{W(k_E)} W(R)$ is the unique E^0-algebra separated and complete for the π-adic topology which is π-torsion free and such that its reduction mod E^{00} is R. The projection $W_{E^0}(R) \to R$ has a unique multiplicative section $a \mapsto [a] = 1\hat{\otimes}(a, 0, \ldots, 0, \ldots)$.

– If E is an ultrametric field such that E^0 is a dvr and if π is a generator of E^{00}, a *perfectoïd E-pair* is a pair (R, I) with R a perfectoïd k_E-algebra and I a principal ideal of $W_{E^0}(R^0)$ with a generator of the form $[\lambda] - \pi\eta$ with λ a pseudo-uniformizer of R and η a unit in $W_{E^0}(R^0)$.

The *tilting functor* associate to any perfectoïd ring A a perfectoïd ring of characteristic p

$$A^\flat = \{a = (a^{(n)})_{n\in\mathbb{N}} \mid a^{(n)} \in A\ ,\ (a^{(n+1)})^p = a^{(n)}\}$$

with $(a+b)^{(n)} = \lim_{m\to+\infty}(a^{(n+m)} + b^{(n+m)})^{p^m}$, $(ab)^{(n)} = a^{(n)}b^{(n)}$ and $|a| = |a^{(0)}|$).

If $\mathrm{char}(A) = p$, the map $a \mapsto a^{(0)}$ identifies $A^\flat$ to A.

Let E be an ultrametric field such that E^0 is a dvr and let π be a generator of E^{00}. If A is a perfectoïd E-algebra, the map

$$\theta_A : W_{E^0}(A^{\flat,0}) \to A^0$$

(here $A^{\flat,0} = (A^\flat)^0$) sending $\sum_{i=0}^{+\infty}[a_i]\pi^i$ to $\sum a_i^{(0)}\pi^i$ is a surjective homomorphism of E^0-algebras and $(A^\flat, \ker\ \theta_A)$ is a perfectoïd E-pair. The map

$$A \to (A^\flat, \ker\ \theta_A)$$

induces an equivalence of categories between perfectoïd E-algebras and perfectoïd E-pairs. A quasi-inverse is the functor

$$(R, I) \mapsto (R, I)_E^\sharp = (W_{E^0}(R^0)/I)[1/\pi]\ .$$

A corollary of this result is Scholze's equivalence of categories between perfectoïd algebras over a perfectoïd field K and perfectoïd $K^\flat$-algebras.

Let E be an ultrametric field, R a perfectoïd k_E-algebra and π (resp. ϖ) a pseudo-uniformizer of E (resp. R). We set

$$B_E^{b,+}(R) = W_{E^0}(R^0)[\tfrac{1}{\pi}] \ \text{ and } \ B_E^b(R) = B_E^{b,+}(R)[\tfrac{1}{[\varpi]}]\ .$$

We fix $q > 1$ and denote $v_\pi : E \to \mathbb{R} \cup \{+\infty\}$ the valuation on E normalized by $v_\pi(\pi) = 1$ and $v_\varpi : R \to \mathbb{R} \cup \{+\infty\}$ the unique map such that $v_\varpi(\varpi) = 1$, $v_\varpi(\varpi^{-1}) = -1$ and that the map $|\cdot|_\varpi : R \to \mathbb{R}_{\geq 0}$ defined by $|a|_\varpi = q^{-v_\pi(a)}$ is a power multiplicative admissible norm.

If $r > 0$ and if $\rho = q^{-r}$, there is a unique norm $|\cdot|_\rho$ on $B_E^b(R)$ such that, if $(a_n)_{n\in\mathbb{N}}$ is a bounded sequence of elements of R and $(\nu_n)_{n\in\mathbb{N}}$ is a sequence of elements of E going to 0 with $v_\pi(\nu_{n+1}) > v_\pi(\nu_n)$ for all n, then

$$\Big|\sum_{n=0}^{+\infty}[a_n]\nu_n\Big|_\rho = q^{-\min_{n\in\mathbb{N}}\{v_\varpi(a_n)+rv_\pi(\nu_n)\}}\ .$$

We denote $B_E(R)$ the completion of $B_E^b(R)$ for these $|\cdot|_\rho$'s. This a Fréchet E-algebra. If $\mathcal{I}$ denote the set of non empty closed intervals contained in $(0,1)$, we have

$$B_E(R) = \varprojlim_{I\in\mathcal{I}} B_{E,I}(R)$$

where $B_{E,I}(R)$ is the Banach algebra completion of $B_E^b(R)$ for the norm $|\cdot|_I$ defined by $|f|_I = \max_{\rho\in I}\{|f|_\rho\}$. Moreover, for each $I \in \mathcal{I}$,

- $B_E(R)$ is a spectral ring,
- if E is a perfectoïd field $B_{E,I}(R)$ is a perfectoïd E-algebra and $(B_E(R)^\flat = B_{E^\flat,I}(R)$,

- $Y_{E,I}^{\text{ad}}(R) := \text{Spa}(B_E(R), B_E^0(R))$ is an adic affinoïd space in the sense of Huber, i.e. the structural presheaf over the associated topological space $|Y_{E,I}^{\text{ad}}(R)|$ is a sheaf.

This last result is, granted to Scholze, a consequence of the second when E is perfectoïd. The general case can be deduced from this special case by a suitable scalar extension followed by a kind of faithfully flat descent. As a consequence, we see, that we can associate to our construction an adic space in the sense of Huber

$$Y_E^{\text{ad}}(R) = \varinjlim_{I \in \mathcal{I}} Y_{E,I}^{\text{ad}}(R) \; .$$

When k_E is finite with q elements, there is a unique continuous E-automorphism φ of $B_E^b(R)$ such that $\varphi([a]) = [a^q]$ for all $a \in R$. It extends uniquely to a continuous automorphism of $B_E(R)$ and defines also an automorphism of $Y_E^{\text{ad}}(R)$. One can define the adic space $X_E^{\text{ad}}(R)$ quotient of $Y_E^{\text{ad}}(R)$ by the group $\varphi^{\mathbb{Z}}(\simeq \mathbb{Z})$.

If E is locally compact and if we chose a Galois extension E_∞ of E which is arithmetically profinite, but not finite, the completion K of E_∞ is a perfectoïd field and $K^\flat$ is the completion of the radical closure of the *field of norms* $\underline{E}$ of the extension E_∞/E. We have canonical identifications

$$|Y_K^{\text{ad}}(R)| = |Y_{K^\flat}^{\text{ad}}(R)| \;\text{ and }\; |Y_{K^\flat}^{\text{ad}}(R)| = |Y_{\underline{E}}^{\text{ad}}(R)|$$

(in Scholze's terminology, for all $I \in \mathcal{I}$, the tilt of the perfectoïd space $Y_{K,I}^{\text{ad}}(R)$ is $Y_{K^\flat,I}^{\text{ad}}(R)$ and this implies they have the same underlying topological space, the second identification results from the fact that $B_{K^\flat,I}(R)$ is nothing but the completion of the radical closure of $B_{\underline{E},I}(R)$). Moreover $\Gamma = \text{Gal}(E_\infty/E)$ acts on $Y_K^{\text{ad}}(R)$ and $|Y_E^{\text{ad}}(R)| = |Y_K^{\text{ad}}(R)|/\Gamma$.

Assume moreover $R = F$ a perfectoïd field. Then $X_E^{\text{ad}}(F)$ can be viewed as some analytisation of the curve $X_E(F)$ I constructed with Laurent Fargues. Assume F algebraically closed and consider the punctured unit disk

$$D^* = \left\{ \lambda \in F \mid 0 < |\lambda| < 1 \right\} \; .$$

For $? = \underline{E}, K^\flat, K, E$ the set Spm $B_?(F)$ of closed maximal ideal of $B_?(F)$ is a subset of $|Y_?^{\text{ad}}(F)|$ and we have identifications

$$D^* = \text{Spm } B_{\underline{E}}(F) = \text{Spm } B_{K^\flat}(F) = \text{Spm } B_K(F) \; , \; D^*/\Gamma = \text{Spm } B_E(F)$$
$$\left\{\text{closed points of } X_E(F)\right\} = D^*/(\Gamma \times \varphi^{\mathbb{Z}}) \; .$$

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Continuity and finiteness of the convergence Newton polygon of a $p-$adic differential equation

Andrea Pulita
(joint work with Jérôme Poineau)

Abstract. We prove the local finiteness of the convergence Newton polygon of a differential equation over a quasi-smooth K-analytic curve, in the sense of Berkovich theory. If $(\mathscr{F},\nabla)$ is the equation, for each $x\in X$, we define the *convergence Newton polygon* $NP^{\mathrm{conv}}(\mathscr{F},x)$ of $\mathscr{F}$, whose first slope is the logarithm of the radius of convergence function of $\mathscr{F}$, and the other slopes are the logarithms of the radii of convergence of all the Taylor solutions of $\mathscr{F}$ at x. The finiteness result means that there exists a locally finite graph $\Gamma(\mathscr{F})\subset X$, together with a canonical retraction $\delta_{\mathscr{F}}:X\to\Gamma(\mathscr{F})$, such that the partial heights of the convergence Newton polygon as functions on X factorize through $\delta_{\mathscr{F}}$. Roughly speaking this result implies that there are only a *finite* number of numerical invariants that one can extract from the slopes of the partial heights of the Newton polygon along the segments of X. As a corollary we have their continuity.

In the ultrametric context the (one variable) radius of convergence function of a differential module M is an important invariant by isomorphisms. It is a function defined over a certain Berkovich space X and its slopes along the segments of X are *numerical invariants* (by isomorphism) of M. If M is a differential module over $K(\!(T)\!)$, where K is trivially valued and of characteristic 0, then from the knowledge of the radius of convergence function of M (and of its submodules) one can recover the B.Malgrange *irregularity* of M, the *Poincaré-Katz rank* of M, and more generally *the entire formal Newton polygon.* The Radius of convergence function is also a major tool in the proof of the p-adic local monodromy theorem (cf. [1], [9], [5]), and more recently of the Sabbah's conjectures (cf. [6]). The radius of convergence function is today one of the most important invariants of an ultrametric differential module. In this paper we prove its *finiteness.* Roughly speaking this implies that the numerical invariants that one can extract from the slopes of the radius of convergence function along the segments of X are *finite* in number. We now explain what this means, and we give an idea of the proof.

Let $(K,|.|)$ be a complete valued ultrametric field of characteristic 0. Let X be a quasi-smooth[1] K-analytic curve, in the sense of Berkovich theory. We recall the Ducro's notion of (weak) triangulation, which is a way of cutting a curve into pieces that are isomorphic to virtual disks or annuli. Let Γ be a locally finite graph (i.e. a finite union of closed segments) in X. If Γ contains the skeleton Γ_S of the triangulation, then the inclusion $\Gamma\subset X$ admits a canonical retraction $\delta_\Gamma:X\to\Gamma$, and X is the topological projective limit of such retractions (cf. [2, Thm.2.20]). If X is an affinoid subset of the affine line $\mathbb{A}^{1,\mathrm{an}}_K$, we provide a set of sufficient conditions that guarantee that given function $f:X\to\mathcal{T}$, where $\mathcal{T}$ is

[1]Quasi-smooth means that Ω_X is locally free, see [4, 2.1.8]. This corresponds to the notion called "rig-smooth" in the rigid analytic setting.

a set, factorizes through such a retraction δ_Γ. In this case $\Gamma = \Gamma(f)$ is called the skeleton of f. Roughly speaking $\Gamma(f)$ is the complement in X of the union of all the disks on which f is constant. The conditions of the criterion are the following:

(C1) For all K-rational point $x \in X$ there exists an open disk containing x on which f is constant.
(C2) f is piecewise linear, continuous, with a finite number of breaks on each closed segment of X.
(C3) There exists a finite connected union of closed segments Γ such that if $\mathrm{D}^-(t,\rho) \cap \Gamma = \emptyset$, then f is log-concave (hence decreasing by (C1)) on the segments inside $\mathrm{D}^-(t,\rho)$.
(C4) The modulus of all possible *non zero* slopes of f at any point is lower bounded by a positive real number $\nu_f > 0$, which is independent on the Berkovich point.
(C5) $\Gamma(f)$ is *directionally finite* at all its bifurcation points i.e. there are a finite number of branches of $\Gamma(f)$ passing through a bifurcation point x of $\Gamma(f)$.
(C6) f is *super-harmonic* outside a finite set $\mathscr{C}(f) \subseteq X$.

Among the functions satisfying these properties there are the functions of $\mathscr{O}(X)$, but also those of the type $\min(|f_1|^{-\alpha_1}, \dots, |f_n|^{-\alpha_n})$, with $\alpha_i > 0$, and many others. These properties are modeled on those satisfied by the partial height of the Newton polygon of a differential operator. The rough idea of the proof is that the super-harmonicity implies that at each bifurcation point of $\Gamma(\mathcal{R})$ the function $\mathcal{R}$ has a break, while the assumption (C2) provides that there are a finite number of breaks, and hence a finite number of bifurcation points.

Let now (M, ∇) be a differential module over the differential ring $(\mathscr{O}(X), \frac{d}{dT})$. Let $Y' = G(T) \cdot Y$, $G \in M_r(\mathscr{O}(X))$, be the differential equation associated to M in a basis. One is allowed to consider Taylor solutions of this equation and test their radius of convergence at each point of $X(\Omega)$, for all complete valued field extension Ω/K. This fact permits to associate to *any* Berkovich point $\xi \in X$ a radius of convergence by testing Taylors solutions at $t_\xi := T(\xi) \in X(\mathscr{H}(\xi))$. Namely denote by $Y(T, t_\xi)$ the Taylor solution of this equation around t_ξ, with initial value $Y(t_\xi, t_\xi) := \mathrm{Id}$. If $Y^{(n)} = G_n(T) \cdot Y$ is the n-th iterate of the equation, then $Y(T, t_\xi) := \sum_{n \geq 0} G_n(t_\xi) \frac{(T - t_\xi)^n}{n!}$. The minimum of the radii of convergence at t_ξ of the entries of $Y(T, t_\xi)$ is given by $\mathcal{R}^Y(\xi) := \liminf_n |\frac{G_n(t_\xi)}{n!}|_\Omega^{-1/n}$. One obtains a function $\mathcal{R}^Y : X \to \mathbb{R}_{>0}$ depending on the chosen basis of M. In order to make this number invariant by base changes in M one sets

$$\mathcal{R}^{\mathrm{M}}(\xi) \ := \ \min(\liminf_n \xi(G_n/n!)^{-1/n}, \rho_{\xi,X}) \ , \tag{1}$$

where $\rho_{\xi,X}$ is the radius of the largest open disk centered at $t_\xi \in X(\mathscr{H}(\xi))$ contained in $X \widehat{\otimes} \mathscr{H}(\xi)$. $\mathcal{R}^{\mathrm{M}} : X \to \mathbb{R}_{>0}$ is called the *radius of convergence function* of M. It represents the smallest radius of convergence of a Taylor solution of M around t_ξ. We now refine this construction by taking in account the other radii. The vector space of germs of convergent solutions at $t_\xi \in X(\mathscr{H}(\xi))$ is naturally filtered by the radius of convergence of its elements. We associate a polygon

$NP^{\text{conv}}(\mathrm{M},\xi)$ to this filtration, called *convergence polygon of* M *at* ξ. Its first slope $s_1^{\mathrm{M}}(\xi) = h_1^{\mathrm{M}}(\xi)$ is equal to $\ln(\mathcal{R}^{\mathrm{M}}(\xi))$. For $i = 1,\ldots,r$ its *i-th slope* is given by $s_i^{\mathrm{M}}(\xi) := \ln(\mathcal{R}_i^{\mathrm{M}}(\xi))$, where $\mathcal{R}_i^{\mathrm{M}}(\xi) \leq \rho_{\xi,X}$ is the radius of the largest open disk centered at t_ξ on which M admits at least $r-i+1$ linearly independent Taylor solutions, where r is the rank of M. This defines univocally $NP^{\text{conv}}(\mathrm{M},\xi)$ as the epigraph[2] of the convex function $h : [0,r] \to \mathbb{R}$ defined by the fact that $h(0) = 0$, and that $h(\xi)$ is linear on $[i-1,i]$ with slope $s_i^{\mathrm{M}}(\xi)$. The values $h_i^{\mathrm{M}}(\xi) := h(i)$ are called the i-th *partial heights*. The main result provides important properties on the behavior of $NP^{\text{conv}}(\mathrm{M},\xi)$ as a function of ξ. Namely we prove that the functions $\mathcal{R}^Y, \mathcal{R}^{\mathrm{M}}, s_i^{\mathrm{M}}, h_i^{\mathrm{M}} : X \to \mathbb{R}_{>0}$ are all *finite functions* i.e. they have a finite skeleton and factorize through it. As a consequence one has their continuity. We precise moreover a family of formal properties enjoyed by them as the piecewise linearity, convexity, super-harmonicity, integrality. Roughly speaking this result means that there are a *finite number* of numerical invariants of M that one can extract from the slopes of $\mathcal{R}^Y, \mathcal{R}^{\mathrm{M}}, s_i^{\mathrm{M}}, h_i^{\mathrm{M}}$ along the branches of X, and that these functions are all *definable* in the sense of [8]. We prove this basically by a classical result due to Young [10] permitting to control "*small*" slopes of the polygon. In the non p-adic case, this is enough to control all the slopes since they are always "*small*". In the p-adic case the "*big*" values of the slopes are reduced to the "*small*" values by using the Frobenius push-forward techniques as in [7] and [3].

In the second step we show how to deduce from this the local finiteness of $\Gamma(\mathscr{F})$ on a general curve. Around each point that does not admit as a neighborhood an affinoid of the affine line we use a slight improvement of A.Ducros result providing a locally étale morphism with values into the affine line with particularly good properties. We give the exact behavior between the polygon of $\mathscr{F}$ and its push-forward on the affine line.

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The Arithmetic Hodge index theorems and arithmetic Berkovich spaces

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(joint work with Xinyi Yuan)

Let us recall the classical Hodge index theorem for a fiberation $\pi : \mathcal{X} \longrightarrow \mathcal{B}$ of a projective and smooth variety over a project and smooth curve of dimension relative dimension $n \geq 1$. Let $L_1, \cdots, L_{n-1}$ be ample line bundles on D, and $D \in \mathrm{N}(X/B)$ a non-zero element satisfying the condition $D_\eta \cdot L_{1,\eta} \cdots L_{n-1,\eta} = 0$ on the generic fiber $\mathcal{X}_\eta$. The classical Hodge index theorem asserts:

$$D^2 \cdot L_1 \cdots L_{n-1} < 0.$$

In [1, 2], Faltings and Hiriljac have proved a Hodge index theorem for arithmetic surfaces. In [3], such a Hodge index theorem has been generalized to high dimensional arithmetic varieties.

In this talk, we give a further extension to projective and flat families $\pi : \mathcal{X}_\mathcal{U} \longrightarrow \mathcal{U}$ over an open variety $\mathcal{U}$ in either algebraic geometry or arithmetic geometry. In the case of algebraic geometry, $\mathcal{U}$ is an open variety over a field k, and in case of arithmetic geometry, we assume that $\mathcal{U}$ is flat and of finite type over $\mathrm{Spec}\mathbb{Z}$.

First we construct a group $\widehat{\mathrm{Pic}}(\mathcal{X}_\mathcal{U})_{\mathrm{int}}$ of *integral metrized line bundles* on $\mathcal{X}_U$ as follows: define a group of *model metrized line bundles*:

$$\widehat{\mathrm{Pic}}(\mathcal{X}_\mathcal{U})_{\mathrm{mod}} := \varinjlim_{\mathcal{X}} \widehat{\mathrm{Pic}}(\mathcal{X})$$

where the limit runs over all projective and flat models $\mathcal{X} \longrightarrow \mathcal{B}$ such that $\mathcal{B}$ is either projective over k in algebraic geometry, or projective and flat over $\mathbb{Z}$ in arithmetic geometry, and $\widehat{\mathrm{Pic}}(\mathcal{X})$ is the usual Picard group in algebraic geometry or the group of Hermitian line bundles in arithmetic geometry. Then we define a topology in this group using an strictly effectively divisor $\widehat{D}$ with support $|D| = \mathcal{X}_0 \setminus \mathcal{U}$. The completion is denoted by $\widehat{\mathrm{Pic}}(\mathcal{X}_\mathcal{U})_{\mathrm{cont}}$. More precisely, an element $\bar{\mathcal{L}}$ in this group is represented by data $(\bar{\mathcal{L}}_i, m_{i,j})$ $(i \geq j \geq 1)$ with a convergence condition, where

(1) $\bar{\mathcal{L}}_i$ is a sequence of line bundles on models $\mathcal{X}_i$ with compatible morphisms $\pi_{i,j} : \mathcal{X}_i \longrightarrow \mathcal{X}_j$ $(i \geq j \geq 0)$ of $\mathcal{U}$-models, and

(2) $\ell_{i,j}$ is a compatible system of rational sections of $\mathcal{L}_i \otimes \pi_{i,j}^* \mathcal{L}_j^{-1}$ whose divisor supports on $\mathcal{X}_i \setminus \mathcal{U}$.

The convergence condition is follows: for any $\epsilon > 0$ there is a i_0 such that for any $i \geq j \geq i_0$, the divisors

$$\epsilon\pi_{i,0}^* \widehat{D} \pm \widehat{\operatorname{div}}\ell_{i,j}$$

are both strictly effective. The group $\widehat{\operatorname{Pic}}(\mathcal{X}_{\mathcal{U}})_{\operatorname{int}}$ of integrate metrized line bundles are of form

$$\bar{L} = (\lim \bar{\mathcal{L}}_n') - (\lim \bar{\mathcal{L}}_n'')$$

with all $\bar{\mathcal{L}}_n'$ and $\bar{\mathcal{L}}_n''$ nef bundles.

Let $X \longrightarrow \operatorname{Spec}K$ denote the generic fiber of $\mathcal{X} \longrightarrow \mathcal{U}$. Then we define

$$\widehat{\operatorname{Pic}}(X)_{\operatorname{int}} = \varinjlim_{U} \widehat{\operatorname{Pic}}(\mathcal{X}_U)_{\operatorname{int}}, \qquad \widehat{\operatorname{Pic}}(K)_{\operatorname{int}} = \varinjlim_{U} \widehat{\operatorname{Pic}}(U)_{\operatorname{int}}.$$

All of these elements can be realized as metrized line bundles in Berkovich spaces X^{an} and $(\operatorname{Spec}K)^{\operatorname{an}}$. We can define (absolute) intersection pairing on

$$\widehat{\operatorname{Pic}}(K)_{\operatorname{int}}^{d+1} \longrightarrow \mathbb{R}, \qquad (\bar{H}_1, \cdots, \bar{H}_{d+1}) \mapsto \bar{H}_1 \cdots \bar{H}_{d+1},$$

and relative intersection pairing

$$\widehat{\operatorname{Pic}}(X)_{\operatorname{int}}^{n+1} \longrightarrow \widehat{\operatorname{Pic}}(K)_{\operatorname{int}} : \qquad (\bar{L}_1, \cdots, \bar{L}_{n+1}) \mapsto \pi_*(\bar{L}_1 \cdots \bar{L}_{n+1}).$$

Definition 9. *Let* $\bar{L} = (L, \|\cdot\|_L), \bar{M} = (M, \|\cdot\|_M) \in \widehat{\operatorname{Pic}}(X)_{\operatorname{int}}$ *and* $\bar{H} \in \widehat{\operatorname{Spec}}(K)_{\operatorname{int}}$.

(1) $\bar{H} \gg 0$*: $\bar{H}$ is big and nef, i.e., it is the limit of nef bundles with positive highest self-intersection.*
(2) $\bar{H} \geq 0$*: $\bar{H}$ is pseudo-effective, i.e., it has non-negative intersection with big and nef bundles.*
(3) $\bar{L} \gg 0$*: L is ample, $\bar{L}$ is nef with positive $\bar{L}^{\dim Y} \cdot Y \gg 0$ for any sub variety Y of X.*
(4) $\bar{M}$ *is $\bar{L}$-finite: there are $\epsilon > 0$ such that*

$$\bar{L} \gg \bar{\epsilon} M \gg -\bar{L}.$$

Theorem 10. *Let $\bar{M}, \bar{L}_1, \cdots, \bar{L}_{n-1}$ be adelic line bundles on X such that every $\bar{L}_i$ are positive and that $M \cdot L_1 \cdots L_{n-1} = 0$ on X, then*

$$\pi_*(\bar{M}^2 \cdot \bar{L}_1 \cdots \bar{L}_{n-1}) \leq 0.$$

Moreover, if $\bar{M}$ is $\bar{L}_i$-finite for each i, then the equality holds if and only if $\bar{M} = f^*\bar{M}'$ for some adelic bundle M' on $\operatorname{Spec}K$.

Remark 11. Let W denote the subspace of $\widehat{\operatorname{Pic}}(X)_{\operatorname{int}}$ of elements which are $\mathcal{L}_i$-finite for all i with pairing defined by

$$\langle \bar{M}_1, \bar{M}_2 \rangle = \bar{M}_1 \cdot \bar{M}_2 \cdot \bar{L}_1 \cdots \bar{L}_{n-1}.$$

Then the theorem implies that the pairing is non-degenerate, that $V := \pi^* \widehat{\operatorname{Pic}}(K)_{\operatorname{int}}$ is a maximal isotropic subspace, and that $V^{\perp}/V$ is negatively defined.

Remark 12. When K is a number field, and $\bar{M}, \bar{L}_1, \cdots, \bar{L}_{n-1}$ are realized on a model $\mathcal{X}$ over $\mathcal{O}_K$ such that $\bar{L}_1 = \cdots = \bar{L}_{n-1}$ is arithmetically ample, the theorem is due to Moriwaki [3].

The same proof will give a local Hodge index theorem for bundles over a local field K. An immediate consequence is the following Calabi–Yau theorem:

Theorem 13. *Let X be a projective variety over a valuation field K of dimension n. Let L be an ample bundle on X with two semi-positive metrics $\|\cdot\|_1$ and $\|\cdot\|_2$. Assume that the induced measures on X^{an} equal to each other:*

$$c_1(L, \|\cdot\|_1)^n = c_2(L, \|\cdot\|_2)^n.$$

Then

$$\|\cdot\|_1 = c\|\cdot\|_2$$

for some constant $c > 0$.

The application to dynamical system is given as follows. Let X be a variety over a field K with two polarized endomorphisms f_1, f_2 in the sense that there are two ample line bundles L_1 and L_2 such that $f_i^* L_i = q_i L_i$ with some $q_i > 1$. Let $\mathrm{Prep}(f_i)$ be the sets of pre periodic points, namely points with finite forward orbits under f_i respectively.

Theorem 14. *If* $\mathrm{Prep}(f_1) \cap \mathrm{Prep}(f_2)$ *is dense in X then* $\mathrm{Prep}(f_1) = \mathrm{Prep}(f_2)$.

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The skeleton and top weight cohomology of $\mathcal{M}_{g,n}$

Sam Payne

(joint work with Dan Abramovich, Lucia Caporaso, and Søren Galatius)

Let $U \subset X$ be an open immersion of smooth complex varieties, where the boundary $\partial X = X \setminus U$ is a divisor with simple normal crossings. The boundary complex $\Delta(\partial X)$ is the dual complex of the boundary divisor. It has vertices v_i corresponding to the irreducible components D_i of ∂X, edges joining v_i to v_j corresponding to the irreducible components of $D_i \cap D_j$, 2-faces spanned by v_i, v_j, and v_k corresponding to the irreducible components of $D_i \cap D_j \cap D_k$, and so on. A fundamental fact is that, if X is complete, the simple homotopy type of the boundary complex $\Delta(\partial X)$ is an invariant of U, independent of the choice of compactification. The rational homology of this complex computes, with a degree shift, the top graded piece of the weight filtration on the cohomology of U. See [4].

The cone over $\Delta(\partial X)$, denoted $\Sigma(X)$ to indicate its role as an analytic skeleton, embeds naturally in the analytification U^{an} with respect to the trivial valuation. It is the space of valuations on the function field of U that are monomial in local coordinates given by the defining equations of the irreducible components of ∂X that contain each stratum, and is canonically identified with the cone complex associated to the toroidal embedding $U \subset X$ in [2]. Thuillier describes a canonical strong deformation retraction from U^{an} onto $\Sigma(X)$ [5]. The preimage of the vertex 0 in $\Sigma(X)$ is the subspace $U^{\beth} \subset U^{\mathrm{an}}$ consisting of points over valued extensions of the complex numbers that are defined over their respective valuation ring. Therefore, the complement $U^{\mathrm{an}} \smallsetminus U^{\beth}$ is independent of the choice of compactification; it may be interpreted as a "deleted tubular neighborhood at infinity." If X is compact, then this deleted tubular neighborhood deformation retracts onto $\Sigma(X) \smallsetminus 0$, and both have the homotopy type of the boundary complex $\Delta(\partial X)$.

We generalize this construction slightly, to smooth toroidal Deligne–Mumford stacks, i.e. stacks that are étale locally like $U \subset X$ as above. Let $\mathfrak{X}$ be a Deligne–Mumford stack with an open substack U. Assume that $\mathfrak{X}$ has an étale cover by a scheme $V \to \mathfrak{X}$ such that $U_V \subset V$ is a toroidal embedding with simple normal crossings, where U_V is the preimage of U. An important example is the Deligne–Mumford–Knudsen compactification of the moduli space of curves with marked points, $\mathcal{M}_{g,n} \subset \overline{\mathcal{M}}_{g,n}$. Given such a cover, let $V_2 = V \times_{\mathfrak{X}} V$. The functorial properties of skeletons of toroidal schemes produce a natural diagram of skeletons $\Sigma(V_2) \rightrightarrows \Sigma(V)$. We show that the colimit of this diagram is independent of the choice of cover, and is canonically a deformation retraction of U^{an}. We call it the *skeleton* of the toroidal embedding, and denote it $\Sigma(\mathfrak{X})$.

For the special case $\mathcal{M}_{g,n} \subset \overline{\mathcal{M}}_{g,n}$, we use local monodromy computations on the boundary strata to produce a canonical identification

$$\Sigma(\overline{\mathcal{M}}_{g,n}) \cong M_{g,n}^{\mathrm{trop}}.$$

This gives a modular interpretation to the skeleton of the Deligne–Mumford–Knudsen modular compactification of $\mathcal{M}_{g,n}$; it parametrizes stable tropical curves of genus g with n marked legs. All of this is joint work with D. Abramovich and L. Caporaso.

In joint work with S. Galatius, I have applied this modular interpretation of the skeleton of $\overline{\mathcal{M}}_{g,n}$ to compute the homotopy type of the deleted tubular neighborhood at infinity in $\mathcal{M}_{1,n}$. We show that it is contractible for $n \leq 2$, and homotopic to a wedge sum of $(n-1)!/2$ spheres of dimension $n-1$, for $n \geq 3$. We conclude that the $2n$th graded piece of $H^k(\mathcal{M}_{1,n}, \mathbb{Q})$ has rank $(n-1)!/2$, for $k = n \geq 3$, and vanishes otherwise, generalizing published results for $n \leq 3$ [1, 3], as well as an unpublished computation of O. Tommasi for $n = 4$.

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Covers of curves and covers of skeleta

Joseph Rabinoff
(joint work with Omid Amini, Matthew Baker and Erwan Brugallé)

1. Triangulated punctured curves

Let K be an algebraically closed field which is complete with respect to a nontrivial, non-Archimedean valuation val. Let R be the valuation ring of K and let k be its residue field. A *triangulated punctured K-curve* consists of the following data:

(1) a smooth, proper, connected K-curve X (regarded either as a scheme or as a K-analytic space),
(2) a finite set of points $D \subset X(K)$ (the set of punctures), and
(3) a finite set $V \subset X$ of type-2 points,

with the requirement that the open analytic domain $X \setminus (V \cup D)$ be a disjoint union of finitely many open annuli, finitely many punctured open balls, and infinitely many open balls. (More precisely, one should refer to such an object as a *semistably* triangulated punctured K-curve for reasons that will become clear below.) The *skeleton* $\Sigma = \Sigma(X, V, D)$ of a triangulated punctured K-curve (X, D, V) is defined to be the union of $V \cup D$ with the skeleta of all of the open annuli and balls in the decomposition $X \setminus (V \cup D)$. The skeleton is the geometric realization of a graph with vertex set $V \cup D$; its open edges are the skeleta of the open annuli and punctured balls. There is a natural metric on $\Sigma \setminus D$ with respect to which the length of the skeleton of an open annulus is the logarithmic modulus of the annulus.

The proof of the following theorem can be found in [3, Propositions 2.2 and 2.3] and [2, Proposition 2.4.4].

Theorem 15 (Berkovich, Bosch-Lütkebohmert)**.** *Let $\mathfrak{X}$ be a semistable formal model of X and let* red $: X \to \mathfrak{X}_k$ *be the reduction map.*

(1) *If $\xi \in \mathfrak{X}_k$ is a generic point then* $\mathrm{red}^{-1}(\xi)$ *is a single type-*2 *point.*
(2) *If $\xi \in \mathfrak{X}_k$ is a node then* $\mathrm{red}^{-1}(\xi)$ *is an open annulus.*
(3) *If $\xi \in \mathfrak{X}_k$ is a smooth closed point then* $\mathrm{red}^{-1}(\xi)$ *is an open ball.*

Let $\mathfrak{X}$ be a semistable formal model such that the points of D reduce to distinct smooth closed points of $\mathfrak{X}_k$. Let V be the set of all points of X that reduce to generic points of $\mathfrak{X}_k$. It follows from Theorem 15 that (X, D, V) is a triangulated punctured curve. In fact this association defines a bijective correspondence between the semistable models of (X, D) of the above form and the set of triangulations (X, D, V). See [4, §5] for a proof of this fact.

Definition 16. *Let (X, D, V) and (X', D', V') be triangulated punctured K-curves. A* finite morphism *from (X', D', V') to (X, D, V) is the data of a finite morphism $f : X' \to X$ such that $f^{-1}(D) = D'$, $f^{-1}(V) = V'$, and $f^{-1}(\Sigma(X, V, D)) = \Sigma(X', V', D')$.*

The following theorem seems well-known to experts. We provide a proof in our paper as none exists in the literature to our knowledge.

Theorem 17. *Let (X, D) and (X', D') be punctured K-curves and let $f : X' \to X$ be a finite morphism such that $D' = f^{-1}(D)$. Then there exist triangulations V, V' such that f becomes a finite morphism of triangulated punctured K-curves.*

One deduces the following Corollary from Theorem 17 using the fact that, if $\mathfrak{X}, \mathfrak{X}'$ are the semistable models corresponding to V, V', then $f : X' \to X$ extends (necessarily uniquely) to a finite morphism $\mathfrak{X}' \to \mathfrak{X}$ if and only if $f^{-1}(V) = V'$.

Corollary 18. *Let X, X' be K-curves and let $f : X' \to X$ be a finite morphism. There exist semistable models $\mathfrak{X}, \mathfrak{X}'$ of X, X', respectively, such that f extends to a finite morphism $\mathfrak{X}' \to \mathfrak{X}$.*

Corollary 18 was proved independently by Coleman [5] and Liu [6] in different contexts. Liu in fact proves much more precise theorems in the case of a discretely-valued base. All of Coleman's and Liu's statements follow formally from Theorem 17, and over more general base fields (using a suitable descent argument).

2. Metrized complexes of curves

Let (X, D, V) be a triangulated punctured K-curve and let Σ be its skeleton. Consider a point $x \in V$. Since x has type 2, the residue field $\widetilde{\mathscr{H}}(x)$ is a finitely generated extension field of k of transcendence degree 1; it is therefore the function field of a canonically determined proper, smooth, connected k-curve C_x. Moreover, there is a canonical injection $\iota_x : T_x(\Sigma) \hookrightarrow C_x(k)$ from the set $T_x(\Sigma)$ of outgoing directions at x to the set of closed points of C_x; see [4, §5] for a definition of ι_x. The data $(\Sigma, \{C_x\}_{x\in V}, \{\iota_x\}_{x\in V})$ is called a *metrized complex of curves*. The metrized complex structure on Σ is intrinsic to the triangulated curve (X, D, V).

Let $f : (X', D', V') \to (X, D, V)$ be a finite morphism of triangulated punctured curves. For $x' \in V'$ with image $x = f(x')$ the field homomorphism $\widetilde{\mathscr{H}}(x) \to \widetilde{\mathscr{H}}(x')$ induces a finite morphism $f_{x'} : C_{x'} \to C_x$. The set-theoretic map $f|_{\Sigma'} : \Sigma' \to \Sigma$ of skeleta along with the maps $(f_{x'})_{x'\in V'}$ satisfy the following properties:

1. f takes vertices to vertices and edges to edges.
2. If e' is an edge of Σ' and $e = f(e')$ then f maps e' bijectively onto e, and there exists an integer $d_f(e') \in \mathbf{Z}_{\geq 1}$, called the *expansion factor* of f along e', such that for all $x', y' \in e'$, the distance from $f(x')$ to $f(y')$ is $d_f(e')$ times the distance from x' to y'.
3. Let $x' \in V'$, let $x = f(x')$, and let e be an edge of Σ adjacent to x. Then the quantity
$$\sum_{\substack{e' \ni x' \\ f(e')=e}} d_f(e')$$
is independent of the choice of edge e adjacent to x.
4. For $x' \in V'$, if e' is an edge representing an outgoing direction $\vec{v}' \in T_{x'}(\Sigma')$ then $d_f(e')$ is the ramification degree of $f_{x'}$ at $\iota_{x'}(\vec{v}')$.

Properties (1) and (2) say that $f : \Sigma' \to \Sigma$ is an integral morphism of metric graphs; property (3) says that f is *harmonic*. It is clear that (3) follows from (4). A *finite morphism of metrized complexes of curves* consists of the data $(f, \{f_{x'}\}_{x' \in V'})$ as above, satisfying (1)–(4).

3. Lifting theorems

Our main results treat the problem of finding a finite morphism of triangulated punctured curves inducing a given finite morphism of abstract metrized complexes of curves on skeleta. The following proposition says that every metrized complex is a skeleton:

Proposition 19. *Let $\Sigma = (\Sigma, \{C_x\}_{x \in V}, \{\iota_x\}_{x \in V})$ be a metrized complex of curves. There exists a triangulated punctured K-curve (X, D, V) such that $\Sigma \cong \Sigma(X, D, V)$ as metrized complexes.*

The proof of Proposition 19 is not difficult; it involves a simple deformation theory argument along with some cutting and pasting of analytic curves. Similar results have appeared in the literature, for instance in [7]. The following theorem is much more substantial.

Theorem 20. *Let (X, D, V) be a triangulated punctured K-curve with skeleton Σ. Let $f : \Sigma' \to \Sigma$ be a finite morphism of metrized complexes of curves such that, for all vertices $x' \in \Sigma'$, the morphism $f_{x'} : C_{x'} \to C_{f(x')}$ is tamely ramified, and every ramification point is in the image of $\iota_{x'}$. Then there exists a triangulated punctured K-curve (X', D', V') and a morphism $f : (X', D', V') \to (X, D, V)$ such that the induced morphism on skeleta is isomorphic to $\Sigma' \to \Sigma$. The morphism $f : X' \to X$ is only branched over D. Moreover, the set of isomorphism classes of such covers $X' \to X$ is finite, and can be explicitly classified, along with the finite group* $\mathrm{Aut}(X'/X)$.

Theorem 20 strengthens and generalizes existing theorems in the literature: see [7] and [8]. Its proof uses the theory of the tamely ramified étale fundamental group as applied to the residue curves C_x to perform a canonical and functorial local lifting procedure; the local lifts are then glued along the edges. Some of the ideas in this proof also appeared in [7].

4. Application: surjectivity of homomorphisms of component groups

In this section K is a complete field equipped with a discrete valuation normalized such that $\mathrm{val}(K^\times) = \mathbf{Z}$. The following question is due to Ken Ribet, who posed it to Matt Baker in personal correspondence:

Question 21. Let $f : X' \to X$ be a finite morphism of smooth, proper, geometrically connected K-curves and let $f_* : \Phi_{X'} \to \Phi_X$ be the induced homomorphism of component groups of special fibers of Néron models of Jacobians. Is f_* surjective when:

(1) the minimal regular model of X' consists of two rational components meeting each other transversally at some number of points, and
(2) X is a Mumford curve of genus at least 2?

By a theorem of Raynaud, the component group Φ_X can be calculated in terms of the skeleton Σ of X associated to a regular semistable model of X. In the language of [1], Raynaud's theorem provides an isomorphism $\Phi_X \cong \mathrm{Jac}(\Sigma)$ of the component group Φ_X with the Jacobian of the graph Σ. This isomorphism remains valid when Σ is the skeleton of a model of X which is semistable but not necessarily regular. Given triangulations of X, X' making f into a finite morphism of triangulated curves, the map $f_* : \Phi_{X'} \to \Phi_X$ coincides with the functorially induced map on skeleta $\mathrm{Jac}(\Sigma') \to \mathrm{Jac}(\Sigma)$. (Here we are suppressing some technical details related to the fact that triangulations and skeleta are geometric notions defined after passing to a complete and algebraically closed valued extension field of K.)

For $\ell_1, \dots, \ell_{g+1} \in \mathbf{Z}_{\geq 1}$ let $B(\ell_1, \dots, \ell_{g+1})$ denote the "banana graph" consisting of two vertices attached by $g+1$ edges of lengths $\ell_1, \dots, \ell_{g+1}$. The skeleton Σ' associated to the minimal regular model of a curve X' as in Question 21 is isomorphic to $B(1, 1, \dots, 1)$. Let $\Sigma' = B(1, 1, 1, 1)$ and $\Sigma' = B(1, 2, 2)$ and consider the harmonic morphism $f : \Sigma' \to \Sigma$ which takes the first two edges of the source to the first edge of the target and the third (resp. fourth) edge of the source to the second (resp. third) edge of the target with expansion factor 2. One checks that the induced map $\mathrm{Jac}(\Sigma') \to \mathrm{Jac}(\Sigma)$ is not surjective. We promote f to a finite morphism of metrized complexes by attaching $\mathbf{P}^1_k$ to each vertex, defining the induced homomorphisms $\mathbf{P}^1_k \to \mathbf{P}^1_k$ by $z \mapsto z^2$, and associating the edges of the source (resp. target) to the points $1, -1, 0, \infty \in \mathbf{P}^1_k(k)$ (resp. $1, 0, \infty \in \mathbf{P}^1_k(k)$). By Proposition 19 there exists a triangulated curve X with skeleton Σ, and by Theorem 20 there exists a triangulated curve X' and a finite morphism of triangulated curves $f : X' \to X$ inducing $f : \Sigma' \to \Sigma$ on skeleta. This procedure can be carried out in such a way that the curves X, X' and the morphism f are defined over K. By the above remarks, the induced homomorphism $\Phi_{X'} \to \Phi_X$ is not surjective. This provides a negative answer to Question 21.

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Tempered Anabelianness of punctured Tate curves

Emmanuel Lepage

Tempered fundamental group

Let K be a complete non-archimedean field and let X be a smooth K-analytic space. In [2, Rem. 6.3.4.(ii)], V. Berkovich defines a notion of étale covers, and in [3], J. de Jong uses this definition to construct an étale fundamental group. Here we will be interested in a smaller category of covers:

Definition 22 ([1, Def. III.2.1.1]). *An étale map $S \to X$ is a tempered cover if there exists a surjective finite étale cover $X' \to X$ such that $S \times_X X' \to X'$ is a topological cover.*

In particular, finite étale covers and topological covers are tempered covers. The category of tempered covers will be denoted by $\mathrm{Cov}^{\mathrm{temp}}(X)$. If $\bar{x}$ is a geometric point, one gets a functor $\mathrm{Cov}^{\mathrm{temp}}(X) \to \mathrm{Set}$ by mapping a tempered cover S to the fiber $S_{\bar{x}}$.

Definition 23. *The tempered fundamental group $\pi_1^{\mathrm{temp}}(X, \bar{x})$ is the group of automorphism of $F_{\bar{x}}$.*

It becomes a topological group by considering as fundamental open neighborhood of 1 the stabilizers of arbitrary elements s of $F_{\bar{x}}(S)$ for arbitrary object S of $\mathrm{Cov}^{\mathrm{temp}}(X)$. If one withdraws the base point $\bar{x}$, $\pi_1^{\mathrm{temp}}(X)$ is well defined up to inner automorphism. The fundamental group $\pi_1^{\mathrm{alg}}(X, \bar{x})$ classifying the finite étale covers is the profinite completion of $\pi_1^{\mathrm{temp}}(X, \bar{x})$.

Example. [1, § III.2.3.2] If $K = \mathbf{C}_p$ and E is an elliptic curve, then
(1) if E has good reduction, $\pi_1^{\mathrm{temp}}(E) \simeq \hat{\mathbf{Z}}^2$;
(2) if E is a Tate curve, $\pi_1^{\mathrm{temp}}(E) \simeq \hat{\mathbf{Z}} \times \mathbf{Z}$.

The tempered fundamental group is much more complicated when X is a hyperbolic curve. For example, it is not locally compact. One may wonder what can be recovered of a hyperbolic curve from its tempered fundamental group. Most precisely,

Question. If X_1 and X_2 are two hyperbolic curves over $\mathbf{C}_p$, is the map

$$\mathrm{Isom}_{\mathbf{Q}_p}(X_1, X_2) \to \mathrm{OutIsom}(\pi_1^{\mathrm{temp}}(X_1^{\mathrm{an}}), \pi_1^{\mathrm{temp}}(X_2^{\mathrm{an}}),$$

given by functoriality of the tempered fundamental group, a bijection ?

The main result of this talk is the following :

Theorem 24 ([4, Theorem 0.3])**.** *Let $q_1, q_2 \neq 0 \in \bar{\mathbf{Q}}_p$ be such that $|q_1|, |q_2| < 1$. Let $E_i = \mathbf{G}_{\mathrm{m}}/q_i^{\mathbf{Z}}$ and $X_i = E_i \backslash \{0\}$ for $i = 1, 2$. If there exists an isomorphism $\pi_1^{\mathrm{temp}}(X_1) \simeq \pi_1^{\mathrm{temp}}(X_2)$, then there exists $\sigma \in \mathrm{Gal}(\bar{\mathbf{Q}}_p/\mathbf{Q}_p)$ such that $q_2 = \sigma(q_1)$.*

Sketch of the proof of Theorem 24

Let ϕ be an isomorphism $\pi_1^{\mathrm{temp}}(X_1) \to \pi_1^{\mathrm{temp}}(X_2)$. It induces an equivalence of categories $\phi^* : \mathrm{Cov}^{\mathrm{temp}}(X_2) \to \mathrm{Cov}^{\mathrm{temp}}(X_1)$.

Step 1. The isomorphism ϕ induces a homeomorphism

$$\phi_* : X_1 \to X_2,$$

which is functorial with respects to tempered cover [4, Theorem 0.1], so that it is compatible with the decomposition groups. The starting idea is to apply the following results of S. Monchizuki to finite étale covers of X_1:

Theorem 25 ([5])**.** *Let Y_1, Y_2 be two hyperbolic curves over $\bar{\mathbf{Q}}_p$ and let ϕ be an isomorphism $\pi_1^{\mathrm{temp}}(Y^{\mathrm{an}}_{1,\mathbf{C}_p}) \simeq \pi_1^{\mathrm{temp}}(Y^{\mathrm{an}}_{2,\mathbf{C}_p})$. Then ϕ induces a natural isomorphism $\mathbb{G}_{Y_1} \to \mathbb{G}_{Y_2}$ of the corresponding graphs of the stable reductions.*

Let Y_2 be a finite Galois cover of X_2 of Galois group G and let $Y_1 = \phi^* Y_2$. Then one gets an isomorphism $\mathbb{G}_{Y_1}/G \to \mathbb{G}_{Y_2}/G$ and $\mathbb{G}_{Y_i}/G$ is a skeleton of the analytic space E_i. The thing is to show that the map $E_i \to \varprojlim_{Y_i} \mathbb{G}Y_i/G$ is a homeomorphism. This is equivalent to the fact that the union $\tilde{V}(X) := \bigcup_{f:Y_i \to X_i} f(V(Y_i))$, where f goes through finite étale covers of X_i and $V(Y_i)$ is the set of vertices of the skeleton of Y_i given by the stable reduction, is dense in E_i.

Step 2. Let $\Omega = \mathbf{G}_{\mathrm{m}} - q^{\mathbf{Z}}$ be the universal topological cover of a punctured Tate curve $X = \mathbf{G}_{\mathrm{m}}/q^{\mathbf{Z}} \backslash \{0\}$.

To $f \in O^\times(\Omega)$, one can associate a μ_{p^n}-torsor $\Omega[f^{1/p^n}]$ of Ω by adding a p^nth root of f. Such an invertible function has an explicite description in terms of an infinite product :

$$f(z) = cz^m \prod_{j<0} \left(\frac{z - q^j}{q^j}\right)^{\alpha_j} \prod_{j \geq 0} \left(\frac{z - q^j}{z}\right)^{\alpha_j},$$

where $m, \alpha_j \in \mathbf{Z}$. The μ_{p^n}-cover $\Omega[f^{1/p^n}]$ descends to a cover of some finite topological cover of X if and only if the sequence is periodic and $\sum \alpha_j = 0$. Moreover $\Omega[f^{1/p^n}]$ depends only on the values of m and $\alpha_j \mod p^n$, and therefore can still be defined when $m, \alpha_j \in \mathbf{Z}_p$. So does the differential form

$$\omega := \frac{df}{f} = m\frac{dz}{z} + \sum_{j<0} \alpha_j \frac{dz}{z - q^j} + \sum_{j \geq 0} \alpha_j \left(\frac{dz}{z - q^j} - \frac{dz}{z}\right).$$

Let $x \in \mathbf{G}_{\mathrm{m}}(\mathbf{C}_p)$ and, for $n \gg 0$ let r_n be the radius of convergence of f^{1/p^n} and let y_n be a preimage in $\Omega[f^{1/p^n}]$ of the Gauss point b_{x,r_n} of the disc of center x and radius r_n. Then the speed of convergence of r_n can be described in terms of $e(x, f) = \mathrm{mult}_x \frac{df}{f}$. Moreover y_n is a point of type 2 and $\widetilde{\mathcal{H}(y_n)}$ is not isomorphic to $\bar{\mathbf{F}}_p(T)$ if $e(x,f) \notin \{p^k-1\}_{k\in\mathbf{N}}$. Therefore if $e(x,f) \notin \{p^k-1\}_{k\in\mathbf{N}}$, y_n must belong to the skeleton of $\Omega[f^{1/p^n}]$, and if the sequence $(\alpha_j)_j$ is periodic, the image in X of $(y_n)_n$ gives a sequence of element in $\tilde{V}(X)$ which tends to x.

Step 3. Step 1 gives us a homeomorphism $\Omega_1 \to \Omega_2$ which extends to an homeomorphism $\phi_* : \mathbf{G}_{\mathrm{m}} \to \mathbf{G}_{\mathrm{m}}$. One can assume that $\phi_*(q_1^j) = q_2^j$ for every $j \in \mathbf{Z}$. If Y_i is a finite topological cover of X_i, let

$$A_{Y_i} = \mathrm{Hom}(\pi_1^{\mathrm{temp}}(Y_i), \mathbf{Z}_p(1)) / \mathrm{Hom}(\pi_1^{\mathrm{top}}(Y_i), \mathbf{Z}_p(1)).$$

The isomorphism ϕ induces an isomorphism

$$\bar{\phi} : \varinjlim_{Y_2} A_{Y_2} \to \varinjlim_{Y_1} A_{Y_1}.$$

The previous step gives a map, which happens to be an isomorphism,

$$\psi_i : \{(m,(\alpha_j)) \in \mathbf{Z}_p \times \mathbf{Z}_p^{\mathbf{Z}}, (\alpha_j) \text{ periodic and } \sum_j \alpha_j = 0\} \to \varinjlim_{Y_i} A_{Y_i}.$$

Lemma 26 ([4, prop. 4.2])**.** *There exists $\beta \in \mathbf{Z}_p^\times$ such that $\bar{\phi}\psi_2 = \beta\psi_1$.*

Step 4. In step 2, one associated to $(m,(\alpha_j)) \in \mathbf{Z}_p \times \mathbf{Z}_p^{\mathbf{Z}}$ a differential form ω_i on Ω_i with logarithmic poles along $\{q_i^j, j \in \mathbf{Z}\}$.

Lemma 27 ([4, lem. 4.3])**.** *If $x \in \mathbf{G}_{\mathrm{m}}(\mathbf{C}_p)$, then* $\mathrm{mult}_x\,\omega_1 = \mathrm{mult}_{\phi_*(x)}\,\omega_2$.

This is mainly a consequence of the fact that the speed of convergence of the radius $r_{1,n}$ is encoded in $\mathrm{mult}_x\,\omega_1$ and that ϕ_* is compatible with decomposition groups.

If $P \in \mathbf{Z}_p[T]$ is a polynomial, there exists an explicit $(m,(\alpha_i)) \in \mathbf{Z}_p \times \mathbf{Z}_p^{\mathbf{Z}}$, not depending on q, such that $\omega(1) = P(q)dz$. The previous lemma applied to this $(m,(\alpha_i))$ and to $x=1$ gives us in particular that

$$P(q_1) = 0 \quad \text{if and only if} \quad P(q_2),$$

which gives the result.

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Berkovich spaces, polyhedra and model theory

Antoine Ducros

This talk was about some results which are proven in the recent paper [4].

Let X be an analytic space over a non-Archimedean, complete field k and let $\mathbf{f} = (f_1, \ldots, f_n)$ be a family of invertible functions on X. Let us recall two results, both of which were proven using de Jong's alterations – but these alterations could have been avoided for 1), which could have been deduced quite formally from a former result by Bieri and Groves (see [2]), based upon explicit computations on Newton polygons.

1) The compact set $|\mathbf{f}|(X)$ is a polytope of the $\mathbb{R}$-vector space $(\mathbb{R}_+^*)^n$ (we use the multiplicative notation) ; this was proven by Berkovich in [1] in the locally algebraic case and has been extended to the general case by the author in [3].

2) If moreover X is Hausdorff and n-dimensional, and if φ denotes the morphism $X \to \mathbb{G}_{m,k}^{n,\mathrm{an}}$ induced by $\mathbf{f}$, then the pre-image of the skeleton S_n of $\mathbb{G}_{m,k}^{n,\mathrm{an}}$ under φ has a piecewise-linear structure making $\varphi^{-1}(S_n) \to S_n$ a piecewise immersion ; this was proven by the author in [3]. (Remind that S_n is the set of semi-norms of the form $\sum a_I \mathbf{T}^I \mapsto \max |a_I| \mathbf{r}^I$).

In the aforementioned paper, we improve 1) and 2), and give a new proof of both of them. Our proofs are based upon the *model theory of algebraically closed, non-trivially valued fields* and don't involve de Jong's alterations.

Let us quickly explain what we mean by improving 1) and 2), and give some precisions about our proofs.

• Concerning 1), we also prove kind of a local avatar of it: if $x \in X$, there exists a compact analytic neighborhood U of x, such that for every compact analytic neighborhood V of x in X, the germs of polytopes $(|\mathbf{f}|(V), |\mathbf{f}|(x))$ and $(|\mathbf{f}|(U), |\mathbf{f}|(x))$ coincide (in other words, the image of a germ is a germ of polyhedron). Moreover if $x \notin \partial X$ the germ $(|\mathbf{f}|(U), |\mathbf{f}|(x))$ is equidimensional, and its dimension can be computed explicitely.

Our new proof of 1), as well as that of its local avatar, is based upon the quantifyer elimination for algebraically closed, non-trivially valued fields. Concerning the local avatar, we have also use Temkin's theory of the *reduction of analytic germs*, which was developed in [6].

• Concerning 2), we prove that the piecewise linear structure on $\varphi^{-1}(S_n)$ is canonical, that is, doesn't depend on the map we choose to write it as a pre-image of the skeleton; we thus answer a question which was asked to us by Temkin.

Moreover, we prove that the pre-image of the skeleton 'stabilizes after a finite, separable ground field extension', and that if $\varphi_1, \ldots, \varphi_m$ are finitely many morphisms from X to $\mathbb{G}_{m,k}^{n,\mathrm{an}}$, the union $\bigcup \varphi_j(S_n)$ also inherits a canonical piecewise-linear structure.

The main model-theoretic result we use instead of de Jong's desingularization to re-prove and extend 2) is a theorem of finiteness (or tameness) by Hrushovski and Loeser, which we will now describe very roughly.

In their recent paper [5] about the homotopy type of Berkovich spaces, Hrushovski and Loeser associate to every morphism $\mathscr{Y} \to \mathscr{X}$ of varieties over a valued field k (the valuation is arbitrary, *i.e.* non necessarily of height 1) a functor $\widehat{\mathscr{Y}/\mathscr{X}}$, from the category of non-trivially valued, algebraically closed extensions of k to that of sets; this functor should be thought of as a model-theoretic avatar of an 'algebra-analytic' object: a fibration whose base would the *algebraic* variety $\mathscr{X}$, and whose fibers would be the *analytification* of the fibers of $\mathscr{Y} \to \mathscr{X}$. They prove that this functor is pro-definable in general, and *definable* when $\mathscr{Y}$ is of relative dimension ≤ 1. This is this definability assertion (whose proof ultimately relates on Riemann-Roch's theorem for algebraic curves) which plays a key role in our proof.

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Model theoretic approaches to non-archimedean geometry

Ehud Hrushovski

(joint work with François Loeser)

This report concerns joint work with François Loeser, [5].

Let V be a variety over an $\mathbb{R}$-valued field F. For simplicity we will assume V is projective, though the result applies to semi-algebraic subsets of V as well.

We require a slight extension of the piecewise-linear category. Let $R_\infty = \mathbb{R} \cup \infty$. Consider subsets P of R_∞^n cut out by linear inequalities $\sum \alpha_i x_i \leq \sum b_i x_i + c$, with $\alpha_i, \beta_i \in \mathbb{Z}$, $c \in \mathbb{R}$, and equalities $x_i = \infty$. Subsets cut out purely by equalities of the latter form will be called *Zariski closed*. A *morphism* $P \to Q$ is a continuous map which is piecewise linear away from ∞, and also on the faces at ∞, in the obvious sense. We refer to this category as ∞PL, and to the objects as extended polyhedra. Compare e.g. [11]. (Actually we use R_∞^w rather than R_∞^n, where w is a finite set with Galois action, so as to represent the mondromy.)

A subset of V^{an} the form U^{an}, with U a closed subvariety of V, is also called *Zariski closed*

Theorem 28. [5] *There exists a canonical directed system P_i of compact extended polyhedra, with ∞PL embedding maps $\phi_{ij} : P_i \to P_j$, retractions $\pi_{ij} : I \times P_j \to P_i$, and homotopies $\alpha_{ij} : P_j \times I \to P_j$ from Id_{P_j} to π_{ij} making P_i an ∞PL strong deformation retract of P_j. We have*

$$V^{an} \cong \varprojlim P_i$$

while the set $V^{an}_{\#}$ of Abhyankar elements is obtained as

$$V^{an}_{\#} = \varinjlim P_i$$

The isomorphisms $V^{an} \cong \varprojlim P_i$ and $V^{an}_{ab} = \varinjlim P_i$ respect a lot more structure. The extended polytope P_i comes together with an Abelian sheaf on the Zariski topology (in the above sense), obtained by pushing forward the sheaf of regular functions on V under the projection $V^{an} \to P_i$. Taken along with these sheaves, the P_i can recover the geometry of V^{an} completely. To begin with, a Zariski closed set is precisely a pullback of a Zariski closed subset of some P_i. Moreover, if Z is Zariski closed and h is a regular map on $V^{an} \setminus Z$, then val$h$ factors through some P_i.

Similar results were proved by many authors, beginning with Berkovich [2], at various levels of generality. See also [8] for a projective limit representation. Our method of obtaining the homotopies is different, however, and likely to have other applications: we work with a slightly different space, the *stable completion* $\widehat{V}$, that permits inductive constructions, and reduces many questions to relative dimension one. The stable completion can be defined over valued fields whose value group may be non-archimedean. We find definable strong deformation retractions to definable extended polyhedra, and as in Theorem 28 the subset $V_{\#}$ of Abhyankar elements is preserved. Whereas $\widehat{V}$ is pro-definable, $V_{\#}$ is a union (direct limit) of definable sets.

One of the consequences of (pro)-definability of $\widehat{V}$, along with the ability to use a non-archimedean value group, is is an automatic uniformity phenomenon. For instance, we obtain the following corollary for Berkovich spaces:

Proposition 29. *Let $f : X \to Y$ be a morphism of quasi-projective varieties, $X_b = f^{-1}(b)$. Then there are finitely many possibilities for the homotopy type of X^{an}_b, as b runs through $Y(F)$. In fact there exist strong deformation retracts P_b of X^{an}_b, whose homeomorphism type is one of finitely many possibilities, as b varies.*

In this talk I will describe the points of the stable completion (called stably dominated types), a valuative criterion for continuity of functions (or homotopies), and a relation to the usual Berkovich space.

0.1. **Imaginaries.** Let $\Gamma = K^*/\mathcal{O}^*$ be the value group of a valued field K with valuation ring $\mathcal{O}$. Such quotients are called *imaginary sorts*. The addition and ordering of Γ pull back to semi-algebraic subsets of K, hence are called *definable*.

It follows from Robinson's quantifier elimination that $+, <$ generate all definable relations on Γ. Thus any subset of Γ^n (or Γ^n_∞) is automatically piecewise linear (∞PL).

Already in [1], it is clear that the value group is treated as a separate sort; the base structure is allowed to have value group elements that are not the values of any field element of the structure.

The quotients $S_n = GL_n(K)/GL_n(\mathcal{O})$ can be viewed as higher-dimensional analogs of Γ. They are closely related to the spaces studied in [10]. Note that $S_n(\mathbb{Q}_p)$ is countable. On the other hand over a base consisting of the field $\mathbb{Q}_p$ and the value group $\mathbb{R}$, one sees essentially a real building interpolating between the p-adic lattices. Such buildings are often built 'by hand' but appear here naturally on their own.

S_n admits a covering T_n by Grassmanian varieties over the residue field, $T_n = GL_n(K)/Ker(GL_n(\mathcal{O}) \to GL_n(k))$. T_1 is the sort RV of [6], and is also equivalent to M. Temkin's graded residue sort [12]. Beyond T_1 we do not explicitly use the sorts T_n, as the topology is skewed towards Γ more than to k; we expect the T_n will be important in extensions of this work restoring the balance between them.

It is shown in [3] that any family of semi-algebraic subsets of K^m can be parametrized by a definable subset of $S_n \times T_n \times K^n$, for some n. It was in this work that the notion of a stably dominated type first arose.

We can view S_n as the family of linear norms on K^n, and this induces a natural topology on S_n, analogous to that topology on V^{an}. When H is a K-space isomorphic to K^n, it will be convenient to use the sort $L_H = M_n(K)/GL_n(\mathcal{O})$ of semi-lattices on H. It can be constructed S_n and S_m, $m < n$, fibered over Grassmanians of H. The elements of L_H can be viewed as linear semi-norms on H (where a seminorm l corresponds to a semi-lattice Λ if $\Lambda = \{h : l(h) \geq 0\}$.)

0.2. **Definition of $\widehat{V}$.** Let F_0 be a valued field; we no longer assume that it is valued in $\mathbb{R}$. Let V be a variety over F_0. We consider valued field extensions F, L, L' of F_0. Let $V_0^{an}(L)$ be the the set of pairs (L', b) with $L' = L(b)$ a valued field extension of L *with* $\Gamma(L) = \Gamma(L')$, $b \in V(L')$; up to the obvious notion of isomorphism over L.

Now for $F \geq F_0$ define $\widehat{V}(F)$ to be the set of functorial sections $L \to V_0^{an}(L)$, i.e. maps $L \mapsto p_L \in V_0^{an}(L))$ defined on all valued field extensions L of F, and compatible with all embeddings $L' \to L$ of valued fields over F. It can be shown that p is determined uniquely by p_F; hence $\widehat{V}(F)$ can be identified with a subset of $V_0^{an}(F)$. The topology and sheaf structures on $\widehat{V}$ are defined analogously to V^{an}.

In fact, if $p \in \widehat{V}(F)$, then p is an *F-definable type.* To explain this notion pass to an affine open subvariety U such that $p \in \widehat{U}(F)$. Fix an affine embedding of U; let H_d be the set of polynomials of degree at most a fixed integer d, viewed as functions on U. Then $\{h \in H_d : \mathrm{val} h(a) \geq 0\}$ is a definable $\mathcal{O}$-submodule H of regular functions on U. Definability implies that there exists $\Lambda_d = \Lambda_d(p) \in L_{H_d}$ such that for any $L \geq F$ and $H \in H(L)$, $h(a) \geq 0$ holds for a realization of p_L

iff $h \in \Lambda_d$. This defines a continuous map $\widehat{V} \to L_d$. It is a continuous map and indeed the linear topology on L_d described above is the quotient topology. Thus for affine V, $\widehat{V}$ embeds into the projective limit of the L_d.

It is easy to describe the image of $\widehat{V}$ in the projective limit (see below), but for fixed d the proof that the image of $\widehat{V}$ in L_d is definable, and that the induced topology is the quotient topology, is rather model theoretic; it would be interesting to see a direct geometric proof.

A third point of view is of Weierstrass domains. If W is a Weierstrass domain in an affinoid, with a unique element in the Shilov boundary, then this element is an Abhyankar type and sits in the stable completion; see [9] §4, [12]. In general a point of $\widehat{V}$ corresponds to a sequence W_d of Weierstrass domains in affine space, corresponding to the lattices Λ_d, such that at the limit the Shilov boundary reduces to a single point.

We omit here the description in terms of domination by the stable sorts, that gives stably dominated types their name; see [4], [5]. Note that a point of $\widehat{V}(F)$ is still a point of $\widehat{V}(F')$ for $F \leq F'$, as is the case for ordinary points of V; for V^{an}, the functoriality reverses direction.

0.3. **Valuative criterion.** In the same way that valuations provide a criterion for compactness in algebraic geometry, iterated valuations provide such a criterion for the stable completion. This should be compared to [7]. We consider algebraically closed valued fields K_2, K_1, K_0 with nontrivial places $K_2 \to K_1 \to K_0$. We obtain three valued fields K_j with residue field K_i $(2 \geq j > i \geq 0.)$ The corresponding value groups fit into an exact sequence $0 \to \Gamma_{10} \to \Gamma_{20} \to \Gamma_{21}$. The theory of (K_2, K_1, K_0) is called ACV^2F; we have $ACV^2F = ACVF \times_{ACF} ACVF$, a fiber product of theories over the two obvious interpretations of ACF in ACVF, as valued field and as residue field. Also, $ACV^2F = ACVF \times_{DOAG} DOAG^2$, where $DOAGTh(\Gamma, +, <)$ and $DOAG^2$ is the theory of a divisible ordered Abelian group with a distinguished convex subgroup.

Let V be a projective variety over the prime field, for simplicity. Then there are natural pro-definable maps

$$\widehat{V}_{10} \longleftarrow \widehat{V}_{20} \longrightarrow \widehat{V}_{21}$$

The right arrow relies on an isomorphism $\widehat{V_{210}} \to \widehat{V_{20}}$, where $\widehat{V_{210}}$ is the set of stably dominated types of V for the theory ACV^2F. (The left arrow depends in addition on $\widehat{\underline{V}(\mathcal{O}_{21})}_{210} = V_{210}$.)

We can now state a criterion for a definable map $f : Y \to V^{\#}$ to extend to a (unique) continuous pro-definable map $F : \widehat{Y} \to \widehat{V}$. (Where again we suppose for simplicity that Y is defined over the prime field.) Namely, f can be interpreted over (K_2, K_0) or over (K_2, K_1). The resulting maps $f_{20} : Y \to \widehat{V}_{20}$ and $f_{21} : Y \to \widehat{V_{21}}$ should commute with the above map $\widehat{V}_{20} \longrightarrow \widehat{V}_{21}$. Similarly f should be compatible with $\widehat{V}_{10} \longleftarrow \widehat{V}_{20}$.

0.4. **Comparison of $\widehat{V}$ with V^{an}.** In [5], functoriality and base change are used for the comparison. We describe here a different method, based on ideas of Poineau. Let F be a field valued in $\mathbb{R}$, and let V be an algebraic variety over a field F. By [5] we obtain a strong definable deformation retraction $H : \widehat{V} \to \Upsilon$, where Υ is definably ∞PL, and various good properties hold. The question is how to transpose this retraction to V^{an}. Since H is F-definable it is Galois invariant, and so we may assume that $F = F^{alg}$. For any base structure $F^c \supset F$, H restricts to a strong deformation retraction $\widehat{V}(F^c) \to \Upsilon(F^c)$. Now $\Upsilon(F^c)$ is ∞PL in the usual sense provided $\Gamma(F^c) = \mathbb{R}$; so it suffices to find a structure F^c such that the natural map $\widehat{V}(F^c) \to V^{an}$ is an isomorphism. This cannot be done if F^c is restricted to the field and value group sorts.

However, as soon as elements of S_n are allowed, there is a natural construction of such an F^c. Namely, let F^c agree with F in the field sort, with $\mathbb{R}$ in the value group sort; and let $\Lambda \in L_n(F^c)$ iff Λ is the intersection of elements $\Lambda_n \in L_n(F)$. We can also define F^c Galois theoretically: let F_{max} be a maximally immediate extension of F^{alg}; then in each sort, F^c is the fixed substructure of $Aut(F_{max}/F)$. It is clear that F^c is unique up to a unique isomorphism, as a structure extending F. Any element of V^{an}, viewed as a type over F, extends uniquely to a type over F^c. Now the following result, essentially from [9], clinches the comparison:

Theorem 30. *Let $F = F^{alg}$ be $\mathbb{R}$-valued. The natural map $\widehat{V}(F^c) \to V^{an}_F$ is an isomorphism.*

A model-theoretic proof would go as follows. *Injectivity* of the map $p \mapsto p|F^c$ is a completely general fact about stably dominated types ([4]), and one sees easily that $p|F \vdash p|F^c$ sot hat $p \mapsto p|F$ is also injective. To prove surjectivity, let q be an element of V^{an}, represented by a valued field extension $L = F(b)$, $b \in V(F)$. We may assume V is affine, and consider $H = H_d$ as above. Then $\{h \in H(F) : vh(b) \geq 0\}$ generates a lattice $\Lambda_d \in L_{H_d}$, with corresponding linear seminorm l_d on H_d We can functorially define p_L by the formulas: $vh(x) = l_d(h)$. The problem is to show consistency: that for any $h_1, \ldots, h_k \in H_d(L)$ there exists x with $vh_i(x) = l_d(h_i)$. Taking an ultrapower (F^*, b) of (F, b) in the sense of continuous logic, it is clear that $b \models p_{F^*}$. (In fact any L with value group $\mathbb{R}$ embeds in such an ultrapower, so p_L is consistent in this case.) It follows that p_L is consistent whenever F^* is an elementary extension of L. But by Robinson's quantifier elimination for ACVF, any $L' \geq F$ admits a common elementary extension with F^*; so $p_{L'}$ is consistent.

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Automorphisms of Drinfeld half-spaces over a finite field

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(joint work with Bertrand Rémy and Annette Werner)

For every integer $n \geqslant 1$, Drinfeld considered in [4] the p-adic analytic space Ω^{n+1} obtained by removing all rational hyperplanes from $\mathbf{P}^{n,\mathrm{an}}_{\mathbf{Q}_p}$. Since then, these spaces are of fundamental importance for understanding representation theory of $\mathrm{PGL}_{n+1}(\mathbf{Q}_p)$ or for realizing (part of) the local Langlands correspondence. In this setting, it was shown by Berkovich that every automorphism of Ω^{n+1} is induced by a projective linear transformation [2]. This result was generalized to products of Drinfeld half-spaces by Alon [1], who also pointed out and corrected a discrepancy in Berkovich's proof. Berkovich's strategy exploits a natural connexion between Ω^{n+1} and the Bruhat-Tits building of the group $\mathrm{PGL}_{n+1}(\mathbf{Q}_p)$.

In his theory of period domains, Rapoport [5] studied both generalizations of Drinfeld spaces over a p-adic field and similar spaces over a finite field. In the latter case, one obtains algebraic varieties defined as suitable open subsets of flag varieties. In particular, if V is a finite dimensional vector space of a finite field k, then the Drinfeld half space $\Omega(\mathrm{V})$ is simply the complement of all (rational) hyperplanes in $\mathbf{P}(\mathrm{V}) = \mathrm{Proj}\ \mathrm{Sym}^{\bullet}\mathrm{V}$. In [3], Dat, Orlik and Rapoport asked if a statement analogous to Berkovich's theorem holds for $\Omega(\mathrm{V})$.

Theorem 31. *Let* V *be a finite dimensional vector space over a finite field* k.

(i) *The restriction map*

$$\mathrm{PGL}(\mathrm{V}) = \mathrm{Aut}_k\big(\mathbf{P}(\mathrm{V})\big) \to \mathrm{Aut}_k\big(\Omega(\mathrm{V})\big), \quad \varphi \mapsto \varphi_{|\Omega(\mathrm{V})}$$

is an isomorphism. Equivalently, every k-automorphism of $\Omega(\mathrm{V})$ *extends to a k-automorphism of* $\mathbf{P}(\mathrm{V})$.

(ii) *For every field extension* K/k *the natural map*

$$\mathrm{PGL}(\mathrm{V}) \longrightarrow \mathrm{Aut}_{\mathrm{K}}\big(\Omega(\mathrm{V})_{\mathrm{K}}\big)$$

is an isomorphism. Equivalently, every K*-automorphism of* $\Omega(\mathrm{V})_{\mathrm{K}}$ *comes by base change from a k-automorphism of* $\mathbf{P}(\mathrm{V})$.

One possible proof would be to adapt Berkovich's and Alon's arguments. However, we adopt a slightly different, and maybe more natural, viewpoint. Thereby, we want to highlight that the true content of this theorem is about *extension* of automorphisms, and that it has in fact very little to do with buildings. This problem is however naturally connected to non-Archimedean analytic geometry, even if it belongs to classical (birational) algebraic geometry.

We consider the blow-up $\pi : \mathrm{X} \to \mathbf{P}(\mathrm{V})$ of $\mathbf{P}(\mathrm{V})$ along the full hyperplane arrangement. The scheme X is projective and smooth over k. It contains $\Omega(\mathrm{V})$ as an open dense subscheme and the complement $\mathrm{D} = \mathrm{X} - \Omega(\mathrm{V})$ is a simple normal crossing divisor whose irreducible components (resp. strata) are naturally indexed by linear subspaces (resp. flags of linear subspaces) in $\mathbf{P}(\mathrm{V})$:

$$\mathrm{D} = \bigcup_{\mathrm{L}} \mathrm{E}_{\mathrm{L}}.$$

First step — Using Berkovich spaces other k endowed with the trivial valuation, one shows that any k-automorphism φ of $\Omega(\mathrm{V})$ extends to a k-automorphism of X. Since D is a (simple) normal crossing divisor on X, the analytic space $\Omega(\mathrm{V})^{\mathrm{an}}$ retracts to a closed subset $\mathfrak{S}(\mathrm{V})$ canonically equipped with the structure of a cone complex [6]. Moreover, in this particular situation, the following additional property holds:

Lemma 32. *The map*

$$\iota : \mathfrak{S}(\mathrm{V}) \to \mathrm{Hom}_{\mathbf{Ab}}\Big(\mathcal{O}\big(\Omega(\mathrm{V})\big)^{\times}, \mathbf{R}_{>0}\Big),\ \mathrm{x} \mapsto (\mathrm{f} \mapsto |\mathrm{f}(\mathrm{x})|)$$

is a closed embedding such that (the images of) distinct cones span distinct linear spaces.

From this, it is easy to deduce that φ induces an automorphism $\mathfrak{S}(\mathrm{V})$ preserving the conical structure, and then that φ extends to an automorphism of X.

Second step — In order to descend φ to a k-automorphism of $\mathbf{P}(\mathrm{V})$, it is enough to check that the following permutation $\widehat{\varphi}$ of the set of (rational) linear subspaces of $\mathbf{P}(\mathrm{V})$:

$$\varphi(\mathrm{E}_{\mathrm{L}}) = \mathrm{E}_{\widehat{\varphi}(\mathrm{L})}$$

preserves the subset of hyperplanes.

One can first check that a hyperplane can only go to a hyperplane or a point by computing $\mathrm{rk}\ \mathrm{CH}^1(\mathrm{E}_{\mathrm{L}})$ and then use induction on dim V to exclude the second possibility, thereby finishing the proof of the theorem. Another way to argue, suggested to us by Carlo Gasbarri, is to observe that the canonical divisor on X defines an element of $\mathrm{CH}^1(\mathrm{X})$ fixed under the natural action of φ which essentially "remembers" the codimension of blown-up linear subspaces.

It is natural wonder whether it is possible to prove this theorem without considering Berkovich spaces. A more interesting problem is to find other "natural" toroidal compactifications satisfying the specific property above (Lemma).

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The Kontsevich-Soibelman skeleton of a degeneration of Calabi-Yau varieties

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(joint work with Mircea Mustaţă)

In [1, §6.6], Kontsevich and Soibelman associate a skeleton to each smooth and projective family of varieties over a punctured disc in the complex plane endowed with a relative differential form of maximal degree. This skeleton is a subset of the non-archimedean generic fiber of the family, which is a smooth and proper analytic space over the field of complex Laurent series. Its construction is motivated by the study of Mirror Symmetry. Kontsevich and Soibelman explain how their skeleton can be computed by extending the family to a projective family over the disc such that the total space is a smooth complex analytic space and the fiber over the center of the disc is a divisor with strict normal crossings. The proof of this result is based on the Weak Factorization Theorem.

Our project consists of several steps. First, we develop and generalize the construction of the Kontsevich-Soibelman skeleton. More precisely, we define a skeleton $\mathrm{Sk}(X,\omega) \subset X^{\mathrm{an}}$ for every complete discretely valued field K, every connected smooth proper K-variety X and every non-zero differential form of maximal degree ω on X, as follows. Let $\mathfrak{X}$ be a (not necessarily proper) regular R-model of X. The differential form ω defines a rational section of the relative canonical line bundle $K_{\mathfrak{X}/R}$ and therefore a divisor $\mathrm{div}_{\mathfrak{X}}(\omega)$ on $\mathfrak{X}$. To each irreducible component E of $\mathfrak{X}_k$, we associate a couple of numerical data $(N,\nu) \in \mathbb{Z}_{>0} \times \mathbb{Z}$ where N is the multiplicity of E in $\mathfrak{X}_k$ and $\nu - 1$ is the multiplicity of E in $\mathrm{div}_{\mathfrak{X}}(\omega)$. We call the quotient μ/N the weight of E with respect to ω. In this way, we obtain a function

weight_ω on the set of divisorial points of X^{an}. The infimum of this function in $\mathbb{R}\cup\{-\infty\}$ is called the weight of X with respect to ω and denoted by $\mathrm{weight}_\omega(X)$. This definition is reminiscent of the definition of the log-canonical threshold in birational geometry.

A divisorial point x on X^{an} is called ω-essential if $\mathrm{weight}_\omega(x) = \mathrm{weight}_\omega(X)$. We define the skeleton $\mathrm{Sk}(X,\omega)$ of X with respect to ω as the closure of the set of ω-essential divisorial points in the set of birational points of X^{an} (points whose image in X is the generic point; they correspond to height one valuations on the function field $K(X)$ that extend the discrete valuation on K). This skeleton is a birational invariant of the pair (X,ω) and it coincides with the skeleton of Kontsevich-Soibelman if $K = \mathbb{C}((t))$ and X and ω are defined over the field $\mathbb{C}\{t,t^{-1}\}$ of germs of meromorphic functions at the origin of the complex plane. The subset $\mathrm{Sk}(X,\omega)$ of X^{an} is invariant under multiplication of ω by elements in $K^\times$. In particular, if X has geometric genus one, then $\mathrm{Sk}(X,\omega)$ does not depend on the choice of ω, and we'll denote it simply by $\mathrm{Sk}(X)$.

From the definition of the skeleton, it is not even clear whether it is non-empty. We can provide an explicit description of $\mathrm{Sk}(X,\omega)$ in terms of a proper regular model with normal crossings, in arbitrary characteristic and without using Weak Factorization. This generalizes the theorem of Kontsevich-Soibelman mentioned above. Let R be the valuation ring of K and k its residue field. Assume that X has a regular proper R-model $\mathfrak{X}$ such that the special fiber

$$\mathfrak{X}_k = \sum_{i\in I} N_i E_i$$

is a divisor with strict normal crossings. The existence of such a model is known when k has characteristic zero or X is a curve. For each i in I we denote by (N_i,ν_i) the couple of numerical data associated to E_i and ω.

The simplicial space $\mathrm{Sk}(\mathfrak{X})$ associated to the reduced special fiber $(\mathfrak{X}_k)_{\mathrm{red}}$ can be canonically embedded into X^{an}. The vertices of this simplicial space correspond bijectively to the irreducible components of $\mathfrak{X}_k$, and their images in X^{an} are the associated divisorial valuations on the function field of X. These divisorial points are connected in $\mathrm{Sk}(\mathfrak{X})$ by means of families of monomial valuations associated to the divisor $\mathfrak{X}_k$. The faces of $\mathrm{Sk}(\mathfrak{X})$ correspond bijectively to the generic points ξ of intersections of irreducible components of $\mathfrak{X}_k$. Such a point ξ is called ω-essential if the following conditions are satisfied:

- if E is an irreducible component of $\mathfrak{X}_k$ passing through ξ and (N,ν) is the associated couple of numerical data, then $\nu/N = \mathrm{weight}_\omega(X)$,
- the point ξ is not contained in the closure of the locus of zeroes of ω on X.

Theorem 33. *The weight of X with respect to ω is given by*

$$\mathrm{weight}_\omega(X) = \min\{\mu_i/N_i \,|\, i \in I\}.$$

Moreover, the skeleton $\mathrm{Sk}(X,\omega)$ *is the union of the closed faces of* $\mathrm{Sk}(\mathfrak{X})$ *corresponding to* ω*-essential points* ξ*. In particular, it is a non-empty compact subspace of* X^{an}*.*

Our main result is the following theorem.

Connectedness Theorem. *Let* k *be a field of characteristic zero, and let* X *be a smooth, proper, geometrically connected* $k((t))$*-variety of geometric genus one. Then the skeleton* $\mathrm{Sk}(X)$ *is connected.*

This result can be viewed as an analog of the Shokurov-Kollár Connectedness Theorem in birational geometry [2, 7.4] and our proof follows similar lines, with some important modifications: while the proof of the Shokurov-Kollár Connectedness Theorem is based on a relative Kawamata-Viehweg vanishing theorem, the key ingredient in our proof is a variant of Kollár's Torsion Freeness Theorem [2, 2.17.4] for schemes over $k[[t]]$. We deduce this variant from Kollár's theorem by means of Greenberg Approximation and Grothendieck's comparison theorem for coherent cohomology of formal schemes.

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