Mathematisches Forschungsinstitut Oberwolfach

Report No. 55/2012

DOI: 10.4171/OWR/2012/55

Mini-Workshop: Efficient and Robust Approximation of the Helmholtz Equation

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25 November – 1 December 2012

Abstract. The accurate and efficient treatment of wave propogation phenomena is still a challenging problem. A prototypical equation is the Helmholtz equation at high wavenumbers. For this equation, Babuška & Sauter showed in 2000 in their seminal SIAM Review paper that standard discretizations must fail in the sense that the ratio of true error and best approximation error has to grow with the frequency. This has spurred the development of alternative, non-standard discretization techniques. This workshop focused on evaluating and comparing these different approaches also with a view to their applicability to more general wave propagation problems.

Mathematics Subject Classification (2000): 65N30,65N12,65N38.

Introduction by the Organisers

The non-standard methods that could overcome the limitations of standard finite difference or finite element methods in the high-frequency regime include

- high order methods;
- Galerkin methods with special ansatz functions (e.g. plane waves);
- Petrov-Galerkin methods with wave-dependent test functions;
- boundary elements with a suitable compression of the integral operator.

Representatives of these methods were discussed at the workshop. In the time-harmonic setting, several talks covered questions of stability, both on the continuous level and of numerical schemes, with the particular emphasis of making the wavenumber-dependence explicit. The topic of *a posteriori* error estimating and,

more generally, adaptivity for this problem class was addressed in several contributions. The iterative solution of the large systems of equations for Helmholtz and Maxwell systems is particularly delicate and therefore the topic of two talks. Since the underlying physical problem is often posed in unbounded domains, several presentations were devoted to questions of coupling different discretizations, to boundary element methods, and to "infinite elements". Also, recent progress on some time-domain formulations such as convolution quadrature techniques was presented.

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Abstracts

FEM/BEM coupling for acoustics in the time-domain

Lehel Banjai
(joint work with V. Gruhne, Ch. Lubich, F. J. Sayas)

In this talk we have discussed the numerical simulation of acoustic wave propagation with localized inhomogeneities. A standard Galerkin finite element method (FEM) in space and leapfrog time-stepping in time was applied on a finite spatial domain containing the inhomogeneities. The equations in the exterior computational domain were dealt with by a time-domain boundary integral (TDBIE) formulation discretized by the Galerkin boundary element method (BEM) in space and convolution quadrature [1] in time.

We have given a stability analysis of the proposed method, starting with the proof of a positivity preservation property of convolution quadrature as a consequence of a variant of the Herglotz theorem. Combining this result with standard energy analysis of the leapfrog discretization of interior equations gives the stability of the method. This is a work in progress and two formulations have been described. For one a complete stability and convergence analysis is available, whereas for the other the analysis is not complete but we have presented some numerical results showing stability in practice. Related work, but for Galerkin discretization of TDBIE, can be found in [2].

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DPG Method as a Non-conforming Discretization Scheme

Leszek Demkowicz
(joint work with J. Gopalakrishnan, M. Melenk, I. Muga, D. Pardo)

This is a "work in progress" report on our attempt to understand the pollution-free behavior of the Discontinuous Petrov Galerkin (DPG) method for linear acoustics equations. Let $\Omega \subset \mathbb{R}^2$ be a domain satisfying Melenk's regularity assumptions [1]. We wish to solve the model linear acoustics problem:

$$\left\{\begin{array}{rll} i\omega u + \nabla p &= g & \text{in } \Omega \\ i\omega p + \operatorname{div} u &= f & \text{in } \Omega \\ p - u_n &= 0 & \text{on } \Gamma = \partial\Omega\,. \end{array}\right. \tag{1}$$

Here p is the pressure, u acoustic velocity, ω the angular velocity and $u_n = u \cdot n$ stands for the normal component of velocity with n denoting the outward normal unit vector. The DPG method is based on a mesh-dependent ultraweak variational formulation:

$$
(2) \qquad \begin{cases} (u,p) \in (L^2(\Omega))^2 \times L^2(\Omega),\ (\hat{u}_n, \hat{p}) \in H^{-1/2}(\Gamma_h^0) \times H^{1/2}(\Gamma_h^0) \\ \underbrace{((u,p), A_h(v,q)) + \langle \hat{u}_n, q\rangle + \langle \hat{p}, v_n\rangle}_{=:b((u,p,\hat{u}_n,\hat{p}),(v,q))} = (g,v) + (f,q) \\ \qquad \forall (v,q) \in H(\mathrm{div}, \Omega_h) \times H^1(\Omega_h),\ q = -v_n \text{ on } \Gamma\,. \end{cases}
$$

Here $A(u,p)$ denotes the first order differential operator corresponding to system (1), index h in $A_h(v,q)$ indicates that the operator is applied element-wise to discontinuous test functions coming from the broken test space $H(\mathrm{div}, \Omega_h) \times H^1(\Omega_h)$, $(\cdot,\cdot)$ stands for $L^2(\Omega)$ inner product and $\langle \cdot, \cdot \rangle$ for the conjugated duality pairing on the mesh skeleton - $H^{-1/2}(\Gamma_h) \times H^{1/2}(\Gamma_h)$ or $H^{1/2}(\Gamma_h) \times H^{-1/2}(\Gamma_h)$.

It has been proved in [4] that, if we equip the traces and fluxes with the minimum energy extension norm corresponding to the graph norm,

$$
(3) \qquad \|(\hat{u}_n, \hat{p})\|_E^2 := \inf\{\|u,p\|^2 + \|A(u,p)\|^2 \,:\, (\hat{u}_n, \hat{p}) = (u,p)|_{\Gamma_h}, u_n = p \text{ on } \Gamma\}
$$

and use the (broken) graph norm for the test space, then the inf-sup constant corresponding to the sesquilinear form $b((u,p,\hat{u}_n,\hat{p}),(v,q))$ is independent of both the mesh and frequency ω. This implies that the DPG method with optimal test functions is *uniformly stable* in frequency ω.

This, unfortunately, does not make it *pollution free* in the engineering sense. A pollution free method is expected to deliver the same relative L^2-error as long as we use the same resolution of wavelength - a fixed number of elements of a specific, fixed order, per wavelength. The stability estimate for the DPG method:

$$
(4) \qquad \begin{aligned} &\|(u-u_h, p-p_h)\| + \|(\hat{u}_n - \hat{u}_{n,h}, \hat{p} - \hat{p}_h)\|_E \\ &\quad \le C \inf_{(w_h, r_h, \hat{w}_n, \hat{r})} \{\|(u-w_h, p-r_h)\| + \|(\hat{u}_n - \hat{w}_{n,h}, \hat{p} - \hat{r}_h)\|_E\} \end{aligned}
$$

implies such a result in 1D only (proved in [3] in a different way). In 1D, traces and fluxes are just numbers so the best approximation of those is always zero and the best approximation error on the right-hand side of (4) reduces to the L^2 best approximation error only. And L^2- projection error is indeed pollution free. In the multidimensional case, however, the minimum energy extension norm hides derivatives (H^1- and $H(\mathrm{div})$-like norms) which do exhibit pollution. This can be intuitively seen from differentiating a plane wave solution $e^{i\omega x}$. The best approximation error in H^1-norm would always be of order $\omega(\omega h)^r$, always one power of ω too much.

The existing convergence analysis indicates thus that, in multidimensions, the DPG method exhibits the same pollution as classical Bubnov-Galerkin scheme.

This does not explain the almost pollution free behavior of the DPG method observed numerically [4]. One perhaps should be more careful with this statement. In a typical 2D experiment with plane waves, ff we compare second order DPG method (four quadratic elements per wavelength to approximate fluxes and traces)

with any other second order method, the DPG method is practically pollution free (experiments with up to 250 wavelengths per domain). On a closer look though, if we count degrees of freedom, the four quadratic DPG elements per wavelength should be compared with four *quintic* standard finite elements. And, with four quintic elements per wave length, the standard Bubnov-Galerkin method delivers pollution-free results as well.

So, are there some higher order, "superconvergence" effects working behind the scenes for the quadratic DPG method ?

A global interpretation of the DPG method.. In practice, the DPG method requires an approximate inverse of the Riesz operator, In *practical DPG method* approximate optimal test functions are obtained by using Bubnov-Galerkin and locally enriched spaces of order $p + \Delta p$ where p is the order of trial shape functions. In practically all computed examples, we have used $\Delta p = 2$.

It was already noticed in [2] that the practical DPG method admits two interpretations. The first one is based on the DPG ultraweak formulation with discontinuous test functions, the second one is based on the ultraweak formulation with *globally conforming test functions*,

$$\left\{\begin{array}{l} (u,p) \in (L^2(\Omega))^2 \times L^2(\Omega) \\ \underbrace{((u,p), A(v,q))}_{=:b((u,p),(v,q))} = (g,v) + (f,q) \\ \forall (v,q) \in H(\mathrm{div},\Omega) \times H^1(\Omega),\ q = -v_n \text{ on } \Gamma \end{array}\right. \tag{5}$$

where the *global optimal test functions* which now solve the global problem,

$$\left\{\begin{array}{l} v \in H(\mathrm{div},\Omega), q \in H^1(\Omega), q = -v_n \text{ on } \Gamma \\ (A(v,q), A(\delta v, \delta q)) + \alpha((v,q),(\delta v,\delta q)) = b((u,p),(\delta v,\delta q)) \\ \qquad\qquad \forall \delta v \in H(\mathrm{div},\Omega), \delta q \in H^1(\Omega), \delta q = -\delta v_n \text{ on } \Gamma\,, \end{array}\right. \tag{6}$$

are approximated with non-conforming ("weakly conforming") Bubnov-Galerkin method where the global continuity of q and normal component v_n have been replaced with L^2-orthogonality of jumps to polynomial spaces used for the discretization of traces and fluxes,

$$\begin{array}{ll} \langle \hat{u}_n, q\rangle = \sum_e \int_e \hat{u}_n [q] = 0 & \forall \hat{u}_n \\ \langle \hat{p}, v_n\rangle = \sum_e \int_e \hat{p} [v_n] = 0 & \forall \hat{p}\,. \end{array} \tag{7}$$

Loosely speaking, if in the first interpretation, we first localize optimal test functions and then approximate them, in the second interpretation, we first approximate and then localize. We emphasize that we are speaking only about the interpretation of the method and not practical computations that are always done locally. In particular, the global interpretation explains why decreasing coefficient α in the test norm produces better results. Setting $\alpha = 0$ in (6), we obtain a non-conforming least squares method for the globally optimal test functions. Notice that in the limiting case $\alpha = 0$, the optimal test functions solve the adjoint

problem which, except for the sign change in the impedance boundary condition, is the very same problem we are trying to solve. One might say thus that working behind the scenes for the DPG method is a non-conforming least squares method. If course, if we were able to solve for the global optimal test functions exactly, the DPG method would reduce to L^2-projection.

But we are not. Restricting ourselves to the theoretical case $\alpha = 0$ (in practice, we must use $\alpha > 0$ to be able to localize the computation of test functions), a Strang's type reasoning leads to the estimate:

(8)

$$\|(u-u_h, p-p_h)\| \leq (1+\frac{1}{\gamma_h}) \inf_{(w_h,r_h)} \|(u-w_h, p-r_h)\| + \frac{1}{\gamma_h} \sup_{(v_h,q_h)} \frac{|((u,p), A_h(v,q))|}{\|A_h(v_h,q_h)\|}$$

Here, γ_h is the discrete inf-sup constant resulting from the approximation of optimal test functions and an approximation of the exact inf-sup constant $\gamma = 1$. The first term represents the best approximation error, the second one a consistency error resulting from the weak enforcement of the interelement continuity. The best approximation error, measured in the L^2 norm, is pollution free.

The challenge now is to estimate γ_h and, first of all, the consistency error. We hope to be able to demonstrate that the consistency error is of order $\omega(\omega h)^r$ where the exponent r is higher that the polynomial order used for traces and fluxes. This would explain the higher order effect in the performance of the DPG method.

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A posteriori error estimation for highly indefinite Helmholtz problems

Willy Dörfler

(joint work with S. Sauter)

One-dimensional Helmholtz equation [1]. We consider the boundary value problem

$$\begin{aligned} L(\gamma)u := -u'' + \gamma u = f \qquad & \text{in } \Omega := (0,1), \\ u = 0 \qquad & \text{in } \{0,1\} \end{aligned}$$

for $u : \Omega \to \mathbb{R}$, where $f \in L^2(\Omega)$ and $\gamma \in L^\infty(\Omega)$, without a condition corresponding its sign. It is assumed that the problem in its weak form formulation

is well-posed in the norm $\|v\|_{\mathcal{H}} := \|v'\|_{L^2(\Omega)} + \|\kappa_* v\|_{L^2(\Omega)}$, with $\kappa := \sqrt{|\gamma|}$ and $\kappa_* := \max\{\kappa, \pi\}$.

This problem is solved on the following approximation space: Choose a piecewise constant approximation $\overline{\gamma}$ to γ with respect to the underlying mesh. The local basis is chosen to be the fundamental system of the operator $L(\overline{\gamma})$. This approximation space can be enhanced by the local "bubbles" $L(\overline{\gamma})\Psi = 1$. Such functions may be highly oscillatory for locally negative γ and can be defined only if $\sqrt{-\gamma}h \neq \pi$ is valid, h the local cell width. The two resulting finite element spaces allow estimates of first and second order in h.

We also provide a posteriori error estimates that allow to compute the solution based on a posteriori controlled mesh refining techniques.

Two-dimensional Helmholtz equation [2]. Let D be a compact domain in $\mathbb{R}^2$ with Lipschitz boundary (containing 0) and B_R be a large ball around zero. The Helmholtz problem on the complement D^c of D with Sommerfeld's radiation condition is approximated by a Helmholtz problem on $\Omega := B_R \setminus \overline{D}$ with an approximated boundary condition. The problem for $u : \Omega \to \mathbb{R}$ reads

$$\begin{aligned} -\Delta u - k^2 u &= f && \text{in } \Omega, \\ u &= g && \text{on } \partial D, \\ \partial_n u &= T_k u && \text{on } \partial B_R. \end{aligned}$$

Here, f and g are given data functions, n is the exterior normal with respect to Ω and k may be a spatially varying function that is however constant and larger 1 outside a ball $B_r \subset B_R$. T_k may be either $u \mapsto \mathrm{i}ku$ or the Dirichlet-to-Neumann operator for the exterior domain $\mathbb{R}^2 \setminus B_R$.

It has been proved in [3] [4] that the weak formulation satisfies an inf-sup-condition and that optimal error estimates hold that only depend on a adjoint approximability constant. Analysing the solutions's structure more closely, it was found that an hp-mesh that obeys $kh/p < 1$ with the moderate dependence $p \sim \log(k)$ leads to a uniform error estimate.

A result of the present paper is, that these conditions will be sufficient to prove that the residual a posteriori error estimator provides a uniform (in k) upper bound for the error in the energy norm. Our analysis also allows variable k.

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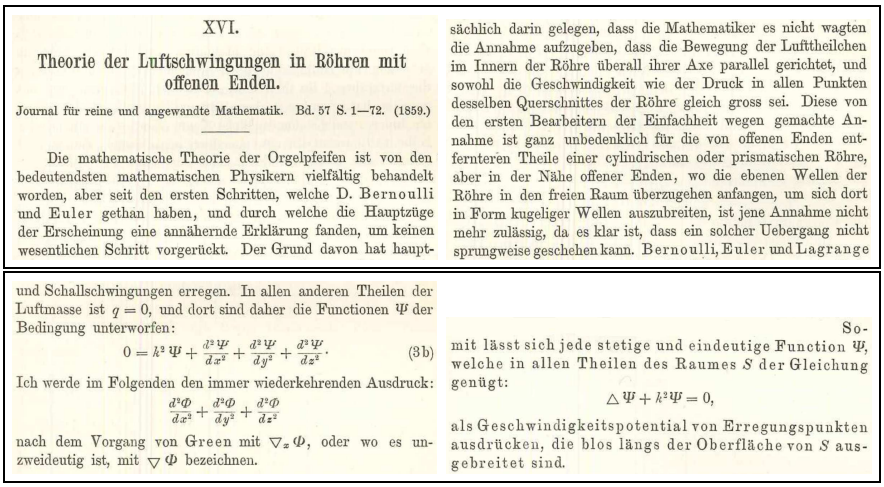
Iterative Methods for Helmholtz and Maxwell Equations

Martin Jakob Gander

There is a certain confusion about which equation is really called the Helmholtz equation, since in the literature on meteorology and climate simulation, also the similar looking equation with the opposite sign on the zeroth order term is called Helmholtz equation, due to early publications using this terminology [2, 3]. Even standard textbooks have adopted this terminology, see for example [4] (sometimes also the eigenvalue problem is called the Helmholtz equation [5]). This summer during a conference, I was asked if maybe Helmholtz himself had already considered both equations, and so looked in the collected works of Helmholtz [6]. The contributions of Helmholtz to advancing science are vast, he worked on hydrodynamics, acoustics (physical and physiological), electrodynamics, galvanism (the contraction of muscles stimulated by electric current), optics (physical and physiological), and even psychology. In a beautiful paper about the understanding of organ pipes [7], see Figure 1, I then found *the Helmholtz equation*. Helmholtz describes in his paper the problem of the open end of the pipe, and using a domain decomposition approach, he connects the outer spherical solution to the inner one in the tube, in order to determine an appropriate boundary condition, a problem that had not been satisfactorily addressed before, as described at the beginning of the paper, see Figure 1.



XVI.

Theorie der Luftschwingungen in Röhren mit offenen Enden.

Journal für reine und angewandte Mathematik. Bd. 57 S. 1—72. (1859.)

Die mathematische Theorie der Orgelpfeifen ist von den bedeutendsten mathematischen Physikern vielfältig behandelt worden, aber seit den ersten Schritten, welche D. Bernoulli und Euler gethan haben, und durch welche die Hauptzüge der Erscheinung eine annähernde Erklärung fanden, um keinen wesentlichen Schritt vorgerückt. Der Grund davon hat hauptsächlich darin gelegen, dass die Mathematiker es nicht wagten die Annahme aufzugeben, dass die Bewegung der Lufttheilchen im Innern der Röhre überall ihrer Axe parallel gerichtet, und sowohl die Geschwindigkeit wie der Druck in allen Punkten desselben Querschnittes der Röhre gleich gross sei. Diese von den ersten Bearbeitern der Einfachheit wegen gemachte Annahme ist ganz unbedenklich für die von offenen Enden entfernteren Theile einer cylindrischen oder prismatischen Röhre, aber in der Nähe offener Enden, wo die ebenen Wellen der Röhre in den freien Raum überzugehen anfangen, um sich dort in Form kugeliger Wellen auszubreiten, ist jene Annahme nicht mehr zulässig, da es klar ist, dass ein solcher Uebergang nicht sprungweise geschehen kann. Bernoulli, Euler und Lagrange

und Schallschwingungen erregen. In allen anderen Theilen der Luftmasse ist $q = 0$, und dort sind daher die Functionen Ψ der Bedingung unterworfen:

$$0 = k^2\Psi + \frac{d^2\Psi}{dx^2} + \frac{d^2\Psi}{dy^2} + \frac{d^2\Psi}{dz^2}. \qquad (3\,b)$$

Ich werde im Folgenden den immer wiederkehrenden Ausdruck:

$$\frac{d^2\Phi}{dx^2} + \frac{d^2\Phi}{dy^2} + \frac{d^2\Phi}{dz^2}$$

nach dem Vorgang von Green mit $\nabla_x \Phi$, oder wo es unzweideutig ist, mit $\nabla\Phi$ bezeichnen.

So-

mit lässt sich jede stetige und eindeutige Function Ψ, welche in allen Theilen des Raumes S der Gleichung genügt:

$$\triangle\Psi + k^2\Psi = 0,$$

als Geschwindigkeitspotential von Erregungspunkten ausdrücken, die blos längs der Oberfläche von S ausgebreitet sind.

Figure 1. Copy of the beginning of the seminal publication of Helmholtz from [6], and an unusual shorthand notation for the Laplacian (the gradient symbol ∇), according to Helmholtz due to Green; only in one place later, Helmholtz uses the now common symbol Δ for the Laplacian (shown on the right).

The numerical approximation of solutions of the Helmholtz equation poses two main difficulties: the first one is related to the approximation problem. Usually, in order to represent a signal accurately, about 10 points per wavelength are sufficient. This is also true for solutions of the Helmholtz equation, but unfortunately, when one discretizes the equation itself on a grid with about 10 points per wavelength, the solution one obtains can be very inaccurate, not because there are not enough points to accurately represent it, but because the discretized operator gives a solution with a substantial phase error. This is the so called pollution effect, see for example [8] and references therein.

The second fundamental difficulty is that iterative methods for the solution of linear systems have historically been derived for discretizations of diffusive problems, especially Laplace's equation, and all the intuition and analysis that went into the development of these methods used fundamental properties of the discretized Laplace equation. Unfortunately, all these intuitions are incorrect for the Helmholtz equation: there is no maximum principle, classical iterative methods are not smoothers for the Helmholtz equation, there is no minimization principle. In [1] one can find detailed explanations why Krylov methods, ILU preconditioners, multigrid and classical domain decomposition methods fail when used for the Helmholtz equation. Only specialized methods for the Helmholtz equation should be used, and in particular a new class of domain decomposition methods, called optimized Schwarz methods, is quite effective [9, 10]. The time harmonic Maxwell's equations present the same two difficulties as the Helmholtz equation, and optimized Schwarz methods have been developed for them, see e.g. [11].

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On shifted Laplace and related preconditioners for finite element approximations of the Helmholtz equation

Ivan G. Graham
(joint work with P. Childs, M. Gander, E. Spence and D. Shanks)

As a model problem for high-frequency wave scattering, we study the boundary value problem

$$\left\{ \begin{array}{rl} -(\Delta + k^2)u &= f \quad \text{in } \Omega, \\ \frac{\partial u}{\partial n} - \mathrm{i}ku &= g \quad \text{on } \partial\Omega, \end{array} \right. \tag{1}$$

where Ω is a bounded domain in $\mathbb{R}^d$ with boundary Γ. Finite element approximations of this problem for high wavenumber k are notoriously hard to solve.

Although they are known to suffer from the pollution effect, low order finite element and finite difference approximations of this problem are still highly utilsed in applications and the iterative solution of the resulting systems with a number of iterations which is independent (or close to being independent) of wave number in general situations remains a problem of great interest in both theory and in practice.

Quite a lot of recent research has focussed on preconditioning (1) using the discretization of the "shifted Laplace" problem

$$\left\{ \begin{array}{rl} -(\Delta + k^2 + \mathrm{i}\epsilon)u &= f \quad \text{in } \Omega, \\ \frac{\partial u}{\partial n} - \mathrm{i}\mu(k,\epsilon)u &= g \quad \text{on } \partial\Omega, \end{array} \right. \tag{2}$$

for some function $\mu(k, \epsilon)$. It is generally observed that if the "absorption" parameter $\epsilon > 0$, is taken large enough, the problem (2) becomes "easier" to solve, but if ϵ is not taken too large then (2) is a good preconditioner for (1).

The use of absorption in preconditioning has been studied in various contexts by several authors. Let A_ϵ denote the system matrix arising from the finite element approximation of (2). Then, for example, Erlangga, Vuik & Oosterlee [3] sought to precondition A_0 with a multigrid approximation of A_ϵ^{-1} and typically used $\epsilon \sim k^2$. Ernst and Gander [4] used Fourier analysis in 1D to show that ϵ needs to be taken to be $\mathcal{O}(k)$ for A_ϵ to be a good preconditioner for A_0 and needs to be taken to be $\mathcal{O}(k^2)$ for multigrid to have a convergence factor less than unity. However the model problem considered in [4] was a very simplified one. Engquist and Ying [5] essentially used $\epsilon = \mathcal{O}(k)$ to enhance the performance of their sweeping preconditioner. However none of these references give rigorous information on how ϵ should be chosen in general to obtain the best performance for preconditioning.

Since the corresponding system matrices are complex and non-Hermitian, general purpose iterative solvers such as GMRES or BiCGStab are required. The analysis of the performance of such solvers is in general hard, since the system matrices are often highly non-normal and so information about the spectrum is not generally enough to provide infomation on the number of iterations. Thus condition number estimates are of limited benefit. In this work we use GMRES and we analyse its convergence using the field of values theory [1].

Using the classical Morawetz multiplier theory from PDE analysis (described in the review article [2]), we estimate the norm of the solution of problem (2) in terms of the data, where the parameters k and ϵ appear explicitly in the estimates. These estimates hold when Ω is a star-shaped Lipschitz domain for small ϵ and for general Lipscitz domains when ϵ is large. The results also hold when Ω is the complement of a star-shaped scatterer in a large ball, with a Dirichlet condition on the scattering surface and either an exact far field condition or a first order absorbing approximation on the surface of the ball. Such stability results for the non-discretized problem (2) can then be used to infer bounds on the spectral norm of A_ϵ^{-1} which are explicit in k and ϵ and these in turn provide information about the field of values of $A_\epsilon^{-1}A_0$, under the assumption that the finite element meshes are shape-regular. This, combined with the field of values theory of GMRES alllows us to prove that the choice ϵ/k sufficiently small ensures $k-$independent convergence for GMRES when A_0 is preconditioned by A_ϵ^{-1}.

In the talk we also presented an analysis of optimised Schwarz domain decomposition methods (see, e.g. [6]) for (2), proving estimates on the rate of convergence explicitly in terms of k and ϵ. We also gave numerical illustrations of the solution of (1), preconditioned by approximations of (2), in 2D, for some constant and variable wavespeed problems.

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Sparse Tensor Edge Elements

Ralf Hiptmair
(joint work with C. Jerez-Hanckes and Ch. Schwab)

As a model problem we consider the variational formulation of the Maxwell cavity source problem [8, Ch. 5]: seek $\mathbf{u} \in V := \boldsymbol{H}_0(\mathbf{curl}, D)$, $D \subset \mathbb{R}^3$ bounded,

$$\underbrace{(\mathbf{curl}\,\mathbf{u}, \mathbf{curl}\,\mathbf{v})_{L^2(D)} - k^2\,(\mathbf{u},\mathbf{v})_{L^2(D)}}_{=:a(\mathbf{u},\mathbf{v})} = (\mathbf{f},\mathbf{v}) \quad \forall \mathbf{v} \in \boldsymbol{H}_0(\mathbf{curl}, D)\;, \tag{1}$$

where the wave number $k > 0$ is supposed to be different from a resonant frequency of D. The source function $\mathbf{f}$ is "stochastic" in the sense that it belongs to $L^2(\Omega, V')$ for a probability space $(\Omega, \mathcal{A}, \mathbb{P})$. Then also the solution $\mathbf{u}$ becomes a V-valued square integrable random variable: $\mathbf{u} \in L^2(\Omega, V)$. Its second moment $\mathcal{M}^2 u = \mathbb{E}(\mathbf{u} \otimes \mathbf{u}) \in L^1(V \otimes V)$, where $\mathbb{E}$ denotes the expectation, can be obtained as the solution of

$$(\mathsf{A} \otimes \mathsf{A}) w^{(2)} = \mathcal{M}^2 \mathbf{f}\;, \tag{2}$$

featuring the tensor product operator $\mathsf{A} \otimes \mathsf{A} : V \otimes V \to V' \otimes V'$ based on the operator $\mathsf{A} : V \to V'$ induced by the bilinear form from (1). Well-known results guarantee existence and uniqueness of solutions of this equation; see [13, Section 1] for a comprehensive exposition.

Knowledge of the second moment $w^{(2)}$ yields key statistic quantities like the variance of the random solution $\mathbf{u}$. A possible approach is probabilistic methods like the Multilevel Monte-Carlo method, where the boundary value problem (1) is solved for many realizations of $\mathbf{f}$ and statistical information is extracted from these samples [1, 7]. Yet, we focus on a deterministic direct approximation of the solution of (2).

A stable Ritz-Galerkin discretization of $\mathsf{A}\mathbf{u} = \mathbf{f}$ by means of a finite dimensional trial space $V_h \subset V$ immediately spawns a stable Ritz-Galerkin discretization of (2), when using the "full tensor product" trial and test space $V_h^{(2)} := V_h \otimes V_h$. Unfortunately, $\dim V_h^{(2)} = (\dim V_h)^2$, whereas the approximation power of $\dim V_h^{(2)}$ is usually not better than that of V_h. This is the notorious "curse of dimensionality".

Taking for granted smoothness of $\mathcal{M}^2\mathbf{u}$, a remedy is offered by sparse tensor Galerkin discretization, using subspaces $\widehat{V}_h^{(2)}$ of $V_h^{(2)}$ with approximation power almost like that of V_h, but dimensions substantially reduced to $\dim \widehat{V}_h^{(2)} \approx \dim V_h$, see [13, Section 1.4], [14, 15, 12, 11], and [6] for an introduction to so-called sparse grids.

However, the stability of sparse tensor Galerkin discretizations can no longer be inferred from that for V_h applied to A, unless A is positive. Non-positive operators are invariably encountered in wave propagation phenomena in frequency domain, and for them stability of the sparse tensor Galerkin discretization has to be established directly. This was done for boundary value problems for the Helmholtz equation $-\Delta u - k^2 u = f$ in [16], see also [13, Sect. 1.4]. In [9] we have tackled the

issue for the Maxwell cavity source problem (1) and its discretization by means of edge elements.

We start from a shape-regular sequence of nested tetrahedral triangulations of D: $\mathcal{T}_0 \prec \mathcal{T}_1 \prec \cdots \prec \mathcal{T}_l \prec \ldots$, as created by successive global regular refinement of $\mathcal{T}_0$. We write $V_l \subset \boldsymbol{H}_0(\mathbf{curl}, D)$ for the edge element space on $\mathcal{T}_l$. Fixing a *base resolution* $L_0 \in \mathbb{N}_0$ and a maximal resolution $L > L_0$, we introduce *sparse tensor edge element spaces*, *cf.* [16, Def. 5.1] and [13, Def. 1.17], [9, Section 4], [3, Sect. 6.4]

$$\boxed{\begin{aligned} \widehat{V}_{L,L_0} &:= \sum_{(l,k)\in\mathcal{S}_{L,L_0}} V_l \otimes V_k \;, \\ \mathcal{S}_{L,L_0} &:= \{(l,k) \in \{0,\ldots,L\}^2,\; l+k \le L+L_0\} \;, \quad 0 \le L_0 \le L \;, \end{aligned}} \tag{3}$$

They yield an asymptotically optimal Galerkin finite element discretization of the second moment problem, if we can establish that is a threshold level $L_0 \in \mathbb{N}$ and $C > 0$ such that the following *inf-sup condition* holds, *cf.* [3, Section 4]

$$\sup_{\widehat{\mathbf{v}}^{(2)}\in\widehat{V}_{L,L_0}} \frac{|\left\langle (\mathsf{A}\otimes\mathsf{A})\widehat{\mathbf{u}}^{(2)}, \widehat{\mathbf{v}}^{(2)}\right\rangle|}{\left\|\widehat{\mathbf{v}}^{(2)}\right\|_V} \ge C \left\|\widehat{\mathbf{u}}^{(2)}\right\|_V \quad \forall \widehat{\mathbf{u}}^{(2)} \in \widehat{V}_{L,L_0} \;, \quad \forall L \ge L_0 \;. \tag{4}$$

A key tool in its proof are uniformly stable projectors $\widehat{\mathsf{F}}^{(2)}_{L,L_0} : V \otimes V \to \widehat{V}_{L,L_0}$ that satisfy special *commuting diagram properties*

$$(\mathbf{curl}\otimes\mathbf{curl}) \circ \widehat{\mathsf{F}}^{(2)}_{L,L_0} = \widehat{\mathsf{G}}^{(2)}_{L,L_0} \circ (\mathbf{curl}\otimes\mathbf{curl}) \;, \tag{5}$$

$$(\mathsf{Id}\otimes\mathbf{curl}) \circ \widehat{\mathsf{F}}^{(2)}_{L,L_0} = \widehat{\mathsf{H}}^{(2)}_{L,L_0} \circ (\mathsf{Id}\otimes\mathbf{curl}) \;, \tag{6}$$

$$(\mathbf{curl}\otimes\mathsf{Id}) \circ \widehat{\mathsf{F}}^{(2)}_{L,L_0} = \widehat{\mathsf{J}}^{(2)}_{L,L_0} \circ (\mathbf{curl}\otimes\mathsf{Id}) \;, \tag{7}$$

with $\widehat{\mathsf{G}}^{(2)}_{L,L_0}$, $\widehat{\mathsf{H}}^{(2)}_{L,L_0}$, and $\widehat{\mathsf{J}}^{(2)}_{L,L_0}$ standing for other stable projectors onto $(\mathbf{curl}\otimes\mathbf{curl})\widehat{\mathsf{F}}^{(2)}_{L,L_0}$, $(\mathsf{Id}\otimes\mathbf{curl})\widehat{\mathsf{F}}^{(2)}_{L,L_0}$, and $(\mathbf{curl}\otimes\mathsf{Id})\widehat{\mathsf{F}}^{(2)}_{L,L_0}$, respectively.

The construction of these projectors turned out to be challenging. The construction that works relies on D. Boffi's *Fortin projectors* [2] $\mathsf{F}_l : V \to V_l$ that are uniformly stable w.r.t. to the $\boldsymbol{H}(\mathbf{curl}, D)$-norm, satisfy a commuting diagram property, and the absorption property $\mathsf{F}_{l-1}\circ\mathsf{F}_l = \mathsf{F}_{l-1} = \mathsf{F}_l \circ \mathsf{F}_{l-1}$ for all $l \in \mathbb{N}_0$. Then we find that p

$$\widehat{\mathsf{F}}^{(2)}_{L,L_0} := \sum_{(l,k)\in\mathcal{S}_{L,L_0}} \Delta\mathsf{F}_l \otimes \Delta\mathsf{F}_k \;, \quad \Delta\mathsf{F}_l := \mathsf{F}_l - \mathsf{F}_{l-1} \;,\; \Delta\mathsf{F}_0 := \mathsf{F}_0 \;. \tag{8}$$

is a projector that delivers suitable discrete candidate functions for (4), see [9, Section 5].

Along the same lines we can deal with k-fold tensor product operators

$$\mathsf{A}^{(k)} = \underbrace{\mathsf{A}\otimes\ldots\otimes\mathsf{A}}_{k \text{ times}} \;, \quad k > 2 \;.$$

though a general treatment will be extremely tedious. For the Helmholtz operator this case of higher moments is discussed in [13, Sect. 1].

The ideas can also be extended to the second moment problem for the electric field integral equation (EFIE). In this case we work in the trace space $V := H^{-\frac{1}{2}}(\mathrm{div}_\Gamma, \Gamma)$ [5, Sect. 2] on a closed orientable polyhedral surface Γ and deal with the non-positive sesqui-linear form, see [5, Sect. 7], and [10, 4],

$$a(\boldsymbol{\xi}, \boldsymbol{\eta}) = \int_\Gamma \int_\Gamma \frac{\exp(-ik|\boldsymbol{x}-\boldsymbol{y}|)}{4\pi|\boldsymbol{x}-\boldsymbol{y}|} \Big(\boldsymbol{\xi}(\boldsymbol{x}) \overline{\boldsymbol{\eta}}(\boldsymbol{y}) - k^{-2} \,\mathrm{div}_\Gamma\, \boldsymbol{\xi}(\boldsymbol{x}) \,\mathrm{div}_\Gamma\, \overline{\boldsymbol{\eta}}(\boldsymbol{y}) \Big) \,\mathrm{d}S(\boldsymbol{y}) \mathrm{d}S(\boldsymbol{x}) \;, \quad \boldsymbol{\xi}, \boldsymbol{\eta} \in V \;,$$

which is discretized using surface edge elements (also known as Raviart-Thomas boundary elements or RWG elements) [5, Sect. 8]. Tensor product saddle point problems are discussed in [3].

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Fourier continuation methods

Mark Lyon

Fourier continuation present a viable and efficient alternative to traditional methods for the resolution of the Gibbs phenomenon. These methods maintain most of the advantages of traditional Fourier approximations and lead to spectral PDE solvers with advantageous properties.

Significant computational problems remain unsolved because the best-known algorithms are not yet efficient enough to obtain sufficiently accurate solutions. Often, when referring to the efficiency of a PDE solution algorithm, the term "points per wavelength" (PPW) is invoked to refer to the resolution of the numerical scheme. It is well known (e.g. [2, 4]), however, that many traditional PDE solvers require increasing numbers of PPW as the number of wavelengths in the problem increases. Thus, the computational cost and memory requirements exhibit super-linear growth with the scale of the problem.

In contrast, Fourier methods and the related Chebyshev methods have excellent properties and completely overcome the difficulties due to pollution error, requiring just a fixed number of points per wavelength. These methods, however, require special conditions rarely met in real-world problems. Fourier methods have traditionally required periodicity of the underlying function and Chebyshev methods are significantly limited by the geometry of the problem. Fourier continuation alleviate both the periodicity and geometric constraints of classical spectral methods.

Figure 1 demonstrates the use of a Fourier continuation (FC) technique (also termed Fourier extension) for overcoming the Gibbs' phenomenon and allowing spectrally accurate approximation of non-periodic functions with Fourier series. These methods have been applied through new fast computational algorithms to the solution of many PDEs in two and three spatial dimensions. The resulting FC-AD solvers ([3, 5]) demonstrate that FC-AD techniques have inherited the pollution free property of Fourier techniques, namely that the required number of PPW does not go up as the size of the problem increases. Recent results with respect to these Fourier Continuations include an increased understanding of the resolution and stability of these techniques, application to non-linear PDEs including the Navier-Stokes equations, and efficient methods for parallelization (e.g. [1]). A recent algorithmic contribution allows for a fast and yet more accurate FC method, which further reduces the PPW requirement significantly (accurate results with just six PPW have been demonstrated). Continuing efforts to improve and build on these PDE solvers include the refinement of the implementations of the boundary conditions and optimal time-stepping for these techniques.

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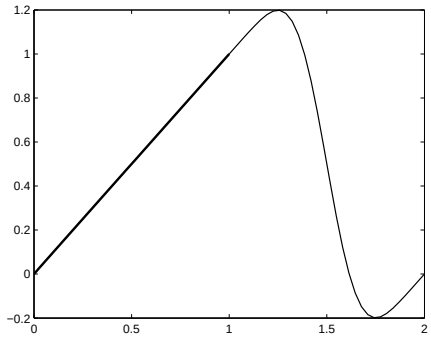

FIGURE 1. Fourier continuation of the function $f(x) = x$.

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$\mathcal{H}$-matrix approximability of the inverses of first kind BEM matrices

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(joint work with M. Faustmann, D. Praetorius)

The boundary element method (BEM) is based on discretizations of boundary integral operators and thus leads to fully populated matrices. Various compression schemes for these BEM matrices have been devised to store and/or realize the matrix-vector multiplication in log-linear complexity. Besides wavelet methods, several clustering-based techniques are available such as multipole expansions, [GR97], panel clustering [HN89], adaptive cross approximation (ACA, [Beb00]), and hybrid cross approximation (HCA, [BG05]). Many of these latter approaches rely on approximating the matrix by a blockwise low-rank matrix and can be understood as specific incarnations of the class of $\mathcal{H}$-matrices introduced in [Hac99, Gra01, Hac09]. In the context of the BEM (e.g., the approximation of the classical single layer, double layer, and hypersingular operator) the various approaches mentioned above exploit that exponential convergence in the block rank can be achieved if the block structure results from clustering algorithms that ensure an admissibilty condition of the form (*) given below. For the single layer operator V, we show that also the inverse of the corresponding BEM matrix $\mathbf{V}$

can be approximated at an exponential rate in the block rank when using the same block structure as that employed to approximate $\mathbf{V}$. Full proofs are given in the forthcoming work [FMP13].

Main results. On the boundary $\partial\Omega$ of a polygonal (for $d = 2$) or polyhedral (for $d = 3$) Lipschitz domains Ω, we consider the single layer operator V associated with the Laplacian and its Galerkin discretization $\mathbf{V}$ that arises from the trial space $S^{0,0}(\mathcal{T}_h) = \operatorname{span}\{\chi_i \,|\, i = 1, \dots, N\}$ of piecewise constant functions on a quasi-uniform mesh $\mathcal{T}_h = \{T_i \,|\, i = 1, \dots, N\}$ with mesh size h. Clusters are subsets of the index set $\mathcal{I} := \{1, \dots, N\}$. A box $B_R \subset \mathbb{R}^d$ of side length R is said to be a bounding box for a cluster σ, if $\operatorname{supp} \chi_i \subset B_R$ for all $i \in \sigma$. Two clusters σ, $\tau \subset \mathcal{I}$ are said to be η-admissible, if there are two bounding boxes B_{R_σ}, B_{R_τ} for the clusters σ, τ such that

$$(*) \qquad \eta \operatorname{dist}(B_{R_\sigma}, B_{R_\tau}) > \min\{\operatorname{diam} B_{R_\sigma}, \operatorname{diam} B_{R_\tau}\};$$

this is a standard admissibility condition underlying many compression schemes for BEM matrices.

Thm. 1: *Fix $\eta > 0$ and $q \in (0, 1)$. Let the cluster pair (σ, τ) be η-admissible. Then, for every $k \in \mathbb{N}$ there are matrices $\mathbf{X}_{\sigma\tau} \in \mathbb{R}^{|\sigma| \times r}$, $\mathbf{Y}_{\tau\sigma} \in \mathbb{R}^{|\tau| \times r}$, of rank $r \le C_{\dim} q^{-d} k^{d+1}$ with*

$$\|\mathbf{V}^{-1}|_{\sigma\times\tau} - \mathbf{X}_{\sigma\tau}\mathbf{Y}_{\tau\sigma}^T\|_2 \le C h^{-(d+2)} q^k,$$

where the constants C, $C_{\dim}$ are independent of h, σ, τ, r, and k.

Thm. 1 yields a blockwise low-rank approximation of $\mathbf{V}^{-1}$. If the block partitioning of $\mathbf{V}^{-1}$ has the structure of $\mathcal{H}$-matrices, then by arguments similar to those of [Gra01, Hac09, GH03, BH03] we can show:

Thm. 2: *Fix $\eta > 0$. Let P be a partition of $\mathcal{I} \times \mathcal{I}$ based on a cluster tree $\mathbb{T}_\mathcal{I}$ as described in* [GH03, Hac09] *with sparsity constant C_{sp},* [Gra01, Hac09, GH03]*. Then, for every $r \in \mathbb{N}$ there is a blockwise rank r matrix $\mathbf{W}_\mathcal{H}$ based on this partitioning P such that*

$$\|\mathbf{V}^{-1} - \mathbf{W}_\mathcal{H}\|_2 \le C C_{\mathrm{sp}} N^{(d+2)/(d-1)} \operatorname{depth}(\mathbb{T}_\mathcal{I}) e^{-b r^{1/(d+1)}},$$

where the constants C, b are independent of N and r.

We mention that typical clustering strategies applied to quasi-uniform meshes with $O(N)$ elements lead to fairly balanced cluster trees $\mathbb{T}_\mathcal{I}$ with $\operatorname{depth}(\mathbb{T}_\mathcal{I}) = O(\log N)$ and feature a sparsity constant C_{sp} that is bounded uniformly in N.

The key step. Thm. 1 is based on the following Thm. 3. In order to formulate Thm. 3, we need to introduce some notation: For given $f \in H^{1/2}(\Gamma)$ we let $\phi_h \in S^{0,0}(\mathcal{T}_h)$ be defined by

$$(**) \qquad (V\phi_h, v)_{L^2(\Gamma)} = (f, v)_{L^2(\Gamma)} \qquad \forall v \in S^{0,0}(\mathcal{T}_h).$$

Thm. 3: *Fix $\eta > 0$ and $q \in (0, 1)$. Then, for each η-admissible cluster pair (σ, τ) with bounding boxes B_{R_σ}, B_{R_τ} the following is true. For each $k \in \mathbb{N}$ there exists a space $V_k \subset S^{0,0}(\mathcal{T}_h)$ with $\dim V_k \le C_{\dim} q^{-d} k^{d+1}$ such that for arbitrary*

$f \in H^{1/2}(\Gamma)$ with $\operatorname{supp} f \subset B_{R_\tau} \cap \Gamma$ *the corresponding solution ϕ_h of (**) satisfies*

$$\min_{\psi_h \in V_k} \|\phi_h - \psi_h\|_{L^2(B_{R_\sigma} \cap \Gamma)} \leq C h^{-2} q^k \|f\|_{L^2(\Gamma)}.$$

The constants C, $C_{\dim}$ are independent of h and the clusters σ, τ.

Extensions and outlook. Applications of the above theorems include the ability to represent factorizations such as the Cholesky factorization and the LU-decomposition of $\mathbf{V}$ in the $\mathcal{H}$-matrix format with exponential accuracy in the block rank. It is also possible to approximate $\mathbf{V}^{-1}$ in the $\mathcal{H}^2$-matrix format with exponential accuracy in the block rank, if "min" is replaced with "max" in (*). Further applications include the ability to represent discretizations of the Poincaré-Steklov operator with exponential accuracy in the block rank, which is relevant for the use of $\mathcal{H}$-techniques for FEM-BEM coupling.

Similar approximation results have previously been shown for discretizations of second order elliptic operators by the FEM in [BH03, Beb05a, Beb07, Sch06, Bör10a, FMP12]. However, the method of proof underlying Thm. 3 and its FEM counterpart in [FMP12] differs from that used in [BH03, Sch06, Bör10a] in that it works in a fully discrete setting directly instead of studying first solution operators on the continuous level and deduce approximation results on the discrete level by an additional projection argument. This difference is the reason for exponential convergence results in Thm. 3 and [FMP12] that are not limited in terms of the mesh size h. The method of proof used for Thm. 3 and in [FMP12] permits generalizations in several directions, which are currently under way: Discretizations by higher order elements, compressibility of the inverses of the (stabilized) hypersingular operator in BEM and BEM operators on open surfaces; Neumann and mixed boundary conditions in FEM.

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Hardy space method for waveguides

Lothar Nannen
(joint work with M. Halla, T. Hohage, J. Schöberl)

Helmholtz equation: We consider for a given positive frequency $\omega > 0$ solutions u to the time-harmonic wave equation

$$-\Delta u(x,y) - \omega^2 u(x,y) = 0 \tag{1}$$

on a simple domain $\Omega = (\mathbb{R} \times D) \setminus K \subset \mathbb{R}^d$. $D \subset \mathbb{R}^{d-1}$ is a bounded Lipschitz-domain with outer normal ν_D and $K \subset (a,0) \times D$ a compact scatterer with $a < 0$ and $\partial K \cap (\mathbb{R} \times \partial D) = \emptyset$.

We assume sound hard boundary conditions $\frac{\partial u}{\partial \nu_D} = 0$ on $\mathbb{R} \times \partial D$, $\frac{\partial u}{\partial \nu_K} = g_K$ on ∂K and a suitable radiation condition for $|x| \to \infty$. In order to solve this problem numerically by finite element methods, we split the domain into a bounded interior domain $\Omega_{\text{int}} := ((a,0) \times D) \setminus K$ and two unbounded exterior domains $\Omega_{\text{ext}}^{\text{left}} := (-\infty, a) \times D$ and $\Omega_{\text{ext}}^{\text{right}} := \mathbb{R}_+ \times D$. While the interior problem can be treated with standard finite element methods, the exterior problems need particular attention. Therefore we consider in the following the right exterior problem

$$-\Delta u - \omega^2 u = 0, \qquad (x,y) \in \Omega_{\text{ext}}^{\text{right}}, \tag{2a}$$

$$\frac{\partial u}{\partial \nu_D} = 0, \qquad (x,y) \in \mathbb{R}_+ \times \partial D, \tag{2b}$$

$$u(0,\bullet) = g_{\text{in}} \qquad y \in D, \tag{2c}$$

$$u \qquad \text{is outgoing for } x \to \infty. \tag{2d}$$

We choose a orthonormal basis of eigenfunctions $(\varphi_n)_{n\in\mathbb{N}}$ to the negative, self-adjoint Neumann-Laplacian $-\Delta : \mathcal{D}(-\Delta) \subset L^2(D) \to L^2(D)$ with eigenvalues

$\lambda_1 := 0 < \lambda_2 \leq \lambda_3 \leq \ldots$. By separation of variables, solutions $u \in H^1_{\mathrm{loc}}(\Omega^{\mathrm{right}}_{\mathrm{ext}})$ to problem (2) are given by

$$u(x,y) = \sum_{n=1}^{\infty} \left(c_n e^{i\kappa_n x} + d_n e^{-i\kappa_n x} \right) \varphi_n(y), \tag{3}$$

with phase velocities $\kappa_n := \sqrt{\omega^2 - \lambda_n}$ for $\omega^2 \geq \lambda_n$ and $\kappa_n := i\sqrt{\lambda_n - \omega^2}$ for $\omega^2 < \lambda_n$. We assume in this paper $\kappa_n \neq 0$, $n \in \mathbb{N}$.

Since $e^{-i\kappa_n x} \to \infty$ for $x \to \infty$ and $\omega^2 < \lambda_n$, a reasonable radiation condition requires $d_n = 0$ for $\omega^2 < \lambda_n$. For the finitely many guided modes (i.e. $\omega^2 > \lambda_n$) it follows by the limiting absorption principle, that for radiating solutions the group velocities $\kappa_n'(\omega)$ have to be positive. In the acoustic problem (2) this is equivalent to positive phase velocities k_n and therefore $d_n = 0$ for all $n \in \mathbb{N}$. This leads to the following radiation condition: A solution u to (2a) is called outgoing, if for each $y \in D$ the Laplace transformed function

$$\hat{u}(s,y) := \int_0^{\infty} e^{-sx} u(x,y), \qquad \Re(s) > 0, \tag{4}$$

has a holomorphic extension to the domain $\mathbb{C}_{\kappa_0} := \{s \in \mathbb{C} \mid \Im(s/\kappa_0) < 0\}$ with a fixed complex parameter κ_0 with $\Re(\kappa_0) > 0$ and $\Im(\kappa_0) > 0$. For the details of the so called pole condition for Helmholtz problems see e.g. [1].

We define the Hardy spaces $H_{\kappa_0} \subset L^2(\kappa_0 \mathbb{R})$ with $\kappa_0 \mathbb{R} := \{\kappa_0 s \mid s \in \mathbb{R}\}$ of functions, which are L^2 boundary values of holomorphic functions v in $\mathbb{C}_{\kappa_0}$ satisfying $\sup_{\epsilon>0} \int_{\mathbb{R}} |v(\kappa_0(s - \epsilon i))|^2 ds < \infty$ (see e.g. [2]). Then a reformulation of the pole condition is simply given by the condition $\hat{u} \in H_{\kappa_0} \otimes L^2(D)$.

A Galerkin method in this tensor product space leads to the Hardy space infinite element method, where the basis functions are tensor products of basis functions ψ_j, $j = -1, ..., N$ in H_{κ_0} and standard finite element basis functions w_l, $l = 1, ..., M$, of $L^2(D)$. For the details of this method and the transformation of (2a) to this space see e.g. [3, 4]. Here, we present only the basis functions in H_{κ_0}:

$$\psi_{-1} := \frac{1}{s - i\kappa_0}, \qquad \psi_j := \frac{2i\kappa}{(s - i\kappa_0)^2} \left(\frac{s + i\kappa_0}{s - i\kappa_0} \right)^j, \qquad j = 0, ..., N, \tag{5}$$

and the basic formula for calculating a mass integral

$$\int_0^{\infty} u(x)v(x)dx \quad \overset{\text{Laplace}}{\longrightarrow} \quad -\frac{i}{2\pi} \int_{\kappa_0 \mathbb{R}} \psi_j(s)\psi_k(-s)ds, \qquad j,k = -1, ..., N. \tag{6}$$

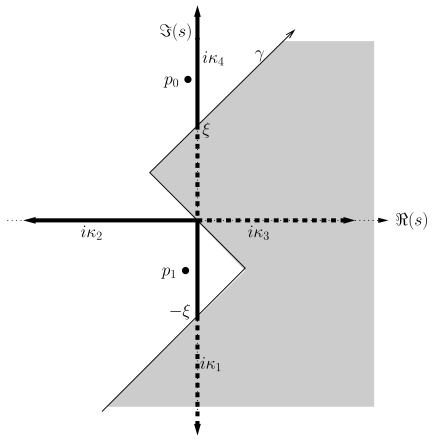
Problems exhibiting phase and group velocities of different signs: E.g. in elastic waveguide problems, the signs of group and phase velocities of modes can be different. Therefore, the standard Hardy space infinite element method as well as the complex scaling/ perfectly matched layer method will lead to wrong solutions, since the radiation condition behind these methods depends only on the phase velocity. As a one-dimensional model problem containing these difficulties we consider

$$\left(-\partial_x^2 - \xi^2 \right)^2 u = \omega^2 \tag{7}$$

with a fixed parameter $\xi > 0$. (3) becomes for this problem

$$u(x) = \sum_{n=1}^{4} c_n e^{i\kappa_n x}, \qquad \kappa_n := \pm\sqrt{\pm\omega^2 + \xi^2}. \tag{8}$$

In the scheme on the right hand side the curves of $i\kappa_n$ when ω goes from 0 (i.e. $\kappa_n \pm \xi$) to ∞ are given. The dashed lines represent the unwanted wavenumbers κ_1 and κ_3 with negative group velocity κ_n'. Note, that a standard pole condition with the given assumptions on κ_0 would enforce $c_1 = c_2 = 0$ for the guided modes (i.e. $\omega < \xi$), but $c_1 = c_3 = 0$ is required. Again we assume in the following that $\omega > 0$ and $\omega \neq \xi$.



If we define the Hardy space H_γ analogously to H_{κ_0} as boundary values of holomorphic functions in the shaded domain of the scheme, the correct radiation condition is the following: u is outgoing, if the Laplace transformed function $\hat{u} := \mathcal{L}\, u$ belongs to the Hardy space H_γ.

Instead of the integration over $\kappa_0\mathbb{R}$ in (6), this modification leads to an integration over γ. The basis (5) shows for the acoustic problem (2) super-algebraic convergence (see [4, 5]), but it will not be useful for the one-dimensional model problem (7) due to the wrong radiation condition for $\omega < \xi$. For this problem, it turned out that the following basis with complex poles p_0 and p_1 (see the scheme for the location) shows exponential convergence for arbitrary $\omega > 0$ (see [6])

$$\begin{aligned}
\psi_{-1}^{p_0}(s) &:= \frac{1}{s - p_0}, \\
\psi_j^{p_0,p_1}(s) &:= \frac{2p_0}{(s-p_0)^2} \left(\frac{|p_0|}{\max(|p_0|,|p_1|)} \frac{s+p_1}{s-p_0} \right)^j, j = 0, ..., N, \\
\psi_j^{p_1,p_0}(s) &:= \frac{2p_0}{(s-p_0)^2} \left(\frac{|p_1|}{\max(|p_0|,|p_1|)} \frac{s+p_0}{s-p_1} \right)^j, j = 0, ..., N.
\end{aligned} \tag{9}$$

Unfortunately, using this basis the condition number of the system matrix grows fast for $N \to \infty$ and is sensitive to the choice of the poles p_0 and p_1.

The biggest advantage of the method is, that the modes of the waveguide are only needed for the theory. The radiation condition and the method itself is independent of the frequency ω and the wavenumbers κ_n. Therefore they are well suited for resonance problems, where the frequency ω is sought. To the authors knowledge, all other numerical methods (e.g. mode matching methods) for waveguide problems with different signs of group and phase velocity need to solve the dispersion relation $\kappa_n(\omega)$ and are therefore frequency dependent.

We gratefully acknowledge financial support from Deutsche Forschungsgemeinschaft (DFG) in programs HO 2551/5-1 and SCH0 1253/2-1.

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Generalized Convolution Quadrature for Hyperbolic Integral Equation

Stefan A. Sauter
(joint work with M. Lopez-Fernandez)

Model Problem. In this extended abstract, we will present the Generalized Convolution Quadrature (GCQ) (see [5]) for solving linear convolution equations of the form

$$k * \phi = g, \tag{1}$$

where $*$ denotes convolution with respect to time, g is a given function, and k is some fixed kernel function/operator. We consider as our model problem the numerical solution of the wave equation in an unbounded exterior domain $\Omega^+ := \mathbb{R}\backslash\overline{\Omega^-}$ for some bounded Lipschitz domain $\Omega^- \subset \mathbb{R}^3$ with boundary $\Gamma := \partial\Omega^-$

$$\partial_{tt} u = \Delta u \qquad \text{in } \Omega^+ \times (0,T)\,,$$

with Dirichlet boundary conditions $u = g$ on $\Gamma \times (0,T)$ and initial conditions $u(0) = \partial_t u(0) = 0$ in Ω^+.

We employ the single layer potential ansatz $u = \mathcal{K}(\partial_t)\,\phi$ in $(0,T) \times H^1(\Omega^+)$, where, for $\zeta \in \mathbb{C}_{\sigma_0} := \{z \in \mathbb{C} \mid \operatorname{Re} z > \sigma_0\}$ for some $\sigma_0 > 0$, we first define the frequency dependent integral operator

$$\mathcal{K}_{-\rho}(\zeta)\,\phi = \int_\Gamma \frac{\mathrm{e}^{-\zeta\|\cdot - y\|}}{4\pi\zeta^\rho \,\|\cdot - y\|}\phi(y)\,d\Gamma_y$$

and via the inverse Laplace transform $\mathcal{L}^{-1}$ the operator $\mathcal{K}(\partial_t)\,\phi = \mathcal{L}^{-1}(\mathcal{K}) \star \phi$. This operator satisfies the wave equation and the initial conditions. The unknown density $\phi : (0,T) \to H^{-1/2}(\Gamma)$ is determined via the Dirichlet boundary conditions

$$\mathcal{K}(\partial_t)\,\phi(t) = g(t) \qquad \text{in } H^{1/2}(\Gamma) \quad \forall t \in (0,T)\,.$$

The solution of the wave equation then is given by $u(t,x) = (\mathcal{K}(\partial_t)\phi)(t,x)$ for all $t \in (0,T)$ and all $x \in \Omega^+$.

Discretization by Generalized Convolution Quadrature. Convolution quadrature has been introduced in [6, 7] for parabolic problems and [8] for hyperbolic ones. The derivation of the method and the error analysis require uniform time stepping and the goal of the *generalized convolution quadrature (GCQ)* is to generalize this approach for non-uniform and adaptive time stepping.[1]

The convolution equation can be rewritten (see, e.g., [1, formulae before (6)]) as a system of integro-differential equations for the density ϕ and an auxiliary function $U : \mathbb{C}_{\sigma_0} \times [0,T]$

$$\begin{bmatrix} -I & \partial_t - \zeta \\ 0 & \mathcal{W}_\rho \end{bmatrix} \begin{pmatrix} \phi \\ U \end{pmatrix} = \begin{pmatrix} 0 \\ \partial_t^\rho g \end{pmatrix} \tag{2}$$

with $\mathcal{W}_\rho U := \dfrac{1}{2\pi\,\mathrm{i}} \displaystyle\int_{\mathcal{C}} \zeta^\rho \mathcal{K}(\zeta)\, U(\zeta,\cdot)\, d\zeta$ and $\rho \in \mathbb{N}$ is a regularization parameter (see [5]).

In the section below, we will present a contour quadrature for the approximation of $\mathcal{W}_\rho U$ which is of the form $\displaystyle\int_{\mathcal{C}} f(\zeta)\, d\zeta \approx \sum_{k=1}^{N_Q} \omega_k f(\zeta_k)$ for some weights/nodes ω_k, ζ_k. As a consequence we need the values of U not at all points $\zeta \in \mathcal{C}$ but only at the quadrature points ζ_k and we set $U_k := U(\zeta_k, \cdot)$.

For the time discretization of the first equation in (2) we introduce time points $0 = t_0 < t_1 < \ldots$ and denote the time steps by $\Delta_n := t_n - t_{n-1}$, $n \geq 1$. We employ the implicit Euler method and obtain the recursive approximation for the first equation $U^{(0)} = 0$ and $U^{(n)}(\zeta) = \dfrac{1}{1-\zeta\Delta_n} U^{(n-1)}(\zeta) + \dfrac{\Delta_n}{1-\zeta\Delta_n}\phi^{(n)}$. For the spatial discretization we employ a boundary element method [9] and denote by h the mesh width of the spatial discretization. The GCQ algorithm with contour quadrature starts with the initialization steps: a) Generate boundary element approximations $\mathcal{K}_{h,k}$ of the operators $\mathcal{K}_k = \mathcal{K}(\zeta_k)$ for all $1 \leq k \leq N_Q$; b) set $\left(U_{h,k}^{(0)}\right)_{k=1}^{N_Q} = \mathbf{0}$; c) solve $\mathcal{K}_h\left(\frac{1}{\Delta_1}\right)\phi_h^{(1)} = (\partial_t^\rho g)_h^{(1)}$. For the recursion $n = 1, 2, \ldots, N$, we assume that $\phi_h^{(n)}$ and $\left(U_{h,k}^{(n-1)}\right)_{k=1}^{N_Q}$ are computed. Then, a) an Euler step is performed: $U_{h,k}^{(n)} = \dfrac{1}{1-\zeta_k\Delta_n} U_{h,k}^{(n-1)} + \dfrac{\Delta_n}{1-\zeta_k\Delta_n}\phi_h^{(n)} \quad \forall 1 \leq k \leq N_Q$; b) the system $\mathcal{K}_h\left(\frac{1}{\Delta_{n+1}}\right)$ and right-hand side $R_h^{(n+1)} := g_h^{(n+1)} - \displaystyle\sum_{k=1}^{N_Q} \omega_k \frac{\mathcal{K}_{h,k} U_{h,k}^{(n)}}{1-\Delta_{n+1}\zeta_k}$ are generated; c) the system $\mathcal{K}_h\left(\frac{1}{\Delta_{n+1}}\right)\phi_h^{(n+1)} = R_h^{(n+1)}$ is solved.

[1] Recently, variable time meshes for space-time Galerkin discretization of retarded potential integral equations have been introduced in [10].

This approach (Laplace transform/integrator for ODE/contour quadrature) allows to use *variable time stepping* (Generalized Convolution Quadrature GCQ) and avoids to store and integrate over the full history.

Contour Quadrature. To approximate the operator $\mathcal{W}_{\rho,h}U^{(n)}$ we fix the contour $\mathcal{C}$ to be the circle in $\mathbb{C}$ about $\Delta_{\min}^{-1}$ with radius $\Delta_{\min}^{-1}$ and propose a parametrization by the composition $\gamma = z \circ u$ of a Möbius transform with the Jacobi elliptic function $\operatorname{sn}(\cdot \mid \lambda)$

$$z(u) := \frac{\Delta_{\min}^{-1}}{q-1}\left(\sqrt{2q-1}\frac{\lambda^{-1/2}+u}{\lambda^{-1/2}-u} - 1\right), \quad u(\sigma) := \operatorname{sn}(\sigma \mid \lambda), \quad \sigma \in J_\lambda,$$

where the parameter $\lambda^2 := \dfrac{q-\sqrt{2q-1}}{q+\sqrt{2q-1}}$ depends on the ratio $q := \Delta_{\max}/\Delta_{\min}$ of the maximal and minimal mesh width. This transformation has been introduced in [2] and has been analyzed in [3] for the class of contour integrals which contains the integrand in $\mathcal{W}_{\rho,h}$.

The error analysis in [5], [3], and [4] implies the following theorem for the convergence of the fully discrete GCQ.

Theorem. *Let $\rho = 3$ and assume that $1 - \Delta_{\max}\sigma_0 > 0$. We further assume that $g \in C^{\rho+3}([0,T])$ and $g^{(\ell)}(0) = 0$ for all $0 \le \ell \le \rho+2$. Let the grading of the time mesh be at most quadratically*[2]*, i.e., $\Delta_{\min} \ge c\Delta_{\max}^2$. We assume for the number of contour quadrature points*

$$N_Q \ge CN\log^2 N.$$

Then, the GCQ algorithm generates an approximate solution $\boldsymbol{\phi}_h = \left(\phi_h^{(n)}\right)_{n=1}^N$ which satisfies

(3)

$$\max_{1\le n\le N}\left\|(\phi(t_n))_{n=1}^N - \boldsymbol{\phi}_h\right\|_{H^{-1/2}(\Gamma)} \le \max_{1\le n\le N}\|\boldsymbol{\phi}-\boldsymbol{\phi}_h\|_{H^{-1/2}(\Gamma)} + C_g\,\mathrm{e}^{\delta_0 T}\,\Delta\left(\log\frac{1}{\Delta}\right),$$

where the constant C_g only depends on the right-hand side g and $\delta_0 = O(1)$.

Remark. *In [5], it was proved that, for sufficiently smooth data and boundary elements of local polynomial order p, the first term in the right-hand side of (3) can be estimated from above by $Ch^{p+3/2}$.*
Results of numerical experiments (cf. [4]) show that for Dirichlet data $g(t)$ which are non-smooth at $t = 0$, quadratic grading of the time mesh can preserve optimal convergence rates $O(\Delta)$ while uniform grading leads to suboptimal convergence rates $O\left(\Delta^{1/2}\right)$.

[2]Generalizations to stronger mesh grading are considered in [4].

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Is the Helmholtz equation really sign indefinite?

Euan A. Spence

(joint work with A. Moiola)

Introduction. The usual variational (or weak) formulations of the Helmholtz equation are sign-indefinite in the sense that the sesquilinear forms cannot be bounded below by a positive multiple of the appropriate norm squared. This is often for a good reason, since in bounded domains under certain boundary conditions the solution of the Helmholtz equation is not unique at certain wavenumbers (those that correspond to eigenvalues of the Laplacian), and thus the variational problem cannot be sign-definite. However, even in cases where the solution is unique for all wavenumbers, the standard variational formulations of the Helmholtz equation are still indefinite when the wavenumber is large.

To illustrate this, consider the *interior impedance problem* for the Helmholtz equation. That is, given a bounded Lipschitz domain $\Omega \subset \mathbb{R}^d$, $f \in L^2(\Omega)$, $g \in L^2(\partial\Omega)$, and $k > 0$, find u such that

$$\mathcal{L}u := \Delta u + k^2 u = -f \quad \text{in } \Omega, \tag{1a}$$

$$\frac{\partial u}{\partial n} - \mathrm{i}ku = g \quad \text{on } \partial\Omega. \tag{1b}$$

This problem can be put in weak (or variational) form by multiplying by the complex conjugate of a test function v and integrating by parts, i.e. using Green's identity

$$\overline{v}\mathcal{L}u = \nabla \cdot [\overline{v}\nabla u] - \nabla u \cdot \overline{\nabla v} + k^2 u\overline{v}. \tag{2}$$

The result is that the boundary value problem (BVP) (1) can be reformulated as:

$$\text{Find } u \in \mathcal{V} \text{ such that } a(u,v) = F(v) \text{ for all } v \in \mathcal{V}. \tag{3}$$

with $\mathcal{V}$ the Hilbert space $H^1(\Omega)$ equipped with norm

$$\|v\|^2_{1,k,\Omega} := \|\nabla v\|^2_{L^2(\Omega)} + k^2 \|v\|^2_{L^2(\Omega)},$$

the sesquilinear form $a(\cdot,\cdot) : \mathcal{V} \times \mathcal{V} \to \mathbb{C}$ given by

$$a(u,v) := \int_\Omega \left(\nabla u \cdot \overline{\nabla v} - k^2 u\overline{v}\right) \mathrm{d}\mathbf{x} - \mathrm{i}k \int_{\partial\Omega} u\overline{v}\, \mathrm{d}s, \tag{4}$$

and the antilinear functional $F : \mathcal{V} \to \mathbb{C}$ given by

$$F(v) := \int_\Omega f\overline{v}\, \mathrm{d}\mathbf{x} + \int_{\partial\Omega} g\overline{v}\, \mathrm{d}s, \tag{5}$$

Given a variational problem of the form (3), ideally one would like to prove that there exist constants $C_c, \alpha > 0$ such that

$$|a(u,v)| \le C_c \|u\|_{\mathcal{V}} \|v\|_{\mathcal{V}} \text{ for all } u,v \in \mathcal{V}, \text{(continuity)}, \tag{6}$$

$$|a(v,v)| \ge \alpha \|u\|^2_{\mathcal{V}} \text{ for all } v \in \mathcal{V}, \text{(coercivity)}. \tag{7}$$

"Sign-definite" is used as a synonym for "coercive" (thus a variational problem is sign-indefinite if and only if it is not coercive). Note that several authors call property (7) "$\mathcal{V}$-ellipticity" (see, e.g., [3, §2.4.1], [2, §1], [7, Equation 2.43]) and use the word "coercivity" for the weaker property of satisfying a Gårding inequality ([3, §2.4.3], [7, Definition 2.1.54]).

If continuity and coercivity can be established then there are three important consequences (i) existence and uniqueness of the solution to the variational problem (3) via the Lax–Milgram theorem, (ii) quasi-optimality of the Galerkin method applied to (3) for *any* finite dimensional subspace of $\mathcal{V}$, and (iii) sign-definiteness of the finite dimensional matrix of the Galerkin method.

Returning to the variational formulation of the interior impedance problem (4) and (5), one can show that if $k^2 \ge \lambda_1$ (the smallest eigenvalue of the negative Laplacian with Dirichlet boundary conditions) then there exists a $v \in H^1(\Omega)$ such that $a(v,v) = 0$; thus $a(\cdot,\cdot)$ is not coercive. This indefiniteness has implications for both the analysis and the practical implementation of finite element methods based on the variational formulation.

A new sign-definite variational formulation of the interior impedance problem. This talk introduced a new sign-definite formulation of the Helmholtz equation posed in the interior of a star-shaped domain with impedance boundary conditions. Like the standard variational formulation, this new formulation

arises just by multiplying the Helmholtz equation by a particular test function and integrating by parts.

Consider the Hilbert space

$$V := \left\{ v : \, v \in H^1(\Omega), \; \Delta v \in L^2(\Omega), \; \nabla v \in \left(L^2(\partial\Omega) \right)^d \right\} \tag{8}$$

with norm

$$\|v\|_V^2 := k^2 \|v\|_{L^2(\Omega)}^2 + \|\nabla v\|_{L^2(\Omega)}^2 + k^{-2} \|\Delta v\|_{L^2(\Omega)}^2 + L\left(k^2 \|v\|_{L^2(\partial\Omega)}^2 + \|\nabla v\|_{L^2(\partial\Omega)}^2 \right),$$

and obvious inner product, where L is the diameter (or some other characteristic length scale) of the domain Ω. Define the sesquilinear form $b : V \times V \to \mathbb{C}$ by

$$\begin{aligned} b(u,v) := & \int_\Omega \left(\nabla u \cdot \overline{\nabla v} + k^2 u \overline{v} + \left(\mathcal{M}u + \frac{1}{3k^2} \mathcal{L}u \right) \overline{\mathcal{L}v} \right) \mathrm{d}\mathbf{x} \\ & - \int_{\partial\Omega} \left(\mathrm{i}ku \, \overline{\mathcal{M}v} + \left(\mathbf{x} \cdot \nabla_{\partial\Omega} u - \mathrm{i}k\beta u + \frac{d-1}{2} u \right) \overline{\frac{\partial v}{\partial n}} \right. \\ & \left. + (\mathbf{x} \cdot \mathbf{n}) \left(k^2 u \overline{v} - \nabla_{\partial\Omega} u \cdot \overline{\nabla_{\partial\Omega} v} \right) \right) \mathrm{d}s, \end{aligned}$$

and antilinear functional $G : V \to \mathbb{C}$ by

$$G(v) := \int_\Omega \left(\overline{\mathcal{M}v} - \frac{1}{3k^2} \overline{\mathcal{L}v} \right) f \, \mathrm{d}\mathbf{x} + \int_{\partial\Omega} \overline{\mathcal{M}v} \, g \, \mathrm{d}s,$$

where β is an arbitrary real constant, d is the spatial dimension,

$$\mathcal{L}u := \Delta u + k^2 u, \quad \mathcal{M}u := \mathbf{x} \cdot \nabla u - \mathrm{i}k\beta u + \frac{d-1}{2} u,$$

and $\nabla_{\partial\Omega}$ is the surface gradient on $\partial\Omega$ (recall that $\nabla_{\partial\Omega}$ is such that if u is differentiable in a neighbourhood of $\partial\Omega$ then $\nabla_{\partial\Omega} u = \nabla u - \mathbf{n} \frac{\partial u}{\partial n}$ on $\partial\Omega$).

The sesquilinear form $b(\cdot,\cdot)$ and functional $G(\cdot)$ are defined in this way because if u is the solution to the BVP (1), then

$$b(u,v) = G(v) \quad \text{for all } v \in V; \tag{9}$$

(this is not immediately obvious, see [4, Proposition 3.2]).

Using the Cauchy–Schwarz inequality it is straightforward to show that the sesquilinear form $b(\cdot,\cdot)$ is continuous on V. In particular, if β is independent of k (as we choose it to be below), then the continuity constant $C_c \sim k$ as $k \to \infty$.

The main novelty of $b(\cdot,\cdot)$ is that, for some domains, it is coercive on V:

Theorem 1. [4, Theorem 3.4] *Let Ω be a Lipschitz domain with diameter L that is star-shaped with respect to a ball, i.e. there exists a $\gamma > 0$ such that $\mathbf{x} \cdot \mathbf{n}(\mathbf{x}) \geq \gamma L$ for all $\mathbf{x} \in \partial\Omega$ such that $\mathbf{n}(\mathbf{x})$ exists. If the arbitrary constant β is chosen such that $\beta \geq \frac{L}{2} \left(1 + \frac{4}{\gamma} + \frac{\gamma}{2} \right)$ then, for any $k > 0$,*

$$\Re b(v,v) \geq \frac{\gamma}{4} \|v\|_V^2 \quad \text{for all } v \in V,$$

i.e. $b(\cdot,\cdot)$ is coercive on V with constant $\gamma/4$.

The idea behind the new formulation. As we saw in above, the standard variational formulation of the interior impedance problem (1) is based on integrating over Ω Green's identity for the Helmholtz equation (2).

The new variational formulation (9) comes from integrating the following identity over Ω

$$\overline{\mathcal{M}v}\mathcal{L}u + \mathcal{M}u\overline{\mathcal{L}v} = \nabla\cdot\big[\overline{\mathcal{M}v}\,\nabla u + \mathcal{M}u\,\overline{\nabla v} + \mathbf{x}(k^2u\overline{v} - \nabla u\cdot\overline{\nabla v})\big] - \nabla u\cdot\overline{\nabla v} - k^2u\overline{v}.$$

where the multiplier $\mathcal{M}$ is defined by $\mathcal{M}v := \mathbf{x}\cdot\nabla v - \mathrm{i}k\beta v + \frac{d-1}{2}v$, and the real number β is chosen to ensure coercivity of the resulting sesquilinear form.

The idea of multiplying the Helmholtz operator, $\mathcal{L}$, by the multiplier $\mathcal{M}$ goes back to Morawetz and Ludwig [6] and was then extensively used by Morawetz in her famous work on the wave equation [5]. (For more history on the use of this type of multiplier, see [1, §5.3].)

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Robust operator estimates

Christian Wieners

(joint work with B. Wohlmuth)

We provide robust stability estimates for first order systems, which then lead to robust estimates for a discretization of least-squares type in operator depending norms. This result transfers the ideas in [DGMZ12] for the Helmholtz equation to the Maxwell problem.

The general setting. Let H be a Hilbert space with norm $\|\cdot\|_H$, let $U \subset H$ be a dense subspace, and let $L\colon U \longrightarrow H$ be a closed linear operator so that U is a Hilbert space in the graph norm $\|u\|_U = \sqrt{\|u\|_H^2 + \|Lu\|_H^2}$. We provide stability estimates of the form $\|u\|_H \le C_L\|Lu\|_H$ for $u \in U$. This shows that L is injective; surjectivity can be show by a corresponding estimate for the adjoint problem.

Maxwell's equations. Let $(0,T)$ be a time interval, and let the permeability μ and permittivity ε be uniformly positive in L_∞. Electro-magnetic waves are determined by the first-order system for the magnetic field $\mathcal{H}$ and the electric field $\mathcal{E}$

$$L(\mathcal{H},\mathcal{E}) = (\partial_t \mathcal{H} + \mu^{-1}\nabla\times\mathcal{E},\ \partial_t\mathcal{E} - \varepsilon^{-1}\nabla\times\mathcal{H})$$

defined on

$$U = \big\{(\mathcal{H},\mathcal{E}) \in \mathrm{L}_2((0,T),U_{\mathrm{D}}) \cap \mathrm{H}^1((0,T),\mathrm{L}_2(\Omega,\mathbb{R}^3\times\mathbb{R}^3))\colon (\mathcal{H}(0),\mathcal{E}(0)) = 0 \text{ in } \Omega\big\}$$

with $U_{\mathrm{D}} = \big\{(h,e)\in U_0\colon h\times n = 0 \text{ on } \partial\Omega_{\mathcal{H}} \text{ and } e\times n = 0 \text{ on } \partial\Omega_{\mathcal{E}}\big\}$ depending on the boundary decomposition $\partial\Omega = \partial\Omega_{\mathcal{E}} \cup \partial\Omega_{\mathcal{H}}$ and the divergence constraint in $U_0 = \big\{(h,e)\in \mathrm{H}(\mathrm{curl},\Omega)^2\colon (\mu h,\nabla\psi)_\Omega = (\varepsilon e,\nabla\psi)_\Omega = 0 \text{ for all } \psi\in C_0^\infty(\Omega)\big\}$. Let H be the closure of U in $\mathrm{L}_2((0,T)\times\Omega,\mathbb{R}^3\times\mathbb{R}^3)$ with the weighted norm

$$\|(\mathcal{H},\mathcal{E})\|_H^2 = \int_0^T \Big(\big(\mu\mathcal{H}(t),\mathcal{H}(t)\big)_\Omega + \big(\varepsilon\mathcal{E}(t),\mathcal{E}(t)\big)_\Omega^2\Big)\,\mathrm{d}t\,.$$

Then, we obtain the stability estimate $\|(\mathcal{H},\mathcal{E})\|_H \le 2T\,\|L(\mathcal{H},\mathcal{E})\|_H$.

The monochromatic Maxwell problem. Now we consider special monochromatic solutions of Maxwell's equation of the form $(\mathcal{H},\mathcal{E})(x,t) = \exp(\mathrm{i}\omega t)(h(x),e(x))$ for given frequency ω; this results into $L(h,e) = (\mathrm{i}\omega h + \mu^{-1}\nabla\times e, \mathrm{i}\omega e - \varepsilon^{-1}\nabla\times h)$. Here, we consider $U_{\mathrm{R}} = \big\{(h,e)\in U_0\colon \langle n\times h,\phi\rangle = \langle n\times e, n\times\phi\rangle \text{ for all } \phi\in C^\infty(\partial\Omega)^3\big\}$. Note that smooth functions with $\nabla\cdot(\varepsilon e) = \nabla\cdot(\mu h) = 0$ in Ω and Robin-type boundary condition $n\times h - (n\times e)\times n = 0$ on $\partial\Omega$ are dense in U_{R}. Here, H denotes the closure of U_{R} in $L_2(\Omega,\mathbb{C}^3\times\mathbb{C}^3)$ with $\|(h,e)\|_H^2 = (h,\mu h)_\Omega + (e,\varepsilon e)_\Omega$.

We consider stability for the case of constant permeability $\mu > 0$ and permittivity $\varepsilon > 0$ on domains with $x\cdot n(x) \ge \alpha > 0$ for a.a. $x\in\partial\Omega$ and $|x|\le R$ for $x\in\Omega$, see also [HMP11, Thm. 3.3] and [Moi11, Thm. 5.4.5]. Then, we obtain

$$\|(h,e)\|_H \quad\le\quad \left(2\sqrt{\varepsilon\mu}R + \frac{(\mu+\varepsilon)R^2}{\alpha}\right)\|L(h,e)\|_H\,.$$

This extends the result for the Helmholtz problem [Mel95, Prop. 8.1.4] to the Maxwell case.

Application. Based on a disjoint partitioning $\bar\Omega = \bigcup_{\tau\in\mathcal{T}}\bar\tau$ into open subdomains $\tau\subset\Omega$ and corresponding subspaces V_τ, we consider a weak formulation

$$(Lu,v)_{H,\tau} \quad=\quad (u,L^*v)_{H,\tau} + \langle\gamma_\tau u,\gamma_\tau^* v\rangle\,,$$

in each subdomain τ, where L^* is the adjoint operator and γ_τ, γ_τ^* and suitable trace mappings on $\partial\tau$. The solution $u\in U$ of $Lu = f$ is then characterized by

$$\sum\nolimits_\tau \langle\gamma_\tau u,\gamma_\tau^* v\rangle \quad=\quad (f,v)_H\,, \qquad v\in\mathrm{kernel}(L^*)\,. \tag{1}$$

Let $\hat u = (\gamma_\tau u)$ be the solution trace on the skeleton $\bigcup\partial\tau$. This is approximated in $\hat U_h = \mathrm{span}\{\hat u_h^k\} \subset \prod\gamma_\tau(V_\tau)$ as follows: For suitable finite element spaces $H_{\tau,h}$

and $V_{\tau,h}$ in τ and an inner product $a_\tau(\cdot,\cdot)$ in V_τ compute a test basis $(v^k_{\tau,h}, u^k_{\tau,h}) \in V_{\tau,h} \times H_{\tau,h}$ with

$$\begin{array}{rcll} a_\tau(v^k_{\tau,h}, v) & + & \left(u^k_{\tau,h}, L^* v\right)_\tau & = & \left\langle \hat{u}^k_{\tau,h}, \gamma^*_\tau\, v \right\rangle, & v \in V_{\tau,h}\,, \\ \left(L^* v^k_{h,\tau}, u\right)_\tau & & & = & 0\,, & u \in H_{\tau,h}\,. \end{array}$$

The Petrov-Galerkin solution $\hat{u}_h = (\hat{u}_{\tau,h}) \in \hat{U}_h$ of (1) is defined by

$$\sum\nolimits_\tau \left\langle \hat{u}_{\tau,h}, \gamma^*_\tau v^k_h \right\rangle \quad = \quad (f, v^k_h)_\Omega\,.$$

Provided a suitable inf-sup constant $\beta > 0$ for the saddle point problems in τ, a stable a priori bound in operator depending norms for this discretization exists depending only β and C_L [WW12].

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