Mathematisches Forschungsinstitut Oberwolfach

Report No. 05/2013

DOI: 10.4171/OWR/2013/05

Integral Geometry and its Applications

Organised by
Semyon Alesker, Tel Aviv
Andreas Bernig, Frankfurt
Franz Schuster, Wien

3 February – 9 February 2013

Abstract. In recent years there has been a series of striking developments in modern integral geometry which has, in particular, lead to the discovery of new relations to several branches of pure and applied mathematics. A number of examples were presented at this meeting, e.g. the work of Bernig, Solanes, and Fu on kinematic formulas on complex projective and complex hyperbolic spaces, that of Schneider and Vedel Jensen on tensor valuations and a series of results on convex body valued valuations by Abardia, Ludwig, Parapatits, and Wannerer.

Mathematics Subject Classification (2010): Primary: 53C65, 52B45, 52A22 Secondary: 53A15, 52A40, 60D05, 43A90.

Introduction by the Organisers

The meeting Integral Geometry and its Applications organized by Semyon Alesker, Andreas Bernig and Franz Schuster, was held from February 3 to February 8, 2013. It was attended by 24 participants (around one third of which were young scientists) working in integral geometry, integral transforms and harmonic analysis. The program contained 15 talks of 50 minutes and 5 talks of 25 minutes. Some highlights of the program were as follows.

A central topic were Hadwiger-type theorems with applications to kinematic formulas. Gil Solanes presented recent progress on kinematic formulas on complex projective and complex hyperbolic spaces. Among other things, it is shown that the algebras of smooth invariant valuations on the complex projective space and the complex hyperbolic space are isomorphic to the algebra of continuous invariant valuations on a hermitian vector space. In the same spirit, Thomas Wannerer studied hermitian analogues of area measures and introduced a module structure on the space of all area measures. Mykhailo Saienko spoke about curvature measures which are invariant under the special unitary group and gave a complete description of them. Dmitry Faifman presented characterization theorems for continuous and generalized valuations which are invariant under the Lorentz group.

Related to these developments were several talks about Minkowski valuations. Monika Ludwig gave a talk about the anisotropic fractional perimeter, which is defined on normed spaces and depends on some parameter $s \in [0,1]$. Using the Blaschke-Petkantschin formula, she showed that the limit behaviour as s tends to 1 is related to the moment body. Alexander Koldobsky showed how the complex intersection body can be defined and used in the solution of Busemann-Petty-type problems in complex vector spaces. Complex versions of difference and projection bodies were classified in Judit Abardia's talk. Lukas Parapatits described several new results on Minkowski valuations, among them a Steiner-type formula. He also presented a joint work with Thomas Wannerer showing that a McMullen-type decomposition for Minkowski valuations does not exist in general without additional assumptions.

Another central topic were tensor valuations. Rolf Schneider talked about his characterization of local tensor valuations on polytopes, which opens the way to attack local kinematic formulas for tensor valuations. Local tensor valuations also play an important role in rotational integral geometry, as was illustrated in Eva Vedel Jensen's talk. In Wolfgang Weil's talk, flag measures were used to construct valuations on polytopes which do not extend by continuity to all convex bodies.

Gestur Ólafsson presented the ideas leading to the computation of the spectrum of the Cos^λ-transform acting on functions on Grassmann manifolds. The same topic was put into a more general context by Boris Rubin, who studied several integral transforms on Stiefel and Grassmann manifolds. Viktor Palamodov studied inversion formulas in the even more general setting of acquisition geometries.

Martina Zähle spoke about a new and short proof of Fu's uniqueness theorem on Legendrian currents. She also sketched invariants for fractal sets which are based on the normal cycle construction. In a related talk, Joseph Fu outlined a framework for proving kinematic formulas for sets which are defined by DC-functions. This approach is based on the recent breakthrough by Rataj-Pokorný, who showed that DC-functions are Monge-Ampère functions.

Liran Rotem presented functional versions of mixed volumes and Alexandrov-type inequalities satisfied by them. New ways to define the addition of convex sets, leading to Orlicz-Brunn-Minkowski theory, were introduced in Daniel Hug's talk. Elisabeth Werner settled some previously open problems by Grünbaum on the space of affinely invariant points. Applications of integral geometry to stochastic geometry were described in Matthias Reitzner's talk on the number of faces of random polytopes.

Workshop: Integral Geometry and its Applications

Table of Contents

Abstracts

Convex body valued valuations in a complex vector space

Judit Abardia
(joint work with Andreas Bernig)

Let V denote a real vector space of dimension n and $\mathcal{K}(V)$ the space of compact convex bodies in V. An operator $Z : \mathcal{K}(V) \to (A, +)$ with $(A, +)$ an abelian semigroup is called a *valuation* if it satisfies the following additivity property

$$Z(K \cup L) + Z(K \cap L) = Z(K) + Z(L),$$

for all $K, L \in \mathcal{K}(V)$ such that $K \cup L \in \mathcal{K}(V)$. If $(A, +)$ is the set of convex bodies with addition the Minkowski sum, then Z is called *Minkowski valuation*. They have been largely studied, see for instance [4, 5, 9, 10, 11, 12, 15, 16, 17].

Two important properties of Minkowski valuations are the covariance and the contravariance with respect to the special linear group $\mathrm{SL}(V, \mathbb{R})$. A valuation $Z : \mathcal{K}(V) \to \mathcal{K}(V^*)$ is $\mathrm{SL}(V, \mathbb{R})$-*contravariant* if

$$Z(gK) = g^{-*}Z(K), \quad \forall g \in \mathrm{SL}(V, \mathbb{R}),$$

where V^* denotes the dual space of V and g^{-*} denotes the inverse of the dual map of g.

A valuation $Z : \mathcal{K}(V) \to \mathcal{K}(V)$ is $\mathrm{SL}(V, \mathbb{R})$-*covariant* if

$$Z(gK) = gZ(K), \quad \forall g \in \mathrm{SL}(V, \mathbb{R}).$$

An example of a continuous, translation invariant Minkowski valuations which is $\mathrm{SL}(V, \mathbb{R})$-contravariant is the projection body operator. For $K \in \mathcal{K}(V)$ the *projection body* ΠK *of* K has support function

$$h(\Pi K, u) = \frac{n}{2} V(K, \dots, K, [-u, u]), \quad u \in V,$$

where $V(K, \dots, K, [-u, u])$ denotes the mixed volume with $(n - 1)$ copies of K and one copy of the segment joining u and $-u$. Ludwig proved in [9, 10] that the projection body operator is the only (up to a constant factor) continuous, translation invariant $\mathrm{SL}(V, \mathbb{R})$-contravariant Minkowski valuation.

For the covariant case, Ludwig proved in [10] that the difference body is the unique (up to a positive constant) continuous Minkowski valuation which is translation invariant and $\mathrm{SL}(V, \mathbb{R})$-covariant. In fact, she classified the continuous, $\mathrm{SL}(V, \mathbb{R})$-covariant Minkowski valuations (not necessarily translation invariant). The *difference body of a convex body* $K \in \mathcal{K}(V)$ is defined by

$$\mathrm{D}K = K + (-K),$$

where $-K$ denotes the reflection of K about the origin.

In the talk, I presented the analog results when the ambient vector space is a complex vector space, that is, we obtain a classification result for the Minkowski valuations in a complex vector space W which are continuous, translation invariant and $\mathrm{SL}(W,\mathbb{C})$-contravariant or $\mathrm{SL}(W,\mathbb{C})$-covariant. Some other results concerning convex bodies in a complex vector space as ambient space can be found in [6, 7, 8].

The classification result we have proved in [2] for the $\mathrm{SL}(W,\mathbb{C})$-contravariant case is the following.

Theorem 1. *Let W be a complex vector space of complex dimension $m \geq 3$. A map $Z : \mathcal{K}(W) \to \mathcal{K}(W^*)$ is a continuous, translation invariant and $\mathrm{SL}(W,\mathbb{C})$-contravariant Minkowski valuation if and only if there exists a convex body $C \subset \mathbb{C}$ such that $Z = \Pi_C$, where $\Pi_C K \in \mathcal{K}(W^*)$ is the convex body with support function*

$$h(\Pi_C K, u) = V(K, \ldots, K, C \cdot u), \quad \forall u \in W,$$

where $C \cdot u = \{cu : c \in C \subset C\}$.

For the $\mathrm{SL}(W,\mathbb{C})$-covariant case the result reads as follows (cf. [1]).

Theorem 2. *Let W be a complex vector space of complex dimension $m \geq 3$. A map $Z : \mathcal{K}(W) \to \mathcal{K}(W)$ is a continuous, translation invariant and $\mathrm{SL}(W,\mathbb{C})$-covariant Minkowski valuation if and only if there exists a convex body $C \subset \mathbb{C}$ such that $Z = \mathrm{D}_C$, where $\mathrm{D}_C K \in \mathcal{K}(W)$ is the convex body with support function*

$$h(\mathrm{D}_C K, \xi) = \int_{S^1} h(\alpha K, \xi) dS(C, \alpha), \quad \forall \xi \in W^*,$$

where $dS(C, \cdot)$ denotes the area measure of C, and $\alpha K = \{\alpha k : k \in K \subset W\}$ with $\alpha \in S^1 \subset \mathbb{C}$.

In the case $\dim_{\mathbb{C}} W = 2$ the previous results are not true, since there exists examples of other Minkowski valuations satisfying those properties.

To prove the classification results, we first use McMullen decomposition's theorem for real-valued valuations [13] and then study each degree of homogeneity by itself. Using the $\mathrm{SL}(W,\mathbb{C})$-contravariance (resp. the $\mathrm{SL}(W,\mathbb{C})$-covariance), it can be proved that, if $\dim_{\mathbb{C}} W \geq 3$ then Z has homogeneity degree $2m-1$ (resp. 1). In order to compute its support function, it is used a characterization result of McMullen [14] (resp. of Goodey and Weil [3]).

References

[1] J. Abardia, *Difference bodies in complex vector spaces*, J. Func. Anal. **263** (2012), 3588–3603.

[2] J. Abardia and A. Bernig, *Projection bodies in complex vector spaces*, Adv. Math. **12** (2011), 445–464.

[3] P. Goodey and W. Weil, *Distributions and valuations*, Proc. London Math. Soc. (3), **49** (1984), 504–516.

[4] C. Haberl, *Minkowski valuations intertwining the special linear group*, J. Eur. Math. Soc., **14** (2012), 1565–1597.

[5] M. Kiderlen, *Blaschke- and Minkowski-endomorphisms of convex bodies*, Trans. Amer. Math. Soc., **358** (2006), 5539–5564.

[6] A. Koldobsky, *Stability of volume comparison for complex convex bodies*, Arch. Math. (Basel), **97** (2011), 91–98.

[7] A. Koldobsky, H. König, and M. Zymonopoulou, *The complex Busemann-Petty problem on sections of convex bodies*, Adv. Math. **218** (2008), 352–367.

[8] A. Koldobsky, G. Paouris, and M. Zymonopoulou, *Complex intersection bodies*, arXiv:1201.0437v1.
[9] M. Ludwig, *Projection bodies and valuations*, Adv. Math., **172** (2002), 158–168.
[10] M. Ludwig. *Minkowski valuations*, Trans. Amer. Math. Soc., **357** (2005), 4191–4213.
[11] M. Ludwig, *Valuations in the affine geometry of convex bodies*, In *Integral geometry and convexity*, World Sci. Publ., Hackensack, NJ, (2006), 49–65.
[12] M. Ludwig, *Minkowski areas and valuations*, J. Differential Geom., **86** (2010), 133–161.
[13] P. McMullen, *Valuations and Euler-type relations on certain classes of convex polytopes*, Proc. London Math. Soc. (3), **35** (1977), 113–135.
[14] P. McMullen, *Continuous translation-invariant valuations on the space of compact convex sets*, Arch. Math. (Basel) **34** (4) (1980), 370–384.
[15] R. Schneider and F. E. Schuster, *Rotation equivariant Minkowski valuations*, Int. Math. Res. Not., Article ID 72894, (2006), 1–20.
[16] F. E. Schuster, *Crofton measures and Minkowski valuations*, Duke Math. J., **154** (2010), 1–30.
[17] F. E. Schuster and T. Wannerer, GL(n) *contravariant Minkowski valuations*, Trans. Amer. Math. Soc., *364* (2012), 815–826.

Invariant valuations for the Lorentz group

Dmitry Faifman
(joint work with Semyon Alesker)

We study translation-invariant continuous valuations on the space of compact convex sets in $\mathbb{R}^n$, denoted $\mathcal{K}(n)$. Those are finitely-additive measures $\phi : \mathcal{K}(n) \to \mathbb{C}$ that are continuous w.r.t. the Hausdorff metric on $\mathcal{K}(n)$. The space of valuations is denoted $Val(\mathbb{R}^n)$. It is naturally a representation of $GL(n)$. By McMullen's theorem, $Val(\mathbb{R}^n)$ decomposes into a direct sum of k-homogeneous valuations $Val_k(\mathbb{R}^n)$, for $0 \le k \le n$. There is a further decomposition $Val_k(\mathbb{R}^n) = Val_k^{ev}(\mathbb{R}^n) \oplus Val_k^{odd}(\mathbb{R}^n)$ into even and odd valuations.

The Lorentz group $G = SO^+(n-1,1)$ is the connected component of the identity in $SO(n-1,1)$. Our goal is to describe all G-invariant elements of $Val(\mathbb{R}^n)$, denoted $Val(\mathbb{R}^n)^G$. By a theorem of Alesker [2], the space of valuations that are invariant under a compact group acting transitively on the space of oriented lines $\mathbb{P}_+^{n-1}$, is finite dimensional, and consists of smooth valuations (in the representation-theoretic sense). The orthogonal group was considered and solved by Hadwiger. More recently, the invariant valuations of several other such groups have been classified, see [4], [8], [9] for $U(n)$, and [5], [6], [7] for some other groups. However, the Lorentz group is neither transitive in its action on $\mathbb{P}_+^{n-1}$, nor is it compact.

The first issue is easily adjusted: the Lorentz group has a finite number of orbits on any partial flag variety. This implies, through the application of Klain's and Schneider's embeddings, that $\dim Val(\mathbb{R}^n)^G < \infty$. However, because of the lack of compactness, the G-invariant valuations turn out to be non-smooth, except for valuations of homogeneity degree 0 and n - those correspond to the Euler characteristic and Lebesgue measure, respectively, which are obviously G-invariant.

Moreover, if one introduces the space of generalized translation-invariant valuations, denoted $Val^{-\infty}(\mathbb{R}^n)$, as the completion of the space of continuous valuations w.r.t. a certain weak topology [1], very similarly to the notion of distributions, then one finds a plenty of Lorentz-invariant generalized valuations. Only the $(n-1)$-homogeneous Lorentz-invariant generalized valuations are in fact continuous.

More preceisely, the following theorems hold for $n \geq 3$; for $n = 2$, the situation is slightly different.

Theorem 1.

$$\dim Val_k^{odd}(\mathbb{R}^n)^G = \begin{cases} 1, & k = n-1 \\ 0, & \textit{otherwise} \end{cases}$$

Theorem 2.

$$\dim Val_k^{ev}(\mathbb{R}^n)^G = \begin{cases} 1, & k = 0, n \\ 2, & k = n-1 \\ 0, & 1 \leq k \leq n-2 \end{cases}$$

Theorem 3.

$$\dim Val_k^{-\infty,ev}(\mathbb{R}^n)^G = \begin{cases} 1, & k = 0, n \\ 2, & 1 \leq k \leq n-1 \end{cases}$$

The odd generalized Lorentz-invariant valuations remain to be classified. From now on, let us restrict to even valuations. We will now explain how those spaces can be described explicitly.

It is a consequence of Alesker's irreducibility theorem [3] that every smooth valuation $\phi \in Val_k^{\infty,ev}(\mathbb{R}^n)$ can be represented by a Crofton formula, which has the form

$$\phi(K) = \int_{Gr(n,n-k)} |Pr_{V/\Lambda}(K)| d\mu(\Lambda)$$

where $\mu \in \mathcal{M}^\infty(Gr(n, n-k))$ is a smooth measure on the Grassmannian $Gr(n, n-k)$. It turns out that every continuous valuation admits a generalized Crofton formula, which has the same form, except that now μ is a generalized measure.

Moreover, a generalized valuation $\phi \in Val_k^{-\infty,ev}(\mathbb{R}^n)$ can be naturally evaluated on certain families of convex bodies, such as the family of smooth convex bodies. Then, there is again a generalized measure $\mu \in \mathcal{M}^{-\infty}(Gr(n, n-k))$ such that the above Crofton formula applies. In particular, we can explicitly describe a generalized valuation by specifying μ. It turns out that for a Lorentz invariant valuation, if one restricts to Lorentz invariant generalized Crofton measures, then the choice of $\mu \in \mathcal{M}^{-\infty}(Gr(n, n-k))^G$ is unique .

We can describe all the generalized Lorentz-invariant Crofton measures explicitly. There is a slight qualitative difference between the cases of even and odd dimension n: as it turns out, for odd n there exists a one-dimensional space of generalized Lorentz-invariant Crofton measures which are supported on the light cone. Here the light cone is the space of all $\Lambda \in Gr(n, n-k)$, s.t. the restriction of the Lorentz quadratic form to Λ is degenerate.

Let us give an example in $\mathbb{R}^3$. Fix the standard Euclidean structure, and denote by α the elevation angle above the $x-y$ coordinate plane, and by θ the azimuth.

For a convex body $K \subset \mathbb{R}^3$, denote by σ_K its surface area measure, and by $h_2((\alpha,\theta);K)$ the area of the projection of K to the plane with normal vector (α,θ). Then

$$\phi(K) = \int_{S^2} |\cos 2\alpha|^{1/2} d\sigma_K(\alpha,\theta) = \frac{d}{d\alpha}\bigg|_{\alpha=\pi/4} \left(\cos\alpha \int_0^{2\pi} h_2((\alpha,\theta);K)d\theta\right)$$

is a 2-homogeneous, Lorentz-invariant continuous valuation, together with its generalized Crofton formula.

References

[1] Alesker, Semyon; The multiplicative structure on continuous polynomial valuations. Geom. Funct. Anal. 14 (2004), no. 1, 1–26.

[2] Alesker, Semyon; On P. McMullen's conjecture on translation invariant valuations. Adv. Math. 155 (2000), no. 2, 239–263.

[3] Alesker, Semyon; Description of translation invariant valuations on convex sets with solution of P. McMullen's conjecture. Geom. Funct. Anal. 11, (2001) no.2, 244–272.

[4] Alesker, Semyon; Hard Lefschetz theorem for valuations, complex integral geometry, and unitarily invariant valuations. J. Differential Geom. 63 (2003), no. 1, 63–95.

[5] Bernig, Andreas; A Hadwiger-type theorem for the special unitary group. Geom. Funct. Anal. 19 (2009), no. 2, 356–372.

[6] Bernig, Andreas; Integral geometry under G_2 and $Spin(7)$. Israel J. Math. 184 (2011), 301–316.

[7] Bernig, Andreas; Invariant valuations on quaternionic vector spaces. Journal de l'Institut de Mathématiques de Jussieu 11 (2012), 467–499.

[8] Bernig, Andreas; Fu, Joseph H. G. Hermitian integral geometry. Ann. of Math. (2) 173 (2011), no. 2, 907–945.

[9] Fu, Joseph H. G.; Structure of the unitary valuation algebra. J. Differential Geom. 72 (2006), no. 3, 509–533.

Recent progress on Monge-Ampère functions

Joe Fu

Roughly speaking, a Monge-Ampère (MA) function $f : \mathbb{R}^n \to \mathbb{R}$ is one for which the determinant of its Hessian $D^2 f$, or of the Hessian of a perturbation of f by a smooth function, is a well-defined signed measure. Formally, a function f with locally L^1 derivative is MA if there exists an integral current of dimension n living in the cotangent bundle $T^*\mathbb{R}^n$ that represents the graph of its differential in a certain precise sense ([2, 4]). Recently D. Pokorný and J. Rataj [1] have settled a longstanding problem by showing that any function f that is expressible as a difference of convex functions is MA, and have used this fact to construct normal cycles for sets that are definable by means of such functions. We conjecture that kinematic formulas are valid for such sets, and outline a framework for a possible proof based on [3].

References

[1] D. Pokorný & J. Rataj, J, *Normal cycles and curvature measures of sets with d.c. boundary.* Preprint arXiv:1209.2305 .

[2] J. H. G. Fu, *Monge-Ampère functions I, II.* Indiana Univ. Math. J. **38** (1989), 745–771, 773–789.

[3] J. H. G. Fu, *Curvature measures of subanalytic sets.* Amer. J. Math.T **116** (1994), 819–880.

[4] R. L. Jerrard, *Some remarks on Monge-Ampère functions.* Singularities in PDE and the Calculus of Variations, CRM Proc. Lecture Notes no. 44, 89–112, 2008.

The Orlicz-Brunn-Minkowski Theory: A general framework, additions, and inequalities

Daniel Hug

(joint work with Richard Gardner, Wolfgang Weil)

Beginning in the late nineteenth century, the classical Brunn-Minkowski theory was developed by Minkowski, Blaschke, Aleksandrov, Fenchel, and others. Combining two concepts, volume and Minkowski addition, it became an extremely powerful tool in convex geometry with significant applications to various other areas of mathematics (cf. [1, 9]).

The Orlicz-Brunn-Minkowski theory, recently introduced by Lutwak, Yang, and Zhang [6, 7], is a new extension of the classical Brunn-Minkowski theory. It represents a generalization of the L_p-Brunn-Minkowski theory [5], analogous to the way that Orlicz spaces generalize L_p spaces. For appropriate convex functions $\varphi : [0,\infty)^m \to [0,\infty)$, a new way of combining arbitrary sets in $\mathbb{R}^n$ is introduced. This operation, called Orlicz addition and denoted by $+_\varphi$, has several desirable properties, but is not associative unless it reduces to L_p addition. A general framework is introduced for the Orlicz-Brunn-Minkowski theory that includes both the new addition and previously introduced concepts, and makes clear for the first time the relation to Orlicz spaces and norms. It is also shown that Orlicz addition is intimately related to a natural and fundamental generalization of Minkowski addition called M-addition. The results obtained show, roughly speaking, that the Orlicz-Brunn-Minkowski theory is the most general possible based on an addition that retains all the basic geometrical properties enjoyed by the L_p-Brunn-Minkowski theory. Along the way, we also extend some results from [2] concerning M-addition of compact, convex sets.

Inequalities of the Brunn-Minkowski type are obtained, both for M-addition and Orlicz addition. The new Orlicz-Brunn-Minkowski inequality implies the L_p-Brunn-Minkowski inequality (cf. [8]). New Orlicz-Minkowski inequalities are obtained that generalize the L_p-Minkowski inequality. One of these has connections with the conjectured log-Brunn-Minkowski inequality of Böröczky, Lutwak, Yang, and Zhang [4], and in fact these two inequalities together are shown to split the classical Brunn-Minkowski inequality.

The talk can be viewed as a continuation of a talk given by Richard Gardner at the Oberwolfach Workshop on "Convex Geometry and its Applications" in

December 2012. Discussions during that workshop at the MFO have led to progress in our joint work on which we report in the current presentation (see [3]).

References

[1] R. J. Gardner, *The Brunn-Minkowski inequality*, Bull. Amer. Math. Soc. **39** (2002), 355–405.

[2] R. J. Gardner, D. Hug, and W. Weil, *Operations between sets in geometry*, J. Europ. Math. Soc., to appear, arXiv:1205.4327, pp. 54.

[3] R. J. Gardner, D. Hug, and W. Weil, *The Orlicz-Brunn-Minkowski Theory: A general framework, additions, and inequalities*, submitted, arXiv:1301.5267, pp. 43.

[4] K. J. Böröczky, E. Lutwak, D. Yang, and G. Zhang, *The log-Brunn-Minkowski inequality*, Adv. Math. **231** (2012), 1974–1997.

[5] E. Lutwak, *The Brunn-Minkowski-Firey theory I: Mixed volumes and the Minkowski problem*, J. Differential Geom. **38** (1993), 131–150.

[6] E. Lutwak, D. Yang, and G. Zhang, *Orlicz projection bodies*, Adv. Math. **223** (2010), 220–242.

[7] E. Lutwak, D. Yang, and G. Zhang, *Orlicz centroid bodies*, J. Differential Geom. **84** (2010), 365–387.

[8] E. Lutwak, D. Yang, and G. Zhang, *The Brunn-Minkowski-Firey inequality for non-convex sets*, Adv. in Appl. Math. **48** (2012), 407–413.

[9] R. Schneider, *Convex Bodies: The Brunn-Minkowski Theory*, Cambridge University Press, Cambridge, 1993.

Rotational integral geometry - with a view to principal rotational formulae

Eva B. Vedel Jensen

(joint work with Markus Kiderlen, Johanna F. Ziegel)

In rotational integral geometry, geometric identities of the following form is considered

$$\int \alpha(K \cap M)\,\mathrm{d}M = \beta(K),$$

where α, β are geometric quantities, K is the spatial object of interest, M is the probe (line, plane, convex body, ...) and $\mathrm{d}M$ is the element of a rotation invariant measure on the set of probes. Here, we will focus on the case, where K is a convex body in $\mathbb{R}^d$ (nonempty, compact, convex subset).

In a series of papers, rotational Crofton formulae have been established where M is a rotating linear subspace of $\mathbb{R}^d$ and α or β is an intrinsic volume or, more generally, a Minkowski tensor, cf. [1, 2, 3, 4]. However, to the best of our knowledge, principal rotational formulae are still largely unexplored. This talk presents such formulae.

An important tool in developing principal rotational formulae is local versions of Minkowski tensors. These tensors can be expressed as integrals with respect to the Lebesgue measure ν_d in $\mathbb{R}^d$ or the support measures of K (generalized curvature measures) $\Lambda_k(K, \cdot)$ which are concentrated on the normal bundle of K, consisting of pairs (x, u) with $x \in \partial K$ and u an outer unit normal at x.

Definition 1. Let K be a convex body in $\mathbb{R}^d$. Let $\mathbb{S}^{d-1}$ be the unit sphere in $\mathbb{R}^d$ and ω_d the surface area of $\mathbb{S}^{d-1}$. For non-negative integers r and s, $k = 0, \dots, d-1$, the local Minkowski tensors are then defined by

$$\Phi_{k,r,s}(K, \psi) := \frac{\omega_{d-k}}{r!\, s!\, \omega_{d-k+s}} \int_{\mathbb{R}^d \times \mathbb{S}^{d-1}} \psi(x, u)\, x^r u^s \Lambda_k(K, \mathrm{d}(x, u))$$
$$\Phi_{d,r,0}(K, \phi) := \frac{1}{r!} \int_K \phi(x)\, x^r \nu_d(\mathrm{d}x),$$

where $\psi : \mathbb{R}^d \times \mathbb{S}^{d-1} \to \mathbb{R}$ and $\phi : \mathbb{R}^d \to \mathbb{R}$. Here, x^r is the tensor of rank r determined by x and $x^r u^s$ is the symmetric tensor product of x^r and u^s.

The classical Minkowski tensors, denoted by $\Phi_{k,r,s}(K)$ and $\Phi_{d,r,0}(K)$, are obtained by choosing the functions ψ and ϕ in the above definition identically equal to 1.

It is possible to derive a principal rotational formula for local Minkowski tensors. We let SO_d be the special orthogonal group in $\mathbb{R}^d$ and $\mathrm{d}R$ the element of the unique rotation invariant *probability measure* on SO_d.

Theorem 2. *For $R \in SO_d$, let $R\psi(x, u) = \psi(R^{-1}x, R^{-1}u)$ and $R\phi(x) = \phi(R^{-1}x)$. Then, for $k = 0, \dots, d-1$,*

$$\int_{SO_d} \Phi_{k,r,s}(K, R\psi)\, \mathrm{d}R \quad = \quad \Phi_{k,r,s}(K, \bar{\psi})$$
$$\int_{SO_d} \Phi_{d,r,0}(K, R\phi)\, \mathrm{d}R \quad = \quad \Phi_{d,r,0}(K, \bar{\phi}),$$

where $\bar{\psi}(x, u) = \int_{SO_d} \psi(Rx, Ru)\, \mathrm{d}R$ and likewise for $\bar{\phi}$.

As a corollary of Theorem 2, we have the following principal rotational formula for curvature measures Φ_k. Recall that for $k = 0, \dots, d-1$, $\Phi_k(K, \cdot) = \Lambda_k(K, \cdot \times \mathbb{S}^{d-1})$ and $\Phi_d(K, \cdot) = \nu_d(K \cap \cdot)$.

Corollary 3. *Let K, M be convex bodies in $\mathbb{R}^d$ and*

$$\phi_M(x) = \frac{\mathcal{H}^{d-1}(M \cap |x|\mathbb{S}^{d-1})}{\mathcal{H}^{d-1}(|x|\mathbb{S}^{d-1})},$$

where $\mathcal{H}^{d-1}$ is the $(d-1)$-dimensional Hausdorff measure. Then, for $k = 0, \dots, d-1$,

$$\int_{SO_d} \Phi_k(K, RM)\, \mathrm{d}R = \int_{\mathbb{R}^d} \phi_M(x) \Phi_k(K, \mathrm{d}x),$$

$$\int_{SO_d} \nu_d(K \cap RM)\, \mathrm{d}R = \int_K \phi_M(x)\, \nu_d(\mathrm{d}x).$$

Theorem 2 may also be used to derive a principal rotational formula where Minkowski tensors are expressed as rotational averages. The result is given in the corollary below.

Corollary 4. *Let K, M be convex bodies in $\mathbb{R}^d$. Suppose M is chosen such that $\mathcal{H}^{d-1}(M \cap |x|\mathbb{S}^{d-1}) > 0$ for all $x \in K$. Let*

$$\phi_{-M}(x) = \frac{\mathcal{H}^{d-1}(|x|\mathbb{S}^{d-1})}{\mathcal{H}^{d-1}(M \cap |x|\mathbb{S}^{d-1})}\mathbf{1}_M(x),$$

if $\mathcal{H}^{d-1}(M \cap |x|\mathbb{S}^{d-1}) > 0$, and $\phi_{-M}(x) = 0$, otherwise. Then

$$\int_{SO_d} \Phi_{k,r,s}(K, R\phi_{-M})\,\mathrm{d}R = \Phi_{k,r,s}(K).$$

References

[1] E.B.V. Jensen, J. Rataj, *A rotational integral formula for intrinsic volumes*, Advances in Applied Mathematics **41** (2008), 530–560.
[2] J. Auneau, E.B.V. Jensen, *Expressing intrinsic volumes as rotational integrals*, Advances in Applied Mathematics **45** (2010), 1–11.
[3] J. Auneau, J. Rataj, E.B.V. Jensen, *Closed form of the rotational Crofton formula*, Mathematische Nachrichten **285** (2012), 164–180.
[4] J. Auneau, J. Ziegel, E.B.V. Jensen, *Rotational integral geometry of tensor valuations*, Advances in Applied Mathematics **50** (2013), 429–444.

Complex intersection bodies

Alexander Koldobsky
(joint work with G.Paouris, M.Zymonopoulou)

The concept of an intersection body was introduced by Lutwak in [7], as part of his dual Brunn-Minkowski theory. In particular, these bodies played an important role in the solution of the Busemann-Petty problem and are also related to the slicing problem. Many results on intersection bodies have appeared in recent years (see [1, 3, 6] and references there), but all of them apply to the real case. The goal of this work is to extend the concept of an intersection body to the complex case.

Origin symmetric convex bodies in $\mathbb{C}^n$ are the unit balls of norms on $\mathbb{C}^n$. We denote by $\|\cdot\|_K$ the norm corresponding to the body K :

$$K = \{z \in \mathbb{C}^n : \ \|z\|_K \le 1\}.$$

In order to define volume, we identify $\mathbb{C}^n$ with $\mathbb{R}^{2n}$ using the standard mapping

$$\xi = (\xi_1, ..., \xi_n) = (\xi_{11} + i\xi_{12}, ..., \xi_{n1} + i\xi_{n2}) \mapsto (\xi_{11}, \xi_{12}, ..., \xi_{n1}, \xi_{n2}).$$

Since norms on $\mathbb{C}^n$ satisfy the equality

$$\|\lambda z\| = |\lambda|\|z\|, \quad \forall z \in \mathbb{C}^n, \ \forall \lambda \in \mathbb{C},$$

Origin symmetric complex convex bodies correspond to those origin symmetric convex bodies K in $\mathbb{R}^{2n}$ that are invariant with respect to any coordinate-wise two-dimensional rotation, namely for each $\theta \in [0, 2\pi]$ and $\xi = (\xi_{11}, \xi_{12}, \dots, \xi_{n1}, \xi_{n2}) \in \mathbb{R}^{2n}$

$$\|\xi\|_K = \|R_\theta(\xi_{11}, \xi_{12}), ..., R_\theta(\xi_{n1}, \xi_{n2})\|_K, \tag{1}$$

where R_θ stands for the counterclockwise rotation of $\mathbb{R}^2$ by the angle θ with respect to the origin. We shall say that K is a *complex convex body in* $\mathbb{R}^{2n}$ if K is a convex body and satisfies equations (1). Similarly, we define complex star bodies.

For $\xi \in \mathbb{C}^n, |\xi| = 1$, denote by

$$H_\xi = \{z \in \mathbb{C}^n : \ (z,\xi) = \sum_{k=1}^{n} z_k \overline{\xi_k} = 0\}$$

the complex hyperplane through the origin, perpendicular to ξ. Under the standard mapping from $\mathbb{C}^n$ to $\mathbb{R}^{2n}$ the hyperplane H_ξ turns into a $(2n-2)$-dimensional subspace of $\mathbb{R}^{2n}$ orthogonal to the vectors

$$\xi = (\xi_{11}, \xi_{12}, ..., \xi_{n1}, \xi_{n2}) \quad \text{and} \quad \xi^\perp = (-\xi_{12}, \xi_{11}, ..., -\xi_{n2}, \xi_{n1}).$$

The orthogonal two-dimensional subspace $H_\xi^\perp$ has orthonormal basis $\left\{\xi, \xi^\perp\right\}$. A star (convex) body K in $\mathbb{R}^{2n}$ is a complex star (convex) body if and only if, for every $\xi \in S^{2n-1}$, the section $K \cap H_\xi^\perp$ is a two-dimensional Euclidean circle with radius $\rho_K(\xi) = \|\xi\|_K^{-1}$.

We introduce complex intersection bodies of complex star bodies using a definition under which these bodies play the same role in complex convexity, as their real counterparts in the real case. We use the notation $|K|$ for the volume of K; the dimension where we consider volume is clear in every particular case.

Let K, L be origin symmetric complex star bodies in $\mathbb{R}^{2n}$. We say that K is the complex intersection body of L and write $K = I_c(L)$ if for every $\xi \in \mathbb{R}^{2n}$

$$|K \cap H_\xi^\perp| = |L \cap H_\xi|. \tag{2}$$

Since $K \cap H_\xi^\perp$ is the two-dimensional Euclidean circle with radius $\|\xi\|_K^{-1}$, (2) can be written as

$$\pi \|\xi\|_{I_c(L)}^{-2} = |L \cap H_\xi|. \tag{3}$$

All the bodies K that appear as complex intersection bodies of different complex star bodies form *the class of complex intersection bodies of star bodies.* The closure of this class in the radial metric forms a more general class of *complex intersection bodies.*

We start with the complex version of Busemann's theorem.

Theorem 1. *Let K be an origin symmetric convex body in $\mathbb{C}^n$ and $I_c(K)$ the complex intersection body of K. Then $I_c(K)$ is also an origin symmetric convex body in $\mathbb{C}^n$.*

Next we extend to the complex case the geometric characterization of intersection bodies due to Goodey and Weil [2].

Theorem 2. *Let K be an origin symmetric complex star body in $\mathbb{R}^{2n}$. Then K is a complex intersection body if and only if $\|\cdot\|_K^{-2}$ is the limit in the radial metric of finite sums $\|\cdot\|_{E_1}^{-2} + \cdots + \|\cdot\|_{E_m}^{-2}$, where E_1,...,E_m are complex ellipsoids in $\mathbb{R}^{2n}$ (i.e. those ellipsoids in $\mathbb{R}^{2n}$ that are complex convex bodies).*

Theorem 3. *An origin symmetric complex star body K in $\mathbb{R}^{2n}$ is a complex intersection body if and only if it is a 2-intersection body in $\mathbb{R}^{2n}$ satisfying (1).*

Theorem 4. *Every origin symmetric complex convex body in $\mathbb{R}^6$ and $\mathbb{R}^4$ is a complex intersection body. The unit ball of every complex finite dimensional subspace of L_p, $0 < p \leq 2$ is a complex intersection body.*

The first part of the latter theorem is no longer true in $\mathbb{R}^8$.

Note that complex intersection bodies played (indirectly) the crucial role in the solution of the complex Busemann-Petty problem in [5]. Finally, we prove a complex version of the hyperplane inequality with arbitrary measures from [4].

Theorem 5. *If K is a complex intersection body in $\mathbb{R}^{2n}$, and γ is an arbitrary measure on $\mathbb{R}^{2n}$ with even continuous density, then*

$$\gamma(K) \leq \frac{n}{n-1} d_n \max_{\xi \in S^{2n-1}} \gamma(K \cap H_\xi) \; |K|^{\frac{1}{n}}.$$

By Theorem 4, this inequality holds for any origin symmetric complex convex body K in $\mathbb{R}^4$ or $\mathbb{R}^6$.

Here

$$d_n = \frac{|B_2^{2n}|^{\frac{n-1}{n}}}{|B_2^{2n-2}|} < 1,$$

and B_2^n is the unit Euclidean ball in $\mathbb{R}^n$.

References

[1] R. J. Gardner, *Geometric Tomography*, Second edition, Cambridge University Press, New York, 2006.

[2] P. Goodey and W. Weil, *Intersection bodies and ellipsoids*, Mathematika **42** (1995), 295–304.

[3] A. Koldobsky, *Fourier analysis in convex geometry*, Mathematical Surveys and Monographs, **116**, AMS, 2005.

[4] A. Koldobsky, *A hyperplane inequality for measures of convex bodies in $\mathbb{R}^n$, $n \leq 4$*, Discrete Comput. Geom. **47** (2012), 538–547.

[5] A. Koldobsky, H. König and M. Zymonopoulou, *The complex Busemann-Petty problem on sections of convex bodies*, Adv. Math. **218** (2008), 352–367.

[6] A. Koldobsky and V. Yaskin, *The interface between convex geometry and harmonic analysis*, CBMS Regional Conference Series in Mathematics, **108**, AMS, Providence, RI, 2008.

[7] E. Lutwak, *Intersection bodies and the dual mixed volumes*, Adv. Math. **71** (1988), 232–261.

Anisotropic fractional perimeters

Monika Ludwig

For a Borel set $E \subset \mathbb{R}^n$ and $0 < s < 1$, the fractional s-perimeter of E is given by

$$P_s(E) = \int_E \int_{E^c} \frac{1}{|x-y|^{n+s}} \, dx \, dy,$$

where E^c denotes the complement of E in $\mathbb{R}^n$ and $|\cdot|$ the Euclidean norm on $\mathbb{R}^n$. The functional P_s is an $(n-s)$-dimensional perimeter on Borel sets on $\mathbb{R}^n$, as $P_s(\lambda E) = \lambda^{n-s} P_s(E)$ for $\lambda > 0$. It is non-local in the sense that it is not determined by the behavior of E in a neighborhood of ∂E.

The limiting behavior of fractional s-perimeters as $s \to 1^-$ and $s \to 0^+$ turns out to be very interesting. A result by Dávila [3], which extends results by Bourgain, Brezis & Mironescu [2], shows that for a bounded Borel set $E \subset \mathbb{R}^n$ of finite perimeter,

$$\lim_{s\to 1^-} (1-s) P_s(E) = \alpha_n P(E), \tag{1}$$

where $P(E)$ is the perimeter of E and α_n is a constant depending on n. The perimeter $P(E)$ coincides with the $(n-1)$-dimensional Hausdorff measure of ∂E when E has smooth boundary. If E is a Borel set of finite Lebesgue measure, then E is of finite perimeter if its characteristic function is in $BV(\mathbb{R}^n)$ and then $P(E)$ is the total variation of the weak gradient of the characteristic function of E. Note that

$$P(E) = \int_{\partial^* E} |\nu_E(x)| \, dH^{n-1}(x), \tag{2}$$

where H^{n-1} denotes $(n-1)$-dimensional Hausdorff measure, $\partial^* E$ the reduced boundary of E and $\nu_E(x)$ the measure theoretic outer unit normal of E at $x \in \partial^* E$.

The limiting behavior for $s \to 0^+$ of fractional Sobolev s-seminorms was determined by Maz′ya & Shaposhnikova [8]. Their result implies that

$$\lim_{s\to 0+} s\, P_s(E) = n\, |B|\, |E| E, \tag{3}$$

for every bounded Borel set $E \subset \mathbb{R}^n$ of finite fractional s-perimeter for all $s \in (0,1)$. Here B is the Euclidean unit ball and $|\cdot|$ is the n-dimensional Lebesgue measure.

Anisotropic perimeter is a natural generalization of the Euclidean notion of perimeter obtained by replacing the Euclidean norm $|\cdot|$ in (2) by an arbitrary norm $\|\cdot\|_L$ with unit ball L. We say that a set $K \subset \mathbb{R}^n$ is a convex body if it is compact and convex and has non-empty interior. For $K \subset \mathbb{R}^n$ an origin-symmetric convex body, the anisotropic perimeter of a Borel set $E \subset \mathbb{R}^n$ with respect to K is

$$P(E, K) = \int_{\partial^* E} \|\nu_E(x)\|_{K^*} \, dx,$$

where $K^* = \{v \in \mathbb{R}^n : v \cdot x \le 1 \text{ for all } x \in K\}$ is the polar body of K. If E is a convex body, then $P(E, K)$ is equal (up to a factor n) to the classical first mixed volume of E and K.

For a Borel set $E \subset \mathbb{R}^n$ and $0 < s < 1$, the anisotropic fractional s-perimeter of E with respect to the origin-symmetric convex body $K \subset \mathbb{R}^n$ is given by

$$P_s(E, K) = \int_E \int_{E^c} \frac{1}{\|x-y\|_K^{n+s}} \, dx\, dy,$$

where $\|\cdot\|_K$ denotes the norm with unit ball K.

A natural question is to study the limiting behavior of anisotropic s-perimeters as $s \to 1^-$ and $s \to 0^+$. While one might suspect that the limit as $s \to 1^-$ of anisotropic s-perimeters with respect to the origin-symmetric convex body K is the anisotropic perimeter with respect to the same convex body, this turns out not to be true in general. In [5], it is proved that for $E \subset \mathbb{R}^n$ a bounded Borel set of finite perimeter,

$$\lim_{s\to 1^-}(1-s)P_s(E,K) = P(E,\mathrm{Z}\,K).$$

Here the convex body $\mathrm{Z}\,K$ is the moment body of K, that is the convex body such that

$$\|v\|_{\mathrm{Z}^*K} = \frac{n+1}{2}\int_K |v\cdot x|\,dx$$

for $v\in\mathbb{R}^n$, where Z^*K is the polar body of $\mathrm{Z}\,K$. For the Euclidean s-perimeter and the Euclidean unit ball B, the convex body $\mathrm{Z}\,B$ is just a multiple of B. Hence (1) is recovered including the value of the constant α_n.

The moment body is closely related to the classical centroid body of K, which is defined as

$$\frac{2}{(n+1)|K|}\,\mathrm{Z}\,K.$$

If we intersect the origin-symmetric convex body K by halfspaces orthogonal to $u\in S^{n-1}$, then the centroids of these intersections trace out the boundary of twice the centroid body of K, which explains the name centroid body. The name moment body comes from the fact that the corresponding moment vectors trace out the boundary (of a constant multiple) of $\mathrm{Z}\,K$. Centroid bodies play an important role within the affine geometry of convex bodies and moment bodies within the theory of valuations on convex bodies.

In [5], it is shown for $E\subset\mathbb{R}^n$ a bounded Borel set of finite perimeter,

$$\lim_{s\to 0^+} s\,P_s(E,K) = n\,|K|\,|E|.$$

The special case when K is the Euclidean unit ball follows from the result by Maz′ya & Shaposhnikova (3). The limiting results for $s\to 1^-$ and $s\to 0^+$ for the anisotropic s-perimeters are both obtained by using the Blaschke-Petkantschin Formula from integral geometry and results on fractional perimeters for subsets of the real line.

An important result for Euclidean fractional s-perimeters is the fractional isoperimetric inequality. For a bounded Borel set $E\subset\mathbb{R}^n$,

$$P_s(E) \ge \gamma_{n,s}\,|E|^{\frac{n-s}{n}}, \tag{4}$$

where $|E|$ is the n-dimensional Lebesgue measure of E and $\gamma_{n,s}>0$ is a constant depending on n and s. Using a symmetrization result by Almgren & Lieb [1], Frank & Seiringer [4] proved that there is equality in (4) precisely for balls (up to sets of measure zero). It is not difficult to see that for a given origin-symmetric convex body K, there is $\gamma_s(K)>0$ such that

$$P_s(E,K) \ge \gamma_s(K)\,|E|^{\frac{n-s}{n}} \tag{5}$$

for every bounded Borel set $E\subset\mathbb{R}^n$. Inequality (5) is the anisotropic fractional isoperimetric inequality. In [5], it is shown that if the minimizers of the anisotropic fractional isoperimetric inequality (5) converge to a bounded Borel set E_1 as $s \to 1^-$, then E_1 is (up to a constant factor) the moment body of K.

Analogues of the results on anisotropic fractional perimeters in the setting of fractional Sobolev spaces are also obtained. Results on the limiting behavior of anisotropic fractional Sobolev seminorms on $BV(\mathbb{R}^n)$ and anisotropic fractional Sobolev inequalities with the sharp constants from (5) are established in [5]. For L_p anisotropic fractional Sobolev seminorms, limiting results are established in [6]. Here L_p moment bodies, which were introduced by Lutwak & Yang [7], determine the results.

References

[1] F. J. Almgren, Jr. and E. H. Lieb, *Symmetric decreasing rearrangement is sometimes continuous*, J. Amer. Math. Soc. **2** (1989), 683–773.

[2] J. Bourgain, H. Brezis, and P. Mironescu, *Another look at Sobolev spaces*, In: Optimal Control and Partial Differential Equations (J. L. Menaldi, E. Rofman and A. Sulem, eds.). A volume in honor of A. Bensoussans's 60th birthday, Amsterdam: IOS Press; Tokyo: Ohmsha, 2001.

[3] J. Dávila, *On an open question about functions of bounded variation*, Calc. Var. Partial Differential Equations **15** (2002), 519–527.

[4] R. Frank and R. Seiringer, *Non-linear ground state representations and sharp Hardy inequalities*, J. Funct. Anal. **255** (2008), 3407–3430.

[5] M. Ludwig, *Anisotropic fractional perimeters*, Preprint.

[6] M. Ludwig, *Anisotropic fractional Sobolev norms*, Preprint.

[7] E. Lutwak and G. Zhang, *Blaschke-Santaló inequalities*, J. Differential Geom. **47** (1997), 1–16.

[8] V. Maz′ya and T. Shaposhnikova, *On the Bourgain, Brezis, and Mironescu theorem concerning limiting embeddings of fractional Sobolev spaces*, J. Funct. Anal. **195** (2002), 230–238.

The $\mathbf{COS}^{\boldsymbol{\lambda}}$-transform and $\mathbf{SL}(\boldsymbol{n}+\mathbf{1},\mathbb{F})$-intertwinors

Gestur Ólafsson

(joint work with Angela Pasquale)

ABSTRACT

We give a short description based on [4] on how to calculate the spectrum of the Cos^λ transform on Grassman manifolds using the representation theory of SL_{n+1}. For simplicity we only discuss the real case but state the final result for $\mathbb{R}$, $\mathbb{C}$ and $\mathbb{H}$. For further details and historical comments see [4, 5].

1. The $\cos^\lambda$ transform

Let X be the space space of p-dimensional subspaces in $\mathbb{K}^{n+1}$. Let $x_o = \{(x_1, \ldots, x_p, 0, \ldots, 0)^T \mid x_j \in \mathbb{K}\}$ be the base point. Let $G = \mathrm{SL}(n+1, \mathbb{K})$.

Then G acts transitively on X by $a \cdot x := \{a(v) \mid v \in x\}$. Let $q = n + 1 - p$. The stabilizer of the basepoint is the maximal parabolic subgroup P

$$\begin{aligned} P &= \left\{ \begin{pmatrix} A & B \\ 0 & C \end{pmatrix} \middle| \begin{array}{c} A \in \mathrm{GL}(p, \mathbb{K}), C \in \mathrm{GL}(q, \mathbb{K}), \det A \det C = 1 \\ B \in \mathrm{M}(p \times q, \mathbb{K}) \end{array} \right\} \\ &\simeq \mathrm{S}(\mathrm{GL}(p, \mathbb{K}) \times \mathrm{GL}(q, \mathbb{K})) \ltimes \mathrm{M}(p \times q, \mathbb{K}) = M' \ltimes N \,. \end{aligned}$$

Write $P = MAN$ where A is then one dimensional center of M' and $M = \mathrm{SL}(p, \mathbb{K}) \times \mathrm{SL}(q, \mathbb{K}))$.

To simplify the notation we will now assume that $\mathbb{K} = \mathbb{R}$. Let $K = \mathrm{SO}(n + 1)$ and $L = \mathrm{S}(\mathrm{O}(p) \times \mathrm{O}(q)) = M \cap K$. Then K acts transitively on X and $X = K/L$.

For $x, y \in X$ fix a convex set $E \subset x$ (containing 0) and $\mathrm{Vol}(E) = 1$. Denote by $\mathrm{pr}_{x,y}$ the orthogonal projection $x \to y$ and define

$$|\mathrm{Cos}(x, y)| := \mathrm{Vol}(\mathrm{pr}_{x,y}(E)) \,.$$

The definition is in fact independent of the choice of E. The Cos^λ-transform is

$$C^\lambda(f)(x) = \int_X |\mathrm{Cos}(x, y)|^{\lambda - \rho} f(y)\, dy \,, \quad f \in L^2(X) \,, \quad \mathrm{Re}(\lambda > (n-1)/2$$

where $\rho = (n+1)/2$ or more generally $\rho = d(n+1)/2$, $d = \dim_{\mathbb{R}}(\mathbb{K})$. As $|\mathrm{Cos}(x, k \cdot y)| = |\mathrm{Cos}(k^{-1} \cdot x, y)|$ it follows that the Cos^λ-transform commutes with the action of K

$$L_k f(x) = f(k^{-1}x) \quad \text{and} \quad C^\lambda(L_k f) = L_k(C^\lambda f) \,.$$

Furthermore $\lambda \mapsto C^\lambda f(x)$ is holomorphic on $\{\lambda \in \mathbb{C} \mid \mathrm{Re}(\lambda) > \rho\}$ and for $f \in C^\infty(X)$ it extends to a meromorphic function on $\mathbb{C}$.

Note, if $p = 1$ and if we lift Cos to a function on $\mathrm{S}^n \times \mathrm{S}^n$ then $|\mathrm{Cos}(x, y)| = |(x, y)| =$ the cosine of the angle between x and y and

$$C^\lambda(f)(x) = \int_{S^n} |(x, y)|^{\lambda - \rho} f(y)\, dy \,, \quad f \in C^\infty(S^n)$$

Hence the name.

The first pole occur at $\lambda = \frac{n-1}{2} = \rho - 1$ and

$$\mathrm{Res}_{\lambda = (n-1)/2} C^\lambda(f)(x) = \int_{(x,y)=0} f(y)\, d\mu(y)$$

where $d\mu$ now stands for the rotational invariant probability measure on the $n-1$-dimensional sphere $\mathrm{S}^{n-1}_x = \{y \in \mathrm{S}^n \mid (x, y) = 0\}$.

2. K-spectrum

To understand the kernel, image and poles of the Cos^λ-transform the first step is to understand the decomposition of $L^2(X)$ (or $C^\infty(X)$) into representations of K (the K-types).

Let us first recall this for the sphere S^n. A polynomial p on $\mathbb{R}^{n+1}$ is said to be *harmonic* if $\Delta p = 0$ where Δ stands for the Laplacian on $\mathbb{R}^{n+1}$. Let $\mathcal{Y}_k$ denote the space of harmonic homogeneous polynomials on $\mathbb{R}^{n+1}$ of degree k restricted to S^n and let π_k denote the irreducible representation of K on $\mathcal{Y}_k$ given by $\pi_k(g)p(x) =$

$p(g^{-1}(x))$. $\mathcal{Y}_k$ can be viewed as a space of functions on X if and only if k is even. Note that π_k and π_m are inequivalent if $k \neq m$. We have $L^2(S^n) \simeq_K \bigoplus_{k\in\mathbb{N}_0}^{\infty} \mathcal{Y}_k$ and $L^2(X) \simeq_K \bigoplus_{k\in 2\mathbb{N}_0}^{\infty} \mathcal{Y}_k$ as a K-representations. It is important to note that each of the representations π_k occurs with multiplicity at most one. The last statement is still correct in the general case except the set of parameters Λ^+ for the irreducible representations in $L^2(X)$ is a multiparameter semigroup of rank p:

$$L^2(X) \simeq_K \bigoplus_{\mu\in\Lambda^+} L^2_\mu(X)$$

with each L^2_μ finite dimensional, irreducible and $L^2_\mu \not\simeq_K L^2_\nu$ if $\mu \neq \nu$. Hence, by Schur's Lemma (Intertwining operator between irreducible reps is a multiplication by a scalar) there exists a meromorphic function $\eta_\mu : \mathbb{C} \to \mathbb{C}$ such that

$$C^\lambda|_{L^2_\mu} = \eta_\mu(\lambda)\mathrm{id}\,.$$

The task is now to determine the functions $\lambda \mapsto \eta_\mu(\lambda)$ for each $\mu \in \Lambda^+$.

There are several ways to determine η_μ. We describe here the ideas from [3] used in [4]. For that we define a continuous representation of G on $L^2(X)$ by

$$\pi_\lambda(g)f(x) = j(g^{-1}, x)^{-\lambda-\rho} f(g^{-1}\cdot x)$$

where $j(g, x)$ is the density determined by

$$\int_X j(g, x)^{-2\rho} f(g\cdot x)\, d\mu(x) = \int_X f(x)\, d\mu(x)\,.$$

The representation π_λ is unitary if and only if $\lambda \in i\mathbb{R}$. Those representations go under the name *generalized principal series representation*. They are just the induced representations

$$\pi_\lambda = \pi^P_\lambda = \mathrm{ind}_P^G 1 \otimes \lambda \otimes 1\,.$$

Define the Cartan involution $\theta : G \to G$ by $\theta(x) = [\overline{x}^{-1}]^T = (x^{-1})^*$. We need the twisted representation $\pi^\theta_\lambda(g) = \pi_\lambda(\theta(g))$. It is equivalent to the principal series representation induced from the opposite parabolic subgroup

$$\overline{P} = \theta(P)\,, \quad \pi^\theta_\lambda = \pi^{\overline{P}}_\lambda := \mathrm{ind}_{\overline{P}}^G 1 \otimes \lambda \otimes 1\,.$$

The following is well known and we refer to [6] for a proof:

Theorem 1. (1) *The representation π_λ (and then also π^θ_λ) is irreducible for almost all $\lambda \in \mathbb{C}$.*

(2) *There exists a meromorphic family of operators $J_\lambda : C^\infty(X) \to C^\infty(X)$ intertwining π_λ and $\pi^\theta_{-\lambda}$ (the Knapp-Stein intertwinors),*

$$J_\lambda \circ \pi_\lambda = \pi^\theta_{-\lambda} \circ J_\lambda\,.$$

Theorem 2 (ÓP)**.** *$C^\lambda \circ \pi_\lambda(g) = \pi^\theta_{-\lambda}(g) \circ C^\lambda$. In fact $C^\lambda = J_\lambda$. In particular $\lambda \mapsto C^\lambda f$, $f \in C^\infty(X)$, extends to a meromorphic function on $\mathbb{C}$.*

The intertwining property had already been observed in [1]. A simple corollary of the intertwining property, irreducibility and Schur's Lemma gives:

Theorem 3. *The* $\cos^\lambda$*-transform is an isomorphism* $C^\infty(X) \to C^\infty(X)$ *for almost all* λ *and there exists a meromorphic function* $\eta(\lambda)$ *such that*

$$C_{-\lambda} \circ C_\lambda = \eta(\lambda)\mathrm{id}\,.$$

Furthermore, $\eta(\lambda) = \eta_0(-\lambda)\eta_0(\lambda)$ *where* $\eta_0(\lambda) = C^\lambda(1)(x_o)$*, an integral that one can compute in terms of* Γ*-functions.*

The Lie algebra $\mathfrak{g} = \mathfrak{sl}(n+1,\mathbb{R})$ acts on $C^\infty(X)$ by

$$\pi_\lambda(X)f = \left.\frac{d}{dt}\right|_{t=0} \pi_\lambda(e^{tX})f, \quad f \in C^\infty(X), X \in \mathfrak{g}\,.$$

Write $\mathfrak{sl}(n+1,\mathbb{R}) = \mathfrak{k} \oplus \mathfrak{s}$ where $\mathfrak{k}$ is the $(+1)$-eigenspace of θ and $\mathfrak{s}$ is the (-1)-eigenspace. The idea is now that we can use $\mathfrak{s}$ to step between different K-types and by that way find an inductive way to calculate $\eta_\mu(\lambda)$ knowing only the function $\eta_0(\lambda)$ were 0 corresponds to the trivial representation of K. Often this is done by using the differential operator $\pi_\lambda(X)$ but the idea of [3] is to use a simple multiplication operator instead. For that let

$$H_o = \begin{pmatrix} \frac{q}{n+1}\,\mathrm{I}_p & 0 \\ 0 & -\frac{p}{n+1}\,\mathrm{I}_q \end{pmatrix} \in \mathfrak{s}\,.$$

Then H_o is M-fixed and we normalize the inner product on $\mathfrak{sl}(n+1,\mathbb{R})$ so that $\langle H_0, H_0\rangle = 1$. Finally $\mathfrak{a} = \mathbb{R}\,H_o$ is the Lie algebra of A. The operator $\mathrm{ad}(H_0)$ has spectrum $\{0,1,-1\}$ and $\mathfrak{n}$, the Lie algebra of N, is the $(+1)$-eigenspace of $\mathrm{ad}(H_0)$. Then define a map $\omega : \mathfrak{s} \to C^\infty(X)$ by

$$\omega(Y)(k) = \langle H_0, \mathrm{Ad}(k^{-1})Y\rangle = \langle \mathrm{Ad}(k)H_0, Y\rangle\,, k \in K, Y \in \mathfrak{s}\,.$$

It is right L-invariant and hence a function on the X.

Next we define a K-intertwining operator $S : L^2(X) \otimes \mathfrak{s}_\mathbb{C} \to L^2(X)$ by $S(f \otimes Y)(k) = \omega(Y)(k)f(k)$. This moves a K-type L^2_μ to K-types in $L^2_\mu \otimes \mathfrak{s} = \bigoplus_{\sigma\in S(\mu)} L^2_\sigma$ (note that this equation defines the set $S(\mu) \subset \Lambda^+$). For the sphere we have $S(k) = \{k-2, k+2\}$. For $\sigma \in S(\mu)$ define $\omega_k(Y) : L^2_\mu \to L^2_\sigma$ by

$$\omega_{\mu,\sigma}(Y) = \mathrm{pr}_\sigma \omega(Y)|_{L^2_\mu}\,.$$

For $\gamma \in \Lambda^+$ let $\omega(\gamma)$ denote the the eigenvalue of the Laplacian Δ on L^2_γ, a well know number that can be evaluated for each γ using explicit formulas. Then one get (see [3]) for $\sigma \in S(\mu)$:

Theorem 4. *Let* $\lambda = \frac{n+1}{n} r$ *. Then for* $\sigma \in S(\mu)$ *we have*

$$\mathrm{pr}_{L^2_\sigma} \circ \pi_\lambda(Y)|_{L^2_\mu} = \frac{1}{2}\left(\omega(\sigma) - \omega(\mu) + 2r\right)\omega_{\mu,\sigma}(Y)\,.$$

Applying C^λ to this gives, after some calculations:

$$\frac{\eta_\sigma(\lambda)}{\eta_\mu(\lambda)} = \frac{2r - \omega(\sigma) + \omega(\mu)}{2r + \omega(\sigma) - \omega(\mu)}$$

which is the recursion relation that we are looking for and it can be solved using Γ-functions and the function $\eta_0(\lambda)$! For $\lambda = (\lambda_1, \dots, \lambda_p)$ and $d = \dim_{\mathbb{R}} \mathbb{K}$ define

$$\Gamma_p(\lambda) = \Gamma_{p,d}(\lambda) := \prod_{j=1}^{p} \Gamma\left(\lambda_j - \frac{d}{2}(j-1)\right).$$

Then, by identifying $\mathbb{C} \ni \lambda \leftrightarrow (\lambda, \dots, \lambda) \in \mathbb{C}^d$, we get:

Theorem 5. $\eta_\mu(\lambda) = (-1)^{|\mu|/2} \dfrac{\Gamma_p\left(\frac{d(n+1)}{2}\right)}{\Gamma_p\left(\frac{dp}{2}\right)} \dfrac{\Gamma_p\left(\frac{\lambda-\rho+dp}{2}\right)\Gamma_p\left(\frac{-\lambda+\rho+\mu}{2}\right)}{\Gamma_p\left(\frac{\rho-\lambda}{2}\right)\Gamma_p\left(\frac{\lambda+\rho+\mu}{2}\right)}.$

References

[1] S. Alesker, *The α-cosine transform and intertwining integrals on real Grassmannians*, Unpublished manuscript, 2003.

[2] S. Alesker and J. Bernstein, *Range characterization of the cosine transform on higher Grassmannians*, Adv. Math. **184** (2004), no. 2, 367–379.

[3] T. Branson, G. Ólafsson, and B. Ørsted, *Spectrum generating operators and intertwining operators for representations induced from a maximal parabolic subgroup*, J. Funct. Anal. **135** (1996), no. 1, 163–205.

[4] G. Ólafsson and A. Pasquale, *The* Cos^λ *and* Sin^λ *transforms as intertwining operators between generalized principal series representations of* $\mathrm{SL}(n+1,\mathbb{K})$, Adv. Math. **229** (2012), no. 1, 267–293.

[5] G. Ólafsson, A. Pasquale, and B. Rubin, Analytic and Group-Theoretic Aspects of the Cosine Transform. Preprint, arXiv:1209.1822, to appear in *Contemporary Math.*

[6] D. A. Vogan, Jr. and N. R. Wallach, *Intertwining operators for real reductive groups*, Adv. Math. **82** (1990), no. 2, 203–243.

Some recent results in integral geometry

Victor Palamodov

1. Introduction

Several tomographic methods are based on reconstruction formulas in integral geometry: X-ray, PET, SPECT, impedance, MRI, thermoacoustic, photoacoustic, Doppler, Compton, ultrasound, seismic tomographies, texture analysis, radar technique etc.

Given a function f defined in a Riemannian manifold (X, g) the problem is to recover f from data of Riemannian integrals

$$\mathrm{R}f(\sigma) = \int_{Z(\sigma)} f \mathrm{d_g} S$$

along a family of varieties $Z(\sigma) \subset X, \sigma \in \Sigma$ against Riemannian area density $\mathrm{d_g}S$. A very short list of authors (1900-1970) is H. Minkowski, H. Lorentz, P. Funk, J. Radon, F. John, A. Cormack, S. Helgason, I. Gelfand.

2. Acquisition geometry

Let X and Σ be manifolds of dimension n, Z is a smooth closed hypersurface in $X \times \Sigma$ such that the natural projections $p : Z \to X,\ \pi : Z \to \Sigma$ are submersions. The manifold Z is called acquisition geometry and a function $\Phi : X \times \Sigma \to \mathbb{R}$ such that $Z = \Phi^{-1}(0)$ is called *generating* function of this geometry. Let $\mathrm{d}V$ be a volume form in X. For a function f with compact support in X the integral

$$Mf(\sigma) = \int \delta(\Phi(x, \sigma)) f \mathrm{d}V = \int_{Z(\sigma)} fq$$

is a corresponding integral transform, where $q \doteq \mathrm{d}V/\mathrm{d}_x\Phi$. In terms of a Riemannian metric g in X it can be written as $Mf(\sigma) = \mathrm{R}\left(|\nabla\Phi|^{-1} f\right)$. A generating function Φ is called *resolved* if $\Sigma = \mathbb{R} \times \Omega$, $\Omega = \mathrm{S}^{n-1}$, $\Phi(x; p, \omega) = \theta(x, \omega) - p,\ p \in \mathbb{R},\ \omega \in \Omega$ and θ is a smooth function $X \times \Omega$.

A resolved generating function is called regular if (**i**)

$$\mathrm{j}(\nabla\theta) \equiv \nabla_x\theta \wedge (\mathrm{d}_\omega \nabla_x \theta)^{\wedge n-1} \neq 0$$

and (**ii**) there are no conjugate points, that is $\theta(x, \omega) - \theta(y, \omega) = 0$ and $\mathrm{d}_\omega(\theta(x, \omega) - \theta(y, \omega)) = 0$ imply $x = y$. The first condition yields that for arbitrary $x \in X$ and for any tangent hyperplane h at x there is a locally unique hypersurface $Z(\sigma)$ through x tangent to h.

Improper integrals. Let f be a real smooth function in a manifold X, n is natural. Define

$$I_n(\rho) = \int_X \frac{\rho}{(f - i0)^n} = \lim_{\varepsilon \searrow 0} \int_X \frac{\rho}{(f - i\varepsilon)^n},$$

for a smooth density ρ with compact support. The limit exists if $\mathrm{d}f \neq 0$ on the zero set of f. For a resolved regular generating function $\Phi = \theta - p$ a singular integral

$$\Theta_n(x, y) = \int_{\not\leq} \frac{\mathrm{d}\Omega}{(\theta(x, \omega) - \theta(y, \omega) - i0)^n},\ x \neq y$$

plays a key role. It converges by (**ii**).

3. Uniform reconstruction

Theorem 1. [1] *Let $\Phi = \theta - p$ be a resolved regular generating function. If $\mathrm{Re}\, i^n \Theta_n(x, y) = 0$ for $x \neq y \in X$, then for any $f \in L_{2\mathrm{comp}}(X)$ and even n*

$$f = \frac{i}{2\pi D_n(x)} M^*\left(Hg^{(n-1)}\right)$$

where $g = Mf$. For odd n

$$f = \frac{1}{2D_n(x)} M^*\left(g^{(n-1)}\right)$$

where $g^{(n-1)} = (\partial/2\pi i\partial p)^{n-1} g$, $Hg(q) = \int g(p)\,\mathrm{d}p/\pi(q-p)$,

$$M^* g(x) = \int_\Omega g(\theta(x,\omega),\omega)\,\mathrm{d}\Omega$$

is a back projection operator. The integrals converge in $L_{2\mathrm{loc}}$ *and*

$$D_n(x) = \frac{1}{|\Omega|}\int_\Omega \frac{\mathrm{d}\Omega}{|\nabla_x \theta(x,\omega)|^n}$$

For even n the inner integral is called filtration, for odd n filtration step is $n-1$-derivative. In both cases the reconstruction is of FBP (filtered backprojection) type.

For checking the condition $\mathrm{Re} i^n \Theta_n(x,y) = 0$ methods of algebraic geometry can be applied.

4. Examples

Several new (as well as many known) FBP inversion formulas are corollaries of this result. In particular, reconstructions for hyperplanes in Euclidean space fully geodesic submanifolds in elliptic and hyperbolic spaces, family of horospheres, equidistant spheres in a hyperbolic ball and also inversions for photoacoustic acquisition geometries follow by specification of the generating function.

Photoacoustic geometries. Let $\xi : \Omega \to \mathbb{R}^n$ be a smooth map. The image $\Gamma = \xi(\Omega)$ is a compact hypersurface called *central* set. A generating function $p - |x - \xi(\omega)|^2$ defines a family of spheres with a center $\xi(\omega)$. This acquisition geometry is of special interest in the photoacoustic (thermoacoustic) tomography. For any ellipsoid and any elliptic paraboloid Γ a FBP reconstruction holds for functions supported in the interior of Γ.

In the case $n = 2$ there are more geometries which allow exact reconstruction formula. Let $\mathbf{C} \subset \mathbb{R}^2$ be a compact closed curve given by equations

$$x_1 = \xi_1(\omega), x_2 = \xi_2(\omega), \ 0 \le \omega < 2\pi$$

where ξ_1, ξ_2 are real trigonometric polynomials of degree $\le k$. A point $x \in \mathbb{R}^2$ is called *hyperbolic* with respect to $\mathbf{C}$ if any straight line L through x meets the curve at $2k$ points (counting with multiplicities). The set H of all hyperbolic points is always open and convex (hyperbolic domain).

If $\mathbf{C}$ is a trigonometric curve in a plane, then a FBP reconstruction formula holds for an acquisition geometry of all circles with centers $\xi \in \mathbf{C}$ and arbitrary function f supported in the hyperbolic domain H.

References

[1] V.Palamodov, *Uniform reconstruction in integral geometry*, Inverse problems **28** (2012) 065014

The Steiner Formula for Minkowski Valuations

Lukas Parapatits
(joint work with Franz E. Schuster, Thomas Wannerer)

Let $\mathcal{K}^n$ denote the set of convex bodies, i.e. nonempty compact convex subsets of $\mathbb{R}^n$. A valuation φ is a map from $\mathcal{K}^n$ to $\mathbb{R}$ that satisfies

$$\varphi(K \cup L) + \varphi(K \cap L) = \varphi(K) + \varphi(L) \tag{1}$$

whenever $K \cup L$ is convex. If φ is an even i-homogeneous continuous translation invariant valuation, then, by a theorem of Hadwiger, φ restricted to an i-dimensional linear subspace E is a multiple of the i-dimensional volume in E. The Klain function of φ, $\mathrm{Kl}_\varphi \colon \mathrm{Gr}_i \to \mathbb{R}$ is defined by the equation

$$\varphi|_E = \mathrm{Kl}_\varphi(E) \cdot \mathrm{vol}_i$$

for all $E \in \mathrm{Gr}_i$. Klain [2] proved that an even i-homogeneous continuous translation invariant valuation is uniquely determined by its Klain function.

The natural question arises, which properties of φ carry over to Kl_φ and vice versa. For example, if φ is nonnegative, then so is Kl_φ. However, the converse is not true. A centrally symmetric convex body is said to belong to the class $\mathcal{K}(i)$ if its i-th projection function is the cosine transform of a nonnegative measure. In particular, $\mathcal{K}(1)$ is the set of zonoids and $\mathcal{K}(n-1)$ is the set of all centrally symmetric convex bodies.

Theorem 1 (P. and Wannerer [5])**.** *Assume that K is a centrally symmetric convex body, then $K \in \mathcal{K}(i)$ if and only if $\varphi(K) \geq 0$ for every even i-homogeneous continuous translation invariant valuation φ with nonnegative Klain function.*

A more general type of valuations are so called Minkowski valuations. These are maps from $\mathcal{K}^n$ to $\mathcal{K}^n$ which satisfy (1) where $+$ is Minkowski addition, i.e. $K + L := \{x + y \colon x \in K, y \in L\}$. With the help of Theorem 1 a Steiner formula for Minkowski valuations can be deduced.

Theorem 2 (P. and Schuster [4])**.** *Assume that Φ is a continuous translation invariant Minkowski valuation, then there exist continuous translation invariant Minkowski valuations $\Phi^{(0)}, \dots, \Phi^{(n)}$ such that*

$$\Phi(K + rB^n) = \sum_{i=0}^{n} r^i \Phi^{(n-i)}(K)$$

for all convex bodies K and $r \geq 0$.

We remark that the original proof of Theorem 2 did not rely on Theorem 1 but rather on a special case that was established independently in [4].

Theorem 2 in turn leads to the following Brunn-Minkowski-type inequality which combines an intrinsic volume V_i with a j-homogeneous Minkowski valuation and generalizes previous results by Lutwak [3], Schuster [6, 7] and Alesker, Bernig and Schuster [1].

Theorem 3 (P. and Schuster [4]). *Let Φ be an* $\mathrm{SO}(n)$*-equivariant j-homogeneous continuous translation invariant Minkowski valuation, $j \in \{2, \dots, n-1\}$, and $1 \le i \le j+1$, then*

$$V_i(\Phi(K+L))^{1/ij} \ge V_i(\Phi K)^{1/ij} + V_i(\Phi L)^{1/ij}$$

for all $K, L \in \mathcal{K}^n$. If K, L are of class C^2_+, then equality holds if and only if K and L are homothetic.

References

[1] S. Alesker, A. Bernig and F. E. Schuster, *Harmonic analysis of translation invariant valuations* Geom. Funct. Anal. **21** (2011), no. 4, 751–773.

[2] D. A. Klain, *Even valuations on convex bodies*, Trans. Amer. Math. Soc. **352** (2000), 71–93.

[3] E. Lutwak, *Inequalities for mixed projection bodies*, Trans. Amer. Math. Soc. **339** (1993), no. 2, 901–916.

[4] L. Parapatits and F. E. Schuster, *The Steiner Formula for Minkowski Valuations*, Adv. Math. **230** (2012), no. 3, 978–994.

[5] L. Parapatits and T. Wannerer, *On the Inverse Klain Map*, Duke Math. J., to appear, arXiv:1206.5370.

[6] F. E. Schuster, *Volume inequalities and additive maps of convex bodies*, Mathematika **53** (2006), 211–234.

[7] F. E. Schuster, *Crofton Measures and Minkowski Valuations*, Duke Math. J. **154** (2010), 1–30.

On the Number of Faces of Random Polytopes

Matthias Reitzner

(joint work with Mareen Beermann)

Let $K \subset \mathbb{R}^d$ be a convex body of unit volume. Choose n random points in K independently and according to the uniform distribution, and denote by P_n the convex hull of these random points. A classical identity due to Efron [2] states that

$$\mathbb{E} f_0(P_n) = n(1 - \mathbb{E} V_d(P_{n-1})).$$

This identity was some years ago generalized by Buchta [1], who proved an identity connecting all moments of $V_d(P_n)$ and $f_0(P_{n+k})$.

$$V_d(P_n)^k = \binom{n+k}{k}^{-1} \mathbb{E} \binom{n+k-f_0(P_{n+k})}{k}$$

Consider the somehow dual case of an isotropic Poisson hyperplane process ζ_t of intensity $t > 0$, which has the property that the number of hyperplanes meeting a convex set $K \subset \mathbb{R}^d$ is Poisson distributed with parameter $tV_1(K)$. This hyperplane process generates a Poisson hyperplane tessellation whose zero cell $Z_{0,t}$ can be defined in the following way. For each $H \in \zeta_t$ denote by H_0 the halfspace bounded by H containing the origin, then the zero cell is defined as

$$Z_{0,t} = \bigcap_{H \in \zeta_t} H_0.$$

Schneider [4] proved identities relating the first moments of the intrinsic volumes and the number of faces,

$$\mathbb{E} f_{d-i}(Z_{0,t}) = c_{d,i} t^i \mathbb{E} V_i(Z_{0,t})$$

with an explicitly given constant $c_{d,i}$ for all $i = 1, \ldots, d$. Schneider's theorem is more general, dealing with the case of non-isotropic Poisson hyperplane processes. In the case $i = 1$ Matheron [3] could even express the moment generating function of $V_1(Z_{0,t})$ to the moments of f_{d-1}, namely

$$\mathbb{E} e^{-tV_1(Z_{0,s})} = \mathbb{E} \left(\frac{s}{s+t} \right)^{f_{d-1}(Z_{0,s+t})}.$$

It is possible to generalize these equations to the case of Poisson polyhedra circumscribed to a convex set K with $V_1(K) = 1$. For each $H \in \zeta_t$ denote by H_K the halfspace bounded by H containing the set K, and if H meets K set $H_K = \mathbb{R}^d$. Then we set

$$Z_{K,t} = \bigcap_{H \in \zeta_t} H_K$$

which is a polyhedron circumscribed to K. Extending the above results of Schneider and Matheron we obtain

$$\mathbb{E} f_{d-1}(Z_{K,t}) = t \mathbb{E}(V_1(Z_{K,t}) - 1)$$

and

$$\mathbb{E} e^{-t(V_1(Z_{0,s})-1)} = \mathbb{E} \left(\frac{s}{s+t} \right)^{f_{d-1}(Z_{0,s+t})}.$$

Our main results carry over these ideas to the case of Poisson polytopes. To this end let η_t be a Poisson point process of intensity $t > 0$. Denote by Π_t the convex hull of $\eta_t \cap K$. Efron's identity reads as

$$\mathbb{E} f_0(\Pi_t) = t(1 - \mathbb{E} V_d(\Pi_t))$$

or equivalently

$$\mathbb{E}(\eta_t(K) - f_0(\Pi_t)) = t \mathbb{E} V_d(\Pi_t)$$

where $\eta_t(K)$ denotes the number of points of η_t in K and thus $\eta_t(K) - f_0(\Pi_t))$ is the number of non-vertices in $\eta_s \cap K$. This can be extended to an identity concerning all moments of $V_d(\Pi_t)$.

$$\mathbb{E}(\eta_t(K) - f_0(\Pi_t))_{(k)} = t^k \mathbb{E} V_d(\Pi_t)^k$$

We use these identities to relate the generating function of the number of non-vertices to the moment generating function of the volume,

$$\mathbb{E}(z+1)^{\eta_t(K) - f_0(\Pi_t)} = \mathbb{E} e^{tzV_d(\Pi_t)}, z \in \mathbb{C}.$$

Further, the ideas of Matheron's result can be used to prove

$$\mathbb{E} e^{-t(1-V_d(\Pi_s))} = \mathbb{E} \left(\frac{s}{s+t} \right)^{f_0(\Pi_{s+t})}$$

for $s, t \geq 0$.

References

[1] C. Buchta, *An identity relating moments of functionals of convex hulls*, Discrete Comput. Geom. **33** (2005), 125–142

[2] B. Efron, *The convex hull of a random set of points*, Biometrika **52** (1965), 331–343

[3] G. Matheron, *Random sets and integral geometry*, Wiley Series in Probability and Mathematical Statistics

[4] R. Schneider, *Weighted faces of Poisson hyperplane tessellations*, Adv. Appl. Prob. **41** (2009), 682–694

Mixed integrals and related inequalities

Liran Rotem

(joint work with Vitali Milman)

Our point of departure will be Minkowski's theorem on mixed volumes:

Theorem 1 (Minkowski). *Fix bodies $K_1, K_2, \ldots, K_m \in \mathcal{K}_c^n$. Then the function $F : (\mathbb{R}^+)^m \to [0, \infty)$, defined by*

$$F(\lambda_1, \lambda_2, \ldots, \lambda_m) = \mathrm{Vol}\left(\lambda_1 K_1 + \lambda_2 K_2 + \cdots + \lambda_m K_m\right),$$

is a homogeneous polynomial of degree n, with non-negative coefficients.

Here $\mathcal{K}_c^n$ is the family of compact and convex bodies in $\mathbb{R}^n$, and the addition operation $+$ is Minkowski addition,

$$A + B = \{a + b : \ a \in A, \ b \in B\}.$$

By standard linear algebra, Minkowski's theorem is equivalent to the existence of a map $V : (\mathcal{K}_c^n)^n \to [0, \infty)$ which is multilinear, symmetric and which satisfies $V(K, K, \ldots, K) = \mathrm{Vol}(K)$. This map is unique, and the number $V(K_1, K_2, \ldots, K_n)$ is known as the mixed volume of $K_1, \ldots, K_n$.

Our goal is to extend Minkowski's theorem to a functional setting. That is, we want to take n functions $f_1, f_2, \ldots, f_n : \mathbb{R}^n \to [0, \infty)$ and define their "mixed volume" $V(f_1, f_2, \ldots, f_n)$. In order to do so we need to choose an appropriate family of functions, a "volume" functional on this family, and an addition operation.

For the family of functions, we choose the class of *quasi-concave* functions. A function $f : \mathbb{R}^n \to [0, \infty)$ is called quasi-concave if for every $x, y \in \mathbb{R}^n$ and every $0 < \lambda < 1$ we have

$$f\left(\lambda x + (1 - \lambda) y\right) \geq \min\left\{f(x), f(y)\right\}.$$

While not always necessary, it is very convenient to assume further that f is upper semicontinuous, that $\max f = f(0) = 1$ and that $f(x) \to 0$ as $|x| \to 0$. Denote this set of functions by $\mathrm{QC}\left(\mathbb{R}^n\right)$.

As a volume, we choose the Lebesgue integral, i.e.

$$\mathrm{Vol}(f) = \int_{\mathbb{R}^n} f(x) dx.$$

Finally, for addition, we define a new addition on quasi-concave functions by

$$(f \oplus g)(x) = \sup_{y \in \mathbb{R}^n} \min \{f(y), g(x-y)\}.$$

We further define the product $\lambda \odot f$ for $f \in \mathrm{QC}(\mathbb{R}^n)$ and $\lambda > 0$ by $(\lambda \odot f)(x) = f\left(\frac{x}{\lambda}\right)$. We briefly comment that these operations emerge as a limit of the natural addition operations on α-concave functions. An explanation of this statement appears in [2] and [4].

Under the above definition, we have to following theorem:

Theorem 2. *Fix $f_1, f_2, \ldots, f_m \in \mathrm{QC}(\mathbb{R}^n)$. Then the function $F : (\mathbb{R}^+)^m \to [0, \infty]$, defined by*

$$F(\lambda_1, \lambda_2, \ldots, \lambda_m) = \int [(\lambda_1 \odot f_1) \oplus (\lambda_2 \odot f_2) \oplus \cdots \oplus (\lambda_m \odot f_m)]$$

is a homogeneous polynomial of degree n, with non-negative coefficients.

The proof of this result appears in [3]. As usual, this is equivalent to the existence of a multilinear, symmetric map $V : \mathrm{QC}(\mathbb{R}^n)^n \to [0, \infty]$ which satisfies $V(f, f, \ldots, f) = \int f$. The number $V(f_1, f_2, \ldots, f_m)$ will be called the *mixed integral* of $f_1, f_2, \ldots, f_m$. The following theorem summarizes some of the basic properties of mixed integrals:

Theorem 3. (1) *$V(K_1, K_2, \ldots, K_n) = V(\mathbf{1}_{K_1}, \mathbf{1}_{K_2}, \ldots, \mathbf{1}_{K_n})$.*
(2) *If $f_i \geq g_i$ for all i, then $V(f_1, f_2, \ldots, f_n) \geq V(g_1, g_2, \ldots, g_n)$.*
(3) *V is rotation and translation invariant.*
(4) *Fix $g_{m+1}, \ldots, g_n \in \mathrm{QC}(\mathbb{R}^n)$, and define*

$$\Phi(f) = V(f[m], g_{m+1}, \ldots, g_n).$$

Φ satisfies a valuation type property: if $f_1, f_2 \in \mathrm{QC}(\mathbb{R}^n)$ and $f_1 \vee f_2 = \max(f_1, f_2) \in \mathrm{QC}(\mathbb{R}^n)$ as well, then

$$\Phi(f_1 \vee f_2) + \Phi(f_1 \wedge f_2) = \Phi(f_1) + \Phi(f_2).$$

Once we have a generalization of the notion of mixed volumes, it makes sense to try and generalize the important inequalities as well. For example, for $f \in \mathrm{QC}(\mathbb{R}^n)$ define its k-th quermassintegral to be

$$W_k(f) = V(\underbrace{f, f, \ldots, f}_{n-k \text{ times}}, \underbrace{\mathbf{1}_D, \mathbf{1}_D, \ldots, \mathbf{1}_D}_{k \text{ times}}),$$

where D is the unit Euclidean ball. This notion of functional quermassintegrals was discovered independently by Bobkov, Colesanti and Fragalà ([1]). In particular, we have the notion of surface area, defined by $S(f) = nW_1(f)$.

We now want to prove a functional isoperimetric inequality. Unfortunately, it turns out that for general quasi-concave functions it is impossible to give a lower bound for $S(f)$ in terms of $\int f$. Surprisingly, however, it is possible to state a functional extension of the isoperimetric inequality:

Theorem 4. *For every $f \in \mathrm{QC}(\mathbb{R}^n)$ we have $S(f) \geq S(f^*)$, where f^* is the symmetric decreasing rearrangement of f.*

Plugging in $f = \mathbf{1}_K$, we see that this theorem really generalizes the isoperimetric inequality.

Using a slightly more complicated notion of a "generalized rearrangement", it is possible to prove functional versions of most of the classic inequalities: Brunn-Minkowski (and its extension to mixed volumes), Alexandrov-Fenchel, and others. As a special case, we have the following extension of Theorem 4:

Theorem 5. *For every $f_1, f_2, \dots, f_n \in \mathrm{QC}(\mathbb{R}^n)$ we have $V(f_1, f_2, \dots, f_n) \geq V(f_1^*, f_2^*, \dots, f_n^*)$.*

For indicator functions, this reduces to the known statement that for every convex bodies $K_1, K_2, \dots, K_n$ in $\mathbb{R}^n$ we have

$$V(K_1, K_2, \dots, K_n) \geq \left(\prod_{i=1}^{n} \mathrm{Vol}(K_i)\right)^{\frac{1}{n}}.$$

Finally, if one is willing to restrict the class of functions, it is possible to prove certain inequalities in a more familiar form. For example, in the class of geometric log-concave functions we have the following Alexandrov type inequalities:

Theorem 6. *Define $g(x) = e^{-|x|}$. For every geometric log-concave function f and every integers $0 \leq k < m < n$ we have*

$$\left(\frac{W_k(f)}{W_k(g)}\right)^{\frac{1}{n-k}} \leq \left(\frac{W_m(f)}{W_m(g)}\right)^{\frac{1}{n-m}},$$

with equality if and only if $f(x) = e^{-c|x|}$ for some $c > 0$.

References

[1] S. Bobkov, A. Colesanti, I. Fragalà, *Quermassintegrals of quasi-concave functions and generalized Prékopa-Leindler inequalities*, preprint.

[2] V. Milman, L. Rotem, *α-concave functions and a functional extension of mixed volumes*, Electronic Research Announcements in Mathematical Sciences **20** (2013), 1–11.

[3] V. Milman, L. Rotem, *Mixed integrals and related inequalities*, Journal of Functional Analysis **264(2)**, 570–604.

[4] L. Rotem, *Support functions and mean width for α-concave functions*, Advances in Mathematics, to appear.

Analytic aspects of the cosine, sine, and Funk transforms on Stiefel and Grassmann Manifolds

Boris Rubin

The cosine, sine, and Funk transforms[1] have a rich history. They arise in integral geometry, harmonic analysis, pseudo-differential operators, group representations, and other branches of mathematics; see the references below. In the classical set-up on the unit sphere S^{n-1} these transforms have the form

$$(\mathcal{C}^\lambda f)(u)=\int_{\mathrm{S}^{n-1}} f(v)\,|\mathrm{Cos}(u,v)|^\lambda\,dv,\quad (\mathcal{S}^\lambda f)(u)=\int_{\mathrm{S}^{n-1}} f(v)\,[\mathrm{Sin}(u,v)]^\lambda\,dv,$$

$$(\mathcal{F}f)(u)=\int_{\{v\in\mathrm{S}^{n-1}|u\cdot v=0\}} f(v)\,d_u v\,,$$

where $\mathrm{Cos}(u,v)=u\cdot v$ and $\mathrm{Sin}(u,v)=(1-|u\cdot v|^2)^{1/2}$ denote the cosine and sine of the angle between the unit vectors u and v, respectively.

Our research addresses the "higher-rank" generalization of these operators for functions on Stiefel and Grassmann manifolds. The main topics are

- Analytic continuation and the structure of the polar sets;
- Connection with the Fourier transform on the space of rectangular matrices;
- Inversion formulas and spectral analysis;
- The group-theoretic realization as intertwining operators between representations of $SL(n,\mathbb{R})$.

More information can be found in [OP, OPR, R].

Let $\mathrm{V}_{n,m}\sim \mathrm{O}(n)/\mathrm{O}(n-m)$ be the Stiefel manifold of $n\times m$ real matrices, the columns of which are mutually orthogonal unit n-vectors. For $v\in\mathrm{V}_{n,m}$, dv stands for the invariant probability measure on $\mathrm{V}_{n,m}$; We write $\mathrm{M}_{n,m}\sim\mathbb{R}^{nm}$ for the space of real matrices $x=(x_{i,j})$ having n rows and m columns and set $|x|_m=\det(x^tx)^{1/2}$, where x^t is the transpose of x. The Siegel gamma function of the cone Ω of positive definite $m\times m$ real symmetric matrices is defined by

$$\Gamma_m(\alpha)=\int_\Omega \exp(-\mathrm{tr}(r))|r|_m^{\alpha-(m+1)/2}dr=\pi^{m(m-1)/4}\prod_{j=0}^{m-1}\Gamma(\alpha-j/2).$$

For $1\le m,k\le n-1$, the higher-rank cosine and sine transforms are defined by

$$(\mathcal{C}^\lambda_{m,k}f)(u)=\int_{\mathrm{V}_{n,m}} f(v)\,|u^tv|^\lambda_m\,dv,\quad (\mathcal{S}^\lambda_{m,k}f)(u)=\int_{\mathrm{V}_{n,m}} f(v)\,|I_m-v^tuu^tv|^\lambda_m\,dv,$$

where $u\in\mathrm{V}_{n,k}$ and I_m denotes the identity $m\times m$ matrix. The corresponding Funk transform has the form

$$(\mathcal{F}_{m,k}f)(u)=\int_{\{v\in\mathrm{V}_{n,m}|\,u^tv=0\}} f(v)\,d_uv,\qquad u\in V_{n,k}.$$

Theorem 1. *Let $1\le m\le k\le n-1$. The following statements hold.*

[1]Don't confuse with the cosine and sine *Fourier* transforms.

(1) *If* $f \in L^1(\mathrm{V}_{n,m})$ *and* $\mathrm{Re}\,\lambda > m-k-1$*, then the integral* $\mathcal{C}^\lambda_{m,k} f$ *converges for almost all* $u \in V_{n,k}$*.*
(2) *If* $f \in C^\infty(\mathrm{V}_{n,m})$*, then for every* $u \in V_{n,k}$*, the function* $\lambda \mapsto (\mathcal{C}^\lambda_{m,k} f)(u)$ *extends to the domain* $\mathrm{Re}\,\lambda \le m-k-1$ *as a meromorphic function with the only poles* $m-k-1, m-k-2, \dots$ *. These poles and their orders are the same as of the gamma function* $\Gamma_m((\lambda+k)/2)$*.*
(3) *The normalized integral* $(\mathcal{C}^\lambda_{m,k} f)(u)/\Gamma_m((\lambda+k)/2)$ *is an entire function of* λ *and belongs to* $C^\infty(\mathrm{V}_{n,k})$ *in the* u*-variable.*

A similar statement, but with different assumptions for λ, is valid for $\mathcal{S}^\lambda_{m,k} f$.

The proof of Theorem 1 relies on the connection between $\mathcal{C}^\lambda_{m,k} f$ and the Fourier transform

$$(\mathcal{F}\varphi)(y) = \int_{\mathrm{M}_{n,m}} e^{\mathrm{tr}(iy^t x)} \varphi(x)\, dx, \qquad y \in \mathrm{M}_{n,m}\ .$$

Let for simplicity $k=m$, $\mathcal{C}^\lambda_m f = \mathcal{C}^\lambda_{m,m} f$. The following theorem holds.

Theorem 2. *Let* $x = vr^{1/2}$ *with* $v \in \mathrm{V}_{n,m}$ *and* $r \in \Omega$ *be the polar decomposition of a matrix* $x \in \mathrm{M}_{n,m}$*. Suppose that* f *is an integrable right* $\mathrm{O}(m)$*-invariant function on* $\mathrm{V}_{n,m}$ *and set* $(E_\lambda f)(x) = |r|_m^{\lambda/2} f(v)$*,* $\lambda \in \mathbb{C}$*. Then for every Schwartz function* ω *for the corresponding distributions we have*

$$\left(\frac{E_\lambda \mathcal{C}^\lambda_m f}{\Gamma_m((\lambda+m)/2)}, \mathcal{F}\omega \right) = c \left(\frac{E_{-\lambda-n} f}{\Gamma_m(-\lambda/2)}, \omega \right),$$

$$c = \frac{2^{m(n+\lambda)}\, \pi^{nm/2}\, \Gamma_m(n/2)}{\Gamma_m(m/2)},$$

where both sides are understood in the sense of analytic continuation.

The next statement shows that the suitably normalized cosine transform, its inverse, and the Funk transform are, in fact, members of the same analytic family.

Theorem 3. *Let* $1 \le m \le k \le n-1$*,* $k+m \le n$*. If* f *is a* C^∞ *right* $\mathrm{O}(m)$*-invariant function on* $\mathrm{V}_{n,m}$*, then for every* $u \in \mathrm{V}_{n,k}$*,*

$$(\mathcal{F}_{m,k} f)(u) = \zeta_\lambda\, (\mathcal{C}^\lambda_{m,k} f)(u)\Big|_{\lambda=-k}$$

in the sense of analytic continuation. If, moreover, $k=m$*, then for every* $u \in \mathrm{V}_{n,m}$*,*

$$(\mathcal{C}^{-\lambda-n}_m \mathcal{C}^\lambda_m f)(u) = \eta_\lambda\, f(u), \qquad \lambda+n, -\lambda \notin \{1,2,3,\dots\}.$$

The coefficients, η_λ *and* ζ_λ *can be explicitly evaluated.*

The cosine transform as an intertwining operator. We introduce the radial and angular components of a matrix $x \in M_{n,m}$ of rank m by

$$\mathrm{rad}(x) = (x^t x)^{1/2} \in \Omega, \qquad \mathrm{ang}(x) = x(x^t x)^{-1/2} \in \mathrm{V}_{n,m},$$

so that $x = \mathrm{ang}(x)\,\mathrm{rad}(x)$. Given $\lambda \in \mathbb{C}$, we define a mapping which assigns to every $g \in \mathrm{GL}(n,\mathbb{R})$ an operator $\pi_\lambda(g)$ acting on measurable functions f on $\mathrm{V}_{n,m}$ by the rule

$$\pi_\lambda(g) f(v) = |\mathrm{rad}(g^{-1}v)|^{-(\lambda+n/2)}\, f(\mathrm{ang}(g^{-1}v)).$$

Clearly, $\pi_\lambda(I_n)$ is an identity operator. One can prove that if f is a measurable right $\mathrm{O}(m)$-invariant function on $\mathrm{V}_{n,m}$, then

$$\pi_\lambda(g_1 g_2)f = \pi_\lambda(g_1)\,\pi_\lambda(g_2)f\,, \qquad g_1, g_2 \in \mathrm{GL}(n,\mathbb{R}).$$

For the restriction of π_λ to $\mathrm{SL}(n,\mathbb{R})$, acting on the space of square integrable right $\mathrm{O}(m)$-invariant functions on $\mathrm{V}_{n,m}$ the following statement holds.

Theorem 4. *Let $\theta : G \to G$ be the involutive automorphism $\theta(g) = (g^{-1})^t$. The cosine transform intertwines representations π_λ and $\pi_{-\lambda} \circ \theta$, namely,*

$$\mathcal{C}_m^\lambda \circ \pi_{\lambda+n/2} = (\pi_{-\lambda-n/2} \circ \theta) \circ \mathcal{C}_m^\lambda,$$

whenever both sides of this equality are analytic functions of λ.

All statements can be reformulated in the language of Grassmann manifolds. Spectral formulas for the higher-rank cosine transforms were obtained in [OP]; see also [OPR].

References

[A] S. Alesker, *The α-cosine transform and intertwining integrals*, Preprint, 2003.

[AB] S. Alesker and J. Bernstein, *Range characterization of the cosine transform on higher Grassmannians*, Advances in Math., **184** (2004), 367–379.

[OP] G. Ólafsson and A. Pasquale, *The Cos$^\lambda$ and Sin$^\lambda$ transforms as intertwining operators between generalized principal series representations of $SL(n+1,K)$*, Advances in Math., **229** (2012), 267–293.

[OPR] G. Ólafsson, A. Pasquale, and B. Rubin, *Analytic and group-theoretic aspects of the cosine transform*, arXiv:1209.1822v1.

[R] B. Rubin, *Funk, cosine, and sine transforms on Stiefel and Grassmann manifolds*, J. of Geom. Anal. (in press).

Hard Lefschetz Theorem for Curvature Measures

Mykhailo Saienko

Let V be a vector space with $\dim V = n$. Consider the so-called *derivation operator* $\mathfrak{L}$ defined as follows

$$\mathfrak{L}\phi(K) := \left.\frac{d}{dt}\right|_{t=0} \phi(K+tB) = \int_{\mathrm{nc}(K)} \mathcal{L}_T\beta,$$

where ϕ is a valuation represented by a form β and $K+tB$ is the Minkowski sum of a convex compact body K and a ball of radius t. $\mathfrak{L}$ decreases the degree of a valuation by 1. The following was shown by Bernig and Bröcker in [1].

Theorem 1 (Hard Lefschetz Theorem for Valuations)**.** *Let $\frac{n}{2} < k \le n$. Then,*

$$\mathfrak{L}^{2k-n} : \mathrm{Val}_k^{sm}(V) \to \mathrm{Val}_{n-k}^{sm}(V)$$

is an isomorphism. In particular, $\mathfrak{L} : \mathrm{Val}_k^{sm}(V) \to \mathrm{Val}_{k-1}^{sm}(V)$ is injective for $k \ge \frac{n+1}{2}$ and surjective for $k \le \frac{n+1}{2}$.

It was conjectured that the space of curvature measures might exhibit a similar sort of symmetry. If one looks at the dimensions of the homogeneous parts of $\mathrm{Curv}^{U(n)}$:

$\mathrm{Curv}_k^{U(n)}$	$n=3$	n=5
0	1	1
1	2	2
2	3	3
3	3	4
4	2	5
5	1	5
6	-	4
7	-	3
8	-	2
9	-	1,

one notices that $\dim \mathrm{Curv}_k^{U(n)} = \dim \mathrm{Curv}_{2n-k-1}^{U(n)}$. The following result generalizes this observation for translation invariant curvature measures.

Theorem 2 (Hard Lefschetz Theorem for Curvature Measures)**.** *Let $\mathfrak{L}$ be the derivation operator on curvature measures defined by the following equation*

$$\mathfrak{L}\Phi(K,S) = \int_{\pi^{-1}(S)\cap nc(K)} \mathcal{L}_T \beta.$$

Let $\frac{n-1}{2} \leq k \leq n-1$ be an integer. Then,

$$\mathfrak{L}^{2k-n+1} : \mathrm{Curv}_k(V) \to \mathrm{Curv}_{n-k-1}(V)$$

is an isomorphism. Particularly, $\mathfrak{L} : \mathrm{Curv}_k(V) \to \mathrm{Curv}_{k-1}(V)$ is injective for $k \geq \frac{n}{2}$ and surjective for $k \leq \frac{n}{2}$.

Compared to the Hard Lefschetz Theorem for valuations, the point of symmetry is $\frac{n-1}{2}$ instead of $\frac{n}{2}$. This implies in particular that $\mathrm{Curv}_0(V) \simeq \mathrm{Curv}_{n-1}(V)$ as opposed to the fact that $\mathrm{Val}_0(V) \simeq \mathrm{Val}_n(V)$. Since $\Omega^{0,n-1}(SV)^V$ is spanned by the volume form of the spherical part of SV, it follows that $\dim \mathrm{Curv}_{n-1}(V) = \dim \mathrm{Curv}_0(V) = 1$.

References

[1] A. Bernig, L. Bröcker *Valuations on manifolds and Rumin cohomology*, J. Differential Geom. **75** (2007), 433–457.

Local tensor valuations

Rolf Schneider

Hadwiger's celebrated characterization theorem for the intrinsic volumes (or quermassintegrals) of convex bodies, which has well-known applications to the proof of integral-geometric formulae, has found analogues and extensions in different directions. For example, there are local counterparts for curvature measures and support measures, and there is Alesker's [2] characterization theorem for the tensor-valued valuations known as Minkowski tensors. The aim of the following is a combination of both extensions, and the final goal is a classification of all local tensor valuations enjoying some natural properties.

In Euclidean space $\mathbb{R}^n$ with a fixed scalar product (also used to identify $\mathbb{R}^n$ with its dual space), we consider the space $\mathcal{K}^n$ of convex bodies with the Hausdorff metric and the vector space $\mathbb{T}^p$ of symmetric tensors of rank $p \in \mathbb{N}_0$. Expressions like ab for symmetric tensors a, b or x^r for vectors x are defined via symmetric tensor products.

The *local Minkowski tensors* are defined by

$$\phi_k^{r,s}(K,\eta) := c_{n,k}^{r,s} \int_\eta x^r u^s \, \Lambda_k(K, \mathrm{d}(x,u))$$

for $r, s, k \in \mathbb{N}_0$ with $k \leq n-1$, convex bodies $K \in \mathcal{K}^n$ and sets η in the Borel σ-algebra $\mathcal{B}(\Sigma)$, where Σ denotes the product space $\mathbb{R}^n \times \mathbb{S}^{n-1}$ of $\mathbb{R}^n$ and the unit sphere $\mathbb{S}^{n-1}$. Here $\Lambda_k(K,\cdot)$ is the kth support measure (or generalized curvature measure) of K. The normalizing factors are given by $c_{n,k}^{r,s} = \omega_{n-k}/r!s!\omega_{n-k+s}$, where $\omega_n = n\kappa_n$ is the surface area of the unit sphere. The (global) *Minkowski tensors* are

$$\Phi_k^{r,s}(K) := \phi_k^{r,s}(K,\Sigma) \quad \text{and} \quad \Psi_r(K) = \Phi_n^{r,0} := \frac{1}{r!}\int_K x^r \, \mathrm{d}x.$$

The tensors $\Phi_k^{r,s}$ appear naturally if the moment tensor Ψ_r is applied to an outer parallel body $K + \rho B^n$ with $\rho > 0$ and the unit ball B^n: there is the Steiner type formula

$$\Psi_r(K + \rho B^n) = \sum_{k=0}^{n+r} \rho^{n+r-k} \kappa_{n+r-k} \sum_{s=\max\{0,r-k\}}^{r} \Phi_{k-r+s}^{r-s,s}(K).$$

The Minkowski tensors are generalized by the local Minkowski tensors, in the same way as the quermassintegrals are localized by the support measures. The mapping $\Gamma : \mathcal{K}^n \times \mathcal{B}(\Sigma) \to \mathbb{T}^p$ defined by $\Gamma(K,\eta) = \phi_k^{r,s}(K,\eta)$ (thus $p = r+s$) has the following properties.

(A) $\Gamma(K,\cdot)$ is a $\mathbb{T}^p$-valued measure, for each $K \in \mathcal{K}^n$.
(B) $\Gamma(\cdot,\eta)$ is a measurable valuation, for each $\eta \in \mathcal{B}(\Sigma)$.
(C) Γ is rotation covariant, that is, $\Gamma(\vartheta K, \vartheta\eta) = \vartheta\Gamma(K,\eta)$ for $\vartheta \in \mathrm{O}(n)$.
(D) Γ is translation covariant of degree p, that is, there are tensors $\Gamma_{p-j}(K,\eta) \in$

$\mathbb{T}^{p-j}$, $j = 0, \dots, p$, such that

$$\Gamma(K+t, \eta+t) = \sum_{j=0}^{p} \Gamma_{p-j}(K, \eta) t^j$$

for $t \in \mathbb{R}^n$.
(E) Γ is locally defined, that is, for $\eta \in \mathcal{B}(\mathbb{R}^n \times \mathbb{S}^{n-1})$ and $K, K' \in \mathcal{K}^n$ with $\eta \cap \operatorname{Nor} K = \eta \cap \operatorname{Nor} K'$ always $\Gamma(K, \eta) = \Gamma(K', \eta)$, where $\operatorname{Nor} K$ denotes the normal bundle of K.
(F) Γ is weakly continuous, that is, for $\lim_{i\to\infty} K_i = K$ always

$$\lim_{i\to\infty} \int_\Sigma f \,\mathrm{d}\Gamma(K_i, \cdot) = \int_\Sigma f \,\mathrm{d}\Gamma(K, \cdot)$$

for all continuous functions $f : \Sigma \to \mathbb{R}$ (the integral is defined coordinate-wise).

Properties (C) and (D) together are known as *isometry covariance.* For the following, it is important to note that also the constant metric tensor $Q \in \mathbb{T}^2$, defined by the scalar product, is isometry covariant.

Although some integral-geometric formulae for tensor valuations in the plane were studied quite early by Müller [6], who took up a suggestion of W. Blaschke, a decisive investigation of tensor valuations on convex bodies started only with the work of McMullen [5]. One of his questions, for an axiomatic characterization of the (global) Minkowski tensors, was answered by Alesker [2] (with the bulk of the work done in [1]).

Theorem (Alesker) *Let $p \in \mathbb{N}_0$. Every continuous isometry covariant valuation on $\mathcal{K}^n$ with values in $\mathbb{T}^p$ is a linear combination, with constant coefficients, of the tensor valuations $Q^m \Phi_k^{r,s}$, with $m, r, s, k \in \mathbb{N}_0$, $k \le n$, $k \le n-1$ if $s \ne 0$, and $2m + r + s = p$.*

For $p = 0$, this is Hadwiger's characterization theorem. The case $p = 1$ was settled in [4], based on work in [7].

Since Hadwiger's theorem has found local counterparts for curvature measures, surface area measures and support measures, it is a natural challenge to prove a local counterpart to Alesker's theorem.

As it turned out, there are more mappings sharing properties (A) – (F) than, as first expected, the linear combinations of the mappings $Q^m \phi_k^{r,s}$. In a first step of investigation, the space $\mathcal{K}^n$ of convex bodies is replaced by the space $\mathcal{P}^n$ of convex polytopes in $\mathbb{R}^n$ (with corresponding alterations of properties (A) – (F)). The following modification of the local Minkowski tensors must be taken into account. For a linear subspace L of $\mathbb{R}^n$, let $Q_L(x, y)$ be the scalar product of the orthogonal projections of the vectors x, y to L. For $P \in \mathcal{P}^n$ and $F \in \mathcal{F}_k(P)$ (the set of k-faces of P, for $k \in \{0, \dots, n-1\}$), let $L(F)$ be the direction space of the face F, that is, the subspace parallel to the affine hull of F. Then we define

$$\phi_k^{r,s,j}(P, \eta) := c_{n,k}^{r,s} \sum_{F \in \mathcal{F}_k(P)} Q_{L(F)}^j \int_{\eta \cap F \times \nu(P,F)} x^r u^s \,\Lambda_k(\mathrm{d}(x, u))$$

for $r, s, j, k \in \mathbb{N}_0$ with $1 \le k \le n-1$, where $\nu(P, F)$ is the set of outer unit normal vectors of P at its face F. Further, $\phi_0^{r,s,j} := \phi_0^{r,s}$.

The following characterization theorem, essentially proved in [8], is the main result.

Theorem *Let $p \in \mathbb{N}_0$. Let $\Gamma : \mathcal{P}^n \times \mathcal{B}(\Sigma) \to \mathbb{T}^p$ be a mapping with the following properties:*

(a) $\Gamma(P, \cdot)$ *is a $\mathbb{T}^p$-valued measure, for each $P \in \mathcal{P}^n$,*

(b) Γ *is isometry covariant,*

(c) Γ *is locally defined.*

Then Γ is a linear combination, with constant coefficients, of the mappings

$$Q^m \phi_k^{r,s,j}$$

with $m, j, r, s \in \mathbb{N}_0$, $k \in \{0, \dots, n-1\}$ and $2m + 2j + r + s = p$.

Conversely, each mapping $Q^m \phi_k^{r,s,j}$ has the properties (a), (b), (c). We stress the fact that (motivated by a characterization theorem for support measures due to Glasauer [3]) we have not assumed here that Γ is a valuation. This property is a consequence: each of the mappings $Q^m \phi_k^{r,s,j}$ is a valuation in its first argument.

We also point out that we have not assumed any continuity property. In fact, this is a delicate point, which is presently the subject of joint investigation with Daniel Hug. We already know that $\phi_k^{r,s,j}$ has a weakly continuous extension to general convex bodies if $k = 0$ (trivially) or $k = n-1$ or $j \le 1$, but that there is no such extension in the other cases. We expect that this will lead to a classification of all mappings $\Gamma : \mathcal{K}^n \times \mathcal{B}(\Sigma) \to \mathbb{T}^p$ with properties (A), (C) – (F), which then also satisfy (B). Integral-geometric applications are also envisaged.

References

[1] S. Alesker, *Continuous rotation invariant valuations on convex sets*, Ann. of Math. **149** (1999), 977–1005.

[2] S. Alesker, *Description of continuous isometry covariant valuations on convex sets*, Geom. Dedicata **74** (1999), 241–248.

[3] S. Glasauer, *A generalization of intersection formulae of integral geometry*, Geom. Dedicata **68** (1997), 101–121.

[4] H. Hadwiger and R. Schneider, *Vektorielle Integralgeometrie*, Elem. Math. **26** (1971), 49–57.

[5] P. McMullen, *Isometry covariant valuations on convex bodies*, Rend. Circ. Mat. Palermo (2) Suppl. **50** (1997), 259–271.

[6] H. R. Müller, *Über Momente ersten und zweiten Grades in der Integralgeometrie*, Rend. Circ. Mat. Palermo (2) **2** (1953), 1–21.

[7] R. Schneider, *Krümmungsschwerpunkte konvexer Körper II*, Abh. Math. Sem. Univ. Hamburg **37** (1972), 204–217.

[8] R. Schneider, *Local tensor valuations on convex polytopes*, Monatsh. Math., published online, DOI 10.1007/s00605-012-0430-9.

Integral Geometry of Complex Space Forms

Gil Solanes
(joint work with Andreas Bernig, Joseph H.G. Fu)

We present some recent progress in the determination of kinematic formulas in isotropic spaces.

Definition 1. A (smooth)*valuation* on a riemannian manifold M is a functional $\mu\colon \mathcal{P}(M) \to \mathbb{R}$, defined on the space of compact manifolds with corners $\mathcal{P}(M)$ by

$$\mu(A) = \int_{N(A)} \omega + \int_A \eta$$

where $N(A)$ is a certain current in the sphere bundle SM called the *normal cycle* of A, and $\omega \in \Omega^{n-1}(SM), \eta \in \Omega^n(M)$ are fixed differential forms. A *curvature measure* on M associates to each $A \in \mathcal{P}(M)$ a Borel measure $\Phi(A, \cdot)$ by

$$\Phi(A, U) = \int_{N(A)\cap\pi^{-1}U} \omega + \int_{A\cap U} \eta, \qquad U \subset M$$

where $\pi : SM \to M$ is the projection.

Curvature measures first appeared in Federer's work on kinematic formulas in euclidean space. Later, Joseph H.G. Fu showed the existence of analogous formulas in other geometries.

Theorem 2 ([6])**.** *Let (M, G) be an isotropic space (i.e. M is a riemannian manifold and G a Lie group acting on M by isometries such that the action on $S(M)$ is transitive). Given a basis $\Phi_1, \dots, \Phi_d$ of the (finite-dimensional) space $\mathcal{C}^G$ of G-invariant curvature measures, there exist constants c_{kij} such that*

$$\int_G \Phi_k(A \cap gB, U \cap gV)dg = \sum_{i,j} c_{kij}\Phi_i(A, U)\Phi_j(B, V),$$

where dg denotes the Haar measure on G.

Of course, by globalizing (i.e. taking $U = V = M$), one gets analogous kinematic formulas at the level of valuations. Kinematic formulas involving curvature measures are called *local*, and those involving valuations are called *global*.

Classically, such kinematic formulas were explicitly known only in the real space forms ($M = \mathbb{R}^n, S^n$ or H^n). More recently, the discovery by Semyon Alesker of an algebra structure on the space of valuations (cf. [2, 3]) has led to a new approach to integral geometry. This new trend, called *algebraic integral geometry*, has allowed the computation of previously unreachable kinematic formulas, such as those of the complex space forms.

The key to this new approach is the so-called *fundamental theorem of algebraic integral geometry*. For simplicity we state it here only for compact manifolds.

Theorem 3. *Let* $\mathcal{V}^G$ *be the space of* G*-invariant valuations on a compact isotropic space* (M,G). *Let* $m : \mathcal{V}^G \otimes \mathcal{V}^G \to \mathcal{V}^G$ *be the Alesker product of valuations. Let* $\mathrm{pd} : \mathcal{V}^G \to (\mathcal{V}^G)^*$ *be the so-called* Alesker-Poincaré duality *given by*

$$\langle \mathrm{pd}(\mu), \varphi \rangle = \frac{m(\mu,\varphi)(M)}{\mathrm{vol}(M)}.$$

Let $k : \mathcal{V}^G \to \mathcal{V}^G \otimes \mathcal{V}^G$ *be the* kinematic operator *defined by*

$$k(\mu)(A,B) = \int_G \mu(A \cap gB)dg,$$

where the Haar measure dg *is normalized so that* $\mathrm{vol}(G) = \mathrm{vol}(M)$. *Then the following diagram commutes*

$$\begin{array}{ccc} \mathcal{V}^G & \xrightarrow{k} & \mathcal{V}^G \otimes \mathcal{V}^G \\ \big\downarrow{\scriptstyle \mathrm{pd}} & & \big\downarrow{\scriptstyle \mathrm{pd}\otimes\mathrm{pd}} \\ (\mathcal{V}^G)^* & \xrightarrow{m^*} & (\mathcal{V}^G)^* \otimes (\mathcal{V}^G)^*. \end{array}$$

Therefore, kinematic formulas can be deduced from the algebra structure of $\mathcal{V}^G$. The first application of this method was [4] where a complete description is given of the global (i.e. at the level of valuations) integral geometry of $\mathbb{C}^n$ under the action of the affine unitary group $\overline{U(n)}$. The algebra structure of the space $\mathrm{Val}^{U(n)}$ of $\overline{U(n)}$-invariant valuations had been previously found in [7]:

$$\mathrm{Val}^{U(n)} = \frac{\mathbb{R}[s,t]}{(f_{n+1}, f_{n+2})}$$

where t denotes the mean width, and $s \in \mathrm{Val}^{U(n)}$ is the average of the projected area on complex lines. The polynomials f_{n+1}, f_{n+2} were explicitly given. This allowed the computation of global kinematic formulas at the level of valuations. However, local kinematic formulas (i.e. at the level of curvature measures) could not be found with this method. A related problem was to find the global kinematic formulas in the non-flat complex space forms: $\mathbb{C}P^n, \mathbb{C}H^n$. Indeed both questions are essentially equivalent by Howard's transfer principle.

Some first results on the integral geometry of $\mathbb{C}P^n$ and $\mathbb{C}H^n$ were obtained in [1]. More recently, the full array of global kinematic formulas in all complex space forms has been found in [5]. From these, the local kinematic formulas were deduced. Next we briefly sketch the main steps leading to these results.

Let M^n_λ denote the complex space form of constant holomorphic curvature $4\lambda \neq 0$, and let G be its isometry group. As in the flat case, the algebra $\mathcal{V}^G$ of invariant valuations in M^n_λ is generated by two elements t, s. The valuation $s \in \mathcal{V}^G$ is naturally defined by integrating the Euler characteristic of intersections with totally geodesic complex hyperplanes. As for $t \in \mathcal{V}^G$, it is defined as the pull-back of the mean width through an arbitrary isometric immersion of M^n_λ into an euclidean space (of sufficiently high dimension). This construction does not depend on the immersion as famously shown by H. Weyl.

Rather surprisingly, the algebras $\mathcal{V}^G$ and $\mathrm{Val}^{U(n)}$ are isomorphic.

Theorem 4. *There exists an isomorphis of algebras* $I : \mathrm{Val}^{U(n)} \to \mathcal{V}^G$ *given by* $I(s) = s, I(t) = t\sqrt{1-\lambda s}$.

By Theorem 3, this allows to compute the kinematic operator $k_\lambda : \mathcal{V}^\lambda \to V^\lambda \otimes V^\lambda$, at least algorithmically. Luckily, the computation turns out to simplify in a very nice way: there exists an explicitly known and rather simple isomorphism $F : \mathrm{Val}^{U(n)} \to \mathcal{V}^G$ of *vector spaces* such that

$$k_\lambda(\chi) = (F \otimes F) \circ k(\chi).$$

In other words, the principal kinematic formulas are formally the same in all complex space forms, when written in terms of suitably chosen bases. At present, the geometric meaning of these bases is partially, but not fully understood.

While the global kinematic formulas are easier to find than the local ones, Howard's transfer principle states that the local kinematic formulas are identical within each family of space forms. As a result, the knowledge of the global kinematic formulas in M_λ^n for all λ has been used in [5] to deduce the local kinematic formulas, common to all λ. In this process, a key tool was the introduction of a module structure on the space of curvature measures over the algebra of valuations. This module structure was computed first in the flat case $\mathbb{C}^n$, and then, as an application of the local kinematic formulas, in the curved spaces $\mathbb{C}P^n$ an $\mathbb{C}H^n$.

References

[1] J. Abardia, E. Gallego, G. Solanes *Gauss-Bonnet theorem and Crofton type formulas in complex space forms*, Israel J. Math. **187** (2012), 287–315.

[2] S. Alesker *The multiplicative structure on polynomial valuations*, Geom. Funct. Anal. **14** (2004), 1–26.

[3] S. Alesker, J. H. G. Fu *Theory of valuations on manifolds III. Multiplicative structure in the general case*, Trans. Amer. Math. Soc. **360** (2008), 1951–1981.

[4] A. Bernig, J.H.G. Fu *Hermitian Integral Geometry*, Annals of Math. **173** (2011), 907–945.

[5] A. Bernig, J.H.G. Fu, G. Solanes, *Integral Geometry of Complex Space Forms*, arXiv:1204.0604.

[6] J.H.G. Fu *Kinematic formulas in integral geometry*, Indiana Univ. Math. J. **39** (1990), 1115–1154.

[7] J.H.G. Fu *Structure of the unitary valuation algebra*, J. Differential Geom. **72** (2006), 509–533.

Integral geometry of hermitian area measures

Thomas Wannerer

A classical result by Schneider [3] states that there exist constants c_{ij}^k such that

$$\int_{SO(n)} S_k(K + gL, \phi \cdot L_g\psi)\, dg = \sum_{i,j=1}^{n-1} c_{ij}^k\; S_i(K,\phi) S_j(K,\psi)$$

for all convex bodies $K, L \subset \mathbb{R}^n$ and bounded Borel functions $\phi, \psi \colon S^{n-1} \to \mathbb{R}$. Here $S_i(K, \,\cdot\,)$, $i = 0, \dots, n-1$, is the area measure of order i of a convex body K and $L_g\psi(v) = \psi(g^{-1}v)$ left translation by an element $g \in SO(n)$.

We prove that such additive kinematic formulas exist also if we replace the rotation group $SO(n)$ by a closed, connected subgroup $G \subset SO(n)$ acting transitively on the unit sphere S^{n-1}. We denote by $\Omega^{n-1}(S\mathbb{R}^n)^G$ the space of smooth $(n-1)$-forms on the sphere bundle $S\mathbb{R}^n$ which are invariant under the group of affine transformations $G \ltimes \mathbb{R}^n$ and let $\pi_1 \colon S\mathbb{R}^n \to \mathbb{R}^n$ and $\pi_2 \colon S\mathbb{R}^n \to S^{n-1}$ denote the natural projections. Furthermore, $N(K)$ denotes the normal cycle of a convex body.

Theorem 1. *Let $\{\omega_1, \dots, \omega_m\}$ be a basis of $\Omega^{n-1}(S\mathbb{R}^n)^G$. If $\omega \in \Omega^{n-1}(S\mathbb{R}^n)^G$, then there exist constants c_{ij} such that*

$$\int_G N(K + gL, \pi_2^*\phi \cdot \pi_2^* L_g\psi \cdot \omega)\, dg = \sum_{i,j=1}^{m} c_{ij} N(K, \pi_2^*\phi \cdot \omega_i) N(L, \pi_2^*\psi \cdot \omega_j)$$

for all convex bodies $K, L \subset \mathbb{R}^n$ and bounded Borel functions $\phi, \psi \colon S^{n-1} \to \mathbb{R}$.

The proof of the above theorem uses a method developed by Fu [2] to the prove the existence of intersectional kinematic formulas in isotropic spaces. In particular, we obtain $\sum c_{ij}\omega_i \otimes \omega_j$ as a certain Gelfand transform of $\omega \wedge dg$.

While Theorem 1 establishes the existence of general additive kinematic formulas, more work needs to be done in order to obtain explicit formulas. We show that in the case $G = U(n)$ and $\psi \equiv const$ we can obtain explicit kinematic formulas from the $\mathrm{Val}^{U(n)}$-module structure on $\mathrm{Area}^{U(n)}$, the space of unitarily invariant area measures, which was introduced in [4]. Let $A_{U(n)} \colon \mathrm{Area}^{U(n)} \to \mathrm{Area}^{U(n)} \otimes \mathrm{Val}^{U(n)}$ denote the semi-local kinematic operator

$$A_{U(n)}(\Psi)(K, L, \phi) = \int_{U(n)} \Psi(K + gL, \phi)\, dg,$$

$a_{U(n)} \colon \mathrm{Val}^{U(n)} \to \mathrm{Val}^{U(n)} \otimes \mathrm{Val}^{U(n)}$ the global kinematic operator

$$A_{U(n)}(\mu)(K, L) = \int_{U(n)} \mu(K + gL)\, dg,$$

and let $* \colon \mathrm{Area}^{U(n)} \otimes \mathrm{Val}^{U(n)} \to \mathrm{Area}^{U(n)}$ denote the action of $\mathrm{Val}^{U(n)}$ on $\mathrm{Area}^{U(n)}$.

Theorem 2.

$$A_{U(n)}(\Psi) = a_{U(n)}(\mathrm{vol}) * (\Psi \otimes \mathrm{vol}).$$

Since both $a_{U(n)}(\mathrm{vol})$ and the $\mathrm{Val}^{U(n)}$-module structure on $\mathrm{Area}^{U(n)}$ are known explicitly (see [1] and [4]), the above theorem yields explicit semi-local kinematic formulas.

References

[1] A. Bernig and J. H. G. Fu, *Hermitian integral geometry*, Ann. of Math. (2) **173** (2011), 907–945.

[2] J. H. G. Fu, *Kinematic formulas in integral geometry*, Indiana Univ. Math. J. **39** (1990), 1115–1154.

[3] R. Schneider, *Kinematische Berührmaße für konvexe Körper und Integralrelationen für Oberflächenmaße*, Math. Ann. **218** (1975), 253–267 (German).

[4] T. Wannerer, *The module of unitarily invariant area measures*, available at `arXiv:1207.6481 [math.DG]`

Flag Measures of Convex Bodies

Wolfgang Weil
(joint work with Daniel Hug et al.)

The classical support measures of convex bodies K can be introduced by a local version of the Steiner formula for intrinsic volumes. The support measures are concentrated on the normal bundle $\mathrm{Nor}(K)$ of K. The curvature measures and the area measures of K appear as images under the projections $(x, u) \mapsto x$ and $(x, u) \mapsto u, (x, u) \in \mathrm{Nor}(K)$. Whereas the curvature measures are used in Integral and Stochastic Geometry, the area measures are helpful in classical Convex Geometry, in particular they are used for certain integral representations of mixed volumes and projection functions and for representations of translation invariant valuations. In order to give more general results of this kind, measures on suitable flag manifolds seem to be appropriate tools.

Such flag measures of convex bodies can be introduced by a local Steiner formula in the affine Grassmannian $A(d, k)$ of k-dimensional flats in $\mathbb{R}^d$. We describe the corresponding approach and summarize some properties of flag measures. We use flag measures to give an integral representation of the mixed volume $V(K[j], M[d-j])$ of two bodies K, M and a formula for the projection function $v_j(K, \cdot) = V_j(K|\cdot)$ of a body K, $j = 0, ..., d$. We show that flag measures help to construct a (translation invariant) valuation on polytopes which has a natural continuity property, but cannot be extended (in a continuous way) to a valuation on all convex bodies. Finally, we report on the construction of mixed flag measures and mention an application to the proximity in a stationary Poisson process of flats.

References

[1] Hinderer, W., Hug, D. and Weil, W.: Extensions of translation invariant valuations on polytopes. In preparation (2013)
[2] Hug, D., Rataj, J. and Weil, W.: A product integral representation of mixed volumes of two convex bodies. *Adv. Geom.* (appeared online on 22.1.2013) doi:10.1515/advgeom-2012-0044
[3] Hug, D., Türk, I. and Weil, W.: Flag measures for convex bodies. In: *Asymptotic Geometric Analysis*, ed. by M. Ludwig et al., Fields Institute Communications **68**, Springer (2013), 145–187

Affine Invariant Points

Elisabeth Werner
(joint work with Mathieu Meyer and Carsten Schütt)

In this paper, we address several question that were left open in Grünbaum's paper [1] on affine invariant points.

Let $\mathcal{K}_n$ be the set of all convex bodies in $\mathbb{R}^n$ (i.e., compact convex subsets of $\mathbb{R}^n$ with nonempty interior), equipped with the Hausdorff metric

$$d_H(K_1, K_2) = \min\{\lambda \geq 0 : K_1 \subseteq K_2 + \lambda B_2^n; K_2 \subseteq K_1 + \lambda B_2^n\},$$

where B_2^n is the Euclidean unit ball centered at 0.

Then a map $p : \mathcal{K}_n \to \mathbb{R}^n$ is called an affine invariant point, if p is continuous and if for every nonsingular affine map $T : \mathbb{R}^n \to \mathbb{R}^n$ one has,

$$p(T(K)) = T(p(K)).$$

An important example of an affine invariant point is the centroid g,

$$g(K) = \frac{\int_K x dx}{|K|}.$$

Let $\mathfrak{P}_n$ be the set of affine invariant points on $\mathcal{K}_n$,

$$\mathfrak{P}_n = \{p : \mathcal{K}_n \to \mathbb{R}^n \big| \ p \ \text{ is affine invariant}\}.$$

Observe that $\mathfrak{P}_n$ is an affine subspace of $C(\mathcal{K}_n, \mathbb{R}^n)$, the continuous functions on $\mathcal{K}_n$ with values in $\mathbb{R}^n$. We denote by $V\mathfrak{P}_n$ the subspace parallel to $\mathfrak{P}_n$. Thus, with the centroid g,

$$V\mathfrak{P}_n = \mathfrak{P}_n - g.$$

Grünbaum [1] posed the problem if there is a finite basis of affine invariant points, i.e. affine invariant points $p_i \in \mathfrak{P}_n$, $1 \leq i \leq l$, such that every $p \in \mathfrak{P}_n$ can be written as

$$p = \sum_{i=1}^{l} \alpha_i p_i, \quad \text{with } \alpha_i \in \mathbb{R} \quad \text{and } \sum_{i=1}^{l} \alpha_i = 1.$$

We answer this question in the negative and prove:

Theorem 1. *$V\mathfrak{P}_n$ is infinite dimensional for all $n \geq 2$.*

In fact, with a suitable norm, $V\mathfrak{P}_n$ is a Banach space. Hence, by Baire's theorem, a basis of $\mathfrak{P}_n$ is not even countable.

For a fixed body $K \in \mathcal{K}_n$, we let

$$\mathfrak{P}_n(K) = \{p(K) : p \in \mathfrak{P}_n\}.$$

Grünbaum also conjectured [1] that for every $K \in \mathcal{K}_n$,

$$\mathfrak{P}_n(K) = \mathfrak{F}_n(K),$$

where $\mathfrak{F}_n(K) = \{x \in \mathbb{R}^n : Tx = x, \text{ for all affine } T \text{ with } TK = K\}$. We give a positive answer to this conjecture, when $\mathfrak{P}_n(K)$ is $(n-1)$-dimensional. Note also that if K has *enough symmetries*, in the sense that $\mathfrak{F}_n(K)$ is reduced to one point x_K, then $\mathfrak{P}_n(K) = \{x_K\}$.

Theorem 2. *Let $K \in \mathcal{K}_n$ be such that $\mathfrak{P}_n(K)$ is (n-1)-dimensional. Then*

$$\mathfrak{P}_n(K) = \mathfrak{F}_n(K).$$

Grünbaum [1] also asked, whether $\mathfrak{P}_n(K) = \mathbb{R}^n$, if $\mathfrak{F}_n(K) = \mathbb{R}^n$. A first step toward solving this problem, is to clarify if there is a convex body K such that $\mathfrak{P}_n(K) = \mathbb{R}^n$. Here, we answer this question in the affirmative and prove that the set of all K such that $\mathfrak{P}_n(K) = \mathbb{R}^n$, is dense in $\mathcal{K}_n$ and consequently the set of all K such that $\mathfrak{P}_n(K) = \mathfrak{F}_n(K)$, is dense in $\mathcal{K}_n$.

Theorem 3. *The set of all $K \in \mathcal{K}_n$ such that $\mathfrak{P}_n(K) = \mathbb{R}^n$ is open and dense in $(\mathcal{K}_n, d_H)$.*

To establish Theorems 1 - 3, we need to introduce new examples of affine invariant points, that have not previously been considered in the literature, among them affine invariant points that are related to the floating body.

Let $K \in \mathcal{K}_n$ and $0 \le \delta < \left(\frac{n}{n+1}\right)^n$. For $u \in \mathbb{R}^n$ and $a \in \mathbb{R}$, $H = \{x \in \mathbb{R}^n : \langle x, u\rangle = a\}$ is the hyperplane orthogonal to u and $H^+ = \{x \in \mathbb{R}^n : \langle x, u\rangle \ge a\}$ and $H^- = \{x \in \mathbb{R}^n : \langle x, u\rangle \le a\}$ are the two half spaces determined by H. Then the (convex) floating body K_δ [2] of K is the intersection of all halfspaces H^+ whose defining hyperplanes H cut off a set of volume at most $\delta\ \mathrm{vol}_n(K)$ from K,

$$K_\delta = \bigcap_{\{H: |H^- \cap K| \le \delta\ \mathrm{vol}_n(K)\}} H^+.$$

Then we get the following new affine invariant points.

Proposition 4. *Let $0 < \delta < \left(\frac{n}{n+1}\right)^n$ and let $p : \mathcal{K}_n \to \mathbb{R}^n$ be an affine invariant point. Then $K \to p(K_\delta)$ is also an affine invariant point. In particular, for the centroid g, $K \mapsto g(K \setminus K_\delta)$ is an affine invariant point.*

References

[1] B. Grünbaum, *Measures of symmetry for convex sets*, Proc. Sympos. Pure Math. 7, (1963), 233–270.

[2] C. Schütt and E. Werner, *The convex floating body*, Math. Scand. **66**, (1990), 275–290.

Curvature notions for singular sets

Martina Zähle

In the first part of the talk some recent joint work with Jan Rataj on general Legendrian cycles in is presented (see [14]).

A *Legendrian cycle* T is an integer multiplicity rectifiable current on $\mathbb{R}^{2d}$ such that spt$T \subset \mathbb{R}^d \times S^{d-1}$, $\partial T = 0$, and $T\lfloor\alpha = 0$ for the contact 1-form α acting as $\langle (u,v), \alpha(x,n)\rangle = u \cdot n$.

An explicit integral representation of the restriction of such currents to the *Lipschitz-Killing $(d-1)$-forms* $\varphi_0, \ldots, \varphi_{d-1}$ admits, in particular, a short proof of a theorem of J. Fu [5]:

Theorem 1. *$T\lfloor\varphi_0 = 0$ implies $T = 0$, i.e., T is uniquely determined by its restriction to the generalized Gauss curvature form φ_0.*

This theorem has various applications in geometric integration theory. Note that the differential form φ_0 can be defined by means of the projection onto the second (spherical) component. The other marginal case φ_{d-1} corresponds with the projection onto the first (spatial) component and leads to a certain counterpart: We call a Legendrian cycle T *full-dimensional* if $\mathcal{H}^{d-1}(W_T \cap \rho(W_T)) = 0$ where $W_T \subset \mathbb{R}^d \times S^{d-1}$ is carrying T and $\rho : (x,n) \mapsto (x,-n)$ denotes the normal reflection. Then the analogue to the above result is the following.

Theorem 2. *Let T be a full-dimensional Legendrian cycle such that $T\lfloor\varphi_{d-1} = 0$. Then $T = 0$.*

The properties of a (general) Legendrian cycle do not necessarily reflect the geometric behaviour of the associated carrying sets. Hence, an additional topological condition - local Gauss-Bonnet formula - is often imposed. In this case $T\lfloor\varphi_k$ agrees with the *k-th Lipschitz-Killing curvature measure* of the underlying geometric set arising from the the projection of spt$T$ onto the spatial component. (See also [6],[7],[1].)

An introduction to fractal versions of these curvature measures is given in the second part of the talk.
For many fractal sets the parallel sets of Lebesgue-a.a. small distances are Lipschitz manifolds of bounded curvature in the sense of [12]. For dimensions $d \leq 3$ this is true for any compact set in $\mathbb{R}^d$ ([5]). Therefore their Lipschitz-Killing curvature-direction measures are determined, which also include information on the anisotropy structure. Under several additional conditions for certain classes of fractals with (local) scaling properties it has been shown that the appropriately rescaled measures of the parallel sets converge (in the average) to some limit measures. The latter are called fractal curvature (-direction) measures, and there exist different methods for proving the existence of these limits as well as the corresponding integral representations. We give a short survey on these developments over the last years, in particular, for the special case of the Minkowski content (cf. [2], [3], [4], [8], [9], [10], [11], [13], [15], [16], [18], [19]).

References

[1] A. Bernig, *Support functions, projections and Minkowski addition of Legendrian cycles*, Indiana Univ. Math. J. **55** (2006), 443–464.

[2] T. Bohl, *Fractal curvatures and Minkowski content of self-conformal sets*, Preprint, arXiv:1211.3421v1, 2012.

[3] T. Bohl, M. Zähle, *Curvature-direction measures of self-similar sets*, Geom. Dedicata (to appear).

[4] U. Freiberg, S. Kombrink, *Minkowski content and local Minkowski content for a class of self-conformal sets*, Geom. Dedicata **159** (2012), 307-325.

[5] J.H.G. Fu: *Tubular neighborhoods in Euclidean spaces*, Duke Math. J. **52** (1985), 1025-1046.

[6] H.J.G. Fu, *Monge-Ampère functions I*, Indiana Univ. Math. J. **38** (1989), 745–771.

[7] J.H.G. Fu, *Curvature measures of subanalytic sets*, Amer. J. Math. **116** (1994), 819–880.

[8] D. Gatzouras, *Lacunarity of self-similar and stochastically self-similar sets*, Trans. Amer. Math. Soc. **352** (2000), 1953–1983.

[9] M. Kesseböhmer, S. Kombrink, *Fractal curvature measures and Minkowski content for self-conformal subsets of the real line*, Adv. Math. **230** (2012), 2474-2512.

[10] M.L. Lapidus, E.P.J. Pearse, and S. Winter, *Minkowski measurability results for self-similar tilings and fractals with monophase generators*, Preprint, arXiv:1104.1641v1, (2011).

[11] J. Rataj, S. Winter, *Characterization of Minkowski measurability in terms of surface area*, J. Math. Anal. Appl. (to appear).

[12] J. Rataj, M. Zähle, *General normal cycles and Lipschitz manifolds of bounded curvature*, Ann. Global Anal. Geom. **27** (2005), 135–156.

[13] J. Rataj, M. Zähle, *Curvature densities of self-similar sets*, Indiana Univ. Math. J. (to appear).

[14] J. Rataj, M. Zähle, *Legendrian cycles and curvature*, Preprint (2013).

[15] S. Winter, *Curvature measures and fractals*, Dissertationes Math. **453** (2008), 1–66.

[16] S. Winter, *Curvature bounds for neighborhoods of self-similar sets*, Comment. Math. Univ. Carolin. **52** (2011), 205-226.

[17] S. Winter, M. Zähle, *Fractal curvature measures of self-similar sets*, Adv. Geom. (to appear), arxiv:1007.0696 (2012).

[18] M. Zähle, *Lipschitz-Killing curvatures of self-similar random sets*, **363** (2011), 2663-2684.

[19] M. Zähle, *Curvature measures of fractal sets*, Contemp. Math. (to appear).

Reporter: Mykhailo Saienko

Participants

Dr. Judit Abardia
Institut für Mathematik
Goethe-Universität Frankfurt
Robert-Mayer-Str. 6-10
60325 Frankfurt am Main
GERMANY

Prof. Dr. Semyon Alesker
Department of Mathematics
School of Mathematical Sciences
Tel Aviv University
P.O.Box 39040
Ramat Aviv, Tel Aviv 69978
ISRAEL

Astrid Berg
Institut f. Diskrete Mathematik &
Geometrie
Technische Universität Wien
Wiedner Hauptstr. 8 - 10
1040 Wien
AUSTRIA

Prof. Dr. Andreas Bernig
Institut für Mathematik
Goethe-Universität Frankfurt
Robert-Mayer-Str. 6-10
60325 Frankfurt am Main
GERMANY

Dmitry Faifman
Department of Mathematics
School of Mathematical Sciences
Tel Aviv University
Ramat Aviv
Tel Aviv 69978
ISRAEL

Prof. Dr. Joseph Fu
Department of Mathematics
University of Georgia
Athens, GA 30602
UNITED STATES

Prof. Dr. Daniel Hug
Institut für Stochastik
Karlsruher Institut f. Technologie (KIT)
Kaiserstr. 89-93
76133 Karlsruhe
GERMANY

Prof. Dr. Alexander Koldobsky
Department of Mathematics
University of Missouri-Columbia
202 Mathematical Science Bldg.
Columbia, MO 65211-4100
UNITED STATES

Prof. Dr. Monika Ludwig
Institut f. Diskr. Mathematik &
Geometrie
Technische Universität Wien
Wiedner Hauptstr. 8-10/104
1040 Wien
AUSTRIA

Eva Ochsenreither
Institut für Stochastik
Karlsruher Institut f. Technologie (KIT)
76128 Karlsruhe
GERMANY

Prof. Dr. Gestur Olafsson
Department of Mathematics
Louisiana State University
Baton Rouge LA 70803-4918
UNITED STATES

Prof. Dr. Victor P. Palamodov
School of Mathematical Sciences
Tel Aviv University
Ramat Aviv
Tel Aviv 69978
ISRAEL

Lukas Parapatits
Institut f. Diskrete Mathematik &
Geometrie
Technische Universität Wien
Wiedner Hauptstr. 8 - 10
1040 Wien
AUSTRIA

Prof. Dr. Matthias Reitzner
FB Mathematik/Informatik
Universität Osnabrück
Albrechtstr. 28a
49076 Osnabrück
GERMANY

Liran Rotem
Department of Mathematics
School of Mathematical Sciences
Tel Aviv University
P.O.Box 39040
Ramat Aviv, Tel Aviv 69978
ISRAEL

Prof. Dr. Boris Rubin
Department of Mathematics
Louisiana State University
Baton Rouge LA 70803-4918
UNITED STATES

Mykhailo Saienko
Institut für Mathematik
Goethe-Universität Frankfurt
Robert-Mayer-Str. 6-10
60325 Frankfurt am Main
GERMANY

Prof. Dr. Rolf Schneider
Mathematisches Institut
Universität Freiburg
Eckerstr. 1
79104 Freiburg
GERMANY

Dr. Ryan Scott
Department of Mathematics
University of Georgia
Athens, GA 30602-7403
UNITED STATES

Prof. Dr. Gil Solanes Farres
Departament de Matematiques
Universitat Autonoma de Barcelona
Campus Universitari
08193 Bellaterra
SPAIN

Prof. Dr. Eva B. Vedel Jensen
Matematisk Institut
Aarhus Universitet
Ny Munkegade 118
8000 Aarhus C
DENMARK

Dr. Thomas Wannerer
Institut für Mathematik
Goethe-Universität Frankfurt
Robert-Mayer-Str. 6-10
60325 Frankfurt am Main
GERMANY

Prof. Dr. Wolfgang Weil
Institut für Algebra u. Geometrie
Karlsruher Institut für Technologie
Kaiserstr. 89-93
76128 Karlsruhe
GERMANY

Prof. Dr. Elisabeth Werner
Department of Mathematics
Case Western Reserve University
Cleveland OH 44106
UNITED STATES

Prof. Dr. Martina Zähle
Mathematisches Institut
Universität Jena
Ernst-Abbe-Platz 2-4
07743 Jena
GERMANY