Mathematisches Forschungsinstitut Oberwolfach

Report No. 08/2013

DOI: 10.4171/OWR/2013/08

Mini-Workshop: The p-Laplacian Operator and Applications

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10 February – 16 February 2013

Abstract. There has been a surge of interest in the p-Laplacian in many different contexts from game theory to mechanics and image processing. The workshop brought together experts from many different schools of thinking to exchange their knowledge and points of view.

Mathematics Subject Classification (2010): 5J92, 35K92, 35B50, 35P30, 49J20, 49R05, 68U10, 91A05, 91A23.

Introduction by the Organisers

The boundary value problem of the p-Laplacian $\Delta_p u = f$ in Ω, $u = 0$ on $\partial\Omega$ is quite benign when $p \in (1, \infty)$, but far from being fully understood when $p = 1$ or $p = \infty$. There are many unsettled existence, uniqueness and regularity issues for these limiting cases. In other contexts, however, the study of the limiting cases has opened a door to a better understanding of the situation $p \in (1, \infty)$. The concept of viscosity solutions is, for example, quite suitable for large values of p. On the other hand, the case $p = 1$ often deals with discontinuous BV solutions, for which variational approaches can be more appropriate.

Moreover, the time dependent equations are also intriguing. Equations of the type

$$u_t - \Delta_p^N u = u_t - \frac{1}{p}|\nabla u|^{2-p}\Delta_p u = f$$

are for example used in image processing. For $p = 1$ they are known as TV-denoising.

The connection between the ordinary Laplace equation ($p = 2$) and Brownian motion is a much-studied classical topic. An analogous connection was found by Peres and Sheffield between tug-of-war games in game theory (replacing the role of the Brownian motion), and equations involving the ∞ or even p-Laplacian. This has paved the way to a deterministic game interpretation of equations involving the p-Laplacian as was done by Kohn and Serfaty.

In the last decade there has also been an abundance of results surrounding the p-Laplacian, some of which juxtapose the extremal cases $p = 1$ and $p = \infty$. Bingham fluids or landslides, for instance, are modelled using 1-Laplacians.

This workshop provided the unique opportunity to bring together experts from different areas, and to discuss their recent developments. The main focus was to encourage an interdisciplinary exchange of knowledge. Every participant gave a talk, with plenty of time left for discussions. Two additional sessions were arranged where the participants presented open problems. There was an additional talk by Rossi about the tug-of-war interpretation. We started on the day the pope resigned.

Mini-Workshop: The p-Laplacian Operator and Applications

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An optimal matching problem for the Euclidean distance

Julio D. Rossi
(joint work with José M. Mazón and Julián Toledo)

We report on recent results, [14], for an optimal matching problem (see [5], [4]) that consists in transporting two commodities (say nuts and screws, we assume that we have the same total number of nuts and screws) to a prescribed location (say factories where we ensemble the nuts and the screws) in such a way that they match there (each factory receive the same number of nuts and of screws) and the total cost of the operation, measured in terms of the Euclidean distance that the commodities are transported, is minimized.

Optimal matching problems for uniformly convex cost where analyzed in [2], [4], [5] and have implications in economic theory (hedonic markets and equilibria), see [5], [6], [7], [8], [4] and references therein. However, when one considers the Euclidean distance as cost new difficulties appear since we deal with a non-uniformly convex cost.

Clearly, the optimal matching problem under consideration is related to the classical Monge-Kantorovich's mass transport problem. Using tools from this theory, we prove the existence of a solution of the optimal matching problem. One of our main contributions in [14] is to perform a method to solve the problem taking limit as $p \to \infty$ in a system of PDE's of $p-$Laplacian type, which allows us to give more information about the matching measure and the Kantorovich potentials for the involved transport. This procedure to solve mass transport problems (taking limit as $p \to \infty$ in a $p-$Laplacian equation) was introduced by Evans and Gangbo in [10] and reveals quite fruitful, see [1], [13], [11]. We have to remark that the limit as $p \to \infty$ in the system requires some care since the system is nontrivially coupled and therefore the estimates for one component are related to the ones for the other, and we believe that it is interesting by its own.

Description of the optimal matching problem. To write the optimal matching problem under consideration in mathematical terms, we fix two non-negative compactly supported functions $f^+,\ f^- \in L^\infty$, with supports $X_+,\ X_-$, respectively, satisfying the mass balance condition

$$M_0 := \int_{X_+} f^+ = \int_{X_-} f^-.$$

We also consider a compact set D (the target set). Then we take a large bounded domain Ω such that it contains all the relevant sets, the supports of f_+ and f_-, X_+, X_- and the target set D. For simplicity we will assume that Ω is a convex $C^{1,1}$ bounded open set. We also assume that

$$X_+ \cap\ X_- = \emptyset, \quad \big(X_+ \cup X_-\big) \cap D = \emptyset \quad \text{and} \quad \big(X_+ \cup X_-\big) \cup D \subset\subset \Omega.$$

Whenever T is a map from a measure space (X, μ) to an arbitrary space Y, we denote by $T\#\mu$ the pushforward measure of μ by T. Explicitly, $(T\#\mu)[B] = \mu[T^{-1}(B)]$. When we write $T\#f = g$, where f and g are nonnegative functions, this means that the measure having density f is pushed-forward to the measure having density g.

For Borel functions $T_\pm : \Omega \to \Omega$ such that $T_+\#f^+ = T_-\#f^-$, we consider the functional

$$\mathcal{F}(T_+, T_-) := \int_\Omega |x - T_+(x)| f^+(x)dx + \int_\Omega |y - T_-(y)| f^-(y)dy.$$

The optimal matching problem can be stated as the minimization problem

$$\min_{(T_+,T_-)\in\mathcal{A}_D(f^+,f^-)} \mathcal{F}(T_+, T_-), \tag{1}$$

where

$$\mathcal{A}_D(f^+, f^-) := \Big\{(T_+, T_-) : T_\pm : \Omega \to \Omega \text{ are Borel functions, } T_\pm(X_\pm) \subset D, \\ \int_{T_+^{-1}(E)} f^+ = \int_{T_-^{-1}(E)} f^- \text{ for all Borel subset } E \text{ of } \Omega\Big\}.$$

If $(T_+^*, T_-^*) \in \mathcal{A}_D(f^+, f^-)$ is a minimizer of the optimal matching problem (1), we shall call the measure $\mu^* := T_+^*\#f^+ = T_-^*\#f^-$ a *matching measure* to the problem. Note that there is no reason why a matching measure should be absolutely continuous with respect to the Lebesgue measure. In fact there are examples of matching measures that are singular (see [14]).

The main result of [14] is the following existence theorem.

Theorem. *The optimal matching problem* (1) *has a solution, that is, there exist Borel functions* $(T_+^*, T_-^*) \in \mathcal{A}_D(f^+, f^-)$ *such that*

$$\mathcal{F}(T_+^*, T_-^*) = \inf_{(T_+,T_-)\in\mathcal{A}_D(f^+,f^-)} \mathcal{F}(T_+, T_-).$$

Moreover, we can obtain a solution $(\tilde{T}_+, \tilde{T}_-)$ *of the optimal matching problem* (1) *with a matching measure supported on the boundary of* D.

Remark We note that the fact that there is an optimal matching measure supported on ∂D greatly simplifies the problem, since it allows to reduce the target set to its boundary.

For the quadratic cost function $c(x, y) = |y - x|^2$, the existence of a matching measure supported on the boundary of D is not true in general, see [5].

Reference [14] contains two different proofs to this existence theorem. The first one is more direct but does not provide a constructive way of getting the optimal matching measure in D, which is one of the unknowns in this problem; consequently, the construction of optimal transport maps (that are proved to exist) remains a difficult task. The main tool in this first proof is the use of ingredients from the classical Monge-Kantorovich theory. The second proof is by approximation of the associated Kantorovich potentials by a system of $p-$Laplacian type problems when p goes to ∞. This approach provides an approximation of the potentials but also allows us to obtain the optimal measure in the limit. In addition we present several examples (that show that, in general, there is no uniqueness of the optimal configuration) and characterize when the optimal matching measure is a Dirac delta.

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The area integral on metric measure spaces

Juha Kinnunen

(joint work with Heikki Hakkarainen and Panu Lahti)

We discus minimizers of the nonparametric area integral

$$\mathcal{F}(u,\Omega) = \int_\Omega \sqrt{1+|Du|^2}\,dx$$

in a metric measure space (X, d, μ) equipped with a doubling measure and a Poincaré inequality. In the Euclidean case minimizers satisfy the corresponding minimal surface equation

$$\sum_{j=1}^{n} D_j \frac{D_j u}{\sqrt{1+|Du|^2}} = 0$$

in an open and bounded subset Ω of $\mathbb{R}^n$. It is well known that an equivalent concept can be obtained as the relaxed area integral

$$\mathcal{F}(u,\Omega) = \inf \left\{ \liminf_{i\to\infty} \int_\Omega \sqrt{1+|Du_i|^2}\, dx \right\},$$

where the infimum is taken over all sequences of functions $u_i \in C^1(\Omega)$ with $u_i \to u$ in $L^1(\Omega)$ as $i \to \infty$. Minimizers are functions of bounded variation with prescribed boundary values, see [5], [6], [7], [8, Chapter 6], [9] and [13]. The advantage of the variational approach is that it can be adapted to the context of metric measure spaces, and it also applies to more general integrals and quasiminimizers with linear growth.

Functions of bounded variation are defined through relaxation in a metric measure space, see [14], [1], [2] and [3]. We recall the definition here. Let Ω be an open subset of X. For $u \in L^1_{\mathrm{loc}}(\Omega)$, the total variation is

$$\|Du\|(\Omega) = \inf \Big\{ \liminf_{i\to\infty} \int_\Omega g_{u_i}\, d\mu : u_i \in \mathrm{Lip}_{\mathrm{loc}}(\Omega), u_i \to u \text{ in } L^1_{\mathrm{loc}}(\Omega) \Big\},$$

where g_{u_i} is the minimal 1-weak upper gradient of u_i and $\mathrm{Lip}_{\mathrm{loc}}(\Omega)$ denotes the class of functions that are Lipschitz continuous on compact subsets of Ω. We refer to [4], [11] and [15] for upper gradients, Poincaré inequalities and Sobolev spaces in the metric setting. A function $u \in L^1(\Omega)$ is of bounded variation, denoted by $u \in \mathrm{BV}(\Omega)$, if $\|Du\|(\Omega) < \infty$. Boundary values of BV-functions is a delicate issue already for domains with a smooth boundary in the Euclidean case. A standard approach is to consider extensions of boundary values to a slightly larger reference domain. Indeed, let Ω and Ω^* be open subsets of X such that the closure of Ω is a compact subset of a bounded open set Ω^*, and assume that $f \in \mathrm{BV}(\Omega^*)$. We define the space $\mathrm{BV}_f(\Omega)$ as the space of functions $u \in \mathrm{BV}(\Omega^*)$ such that $u = f$ μ-almost everywhere in $\Omega^* \setminus \Omega$. Next we give a definition of the minimizer of a relaxed area integral with prescribed boundary values in metric measure spaces. For $u \in \mathrm{BV}_f(\Omega)$, the generalized surface area functional is

$$\mathcal{F}(u,\Omega) = \inf \Big\{ \liminf_{i\to\infty} \int_{\Omega^*} \sqrt{1+g_{u_i}^2}\, d\mu : u_i \in \mathrm{Lip}_{\mathrm{loc}}(\Omega^*), u_i \to u \text{ in } L^1_{\mathrm{loc}}(\Omega^*) \Big\},$$

where g_{u_i} is the minimal 1-weak upper gradient of u_i. A function $u \in \mathrm{BV}_f(\Omega)$ is a minimizer of the generalized surface area functional with the boundary values f, if

$$\mathcal{F}(u,\Omega) = \inf \mathcal{F}(v,\Omega),$$

where the infimum is taken over all $v \in \mathrm{BV}_f(\Omega)$.

The direct methods in the calculus of variations can be applied to show that a minimizer exists for an arbitrary bounded domain with BV-boundary values.

The necessary compactness result can be found in [14], and the lower semicontinuity property of the area integral is shown in [10]. In the Euclidean case with the Lebesgue measure, minimizers can be shown to be smooth. However, it is somewhat unexpected that the regularity fails even for continuously differentiable weights in the Euclidean case. In [10] we give an explicit example of a minimizer that is discontinuous at an interior point of the domain. Similar examples for a slightly different functionals are presented in [5, p. 132]. This phenomenon occurs only in the case when the variational integral has linear growth. For variational integrals with superlinear growth, the minimizers are locally Hölder continuous even in the metric setting by [12]. In particular, these examples show that there does not seem to be hope to extend the regularity theory of minimizers of functionals with linear growth to the metric setting.

The main result of [10] shows that the minimizers are locally bounded, and the previously mentioned examples show that this result is essentially the best possible that can be obtained in this generality. We prove the main result by purely variational techniques without referring to the minimal surface equation. Indeed, the minimizers satisfy a De Giorgi type energy estimate, and the local boundedness follows from an iteration scheme. This point of view may be interesting already in the Euclidean case and it also applies to quasiminimizers of the area integral.

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BMO-estimates for the p-Laplacien

Sebastian Schwarzacher
(joint work with Lars Diening and Petr Kaplický)

My talk was about local regularity properties of the inhomogeneous p-Laplace system: We looked at local solutions $u \in W^{1,p}(\Omega;\mathbb{R}^N)$ of

$$-\mathrm{div}(A(\nabla u) = -\mathrm{div}F, \tag{1}$$

where $A(\nabla u) := |\nabla u|^{p-2}\nabla u$ and Ω is a domain. The problem is well posed if $F \in W^{1,p'}(\Omega)$ and a unique solution exists under suitable boundary conditions; we will not discuss boundary matter, as our results are local or in the whole space.

What I discussed, is how ∇u is influenced by F being in a different function space. This is motivated by the linear Calderón-Zygmund theory. Indeed, if $p = 2$ the inhomogeneous 2-Laplace is nothing else but Poisson's equation:

$$-\Delta u := -\mathrm{div}(\nabla u) = -\mathrm{div}\, F. \tag{2}$$

For these solutions $F \mapsto \nabla u$ is a singular integral operator. These operators have good continuity properties which can be used to prove regularity:

Theorem: Calderón-Zygmund, singular integrals. *Let u be a solution to (2) and $1 < q < \infty$. If $F \in L^q(\mathbb{R}^n)$, then $\nabla u \in L^q(\mathbb{R}^n) : \|\nabla u\|_q \leq c\, \|F\|_q$.*

This result could be transferred to the p-Laplace case by Iwaniec [4], for $p' \leq q < \infty$, look also [2] and [7].

If $p \neq 2$ things are much more difficult, when $1 < q < p'$. In this case $\nabla u \notin L^p_{\mathrm{loc}}(\Omega)$, which excludes Cacciopolli estimates. However, in the case $p - \epsilon < q \leq p$, for a small $\epsilon > 0$ depending on n, N, p the estimates are proved. It was done by Iwaniec [5] in the first place and later by Kinnunen and Lewis [6]. The integrability properties of gradients of the p-Laplace is:

Theorem: Non-linear Calderón-Zygmund theorey. *Let u be a solution to (1) and $p' - \epsilon < q < \infty$. If $F \in L^q(\mathbb{R}^n)$, then $A(\nabla u) \in L^q(\mathbb{R}^n) : \|A(\nabla u)\|_q \leq c\, \|F\|_q$.*

The next natural question that arises is, what happens for $q \to \infty$? And how about finer regularity like modulus of continuity?

To answer this question we have to look at the maximal regularity available. This is of course the case, when $F \equiv 0$. We call solutions of (1), with $F \equiv 0$ p-harmonic. Up to now the maximal regularity known is $\nabla u \in C^{\alpha}_{\mathrm{loc}}(\Omega)$. This is the famous result by Ural'tseva [9] for equations and for vector valued solutions by Uhlenbeck [8]. These results are sharp, which is showed the 2-dimensional case. Iwaniec and Manfredi showed in [3], the exact maximal regularity for solutions in the plane. However, one observes, that in the 2-Dim case the quantity

$V(\nabla u) = |\nabla u|^{\frac{p-2}{2}}\nabla u$ is always in C^1_{loc}! The quantity $V(\nabla u)$ is a natural quantity; heuristically it is the L^2 substitute for ∇u which has natural power p. This motivates the conjecture, that $V(\nabla u)$ is always in C^1_{loc} for all dimensions.

Let us get back to our previous question: What happens for $q \to \infty$ and beyond?

We know by the linear Calderón-Zygmund theory for singular integral operators for Poisson's equation, that $F \in L^\infty$ does not imply, that $\nabla u \in L^\infty$. It turns out, that the correct borderline space is the space of bounded mean oscillations as for Poisson's equation we have: $F \in \text{BMO}$ implies that $\nabla u \in \text{BMO}$.

We say that $w \in \text{BMO}(\Omega)$, if $\|w\|_{\text{BMO}(\Omega)} = \sup_{B \subset \Omega \text{ is ball}} \fint_B |w - (w)_B|\,dx < \infty.$

The first result for the p-Laplace was done by DiBenedetto and Manfredi in [2]. They proved in the case that $p \geq 2$, that if $F \in \text{BMO}$, then $\nabla u \in \text{BMO}$.

We were able to extend this result to all $1 < p < \infty$, by proving the following local estimate:

Theorem: BMO Estimate. *Let u be a solution to* (1)*. If $F \in BMO_{loc}(\Omega)$, then $A(\nabla u) \in BMO_{loc}(\Omega)$. Moreover*

$$\|A(\nabla u)\|_{BMO(B)} \leq c \fint_{2B} |(A(\nabla u)) - (A(\nabla u))_{2B}|\,dx + c\|F\|_{BMO(2B)},$$

where $A(\nabla u) = |\nabla u|^{p-2}\nabla u$.

Remark, that this estimate is linear in A. It was proved by Diening, Kaplický and myself and can be found in [1].

Concerning the finer regularity, beyond BMO, we got the following estimate:

Theorem: Hölder Continuity.

$$\|A(\nabla u)\|_{C^\beta(B_R)} \leq \frac{c}{R^\beta} \fint_{2B_R} |(A(\nabla u)) - (A(\nabla u))_{2B_R}|\,dx + c\|F\|_{C^\beta(2B_R)},$$

for all β which are below the Hölder exponent of p-harmonic functions.

This closes the gap between transferring integrability from F to ∇u and transferring continuity from F to ∇u.

Finally let me point out, that the interested reader can find more general versions of the previous estimates for a much more general class of solutions in [1].

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On a singular elliptic equation involving the 1-Laplacian

Florian Krügel

In [5] the functional

$$F(u) := \int_\Omega |Du| + \frac{1}{p}\int_\Omega |Du|^p - \int_\Omega au$$

is considered, where $a > 0$ is a constant, $p \in (1, \infty)$, and $\Omega \subset \mathbb{R}^n$ is a bounded domain. The functional F appears in the context of *Bingham fluids*, see e.g. [2] or [3]; for $n = 2$ and $p = 2$ the solution represents the flow velocity in the steady state of the laminar flow a Bingham fluid which is pressed through a cylindrical pipe of cross section Ω when an exterior pressure a is applied. In addition the functional is studied in [6] in a different context, namely elliptic regularization of the equation of prescribed mean curvature.

The formal Euler-Lagrange equation $-\Delta_1 u - \Delta_p u = a$ for minimizers of F does not make sense in critical points, and we define solutions of the equation to be minimizers of F with respect to compactly supported perturbations.

Under Dirichlet boundary conditions (in the space $b + W_0^{1,p}(\Omega)$) the functional has a unique minimizer. Taking the boundary values to be zero, the minimizer u has the following properties:

- u can be zero for small a. More precisely, u is nontrivial if and only if $a > \nu(\Omega)$ (the Cheeger constant of Ω).
- If u is nontrivial, u has a maximum set M with $|M| > 0$ (a *plateau*). Furthermore $\frac{P(M)}{|M|} = a$ (where P denotes the perimeter) and for all $G \subset\subset \operatorname{int}(M)$, $\frac{P(G)}{|G|} \geq a$. If this inequality can be extended to all measurable $G \subset M$ (this works for example if M is convex), $\nu(M) = a$, and M is a Cheeger set in itself.
- u is Lipschitz continuous in Ω, and continuous on $\overline{\Omega}$ if Ω has a Lipschitz boundary.

It is not known if M is actually convex, although it seems to be a reasonable conjecture if Ω is convex.

There exists a suitable notion of subsolutions and supersolutions of $-\Delta_1 u - \Delta_p u = a$. The definition is as follows:

A function $u \in W^{1,p}_{\mathrm{loc}}(\Omega)$ is called a *supersolution* if for all nonnegative test functions $\phi \in C_0^\infty(\Omega)$,

$$\int_{\{Du\neq 0\}} \left(|Du|^{p-2}Du + \frac{Du}{|Du|}\right)\cdot D\phi + \int_{\{Du=0\}} |D\phi| \geq \int_\Omega a\phi$$

and a *subsolution* if for all nonnegative test functions $\phi \in C_0^\infty(\Omega)$,

$$\int_{\{Du\neq 0\}} \left(|Du|^{p-2}Du + \frac{Du}{|Du|}\right)\cdot D\phi - \int_{\{Du=0\}} |D\phi| \leq \int_\Omega a\phi$$

(all integrals are with respect to Lebesgue measure).

These supersolutions and subsolutions have all the expected properties, and a theory analogous to the one for p-Laplace equations as described in [4] can be developed. Most notably:

- A function is a solution (in the sense of being a local minimizer, described above) if and only if it is a subsolution and a supersolution.
- The comparison principle holds.
- A characterization of supersolutions by a comparison principle with solutions can be given, which is analogous to the notion of the $\mathcal{A}$-superharmonic functions from [4].

One can give explicit rotation-invariant solutions: first a solution on $\mathbb{R}^n$ which is constant in a ball with radius $\frac{n}{a}$, which is in C^1 and together with a numerical solution obtained in [6] raises the conjecture that solutions are in C^1 in general; and second a solution on $\mathbb{R}^n \setminus \{0\}$ for $a = 0$ that is constant outside of a ball. This solution is instructive because putting two of these solutions on two separated balls yields a function which may or may not be a solution, depending on the distance of the two balls.

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Limit as $p \to \infty$ of the $p-$Laplacian ground state and optimal transportation problems

Luigi De Pascale
(joint work with Thierry Champion and Chloé Jimenez)

The ground state of the p-Laplacian is the eigenfunction u_p corresponding to the first eigenvalue λ_p i.e. a solution of the Dirichlet problem

$$(1) \qquad \begin{cases} -div(|Du_p|^{p-2}Du_p) = \lambda_p u_p^{p-1} & \text{in } \Omega, \\ u_p = 0 & \text{on } \partial\Omega, \\ u_p \geq 0 & \text{in } \Omega, \end{cases}$$

It is known that the first eigenfunction is positive and unique. In [2] it was proved that as $p \to \infty$, up to subsequences, $u_p \to u_\infty$ uniformly and this last function is a viscosity solution of

$$(2) \qquad \min\{|Du_\infty| - \Lambda_\infty u_\infty, -\Delta_\infty u_\infty\} = 0,$$

with $\Lambda_\infty = \dfrac{1}{\max\{d(x,\partial\Omega) \mid x \in \overline{\Omega}\}} := \dfrac{1}{R_1}$ and 0 boundary data. However viscosity solutions of this last equations may be not unique [4]. Then one asks if the limiting procedure above select a special solution of the limit problem [3]. In this quest for informations we consider three more quantities related to problem (1) and we investigate their limits and some related variational problems. The main tools are Γ-convergence and convex duality theory and the details are contained in [1]. Before introducing the main players let us recall that

$$\lambda_p = \min_{W_0^{1,p}(\Omega)\setminus\{0\}} \frac{\|Du\|_p^p}{\|u\|_p^p} = \min_{W_0^{1,p}(\Omega)\cap\{\|u\|_p=1\}} \|Du\|_p^p,$$

and we will always consider the ground state normalized so that $\|u\|_p = 1$.

We introduce the following measures:

$$(3) \qquad \sigma_p := \frac{|\nabla u_p|^{p-2}\nabla u_p}{\lambda_p}dx, \quad f_p := u_p^{p-1}dx, \quad \mu_p := \frac{|\nabla u_p|^{p-2}}{\lambda_p}dx.$$

The above measures satisfy the following inequalities for $p > 2$:

$$\int_\Omega \left|\frac{\nabla u_p}{\lambda_p^{1/p}}\right|^p dx = 1, \quad \int_\Omega d|f_p| \leq |\Omega|^{1/p},$$
$$\int_\Omega d|\mu_p| \leq (\frac{|\Omega|}{\lambda_p})^{2/p}, \quad \int_\Omega d|\sigma_p| \leq (\frac{|\Omega|}{\lambda_p})^{1/p}.$$

Then there exists $u_\infty \in \mathrm{Lip}(\Omega) \cap \mathcal{C}_0(\Omega)$ with $\|u_\infty\|_\infty = 1$, $f_\infty \in \mathcal{M}_b^+(\overline{\Omega})$ a probability measure, $\mu_\infty \in \mathcal{M}_b^+(\overline{\Omega})$ and $\xi_\infty \in L^1_{\mu_\infty}(\Omega)^d$ such that, up to subsequences:

$$u_p \to u_\infty \text{ uniformly on } \overline{\Omega}, \quad f_p \overset{*}{\rightharpoonup} f_\infty \text{ in } \mathcal{M}_b(\overline{\Omega}),$$
$$\mu_p \overset{*}{\rightharpoonup} \mu_\infty \text{ in } \mathcal{M}_b^+(\overline{\Omega}), \quad \sigma_p \overset{*}{\rightharpoonup} \sigma_\infty := \xi_\infty\mu_\infty \text{ in } \mathcal{M}_b(\overline{\Omega}, \mathbb{R}^N).$$

With these notations we may write the problem

$$(\mathcal{P}_p) \qquad \min_{u\in W_0^{1,p}(\Omega)} \left\{ \frac{1}{p\lambda_p}\int_\Omega |\nabla u(x)|^p\ dx - \langle f_p, u\rangle \right\}.$$

and by definitions of f_p, it follows that u_p is the unique minimizer of $(\mathcal{P}_p)$. Moreover, since the solution set of the problem (1) is spanned by u_p, we may consider $(\mathcal{P}_p)$ as a variational formulation of (1) for $\lambda = \lambda_p$. Passing to the limit (in variational sense) problem $(\mathcal{P}_p)$ and its dual as $p \to \infty$ we obtain

$$(\mathcal{P}_\infty) \quad \min\{- < f_\infty, u >:\ u \in \mathrm{Lip}(\Omega),\ |\nabla u| \le \Lambda_\infty \text{ a.e., } u = 0 \text{ on } \partial\Omega\},$$

$$(\mathcal{P}^*_\infty) \qquad \min_{\sigma\in\mathcal{M}_b(\mathbb{R}^N)^N} \{\Lambda_\infty \int_{\overline{\Omega}} |\sigma| :\ \mathrm{spt}(\sigma)\subset\overline{\Omega},\ -\mathrm{div}(\sigma)\in\mathcal{M}_b(\mathbb{R}^N) \text{ and } - \mathrm{div}(\sigma) = f_\infty \text{ in } \Omega\}.$$

It is easy to recognize in $(\mathcal{P}_\infty)$ and $(\mathcal{P}^*_\infty)$ two possible dual formulations of the Monge-Kantorovich problem described below in a version adapted to this setting. Given two probability measures $\alpha \in \mathcal{P}(\Omega)$ and $\nu \in \mathcal{P}(\overline{\Omega})$ we consider

$$\min\left\{ \int_{\overline{\Omega}\times\overline{\Omega}} |x-y| d\gamma : \pi^1_\sharp\gamma = \alpha,\ \pi^2_\sharp\gamma = \nu \right\}. \tag{4}$$

A measure γ such that $\pi^1_\sharp\gamma = \alpha,\ \pi^2_\sharp\gamma = \nu$ is called a transport plan of α to ν. Notice that by the direct method of the Calculus of Variations the minimum in (4) is achieved. The minimal value is usually called Wasserstein distance of α and ν and it is denoted by $\mathcal{W}_1(\nu, \alpha)$.

We have that $f_\infty \in \mathcal{P}(\Omega)$ so that we can consider its Wasserstein distance from $\mathcal{P}(\partial\Omega)$, i.e. the following variational problem defined on $\mathcal{P}(\partial\Omega)$

$$\inf_{\nu\in\mathcal{P}(\partial\Omega)} \mathcal{W}_1(f_\infty, \nu). \tag{5}$$

With the usual abuse of notations, we shall denote by $\mathcal{W}_1(f_\infty, \mathcal{P}(\partial\Omega))$ the infimum in (5). We can also rewrite it as

$$\mathcal{W}_1(f_\infty, \mathcal{P}(\partial\Omega)) \ = \ \inf\left\{ \int_{\overline{\Omega}\times\overline{\Omega}} |x-y| d\gamma : \pi^1_\sharp\gamma = f_\infty,\ \pi^2_\sharp\gamma \in \mathcal{P}(\partial\Omega) \right\} \tag{6}$$

The limits u_∞, f_∞ and Λ_∞ satisfies the following:

(i) f_∞ maximizes $\mathcal{W}(\cdot, \mathcal{P}(\partial\Omega))$ in $\mathcal{P}(\Omega)$,
(ii) $\Lambda_\infty = \frac{1}{R_1}$,
(iii) $spt(f_\infty) \subset argmax\ u_\infty \subset argmax\ d_\Omega$.

Finally let us conclude with an open problem. It would be interesting and useful to prove that in (iii) above equalities hold.

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Nonuniqueness of ∞-ground states

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(joint work with Ryan Hynd and Charles K. Smart)

I will talk about the construction of a dumbbell domain for which the associated principal ∞-eigenvalue is not simple. This gives a negative answer to the outstanding problem posed in [1] and [2]. It remains a challenge to determine whether simplicity holds for convex domains.

Let Ω be a bounded open set in $\mathbb{R}^n$. According to Juutinen-Lindqvist-Manfredi [1], a continuous function $u \in C(\bar{\Omega})$ is said to be an *infinity ground state in* Ω if it is a positive viscosity solution of the following equation:

$$\begin{cases} \max\left\{\lambda_\infty - \frac{|Du|}{u},\ \Delta_\infty u\right\} = 0 & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

Here

$$\lambda_\infty = \lambda_\infty(\Omega) = \frac{1}{\max_\Omega d(x, \partial\Omega)}$$

is the principal ∞-eigenvalue, and Δ_∞ is the infinity Laplacian operator, i.e,

$$\Delta_\infty u = u_{x_i} u_{x_j} u_{x_i x_j}.$$

The above equation is the limit as $p \to +\infty$ of the equation

$$\begin{cases} -\mathrm{div}(|Du|^{p-2}Du) = \lambda_p^p |u|^{p-2}u & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

which is the Euler-Lagrange equation of minimizing the nonlinear Rayleigh quotient

$$\inf_{\phi \in W_0^{1,p}(\Omega)} \frac{\int_\Omega |D\phi|^p\, dx}{\int_\Omega |\phi|^p\, dx}$$

and λ_p is the principal eigenvalue of p-Laplacian. Precisely speaking, let u_p be a positive solution of equation (1.2) satisfying

$$\int_\Omega u_p^p\, dx = 1.$$

If u_∞ is a limiting point of $\{u_p\}$, i.e, there exists a subsequence $p_j \to +\infty$ such that

$$u_{p_j} \to u_\infty \quad \text{uniformly in } \bar{\Omega},$$

it was proved in [1] that u_∞ is a viscosity solution of the equation (1.1) and

$$\lim_{p\to+\infty} \lambda_p = \lambda_\infty.$$

We say that u is *a variational infinity ground state* if it is a limiting point of $\{u_p\}$.

A natural problem regarding equation (1.1) is to deduce whether or not infinity ground states in a given domain are unique up to a multiplicative factor; in this case, λ_∞ is said to be *simple*. The simplicity of λ_∞ has only been established for those domains where the distance function $d(x, \partial\Omega)$ is an infinity ground state ([3]). Such domains includes the ball, stadium, and torus. It has been a significant outstanding open problem to verify if simplicity holds in general domains or to exhibit an example for which simplicity fails. In this paper, we resolve this problem by constructing a planar domain where simplicity fails to hold. It is not clear to us whether variational infinity ground states are unique. Our result, however, shows that variational infinity ground states in general are not continuous with respect to domain. A somewhat similar nonuniqueness result has been proved very recently for the nonlocal infinity eigenvalue problem ([4]). Surprisingly, the nonlocal version is much simpler. Its ground states possess several interesting properties which are not true in the local case. In particular, nonlocal infinity ground states even have explicit represenation formulas.

For $\delta \in (0,1)$, denote the dumbbell

$$D_0 = B_1(\pm 5e_1) \cup R$$

for $R = (-5,5)\times(-\delta,\delta)$ and $e_1 = (1,0)$. Throughout this paper, $B_r(x)$ represents the open ball centered at x with radius r.

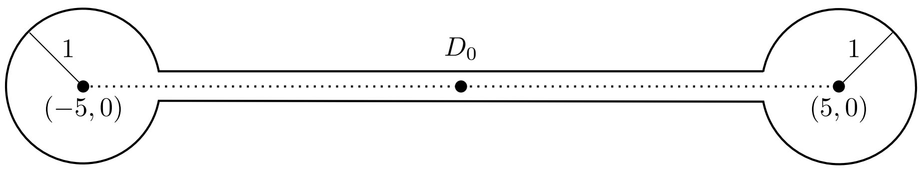

FIGURE 1. The dumbbell domain D_0.

The following is our main result.

Theorem.
There exists $\delta_0 > 0$ such that when $\delta \le \delta_0$, the dumbbell D_0 possesses an infinity ground state u_∞ which satisfies $u_\infty(5,0) = 1$ and $u_\infty(-5,0) \le \frac{1}{2}$. In particular, u is not a variational ground state and $\lambda_\infty(D_0)$ is not simple.

We remark that the infinity ground state described in the theorem is nonvariational simply because it is not symmetric with respect to the x_2-axis, which variational ground states can be showed to be. This immediately follows from the fact that λ_p is simple, which implies any solution u_p of (1.2) on $\Omega = D_0$ must be symmetric with respect to the x_2-axis. We also remark that the number "$\frac{1}{2}$" in the above theorem is not special. By choosing a suitable δ_0, we can in fact make $u_\infty(-5, 0)$ less than any positive number.

Idea of the proof: Consider the union of two disjoint balls with distinct radius $U_\epsilon = B_1(5e_1) \cap B_{1-\epsilon}(-5e_1)$ for $\epsilon \in (0, 1)$. If u is an infinity ground state of U_ϵ, the uniqueness of λ_∞ ([1]) immediately implies that $u \equiv 0$ in $B_{1-\epsilon}(-5e_1)$. A similar conclusion also holds for the principal eigenfunction of Δ_p. It is therefore natural to expect that such a degeneracy of u on the smaller ball may change very little if we add a narrow tube connecting these two balls. The key is to get uniform control of the width of the tube as $\epsilon \to 0$ for variational infinity ground states in an asymmetric perturbation of D_0.

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Eigenvalue problem for the 1-Laplace operator

Friedemann Schuricht

(joint work with Bernd Kawohl and Zoja Milbers)

The eigenvalue problem for the p-Laplace operator

$$-\text{div}|Du|^{p-2}Du = \lambda|u|^{p-2}u \text{ on } \Omega\,, \quad u = 0 \text{ on } \partial\Omega$$

is related to the variational problem

$$\int_\Omega |Du|^p\,dx \;\to\; \text{Min! in } W_0^{1,p}(\Omega) \text{ subject to } \int_\Omega |u|^p\,dx = 1\,.$$

While this problem is intensively studied for $p > 1$, the limit case $p = 1$ is a natural generalization. Here a first observation is, that there are no solutions in $W_0^{1,1}(\Omega)$ in general. One rather has to study the problem in $BV(\Omega)$ with a weaker notion of homogeneous boundary conditions and is lead to

$$(1) \qquad \int_\Omega d|Du| + \int_{\partial\Omega} |u|\,d\mathcal{H}^{n-1} \;\to\; \text{Min! in } BV(\Omega) \text{ subject to } \int_\Omega |u|\,dx = 1$$

where we always assume that $\Omega \subset \mathbb{R}^n$ is open and bounded with Lipschitz boundary. Then the existence of a minimizer follows by standard direct methods in $BV(\Omega)$.

Since typical minimizers are characteristic functions (being zero on a positive set), the formal Euler-Lagrange equation

$$-\text{div}\frac{Du}{|Du|} = \lambda \frac{u}{|u|} \tag{2}$$

has no direct meaning and needs some appropriate interpretation. For that we derive a necessary minimizing condition by applying a nonsmooth Lagrange multiplier rule and by computing the convex subdifferentials of the functions entering (1). This way we get for a minimizer $u \in BV(\Omega)$ (by using Sgn for the set-valued sign function) that

$$\exists\, s(x) \in \text{Sgn}(u(x)) \text{ a.e. on } \Omega \text{ and } \exists\, z \in L^\infty(\Omega, \mathbb{R}^n) \text{ with} \tag{3}$$

$$|z(x)| \le 1 \text{ a.e., } \text{div}z \in L^\infty(\Omega),\ E(u) = -\int_\Omega u\text{div}z\, dx \tag{4}$$

such that

$$-\text{div}z = \lambda s \text{ a.e. on } \Omega\,, \qquad \lambda = E(u)\,.$$

Here s replaces $\frac{u}{|u|}$ and z replaces $\frac{Du}{|Du|}$ in the formal equation (2). A more refined analysis even gives that

$$\forall\, s(x) \in \text{ Sgn}(u(x)) \text{ a.e. on } \Omega \quad \exists\, z \in L^\infty(\Omega, \mathbb{R}^n) \text{ with (4)} \tag{5}$$

such that equation (1) is satisfied, i.e. we have infinitely many equations as necessary condition (cf. Kawohl and Schuricht [3], Schuricht [7]). We call (1) *single* equation if it is based on (3), (4) and we call it *multiple* equation if it is related to (5). In Milbers & Schuricht [5] it is shown that the vector field z corresponding to a selection s doesn't have to be unique.

It turns out that the definition of higher eigensolutions cannot be done by means of an eigenvalue equation as in the classical case. While the single equation (1) has too many solutions, the multiple equation seems to be satisfied merely for minimizers. Therefore we define eigensolutions as critical points of the variational problem while, due to the nonsmoothness of the problem, critical points are taken in the sense of weak slope (cf. Degiovanni & Marzocchi [1]). By a minimax principle based on the weak slope the existence of a sequence of eigenfunctions u_k with eigenvalues $\lambda_k \to \infty$ can be shown (see Chang [2] and Milbers & Schuricht [4]).

The eigensolutions (u_k, λ_k) satisfy (1) as single equation. Since the single equation has solutions that might not be eigensolutions, a further necessary condition for eigenfunctions u by means of inner variations has been derived as

$$\int_\Omega \langle z, D\eta \cdot z\rangle - \text{div}\eta\, d|Du| = -\lambda \int_\Omega |u| \text{div}\eta\, dx \text{ for all } \eta \in C_0^\infty(\Omega) \tag{6}$$

with $\lambda = E(u)$ and z according to the polar decomposition $Du = z|Du|$ of the total variation measure (cf. Milbers & Schuricht [6]). This additional condition rules out solutions of the single equation that didn't seem to be critical points. However, it is not yet clear whether all solutions of the single equation (1) combined with (6) are also eigensolutions of the 1-Laplace operator.

Let us finally mention that the weak slope, and thus our notion of eigensolutions, depends on the underlying topology. We used the L^1-topology but also the BV-topology might be considered. Moreover, alternative notions of slope as e.g. the strong slope due to DeGiorgi could be taken for the definition of eigensolutions. In the case $\Omega = (0,1) \subset \mathbb{R}$ the eigensolutions can be given quite explicitly for different choices of topology and slope (cf. Chang [2], Milbers & Schuricht [6]). Here it turns out that the set of eigensolutions really differs for different choices. However it seems that the definition of eigensolutions by means of the weak slope with the L^1-topology is a suitable approach for eigensolutions of the 1-Laplace operator.

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A nonvariational Dirichlet problem with the p-Laplacian and a convection term

Peter Takáč

(joint work with Jorge García-Melián and José Sabina de Lis)

We study the following nonvariational $(p-1)$-homogeneous eigenvalue problem with **a convection term**:

$$(\mathbf{P}) \qquad \begin{cases} -\Delta_p u + B(x, \nabla u) = \lambda\,|u|^{p-2}u + h(x) & \text{in } \Omega\,; \\ u = 0 & \text{on } \partial\Omega\,, \end{cases}$$

in a bounded domain $\Omega \subset \mathbb{R}^N$ with smooth boundary $\partial\Omega$, where $\Delta_p u \stackrel{\text{def}}{=} \operatorname{div}(|\nabla u|^{p-2}\nabla u)$ with $1 < p < \infty$, $\lambda \in \mathbb{R}$ (spectral parameter), and $h \in L^\infty(\Omega)$.

The **convection term** $B(x, \nabla u)$, $B\colon\, \Omega \times \mathbb{R}^N \to \mathbb{R}$, is a continuous function assumed to be also homogeneous of degree $(p-1)$ and odd with respect to the second variable $\boldsymbol{\eta} = \nabla u$. A canonical example is $B(x, \boldsymbol{\eta}) = (\mathbf{a}(x) \cdot \boldsymbol{\eta})|\boldsymbol{\eta}|^{p-2}$ with a given vector field $\mathbf{a} \in [L^\infty(\Omega)]^N$, for $(x, \boldsymbol{\eta}) \in \Omega \times \mathbb{R}^N$.

Finally, we assume that $B\colon\, \Omega \times (\mathbb{R}^N \setminus \{\mathbf{0}\}) \to \mathbb{R}$ is locally Lipschitz-continuous with respect to the variable $\boldsymbol{\eta} = \nabla u$ throughout the domain $\Omega \times (\mathbb{R}^N \setminus \{\mathbf{0}\})$.

We consider a nonvariational linear 2^{nd}-order operator $\mathcal{L}$, see H. Berestycki, L. Nirenberg, and S. R. S. Varadhan (1994), with a **variational formula** for the principal eigenvalue:

$$(\mathbf{L}) \qquad \begin{cases} -\mathcal{L}u \geq \lambda u & \text{and} \quad u > 0 \text{ in } \Omega\,; \\ u = 0 & \text{on } \partial\Omega\,. \end{cases}$$

The principal eigenvalue is obtained from

$$\lambda_1 = \sup_{u>0} \lambda = \sup_{u>0} \inf_{x\in\Omega} \frac{-\mathcal{L}u}{u} = -\inf_{u>0} \sup_{x\in\Omega} \frac{\mathcal{L}u}{u}\,.$$

For Problem **(P)** in dimension $N = 1$, see [1].

We write "$u \ll v$" if and only if

$$u(x) < v(x) \ \text{ in } \Omega \quad \text{ and } \quad \frac{\partial u}{\partial \nu}(x_0) > \frac{\partial v}{\partial \nu}(x_0) \ \text{ on } \partial\Omega\,.$$

Theorem 1. *The eigenvalue problem for* **(P)** *(i.e., $h \equiv 0$ in Ω) possesses a unique eigenvalue $\lambda_1 \in \mathbb{R}$ associated to a positive eigenfunction $\varphi_1 \in W_0^{1,p}(\Omega)$, $\varphi_1 \gg 0$ in Ω (the Hopf maximum principle). Furthermore, we have $\lambda_1 > 0$ and $\lambda > \lambda_1$ holds for any other eigenvalue $\lambda \in \mathbb{R}$ of problem* **(P)**. *Finally, if the boundary $\partial\Omega$ is connected, then λ_1 is also simple and isolated, i.e., there is a constant $\delta > 0$ such that $\lambda \geq \lambda_1 + \delta$ holds for any eigenvalue $\lambda \in \mathbb{R}$ of problem* **(P)** *associated to an eigenfunction $\varphi \in W_0^{1,p}(\Omega)$ satisfying $\varphi/\varphi_1 \not\equiv \mathrm{const}$ in Ω.*

Theorem 2. *Now let $h \in L^\infty(\Omega)$ with $h \not\equiv 0$ in Ω. Then, for every $\lambda < \lambda_1$, problem* **(P)** *has a solution $u \in W_0^{1,p}(\Omega)$. If $\lambda < 0$ then this solution is unique. Finally, if $0 \leq \lambda < \lambda_1$ and $h \in L^\infty(\Omega) \cap C(\Omega)$ with $h \geq 0$ and $h \not\equiv 0$ in Ω, then $u \gg 0$ in Ω (the Hopf maximum principle) holds and this solution is also unique.*

Remark (Theorem 2). (a) It follows from the proof of Theorem 2 that there exists $\delta > 0$ such that problem **(P)** possesses at least one solution also for $\lambda \in \mathbb{R}$ satisfying $\lambda_1 < \lambda < \lambda_1 + \delta$.

(b) We will see that if $h \geq 0$ in Ω then there is no nonnegative solution $u \geq 0$ to problem **(P)** when • either $\lambda > \lambda_1$, • or else $\lambda = \lambda_1$ and $h \not\equiv 0$ in a neighborhood of $\partial\Omega$.

The proofs of the two main theorems are based on some comparison principles, cf. P. Tolksdorf (1983), M. Cuesta and P. Takáč (1998, 2000), and J. García-Melián (2008).

However, the following two **comparison propositions** are new:

Proposition 3. *Let $1 < p < \infty$ and assume that $B\colon \Omega \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ is a continuous function, $B = B(x, s, \boldsymbol{\eta})$, which is strictly monotone increasing in the second variable $s \in \mathbb{R}$ for a.e. $x \in \Omega$ and for every $\boldsymbol{\eta} \in \mathbb{R}^N$. Assume that $u, v \in C^1(\Omega) \cap C(\overline{\Omega})$ verify*

$$\begin{aligned} -\Delta_p u + B(x, u, \nabla u) &= h(x) \quad \text{in } W^{-1,p'}(\Omega)\,, \\ -\Delta_p v + B(x, v, \nabla v) &= h'(x) \quad \text{in } W^{-1,p'}(\Omega)\,, \end{aligned}$$

in the weak sense in the dual space $W^{-1,p'}(\Omega)$ *of the Sobolev space* $W_0^{1,p}(\Omega)$, $p' = p/(p-1) \in (1,\infty)$, *where* $h, h' \in L^\infty(\Omega)$ *satisfy* $h \le h'$ *a.e. in* Ω. *If* $h - h'$ *is continuous in* Ω *and* $u \le v$ *holds on* $\partial\Omega$, *then* $u \le v$ *holds also throughout* Ω.

When the strict monotonicity condition on $B = B(x, s, \boldsymbol{\eta})$ in s is dropped, it might be difficult to obtain a similar result. In most cases, we need only $\tilde{B}(x, s, \boldsymbol{\eta}) = B(x, \boldsymbol{\eta}) + \mu\, |u|^{p-2}u$ with $\mu > 0$.

However, for direct applications in our proofs it will be sufficient to consider the particular case in which $\mu = 0$ and $B = B(x, \boldsymbol{\eta})$ is independent from $s \in \mathbb{R}$ and homogeneous of degree $(p-1)$ in the gradient variable $\boldsymbol{\eta} \in \mathbb{R}^N$ $(\boldsymbol{\eta} = \nabla u)$.

We also need to impose a local Lipschitz condition with respect to $\boldsymbol{\eta}$, but only away from the origin $\mathbf{0} \in \mathbb{R}^N$. This will allow us to cover the prototype case $B(x, \boldsymbol{\eta}) = (\mathbf{a}(x) \cdot \boldsymbol{\eta})|\boldsymbol{\eta}|^{p-2}$ also for $1 < p < 2$.

Proposition 4. *Let* $1 < p < \infty$ *and assume that* $B\colon \Omega \times \mathbb{R}^N \to \mathbb{R}$ *satisfies the hypotheses stated at the beginning. Assume that* $u, v \in C^1(\overline{\Omega})$ *verify*

$$\begin{aligned} &-\Delta_p u + B(x, \nabla u) = h(x) \quad \text{in } W^{-1,p'}(\Omega)\,, \\ &-\Delta_p v + B(x, \nabla v) = h'(x) \quad \text{in } W^{-1,p'}(\Omega)\,, \end{aligned}$$

in the weak sense, where $h, h' \in L^\infty(\Omega) \cap C^{\alpha'}(\Omega)$ *satisfy* $h \le h'$ *and* $h' > 0$ *in* Ω. *If* $u \le v = 0$ *hold on* $\partial\Omega$, *then* $u \le v$ *holds also throughout* Ω.

In order to prove Propositions 3 and 4, the following two Lemmas on weak and strong comparison principles are needed.

Lemma (Proposition 3). *Let* $1 < p < \infty$, $f, g \in L^\infty(\Omega)$, *and let* $B\colon \Omega \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ *be as in* **Proposition 3**.
Assume that $u, v \in C^1(\Omega) \cap C(\overline{\Omega})$ *are weak solutions to* $\Delta_p u = f$ *and* $\Delta_p v = g$ *in* Ω *in the sense of distributions, such that* $u \le v$ *in* Ω *and* $u(x^0) = v(x^0)$ *at some point* $x^0 \in \Omega$. *Assume that also* $u < v$ *on* $\partial\Omega$ *and* $f - g$ *is continuous in* Ω. *Then there exists a point* $x_0 \in \Omega$ *such that* $u(x_0) = v(x_0)$, $\nabla u(x_0) = \nabla v(x_0)$, *and* $f(x_0) \le g(x_0)$.

In the proof of **Proposition 4** we need another important "local" result which is used also later. It generalizes some known strong comparison theorems in a nondegenerate setting, cf. M. Cuesta and P. Takáč (1998, 2000).

Lemma (Proposition 4). *Let* $1 < p < \infty$, $f, g \in L^\infty(\Omega)$, *and let* $B\colon \Omega \times \mathbb{R}^N \to \mathbb{R}$ *be as specified at the beginning. Assume that* $u, v \in C^1(\overline{\Omega})$ *satisfy*

$$-\Delta_p u + B(x, \nabla u) = f(x) \le g(x) = -\Delta_p v + B(x, \nabla v) \quad \text{in } \Omega$$

in the sense of distributions, $u \le v$ *in* Ω, *and* $\nabla v \ne \mathbf{0}$ *holds in* $\overline{\mathcal{V}}$ *together with* $u, v \in C^2(\overline{\mathcal{V}})$, *where* $\mathcal{V} \subset \Omega$ *is some open set (*not *necessarily connected). Then we have either* $u \equiv v$ *in some connected component of* $\mathcal{V}$, *or else* $u < v$ *throughout* $\mathcal{V}$.

If the latter alternative holds, if the hypothesis $\nabla v \neq \mathbf{0}$ *in* $\overline{\mathcal{V}}$ *is replaced by* $u(x_0) = v(x_0)$ *and* $\partial v/\partial \boldsymbol{\nu}_0(x_0) \neq 0$ *for some point* $x_0 \in \partial\mathcal{V}$*, and if the boundary* $\partial\mathcal{V}$ *of* $\mathcal{V}$ *is of class* C^2 *in a open neighborhood* $\mathcal{V}_0 \subset \partial\mathcal{V}$ *of* x_0*, then we have*

$$\frac{\partial u}{\partial \boldsymbol{\nu}_0}(x_0) > \frac{\partial v}{\partial \boldsymbol{\nu}_0}(x_0) \tag{1}$$

where $\boldsymbol{\nu}_0 \equiv \boldsymbol{\nu}|_{\mathcal{V}_0} \colon \mathcal{V}_0 \to \mathbb{R}^N$ *denotes the unit outer normal vector field on the boundary portion* $\mathcal{V}_0 \subset \partial\mathcal{V}$*.*

In the proof of **Proposition 4** we first take advantage of the strong maximum and boundary point principles (P. Tolksdorf (1983), J. L. Vázquez (1984)) to obtain

$$v > 0 \ \text{ in } \Omega \quad \text{ and } \quad \frac{\partial v}{\partial \boldsymbol{\nu}} < 0 \ \text{ on } \partial\Omega\,.$$

This brings us to a "nondegenerate setting" in Ω near its boundary $\partial\Omega$. There, we investigate the inequality $u \leq \theta v$ for

$$\theta \stackrel{\text{def}}{=} \sup_{\Omega} \frac{u}{v} \in \mathbb{R}\,.$$

If $\theta \leq 1$, we are done. Otherwise, if $1 < \theta < \infty$, we improve it by $u \leq \vartheta v$ near $\partial\Omega$ with some $\vartheta < \theta$. Finally, we apply Lemma (Proposition 3) in order to conclude that the difference $\theta v - u$ cannot have a positive maximum inside Ω. This forces $\theta \leq 1$.

For any $\mu > 0$ large enough ($\mu \geq \mu_0 > 0$) and any $\lambda \in \mathbb{R}$, the left-hand side of the following fixed point problem is coercive,

$$(\mathbf{P}_{\mu,\lambda}) \qquad \begin{cases} -\Delta_p u + B(x, \nabla u) + \mu\, |u|^{p-2} u & \\ \qquad = (\mu + \lambda)\, |\tilde{u}|^{p-2}\tilde{u} + h(x) & \text{in } \Omega\,; \\ u = 0 & \text{on } \partial\Omega\,. \end{cases}$$

Let $\tilde{u} \in L^\infty(\Omega)$. By **Proposition 3** and regularity, the solution $u \in W_0^{1,p}(\Omega)$ is unique. We have $u \in C^{1,\alpha}(\overline{\Omega})$ for some $\alpha \in (0,1)$.

The fixed point mapping $\tilde{u} \mapsto u \colon C^1(\overline{\Omega}) \to C^1(\overline{\Omega})$ is continuous, order-preserving, and compact. Our proofs of **Theorems 1** and **2** take advantage of these properties in an essential way [2].

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Three Nonlinear Eigenvalue Problems

Peter Lindqvist

1. The p-Laplace eigenvalue problem

The minimization of the Rayleigh quotient

$$\frac{\int_\Omega |\nabla u|^p\,dx}{\int_\Omega |u|^p\,dx} \qquad (1 < p < \infty)$$

among all $u \in W_0^{1,p}(\Omega)$, $u \not\equiv 0$, yields the Euler-Lagrange equation

$$\operatorname{div}(|\nabla u|^{p-2}\nabla u) + \lambda |u|^{p-2}u \;=\; 0$$

in the bounded domain Ω in $\mathbb{R}^n$. The solutions are interpreted in the weak sense. It is known that the minimum, say $\lambda = \lambda_1$, is a simple eigenvalue and that

$$\lambda_2 = \inf\{\lambda |\, \lambda > \lambda_1, \lambda = \text{an eigenvalue}\}$$

is an eigenvalue and $\lambda_2 > \lambda_1$. In other words, λ_1 is *isolated*. There are arbitrarily large eigenvalues. The main open problem is the discreteness of the spectrum. (This is clear in the linear case $p = 2$, when the equation reads $\Delta u + \lambda u = 0$, and in the one dimensional case $n = 1$.) To the best of my knowledge, it has not yet been proved even that there exists one number $\lambda > \lambda_2$ that is not an eigenvalue! Neither has it been proved that the higher eigenvalues have finite multiplicity. The conjecture is open even for a disc in the plane.

The asymptotic cases $p = \infty$ and $p = 1$ are fascinating. For $p = \infty$ the Euler-Lagrange equation becomes

$$\max\left\{\Lambda - \frac{|\nabla u|}{u},\ \sum_{i,j=1}^{n} \frac{\partial u}{\partial x_i}\frac{\partial u}{\partial x_j}\frac{\partial^2 u}{\partial x_i \partial x_j}\right\} \;=\; 0$$

and it has a *positive* solution $u \in W_0^{1,p}(\Omega)$ if and only if

$$\Lambda = \Lambda_\infty \equiv \frac{1}{\max\limits_{x\in\Omega} \operatorname{dist}(\mathrm{x}, \partial\Omega)}.$$

R. Hynd, C. Smart, and Y. Yu have recently exhibited an example showing that the first eigenvalue Λ_∞ is not simple: two independent positive solutions are constructed in a dumbbell shaped domain, cf. [6]. The question remains, whether the limit of the p-eigenfunctions u_p, nonetheless, is unique. —The problem has connections to optimal mass transportation, cf. [2], and perhaps to sphere packing.

2. A fractional p-Laplace eigenvalue problem

This comes from the Rayleigh quotient

$$\frac{\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{|u(y)-u(x)|^p}{|y-x|^{\alpha p}}\,dx\,dy}{\int_{\mathbb{R}^n} |u(x)|^p\,dx}$$

over all $u \in W_0^{s,p}(\Omega)$, $s = \alpha - \frac{s}{p}$, where at least $n < \alpha p < n + p$. (We have $u = 0$ in the complement $\mathbb{R}^n \setminus \Omega$.) Again, the first eigenvalue is simple and isolated, see also [1]. The Euler-Lagrange equation is derived in [8]. Due to its non-local nature rather strange phenomena occure among the higher eigenfunctions.

The asymptotic case $p = \infty$ leads to the interesting equation

$$\max\{\mathcal{L}_\infty^- u + \lambda u,\, \mathcal{L}_\infty u\},$$

where

$$\mathcal{L}_\infty u(x) = \sup_y \frac{u(y) - u(x)}{|y - x|^\alpha} + \inf_y \frac{u(y) - u(x)}{|y - x|^\alpha}$$

and the last infimum term defines the operator $\mathcal{L}_\infty^- u(x)$. The equation has to be interpreted in the viscosity sense. It has a positive solution in $W_0^{\alpha,\infty}(\Omega)$ if and only if $\lambda = \Lambda_\infty^\alpha$. A central part of the domain, the High Ridge Γ is defined as the points where the distance function $\delta(x) = \mathrm{dist}(\mathrm{x}, \partial\Omega)$ takes its maximum $R = \max\{\delta(x)\}$, i.e.

$$\Gamma = \{x \in \Omega |\, \delta(x) = R\}.$$

If $\Gamma_1 \subset \Gamma$ and $\varrho_1(x) = \mathrm{dist}(\mathrm{x}, \Gamma_1)$, then

$$u(x) = \frac{\delta(x)^\alpha}{\delta(x)^\alpha + \varrho_1(x)^\alpha}, \quad u(x) = 0 \text{ in } \mathbb{R}^n \setminus \Omega,$$

is a positive eigenfunction in Ω, indeed. In general, uniqueness is lost and the first eigenvalue is not simple. There are several open problems. Does the formula produce all first eigenfunctions? Is the maximal solution (the one with $\Gamma_1 = \Gamma$) the limit of the fractional p-eigenfunctions u_p?

3. Rayleigh quotient in the Luxemburg norm

Let the variable exponent $p(x)$ be a smooth function and $1 < p(x) < \infty$. Consider the Luxemburg norm

$$\|f\|_{p(x)} = \inf\left\{ t > 0 \,|\, \int_\Omega \left|\frac{f(x)}{t}\right|^{p(x)} \frac{dx}{p(x)} \le 1 \right\}.$$

The minimization of the Rayleigh quotient

$$\frac{\|\nabla u\|_{p(x)}}{\|u\|_{p(x)}}$$

leads to a peculiar Euler-Lagrange equation, see [3]. (The reason for not directly considering the ratio $\int_\Omega |\nabla u|^{p(x)}\, dx : \int_\Omega |u|^{p(x)}\, dx$ is that the minimum usually is zero, cf. [4]. A constraint like $\int_\Omega |u|^{p(x)}\, dx = C$ leads to other undesirable effects, cf. [3].) We do not know whether the first eigenvalue is simple.

Also now one can study the case $p(x) \to \infty(x)$. We let $p(x)$ approach infinity via the sequence $p(x), 2p(x), 3p(x), \ldots$ The limit equation is rather difficult and has a nonnegative viscosity solution in $W_0^{1,\infty}(\Omega)$, if the eigenvalue involved is exactly the Λ_∞ above. We do not know whether this is the only possible first eigenvalue. Neither do we know about its simplicity. But a *local* uniqueness holds:

in a sufficiently small interior subdomain one cannot perturb the positive viscosity solution(s).

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Update on Nonlinear Potential Theory

Giuseppe Mingione

0.1. **Generalities.** In classical potential theory representation formulas and estimates via fundamental solutions are of basic importance. For the Poisson equation in $\mathbb{R}^n$, that is $-\triangle u = \mu$ (we take μ to be a Radon measure with compact support) we have

$$|u(x)| \leq c\mathbf{I}_2^{|\mu|}(x,R) + c \fint_{B(x,R)} (|u| + Rs)\, dy \qquad (n \geq 3)$$

and

$$|Du(x)| \leq c\mathbf{I}_1^{|\mu|}(x,R) + c \fint_{B(x,R)} (|u| + Rs)\, dy\,,$$

where on the right hand side it appears the standard (truncated) Riesz potential

$$\mathbf{I}_\beta^{|\mu|}(x,R) := \int_0^R \frac{|\mu|(B(x,\varrho))}{\varrho^{n-\beta}}\, \frac{d\varrho}{\varrho} \qquad \beta \in (0,n]\,.$$

It happens that, although the previous estimates at a first sight appear to be extremely linked to the linear nature of the Poisson equation, sharp analogs can be derived in the case of nonlinear equations as well. We shall therefore consider quasilinear, possibly degenerate equations of the type

$$-\operatorname{div} a(Du) = \mu \qquad \text{in } \Omega\,, \tag{1}$$

where $\Omega \subset \mathbb{R}^n$ is an open subset and the vector field $a\colon \mathbb{R}^n \to \mathbb{R}^n$ is of class C^1 and satisfies the following growth and ellipticity assumptions:

$$\begin{cases} |a(z)| + (|z|^2+s^2)^{1/2}|\partial a(z)| \le L(|z|^2+s^2)^{(p-1)/2} \\ \nu(|z|^2+s^2)^{(p-2)/2}|\lambda|^2 \le \langle \partial a(z)\lambda, \lambda\rangle \end{cases} \tag{2}$$

whenever $z, \lambda \in \mathbb{R}^n$. Here it is $0 < \nu \le L$ and $s \ge 0$. Assumptions (2) are standard after the work of Ladyzhenskaya and Ural'tseva and are fulfilled - with the choice $s = 0$ - by the classical p-Laplacean operator given by $-\triangle_p u := -\mathrm{div}\,(|Du|^{p-2}Du)$, to which a huge literature has been devoted.

0.2. **Nonlinear potential estimates.** The first nonlinear potential estimate has been obtained by Kilpeläinen & Malý [4] (see also [14, 5, 2] for later proofs) and involves so called Wolff potentials, which are defined by

$$\mathbf{W}^{\mu}_{\beta,p}(x,R) := \int_0^R \left(\frac{|\mu|(B(x,\varrho))}{\varrho^{n-\beta p}}\right)^{1/(p-1)} \frac{d\varrho}{\varrho} \qquad \beta \in (0, n/p]\,.$$

These are a sort of nonlinear Riesz potentials and the resulting inequality, which is sharp, is

$$|u(x)| \le c\mathbf{W}^{\mu}_{1,p}(x,R) + c \fint_{B(x,R)} (|u| + Rs)\, dy\,.$$

Let us remark that this result is surprising already in the non degenerate case $p \neq 2$ - when $\mathbf{W}^{\mu}_{1,p}$ coincides with the standard Riesz potential $\mathbf{I}^{|\mu|}_1$ - since the main pout here is *passing from linear to nonlinear equations.* Since [4], it has been an open problem to determine whether or not similar potential estimates were possible for the gradient of solutions, and we are going to present the answer to this question. There's anyway a twist here: although the standard orthodoxy of nonlinear potential theory prescribes that Wolff potentials should replace everywhere Riesz potentials when considering the case $p \neq 2$, it turns out that, surprisingly enough, this is not the case when looking at the gradient, and the theory linearizes. It indeed holds the following theorem, that we state in the form a priori estimate for more regular solutions, while general cases can be achieved via approximation:

Theorem 1 ([12, 7, 3])**.**
Let $u \in C^1(\Omega)$ be a solution to (1) for $p \ge 2 - 1/n$. Then there exists a constant $c \equiv c(n,p,\nu,L)$ such that

$$|Du(x)|^{p-1} \le c\mathbf{I}^{|\mu|}_1(x,R) + c\left(\fint_{B(x,R)} (|Du| + s)\, dy\right)^{p-1} \tag{3}$$

holds whenever $B(x,R) \subseteq \Omega$ is a ball centered at x and with radius R.

Needless to say, when $\Omega \subset \mathbb{R}^n$ and $u \in W^{1,1}(\mathbb{R}^n)$, letting $R \to \infty$ in (3) yields the classical Riesz potential bound

$$|Du(x)|^{p-1} \le c\int_{\mathbb{R}^n} \frac{d|\mu|(y)}{|x-y|^{n-1}}\,.$$

Theorem 1 allows to reduce, up to the $W^{1,\infty}$-level, the analysis of solutions to quasilinear degenerate equations to the analysis of Riesz potentials and therefore, ultimately unifies the linear and the nonlinear theories. As a matter of fact the following theorem shows that the analogy carries on up to the C^1-level:

Theorem 2 ([7]).
Let u be as in Theorem 1. If

$$\lim_{R\to 0} \mathbf{I}_1^{|\mu|}(x,R) = 0$$

holds locally uniformly in Ω w.r.t. x then Du is continuous in Ω.

A relevant corollary of Theorem 2 provides the nonlinear analog of a classical result of Stein [13] concerning the borderline case of Soboloev-Morrey embedding theorem in terms of Lorentz spaces. Given a Sobolev function f, the condition $Df \in L(n,1)$ implies that f is continuous. The implications for the Poisson equation are immediate: $\triangle u \in L(n,1)$ implies that Du is continuous (see also [1] and references therein). Now, Theorem 2 yields that $-\triangle_p u \in L(n,1)$ implies that Du is continuous; this result, by mean of ad hoc vectorial arguments, remains true for the p-Laplacean system [11]. Let us now turn to the parabolic case

$$u_t - \operatorname{div} a(Du) = \mu \qquad \text{in} \quad \Omega_T := \Omega \times (-T,0)\,, \tag{4}$$

that involves relevant additional difficulties and requires new techniques and ideas. This time we consider caloric Riesz potentials which are naturally built starting by standard parabolic cylinders $Q_r(x_0,t_0) := B(x_0,r)\times(t_0-r^2,t_0)$, and that are therefore defined by

$$\mathbf{I}_\beta^\mu(x_0,t_0;r) := \int_0^r \frac{|\mu|(Q_\varrho(x_0,t_0))}{\varrho^{N-\beta}}\,\frac{d\varrho}{\varrho}\,, \qquad\qquad 0<\beta\le N := n+2\,.$$

The number N is usually called the parabolic dimension. It then holds

Theorem 3 ([4]).
Let u be a solution to (4) with $p\ge 2$. There exists a constant c, depending only on n,p,ν,L, such that

$$|Du(x_0,t_0)| \le c\mathbf{I}_1^{|\mu|}(x_0,t_0;r) + c\,⨍_{Q_r(x_0,t_0)} (|Du|+s+1)^{p-1}\,dx\,dt$$

holds whenever $(x_0,t_0)\in\Omega_T$ is a Lebesgue point of Du and whenever $Q_r(x_0,t_0)\subset \Omega_T$ is a standard parabolic cylinder.

The previous gradient bound is exactly the same of the one that appears when considering the heat equation; we refer also to [2] for the special case in which $p=2$ and to [9] for the case $p<2$. The proof of Theorem 3 in turn heavily relies on a more general potential estimate that involves a new class of “intrinsic potentials” built to match the classical intrinsic geometry of the evolutionary p-Laplacean operator; see [6, 8, 10] for a detailed discussion.

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Boundedness of the gradient of solutions to Uhlenbeck type elliptic systems

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(joint work with Vladimir Maz'ya)

We deal with regularity properties of solutions to Uhlenbeck type nonlinear elliptic systems, involving differential operators whose coefficients only depend on the modulus of the gradient. The local regularity theory for this kind of systems has its roots in the paper [7], whereas the scalar case had earlier been considered in [8]. The study of global regularity (i.e. up to the boundary) in boundary value problems has been initiated in [1].

In the present note, we report on some recent global regularity results established in [5] (see also [3] for the scalar case). Our concern are minimal assumptions on the boundary of the domain, and on the right-hand side of an Uhlenbeck type system, ensuring global boundedness of the gradient of solutions to the associated Dirichlet or Neumann boundary value problems. Although quite general ellipticity and growth conditions on the differential operator, non-necessarily of power type, can be treated via our approach, here we limit ourselves to discussing p-Laplacian operators. Specifically, we deal with systems of the form

$$-\mathbf{div}(|\nabla \mathbf{u}|^{p-1}\,\nabla \mathbf{u}) = \mathbf{f}(x) \quad \text{in } \Omega\,, \tag{1}$$

coupled with the homogeneous boundary condition

$$\mathbf{u} = 0 \text{ on } \partial\Omega\,, \quad \text{or} \quad \frac{\partial \mathbf{u}}{\partial \nu} = 0 \text{ on } \partial\Omega\,. \tag{2}$$

Here, Ω is a domain, namely an open bounded connected set in $\mathbb{R}^n$, with $n \geq 3$, the exponent $p \in (1, \infty)$, $\mathbf{u} : \Omega \to \mathbb{R}^N$, $N \geq 1$, is a vector-valued unknown function, $\nabla\mathbf{u} : \Omega \to \mathbb{R}^{Nn}$ denotes its matrix-valued gradient, $\mathbf{f} : \Omega \to \mathbb{R}^N$ is a datum, $\mathbf{div}$ stands for the $\mathbb{R}^N$-valued divergence operator, and ν for the outward unit normal to $\partial\Omega$. Clearly, in the case of Neumann boundary conditions, the function $\mathbf{f}$ has to fulfil the compatibility condition $\int_\Omega \mathbf{f}(x)\, dx = 0$.

The right-hand side $\mathbf{f}$ in (1) is assumed to belong to the Lorentz space $L^{n,1}(\Omega, \mathbb{R}^N)$. Recall that $L^{n,1}(\Omega, \mathbb{R}^N)$ is a borderline space for the family of Lebesgue spaces $L^q(\Omega, \mathbb{R}^N)$ with $q > n$, in the sense that $L^q(\Omega, \mathbb{R}^N) \subsetneqq L^{n,1}(\Omega, \mathbb{R}^N) \subsetneqq L^n(\Omega, \mathbb{R}^N)$ for every $q > n$.

As far as the domain is concerned, we either impose a local regularity property of $\partial\Omega$, or a global geometric property on Ω. The relevant regularity assumption amounts to requiring that $\partial\Omega \in W^2L^{n-1,1}$, namely that Ω is locally the subgraph of a function of $n-1$ variables whose second-order distributional derivatives belong to the Lorentz space $L^{n-1,1}$. This is the weakest possible integrability condition on the second-order derivatives of such a function for its first-order derivatives to be continuous, and hence for $\partial\Omega \in C^{1,0}$. By contrast, customary results, in the existing literature, concerning regularity at the boundary, require that $\partial\Omega \in C^{1,\alpha}$ for some $\alpha \in (0, 1]$.

Theorem 1.
Let Ω be a domain in $\mathbb{R}^n$, $n \geq 3$, such that $\partial\Omega \in W^2L^{n-1,1}$. Assume that $\mathbf{f} \in L^{n,1}(\Omega, \mathbb{R}^N)$. Let $\mathbf{u}$ be either the (unique) weak solution to (1) subject to homogeneous Dirichlet boundary conditions, or the (unique up to additive constant vectors) weak solution to (1) subject to homogeneous Neumann boundary conditions. Then there exists a constant $C = C(p, \Omega)$ such that

$$\|\nabla\mathbf{u}\|_{L^\infty(\Omega,\mathbb{R}^{Nn})} \leq C\|\mathbf{f}\|^{\frac{1}{p-1}}_{L^{n,1}(\Omega,\mathbb{R}^N)}. \tag{3}$$

In particular, $\mathbf{u}$ is Lipschitz continuous in Ω.

A global geometric assumption on Ω under which regularity of $\partial\Omega$ can be dispensed with is convexity.

Theorem 2.
The same conclusion as in Theorem 1 holds if Ω is any convex domain in $\mathbb{R}^n$, $n \geq 3$.

Note that, since (1) is the Euler equation of the minimization problem for the strictly convex functional

$$J(u) = \int_\Omega \left(\tfrac{1}{p}|\nabla\mathbf{u}|^p - \mathbf{f}\cdot\mathbf{u}\right) dx\,, \tag{4}$$

Theorem 2 provides a version in the vectorial case ($N > 1$) of the so called semi-classical Hilbert-Haar theory of minimization of strictly convex scalar integral functionals of the modulus of gradient on convex domains in classes of Lipschitz functions.

Let us briefly comment on our hypotheses on $\mathbf{f}$ and Ω. The assumption $\mathbf{f} \in L^{n,1}(\Omega, \mathbb{R}^N)$ is sharp for $\nabla \mathbf{u} \in L^\infty(\Omega, \mathbb{R}^{Nn})$, for instance in the linear, scalar case ($N = 1$). This is demonstrated by the Dirichlet problem for the Poisson equation in a ball B [2]. We emphasize that the datum $\mathbf{f}$ (and hence u) showing the optimality of $L^{n,1}(B, \mathbb{R})$ is not not radially symmetric. Let us mention that the local boundedness of the gradient for local solutions to Uhlenbeck type elliptic systems with $\mathbf{f} \in L^{n,1}_{\mathrm{loc}}(\Omega, \mathbb{R}^N)$ is proved in [6].

The assumption $\partial\Omega \in W^2L^{n-1,1}$ is optimal in Theorem 1, as far as integrability properties of the curvature of $\partial\Omega$ are concerned. This is shown, even for scalar problems, by examples of Dirichlet and Neumann problems for the p-Laplace equation in domains whose boundaries have conical singularities – see e.g. [4]. Similar examples also demonstrate that the convexity of Ω is a sharp global assumption in Theorem 2, since its conclusion need not hold even under slight local (non-smooth) perturbations of convex domains [4].

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Fast diffusion and geometry of domain

Shigeru Sakaguchi

Let Ω be a domain in $\mathbb{R}^N$, where $N \geq 2$ and $\partial\Omega$ is not necessarily bounded. We consider two fast diffusion equations of the forms $\partial_t u = \mathrm{div}(|\nabla u|^{p-2}\nabla u)$ and $\partial_t u = \Delta u^m$, where $1 < p < 2$ and $0 < m < 1$. Let $u = u(x,t)$ be the bounded solution of either the initial-boundary value problem:

$$\partial_t u = \mathrm{div}(|\nabla u|^{p-2}\nabla u) \quad \text{in } \Omega \times (0,\infty), \tag{1}$$

$$u = 1 \quad \text{on } \partial\Omega \times (0,\infty), \tag{2}$$

$$u = 0 \quad \text{on } \Omega \times \{0\}, \tag{3}$$

or the Cauchy problem:

(4) $\partial_t u = \mathrm{div}(|\nabla u|^{p-2}\nabla u)$ in $\mathbb{R}^N \times (0,\infty)$ and $u = \mathcal{X}_{\Omega^c}$ on $\mathbb{R}^N \times \{0\}$,

where $\mathcal{X}_{\Omega^c}$ denotes the characteristic function of the set $\Omega^c = \mathbb{R}^N \setminus \Omega$. The first theorem tells us about the interaction between fast diffusion and geometry of domain for $\partial_t u = \mathrm{div}(|\nabla u|^{p-2}\nabla u)$.

Theorem 1. ([9]) *Let u be the solution of either problem* (1)-(3) *or problem* (4). *Let $\alpha > \frac{(N+1)(2-p)}{2p}$ and $x_0 \in \Omega$. Assume that the open ball $B_R(x_0)$ centered at x_0 and with radius $R > 0$ is contained in Ω and such that $\overline{B_R(x_0)} \cap \partial\Omega = \{y_0\}$ for some $y_0 \in \partial\Omega$ and $\partial\Omega \cap B_\delta(y_0)$ is of class C^2 for some $\delta > 0$.*

Then we have:

$$\lim_{t\to 0^+} t^{-\frac{N+1}{2p}} \int_{B_R(x_0)} (u(x,t))^\alpha \, dx = c(p,\alpha,N) \left\{ \prod_{j=1}^{N-1} \left[\frac{1}{R} - \kappa_j(y_0) \right] \right\}^{-\frac{1}{2}}. \tag{5}$$

Here, $\kappa_1(y_0), \ldots, \kappa_{N-1}(y_0)$ denote the principal curvatures of $\partial\Omega$ at y_0 with respect to the inward normal direction to $\partial\Omega$ and $c(p,\alpha,N)$ is a positive constant depending only on p, α and N (of course, $c(p,\alpha,N)$ depends on the problems (1)-(3) *or* (4)). *When $\kappa_j(y_0) = \frac{1}{R}$ for some $j \in \{1, \cdots, N-1\}$, the formula* (5) *holds by setting the right-hand side to ∞ (notice that $\kappa_j(y_0) \le \frac{1}{R}$ for every $j \in \{1, \cdots, N-1\}$)*

Concerning $\partial_t u = \Delta u^m$ with $0 < m < 1$, let $u = u(x,t)$ be the bounded solution of either the initial-boundary value problem:

$$\partial_t u = \Delta u^m \quad \text{in } \Omega \times (0,\infty), \tag{6}$$

$$u = 1 \quad \text{on } \partial\Omega \times (0,\infty), \tag{7}$$

$$u = 0 \quad \text{on } \Omega \times \{0\}, \tag{8}$$

or the Cauchy problem:

(9) $\partial_t u = \Delta u^m$ in $\mathbb{R}^N \times (0,\infty)$ and $u = \mathcal{X}_{\Omega^c}$ on $\mathbb{R}^N \times \{0\}$.

The second theorem tells us about the interaction between fast diffusion and geometry of domain for $\partial_t u = \Delta u^m$.

Theorem 2. ([9]) *Let u be the solution of either problem* (6)-(8) *or problem* (9). *Let $\alpha > \frac{(N+1)(1-m)}{4}$ and $x_0 \in \Omega$. Assume that the open ball $B_R(x_0)$ centered at x_0 and with radius $R > 0$ is contained in Ω and such that $\overline{B_R(x_0)} \cap \partial\Omega = \{y_0\}$ for some $y_0 \in \partial\Omega$ and $\partial\Omega \cap B_\delta(y_0)$ is of class C^2 for some $\delta > 0$.*

Then we have:

$$\lim_{t\to 0^+} t^{-\frac{N+1}{4}} \int_{B_R(x_0)} (u(x,t))^\alpha \, dx = c(m,\alpha,N) \left\{ \prod_{j=1}^{N-1} \left[\frac{1}{R} - \kappa_j(y_0) \right] \right\}^{-\frac{1}{2}}. \tag{10}$$

Here, $\kappa_1(y_0), \ldots, \kappa_{N-1}(y_0)$ denote the principal curvatures of $\partial\Omega$ at y_0 with respect to the inward normal direction to $\partial\Omega$ and $c(m,\alpha,N)$ is a positive constant *depending only on* m, α *and* N (*of course,* $c(m, \alpha, N)$ *depends on the problems* (6)-(8) *or* (9)). *When* $\kappa_j(y_0) = \frac{1}{R}$ *for some* $j \in \{1, \cdots, N-1\}$*, the formula* (10) *holds by setting the right-hand side to* ∞.

When $p > 2, m > 1$ and $\alpha = 1$, the same formulas (5) and (10) were obtained for problems (1)-(3) and (6)-(8) in [4]. With the help of the techniques employed in [6], one can easily see that the formulas (5) and (10) also hold true for problems (4) and (9). Moreover, in [6], the nonlinear diffusion equation of the form $\partial_t u = \Delta\phi(u)$ where $\delta_1 \leq \phi'(s) \leq \delta_2$ $(s \in \mathbb{R})$ for some positive constants δ_1 and δ_2 was also dealt with. By a little more observation, we see that any $\alpha > 0$ is OK for these cases. In Theorems 1 and 2, if p is close to 1 or if $N \geq 4$ and m is close to 0, then $\alpha = 1$ can not be chosen. Indeed, when $\alpha = \frac{(N+1)(2-p)}{2p}$ or $\alpha = \frac{(N+1)(1-m)}{4}$, $c(p, \alpha, N) = \infty$ or $c(m, \alpha, N) = \infty$. The main ingredients of the proofs of the formulas (5) and (10) consist of two steps. One is the reduction to the case where $\partial\Omega$ is bounded and of class C^2 with the aid of the comparison principle. The other is the construction of appropriate supersolutions and subsolutions to the problems near $\partial\Omega$ in a short time. In fact, in [4], such barriers were constructed in a set $\Omega_\rho \times (0, \tau]$, with

$$\Omega_\rho = \{x \in \Omega \ : \ \operatorname{dist}(x, \partial\Omega) < \rho\}, \tag{11}$$

where ρ and τ were chosen sufficiently small. When $p > 2$ or $m > 1$, the property of finite speed of propagation of disturbances from rest yields that both the solution u and the barriers equal zero on $\Gamma_\rho \times (0, \tau]$, where

$$\Gamma_\rho = \{x \in \Omega \ : \ \operatorname{dist}(x, \partial\Omega) = \rho\}. \tag{12}$$

This property does not occur when $1 < p < 2$ or $0 < m < 1$, because of the property of *infinite* speed of propagation of disturbances from rest. Also in [6], the equation $\partial_t u = \Delta\phi(u)$ has the property of infinite speed of propagation of disturbances from rest. To compare the solution with the barriers on $\Gamma_\rho \times (0, \tau]$, in [6], the result of Atkinson and Peletier [1] concerning the asymptotic behavior of one-dimensional similarity solutions and the following short time behavior of u obtained by [5] plays a key role:

$$\lim_{t \to 0^+} -4t\Phi(u) = \operatorname{dist}(x, \partial\Omega)^2 \ \text{ uniformly on every compact subset of } \Omega, \tag{13}$$

where $\Phi(s) = \int_1^s \frac{\phi'(\xi)}{\xi} d\xi$ for $s > 0$. However, when $1 < p < 2$ or $0 < m < 1$, the short time behavior of u is not controlled by the distance function in such a way. To overcome this difficulty in the proofs of Theorems 1 and 2, we use the fact that the short time behavior of the solution u is described by the boundary blow-up solutions given in [8, 2]. The details will be given by [9].

Let us mention **related open problems**. Suppose that Ω is bounded. Consider problems (1)-(3), (6)-(8), (4) and (9), where $p \neq 2$ and $m \neq 1$. The case where $p > 2$ or $m > 1$ is also considered. Suppose that there exists a C^1 domain D with $\overline{D} \subset \Omega$ such that

$$u(x, t) = a(t) \ \text{ for every } (x, t) \in \partial D \times (0, \infty) \tag{14}$$

for some function $a : (0, \infty) \to [0, \infty)$. Then, must $\partial\Omega$ be a sphere? See [3, 5, 7] for such problems. One easy remark is that if the initial conditions in problems (6)-(8) and (9) are replaced with

$$u = \varepsilon \text{ on } \Omega \times \{0\} \text{ and } u = \mathcal{X}_{\Omega^c} + \varepsilon \mathcal{X}_\Omega \text{ on } \mathbb{R}^N \times \{0\}$$

for some $\varepsilon \in (0, 1)$, respectively, then it follows from the results of [5, 7] that $\partial\Omega$ must be a sphere for problems (6)-(8) and (9).

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Boundary estimates for non-negative solutions to non-linear parabolic equations

Kaj Nyström

(joint work with Håkan Persson and Olow Sande)

In this report we briefly mention parts of the main results establish in joint work with Håkan Persson and Olow Sande, see [6], concerning the boundary behaviour of non-negative solutions to certain non-linear parabolic equation in space-time cylinders $\Omega_T = \Omega \times (0, T)$, $T > 0$, where $\Omega \subset \mathbb{R}^n$ is a bounded domain, i.e., an open, connected and bounded set. Given p, $1 < p < \infty$, fixed, recall that general equations of p-parabolic type are equations of the form

$$Hu = \partial_t u - \nabla \cdot A(x, t, \nabla u) = 0 \tag{1}$$

where $A(x, t, \eta) = (A_1(x, t, \eta), ..., A_n(x, t, \eta)) : \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$ is measurable, $A(x, t, \eta)$ is, for almost all fixed $(x, t) \in \mathbf{R}^n$, continuous in η_k, for every $k \in \{1, ..., n\}$ and whenever $\eta \in \mathbb{R}^n$, and the following conditions are satisfied, whenever $(x, t, \eta) \in \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^n$, $\xi \in \mathbb{R}^n$, and for some β, $1 \leq \beta < \infty$:

$$\begin{aligned} &(i) \quad |A(x, t, \eta)| \leq \beta |\eta|^{p-1}, \\ &(ii) \quad \big(A(x, t, \eta) - A(x, t, \xi)\big) \cdot \big(\eta - \xi\big) \geq \beta^{-1}(|\eta| + |\xi|)^{p-2} |\eta - \xi|^2. \end{aligned} \tag{2}$$

In the special case $p = 2$, $\nabla \cdot A(x,t,\eta) = \partial_{x_i}(a_{ij}(x,t)\eta_j)$ and when the matrix $A(x,t) = \{a_{ij}(x,t)\}$ is real, symmetric, and such that $A(x,t,\eta)$ satisfies (2), the problems studied in [6] has a long and rich history, see [1], [2], [5], [7]. On the contrary for $p \neq 2$, $1 < p < \infty$, very little is know and we refer to [4]. In [6] we consider the case which is in between these two situations as we consider general equations as in (1), assuming (2), but with the important extra assumption that $p = 2$. In particular, we consider non-linear parabolic equations with linear growth. In this special case we are able to establish boundary Harnack type inequalities for non-negative solutions vanishing on a portion of the lateral boundary of Ω_T.

Given $x \in \mathbb{R}^n$ and $r > 0$, let $B(x,r) = \{y \in \mathbb{R}^n : |x-y| < r\}$. For $(x,t) \in \mathbb{R}^{n+1}$ and $r > 0$ we let $C_r(x,t) = B(x,r) \times (t - r^2, t + r^2)$. Furthermore, we let $d_p(x,t,y,s) = (|x-y|^2 + |t-s|)^{1/2}$ denote the parabolic distance between $(x,t),(y,s) \in \mathbb{R}^{n+1}$. If $O \subset \mathbb{R}^n$ is open and $1 \leq q \leq \infty$, then by $W^{1,q}(O)$, we denote the space of equivalence classes of functions f with distributional gradient $\nabla f = (f_{x_1}, \ldots, f_{x_n})$, both of which are q-th power integrable on O. Let

$$\|f\|_{W^{1,q}(O)} = \|f\|_{L^q(O)} + \|\,|\nabla f|\,\|_{L^q(O)}$$

be the norm in $W^{1,q}(O)$ where $\|\cdot\|_{L^q(O)}$ denotes the usual Lebesgue q-norm in O. $C_0^\infty(O)$ is the set of infinitely differentiable functions with compact support in O. By $\nabla\cdot$ we denote the divergence operator. Given $t_1 < t_2$ we denote by $L^q(t_1, t_2, W^{1,q}(O))$ the space of functions such that for almost every t, $t_1 \leq t \leq t_2$, the function $x \to u(x,t)$ belongs to $W^{1,q}(O)$ and

$$\|u\|_{L^q(t_1,t_2,W^{1,q}(O))} := \left(\int_{t_1}^{t_2}\int_O \left(|u(x,t)|^q + |\nabla u(x,t)|^q\right) dxdt\right)^{1/q} < \infty.$$

Given a bounded domain $G \subset \mathbb{R}^n$ and $t_1 < t_2$ we let $G_{t_1,t_2} := G \times \{t : t_1 < t < t_2\}$. We say that u is a weak solution to (1) in G_{t_1,t_2} if, for all open sets $G' \subseteq G$ and $t_1 < t_1' < t_2' < t_2$, we have $u \in L^2(t_1', t_2', W^{1,2}(G'))$ and

$$\begin{aligned} 0 \;&=\; -\int_{t_1'}^{t_2'}\int_{G'} A(x,t,\nabla u)\cdot\nabla\theta\, dxdt + \int_{t_1'}^{t_2'}\int_{G'} u\partial_t\theta\, dxdt \\ &\quad -\int_{G'} u(x,t_2')\theta(x,t_2')dx + \int_{G'} u(x,t_1')\theta(x,t_1')dx \end{aligned} \tag{3}$$

whenever $\theta \in C(t_1, t_2, C_0^\infty(G'))$. We consider non-negative weak solutions to the equation in (1) in cylindrical domains $\Omega_T = \Omega \times (0,T)$, where $\Omega \subset \mathbb{R}^n$ is a bounded domain and $T > 0$. Concerning the domain Ω we assume that Ω is a NTA-domain (non-tangentially accessible domain), with parameters M, r_0, in the sense of [3]. Assuming that Ω is a NTA-domain one can prove that all points on the parabolic boundary

$$\partial_p\Omega_T = S_T \cup (\Omega \times \{0\}), \quad S_T = \partial\Omega \times (0,T),$$

of the cylinder Ω_T are regular for the Dirichlet problem for the operator H in (1). If Ω is a given NTA-domain, with parameters M and r_0, then there exists, for any $x_0 \in \partial\Omega$, $0 < r < r_0$, a non-tangential corkscrew point, i.e., a point $A_r(x_0) \in \Omega$, such that

$$M^{-1}r < d(x_0, A_r(x_0)) < r, \quad \text{and} \quad d(A_r(x_0), \partial\Omega) \geq M^{-1}r.$$

We let $A_r(x_0, t_0) = (A_r(x_0), t_0)$ whenever $(x_0, t_0) \in S_T$ and $0 < r < r_0$. In [6] we first establish the following theorem.

Theorem 1.
Let H be as in (1) and assume (2) with $p = 2$. Let $\Omega_T = \Omega \times (0, T)$, where $\Omega \subset \mathbb{R}^n$ is a bounded NTA-domain with parameters M, r_0, and $T > 0$. Let u be a non-negative solution of $Hu = 0$ in Ω_T vanishing continuously on S_T. Let $0 < \delta \ll \sqrt{T}$ be a fixed constant, let $(x_0, t_0) \in S_T$, $\delta^2 \leq t_0 \leq T - \delta^2$, and assume that $r < \min\{r_0/2, \sqrt{(T - t_0 - \delta^2)/4}, \sqrt{(t_0 - \delta^2)/4}\}$. Then, there exists $c = c(H, M, r_0, diam(\Omega), T, \delta)$, $1 \leq c < \infty$, such that

$$u(x, t) \leq cu\big(A_r(x_0, t_0)\big)$$

whenever $(x, t) \in \Omega_T \cap C_{r/4}(x_0, t_0)$.

Let Ω, Ω_T, u be as in the statements of Theorem 1. Assume that u is continuous on the closure of Ω_T. Then $u(x, 0) = 0$ whenever $x \in \partial\Omega$. Extend u to $\mathbb{R}^n \times [0, T]$ by putting $u \equiv 0$ on $(\mathbb{R}^n \setminus \Omega) \times [0, T]$. Then there exists, see [6], a unique locally finite positive Borel measure μ on $\mathbb{R}^n \times [0, T]$, with support in $\partial\Omega \times [0, T]$, such that,

$$-\int_{t_1}^{t_2}\int_{\mathbb{R}^n} A(x, t, \nabla u) \cdot \nabla\theta dxdt + \int_{t_1}^{t_2}\int_{\mathbb{R}^n} u\partial_t\theta dxdt = \iint \theta d\mu \tag{4}$$

whenever $0 < t_1 < t_2 < T$, $\theta \in C_0^\infty(\mathbb{R}^n \times (t_1, t_2))$. In [6] we establish the following theorem concerning the doubling property of the measure μ.

Theorem 2.
Let H be as in (1) and assume (2) with $p = 2$. Let $\Omega_T = \Omega \times (0, T)$, where $\Omega \subset \mathbb{R}^n$ is a bounded NTA-domain with parameters M, r_0, and $T > 0$. Let u be a non-negative solution of $Hu = 0$ in Ω_T, assume that u is continuous on the closure of Ω_T and that u vanishes continuously on S_T. Let μ be the measure associated to u, with support in $\partial\Omega \times [0, T]$, as in (4). Let $0 < \delta \ll \sqrt{T}$ be a fixed constant, let $(x_0, t_0) \in S_T$, $\delta^2 \leq t_0 \leq T - \delta^2$, and assume that

$$r < \min\{r_0/2, \sqrt{(T - t_0 - \delta^2)/4}, \sqrt{(t_0 - \delta^2)/4}\}.$$

Then, there exists $c = c(H, M, r_0, \mathrm{diam}(\Omega), T, \delta)$, $1 \leq c < \infty$, such that

$$\mu\big(\Delta(x_0, t_0, 2r)\big) \leq c\mu\big(\Delta(x_0, t_0, r)\big).$$

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Some Free Boundary Problems Associated With The *p*-Laplacian

Peiyong Wang

In this presentation, I reviewed the formulation of the one-phase and two-phase free boundary problems of phase transition for the p-Laplacian, and presented some recently-proved uniqueness results. They constitute a nonlinear version of a uniqueness theory for the linear elliptic and parabolic phase transition problems established in [2] and [1].

Following variational principle, one may propose to minimize the functionals

$$J_p^1(u) = \int_\Omega \frac{1}{p}|\nabla u|^p + Q^p(x)\mathbf{1}_{\{u>0\}}(u)dx$$

for the one-phase problem, and

$$J_p^2(u) = \int_\Omega \frac{1}{p}|\nabla u|^p + Q^p(x)\lambda^p(u)dx$$

for the two-phase problem of phase transition, where $1 < p < \infty$, $u \in W^{1,p}(\Omega)$, $\inf_\Omega Q(x) > 0$, and

$$\lambda(s) = \begin{cases} \lambda_1 & \text{if } s < 0 \\ \lambda_2 & \text{if } s > 0 \end{cases}$$

with $0 < \lambda_1 < \lambda_2$. The Dirichlet boundary condition, $u - \sigma \in W_0^{1,p}(\Omega)$ for a given data $\sigma \in W^{1,p}(\Omega)$, is usually assumed.

The free boundary conditions for the one-phase and two-phase problems are respectively $u_\nu^+ = (\frac{p}{p-1})^{\frac{1}{p}} Q(x)$ and $(u_\nu^+)^p - (u_\nu^-)^p = \frac{p}{p-1} Q^p(x)(\lambda_2^p - \lambda_1^p)$, which are verified in the weak sense. Here ν is the interior unit normal to the boundary of the positive domain.

The viscosity formulation of the above phase transition problems reads

$$-\triangle_p u(x) = 0 \text{ in } \Omega^+$$

for the one-phase problem and

$$-\triangle_p u(x) = 0 \text{ in } \Omega^+ \text{ and in } \Omega^-$$

for the two-phase problem, together with the above respective free boundary condition. Here $\triangle_p u = (p-2)|\nabla u|^{p-4} < D^2 u \nabla u, \nabla u > +|\nabla u|^{p-2} \triangle u$ is the p-Laplacian of u. Both the partial differential equations and free boundary conditions are verified in the viscosity sense of employing sub- and super-solution tests.

A well-known existence result for minimizing functionals such as J_p^1 and J_p^2 is the following theorem.

Theorem.

If $J_p(\sigma) < \infty$, there exists a minimizer of the variational problem such as minimizing J_p^1 or J_p^2.

There I deviate my talk to the formulation of the variational problem for ∞-Laplacian. The main difficulty in the formulation lies in the appropriate formula for the free boundary condition. Intuitively, one would send p to ∞ in the above problems for p-Laplacian to obtain the formulation for the ∞-Laplacian. For the one-phase problem, the limiting free boundary condition is $u_\nu^+ = Q(x)$ for a given positive function Q, which seems quite natural. However, for the two-phase problem, the limiting free boundary condition is $u_\nu^+(x) = Q(x)\lambda_2$ which is a one-phase condition. It is not clear at this time if it is a reasonable free boundary condition for a two-phase problem.

On the other hand, the uniqueness of a viscosity solution or of a minimizer fails for small boundary data for the elliptic one-phase or two-phase problem. It is typically a bifurcation phenomenon which is described for the linear problems in [1] and [2]. In order to analyze this non-uniqueness phenomenon, we resort to study of the corresponding evolution. In fact, we start with a 'fattened' free boundary problem of minimizing the functional of one-phase, for simplicity

$$J_\varepsilon^1(u) = \int_\Omega \frac{1}{p}|\nabla u(x)|^p + Q(x)\Gamma_\varepsilon(u(x))dx,$$

for $u \in W^{1,p}(\Omega)$ subject to the boundary condition $u(x) = \sigma(x)$ on $\partial\Omega$ in the sense of traces of distributions, where $\Gamma_\varepsilon(s)$ is a smooth function which approximates the characteristic function $\mathbf{1}_{\{u>0\}}$, and which is defined by $\Gamma_\varepsilon(s) = \Gamma(\frac{s}{\varepsilon})$ and by

$$\Gamma(s) = \begin{cases} 0 & \text{if } s \le -1 \\ 1 & \text{if } s \ge 1, \end{cases}$$

Then we can prove there is a unique solution of the evolutionary problem

$$\begin{cases} H^\varepsilon w = w_t - \triangle_p w + \beta_\varepsilon(w) = 0 & \text{in } \Omega \times (0,T) \\ w(x,t) = \sigma(x) & \text{on } \partial\Omega \times (0,T) \\ w(x,0) = w_0(x) & \text{on } \bar{\Omega}, \end{cases}$$

where $\beta_\varepsilon = \Gamma'_\varepsilon$.

Define $\mathfrak{S}$ to be the set of solutions of the evolutionary problem. Define

$$\bar{u}(x) = \inf_{u \in \mathfrak{S}, u \ge u_0, u \ne u_0} u(x), \; x \in \bar{\Omega}, \tag{1}$$

where u_0 is the minimizer of the functional J_p^1.
Theorem.
If $w_0 \le \bar{u}$ on $\bar{\Omega}$, then

$$\lim_{t\to+\infty} w(x,t) = u_0(x)$$

locally uniformly for $x \in \bar{\Omega}$.

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Lipschitz truncation and the p-Laplace

Lars Diening

In this lecture I present an overview of the Lipschitz truncation technique and its application for problems with p-Laplace structure both stationary and instationary. The Lipschitz truncation method goes back to Acerbi and Fusco [1] and allows to approximate a Sobolev function $u \in W^{1,p}$ by Lipschitz functions. An important feature of this approximation is that the function is only changed on a small set.

The best way to construct such approximations is to proceed similar to the Calderón-Zygmund decomposition of L^p functions but now in the Sobolev space $W^{1,p}$. The Lipschitz truncation u_λ can be seen as the good part of the corresponding Calderón-Zygmund decomposition.

My first example is from the context of incompressible, generalized Newtonian fluids. The motion of such fluids can be described by a system of the following structure

$$\begin{aligned} \partial_t v - \operatorname{div}\big(S(\epsilon(v))\big) + [\nabla v]v + \nabla q &= f \qquad \text{on } \Omega,\\ \operatorname{div} v &= 0 \qquad \text{on } \Omega,\\ v &= 0 \qquad \text{on } \partial\Omega, \end{aligned}$$

where v is the velocity, q the pressure, $S(\epsilon(v)) = (1 + |\epsilon(v)|)^{p-2}\epsilon(v)$ is the extra stress with $1 < p < \infty$, f the external force and $\epsilon(v) = \frac{1}{2}(\nabla v + (\nabla v)^T)$ is the symmetric part of the gradient. In particular, the main operator $\operatorname{div}(S(\epsilon(v)))$ has similar structure as the p-Laplace, where gradients are replaced by its symmetric part $\epsilon(v)$. For small values of p the convective term $[\nabla v]v$ is a non-compact perturbation of the monotone main term $\operatorname{div}(S(\epsilon(v)))$. This makes the construction of weak, long-time solutions very difficult. The difficulty is the passage to the weak limit in the non-linear term $S(\epsilon(v))$. I show, how to overcome this problem by use of the Lipschitz truncation technique [2, 3, 6, 7, 11].

The second application is concerned with harmonic and caloric approximation lemmas, see [10] for an excellent overview. This concept goes back to De Giorgi [4]. A function u is said to be almost harmonic on a ball B, if

$$\int \nabla u \nabla \xi \, dx \leq \delta \int |\nabla u| \, dx \, \|\nabla \xi\|_\infty$$

for all $\xi \in C_0^\infty(B)$. Now, the approximation lemma of De Giorgi says that we can find a harmonic function on B, which is close to u in the L^2-sense. Such type of results are for example useful in the theory of partial regularity. The whole concept was generalized by Duzaar and Mingione [9] to the setting of (almost) p-harmonic functions. I will explain the approach of [5, 8], which allows to preserve the boundary values of u and to get also closeness of the gradients. This allows to apply the approximation lemma more efficiently. I will show that both - the harmonic and the p-harmonic approximation lemma - are useful in the study for systems with p-growth even in the quasi-convex case.

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Tug-of-War games and PDEs

Julio D. Rossi

We report on recent results concerning Tug-of-War games and PDEs, see [10, 11, 6, 7, 8, 9, 2] and the survey [12]. We will look for a probabilistic approach to approximate solutions to the $\infty-$Laplacian. This is the nonlinear degenerate elliptic operator, usually denoted by Δ_∞, given by,

$$\Delta_\infty u := \left(D^2u \, \nabla u\right) \cdot \nabla u = \sum_{i,j=1}^{N} \frac{\partial u}{\partial x_i} \frac{\partial u}{\partial x_j} \frac{\partial^2 u}{\partial x_i x_j},$$

and arises from taking limit as $p \to \infty$ in the p-Laplacian operator in the viscosity sense, see [1]. In fact, let us present a formal derivation. First, expand (formally) the $p-$laplacian:

$$\begin{aligned} \Delta_p u &= \operatorname{div}\left(|\nabla u|^{p-2}\nabla u\right) = \\ &= (p-2)|\nabla u|^{p-4} \left\{ \frac{1}{p-2}|\nabla u|^2 \Delta u + \sum_{i,j} u_{x_i} u_{x_j} u_{x_i,x_j} \right\} \end{aligned}$$

and next, using this formal expansion, pass to the limit in the equation $\Delta_p u = 0$, to obtain

$$\Delta_\infty u = \sum_{i,j} u_{x_i} u_{x_j} u_{x_i,x_j} = Du \cdot D^2u \cdot (Du)^t = 0.$$

Note that this calculation can be made rigorous in the viscosity sense.

The ∞-laplacian operator appears naturally when one considers absolutely minimizing Lipschitz extensions of a boundary function F; see [3] and also the survey [1]. A fundamental result of Jensen [3] establishes that the Dirichlet problem for Δ_∞ is well posed in the viscosity sense. Solutions to $-\Delta_\infty u = 0$ (that are called infinity harmonic functions) are also used in several applications, for instance, in optimal transportation and image processing.

0.1. **Description of the game.** We follow [10] and [2], but we restrict ourselves to the case of a game in a bounded smooth domain $\Omega \subset \mathbb{R}^N$ (the results presented in [10] are valid in general length spaces).

A Tug-of-War is a two-person, zero-sum game, that is, two players are in contest and the total earnings of one are the losses of the other. Hence, one of them, say Player I, plays trying to maximize his expected outcome, while the other, say Player II is trying to minimize Player I's outcome (or, since the game is zero-sum, to maximize his own outcome). Now consider a bounded domain $\Omega \subset \mathbb{R}^N$, and take $\Gamma_D \subset \partial\Omega$ and $\Gamma_N \equiv \partial\Omega \setminus \Gamma_D$. Let $F : \Gamma_D \to \mathbb{R}$ be a Lipschitz continuous function. At an initial time, a token is placed at a point $x_0 \in \overline{\Omega} \setminus \Gamma_D$. Then, a (fair) coin is tossed and the winner of the toss is allowed to move the game position to any $x_1 \in B_\epsilon(x_0) \cap \overline{\Omega}$. At each turn, the coin is tossed again, and the winner chooses a new game state $x_k \in B_\epsilon(x_{k-1}) \cap \overline{\Omega}$. Once the token has reached some $x_\tau \in \Gamma_D$, the game ends and Player I earns $F(x_\tau)$ (while Player II earns $-F(x_\tau)$). This is the reason why we will refer to F as the *final payoff function.*

This procedure yields a sequence of game states $x_0, x_1, x_2, \dots, x_\tau$, where every x_k except x_0 are random variables, depending on the coin tosses and the strategies adopted by the players.

Now, if S_I and S_{II} denote the strategies adopted by Player I and II respectively, we define the expected payoff for player I as

$$V_{x_0,I}(S_I, S_{II}) = \begin{cases} \mathbb{E}^{x_0}_{S_I,S_{II}}[F(x_\tau)], & \text{if the game terminates a.s.} \\ -\infty, & \text{otherwise.} \end{cases}.$$

Analogously, we define the expected payoff for player II as

$$V_{x_0,II}(S_I, S_{II}) = \begin{cases} \mathbb{E}^{x_0}_{S_I,S_{II}}[F(x_\tau)], & \text{if the game terminates a.s.} \\ +\infty, & \text{otherwise.} \end{cases}.$$

We define the *ϵ-value for Player I* as $u^\epsilon_I(x_0) = \sup_{S_I} \inf_{S_{II}} V_{x_0,I}(S_I, S_{II})$, and the ϵ-value for Player II as $u^\epsilon_{II}(x_0) = \inf_{S_{II}} \sup_{S_I} V_{x_0,II}(S_I, S_{II})$. In some sense, $u^\epsilon_I(x_0), u^\epsilon_{II}(x_0)$ are the least possible outcomes that each player expects to get when the ϵ-game starts at x_0. Notice that we penalize severely the games that never end. If $u^\epsilon_I = u^\epsilon_{II} := u_\epsilon$, we say that *the game has a value.* In [10] it is shown that, under very general hypotheses, that are fulfilled in the present setting, the ϵ-Tug-of-War game has a value.

Theorem. [10], [2] *There exists a uniform limit $u(x) = \lim_{\varepsilon\to 0} u_\epsilon$ that is characterized as the unique viscosity solution to the mixed boundary value problem*

$$\begin{cases} -\Delta_\infty u(x) = 0 & \text{in } \Omega, \\ \dfrac{\partial u}{\partial n}(x) = 0 & \text{on } \Gamma_N, \\ u(x) = F(x) & \text{on } \Gamma_D. \end{cases}$$

There is also a version of this result for the $p-$Laplacian, [11], [8]. This version was used in [6], see also [4], to obtain characterizations of being $p-$harmonic in terms of asymptotic mean value formulas.

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Variations on the p-Laplacian

Bernd Kawohl

In this survey lecture I reported first on the use of p-Laplacian operators in the context of image processing and showed some numerical results. I explained that the Tortorelli-Ambrosio approximation of the Mumford-Shah functional, once discretized, has many features in common with discretised Perona-Malik flow. Then I derived evolution equations involving the normalized or game-theoretic p-Laplacian

$$u_t - \Delta_p^N u := u_t - \tfrac{1}{p}|Du|^{2-p}\mathrm{div}\left(|Du|^{p-2}Du\right) = 0.$$

For $p = 1$ this gives the level set formulation of mean curvature flow treated by Evans and Spruck, while for $p = \infty$ the equation was treated in [4]. The case $p \in (1, \infty)$ is nondegenerate and more benign in the spatial variable and can be found in [2].

Then I turned to Dirichlet problems involving Δ_p for homogeneous and inhomogeneous data. While for $p \in (1, p)$ these are uniquely solvable in a weak and viscosity sense, there are cases of nonuniqueness in the variational sense for $p = \infty$ and nonuniqueness in the viscosity sense for $p = 1$. Then I compared the p–Laplacian with its normalized version $\Delta_p^N u := \frac{1}{p}|\nabla u|^{2-p}\Delta_p u$ and studied equations like $-\Delta_p u = 1$ or $-\Delta_p^N u = 1$ as $p \to 1$ or $p \to \infty$, also with overdetermined boundary conditions, in which case solutions can only exist on balls unless $p = \infty$. In that case the ridge of the domain plays a crucial role, see [1].

Harmonic functions are known to have mean value properties, and so do p-harmonic ones [9], except that the mean value is taken over sets different from balls. For various domains e.g. ellipsoids with eccentricities depending on p, and orientation of the principal axis depending on $Du(x)/|Du(x)|$ as well as for more general quasilinear equations this was done in detail in [3].

Finally I presented results and open problems on the eigenvalue problem $-\Delta_p u = \lambda|u|^{p-2}u$. During the workshop F.Schuricht and I were able to show that the discontinuous first eigenfunction of the 1-Laplacian can be interpreted as a viscosity solution of the corresponding discontinuous differential equation, if one is willing to accept the notion that their upper and lower semicontinous representatives are used in the appropriate way. In this written report I add the announcement that for the normalized p-Laplacian on a ball, and for the case of radial functions, one can compute a complete orthonormal system of (radial) eigenfunctions in a canonically weighted L^2 space. In fact in this situation the normalized p-Laplacian operator transforms essentially into an ODE that can be interpreted as (linear) Laplace operator in a fractional dimension. The corresponding Bessel equation is well understood, and the details are subject of an ongoing master thesis of Jannis Kurtz in Cologne.

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Open problem session

De Pascale, Diening, Kawohl, Lindqvist, Rossi, Takáč

1. Characterization of eigenvalues

It is currently unclear how many eigenvalue the p-Laplacian has. Two sequences of eigenvalues have been introduced through the years. Both of them are obtained by a $min - max$ procedure. Denote by

$$\mathcal{K}_n = \{K \subset \{u \in L^p \ : \ \|u\|_p = 1\} \ : \ K \text{ is compact, symmetric, and } \gamma(K) \geq n\},$$

where $\gamma(G) := \inf\{m \ : \ \exists\varphi : G \to \mathbb{R}^m \setminus \{0\}, \text{continuous and odd}\}$ denotes the Krasnoselski genus. In [5] it is proved that

$$\lambda_n := \inf_{G \in \mathcal{K}_n} \sup_{u \in G} \|\nabla u\|_p^p$$

is a sequence of eigenvalue of the p-Laplacian.

Another sequence of eigenvalues was built in [7] as follows. Denote by

$$\Sigma_{k-1} = \{\varphi \in \mathcal{C}^1(S^{k-1}, \{u \in L^p \ : \ \|u\|_p = 1\}), \ \varphi \text{ is odd}\},$$

then

$$\mu_k := \inf_{\varphi \in \Sigma_{k-1}} \sup_{u \in \varphi(S^{k-1})} \|\nabla u\|_p^p$$

is a sequence of eigenvalue for the p-Laplacian.

From some basic remarks one obtains that $\lambda_1 = \mu_1$ and $\lambda_2 = \mu_2$ (see [7] for the details). Moreover, λ_1 is the smallest eigenvalue of the p-Laplacian and it is simple and isolated, λ_2 is the second eigenvalue. Unfortunately for higher eigenvalues the situation is complicated by the higher topological dimension of the sets involved in the $min - max$ procedure, so that we only know that

$$\lambda_k \leq \mu_k.$$

Does equality hold? (Contributed by Luigi De Pascale).

2. Construction of certain p-harmonic functions in higher dimensions

Let u be a p-harmonic scalar function in $\mathbb{R}^n$. Then it is well known, that ∇u is C^α-continuous for some exponent $\alpha = \alpha(n,p) > 0$. In the case of the plane the optimal choice of $\alpha(2,p)$ is known due to the works for Aronsson [1], Bojarski, Iwaniec, Manfredi [12]. However, the case $n \geq 3$ is completely open and we only know $\alpha(n,p) > 0$.

It is less known that the solution in the plane with this lowest regularity can be found by a very simple construction, which is due to Dobrowolski [6]. Dobrowolski studied the corner regularity of p-harmonic function with zero boundary values in the plane. He observed that if you take the solution on the first quadrant with zero boundary values on the axis and use odd-reflection to the other quadrants, then you obtain a global p-harmonic function with the lowest possible regularity. It is an interesting question, if a similar construction can be used to construct certain p-harmonic functions also in higher dimensions. The hope is that these example have the lowest possible regularity. So far, we found an easier (yet formal) access to the calculations of Dobrowolski or Aronsson. If we use the transformation $q := \nabla_x u$ and $v(q) := q \cdot x - u(x)$, then we get a simplified equation for v, namely $\operatorname{tr}(M^{-1}) = 0$ for

$$M := \nabla_q^2 v\Big(\mathrm{Id} + (\mathrm{p}' - 2)\frac{\mathrm{q}}{|\mathrm{q}|} \otimes \frac{\mathrm{q}}{|\mathrm{q}|}\Big).$$

Hence, the coefficient of t of the characteristic polynomial $\chi_M(t) := \det(t - M)$ is zero. For $n = 2$ this is equivalent to $\operatorname{tr}(M) = 0$ and we get a simple linear equation, which can be solved for example with the ansatz $v(q) = |q|^{\beta(p)-2} q_1 q_2$. Note that the only dependence of v on p is via the exponent $\beta(p)$. Maybe, a similar observation holds for $n \geq 3$. Once, the solution v is found, we can reconstruct u by $u(x) = x \cdot q - v(q)$ and $x = \nabla_q v$.

What kind of regularity do we expect? To be honest, we don't know. However, for $n = 2$ we have the following interesting observation: If we define $V(\nabla u) := |\nabla u|^{\frac{p-2}{2}} \nabla u$, then at least $V(\nabla u) \in C^1$ for all $p \in (1, \infty)$. So maybe, this is true also for $n \geq 3$. Moreover, for $p = \infty$ and $n = 2$ we formally have $v(q) = |q|^2 q_1 q_2$, which recovers for ∇u the regularity $C^{0,1/3}$, since $\alpha = \frac{1}{\beta - 1}$.

(Contributed by L. Diening)

3. Nodal patterns

Numerical simulations by J. Horák [9, 10] suggest that the second eigenfunction for the Dirichlet-p-Laplacian on a disc or a square in $\mathbb{R}^2$ has the following nodal pattern. For the disc the nodal line is a diameteter. For the square and $p \in (1,2]$ it divides the domain into congruent rectangles, while for a square and $p \in [2,\infty)$ the nodal line is diagonal, and only a diagonal if $p > 2$. Prove this. (Contributeted by B. Kawohl)

4. Strong Comparison

To the best of my knowledge the following principle has been proved only in special cases: Suppose that u_1 and u_2 are two solutions of the p-Laplace equation

$$\operatorname{div}(|\nabla u|^{p-2}\nabla u) = 0$$

in the domain Ω in n-dimensional space. If $u_2 \geq u_1$ in Ω, is it then true that either $u_2 \equiv u_1$ or $u_2 > u_1$ in Ω? The plane case $n = 2$ was settled in [12]. Also the linear case $p = 2$ and the one dimensional case are clear. —It is likely that this is simpler than the celebrated unique continuation problem.

(Contributed by Peter Lindqvist)

5. Regularity of the first eigenvalue for the infinity Laplacian under domain perturbations

Consider a continuous vector field $V : \mathbb{R}^N \to \mathbb{R}^N$, the deformation field, and for small $t \in \mathbb{R}$, the perturbed domains $\Omega_t = (Id + tV)(\Omega) = \{x + tV(x),\ x \in \Omega\}$. For the first variation with respect to a vector field (also called shape derivative) of the first eigenvalue of the infinity Laplacian one can show the following.

(Navarro–Rossi–Saintier–San Antolin) *The first eigenvalue $\lambda_1(\Omega_t)$ is Lipschitz continuous with respect to t at $t = 0$. More precisely, there exists a constant $C = C(\Omega, V)$ such that*

$$|\lambda_1(\Omega_t) - \lambda_1(\Omega)| \leq C|t|$$

for every t small enough. However, $\lambda_1(\Omega_t)$ may be not differentiable with respect to t at $t = 0$ when one considers deformations of the domain driven by a vector field. Indeed, there exists a domain Ω (a square) and a vector field V such that

$$\lim_{t\to 0+} \frac{\lambda_1(\Omega_t) - \lambda_1(\Omega)}{t} \neq \lim_{t\to 0-} \frac{\lambda_1(\Omega_t) - \lambda_1(\Omega)}{t}.$$

This lack of differentiability has to be contrasted with the differentiability of the first eigenvalue of the $p-$Laplacian with respect to the domain.

However, there are explicit examples for which $\lambda_1(\Omega_t)$ is differentiable (like triangles) and, moreover, in some cases the derivative can be explicitly computed.

Problems: a) Find a large class of sets and vector fields for which $\lambda_1(\Omega_t)$ is differentiable at $t = 0$ and compute the derivative. **b)** Prove or disprove that the lateral derivatives always exist. (Contributed by J. D.Rossi)

6. Strong Comparison Principle

The stationary (elliptic) problem: Consider the Dirichlet problem(s) for the unknown functions $u, v \in W_0^{1,p}(\Omega)$,

$$\begin{cases} -\Delta_p u = f(x) & \text{in } \Omega; \quad u|_{\partial\Omega} = 0\,, \\ -\Delta_p v = g(x) & \text{in } \Omega; \quad v|_{\partial\Omega} = 0\,, \end{cases} \tag{1}$$

where $f, g \in L^\infty(\Omega)$ are given and $f \leq g$ a.e. in Ω. Here $\Omega \subset \mathbb{R}^N$ is a bounded domain with boundary $\partial\Omega$ of class $C^{2+\alpha}$. It is easy to see that then $u \leq v$ a.e. in Ω, by proving that $\nabla(u-v)^+ = 0$ a.e. in Ω.

Problem: Does $f \not\equiv g$ in Ω imply the (Hopf-type) *strong comparison principle*,

$$u < v \text{ in } \Omega \quad \text{and} \quad \frac{\partial u}{\partial \nu} > \frac{\partial v}{\partial \nu} \text{ on } \partial\Omega\ ? \tag{2}$$

This conjecture is true for $\Omega = (-R, R) \subset \mathbb{R}^1$ – a bounded interval ($N = 1$) and for any $1 < p < \infty$; see [3], by a rather "straightforward" calculation. Of course, the case $p = 2$ is nothing else than the (Hopf-type) strong maximum principle for $v - u$. For $N \geq 2$, several important special cases of (2) have been proved, e.g., in [3, 4, 8] and [11]. A counterexample to a "loosely" related problem is given in [4].

Hardly anything is known for the corresponding *time-dependent (parabolic) problem*: Consider the Dirichlet initial-boundary value problem(s)

$$\begin{cases} \dfrac{\partial u}{\partial t} - \Delta_p u = f(x,t) & \text{in } \Omega \times (0,T); \quad u(\,\cdot\,,t)|_{\partial\Omega} = 0\,; \\ \quad u(x,0) = u_0(x) & \text{in } \Omega\,, \\ \dfrac{\partial v}{\partial t} - \Delta_p v = g(x,t) & \text{in } \Omega \times (0,T); \quad v(\,\cdot\,,t)|_{\partial\Omega} = 0\,; \\ \quad u(x,0) = u_0(x) & \text{in } \Omega\,, \end{cases} \tag{3}$$

for the weak solutions $u, v \colon [0,T] \to W_0^{1,p}(\Omega)$, with given $f, g \in L^\infty(\Omega \times (0,T))$ and $u_0, v_0 \in W_0^{1,p}(\Omega)$ satisfying $f \leq g$ a.e. in $\Omega \times (0,T)$ and $u_0 \leq v_0$ a.e. in Ω.

Problem: Does $f \not\equiv g$ in $\Omega \times (0,T)$ (or, alternatively, $u_0 \not\equiv v_0$ in Ω) imply the (Hopf-type) *strong comparison principle*,

$$u(\,\cdot\,,T) < v(\,\cdot\,,T) \text{ in } \Omega \quad \text{and} \quad \frac{\partial u}{\partial \nu}(\,\cdot\,,T) > \frac{\partial v}{\partial \nu}(\,\cdot\,,T) \text{ on } \partial\Omega\ ? \tag{4}$$

If needed, the condition $f \not\equiv g$ in $\Omega \times (0,T)$ may be replaced by the stronger condition $f(\,\cdot\,,t) \not\equiv g(\,\cdot\,,t)$ in Ω for a.e. $t \in (0,T)$. The latter condition would suffice to derive the stationary case (2) from (4).

Analogous questions concern the more general nonlinearities $f = f(x,t,u)$, $g = g(x,t,u)$ and even $f = f(x,t,u,\nabla u)$, $g = g(x,t,v,\nabla v)$ for which also the problem of the (standard) *weak comparison principle*, i.e., $u \leq v$ in Ω or $\Omega \times (0,T)$, is still open in a number of interesting cases; see [13] for the elliptic problem in Ω.

(Contributed by P. Takáč)

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