Mathematisches Forschungsinstitut Oberwolfach

Report No. 09/2013

DOI: 10.4171/OWR/2013/09

Mini-Workshop: Stochastic Analysis for Poisson Point Processes: Malliavin Calculus, Wiener-Ito Chaos Expansions and Stochastic Geometry

Organised by
Matthias Reitzner, Osnabrück
Giovanni Peccati, Luxembourg

10 February – 16 February 2013

Abstract. Malliavin calculus plays an important role in the stochastic analysis for Poisson point processes. This technique is tightly connected with chaotic expansions, that were introduced in the first half of the last century by Itô and Wiener. These techniques found an increasing number of applications, in particular in the field of stochastic geometry. This in turn inspired new research in stochastic analysis. Leading experts and young researchers of both fields met for a week for fruitful discussions and new cooperations.

Mathematics Subject Classification (2010): 60F05, 60D05, 60H07, 60G57.

Introduction by the Organisers

The meeting 'Stochastic Analysis for Poisson Point Processes: Malliavin Calculus, Wiener-Itô Chaos Expansions and Stochastic Geometry' organised by Giovanni Peccati and Matthias Reitzner, was held in Oberwolfach from February 10 to February 15, 2013. It was attended by 16 participants, roughly half of them working in stochastic analysis on Poisson point processes, the others working in stochastic geometry; 5 were younger participants. The most important part of the program consisted of informal discussions between participants of the two different groups for initiating new research cooperations on the border between stochastic analysis and stochastic geometry.

The official program involved 7 long lectures (composed of two connected one hour talks), and 9 one hour lectures. We now provide a short description of the contributions.

Daniel Hug and Mathew Penrose presented old and new problems from stochastic geometry and random graphs, with emphasis on recent developments. Christoph Thäle reported on modern results for random mosaics, an extremely important area at the heart of stochastic geometry.

Nicolas Privault and Laurent Decreusefond provided a very inspiring introduction to the Malliavin calculus for point processes and moment identities. Several connections were made with optimal transport and the general theory of determinantal point processes. These two topics are actually among the most active of modern stochastic analysis: it is reasonable to expect that the workshop will stimulate new research in these directions. Giovanni Peccati provided an accessible introduction to the Stein's and Chen-Stein method, as well as to their many applications to limit theorems on the Poisson space. Mark Podolskij gave an interesting talk on his recent research on limit theorems for ambit processes. The use of Malliavin calculus for Lévy processe and general completely random measures was discussed by Frederic Utzet in his talk on multiple Stratonovich integrals and the Hu-Meyer formula.

One should note that many of these talks aimed at generalising the existing tools of stochastic analysis on the Poisson space to more general point processes. This should be of high importance for future applications in stochastic geometry. Indeed, geometric applications often require more flexible models than just Poisson point processes.

One of the principal emphasis of the workshop was on limit theorems where the error term and the speed of convergence is described in terms of Malliavin operators. Domenico Marinucci reported on his recent work (jointly with Durastanti and Peccati) on limit theorems for Wavelet Coefficients of Spherical Poisson Fields. Raphaël Lachièze-Rey provides an ideal introduction to limit theorems for U-statistics associated with marked point processes, together with a clear presentation of several powerful new results in the domain (this is connected to joint work with G.Peccati). Solesne Bourguin spoke about mixed limits on the Poisson space. This is very recent work (jointly with Peccati) offering a possibility to prove far-reaching multivariate limit theorems, which automatically show the asymptotic independence of the coordinates of the investigated random vector.

Günter Last gave talks about joint work with Hug, Penrose and Schulte on the covariance structure of the intrinsic volumes of the Boolean model. This has been out of reach for many years and could be described in full generality only very recently using the Malliavin calculus, thus providing a breakthrough in stochastic geometry.

Ilya Molchanov and Sergei Zuyev spoke about their work on variational calculus for Poisson processes. This was published ten years ago and now turns out to fit perfectly into the framework of Malliavin calculus. Further interesting applications of stochastic analysis in stochastic geometry have been described by Matthias Schulte who presented his central limit theorems for the Poisson-Voronoi approximation, and Matthias Reitzner who described recent results for the disk graph. This last talk also displayed an inspiring discussion about an important open question, namely how to prove effective concentration inequalities for non-Lipschitz functionals on the Poisson space.

Mini-Workshop: Stochastic Analysis for Poisson Point Processes: Malliavin Calculus, Wiener-Ito Chaos Expansions and Stochastic Geometry

Table of Contents

Abstracts

From processes of flats to random tessellations and from particle processes to Boolean models

Daniel Hug

The aim of these two talks is to provide an introduction to some basic models of stochastic geometry in the framework of Poisson processes. These models will be relevant for subsequent talks. We begin by recalling the notion of a general Poisson process without requiring the intensity measure to be diffuse. A ubiquitous tool and key result in this context is J. Mecke's characterization of Poisson processes, along with a multivariate extension which holds for Poisson processes. Moreover, we indicate by several examples throughout the talks how these results turn out to be useful.

Then we explore the effect of invariance assumptions on geometric point processes (see [6]). In the case of a stationary process of k-flats, for instance, the intensity measure decomposes partly, which has useful consequences for stationary Poisson processes of flats. Then we describe the intensity measure of the section process of a stationary (but not necessarily isotropic) process of flats with a fixed subspace. Moreover, we analyze the intersection processes of a given order for a stationary Poisson process of hyperplanes. Then we indicate the connection to recent work on asymptotic variances and central limit theorems obtained by the Malliavin-Stein method (see [2, 3, 4]). In a somewhat dual situation, the notion of proximity is recalled, which was originally introduced by R. Schneider [5] and which quantifies how dense the flats in a stationary Poisson random field of k-flats are, if k is smaller than half the dimension of the space (so that almost surely any two flats in a realization will not intersect each other). This notion has been further explored in [7], recent work in progress with Ch. Thäle and W. Weil investigates related inverse problems, that is, the reconstruction of the direction distribution of a process of flats from the direction distribution of an associated segment process.

A hyperplane process induces a hyperplane tessellation, and a point process leads to a Voronoi tessellation in a canonical way. We provide a common framework to study the zero cell of the former and the typical cell of the latter in a Poisson setting, that is, for a (not necessarily stationary) Poisson process of hyperplanes and for a stationary Poisson point process. Then we describe some of the results of recent joint work with Julia Hörrmann [1], on the volume of the zero cell in arbitrary dimensions, with a particular emphasis on high-dimensional asymptotics. The results described here include sharp bounds for the variance of the volume of the respective random cell. In addition, we highlight the connection to the hyperplane conjecture.

In the second talk, we start with a description of the intensity measure of a stationary particle process (see [6]). Then we recall the notion of a density of a geometric functional for a given stationary particle process, and we provide alternative descriptions for theses densities. The union set of a Poisson particle process is a Boolean model. It is well known that a Boolean model can also be obtained from a Poisson point process by independent marking. We describe how first order properties, that is, relations between mean values for functionals of a Boolean model in a given observation window and densities of the underlying Poisson particle process can be derived. Here methods of integral geometry are crucial. In particular, local (measure-valued) versions of classical functionals such as the intrinsic volumes and mixed functionals and measures are required. This topic will be described in more detail in a subsequent talk by Günter Last who will report on joint recent work (by D. Hug, G. Last and M. Schulte) concerning second order properties (covariances of geometric functionals) and central limit theorems. These results are again based on the Malliavin-Stein method, but also on new applications of the aforementioned integral geometric methods.

References

[1] J. Hörrmann, D. Hug, *On the volume of the zero cell of a class of isotropic Poisson hyperplane tessellations*, arXiv:1208.3325

[2] G. Last, M. Penrose, M. Schulte, C. Thäle, *Moments and central limit theorems for some multivariate Poisson functionals*, arXiv:1205.3033

[3] M. Reitzner, M. Schulte, *Central limit theorems for U-statistics of Poisson point processes*, Ann. Probab. (to appear) (2012), arXiv:1104.1039

[4] M. Schulte, C. Thäle, *Exact and asymptotic results for intrinsic volumes of Poisson k-flat processes*, arXiv:1011.5777

[5] R. Schneider, *A duality for Poisson flats*, Adv. Appl. Probab. **31** (1999), 63–68.

[6] R. Schneider, W. Weil, *Stochastic and Integral Geometry.* Springer, Berlin, 2008.

[7] M. Schulte, C. Thäle, *Distances between Poisson k-flats*, arXiv:1206.4807

Random Graphs

Mathew D. Penrose

Two survey talks on random graphs. The first talk discussed the Erdös-Rényi model $G(n, p)$. We described threshold functions for the following: existence of a subgraph isomorphic to a specified finite graph; non-existence of isolated vertices; connectivity. Also Poisson limit theorems for number of copies of a specified finite graph, and for number of isolated vertices. Also discussed giant component/phase transition phenomenon, and two-point concentration phenomenon for the maximum degree and for the clique number.

The second talk described graphs created on a randomly scattered set of vertices in Euclidean d-space, by placing an edge between any two vertices at most unit distance apart. When the underlying set of vertices is an infinite homogeneous Poisson process, this graph is called the Gilbert graph; when it is a finite set of points in a cube, it is called the random geometric graph. We discuss basic notions notions for the Gilbert graph such as cluster size distribution, and phase transition for infinite components. We related these notions to asymptotic properties of the random geometric graph.

Various limit theorems for ambit processes

Mark Podolskij

(joint work with Ole E. Barndorff-Nielsen, Andreas Basse-O'Connor, Jose Manuel Corcuera, Mikko Pakkanen)

In a recent paper Barndorff-Nielsen and Schmiegel [3] introduced the class of *ambit processes* as a time-spatial model for the velocity field in a turbulent flow:

$$X_t(z) = \mu + \int_{A_t(z)} g(\xi, s, z, t)\sigma_s(\xi)L(d\xi, ds) + \int_{D_t(z)} q(\xi, s, z, t)a_s(\xi)d\xi ds,$$

where $A_t(z)$ and $D_t(z)$ are ambit sets, g and q are deterministic weight functions, b represents the intermittency field, a is the drift field and L denotes a Lévy basis. This type of models has a very rich probabilistic structure and a direct physical interpretation. The stochastic field X represents the velocity, while σ^2 is the energy dissipation under suitable normalization. The ambit set $A_t(z)$ (resp. $D_t(z)$) determines the influence area of the intermittency field σ (resp. the drift field a) on the velocity $X_t(z)$. A particular choice of models for g, q, a, b may reproduce the stylized facts of a turbulent flow, such as stationarity, isotropy, skewness, scaling behaviour and aggregational Gaussianity.

In this work we are interested in determining the asymptotic behaviour of certain high frequency functionals for various subclasses of the introduced ambit fields. We will see that the limit theory strongly depends on the weight function g (in the following we ignore the Lebesgue drift of the original process) and on the driving motion L.

We start with a *Brownian semistationary process* without drift that are defined as

$$X_t = X_0 + \int_{-\infty}^{t} g(t-s)\sigma_s dW_s,$$

where σ is a smooth stochastic process and the function g has the representation

$$g(x) = x^\alpha f(x), \qquad \alpha \in (-1/2, 1/2)$$

such that $f : [0, \infty)$ is continuously differentiable function with exponential decay and $f(0) \neq 0$. Such processes have a small scale behaviour similar to that of the fractional Brownian motion B^H with a Hurst parameter $H = \alpha + 1/2 \in (0, 1)$. In the following we will study the asymptotic behaviour of the functional

$$V(X)_t^n = \sum_{i=1}^{[t/\Delta_n]} |\Delta_i^n X|^2, \qquad \Delta_i^n X = X_{i\Delta_n} - X_{(i-1)\Delta_n}$$

as $\Delta_n \to 0$, although more general statistics may be considered (cf. [1, 2]). Setting $G_t := \int_{-\infty}^t g(t-s)dW_s$ we define the scaling constant $\tau_n^2 := \mathbb{E}[(G_{t+\Delta_n} - G_t)^2]$. Our first result has been proved in [1]:

$$\Delta_n \tau_n^{-2} V(X)_t^n \xrightarrow{ucp} \int_0^t \sigma_s^2 ds.$$

The associated stable central limit theorem, which holds for $\alpha \in (-1/2, 0]$ and if σ is Hölder continuous of order $\gamma > 1/2$, is as follows:

$$\Delta_n^{-1/2}\Big(\Delta_n \tau_n^{-2} V(X)_t^n - \int_0^t \sigma_s^2 ds\Big) \xrightarrow{d_{st}} \rho \int_0^t \sigma_s^2 dW_s',$$

where W' is a new Brownian motion (independent of everything) and ρ is a certain constant that depends only on α. This result is proved via a combination of Malliavin calculus and some techniques from stochastic integration. We remark that for general weight function g more complicated limits may appear. In general, we may expect law of large numbers of the type

$$\Delta_n \tau_n^{-2} V(X)_t^n \xrightarrow{ucp} \int_{\mathbb{R}^+} \left(\int_{-\theta}^{t-\theta} \sigma_s^2 ds\right) \pi(d\theta),$$

where π is a probability measure on $\mathbb{R}^+$. This situation appears when g has many singularities.

For processes driven by a pure jump Lévy motion the limiting behaviour of the statistic $V(X)_t^n$ is completely different. Let us consider a simple Lévy moving average process

$$X_t = X_0 + \int_{-\infty}^t g(t-s) dL_s,$$

where L is a β-stable process with $\beta \in (0,2)$ or a compound Poisson process. Again the function g is assumed to be of the form

$$g(x) = x^\alpha f(x), \qquad \alpha \in (0,1),$$

with $f : \mathbb{R}^+ \to \mathbb{R}$ being a smooth function with exponential decay and $f(0) \neq 0$. In this case the process X is continuous, but it is not a semimartingale. We obtain the following result: Assume that L is a compound Poisson process and $\alpha \in (0, 1/2)$, then we obtain for any $t > 0$

$$\Delta_n^{-2\alpha} V(X)_t^n \xrightarrow{d_{st}} f^2(0) \sum_{k:\ T_k \in [0,t]} |\Delta L_{T_k}|^2 \left(\sum_{l=1}^\infty |(l-U_k)_+^\alpha - (l-1-U_k)_+^\alpha|^2\right),$$

where (T_k) are jump times of L, ΔL_T denotes the jump size at time T and $(U_k)_{k\geq 1}$ is a sequence of iid $\mathcal{U}([0,1])$-distributed random variables. We conjecture that similar asymptotic result holds for a rather general pure jump semimartingale driver (given sufficient conditions).

Finally, we present first limit theory for the case of continuous ambit fields observed along a curve. We consider a two dimensional ambit field $(X_t)_{t\in\mathbb{R}^2}$:

$$Y_{t_1,t_2} = \int_{-\infty}^{t_1} \int_{-\infty}^{t_2} g(t_1 - s_1, t_2 - s_2) \sigma_{s_1,s_2} W(ds_1, ds_2),$$

where W denotes a white noise process and $g : (\mathbb{R}^+)^2 \to \mathbb{R}$ is given as

$$g(x) = \|x\|^\alpha f(x), \qquad \alpha \in (-1,0).$$

Again $f : (\mathbb{R}^+)^2 \to \mathbb{R}$ is assumed to be a smooth function with exponential decay and $f(0) \neq 0$. We observe the random field Y along a curve $z : [0,t] \to \mathbb{R}^2$, $z(s) = (z_1(s), z_2(s))$, i.e.

$$X_s = Y_{z(s)}.$$

Furthermore, we assume that z'_k, $k = 1, 2$, has a constant sign. In this framework we obtain the following asymptotic result: Assume that $\alpha \in (-1, -1/2)$, then it holds that

$$\Delta_n^{-1-\alpha} V(X)_t^n \xrightarrow{ucp} \int_0^t \phi_s^2 \sigma_s^2 ds,$$

where ϕ_s describes a certain space deformation; in particular, it depends on z' and $f(0)$. Furthermore, it holds that

$$\Delta_n^{-1/2} \left(\Delta_n^{-1-\alpha} V(X)_t^n - \int_0^t \phi_s^2 \sigma_s^2 ds \right)$$

$$\xrightarrow{d_{st}} \lambda \int_0^t \phi_s^2 \sigma_s^2 dW'_s,$$

where W' is a Brownian motion independent of everything and λ is a known constant.

References

[1] O. E. Barndorff-Nielsen, J. M. Corcuera and M. Podolskij, *Multipower variation for Brownian semi-stationary processes*, Bernoulli **17** (2011), 1159–1194.

[2] O. E. Barndorff-Nielsen, J. M. Corcuera and M. Podolskij, *Limit theorems for functionals of higher order differences of Brownian semi-stationary processes*, to appear in *Prokhorov and Contemporary Probability Theory*, Springer.

[3] O. E. Barndorff-Nielsen and J. Schmiegel, *Ambit processes; with applications to turbulence and cancer growth*, in F.E. Benth, Nunno, G.D., Linstrom, T., Oksendal, B. and Zhang, T. (Eds.): *Stochastic Analysis and Applications: The Abel Symposium 2005.* Heidelberg: Springer, 93–124.

Combinatorics, moments and quasi-invariance for Poisson random integrals

Nicolas Privault

1. Moment identities

Consider a Poisson point process with σ-finite diffuse measure $\sigma(dx)$ on a σ-compact metric space X. The underlying probability space Ω is a space of configurations whose elements $\omega \in \Omega$ are identified with the Radon point measures $\omega = \sum_{x\in\omega} \epsilon_x$, where ϵ_x denotes the Dirac measure at $x \in X$ and the Poisson probability measure with intensity σ on Ω is denoted by π_σ. The isometry formula for the multiple compensated Poisson stochastic integrals $I_k(f_k)$ of symmetric square-integrable functions $f_k : X^k \to \mathbb{R}$ in k variables shows that

$$(1) \qquad \mathbb{E}\left[I_k(f^{\otimes k})F\right] = \mathbb{E}\left[\int_{X^k} f(x_1)\cdots f(x_k) D_{x_1}\cdots D_{x_k} F \sigma(dx_1)\cdots\sigma(dx_k)\right]$$

where $F : \Omega \to \mathbb{R}$ is a finite sum of multiple stochastic integrals and D_x is the finite difference operator defined by

$$D_x F := \varepsilon_x^+ F(\omega) - F(\omega) = F(\omega \cup \{x\}) - F(\omega), \qquad \omega \in \Omega, \quad x \in X.$$

Next, using the relation

$$\mathcal{E}(g) := \exp\left(-\int_0^\infty g(x)dx\right)\prod_{x\in\omega}(1+g(x)) = \sum_{n=0}^\infty \frac{1}{n!} I_n(g^{\otimes n})$$

with $g = e^f - 1$ we find, by the Faà di Bruno formula applied to the exponential function,

$$\begin{aligned}
\sum_{n=0}^\infty \frac{1}{n!}\mathbb{E}\left[F\left(\int_X f d\omega\right)^n\right] &= \mathbb{E}[F e^{\int_X f d\omega}] = e^{\int_X (e^f-1)d\sigma}\mathbb{E}\left[F\mathcal{E}(e^f-1)\right] \\
&= \sum_{k=0}^\infty \frac{1}{k!}\int_{X^k}(e^{f(x_1)}-1)\cdots(e^{f(x_n)}-1)\mathbb{E}\left[\varepsilon_{\mathfrak{x}_k}^+ F\right]\sigma(dx_1)\cdots\sigma(dx_k) \\
&= \sum_{n=0}^\infty \frac{1}{n!}\sum_{P_1,\ldots,P_k\subset\{1,\ldots,n\}}\int_{X^k} f^{|P_1|}(x_1)\cdots f^{|P_k|}(x_k)\mathbb{E}\left[\varepsilon_{\mathfrak{x}_k}^+ F\right]\sigma(dx_1)\cdots\sigma(dx_k),
\end{aligned}$$

with $\varepsilon_{\mathfrak{x}_k}^+ = \varepsilon_{x_1}^+\cdots\varepsilon_{x_k}^+$, for $F : \Omega \to \mathbb{R}$ a bounded random variable, where the sum runs over all partitions $\{P_1,\ldots,P_k\}$ of $\{1,\ldots,n\}$, hence the relation

$$\begin{aligned}
&\mathbb{E}\left[F\left(\int_X f d\omega\right)^n\right] \\
&\qquad = \sum_{P_1,\ldots,P_k\subset\{1,\ldots,n\}}\int_{X^k} f^{|P_1|}(x_1)\cdots f^{|P_k|}(x_k)\mathbb{E}\left[\varepsilon_{\mathfrak{x}_k}^+ F\right]\sigma(dx_1)\cdots\sigma(dx_k),
\end{aligned}$$

which extends as the moment identity

$$(2) \qquad \mathbb{E}\left[\left(\int_X u_x(\omega)\omega(dx)\right)^n\right] = \sum_{P_1,\ldots,P_k}\mathbb{E}\left[\int_{X^k}\varepsilon_{\mathfrak{x}_k}^+(u_{x_1}^{|P_1|}\cdots u_{x_k}^{|P_k|})\sigma(dx_1)\cdots\sigma(dx_k)\right],$$

for $u(x,\omega)$ a sufficiently integrable random process on $X \times \Omega$, cf. Prop. 3.1 of [6]. From the relation

$$\varepsilon_{\mathfrak{x}_k}^+(u_{x_1}\cdots u_{x_k}) = \varepsilon_{x_1,\ldots,x_k}^+(u_{x_1}\cdots u_{x_k}) = \sum_{\Theta\subset\{1,\ldots,k\}} D_\Theta(u_{x_1}\cdots u_{x_k}),$$

where $D_\Theta = D_{x_1} \cdots D_{x_l}$ when $\Theta = \{1, \ldots, l\}$, we deduce that

$$\mathbb{E}\left[\left(\int_X u_x(\omega)\omega(dx)\right)^n\right] = \sum_{P_1,\ldots,P_k} \mathbb{E}\left[\int_{X^k} \varepsilon^+_{\mathfrak{x}_k}(u_{x_1}^{|P_1|} \cdots u_{x_k}^{|P_k|})\sigma(dx_1)\cdots\sigma(dx_k)\right]$$
$$= \sum_{P_1,\ldots,P_k} \sum_{\Theta\subset\{1,\ldots,k\}} \mathbb{E}\left[\int_{X^k} D_\Theta(u_{x_1}^{|P_1|} \cdots u_{x_k}^{|P_k|})\sigma(dx_1)\cdots\sigma(dx_k)\right].$$

Under the cyclic condition $D_{x_1}u_{x_2}\cdots D_{x_k}u_{x_1} = 0$ we get $D_{x_1}\cdots D_{x_k}(u_{x_1}\cdots u_{x_k}) = 0$, $x_1,\ldots,x_k \in X$, $\omega \in \Omega$, cf. [3], [5], and provided in addition that the moment $\int_X u^k(s)\sigma(ds)$ is deterministic, $k \geq 1$, a decreasing induction shows that

$$\mathbb{E}\left[\left(\int_X u_x(\omega)\omega(dx)\right)^n\right] = \sum_{P_1,\ldots,P_k} \int_{X^k} u_{x_1}^{|P_1|} \cdots u_{x_k}^{|P_k|}\sigma(dx_1)\cdots\sigma(dx_k), \qquad n \geq 1,$$

i.e. $\int_X u_x(\omega)\omega(dx)$ has a compound Poisson distribution. See [7] for related consequence for the mixing of random transformations of Poisson measures. Such results have been recently extended to point processes with Papangelou intensities in [1].

2. Quasi-invariance

Formula (1) can be extended to indicator functions $\mathbf{1}_{A(\omega)}$ over random sets $A(\omega)$, as

$$\mathbb{E}\left[FI_n(\mathbf{1}_A^{\otimes n})\right] = \mathbb{E}\left[FC_n(\omega(A),\sigma(A))\right] \tag{3}$$
$$= \mathbb{E}\left[\int_{X^n} D_{x_1}\cdots D_{x_n}\left(F\prod_{p=1}^n \mathbf{1}_A(x_p)\right)\sigma(dx_1)\cdots\sigma(dx_n)\right],$$

via a pathwise extension of the multiple stochastic integral, by application of Stirling inversion to (2) and to the Charlier polynomial $C_n(x,\lambda)$ of order $n \in \mathbb{N}$ with parameter $\lambda > 0$, cf. [4]. As a consequence, if $\tau : \Omega \times X \to Y$ satisfies the cyclic condition $D_{t_1}\tau(\omega,t_2)\cdots D_{t_k}\tau(\omega,t_1) = 0$, $t_1,\ldots,t_k \in X$, $\omega \in \Omega$, for all $k \geq 1$, and $g : Y \to \mathbb{R}$ is sufficiently integrable we get

$$\mathbb{E}\left[e^{-\int_X g(\tau(\omega,x))\sigma(dx)} \prod_{x\in\omega}(1 + g(\tau(\omega,x)))\right] = 1.$$

Denoting by $\tau_* : \Omega \to \Omega$ the mapping defined by shifting configuration points according to τ, this implies the non-adapted Girsanov identity

$$\mathbb{E}\left[F(\tau_*(\omega))e^{-\int_X \phi(\omega,x)\sigma(dx)} \prod_{x\in\omega}(1 + \phi(\omega,x))\right] = \mathbb{E}[F], \qquad F \in L^1(\Omega),$$

provided $\tau(\omega,\cdot) : X \to X$ is invertible on X for all $\omega \in \Omega$, and the density

$$\phi(\omega,x) := \frac{d\tau_*^{-1}(\omega,\cdot)\sigma}{d\sigma}(x) - 1, \qquad x \in X,$$

exists for all $\omega \in \Omega$. If $\tau_* : \Omega \to \Omega$ is invertible then the random transformation $\tau_*^{-1} : \Omega \to \Omega$ is absolutely continuous with respect to π_σ, with density

$$\frac{d\tau_*^{-1}\pi_\sigma}{d\pi_\sigma} = e^{-\int_X \phi(\omega,x)\sigma(dx)} \prod_{x\in\omega}(1+\phi(\omega,x)). \tag{4}$$

3. Examples and stopping sets

Examples can be constructed when $A(\omega)$ is a stopping set, i.e. a random set such that $\{A \subset U\} \in \mathcal{F}_K$ for all $U \subset K$, where $\mathcal{F}_K$ denotes the σ-algebra generated by points inside K, cf. [8] and Def. 2.27 in [2]. In this case (3) shows that $\mathbb{E}[I_n(\mathbf{1}_A^{\otimes n})] = 0$. Examples of transformations $\tau(\omega,x)$ can be defined by leaving $A(\omega)$ invariant and by shifting $x \mapsto \tau(\omega,x)$ depending only on those points of ω that belong to $A(\omega)$. Specific examples include A the smallest ball containing the n points closest to the origin when $X = \mathbb{R}^d$, or $A = [0, T_n]$ when $X = \mathbb{R}_+$ and T_n is the nth Poisson jump time, and A the complement of the open convex hull of the points of ω that belong to the unit ball.

References

[1] L. Decreusefond and I. Flint. Moment formulae for general point processes. Preprint arXiv:1211.4811, 2012.
[2] I. Molchanov. *Theory of random sets*. Probability and its Applications (New York). Springer-Verlag London Ltd., London, 2005.
[3] N. Privault. Moment identities for Poisson-Skorohod integrals and application to measure invariance. *C. R. Math. Acad. Sci. Paris*, 347:1071–1074, 2009.
[4] N. Privault. Girsanov identities for Poisson measures under quasi-nilpotent transformations. *Ann. Probab.*, 40(3):1009–1040, 2012.
[5] N. Privault. Invariance of Poisson measures under random transformations. *Ann. Inst. H. Poincaré Probab. Statist.*, 48(4):947–972, 2012.
[6] N. Privault. Moments of Poisson stochastic integrals with random integrands. 32(2):227–239, 2012.
[7] N. Privault. Mixing of Poisson random measures under interacting transformations. arXiv:1301.0672, 2013.
[8] S. Zuyev. Stopping sets: gamma-type results and hitting properties. *Adv. in Appl. Probab.*, 31(2):355–366, 1999.

Towards a Malliavin calculus for general point processes

Laurent Decreusefond
(joint work with I. Flint)

Malliavin calculus was initially conceived for the standard Brownian motion, to prove that solutions of some stochastic differential equations do have a density with respect to the Lebesgue measure. Later on, it was extended to Poisson point processes [1, 9] and more general Gaussian processes [8, 11]. The generalization of Itô integral it yielded, paved the way to anticipative calculus and integration with respect to processes which are not semi-martingales like the fractional Brownian motion [3]. The differential structure inherited from these considerations appeared to be very useful for Greeks computation in mathematical finance for instance [6]. More recently, there have been a tremendous activity around the application of the Malliavin calculus main formula which is the integration by parts formula to the Stein method. Combining the two approaches, it is possible to estimate finely distances between some probability measures and Gaussian or Poisson measures [2, 7]. In order to extend the applicability of this approach, it is thus important to have integration by parts formula for some other processes. That is the object of the present presentation.

In a recent paper [10], Privault gave some new formulas for the moments of any order of stochastic integrals with respect to a Poisson process. These formulas were the starting point of the following considerations. For a locally finite point process of distribution μ, on a Polish space E, it is said to have a Papangelou intensity if there exists c such that

$$\int_{\Gamma_E} \sum_{x\in\omega} u(x,\omega\backslash x)\ \mathrm{d}\mu(\omega) = \int_{\Gamma_E}\int_E u(x,\omega)\, c(x,\omega)\ \mathrm{d}\lambda(x)\ \mathrm{d}\mu(\omega),$$

where Γ_E is the set of locally finite configurations over E, equipped with the vague topology. We also introduce the compound Papangelou intensity $\hat{c}$: $\hat{c}(x,\,\omega) = c(x,\,\omega)$, $\quad \hat{c}(\nu\cup\eta,\omega) = \hat{c}(\nu,\eta\cup\omega)\,\hat{c}(\eta,\omega)$. Informally, $\hat{c}(\nu,\omega)$ is the probability to observe $\omega\cup\nu$ given the configuration ω is already observed. Let us introduce a few notations: For $\mathcal{P} = P_1,\,\cdots,\,P_k$ a partition of $\{1,\,\cdots,\,n\}$ into k subsets (whose set is denoted by $\mathcal{T}_n^k$),

$$u^{\mathcal{P}}(x,\omega) = \prod_{l=1}^{k}\prod_{i\in P_l} u_i(x_l,\omega).$$

In [5], we proved the following identity, valid for any F and u_k sufficiently integrable:

$$(1)\quad \mathbf{E}[F(\omega)\prod_{k=1}^{n}\int u_k(y,\omega)\,\omega(\,\mathrm{d}y)] = \sum_{k=1}^{n}\sum_{\mathcal{P}\in\mathcal{T}_n^k}\mathbf{E}\Big[\int_{E^k} F(\omega\cup x)u^{\mathcal{P}}(x,\omega\cup x)\hat{c}(x,\omega)\,\lambda^{\otimes(k)}(dx)\Big]$$

where $\lambda(dx)$ is the intensity measure: $\lambda(A) = \mathbf{E}[\omega(A)]$. A consequence of (1) is the following identity: For any bounded function F on Γ_E,

$$(2)\qquad \mathbf{E}[F\delta(u)] = \mathbf{E}[\int_E D_zF(\omega)u(z,\omega)c(z,\omega)\ \mathrm{d}\lambda(z)],$$

where D is the usual difference operator: For $F:\Gamma_E\to\mathbf{R}$,

$$\begin{array}{llll} DF: & E\times\Gamma_E & \longrightarrow & \mathbf{R} \\ & (x,\omega) & \longmapsto & D_xF(\omega) = F(\omega\cup x) - F(\omega\backslash x), \end{array}$$

and δ is a newly defined divergence operator: If $\mathbf{E}[\int |u(y,\omega)|\, c(y,\omega)\, \lambda(dy)] < \infty$,

$$\delta(u) = \int u(y, \omega\backslash y)\ \mathrm{d}\omega(y) - \int u(y,\omega)\, c(y,\omega)\ \mathrm{d}\lambda(y).$$

This extends the well known integration by parts formula for Poisson processes (obtained by taking $c \equiv 1$), which is in fact a simple consequence of the Mecke formula. Thus, (2) is the first step towards a Malliavin calculus for general point processes. In particular, we proved a generalization of the Skorokhod isometry formula:

$$\begin{aligned}\mathbf{E}[\delta(u)^2] = \mathbf{E}[\int u(y,\omega)^2 c(y,\omega)\ \mathrm{d}\lambda(y)]& \\ + \mathbf{E}[\iint D_y u(z,\, \omega) D_z u(y,\, \omega) \hat{c}(\{y,\, z\},\, \omega)\ \mathrm{d}\lambda(y)\ \mathrm{d}\lambda(z)]& \\ - \mathbf{E}[u(z,\, \omega)u(y,\, \omega)\Big(\hat{c}(\{y,\, z\},\, \omega) - c(y,\, \omega)c(z,\, \omega)\Big)\ \mathrm{d}\lambda(y)\ \mathrm{d}\lambda(z)]& \\ := A_1 + A_2 + A_3.&\end{aligned}$$

A_1 is the term we expect for an isometry formula. The term A_2 does exist for Poisson process but for u deterministic or satisfying the cyclicity condition $D_y u(z,\, \omega) D_z u(y,\, \omega) = 0$, it vanishes. As to the term A_3, it corresponds to the term added by the correlation between the particles.

Another kind of gradient can be defined for point processes if E is a space with a notion of differentiation, for instance $E = \mathbf{R}^d$ or a manifold. Then, for any vector field v, we set

$$\begin{aligned}&\nabla_v f(\int_E h_1\ \mathrm{d}\omega, \cdots, \int_E h_k\ \mathrm{d}\omega) \\ &\qquad = \sum_{j=1}^{n} (\partial_j) f(\int_E h_1\ \mathrm{d}\omega, \cdots, \int_E h_k\ \mathrm{d}\omega)\ \langle \nabla^E h_j, v\rangle,\end{aligned}$$

provided that f and the h_k are sufficiently regular. For $E = \mathbf{R}^d$, one can establish an integration by parts formula which reads as

$$\begin{aligned}&\mathbf{E}[\nabla_v F\, G] = -\mathbf{E}[F\, \nabla_v G] \\ &\qquad + \mathbf{E}[F\, G\ \Big(\int \mathrm{div}_\lambda\, v(x) + \langle \nabla^E \ln c(x, \omega\backslash x),\ v(x)\rangle\ \mathrm{d}\omega(x)\Big)],\end{aligned}$$

where div_λ is the adjoint of the gradient on $\mathbf{R}^d$ with respect to the intensity measure λ. At the opposite of the Gaussian case where there is only one gradient, for point processes, there are two rather different gradients with complementary properties. For instance, it is shown in [4] that a functional of configurations is Lipschitz with respect to the total variation distance on the configuration space whenever its discrete gradient is bounded, whereas a bounded differential gradient induces that the functional is Lipschitz with respect to the Wasserstein distance between configurations.

References

[1] N. Bouleau and F. Hirsch. *Dirichlet Forms and Analysis on Wiener Space*. De Gruyter Studies in Mathematics, 1992.
[2] L. Coutin and L. Decreusefond. Stein's method for Brownian approximations. *Submitted*, http://hal.archives-ouvertes.fr/hal-00717812, July 2012.
[3] L. Decreusefond. Stochastic calculus with respect to Volterra processes. *Annales de l'Institut Henri Poincaré (B) Probability and Statistics*, 41:123–149, 2005.
[4] L. Decreusefond, A. Joulin, and N. Savy. Upper bounds on Rubinstein distances on configuration spaces and applications. *Communications on stochastic analysis*, 4(3):377–399, 2010.
[5] Laurent Decreusefond and Ian Flint. Moment formulae for general point processes. *Submitted* http://hal.archives-ouvertes.fr/hal-00753801, November 2012.
[6] E. Gobet and A. Kohatsu-Higa. Computation of Greeks for barrier and look-back options using Malliavin calculus. *Electron. Comm. Probab.*, 8:51–62 (electronic), 2003.
[7] I. Nourdin and G. Peccati. *Normal Approximations with Malliavin Calculus: From Stein's Method to Universality*. Cambridge University Press, 2012.
[8] D. Nualart. *The Malliavin Calculus and Related Topics*. Springer–Verlag, 1995.
[9] N. Privault. *Stochastic analysis in discrete and continuous settings with normal martingales*, volume 1982 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin, 2009.
[10] Nicolas Privault. Girsanov identities for poisson measures under quasi-nilpotent transformations. *Annals of Probability*, 40(3):1009–1040, 05 2012.
[11] A. S. Üstünel. *An Introduction to Analysis on Wiener Space*, volume 1610 of *Lectures Notes in Mathematics*. Springer-Verlag, 1995.

Normal Approximations for Wavelet Coefficients on Spherical Poisson Fields

Domenico Marinucci
(joint work with Claudio Durastanti, Giovanni Peccati)

A classical problem in asymptotic statistics is the assessment of the speed of convergence to Gaussianity (that is, the computation of explicit Berry-Esseen bounds) for parametric and nonparametric estimation procedures. In this area, an important novel development is given by the derivation of effective Berry-Esseen bounds by means of the combination of two probabilistic techniques, namely the *Malliavin calculus of variations* and the *Stein's method* for probabilistic approximations. The fact that one can use Malliavin calculus to deduce normal approximation bounds (in total variation) for functionals of Gaussian fields was first exploited in [5] – where one can find several quantitative versions of the fourth moment theorem for chaotic random variables proved in [7]. Lower bounds can also be computed, entailing that the rates of convergence provided by these techniques are sharp in many instances – see again [6].

In a recent series of contributions, the interaction between Stein's method and Malliavin calculus has been further exploited for dealing with the normal approximation of functionals of a general Poisson random measure. These findings have recently found a wide range of applications in the field of stochastic geometry – see [1, 2, 3] for a sample of geometric applications.

The purpose of this paper is to apply and extend the main findings of [8, 9] in order to study the multidimensional normal approximation of the elements of the first Wiener chaos of a given Poisson measure. Our main goal is to deduce bounds that are well-adapted to deal with applications where the dimension of a given statistic increases with the number of observations. This is a framework which arises naturally in many relevant fields of modern statistical analysis; in particular, our principal motivation originates from the implementation of wavelet systems on the sphere. In these circumstances, when more and more data become available, a higher number of wavelet coefficients is evaluated, as it is customarily the case when considering, for instance, thresholding nonparametric estimators. We shall hence be concerned with sequences of Poisson fields, whose intensity grows monotonically. We then exploit the wavelets localization properties to establish bounds that grow linearly with the number of functionals considered; we are then able to provide explicit recipes, for instance, for the number of joint testing procedures that can be simultaneously entertained ensuring that the Gaussian approximation may still be shown to hold, in a suitable sense.

References

[1] Lachièze-Rey, R. and Peccati, G. (2011). *Fine Gaussian fluctuations on the Poisson space I: contractions, cumulants and geometric random graphs*, Electron. J. Probab. 18 (2013), no. 32, 1-32. .
[2] Lachièze-Rey, R. and Peccati, G. (2012). Fine Gaussian fluctuations on the Poisson space II: rescaled kernels, marked processes and geometric U-statistics. Preprint.
[3] Last, G., Penrose, M., Schulte, M., Thäle, C. (2012). Moments and central limit theorems for some multivariate Poisson functionals. Preprint.
[4] Marinucci, D. and Peccati, G. (2011), *Random Fields on the Sphere. Representation, Limit Theorems and Cosmological Applications*. Lecture Notes of the London Mathematical Society, 389. Cambridge University Press.
[5] Nourdin, I. and Peccati, G. (2009), Stein's method on Wiener chaos, *Probability Theory Related Fields* 145, no. 1-2, 75–118.
[6] Nourdin, I. and Peccati, G. (2012), *Normal approximations using Malliavin calculus: from Stein's method to universality*. Cambridge University Press, Cambridge.
[7] Nualart, D. and Peccati, G. (2005), Central limit theorems for sequences of multiple stochastic integrals, *Annals of Probability* 33, no. 1, 177–193.
[8] Peccati, G., Solé, J.-L. , Taqqu, M.S. and Utzet, F. (2010), Stein's method and normal approximation of Poisson functionals. *The Annals of Probability* 38 (2), 443-478.
[9] Peccati, G. and Zheng, C. (2010), Multi-dimensional Gaussian fluctuations on the Poisson space. *Electronic Journal of Probabability* 15 (48), 1487-1527.

Stein's method on the Poisson space

Giovanni Peccati

I delivered two talks, concerning the connections between two powerful probabilistic techniques, namely the Stein's and Chen-Stein methods for probabilistic approximations, and the Malliavin calculus of variations on the Poisson space. The connection between these two topics was first developed on the Gaussian space by I. Nourdin and myself (see the monograph [4]). The staple of the talks was the content of the papers [6, 7], where the Malliavin calculus on the Poisson space (in a form due to Nualart and Vives [5]) was first combined with the Stein's and Chen-Stein methods, with specific emphasis on limit theorems for chaotic random variables. I also discussed some of the recent developments obtained in [1, 2, 3], where several applications in stochastic geometry have been developed. One important point developed in the talks is that the use of Malliavin calculus allows one to deal quite easily with the asymptotic independence of random variables, often achieving conclusions that are not obtainable by other techniques.

References

[1] Bourguin, S. and Peccati, G.: *Portmanteau inequalities on the Poisson space: mixed limits and multidimensional clustering*, arXiv: 1209.3098 [math.PR] (2012).
[2] Lachièze-Rey, R. and Peccati, G.: *Fine Gaussian fluctuations on the Poisson space I*, Electron. J. Probab. 18 (2013), no. 32, 1-32.
[3] Lachièze-Rey, R. and Peccati, G.: *Fine Gaussian fluctuations on the Poisson space II*, arXiv: 1205.0632 [math.PR] (2012).
[4] Nourdin, I. and Peccati, G.: *Normal Approximations with Malliavin Calculus*, Cambridge University Press, Cambridge (2012).
[5] Nualart, D. and Vives, J.: *Anticipative calculus for the Poisson process based on the Fock space*, Lecture Notes in Mathematics **1426**, 154–165 (1990).
[6] Peccati, G.: *The Chen-Stein method for Poisson functionals*, arXiv: 1112.5051 [math.PR] (2011).
[7] Peccati, G.; Solé, J.L., Taqqu, M.S. and Utzet, F.: *Stein's method and normal approximation of Poisson functionals*, Ann. Probab. **38**, 443–478 (2010).

Variational analysis of Poisson processes

Ilya Molchanov, Sergei Zuyev

In the two lectures we summarise the basic ideas about optimisation methods for Poisson point processes.

Given a rather general phase space X, we consider the family of finite Poisson process distributions. Given a functional F of configurations Π, we treat the expectation $\mathbf{E}_\mu F(\Pi)$ as a function $f(\mu)$ of a finite measure μ which is the intensity measure of a corresponding Poisson point processes. Thus the domain of f is the set $\mathbb{M}_+$ of (non-negative) finite measures which is a cone in the Banach space $\mathbb{M}$ of all signed measures with a finite total variation norm. By explicitly developing the expectation $\mathbf{E}_{\mu+\eta} F(\Pi)$ for $\eta \in \mathbb{M}$ such that $\mu + \eta \in \mathbb{M}_+$ one can show that under rather mild assumptions on F the function f is analytic. The corresponding derivatives have explicit form, in particular,

$$\mathbf{E}_{\mu+\eta} F - \mathbf{E}_\mu F = \int_X \bar{\Delta}_\mu(x)\, \eta(dx) + o(\|\eta\|)\,,$$

where δ_x is the Dirac measure concentrated on $\{x\}$. The function $\bar{\Delta}_\mu(x) = \mathbf{E}_\mu[F(\Pi+\delta_x)-F(\Pi)]$ is called the expected first difference and has the meaning of the gradient, [7]. Varying intensity measure allows us to establish Margulis-Russo type formula for Poisson process which has proved extremely useful, e.g., in percolation theory, since it links the geometry and the probability in a nice form, [11]. Another application of the variation approach is the Gamma-type result according to which the volume of a stopping set (a generalisation of a stopping time) given the number of Poisson process points in it, has Gamma distribution under some scalability condition, see [12].

Problem of finding an optimal distribution for a certain criterion could be formulated as an abstract constrained optimisation problem on the Banach space $\mathbb{M}$: $f(\mu) \to \inf$ over $\mu \in \mathbb{M}_+ \cap \mathring{A}$, where $\mathring{A} \subseteq \mathbb{M}_+$. The first order optimality condition involves characterising the tangent cone to the constrained set. This was done for many practically interesting cases in [6]. In particular, if μ^* is minimising $\mathbf{E}_\mu F$ over μ with a fixed total mass, then there exists u such that $\bar{\Delta}_{\mu^*}(x) \geq u$ for all $x \in X$ and $\bar{\Delta}_{\mu^*}(x) = u$ μ^*-almost everywhere, see [7].

The second lecture concerns applications of variational methods, most importantly, the necessary condition for extremum. Consider approximation of convex twice differentiable function g on $[a, b]$ by means of a linear spline determined by the values of g at points of a Poisson process Π_μ on $[a, b]$ with intensity measure μ. The objective function $f(\mu)$ is the mean approximation error in L^1-metric. The expected first difference can be explicitly written as

$$\bar{\Delta}_\mu(x) = -g(x)\mathbf{E}_\mu(r_x^- + r_x^+) + \mathbf{E}_\mu r_x^- \mathbf{E}_\mu g(x + r_x^+) + \mathbf{E}_\mu r_x^+ \mathbf{E}_\mu g(x - r_x^-)\,,$$

where $r_x^\pm$ are distances from x to its left and right neighbours from Π_μ. The optimality condition over intensity measures μ with the fixed total mass a requires that the expected first difference is a constant on the support of μ, and eventually leds to a differential equation that is to be solved numerically.

However, considerably more interesting questions appear in the high intensity setting, as the total mass $a \to \infty$. The local nature of the expected first difference makes it possible to transform the problem locally to its stationary counterpart in a neighbourhood of x. Heuristically, this amounts to looking at the asymptotics of $r_x^\pm$ and f near x, so that $\bar{\Delta}_\mu(x) \propto a^{-3}p(x)^{-3}f''(x)$. Then we obtain that the density of μ in the high intensity case should be proportional to $(f'')^{1/3}$.

This heuristic idea can be formalised for Poisson processes in $\mathbb{R}^d$ using the concept of stopping set, noticing that the first order difference depends on the configuration through the corresponding stopping set. These tools can be easily used to fid optimal Poisson approximation of convex sets by inscribed polytopes and of multivariate convex functions by tangent planes and by triangulated Bezier surfaces.

At the next step relationships to the corresponding deterministic problems are explained using the example of optimal quantisation. The asymptotic behaviour of errors is also determined locally using tessellations of the space (which in the optimal quantisation example is the Voronoi tessellation), see [1, 2]. The optimal quantisation setting is closely related to the clustering problem in statistics. In this problem, replacing deterministic set of points with a Poisson process turns a non-convex optimisation problem into a convex one that is possible to solve either asymptotically in the high intensity setting or numerically by the steepest descent approach for any given total intensity. Its variant for optimal placement of telecommunication stations has been considered in [7]. Note that the Poissonisation of deterministic optimal quantisation problem has been studied in [3].

Finally, relations to optimisation issues in experimental design [5] and design of random materials with required structural properties [4] are mentioned.

References

[1] S. Graf and H. Luschgy. *Foundations of Quantization for Probability Distributions*, volume 1730 Lect. Notes Math. **1730** (2000), Springer, Berlin.

[2] I. Molchanov and N. Tontchev. *Optimal approximation and quantisation*, J. Math. Anal. Appl. **325** (2007), 1410–1429.

[3] I. Molchanov and N. Tontchev. *Optimal Poisson quantisation*, Statist. Probab. Lett. **77** (2007), 1123–1132.

[4] I. S. Molchanov, S. N. Chiu, and S. A. Zuyev. *Design of inhomogeneous materials with given structural properties*, Phys. Rev. E **62** (2000), 4544–4552.

[5] I. S. Molchanov and S. Zuyev. *Optimisation in space of measures and optimal design*, ESAIM Probab. Stat., **8** (2004), 12–24.

[6] I. S. Molchanov and S. A. Zuyev. *Tangent sets in the space of measures: with applications to variational calculus*, J. Math. Anal. Appl. **249** (2000), 539–552.

[7] I. S. Molchanov and S. A. Zuyev. *Variational analysis of functionals of a Poisson process*, Math. Oper. Res. **25** (2000), 485–508.

[8] I. S. Molchanov and S. A. Zuyev. *Steepest descent algorithms in space of measures*, Statist. Comput. **12** (2002), 115–123.

[9] J. Møller and S. Zuyev. *Gamma-type results and other related properties of Poisson processes*, Adv. Appl. Probab. **28** (1996), 662–673.

[10] M. N. M. van Lieshout, I. S. Molchanov, and S. A. Zuyev. *Clustering methods based on variational analysis in the space of measures*, Biometrika **88** (2001), 1021–1033.

[11] S. Zuyev. Russo's Formula for the Poisson Point Processes and its Applications, *Discrete Math. and Applications*, **3** (1993), 63–73.

[12] S. Zuyev. *Stopping sets: Gamma-type results and hitting properties*, Adv. Appl. Probab. **31** (1999), 355–366.

Geometric functionals of the Boolean model: mean values, covariance structure and beyond

Günter Last

(joint work with Daniel Hug, Matthias Schulte)

Let η be a *stationary Poisson process* on the space $\mathcal{K}^d$ of *convex bodies* in $\mathbf{R}^d$, that is, on the space of compact, convex subsets of $\mathbf{R}^d$. All random objects occurring in this talk are defined on an abstract probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Stationarity implies that the *intensity measure* Λ of η has to be translation invariant. We also assume that Λ is *locally finite*, so that Theorem 4.1.1 in [9] implies that

$$\Lambda(\cdot) = \gamma \iint \mathbf{1}\{K + x \in \cdot\} dx \mathbb{Q}(dK).$$

where the *intensity* γ is positive and finite and where $\mathbb{Q}$ is a probability measure on $\mathcal{K}^d$. We assume $\gamma > 0$. The *Boolean model* associated with η is defined by

$$Z := \bigcup_{K \in \eta} K,$$

where we have identified η with its support. This is a fundamental model of stochastic geometry and continuum percolation with many applications in materials science and physics. Let Z_0 denote a *typical grain*, that is, a random convex set with distribution $\mathbb{Q}$. Since Λ is locally finite, the mean values

$$v_i := \mathbb{E}\, V_i(Z_0), \quad i \in \{0, \dots, d\},$$

are finite, where $V_i(K)$, $i = 0, \dots, d$, are the *intrinsic volumes* of a convex body K, or more generally of a set K in the *convex ring* $\mathcal{R}^d$; see [9] for more detail.

In this talk we present a systematic mathematical treatment of second order properties of the random vector $(V_0(Z \cap W), \dots, V_d(Z \cap W))$ of the intrinsic volumes of Z, observed within the compact, convex window W. More details and full proofs will be given in [4]. The volume functional was first studied in [1, 6], while in [2] Berry-Esseen bounds and large deviation inequalities were established. Volume and surface were treated (in a more general setting) in [3]. A mathematical nonrigorous approach to second moments of curvature measure with a very interesting application to morphological thermodynamics was presented in [7].

Throughout we assume that the typical grain Z_0 satisfies

$$\mathbb{E}\, R(Z_0)^{3d} < \infty,$$

where $R(K)$ is the radius of the circumball of a convex set K. Our first result is the existence of the asymptotic covariances

$$\sigma_{i,j} := \lim_{r(W) \to \infty} \frac{\mathbb{C}\mathrm{ov}(V_i(Z \cap W), V_j(Z \cap W))}{V_d(W)}, \quad i, j \in \{0, \dots, d\},$$

where $r(W)$ denotes the *inradius* of a convex set W. It turns out that this convergence happens at rate $r(W)^{-1}$. Moreover, the limits are given by

$$\sigma_{i,j} = \gamma \sum_{n=1}^{\infty} \frac{1}{n!} \iint V_i^*(K_1 \cap K_2 \cap \ldots \cap K_n) V_j^*(K_1 \cap K_2 \cap \ldots \cap K_n) \Lambda^{n-1}(d(K_2, \dots, K_n)) \mathbb{Q}(dK_1), \tag{1}$$

where

$$V_i^*(K) := \mathbb{E}\, V_i(Z \cap K) - V_i(K), \quad K \in \mathcal{K}^d.$$

In proving these results we make crucial use of the explicit Fock space representation of Poisson functionals derived in [5].

Closely related with the asymptotic covariances are the numbers

$$\rho_{i,j} := \gamma \sum_{n=1}^{\infty} \frac{1}{n!} \iint V_i(K_1 \cap \cdots \cap K_n) V_j(K_1 \cap \cdots \cap K_n) \Lambda^{n-1}(d(K_2, \dots, K_n)) \mathbb{Q}(dK_1).$$

We show that

$$\rho_{i,j} = \int e^{\gamma C_d(x)} H_{i,j}(dx), \quad i,j \in \{0,\dots,d\},$$

where

$$C_d(x) := \mathbb{E}\, V_d(Z_0 \cap (Z_0 + x)), \quad x \in \mathbf{R}^d,$$

is the mean *covariogram* of the typical grain, and where the $H_{i,j}$ are finite measures on $\mathbf{R}^d$. These measures are derived from curvature measures associated with the typical grain. The exact definition is not given here.

Define for any $j \in \{0,\dots,d-1\}$ and $l \in \{j,\dots,d\}$ a polynomial $P_{j,l}$ on $\mathbf{R}^{d-j}$ of degree $l-j$ by

$$P_{j,l}(t_j,\dots,t_{d-1}) := \mathbf{1}\{l=j\} + c_j^l \sum_{s=1}^{l-j} \frac{(-1)^s}{s!} \sum_{\substack{m_1,\dots,m_s=j \\ m_1+\dots+m_s=sd+j-l}}^{d-1} \prod_{i=1}^{s} c_d^{m_i} t_{m_i},$$

where, as in [9, (5.4)],

$$c_j^m := \frac{m!\kappa_m}{j!\kappa_j}, \quad m,j \in \{0,\dots,d\}.$$

If Z_0 is *isotropic* (i.e. distributionally invariant under rotations), then a classical result of stochastic geometry says that, for $i \in \{1,\dots,d\}$ and $K \in \mathcal{K}^d$,

$$V_i^*(K) = -(1-p) \sum_{k=i}^{d} V_k(K) P_{i,k}(\gamma v_i,\dots,\gamma v_{d-1}). \tag{2}$$

We refer to [9] for a proof and a discussion of the history of this result. Combining (2) with (1), yields integral representations of the asymptotic covariances. In the two-dimensional (isotropic) case these formulas are rather explicit.

We also show that many of our results can be derived for general additive and translation invariant geometric functionals. Moreover, using the Stein-Malliavin approach from [8] we prove a multivariate central limit theorem with Berry-Esseen bounds.

References

[1] A. Baddeley, *A limit theorem for statistics of spatial data*, Adv. in Appl. Probab. **12** (1980), 447–461.

[2] L. Heinrich, *Large deviations of the empirical volume fraction for stationary Poisson grain models*, Ann. Appl. Probab. **15** (2005), 392–420.

[3] L. Heinrich, I.S. Molchanov, *Central limit theorems for a class of random measures associated with germ-grain models*, Adv. in Appl. Probab. **31** (1999), 283–314.

[4] D. Hug, G. Last, M. Schulte, *Second order properties and central limit theorems for intrinsic volumes of Boolean models*, in preparation, (2013).

[5] G. Last, M.D. Penrose, *Fock space representation, chaos expansion and covariance inequalities for general Poisson processes*, Probab. Theory and Related Fields **150** (2011), 663–690.

[6] S. Mase, *Asymptotic properties of stereological estimators for stationary random sets*, J. Appl. Probab. **19** (1982). 111–126.

[7] K.R. Mecke, *Exact moments of curvature measures in the Boolean model*, Journal of Statistical Physics **102** (2001), 1343–1381.

[8] G. Peccati, C. Zheng, *Multi-dimensional Gaussian fluctuations on the Poisson space.* Electron. J. Probab. **15** (2010), 1487–1527.
[9] R. Schneider, W. Weil, *Stochastic and Integral Geometry.* Springer, Berlin, (2008).

Recent Trends in Stochastic Geometry

Christoph Thäle
(joint work with Tomasz Schreiber and Matthias Schulte)

Modern stochastic geometry is concerned with the development of new models for spatial random structures and with their exact and asymptotic analysis. In the first talk, a relatively new model for a random tessellation of the d-dimensional Euclidean space is presented and its first- and second-order properties are investigated. Limit theorems with a limiting Weibull point process on the real half-line are in the focus of the second talk.

1. Iteration Stable Tessellations

Imagine that we are given a polytope $W \subset \mathbb{R}^d$ and some $t > 0$. We then assign to W a random lifetime, which is exponentially distributed with mean $\Lambda([W])^{-1}$, where Λ is the isometry invariant measure on the space of hyperplanes in $\mathbb{R}^d$ and $[W]$ stands for the collection of all hyperplanes that have non-empty intersection with W. When the lifetime of W has run out, we choose a hyperplane H according to the conditional distribution $\Lambda(\,\cdot\,|[W])$, which cuts W into the two sub-polytopes $W^{\pm} = W \cap H^{\pm}$, where $H^{\pm}$ are the two half-spaces determined by H. The described construction continues now recursively in $W^{\pm}$ and we stop it when time t is reached. The outcome of this procedure is a random tessellation $Y(t,W)$ of W, see [2].

It is a crucial observation that the random process $(Y(t,W))_{t>0}$ is a pure-jump Markov process in the space of tessellations of W. Its generator is given by

$$\mathbb{A}F(Y(s,W)) = \int_{[W]} \sum_{f \text{ a cell of } \mathrm{Y(s,W)\cap H}} [F(Y(s,W) \cup f) - F(Y(s,W))]\,\Lambda(dH),$$

where F is a bounded measurable function on the space of tessellations of W and where $Y(s,W) \cup f$ is the tessellation $Y(s,W)$ with the new facet f added, see [5]. Consequently, $F(Y(t,W)) - \int_0^t \mathbb{A}F(Y(s,W))\,ds$ is a martingale wrt. the natural filtration induced by $(Y(t,W))_{t>0}$. This can be used to deduce first-order properties of $Y(t,W)$. Taking for example for F the surface area $S = V_{d-1}$, we see that

$$\begin{aligned}\mathbb{E}S(Y(t,W)) &= \int_0^t \int_{[W]} \sum_{f \text{ a cell of } \mathrm{Y(t,W)\cap H}} S(f)\,\Lambda(dH)\,ds \\ &= \int_0^t \int_{[W]} S(W \cap H)\,\Lambda(dH)\,ds = t\,V(W),\end{aligned}$$

where $V(W) = V_d(W)$ is the volume of W. Using a similar martingale approach, one can also investigate second-order properties of $Y(t,W)$. For example, the following has been derived in [4]:

$$\mathbb{V}S(Y(t,W)) = \frac{d-1}{2}\int_W\int_W \frac{1-e^{-\frac{2\kappa_{d-1}}{d\kappa_d}t\|x-y\|}}{\|x-y\|^2}\,dx\,dy.$$

Turning to the asymptotic regime, we consider a fixed time $t>0$ and the rescaled polytopes $W_R = RW$. Let us write $\sim$ for the asymptotic equivalence of functions. Then (see [4])

$$\mathbb{V}S(Y(t,W_R)) \sim \pi V(W) R^2 \log R$$

if $d=2$, and for $d \geq 3$ we have that

$$\mathbb{V}S(Y(t,W_R)) \sim \frac{d-1}{2}R^{2(d-1)}\int_W\int_W \|x-y\|^{-2}\,dx\,dy.$$

Besides, first- and second-order structure of $Y(t,W)$ (resp. $Y(t,W_R)$), also the central limit problem (and also its functional counterpart) has been considered in [6], showing that Gaussian limits only appear in space dimension $d=2$, whereas in all higher space dimensions a non-Gaussian limiting distribution shows up.

2. Poisson Limit Theorems

Fix some standard Borel space $(\mathbb{Y},\mathcal{Y})$ with a non-atomic σ-finite measure λ. By η_t we denote a Poisson point process on $\mathbb{Y}$ with intensity measure $\lambda_t = t\lambda$ and $\eta^k_{t,\neq}$, $k \geq 1$, stands for the set of all k-tuples of distinct points of η_t. Let further $f:\mathbb{Y}^k \to \mathbb{R}$ be a non-negative measurable function that is invariant under permutations of its arguments and satisfies $\lambda^k(f^{-1}([0,x])) < \infty$ for all $x>0$. The Poisson point process η_t and the function f induce a collection of points $\xi_t = \{f(y_1,\dots,y_k) : (y_1,\dots,y_k) \in \eta^k_{t,\neq}\}$ on the positive real half-axis $\mathbb{R}_+$. In fact, by our assumptions on f, ξ_t is a locally finite point process on $\mathbb{R}_+$. Because of the symmetry of f, every $f(y_1,\dots,y_k)$ also occurs for permutations of the argument $(y_1,\dots,y_k)$. However, we count the point $f(y_1,\dots,y_k)$ for every subset $\{y_1,\dots,y_k\} \subset \eta_t$ only once. Nevertheless, the point process ξ_t might still have multiple points if there are several subsets having the same value under f.

For $\gamma>0$, $t\geq 1$ and $x>0$ let us introduce

$$\alpha_t(x) = \frac{1}{k!}\,\mathbb{E}\sum_{(y_1,\dots,y_k)\in\eta^k_{t,\neq}} \mathbf{1}(f(y_1,\dots,y_k) \leq xt^{-\gamma})$$

and

$$r_t(x) = \sup_{\substack{y_1,\dots,y_{k-j}\in\mathbb{Y}\\ 1\leq j\leq k-1}} \lambda_t^j(\{(\hat{y}_1,\dots,\hat{y}_j)\in\mathbb{Y}^j : f(\hat{y}_1,\dots,\hat{y}_j,y_1,\dots,y_{k-j}) \leq xt^{-\gamma}\}).$$

In the following, we investigate the rescaled point process $t^\gamma\xi_t := \{t^\gamma f(y_1,\dots,y_k) : (y_1,\dots,y_k)\in\eta^k_{t,\neq}\}$.

Assume that there are constants $\beta, \tau > 0$ such that $\lim_{t\to\infty} \alpha_t(x) = \beta x^\tau$ and that $\lim_{t\to\infty} r_t(x) = 0$ for any $x > 0$. Then $t^\gamma \xi_t$ converges as $t \to \infty$ in distribution to ξ, where ξ is a Poisson (Weibull) point process on $\mathbb{R}_+$ with intensity measure

$$\nu(B) = \beta\tau \int_B u^{\tau-1}\, du, \qquad B \subset \mathbb{R}_+ \text{ Borel.}$$

A proof of this (and some local refinements) is given in [7]. It is based on recent findings that combine the classical Chen-Stein method for Poisson approximation with the Malliavin calculus of variations on the Poisson space [3].

The general result can be used to study a number of problems arising in stochastic geometry. For example, let $K \subset \mathbb{R}^d$ be a convex body and η_t be the restriction of a stationary Poisson point process of intensity $t \geq 1$ to K. Consider the family ξ_t of all volumes of d-dimensional simplices that can be formed by $d+1$ points of η_t, i.e. $\xi_t = \{V([y_1, \dots, y_{d+1}]) : (y_1, \dots, y_{d+1}) \in \eta^{d+1}_{t,\neq}\}$, where $[y_1, \dots, y_{d+1}]$ stands for the simplex with vertices $y_1, \dots, y_{d+1}$. Using the above result and methods from integral geometry (in particular, the Blaschke-Petkantschin formula) one can show that the point processes $t^{d+1}\xi_t$ converge in distribution as $t \to \infty$ to a homogeneous Poisson point process on $\mathbb{R}_+$ with intensity β, where

$$\beta = \frac{d\kappa_d}{d+1} \int_{[K]} V_{d-1}(K \cap H)^{d+1}\, dH.$$

In the particular case $d = 2$ we have the simple expression $\beta = 2V(K)^2$. This generalizes and extends a problem previously studied in [1].

Similar methods can also be used to study the proximity of Poisson k-flat processes in $\mathbb{R}^2$ with $d - 2k > 0$, intrinsic volumes of intersection processes of Poisson k-flats in $\mathbb{R}^d$ if $d - 2k \leq 0$, edge lengths in a Poisson random polytope on the sphere, edge lengths in a random geometric graph or a Delauney graph as well as small cells in a Poisson line tessellation, see [7].

References

[1] G. Grimmett, S. Janson, *On smallest triangles*, Rand. Struct. Alg. **23**, 206–223, (2003).

[2] W. Nagel, V. Weiß, *Crack STIT Tessellations: Characterization of stationary random tessellations stable with respect to iteration*, Adv. Appl. Probab. **37**, 859-883 (2005).

[3] G. Peccati, *The Chen-Stein method for Poisson functionals*, arXiv: 1112.5051 [math.PR] (2011).

[4] T. Schreiber, C. Thäle, *Second-order theory for iteration of stable tessellations*, Probab. Math. Stat. **32**, 281-300 (2012).

[5] T. Schreiber, C. Thäle, *Geometry of iteration stable tessellations: Connection with Poisson hyperplanes*, accepted for publication in Bernoulli (2012+).

[6] T. Schreiber, C. Thäle, *Limit theorems for iteration stable tessellations*, accepted for publication in The Annals of Probability (2012+).

[7] M. Schulte, C. Thäle, *The scaling limit of Poisson-driven order statistics with applications in geometric probability*, Stoch. Proc. Appl. **122**, 4096-4120 (2012).

Scaling regimes for geometric Poisson functionals

Raphaël Lachièze-Rey
(joint work with Giovanni Peccati)

In a recent series of papers, the combination of Stein's method and Malliavin calculus in the Poisson framework allowed for the asymptotic study of many different random geometric functionals. We show in this talk how the behaviour of a certain kind of variables, called Poisson U-statistics, and more generally of finite Wiener-Itô series, can be well described in this framework. In particular the scale of interaction between the particles of the process within the functional, that can be informally defined as the radius of influence of a given particle, is of great importance with respect to the limit behaviour. The corresponding results are contained in [1, 2].

More precisely, let $\lambda > 0$, η_λ be a Poisson measure on some Polish space E, and $F_\lambda = F(\eta_\lambda)$ a functional under the form

$$F_\lambda := F(\eta_\lambda) = \mathbb{E}F_\lambda + \sum_{q=1}^{q} I_q(h_{\lambda,q})$$

where I_q is the q-th multiple Wiener-Ito integral and $h_{\lambda,q}$ is a kernel $E^q \mapsto \mathbb{R}$ satisfying adapted intergability conditions. The aim of our work is to study the asymptotic behaviour of

$$\tilde{F}_\lambda = \frac{F_\lambda - \mathbb{E}F_\lambda}{\mathrm{var}F_\lambda}.$$

In most geometric applications, E is assumed to be under the form $E = \mathbb{R}^d \times M$ where M is a locally compact space, called the *marks space*, endowed with a probability distribution, and η_λ is a homogeneous Poisson measure restricted to $[-\lambda^{1/d}, \lambda^{1/d}] \times M$. Each point $x = (t, m) \in E$ contains a spatial variable $t \in \mathbb{R}^d$ and a mark $m \in M$.

Our point of view was to introduce a geometric assumption of the form of the kernels $h_{\lambda,q}$, namely that each can be put under the form

$$h_{\lambda,q}(x_1, \ldots, x_q) = \gamma_{\lambda,q} h(\alpha_\lambda(x_1, \ldots, x_q)),\ x_1, \ldots, x_q \in E,$$

where α_λ denotes the scalar multiplication of the spatial variables, h_q is a function on E^q assumed to be stationary, i.e. invariant under spatial translations, and $\gamma_{\lambda,q} \in \mathbb{R}$. The stationary assumption is motivated by many problems arising from stochastic geometry, and the factor α_λ denotes a change of scale in the problem, i.e. a zoom-in or zoom-out situation. For $1 \leq q \leq k$, h_q is furthermore assumed to decrease sufficiently fast from the diagonal, which is formalized by an explicit integrability condition on E^q.

Under these assumptions one can give explicit variance asymptotics and bounds on the distance between F_λ and the limit law, with the help of the *contractions*, an analytic tool arising from the multiplication formula. An interesting particular case is when F_λ is a Poisson U-statistic, as introduced in [4] , i.e.

$$F_\lambda = \sum_{(x_1,\dots,x_k)\in\eta_\lambda^{\neq}} h(x_1,\dots,x_k),$$

is the sum over all the k-tuples of distinct points of η_λ for a given kernel h.

Depending on $\{\alpha_\lambda; \lambda > 0\}$, several different limit regimes appear in the limit, such as a Gaussian limit, a Poisson limit, or a χ^2-limit. The case $\alpha_\lambda \sim \lambda^{1/d}$ coincides with geometric U-statistics, i.e. U-statistics which kernel does not depend on λ, and in this case we were able to complete the work of [4] and completely characterize the limit law, living in a Gaussian chaos. The case $\alpha_\lambda \sim 1$ meets and generalises some results about random graphs from [3], telecommunication networks, or the boolean model. For smaller α_λ, the speed of convergece to the normal law decreases until the limit law undergoes a Gauss/Poisson transition, illustrated in [1] in an example involving geometric random graphs.

References

[1] Lachièze-Rey, R. and Peccati, G. (2011). *Fine Gaussian fluctuations on the Poisson space I: contractions, cumulants and geometric random graphs*, Electron. J. Probab. 18 (2013), no. 32, 1-32.

[2] Lachièze-Rey, R. and Peccati, G. (2012). *Fine Gaussian fluctuations on the Poisson space II: rescaled kernels, marked processes and geometric U-statistics.* Preprint.

[3] M. Penrose, *Random Geometric Graphs*, Oxford Studies in Probability, Oxford University Press 2003.

[4] M. Reitzner and M. Schulte, *Central Limit Theorems for U-Statistics of Poisson Point Processes*, Ann. Probab., to appear

Mixed limits and asymptotic independence on the Poisson space

Solesne Bourguin

(joint work with Giovanni Peccati)

In a recent series of papers, it has been shown that one can combine Malliavin calculus with Stein's method on a Gaussian space, in order to obtain limit theorems and explicit bounds in the normal approximation of non–linear functionals of the underlying Gaussian field. The seminal paper [2] was the first to bring together Stein's method and Malliavin calculus. The reference [4] extends this paper to the multidimensional case, and [3] uses the techniques developed in the two previous references to study universality properties on Gaussian Wiener chaos. These references also contain examples of applications of these approximation results to the derivation of Berry–Esséen bounds in the Breuer–Major theorem as well as new second order Poincaré inequalities. Later on, the authors of [6] and [7] combined a discrete version of the Malliavin calculus on the Poisson space with Stein's method in order to obtain limit theorems and explicit bounds in the normal approximation of a non–linear functional F of an underlying Poisson random measure. A general bound of the form, where $Z \sim \mathcal{N}(0,1)$,

$$\begin{aligned} d_{\mathcal{H}}(F,Z) &\leq \frac{1}{2}\sum_{i,j=1}^{d} \mathbf{E}\left|C(i,j) - \left\langle DF_i, -DL^{-1}F_j\right\rangle_{L^2(\mu)}\right| \\ &\quad + \frac{1}{4}\mathbf{E}\int_Z \left(\sum_{i=1}^{d} |D_zF_i|\right)^2 \left(\sum_{i=1}^{d} \left|D_zL^{-1}F_i\right|\right)\mu(dz). \end{aligned}$$

was obtained, encompassing the original one–dimensional result of [6] and where the operator D and L^{-1} are differential operator of the Malliavin calculus. These results allowed, among other applications, for the derivation of central limit theorems for non–linear functionals of Lévy moving averages (such as the Ornstein–Uhlenbeck Lévy process). Recently, a new approximation framework was proposed in [5] that combined the Malliavin calculus on the Poisson space with the Chen–Stein method in order to obtain limit theorems and explicit bounds in the Poisson approximation of non–linear functionals of the underlying Poisson random measure. This new Poisson approximation result comes in the form of a bound on the distance in total variation between the laws of a general functional F of a Poisson random measure and a Poisson random variable Z with parameter λ, and can be stated as follows.

$$\begin{aligned} d_{TV}(F,Z) &\leq \frac{1-e^{-\lambda}}{\lambda}\mathbf{E}\left|\lambda - \left\langle DF, -DL^{-1}F\right\rangle_{L^2(\mu)}\right| \\ &\quad + \frac{1-e^{-\lambda}}{\lambda^2}\mathbf{E}\int_Z \left|D_zF(D_zF-1)D_zL^{-1}F\right|\mu(dz). \end{aligned}$$

In this talk, we present recent results based on the reference [1] originating from a new general inequality on the Poisson space obtained by combining the Chen–Stein method with Malliavin calculus on the Poisson space. Additionally, new results are obtained such as multidimensional Poisson approximations, stable and mixed limit theorems, as well as a characterisation of asymptotic independence for U–statistics. Applications to limit theorems involving the joint convergence of vectors of subgraph–counting statistics exhibiting both a Poisson and a Gaussian behaviour are presented.

References

[1] S. Bourguin and G. Peccati, *Portmanteau inequalities on the Poisson space: mixed regimes and multidimensional clustering*, arXiv:1209.3098v2, (2012).

[2] I. Nourdin and G. Peccati, *Stein's method on Wiener chaos*, Probab. Theory Related Fields, **145**, (2009), 75–118.

[3] I. Nourdin, G. Peccati and G. Reinert, *Invariance principles for homogeneous sums: universality of Gaussian Wiener chaos*, Ann. Probab., **38**, (2010), 1947–1985.

[4] I. Nourdin, G. Peccati and A. Réveillac, *Multivariate normal approximation using Stein's method and Malliavin calculus*, Ann. Inst. Henri Poincaré Probab. Stat., **46**, (2010), 45–58.

[5] G. Peccati, *The Chen–Stein method for Poisson functionals*, arXiv:1112.5051v2, (2012).

[6] G. Peccati, J.L. Solé, M.S. Taqqu and F. Utzet, *Stein's method and normal approximation of Poisson functionals*, Ann. Probab., **38**, (2010), 443–478.

[7] G. Peccati and C. Zheng, *Multi–dimensional Gaussian fluctuations on the Poisson space*, Electron. J. Probab., **15**, (2010), 1487–1527.

Malliavin calculus for completely random measures

Frederic Utzet

The purpose of this talk is to extend Malliavin calculus for Poisson processes based on the *difference operator* or *add one cost operator*,

$$D_z F(\omega) = F(\omega + \delta_z) - F(\omega),$$

to the functionals of a *completely random measure* without fixed atoms, and it is an adaptation of the ideas of Solé *et al.* [10] to the random measures context.

Let Z be a complete and separable metric space and $\mathcal{Z}$ be its Borel σ–field. Following the nomenclature of Daley and Vere-Jones [1], a measure μ on $(Z, \mathcal{Z})$ is said to be *finitely bounded* if $\mu(A) < \infty$ for every bounded set $A \in \mathcal{Z}$; note that such a measure is σ–finite thanks to the separability properties of Z. A random measure on $(Z, \mathcal{Z})$ is a measurable mapping from a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ into the set of finitely bounded measures on $(Z, \mathcal{Z})$ endowed with the σ–field induced by the evaluation maps $\mu \to \mu(A)$, for every finitely bounded measure μ and $A \in \mathcal{Z}$.

We will assume that the intensity measure (or first moment measure) of a random measure ξ on $(Z, \mathcal{Z})$:

$$\lambda(A) := \mathbb{E}\,\xi(A),\ \ A \in \mathcal{Z}$$

is finitely bounded (this is a general assumption in the theory of random measures, see Daley and Vere-Jones [1, Page 65] or in its applications to Stochastic Geometry, see Schneider and Weyl [9, Page 57]); in particular, as we commented, we assume that λ is σ–finite. A point $s \in S$ is called a *fixed atom* of the random measure ξ if $\mathbb{P}\big\{\xi\{s\} > 0\big\} > 0$. If ξ has no fixed atoms, then for every $s \in S$, $\lambda\{s\} = \mathbb{E}\,\xi\{s\} = 0$, so its intensity measure is non-atomic.

A random measure ξ with intensity mesure λ is said to be completely random if for any family of disjoint sets $A_1, \dots, A_k \in Z$, with $\lambda(A_i) < \infty,\ i = 1, \dots, k$, the random variables $\xi(A_1), \dots, \xi(A_k)$ are independent.

Let ξ be a a completely random measure without fixed atoms with intensity measure λ. For $A \in \mathcal{Z}$ with $\lambda(A) < \infty$, it is proved that the random variable $\xi(A)$ has an infinitely divisible law concentrated on $[0, \infty)$, and this gives rise to the representation of ξ in terms of a Poisson process on $Z \times (0, \infty)$ called Kingman representation (see Kigman [4, 5], Daley and Vere–Jones [1], Kallenberg [3]); we have that

$$\xi(A) = \beta(A) + \iint_{Z \times (0,\infty)} \mathbf{1}_A(t) x \, dN(t, x),$$

where β is a (deterministic) finitely bounded measure, and N is a Poisson process on $Z \times (0, \infty)$ with intensity given by a certain measure ν, which is σ–finite and non-atomic; however, ν is not necessarily finitely bounded, and in many interesting cases it may be infinity on sets of the form $A \times (0, \varepsilon)$. We will assume that the deterministic measure β is 0. Hence, from now on, we can take Ω to be the canonical space of the Poisson process N,

$$\Omega = \{\omega = \sum_{i=1}^{n} \delta_{(t_i,x_i)},\ t_i \in Z,\ x_i > 0, n \in \mathbb{N} \cup \{\infty\}\},$$

and, for $\omega = \sum_{i=1}^{n} \delta_{(t_i,x_i)}$, and $A \in \mathcal{Z}$

$$\xi(\omega)(A) = \sum_{i=1}^{n} x_i \delta_A(t_i).$$

Following Itô [2], the completely random measure ξ defined on Z is extended to a completely random measure M on $Z \times (0,\infty)$, now in the sense of vector measures (see, Peccati and Taqqu [8, Chapter 5]) using the compensate Poisson process

$$\widehat{N} = N - \nu.$$

To this end, define a measure μ on $Z \times (0,\infty)$ by

$$d\mu(t,x) = x^2\, d\nu(t,x).$$

For $C \in \mathcal{Z} \times \mathcal{B}(0,\infty)$ such that $\mu(C) < \infty$, we have $\mathbf{1}_C(t,x)x \in L^2(Z \times (0,\infty), \nu)$, hence the following random variable is well defined as a limit in $L^2(\Omega)$:

$$M(C) := \iint_{Z\times(0,\infty)} \mathbf{1}_C(t,x) x \widehat{N}(dt,dx).$$

It turns out that $M(C)$ is centered, and if C_1 and C_2 satisfy $\mu(C_1) < \infty$ and $\mu(C_2) < \infty$, then

$$\mathbb{E}\big[M(C_1)M(C_2)\big] = \mu(C_1 \cap C_2).$$

Moreover, M is a completely random measure on $Z \times (0,\infty)$ with control measure μ.

Then we can construct the usual *Itô multiple integral* of order n of functions of $L_n^2 := L^2\Big(\big(Z \times (0,\infty)\big)^n, \mu^{\otimes n}\Big)$, satisfying all the usual properties (see Peccati and Taqqu [8]).

Moreover, following again Itô steps [2], every square integrable random variable, measurable with respect to the σ–field generated by ξ admits a chaos expansion: If $F \in L^2(\Omega)$, then

$$F = \sum_{n=0}^{\infty} I_n(f_n),\ f_n \in L_n^2.$$

From this point, it is possible to apply the machinery of the annilation operators (Malliavin derivative) and creation operators (Skorohod integral) on a Fock space as was exposed by Nualart and Vives [6].

So, on the one hand, for

$$F = \sum_{n=0}^{\infty} I_n(f_n),\ f_n \in L_n^2.\ f_n \text{ symmetric}$$

such that

$$\sum_{n=1}^{\infty} n n! \|f_n\|^2_{L^2_n} < \infty,$$

the Malliavin derivative of F is defined as

$$D_{(t,x)}F = \sum_{n=1}^{\infty} n I_{n-1}\Big(f_n\big((t,x),\cdot\big)\Big),$$

convergence in $L^2(Z \times (0,\infty) \times \Omega, \mu \otimes \mathbb{P})$.

On the other hand, given a random variable F and $(t,x) \in Z \times (0,\infty)$, we can define a *quotient operator*

$$D'_{(t,x)}F(\omega) = \frac{1}{x}\big(F(\omega + \delta_{(t,x)}) - F(\omega)\big).$$

Under some (quite general) conditions, D' and D coincide.

Furthermore, the operators

- Skorohod integral δ,
- Ornstein–Uhlenbeck generator L,
- Inverse of L

are constructed in the standard way using chaos expansions.

Unfortunately, the bounds for the Wasserstein distance to the normal law in terms of the Malliavin operators of the Poisson case given in Peccati *et al.* [7] do not transfer directly to that context of random measures.

Acknowledgement. The author has been partially supported by grant MTM2009-08869 Ministerio de Ciencia e Innovación and FEDER.

References

[1] D.J. Daley, and D. Vere–Jones,, *An Introduction to the Theory of Point Processes, Volume II: General Theory and Structure*, 2nd. edition, Springer, 2008.

[2] K. Itô, *Spectral type of the shift transformation of differential processes with stationary increments*, Trans. Amer. Math. Soc., **81** (1956) 253–263.

[3] O. Kallenberg, *Foundations of Modern Probabilty*, 2nd. edition, Springer (2010)

[4] J.F.C. Kingman, *Completely random measures*, Pacific Journal of Mathematics, **21** (1967) 59–78.

[5] J.F.C. Kingman, *Poisson Processes*, Oxford Science Publications, Clarendon Press, Oxford, 1993.

[6] D. Nualart and J. Vives, *Anticipative calculus for the Poisson process based on the Fock space*, In: Seminaire de Probabilites XXIV, Lecture Notes in Matehmatics, Vol 1426, Springer, 1990, 154–165.

[7] G. Peccati, J.Ll. Solé, M.S. Taqqu and F. Utzet, *Stein's method and normal approximation of Poisson functionals*, Ann. Probab. **38** (2010), 443–478.

[8] G. Peccati and M.S. Taqqu, *Wiener Chaos: Moments, Cumulants and Diagrams*, Bocconi University Press, Springer, Milano, 2011.

[9] R. Schneider, and W. Weil, Stochastic and Integral Geometry, Springer, 2008.

[10] J.Ll. Solé, F. Utzet and J. Vives,*Canonical Lévy process and Malliavin calculus*, Stochastic Processes and their Applications **117** (2007), 165–187.

The Poisson-Voronoi approximation

Matthias Schulte

The Poisson-Voronoi approximation is a random approximation that can be used to reconstruct an unknown set. Let $K \subset \mathbb{R}^d$ be a compact convex set and let η_t be a stationary Poisson point process of intensity $t > 0$ in $\mathbb{R}^d$. For a point $x \in \eta_t$ the Voronoi cell of x is given by

$$C(x, \eta_t) = \{z \in \mathbb{R}^d : \|z - x\| \leq \|z - y\| \text{ for all } y \in \eta_t\},$$

and x is called nucleus of $C(x, \eta_t)$. The Voronoi cells $(C(x, \eta_t))_{x \in \eta_t}$ form a tessellation of $\mathbb{R}^d$, where every cell contains all points of $\mathbb{R}^d$ such that the nucleus is the closest point of η_t. The Poisson-Voronoi approximation of K is defined as

$$\mathrm{A}_t(K) = \bigcup_{x \in \eta_t \cap K} C(x, \eta_t).$$

The underlying idea is that we want to reconstruct an unknown set K, but the only information available is a kind of oracle which tells us for every point of η_t whether it belongs to K or not. Now we construct the Voronoi tessellation with respect to η_t and approximate K by the union of all Voronoi cells with nucleus in K.

Since the Poisson-Voronoi approximation can be used to estimate the volume of an unknown set, one is interested in its volume. A short computation shows that

$$\mathbb{E}\,\mathrm{Vol}(\mathrm{A}_t(K)) = \mathrm{Vol}(K)$$

so that the volume of the Poisson-Voronoi approximation of K is an unbiased estimator for the volume of K. For its variance one has the lower and upper bounds (see [3, 5])

$$\underline{C}\,\kappa_1 V_{d-1}(K)\, t^{-1-\frac{1}{d}} \leq \mathrm{Var}\,\mathrm{Vol}(\mathrm{A}_t(K)) \leq \overline{C} \sum_{i=0}^{d-1} \kappa_{d-i} V_i(K)\, t^{-2+\frac{i}{d}}$$

for $t \geq (2/r(K))^d$ with constants $\underline{C}, \overline{C} > 0$ only depending on the dimension d. Here, $V_i(K)$, $i = 0, \ldots, d$, stand for the intrinsic volumes of K, $r(K)$ denotes the inradius of K, and κ_i is the volume of the unit ball in $\mathbb{R}^i$. It is also possible to derive the asymptotic result

$$\lim_{t \to \infty} \frac{\mathrm{Var}\,\mathrm{Vol}(\mathrm{A}_t(K))}{t^{-1-\frac{1}{d}}} = c_d \kappa_1 V_{d-1}(K)$$

with a constant $c_d > 0$ only depending on the dimension d. This means that the asymptotic variance of the volume of the Poisson-Voronoi approximation of K for increasing intensity t depends only on the surface area of K.

Knowing the asymptotic behaviour of the variance, one can consider the limiting distribution. It can be shown (see [5]) that

$$\frac{\mathrm{Vol}(\mathrm{A}_t(K)) - \mathrm{Vol}(K)}{\sqrt{\mathrm{Var}\,\mathrm{Vol}(\mathrm{A}_t(K))}} \to N \ \text{ in distribution as } \ t \to \infty,$$

where N stands for a standard Gaussian random variable.

The proofs of the variance bounds and the central limit theorem in [5] make use of the Wiener-Itô chaos expansion and the Malliavin-Stein method. As a random variable depending on a Poisson point process the volume of the Poisson-Voronoi approximation has a so called Wiener-Itô chaos expansion

$$\mathrm{Vol}(\mathrm{A}_t(K)) = \mathrm{Vol}(K) + \sum_{n=1}^{\infty} I_n(f_n),$$

where $I_n(\cdot)$ stands for the n-th multiple Wiener-Itô integral and the square integrable symmetric functions $f_n \in L_s^2((\mathbb{R}^d)^n)$, $n \in \mathbb{N}$, are given by formulas reflecting the effect of adding points to the Poisson point process (see [2]). Moreover, its variance has the representation

$$\mathrm{Var}\,\mathrm{Vol}(\mathrm{A}_t(K)) = \sum_{n=1}^{\infty} n!\,\|f_n\|_n^2.$$

Together with some estimates for the functions f_n, $n \in \mathbb{N}$, this yields lower and upper bounds for the variance. The central limit theorem is based on a bound for the normal approximation of Poisson functionals in terms of Malliavin operators (see [1]), which are defined via their Wiener-Itô chaos expansions. A technical computation shows that this bound is less than a sum of deterministic integrals depending on the functions f_n, $n \in \mathbb{N}$. For the volume of the Poisson-Voronoi approximation these integrals vanish for increasing intensity, which implies the central limit theorem.

The error of the Poisson-Voronoi approximation can be measured by the volume of the symmetric difference of $\mathrm{A}_t(K)$ and K. For the volume of the symmetric difference one can also show variance bounds and a central limit theorem. The upper bound and the central limit theorem can be followed directly from the proofs for the volume of the Poisson-Voronoi approximation.

The assumption that K is convex is not necessary for the construction of the Poisson-Voronoi approximation but simplifies some arguments in the proofs. It can be replaced by some more general conditions. For another possible generalization of the class of approximated sets see [4], where sets of finite perimeter are considered and bounds for moments are derived.

References

[1] G. Peccati, J. L. Solé, M. S. Taqqu and F. Utzet, *Stein's method and normal approximation of Poisson functionals*, Ann. Probab. **38** (2010), 443–478.

[2] G. Last and M. D. Penrose, *Poisson process Fock space representation, chaos expansion and covariance inequalities*, Probab. Theory Related Fields **150** (2011), 663–690.

[3] M. Heveling and M. Reitzner, *Poisson-Voronoi approximation*, Ann. Appl. Probab. **19** (2009), 719–736.

[4] M. Reitzner, E. Spodarev and D. Zaporozhets, *Set reconstruction by Voronoi cells*, Adv. in Appl. Probab. **44** (2012), 938–953.

[5] M. Schulte, *A central limit theorem for the Poisson-Voronoi approximation*, Adv. in Appl. Math. **49** (2012), 285–306.

On some Large Deviation inequality

Matthias Reitzner

Let ω_t be a Poisson point process in $\mathbb{R}^d$ of constant intensity $t > 0$.To define the disk graph $G(\omega_t, \delta_t)$ we take all points of ω_t to be the vertices of $G(\omega_t, \delta_t)$ and connect two points $x, y \in \omega_t$ by an edge if

$$\|x - y\| \leq \delta_t.$$

The resulting graph $G(\omega_t, \delta_t)$ is a random geometric graph, called disk graph, or sometimes Gilbert graph, interval graph (for $d = 1$) or distance graph. The disk graph is the maybe most natural construction of a random geometric graph, see e.g. the book by Penrose [5].

We are interested in the local behaviour of the disk graph within a convex body W, when $t \to \infty$ and $\delta_t \to 0$ in such a way that $t^2 \delta_t^d$ converges to a constant. Define the number of edges of $G(\xi, \delta)$ in the window W given by

$$N_t = N(\omega_t, \delta_t) = \frac{1}{2} \sum_{(x,y) \in (\omega_t \cap W)^2_{\neq}} \mathbf{1}(\|x - y\| \leq \delta_t).$$

Classical results are the expectations of these quantities which are proved using the Slivnyak-Mecke theorem

$$\mathbb{E} N_t \approx \frac{t^2}{2} \big(V(W) \, \kappa_d \, \delta_t^d$$

and

$$\mathbb{V} N_t \approx V(W) \, t^2 \delta_t^d (\kappa_d^2 \, t \delta_t^d + \frac{\kappa_d}{2})$$

for $\delta_t \to 0$. Limit theorems showing that the normalized random variable converges to a Poisson (or normal) distributed random variable are e.g. due to Penrose [5], Lachiéze-Rey and Peccati [3] [4], Bourguin and Peccati [2], Schulte [7], and others.

A deeper understanding of the behaviour of the disk graph can be obtained by ordering all distances between two points in the convex body W. This yields the point set

$$\xi_t = \{\|x_1 - x_2\| : \ (x_1, x_2) \in (\omega_t \cap W)^2_{\neq}\}$$

on the positive real line. It was proved by Schulte and Thäle [8] that $t^{2/d} \xi_t$ converges in distribution to a Poisson point process, and that the shortest distance between two points is Weibull distributed.

The main part of the talk is a proof of a large deviation inequality in this case. Assume $K \in \mathbb{N}$, $\alpha > 0$ and $\delta_t = \alpha t^{-\frac{2}{d}}$ (sparse regime). There exists a $c = c(K, \alpha, W)$ such that

$$\mathbb{P}(N_t \geq K \ln t) \quad \leq \quad c t^{-K+1}.$$

The proof of this inequality uses Talagrand's large deviation inequality for the convex distance function and an Poissonization argument. It would be interesting to obtain a direct proof in the Poisson setting using the Malliavin calculus.

As an application we present a connection to a question concerning empty triangles. Given a finite point set X in the plane, the degree of a pair $\{x, y\} \subset X$ is the number of *empty triangles* $t = \mathrm{conv}\{x, y, z\}$, where empty means $t \cap X = \{x, y, z\}$. Define $\deg X$ as the maximal degree of a pair in X.

Here we take X to be the intersection of a Poisson point process with the convex body W. Observe that for any pair $(x, y) \in (\omega_t \cap W)^2_{\neq}$ the degree is clearly bounded by the number of points $\omega_t \cap W$ which has expectation $tV_d(W)$. It turns out that the degree of X is close to this trivial upper bound. It is proved in Bárány, Marckert and Reitzner [1] that for $X = \omega_t \cap W$ there is a constant $c > 0$ such that

$$\mathbb{E}(\deg X) \geq \frac{c}{\ln t}\, t.$$

The proof uses essentially a large deviation inequality for the length of the disk graph.

References

[1] I. Bárány, J.-F. Marckert and M. Reitzner, *Many empty triangles have a common edge*, Manuscript.

[2] S. Bourguin and G. Peccati, *Portmanteau inequalities on the Poisson space: mixed limits and multidimensional clustering*, Manuscript

[3] R. Lachiéze-Rey and G. Peccati, *Fine Gaussian fluctuations on the Poisson space, I: contractions, cumulants and geometric random graphs*, Electron. J. Probab. 18 (2013), no. 32, 1-32.

[4] R. Lachiéze-Rey and G. Peccati, *Fine Gaussian fluctuations on the Poisson space, II: rescaled kernels, marked processes and geometric U-statistics*. Manuscript

[5] M. Penrose, *Random Geometric Graphs*, Oxford Studies in Probability, Oxford University Press 2003.

[6] M. Reitzner and M. Schulte, *Central Limit Theorems for U-Statistics of Poisson Point Processes*, Ann. Probab., to appear

[7] M. Schulte, *Malliavin-Stein Method in Stochastic Geometry*, PhD Thesis (2013)

[8] M. Schulte and C. Thäle, *The scaling limit of Poisson-driven order statistics with applications in geometric probability*, Stochastic Processes and their Applications **122** (2012), 4096-4120

Reporter: Raphael Lachieze-Rey

Participants

Dr. Solesne Bourguin
University of Luxembourg
Mathematics Research Unit, FSTC
Campus Kirchberg
6, rue Richard Coudenhove-Kalergi
1359 Luxembourg
LUXEMBOURG

Prof. Dr. Laurent Decreusefond
Telecom ParisTech
23, Avenue d'Italie
75013 Paris
FRANCE

Prof. Dr. Daniel Hug
Institut für Stochastik
Karlsruher Institut f. Technologie (KIT)
Kaiserstr. 89-93
76133 Karlsruhe
GERMANY

Dr. Raphael Lachieze-Rey
C.N.R.S.
Laboratoire MAP 5
Université Paris Descartes
45, rue des Saints-Peres
75270 Paris Cedex 06
FRANCE

Prof. Dr. Günter Last
Institut für Stochastik
Karlsruher Institut f. Technologie (KIT)
Kaiserstr. 89-93
76133 Karlsruhe
GERMANY

Prof. Dr. Domenico Marinucci
Dipartimento di Matematica
Universita degli Studi di Roma II
Tor Vergata
Via della Ricerca Scientifica
00133 Roma
ITALY

Prof. Dr. Ilya S. Molchanov
Institut f. Mathematische Statistik
Universität Bern
Sidlerstr. 5
3012 Bern
SWITZERLAND

Prof. Dr. Giovanni Peccati
Université du Luxembourg
Unite de Recherche en Mathématiques
6, Rue Richard Coudenhove-Kalergi
1359 Luxembourg
LUXEMBOURG

Prof. Dr. Mathew Penrose
Dept. of Mathematical Sciences
University of Bath
Claverton Down
Bath BA2 7AY
UNITED KINGDOM

Prof. Dr. Mark Podolskij
Institut für Angewandte Mathematik
Universität Heidelberg
Im Neuenheimer Feld 294
69120 Heidelberg
GERMANY

Prof. Dr. Nicolas Privault
Division of Mathematical Sciences
Nanyang Technological University
21 Nanyang Link
Singapore 637371
SINGAPORE

Prof. Dr. Matthias Reitzner
FB Mathematik/Informatik
Universität Osnabrück
Albrechtstr. 28a
49076 Osnabrück
GERMANY

Dr. Matthias Schulte
Fachbereich Mathematik/Informatik
Universität Osnabrück
Albrechtstr. 28A
49076 Osnabrück
GERMANY

Dr. Christoph Thaele
Fachbereich Mathematik/Informatik
Universität Osnabrück
Albrechtstr. 28A
49076 Osnabrück
GERMANY

Prof. Dr. Frederic Utzet
Departament de Matematiques
Universitat Autonoma de Barcelona
Campus Universitari
08193 Bellaterra
SPAIN

Prof. Dr. Sergei Zuyev
Department of Mathematical Sciences
Chalmers University of Technology
41296 Göteborg
SWEDEN