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Representations of Lie Groups and Supergroups

Organised by
Joachim Hilgert, Paderborn
Toshiyuki Kobayashi, Tokyo
Karl-Hermann Neeb, Erlangen
Tudor Ratiu, Lausanne

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Abstract. The workshop focussed on recent developments in the representation theory of group objects in several categories, mostly finite and infinite dimensional smooth manifolds and supermanifolds. The talks covered a broad range of topics, with a certain emphasis on benchmark problems and examples such as branching, limit behavior, and dual pairs. In many talks the relation to physics played an important role.

Mathematics Subject Classification (2000): 22E, 32M, 46G20 53D, 58B, 58C.

Introduction by the Organisers

The workshop *Representations of Lie groups and supergroups* was organized by Joachim Hilgert (Paderborn), Toshiyuki Kobayashi (Tokyo), Karl-Hermann Neeb (Erlangen), and Tudor Ratiu (Lausanne).

From the very beginning applications in physics were a major motivation for the study of representations of Lie groups. Later also number theory, specifically the Langlands program, became a driving force. A lot of effort has been invested in the classification of unitary representations during the last two decades of the 20th century. There is a huge body of information available, but the central classification problems are still not completely solved. At the moment research in that direction is concentrated with some American research teams.

The majority of research nowadays is focused on benchmark problems and examples, such as branching, limit behavior, and dual pairs. Moreover, the extension of the scope of representation theory to infinite dimensional groups on the one hand, and supergroups on the other, plays an important role. A common feature of these efforts is the identification of relevant classes of examples which are general enough to be interesting, but at the same time have enough restrictions to allow a general theory. In many cases the choice of these benchmark examples is guided by problems from theoretical physics. The focus of this workshop was on these recent developments.

The meeting was attended by 51 participants from many European countries, Canada, the USA, and Japan. The meeting was organized around a series of 23 lectures each of 50 minutes duration. The set of speakers chosen was a mix of researchers in all stages of their careers, from very promising young post-docs to senior scientists who have been contributing key results to the field over the last 45 years.

We feel that the meeting was exciting and highly successful. The quality of the lectures was outstanding and the intensity of discussions was exceptional even for Oberwolfach standards. A good indicator for this observation is the fact that all the available blackboards were occupied by discussion groups until late every evening of the meeting. What is even more remarkable is that the composition of these discussion groups changed every day. In particular, the researchers who have been focussing on either finite dimensional, infinite dimensional, or super contexts in their recent research did not stay among themselves. New collaborations have been started, and established research partners from different continents had the opportunity to discuss further projects in person.

Without going too much into detail, let us mention some important new developments.

In the area of *infinite-dimensional Lie groups* things are moving on two mutually interacting levels, one is the analytic theory of unitary representations and the other deals with geometric structures (symplectic, Poisson etc.) on manifolds with group actions. Based on new systematic approaches to specific classes of representations, we have seen precise classification results for various classes of groups such as oscillator groups, gauge groups and diffeomorphism groups (Janssens, Goldin, Zellner). Particulary interesting new directions are concerned with the combination of methods from stochastic analysis and quantum field theory with Lie theory (Gordina, Jorgensen, Vershik) and it also appears that, for certain classes of infinite-dimensional Lie algebras the global categorical perspective can provide deep new insights (Penkov). On the geometric side the powerful method of dual pairs is now emerging for important classes of infinite dimensional Hamiltonian systems (Gay-Balmaz), invariant theory for gauge groups is connected to singularity theory (Iohara) and new regularity results for differential equations on infinite dimensional groups have been obtained (Glöckner).

The analytic representation theory of *Lie supergroups* (as opposed to the algebraic representation theory of Lie superalgebras, which is also a thriving field but was not within the scope of this workshop) has made substantial progress in recent years, fueled in particular by questions of harmonic analysis originating from mesoscopic physics. Through this development a rapprochement of the representation theory of supergroups and traditional representation theory of Lie groups can be observed. A similar effect can be observed for the interplay between representations of supergroups and Clifford analysis (Alldridge, de Bie, Przebinda, Wurzbacher).

The main focus of the representation theory of *finite dimensional Lie groups* has shifted from the classification problem of the unitary dual of reductive groups (which is still unsolved) to structural results of representations such as branching problems to subgroups, and analysis on minimal representations of reductive groups. The interactions of infinite dimensional representations with global analysis on non-compact manifolds have been also actively studied, which often bring us new geometric insights. Interesting progress includes generalized Cartan decompositions for visible actions on complex manifolds (A. Sasaki) and for real spherical varieties (B. Krötz), analysis on branching laws to non-compact subgroups, and conformally equivariant differential systems (T.Kubo).

More specific information is contained in the abstracts which follow in this volume.

Workshop: Representations of Lie Groups and Supergroups

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Abstracts

Closed formulas for integral kernels of generalized Fourier transforms

Hendrik De Bie

Harmonic analysis in $\mathbb{R}^m$ is governed by the following three operators

$$\Delta := \sum_{i=1}^{m} \partial_{x_i}^2, \qquad |x|^2 := \sum_{i=1}^{m} x_i^2, \qquad \mathbb{E} := \sum_{i=1}^{m} x_i \partial_{x_i}$$

with Δ the Laplace operator and $\mathbb{E}$ the Euler operator. As observed in [8, 9], the operators $E = |x|^2/2$, $F = -\Delta/2$ and $H = \mathbb{E} + m/2$ are invariant under $O(m)$ and generate the Lie algebra $\mathfrak{sl}_2$:

$$\big[H, E\big] = 2E, \qquad \big[H, F\big] = -2F, \qquad \big[E, F\big] = H.$$

Recently, there has been a lot of interest in other differential or difference operator realizations of $\mathfrak{sl}_2$ or other Lie (super)algebras, such as $\mathfrak{osp}(1|2)$. The focus is in particular on the generalized Fourier transforms that subsequently arise. We mention the Dunkl transform [7], various discrete Fourier transforms [1, 10], Fourier transforms in Clifford analysis [3, 5, 6], etc. For a detailed review, we refer the reader to [4].

A hard problem in this context is to find explicit closed formulas for the integral kernel of the associated Fourier transforms. Here we are concerned with a partial solution of this problem for one of the most important new realizations of this type.

The set up is as follows. It can be observed that the $\mathfrak{sl}_2$ relations also hold for the generalized operators $|x|^a$, $|x|^{2-a}\Delta$ and $\mathbb{E} + \frac{a+m-2}{2}$, with $a > 0$ a real parameter. One then has the following commutators

$$\begin{aligned}
\big[|x|^{2-a}\Delta, |x|^a\big] &= 2a\,(\mathbb{E} + \frac{a+m-2}{2})\\
\left[|x|^{2-a}\Delta, \mathbb{E} + \frac{a+m-2}{2}\right] &= a\,|x|^{2-a}\Delta\\
\left[|x|^{a}, \mathbb{E} + \frac{a+m-2}{2}\right] &= -a\,|x|^{a}.
\end{aligned}$$

This was first observed, in the context of minimal representations, for $a = 1$ in [11, 12] and subsequently generalized to arbitrary a in [2].

This realization yields the so-called radially deformed Fourier transform

$$\mathcal{F}_a = e^{\frac{i\pi(m+a-2)}{2a}} e^{\frac{i\pi}{2a}(|x|^{2-a}\Delta - |x|^a)},$$

where a suitable normalization has been added to make the transform unitary. This transform can be written as an integral transform

$$\mathcal{F}_a(f)(y) = \int_{\mathbb{R}^m} K_a(x,y) f(x) |x|^{a-2} dx$$

defined on the function space $L_2(\mathbb{R}^m, |x|^{a-2}dx)$. A series expansion of $K_a(x,y)$ was obtained in [2].

Note that for $a = 2$ the kernel $K_a(x,y)$ reduces to the usual exponential kernel of the ordinary Fourier transform. Also when $a = 1$, a closed form is known, see [11, 12]. For arbitrary a such a closed form is not available. Moreover, there are no bounds known on $K_a(x,y)$ for $a \neq 1$ or $\neq 2$. Also a characterization of the kernel $K_a(x,y)$ as the unique eigenfunction of a system of PDEs is not known.

The strategy we will follow to determine an explicit formula for the series expansion depends on two essential steps:

- find a recursion property on the dimension
- use a trick to find the explicit formula in dimension 2.

Using these two steps, we were able to find an explicit formula for the kernel of the radially deformed Fourier transform in even dimension for $a = 2/n$ with $n \in \mathbb{N}$.

A comparison will also be made with similar results obtained for other classes of transforms, in particular with the Clifford-Fourier transform [6].

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Weyl Calculus and Dual Pairs

Tomasz Przebinda
(joint work with Mark McKee and Angela Pasquale)

Let W be a vector space of finite dimension $2n$ over $\mathbb{R}$ with a non-degenerate symplectic form $\langle \cdot, \cdot \rangle$. Let $\mathrm{W} = \mathsf{X} \oplus \mathsf{Y}$ be a complete polarization. Fix an element $J \in \mathfrak{sp}$ such that $J^2 = -I$ and the symmetric bilinear form $\langle J\cdot, \cdot \rangle$ is positive definite. Let dw be the Lebesgue measure on W such that the volume of the unit cube with respect to this form is 1. Similarly we normalize the Lebesgue measures on X and on Y.

Each element $K \in \mathcal{S}^*(\mathsf{X} \times \mathsf{X})$ defines an operator $\mathrm{Op}(K) \in \mathrm{Hom}(\mathcal{S}(\mathsf{X}), \mathcal{S}^*(\mathsf{X}))$ by

$$\mathrm{Op}(K)v(x) = \int_{\mathsf{X}} K(x, x')v(x')\, dx'.$$

The map $\mathrm{Op} : \mathcal{S}^*(\mathsf{X} \times \mathsf{X}) \to \mathrm{Hom}(\mathcal{S}(\mathsf{X}), \mathcal{S}^*(\mathsf{X}))$ is an isomorphism of linear topological spaces. This is known as the Schwartz Kernel Theorem, [1, Theorem 5.2.1].

Fix the unitary character $\chi(r) = e^{2\pi i r}$, $r \in \mathbb{R}$, and recall the Weyl transform $\mathcal{K} : \mathcal{S}^*(\mathrm{W}) \to \mathcal{S}^*(\mathsf{X} \times \mathsf{X})$ given by

$$\mathcal{K}(f)(x, x') = \int_{\mathsf{Y}} f(x - x' + y)\chi(\frac{1}{2}\langle y, x + x' \rangle)\, dy.$$

The Weyl symbol of the operator $\mathrm{Op} \circ \mathcal{K}(f)$ is the symplectic Fourier transform, $\widehat{f}$, of f defined by

$$\widehat{f}(w') = 2^{-n} \int_{\mathrm{W}} f(w)\chi(\frac{1}{2}\langle w, w' \rangle\, dw \qquad (w' \in \mathrm{W}).$$

A theorem of Calderon and Vaillancourt asserts that the operator $\mathrm{Op} \circ \mathcal{K}(f)$ is bounded on $\mathrm{L}^2(\mathsf{X})$ if its Weyl symbol and all its derivatives are bounded functions on W, [2, Theorem 3.1.3]. One motivation for our work is to compute the Weyl symbols of some obvious bounded operators which come from the Representation Theory of Real Reductive Groups. Many of these symbols turn out to be singular distributions. In order to introduce them we recall Weil Representation.

Denote by $\mathrm{Sp} \subseteq \mathrm{GL}(\mathrm{W})$ the symplectic group. For an element $g \in \mathrm{Sp}$ let $J_g = J^{-1}(g - 1)$. Then the adjoint with respect to the form $\langle J\cdot, \cdot \rangle$ is $J_g^* = Jg^{-1}(1 - g)$. In particular both have the same kernel. Thus the image of J_g is

$$J_g\mathrm{W} = (KerJ_g^*)^{\perp} = (KerJ_g)^{\perp}.$$

Hence, the restriction of J_g to $J_g\mathrm{W}$ defines an invertible element. Thus it makes sense to talk about $\det(J_g)^{-1}_{J_g\mathrm{W}}$. Let

$$\widetilde{\mathrm{Sp}} = \{\tilde{g} = (g, \xi) \in \mathrm{Sp} \times \mathbb{C}, \quad \xi^2 = i^{\dim(g-1)\mathrm{W}} \det(J_g)^{-1}_{J_g\mathrm{W}}\}$$

and let

$$\chi_{c(g)}(u) = \chi(\frac{1}{4}\langle (g + 1)w, (g - 1)w \rangle \qquad (u = (g - 1)w,\ w \in \mathrm{W}).$$

For $\tilde{g} = (g, \xi) \in \widetilde{\mathrm{Sp}}$ define

$$\Theta(\tilde{g}) = \xi, \qquad T(\tilde{g}) = \Theta(\tilde{g})\chi_{c(g)}\mu_{(g-1)\mathrm{W}}, \qquad \omega(\tilde{g}) = \mathrm{Op} \circ \mathcal{K} \circ T(\tilde{g}),$$

where $\mu_{(g-1)\mathrm{W}}$ is the Lebesgue measure on the subspace $(g-1)\mathrm{W}$ normalized so that the volume of the unit cube with respect to the form $\langle J\cdot, \cdot\rangle$ is 1. In these terms $(\omega, \mathrm{L}^2(\mathsf{X}))$ is the Weil representation of $\widetilde{\mathrm{Sp}}$ attached to the character χ. A proof of this fact based on previous work of [4] may be found in [5]. Conversely, one may take the above definition of ω and check directly that we get a representation with all the required properties. This was done in [7].

We consider a dual pair $(\mathcal{G}, \mathcal{G}')$, in the symplectic group Sp, with $\mathcal{G}$ compact. Let $\widetilde{\mathcal{G}}$ be the preimage of $\mathcal{G}$ in the metaplectic group equipped with the Haar measure of total mass 1. Fix an irreducible unitary representation Π of $\widetilde{\mathcal{G}}$. Then the operator

$$\omega(\check{\Theta}_\Pi) = \int_{\widetilde{\mathcal{G}}} \Theta_\Pi(\tilde{g}^{-1})\omega(\tilde{g})\, d\tilde{g}$$

is the orthogonal projection $\mathrm{L}^2(\mathsf{X}) \to \mathrm{L}^2(\mathsf{X})_\Pi$ onto the Π-isotypic component of $\mathrm{L}^2(\mathsf{X})$. The Weyl symbol of this projection is equal to a constant multiple of

$$T(\check{\Theta}_\Pi) = \int_{\widetilde{\mathcal{G}}} \Theta_\Pi(\tilde{g}^{-1})T(\tilde{g})\, d\tilde{g}.$$

For example, if $\mathcal{G} = \mathrm{O}_1 = \{\pm 1\}$ and $\mathcal{G}' = \mathrm{Sp}$, then

$$\widetilde{\mathcal{G}} = \{(1,1),\ (1,-1),\ (-1, i^n 2^{-n}),\ (-1, -i^n 2^{-n})\}$$

with the multiplication given by $(g_1, \xi_1)(g_2, \xi_2) = (g_1 g_2, \xi_1 \xi_2 C(g_1, g_2))$, where $C(1, \pm 1) = C(\pm 1, 1) = 1$ and $C(-1,-1) = 2^{2n}$. Hence, $\widetilde{\mathcal{G}}$ is isomorphic to the four element group $\{(1,1),\ (1,-1),\ (-1, i^n),\ (-1, -i^n)\}$ with the coordinate-wise multiplication. In these terms, the following two one dimensional representations of $\widetilde{\mathcal{G}}$ occur in ω.

$$\Pi_+(g, \eta) = \eta, \quad \Pi_-(g, \eta) = g\eta$$

A straightforward computation shows that

$$T(\check{\Theta}_{\Pi_\pm}) = \frac{1}{2}\left(\delta \pm 2^{-n} dw\right), \tag{1}$$

where δ is the Dirac delta at the origin in W.

In general, Classical Invariant Theory says that the space $\mathrm{L}^2(\mathsf{X})_\Pi$ is irreducible under the joint action of $\widetilde{\mathcal{G}}$ and $\widetilde{\mathcal{G}}'$, [3]. Hence $\mathrm{L}^2(\mathsf{X})_\Pi = \mathrm{L}^2(\mathsf{X})_{\Pi \otimes \Pi'}$ for an irreducible unitary representation Π' of $\widetilde{\mathcal{G}}'$. We are interested in the character $\Theta_{\Pi'}$ of Π'.

Since the map τ' is quadratic and has compact fibers, the pull-back $\mathcal{S}(\gamma') \ni \psi \to \psi \circ \tau' \in \mathcal{S}(\mathrm{W})$ is well defined and continuous. Hence by dualizing we get a push-forward of distributions $\tau'_* : \mathcal{S}^*(\mathrm{W}) \to \mathcal{S}^*(\gamma')$. Then, for an appropriately defined Fourier transform $\mathcal{F}$ on γ',

$$\frac{1}{\Theta \circ \tilde{c}} \tilde{c}_-^* \Theta_{\Pi'} = \frac{(\text{central character of } \Pi)(\tilde{c}(0))}{\dim \Pi} \mathcal{F}(\tau'_*(T(\check{\Theta}_\Pi))), \tag{2}$$

as shown in [6, Theorem 6.7]. For example, if $\mathcal{G} = \mathrm{O}_1$ then

$$\frac{1}{\Theta \circ \tilde{c}} \tilde{c}_-^* \Theta_{\Pi'_\pm} = \Pi_\pm(\tilde{c}(0)) \frac{1}{2}(\delta \pm \mu_{\mathcal{O}_{min}}),$$

where $\mu_{\mathcal{O}_{min}} = \tau'_*(2^{-n} dw)$ is an invariant measure on one of the two non-zero minimal nilpotent orbits in $\gamma' = \mathfrak{sp}$ and $\Pi'_\pm$ are the corresponding two irreducible pieces of the Weil representation ω.

In this paper we compute explicitly the distribution $T(\check{\Theta}_\Pi)$ in terms of the $\mathcal{G}\mathcal{G}'$ orbital integrals on W. In particular we see that $T(\check{\Theta}_\Pi)$ is a smooth function if and only if $(\mathcal{G}, \mathcal{G}')$ is a pair of compact unitary groups. Also, modulo a few exceptions, $T(\check{\Theta}_\Pi)$ is a locally integrable function if and only if the rank of $\mathcal{G}$ is greater or equal to the rank of $\mathcal{G}'$.

Let $\tau : \mathrm{W} \to \gamma^*$ be the unnormalized moment map given by $\tau(w)(x) = \langle xw, w\rangle$. Similarly we have $\tau' : \mathrm{W} \to \gamma'^*$. The variety $\tau^{-1}(0) \subseteq \mathrm{W}$ is the closure of a single orbit $\mathcal{O}$. There is a positive $\mathcal{G}\mathcal{G}'$ - invariant measure $\mu_{\mathcal{O}}$ on this orbit which defines a tempered distribution, homogeneous of degree $\deg \mu_{\mathcal{O}}$. Let $M_t(w) = tw$, $w \in \mathrm{W}$. Denote by M_t^* the corresponding pullback of distributions. In particular $M_t^* \mu_{\mathcal{O}} = t^{\deg \mu_{\mathcal{O}}} \mu_{\mathcal{O}}$. We show that

$$t^{\deg \mu_{\mathcal{O}}} M_{t^{-1}}^* T(\check{\Theta}_\Pi) \underset{t \to 0}{\to} const\, \mu_{\mathcal{O}},$$

where *const* is a non-zero constant. This last statement leads to an elementary proof of the equality $WF_1(\Theta_{\Pi'}) = \tau'\tau^{-1}(0)$. This equality was already verified in [6, Theorem 6.11], but the proof used some strong results specific for the representation theory of real reductive groups, [8].

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Restriction of complementary series representations of $O(1,N)$ to symmetric subgroups

Jan Möllers
(joint work with Yoshiki Oshima and Bent Ørsted)

We decompose complementary series representations of the rank one group $G = SO_0(1, n+1)$, $n \in \mathbb{N}$, under the restriction to symmetric subgroups. For details we refer to our paper [4].

All irreducible unitary representations of G are obtained as subrepresentations of representations induced from a parabolic subgroup $P = MAN$ on the level of $(\mathfrak{g}, K)$-modules. Up to conjugation P is unique and $M \cong SO(n)$, $A \cong \mathbb{R}_+$ and $N \cong \mathbb{R}^n$. We restrict our attention to representations induced from characters of P. Denote by π_σ^G the representation of G, which is induced from the character of P which is trivial on M and N and given by the character $\sigma \in \mathfrak{a}_\mathbb{C}^*$ on A (normalized parabolic induction). We identify $\mathfrak{a}_\mathbb{C}^* \cong \mathbb{C}$ such that π_σ^G is irreducible and unitarizable if and only if $\sigma \in i\mathbb{R} \cup (-n, n)$. By abuse of notation we denote by π_σ^G also the corresponding irreducible unitary representations. For $\sigma \in i\mathbb{R}$ the representations π_σ^G are called unitary principal series representations and for $\sigma \in (-n, 0) \cup (0, n)$ they are called complementary series representations. We have natural isomorphisms $\pi_{-\sigma}^G \cong \pi_\sigma^G$ for $\sigma \in i\mathbb{R} \cup (-n, n)$.

Apart from the maximal compact subgroup any symmetric subgroup $H \subseteq G$ is (up to connected components) of the form $H = SO_0(1, m+1) \times SO(n-m)$, $0 \leq m \leq n$. We find the decomposition of $\pi_\sigma^G|_H$ into irreducible H-representations. In the formulation of the branching law we use the convention $[0, \alpha) = \emptyset$ for $\alpha \leq 0$. Further denote by $\mathcal{H}^k(\mathbb{R}^{n-m})$ the representation of $SO(n-m)$ on the space of solid spherical harmonics of degree k on $\mathbb{R}^{n-m}$. Note that $\mathcal{H}^k(\mathbb{R}^{n-m})$ is irreducible for $n - m > 2$ and otherwise decomposes possibly into at most two irreducible components.

Theorem. *For $\sigma \in i\mathbb{R} \cup (-n, n)$ the representation π_σ^G of $G = SO_0(1, n+1)$ decomposes into representations of $H = SO_0(1, m+1) \times SO(n-m)$, $0 \leq m < n$, as follows:*

$$\pi_\sigma^G\big|_H \cong \sum_{k=0}^{\infty}{}^{\oplus} \left(\int_{i\mathbb{R}_+}^{\oplus} \pi_\tau^{SO_0(1,m+1)} \, \mathrm{d}\tau \oplus \bigoplus_{j \in \mathbb{Z} \cap \left[0, \frac{|\operatorname{Re}\sigma| - n + m - 2k}{4}\right)} \pi_{|\operatorname{Re}\sigma| - n + m - 2k - 4j}^{SO_0(1,m+1)} \right) \boxtimes \mathcal{H}^k(\mathbb{R}^{n-m}).$$

First of all, the restriction $\pi_\sigma^G|_H$ is decomposed with respect to the action of $SO(n-m)$, the second factor of H. Then the decomposition of each $\mathcal{H}^k(\mathbb{R}^{n-m})$-isotypic component into irreducible representations of $SO(1, m+1)$ contains continuous and discrete spectrum in general. The continuous part is a direct integral of unitary principal series representations $\pi_\tau^{SO_0(1,m+1)}$. The discrete part appears if and only if $k < \frac{|\operatorname{Re}\sigma|-n+m}{2}$ and is a finite direct sum of complementary series representations. Therefore the whole branching law of $\pi_\sigma^G|_H$ contains only finitely many discrete components and the discrete part is non-trivial if and only if $|\operatorname{Re}\sigma| > n-m$. In particular for $m > 0$ there is always at least one discrete component if σ is sufficiently close to the first reduction point n or $-n$.

For $\sigma \in i\mathbb{R}$ the decomposition is purely continuous. In this case the branching law is actually equivalent to the Plancherel formula for the Riemannian symmetric space $SO_0(1,m+1)/SO(m+1)$ and therefore easy to derive. We remark that a similar method was used in [3] for the branching laws of the most degenerate principal series representations of $GL(n,\mathbb{R})$ with respect to symmetric pairs. However, for the complementary series representations, i.e. $\sigma \in (-n,0)\cup(0,n)$, the decomposition cannot be obtained in the same way.

Our proof of the explicit Plancherel formula works uniformly for $\sigma \in i\mathbb{R} \cup (-n,n)$. It uses the "Fourier transformed realization" of π_σ^G on $L^2(\mathbb{R}^n, |x|^{-\operatorname{Re}\sigma}\,\mathrm{d}x)$ (see e.g. [1, 7]). For this consider first the non-compact realization on the nilradical $\overline{N}$ of the parabolic subgroup $\overline{P}$ opposite to P. We then take the Euclidean Fourier transform on $\overline{N} \cong \mathbb{R}^n$ to obtain a realization of π_σ^G on $L^2(\mathbb{R}^n, |x|^{-\operatorname{Re}\sigma}\,\mathrm{d}x)$. The advantage of this realization is that the invariant form is simply the L^2-inner product. The Lie algebra action in the Fourier transformed picture is given by differential operators up to order two, the crucial operators being the second order Bessel operators studied in [1]. Using these operators we reduce the branching law to the spectral decomposition of an ordinary differential operator of hypergeometric type on $L^2(\mathbb{R}_+)$. The spectral decomposition of this operator is derived using the Weyl–Titchmarch–Kodaira method and gives the the branching law and the explicit Plancherel formula.

Up to now only partial results regarding the branching of π_σ^G, $\sigma \in (-n,n)$, to H were known:

- For $n = 2$ and $m = 1$ the full decomposition was given by Mukunda [5] using the non-compact picture. This case corresponds to the branching law $SL(2,\mathbb{C}) \searrow SL(2,\mathbb{R})$.
- For $n \geq 2$ and $m = n-1$ Speh–Venkataramana [6, Theorem 1] proved the existence of the discrete component $\pi_{\sigma-1}^{SO_0(1,n)}$ in $\pi_\sigma^{SO_0(1,n+1)}$ for $\sigma \in (1,n)$ (special case $j = k = 0$ in our theorem). They also use the Fourier transformed picture for their proof. This is a special case of their more general result for complementary series representations of G on differential forms, i.e. induced from more general (possibly non-scalar) P-representations.
- The same special case was obtained by Zhang [8, Theorem 3.6]. He actually proved that for all rank one groups $G = SU(1,n+1;\mathbb{F})$, $\mathbb{F} = \mathbb{R},\mathbb{C},\mathbb{H}$, resp. $G = F_{4(-20)}$ certain complementary series representations of $H =$

$SU(1,n;\mathbb{F})$ resp. $H = Spin(8,1)$ occur discretely in some spherical complementary series representations of G. His proof uses the compact picture and explicit estimates for the restriction of K-finite vectors.

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Diffeomorphism group representations and quantum mechanics: Some current directions

Gerald A. Goldin

This talk reviews how unitary representations of diffeomorphism groups unify the description of a wide variety of quantum-mechanical systems and predict some new possibilities. Some directions for ongoing research are suggested.

Local current groups or gauge groups, diffeomorphism groups, and (sometimes) their extensions describe *local symmetry*. Diffeomorphism groups associate space or spacetime transformations with *regions* in a C^∞ manifold M, according to the support of the diffeomorphism. They can describe local kinematical symmetry.

Let M be the manifold of physical space, with $\dim[M] = d$; let G be the group of compactly-supported C^∞ diffeomorphisms of M under composition; and let S be the additive group of compactly-supported, C^∞ real-valued functions on M. Among the systems studied over several decades that correspond to inequivalent unitary representations of G, or its natural semidirect product with S, are:

(a) N-particle quantum mechanics, with particles distinguished by their masses; (b) systems of particles obeying *Bose or Fermi exchange statistics*, for $d \geq 2$; (c) systems of particles obeying *intermediate, or anyon statistics*, for $d = 2$; (d) systems of particles obeying *parastatistics*, for $d \geq 2$; (e) systems of *nonabelian anyons*, for $d = 2$; (f) systems of *tightly bound charged particles* – point dipoles, quadrupoles, etc.; (g) particles with spin, arranged in *spin towers*, for $d = 3$, according to representations of $GL(3,\mathbf{R})$; (h) particles with *fractional spin*, for

$d = 2$; (i) systems of *infinitely many particles in locally finite configurations*, corresponding to a free or interacting Bose gas, Fermi gas, or more exotic possibilities; (j) systems of *infinitely many particles with accumulation points*; (k) quantized *vortex* filaments when $d = 2$, or ribbons when $d = 3$, obtained from representations of area- (resp., volume-) preserving diffeomorphisms; (l) configurations of *extended objects*, including loops and strings, knotted configurations, and submanifolds having nontrivial topology and nontrivial internal symmetry; and (m) quantum particles having *nonlinear time-evolutions*.

The unitary group representations give (under technical conditions) self-adjoint representations of the "current algebra" of quantum mechanics — mass and momentum density operators in the physical space M, averaged with scalar functions and vector fields respectively. The most elementary representation describes a single particle, where we recover the usual quantum probability density and flux as expectation values. For multiparticle systems, recovering probability density and flux on configuration space involves expectation values of *products* of density and current operators on M (i.e., correlation functionals).

Under general conditions, a unitary representation of G (or its semidirect product with S) may be written by specifying a *configuration space* Δ derived somehow from M, on which G acts naturally; a *measure* μ on Δ that is *quasi-invariant* under the action of G, and a *unitary* 1*-cocycle* acting on the complex numbers (or, more generally, on a complex inner product space). Understanding the representations thus involves classifying possible configuration spaces and cocycles.

Let M be Euclidean space. For N-particle configuration space Δ, inequivalent cocycles are obtained by *inducing* (generalizing Mackey's method) from *inequivalent unitary representations of the fundamental group* $\pi_1[\Delta]$. When $d \geq 3$, $\pi_1[\Delta]$ is the *symmetric group* S_N; alternating representations lead to fermions, and higher-dimensional representations lead to the parastatistics of Greenberg and Messiah. But when $d = 2$, the fundamental group is Artin's *braid group* B_N, whose 1-dimensional unitary representations (characters) lead to particles obeying intermediate, or *anyon* statistics (where a counterclockwise exchange introduces a fixed phase $\exp i\theta$ in the wave function), confirming a conjecture of Leinaas and Myrheim. The theoretical possibility of *nonabelian anyons*, also called *plektons*, first arose from local current algebra (in joint work with Menikoff and Sharp); such systems are obtained (with $d = 2$) from the higher-dimensional representations of B_N. These ideas now find application in quantum computing as well as in condensed matter physics. Of course, even more possibilities of interest occur when M itself is multiply connected.

We consider various classes of configuration spaces: (A) Infinite but locally finite subsets of M describe the physics of gases. This space is standard in continuum classical or quantum statistical mechanics. Quasi-invariant measures include *Poisson measures*, associated with gases of noninteracting particles at fixed average density, and *Gibbs measures*, associated with translation-invariant two-body interactions. These describe equilibrium states and correlation functions in the classical case, and specify the current algebra representations in the quantum theory. (B)

Generalized functions or *distributions* over M include derivatives of δ-functions and other configurations. (C) Embeddings or immersions, parameterized or unparameterized, of another manifold N in M, describe extended configurations such as loops or branes. For $N = S^1$ and $d = 3$, we have classes of orbits for different *knots*. (D) Configuration spaces of *closed subsets* of M, as proposed by Ismagilov, include the unparameterized (but not the parameterized) embeddings or immersions, as well as infinite point configurations. (E) The space of *countable subsets* of M (with no condition of local finiteness), that Moschella and I considered, generalizes locally finite configurations to allow accumulation points. It also generalizes closed subsets, in that (for M separable) a closed subset can be recovered as the closure of different countable subsets. (F) Coadjoint orbits of the diffeomorphism group, and their unions, provide another context for configuration spaces. In the spirit of geometric quantization, the *symplectic structure* on coadjoint orbits provides a systematic way to obtain cocycles. (G) Considering the internal symmetry of point particles motivates the more general category of *bundles* over M to which one can lift the action of diffeomorphisms of M. In defining *marked configuration spaces*, one identifies a manifold Q for the internal degrees of freedom, and associates to each point in an ordinary configuration a value or "mark" in Q.

Each method of characterizing configuration spaces has some significant literature; some are associated with a point of view about quantization or about quantum mechanics. Each has its own open questions, sometimes of a technical nature. The diffeomorphism group approach helps understand these distinct but overlapping spaces in quantum theory. The overarching question remains of selecting just one as a "universal configuration space." for quantum mechanics.

Now a certain 1-cocycle leads to a modified 1-particle unitary representation of the diffeomorphism group of $\mathbf{R}^d$, and there results an interesting general nonlinear (derivative) Schrödinger equations. This equation, discovered in joint work with Doebner, includes or intersects with numerous specific but often *ad hoc* proposals that have been made over the years for nonlinear modifications of quantum mechanics — proposals by Kibble, Guerra and Pusterla, Sabatier and Auberson, Bialynicki-Birula and Mycielski, Kostin, Haag and Bannier, and Schuch. It also has new features. To interpret it, we adopt a measurement theory suggested by Mielnik, and determine the group of *nonlinear gauge transformations*. The notion of *physical equivalence classes* (under gauge transformation) of the nonlinear quantum time-evolutions provides a further unification of the many historically-proposed nonlinear quantum theories, together with some newer ones. Finally, we generalize Madelung's 1927 formulation to obtain gauge-invariant hydrodynamical equations for nonlinear quantum mechanics. Among new features, the quantum potential term is now governed by two independent coefficients instead of one; there is a term that moves us from Euler to Navier-Stokes hydrodynamics; and forces can be exerted by two additional external vector fields. An explicit frictional term derives from Kostin's version of nonlinear quantum mechanics.

Directions for research include the following. (1) We need a general theory of quasi-invariant measures on spaces of embeddings and immersions. (2) As far as I know we do not have a full theory of densely defined vector fields on Hilbert manifolds defining flows, to generalize Stone's theorem. (3) What are the best special and general relativistic generalizations of the role played by the diffeomorphism group in quantum mechanics? (4) Increasingly general nonlinear gauge transformations can move quantum states out of Hilbert space; this needs development. (5) An open question is how to extend nonlinear quantum time-evolutions to composite systems, with or without the "no signal" property. (6) Methods of differential geometry should be applied to explore generalized quantum hydrodynamics as a (sometimes dissipative) dynamical system (this is a topic of ongoing work with Gay-Balmaz and Ratiu).

The references below contain extensive bibliographic citations to the topics reviewed here [1, 2, 3].

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Osterwalder-Schrader positivity in representation theory, in physics, in stochastic processes, and in harmonic analysis

Palle E. T. Jörgensen

1. INTRODUCTION.

In the literature [2], the notion of Osterwalder-Schrader positivity (OS-positivity) have come to refer to a condition for stochastic processes, for quantum fields, and for representations. In the three contexts the respective positivity conditions in the different contexts are equivalent, and the interconnections was the focus of this Oberwolfach presentations.

While the subject of Osterwalder-Schrader positivity started in the seventies [12, 13], and its ramifications have proved successful in the diverse areas, the subject is now also part of harmonic analysis.

For example it is known that covariance functions of OS-positive stochastic processes indexed by time may be characterized by an explicit integral representation, [1, 3, 4, 7, 8, 9] .

In the setting of Wightman axioms, relativistic fields are operator valued tempered distributions which satisfy invariance under the Poincare group; the symmetry group of space time. In the early decades in the subject, it became clear that it would be difficult to realize concrete interacting fields directly in terms of the Wightman axioms; some of the difficulties having to do with families of non-commuting unbounded operators.

OS-positivity suggests the possibilities of replacing the non-commuting operators in the Wightman system with commuting random variables, and also replace the Poincare group with the Euclidean group, thus instead one is aiming for a model of stochastic processes having Euclidean invariance. When time is continued to the imaginary line, the solution to the Euclidean model then turns into a solution to the Wightman axioms.

As it turned out, this analytic continuation was successful in realizing solutions to quantum fields in the setting of the Haag-Kastler axioms; i.e., von Neumann algebras of local quantum fields.

Among the problems to be resolved in linking the two worlds of Euclidian covariance and relativistic covariance is an analytic continuation of unitary representations of the Euclidean group into unitary representations of the Poincare group. At a formal level, one thinks of a continuation from real time to purely imaginary time, and the initial paper by Osterwalder-Schrader gave a successful framework for this, as well as a constructive procedure. This was the initial formulation of what is now known as OS-positivity, and it deals with a property that some representations of the Euclidean group may or may not have. When the property holds we speak of OS-positivity.

In work between the author and Olafsson, [5, 6], it was suggested that OS-positivity should be part of the theory of unitary representations of Lie groups in general; hence we examined classes of unitary representations of Lie groups G which allow OS-positivity. When it is made precise, we end up with a pair of Lie groups G and G^c, where G^c is called the c-dual. With OS-positivity, we proved that the unitary representations of G will then have c-duals, arising via OS-positivity reflection as unitary representations of G^c; much in the same way we get unitary representations of the Poincaré group from Osterwalder-Schrader's original work. In the papers by Jorgensen-Olafsson this was illustrated as follows: we obtained the highest weight representations of semisimple Lie groups as reflections resulting as c-duals of complementary series representations.

The aim of the talk was to link this insight to unitary representations with the associated stochastic processes. Previously OS-positive stochastic processes had been restricted to the initial setting of Osterwalder-Schrader in the context of the two groups, the Euclidian group, and its c-dual, the Poincaré group.

One reason for why the adaptation of Osterwalder-Schrader to other Lie groups has been slow in coming is that in its initial setup Osterwalder-Schrader relies of a certain contractive selfadjoint representation of an abelian semigroup; and for the corresponding problems of stochastic processes in the context of c-dual Lie groups one is forced to work out the harmonic analysis of instead certain non-abelian semigroups. The latter are part of the theory of non-compact causal symmetric spaces.

Some of these difficulties have been resolved in recent work between Neeb and Olafsson [10, 11], and even more recent joint work between these two authors and the speaker.

A key point in the new developments is carrying over the more traditional OS-positivity analysis to Lie groups G with symmetry, and their unitary representations associated to causal symmetric spaces. In this context, one must deal with non-abelian subsemigroups S in G; and OS-positivity for both representations and for stochastic processes will be defined relative to this semigroup and its representations.

2. The mathematical setting

We will use the following notation:

G: Lie group,
$\mathfrak{g}$: Lie algebra,
τ: period-2 automorphism in G, or in $\mathfrak{g}$,
$H := \{h \in G;\, \tau(h) = h\}$,
$\mathfrak{h} = \{x \in \mathfrak{g};\, \tau(x) = x\}$, $\mathfrak{q} = \{y \in \mathfrak{g};\, \tau(y) = -y\}$,
$\mathcal{H}$: Hilbert space,
J: $\mathcal{H} \to \mathcal{H}$, unitary such that $J^2 = I$,
$\pi \in \mathrm{Rep}(G, \mathcal{H})$ is a unitary representation, such that
$J\pi(g)\, J = \pi(\tau(g))$, $\forall g \in G$,
$C \subset \mathfrak{g}$, cone such that $C \cap (-C) = \{0\}$,
$C - C = \mathfrak{g}$, $Ad_H C \subseteq C$.
Then $S := H \exp C$ is a semigroup in G.

Theorem 2.1. [5] *If there is a closed subspace $\mathcal{H}_+ \subset \mathcal{H}$ invariant under $\pi(S)$ such that*

$$\langle f_+, Jf_+\rangle_{\mathcal{H}} \geq 0,\ \forall f_+ \in \mathcal{H}_+, \tag{1}$$

then

$$\mathfrak{g}^c := \mathfrak{h} + i\mathfrak{q}, \ \text{and}$$
$$\pi^c(x + iy) = \pi(x) + i\pi(y),\ x \in \mathfrak{h},\ y \in \mathfrak{q}$$

exponentiates to a unity representation of the simply connected Lie group G^c with Lie algebra $\mathfrak{g}^c$.

A useful tool in the study of these representations, and the associated stochastic processes, is the operation

$$s^{\#} = \tau(s)^{-1},\ s \in S.$$

A key lemma states that if (1) holds then $S \ni s \mapsto \pi(s)$ turns into a $\#$-semigroup relative to the new inner product $\langle \cdot, \cdot \rangle_{new}$ from (1); i.e., in the Hilbert space

$$\mathcal{H} := (\mathcal{H}_+ \text{ with } (1)/\{f_+ \in \mathcal{H}_+; \langle f_+, Jf_+\rangle = 0\})^{\sim}$$

where $\sim$ stands for Hilbert-completion.

Indeed, $\langle f_+, \pi(s)h_+\rangle_{new} = \langle \pi(s^\#)f_+, h_+\rangle_{new}$, $\forall f_+, h_+ \in \mathcal{H}_+$, and $\forall s \in S$.

The issue about the unbounded operators $\pi(x)$, $x \in \mathfrak{g}$ coming from a unitary representation $\pi \in \mathrm{Rep}(G, \mathcal{H})$ is subtle: Since $\mathfrak{g} = \text{Lie alg } G$; if $x \in \mathfrak{g}$, then $\pi(x)$ is skew-adjoint with respect to $\mathcal{H}$, i.e., $\pi(x)^* = -\pi(x)$; but changing the inner product, for $y \in \mathfrak{q}$, $\pi(y)$ becomes selfadjoint with respect to $\langle f_+, h_+\rangle_{new} := \langle f_+, Jh_+\rangle$, $f_+, h_+ \in \mathcal{H}_+$.

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A Natural Limit of Bruhat Poisson Structures on G/B

Arlo Caine

(joint work with Sam Evens)

Let G be a complex semi-simple Lie group and let B^+ be a Borel subalgebra of G. This project concerns real algebraic Poisson structures on the flag variety $X = G/B^+$. The Bruhat cells in X are the B^+ orbits on X. Each Bruhat cell is of the form $X_w = B^+wB^+/B^+$ where w is an element of the Weyl group $W = N_G(H)/H$. Choosing a maximal complex torus H in B^+ determines a unique Borel subgroup B^- such that $B^- \cap B^+ = H$. The B^- orbits on X have the form $X^v = B^-vB^+/B^+$, for $v \in W$, and are called opposite Bruhat cells. The intersections $X^v \cap X_w$, known as a Richardson varieties, are non-empty whenever $v \leq w$ in the Bruhat order on W and are smooth and irreducible in X ([5]). Let K be a compact real form of G. Then $H = TA$ under the Cartan decomposition associated to K. The Bruhat Poisson structure Π_0 on X ([1],[2]) is real-algebraic, T-invariant, and has symplectic leaves which are precisely the Bruhat cells X_w, for all $w \in W$. We evolve this Poisson structure through a 1-parameter family of T-invariant real algebraic Poisson structures Π_t, all admitting the same symplectic foliation, and show that this family limits to a real algebraic Poisson structure Π_∞ on X. The real structure Π_∞ turns out to be invariant under the action of the complex torus H. Our main result is the following theorem.

Theorem. *The symplectic leaves of* (X, Π_∞) *are precisely the Richardson varieties* $X^v \cap X_w$*, for all* v *and* w *with* $v \leq w$ *in* W*.*

Typically, the Bruhat Poisson structure on X is constructed using the model for X as a real quotient K/T instead of the complex quotient G/B^+. These two models are equivalent because of the Iwasawa decomposition $G = KAN^+$, where N^+ is the nilpotent radical of B^+. This decomposition defines two factorization maps $\mathbf{k}\colon G \to K$ and $\mathbf{d}^+\colon G \to D^+$ (where $D^+ = AN^+$) by $g = \mathbf{k}(g)\mathbf{d}^+(g)$ for each $g \in G$. The former induces a diffeomorphism $\mathbf{k}\colon G/B^+ \to K/T$ whose inverse is induced from the inclusion $K \to G$. A standard construction using the root data from the choice of B^+ and H produces a Poisson Lie group structure π_K on K which is left and right T-invariant. The quotient map $(K, \pi_K) \to K/T$ then co-induces a T-invariant Poisson structure Π_K on K/T. The dual group to (K, π_K) is D^+ and the dressing action of D^+ on K/T coincides with the action of D^+ on G/B^+ projected to K/T. Since the D^+-orbits coincide with B^+-orbits on G/B^+, the symplectic leaves of $(K/T, \Pi_K)$ are the Bruhat cells $\mathbf{k}(X_w)$, for $w \in W$.

Let $\mathfrak{d}$ denote the Lie algebra $\mathfrak{g}$ of G viewed as a real Lie algebra. Then the Lie algebras $\mathfrak{k}$ of K and $\mathfrak{d}^+$ of D^+ are Lagrangian subalgebras of $\mathfrak{d}$ with respect to the imaginary part of the Killing form on $\mathfrak{g}$. Using ideas in [3], we reinterpret the construction of Π_K by constructing a T-invariant Poisson structure $\Pi_0 = \Pi_{\mathfrak{k},\mathfrak{d}^+}$ on G/B^+ depending on the choices of K and D^+ which makes $k\colon (G/B^+, \Pi_{\mathfrak{k},\mathfrak{d}^+}) \to (K/T, \mathfrak{k}_K)$ into a Poisson equivalence. This allows us to compare Bruhat Poisson structures associated to different choices of $\mathfrak{k}$ on the same geometric model G/B^+ of the flag variety. A one-parameter subgroup of A, generated by an element on which each of the positive roots take negative values, acts by conjugation on the compact real forms in the variety of Lagrangian subalgebras $\mathcal{L}(\mathfrak{d})$ carrying $\mathfrak{k}$ to $\mathfrak{k}_t$. Correspondingly, the left action of this group on G/B^+ induces a flow carrying the Bruhat Poisson structure Π_0 to another $\Pi_t = \Pi_{\mathfrak{k}_t,\mathfrak{d}^+}$. In the limit as $t \to \infty$ the compact real forms $\mathfrak{k}_t$ contract to $\mathfrak{d}^- = \mathfrak{n}^- + \mathfrak{t}$ in the variety of Lagrangian subalgebras $\mathcal{L}(\mathfrak{d})$ and the corresponding Poisson structures converge to $\Pi_\infty = \Pi_{\mathfrak{d}^-,\mathfrak{d}^+}$ on X.

Although Π_∞ is co-induced from a Poisson Lie group structure $\pi_{\mathfrak{d}^-,\mathfrak{d}^+}$ on G via the quotient map $G \to G/B^+$, the symplectic leaves are not the orbits of an action of the dual group D^+ since the dressing vector fields are incomplete. Instead, we apply a Lie theoretic formula from [4] to compute the rank of Π_∞ at each point of X and show that it agrees with the dimension of the Richardson variety passing through that point. Since we can argue that the leaves are contained in the Richardson varieties and the Richardson varieties are connected, the result follows.

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Dual pairs in fluid dynamics and central extensions of diffeomorphism groups

François Gay-Balmaz

(joint work with Cornelia Vizman)

The concept of dual pair, formalized by [W], is an important notion in Poisson geometry and has many applications in the context of momentum maps and reduction theory, see e.g. [OR] and references therein. Let (M,ω) be a finite dimensional symplectic manifold and let P_1, P_2 be two finite dimensional Poisson manifolds. A pair of Poisson mappings

$$P_1 \xleftarrow{\mathbf{J}_1} (M,\omega) \xrightarrow{\mathbf{J}_2} P_2$$

is called a *dual pair* if $\ker T\mathbf{J}_1$ and $\ker T\mathbf{J}_2$ are symplectic orthogonal complements of one another, where $\ker T\mathbf{J}_i$ denotes the kernel of the tangent map $T\mathbf{J}_i$ of $\mathbf{J}_i$. Dual pair structures arise naturally in classical mechanics. In many cases, the Poisson maps $\mathbf{J}_i$ are momentum mappings associated to Lie algebra actions on M. For example, in [M] (see also [CR], [GoS] and [Iw]) it was shown that the concept of dual pair of momentum maps can be useful for the study of bifurcations in Hamiltonian systems with symmetry.

We consider two fundamental "dual pairs" of momentum mappings arising in fluid dynamics. Both of them have attractive properties but present additional difficulties due to fact that the manifolds involved are infinite dimensional. This is why we first describe them only at a formal level and we use the word "dual pair" (with quotation mark) in a formal sense. The first "dual pair" is associated to the Euler equations of an ideal fluid and was discovered by [MW] in the context of Clebsch variables; the second one is associated to the n-dimensional Camassa-Holm equations and was discovered by [HoM] in the context of singular solutions.

(1) Consider the Euler equations for an ideal fluid on a domain S,

$$\partial_t u + u \cdot \nabla u = -\operatorname{grad} p, \quad \operatorname{div} u = 0,$$

where u is the velocity and p is the pressure. As shown in [A], the flow of the Euler equations describe geodesics on the group of volume preserving diffeomorphisms of S relative to the right invariant L^2-metric. In [MW], the "dual pair" for Euler equations is described as follows. Consider a symplectic manifold (M,ω), a volume manifold (S,μ), and let $\mathcal{F}(S,M)$ be the space of smooth maps from S to M. The left action of the group $\operatorname{Diff}(M,\omega)$ of symplectic diffeomorphisms and the right action of the group $\operatorname{Diff}(S,\mu)$ of volume preserving diffeomorphisms are two commuting symplectic actions on $\mathcal{F}(S,M)$. Their momentum maps $\mathbf{J}_L$ and $\mathbf{J}_R$ form the "dual pair" for the Euler equation:

$$\mathfrak{X}(M,\omega)^* \xleftarrow{\mathbf{J}_L} \mathcal{F}(S,M) \xrightarrow{\mathbf{J}_R} \mathfrak{X}(S,\mu)^*.$$

While the right leg represents Clebsch variables for the Euler equations, the left leg is a constant of motion for the induced Hamiltonian system on $\mathcal{F}(S,M)$.

(2) Consider the n-dimensional Camassa-Holm equations on a domain M,

$$\partial_t m + u \cdot \nabla m + \nabla u^{\mathsf{T}} \cdot m + m \operatorname{div} u = 0, \quad m = (1-\alpha^2\Delta)u,$$

whose flow describes geodesics on the group of all diffeomorphisms of M relative to a right invariant H^1 metric, see [HoM]. For more general choices for m, these equations are known under the generic name of EPDiff equations (standing for the Euler-Poincaré equations associated with the diffeomorphism group). In [HoM], the associated dual pair is described as follows. Let $\operatorname{Emb}(S,M)$ be the space of embeddings of S into M and consider the left action of the diffeomorphism group $\operatorname{Diff}(M)$ and the right action of the diffeomorphism group $\operatorname{Diff}(S)$. The dual pair consists of the momentum maps associated to the induced actions on the cotangent bundle $T^*\operatorname{Emb}(S,M)$ endowed with canonical symplectic form:

$$\mathfrak{X}(M)^* \xleftarrow{\mathbf{J}_L} T^*\operatorname{Emb}(S,M) \xrightarrow{\mathbf{J}_R} \mathfrak{X}(S)^*.$$

The left leg provides singular solutions of the EPDiff equation, whereas the right leg is a constant of motion associated to the collective motion on $T^*\operatorname{Emb}(S,M)$. In the one-dimensional case, $\mathbf{J}_L$ recovers the peakon solutions of the one-dimensional Camassa-Holm equation, [CaH].

More recently, the ideal fluid "dual pair" has been shown to apply for the Vlasov equation in kinetic theory [HoT2]. On the other hand, the EPDiff dual pair has been extended to the case of the Euler-Poincaré equations associated with the automorphism group of a principal bundle in [GBTV], needed for the study of the singular solutions of the two-component Camassa-Holm equation and its generalizations, see [HoT1], [GBTV], and references therein.

As we mentioned above, the reader should be warned that we use here the word "dual pair" (with quotation mark) in a formal sense for many reasons. Firstly, these examples are infinite dimensional and the concept of dual pair in infinite dimensions presents several difficult points that deserve further investigation. Secondly, the dual pair properties need to be shown in a rigorous way for the two situations. This means that one has to prove that the left action is transitive on the level subset of the right momentum map, and the right action is transitive on the level subset of the left momentum map. Finally, the ideal fluid "dual pair" reveals additional difficulties when one wants to check the momentum map properties of its two legs in a rigorous way. In fact, a reformulation is needed in this case. Interestingly enough, this reformulation leads naturally to well-known central extensions of groups of diffeomorphisms. Our main goal is to overcome these difficulties in order to rigorously show the dual pair properties. We proceed in several steps.

(I) We define the concepts of weak dual pairs and dual pairs, appropriate in the infinite dimensional setting, and give criteria for weak dual pair and dual pair properties.

(II) We provide a reformulation of the ideal fluid dual pair that allows us to show in a rigorous way that the two legs are momentum mappings. More precisely, we replace the groups $\mathrm{Diff}(M,\omega)$ and $\mathrm{Diff}(S,\mu)$ by the subgroups $\mathrm{Diff}_{ham}(M,\omega)$ and $\mathrm{Diff}_{ex}(S,\mu)$ of Hamiltonian and exact volume preserving diffeomorphisms, respectively. This allows us to show the existence of nonequivariant momentum maps.

(III) In order to have equivariance, required from the dual pair properties, we need to consider central extensions. In this context the group of quantomorphisms, central extension of $\mathrm{Diff}_{ham}(M,\omega)$, and the Ismagilov central extension of the group $\mathrm{Diff}_{ex}(S,\mu)$ appear naturally.

(IV) We show that the pair of momentum maps obtained above forms a weak dual pair. When restricted to the space of embeddings and under the condition $H^1(S)=0$, the weak dual pair is shown to be a dual pair.

This study yields several new developments in infinite dimensional symplectic geometry. For example, we are currently using the dual pair property to obtain a coadjoint orbit correspondence and describe new classes of symplectic Grassmannians.

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Groupoids and algebroids associated to W^*-algebras

Anatol Odzijewicz

The concepts of von Neumann algebra (W^*-algebra), groupoid and algebroid play the significant role in contemporary mathematics and mathematical physics. It is well known that there exists an interesting relationship of this concepts with others branches of mathematics. As examples let us mention the groupoid approach in topology [1] or the application of algebroid language in differential geometry [2].

In this presentation, based on papers [3] and [4], we investigate the groupoid and algebroid arising in a natural way from the structure of W^*-algebra. The results of this investigation we present in the series of statements. We also point out their connection with the Banach-Poisson geometry [5].

Let us recall that

(1) Groupoid is a small category with inverse morphisms.

(2) An algebroid on manifold M is a vector bundle (A, q, M) with a vector bundle map $a : A \to TM$ over M (anchor map) and a bracket $[\ ,\ ] : \Gamma A \times \Gamma A \to \Gamma A$ which is
  (a) $\mathbb{R}$-bilinear, alternating, and satisfing the Jacobi identity;
  (b) $[X, uY] = u[X, Y] + a(X)(u)Y$;
  (c) $a([X, Y]) = [a(X), a(Y)]$;
  for $X, Y \in \Gamma A,\ u \in C^\infty(M)$.

(3) W^*-algebra is a C^*-algebra which posses a predual Banach space $\mathfrak{M}_*$, [6]. An element $p \in \mathfrak{M}$ is called a projection if $p^* = p = p^2$. The set of projections of the W^*-algebra $\mathfrak{M}$ is the lattice which we will denote by $\mathcal{L}(\mathfrak{M})$. An element $u \in \mathfrak{M}$ is called a partial isometry if uu^* (or equivalently u^*u) is a projection. The set of partial isometries of the W^*-algebra $\mathfrak{M}$ we will denote by $\mathcal{U}(\mathfrak{M})$.
  The left support $l(x) \in \mathcal{L}(\mathfrak{M})$ (right support $r(x) \in \mathcal{L}(\mathfrak{M})$) of $x \in \mathfrak{M}$

is the least projection in $\mathfrak{M}$, such that $l(x)x = x$ (resp. $x\ r(x) = x$). If $x \in \mathfrak{M}$ is selfadjoint, then support $s(x) := l(x) = r(x)$. The polar decomposition of $x \in \mathfrak{M}$ is $x = u|x|$, where $u \in \mathfrak{M}$ is partial isometry and $|x| := \sqrt{x^*x} \in \mathfrak{M}^+$. One has

$$l(x) = s(|x^*|) = uu^*, \quad r(x) = s(|x|) = u^*u.$$

Let $G(p\mathfrak{M}p)$ be the group of all invertible elements in the W^*-subalgebra $p\mathfrak{M}p \subset \mathfrak{M}$. We define the subset $\mathcal{G}(\mathfrak{M})$ of partially invertible elements in $\mathfrak{M}$ by

$$\mathcal{G}(\mathfrak{M}) := \{x \in \mathfrak{M};\ \ |x| \in G(p\mathfrak{M}p),\ \text{where } p = s(|x|)\} \subsetneq \mathfrak{M}.$$

Proposition 1. *The set of partially invertible elements $\mathcal{G}(\mathfrak{M})$ with*

(1) *the source and target maps $\boldsymbol{s}, \boldsymbol{t} : \mathcal{G}(\mathfrak{M}) \to \mathcal{L}(\mathfrak{M})$*

$$\boldsymbol{s}(x) := r(x), \qquad \boldsymbol{t}(x) := l(x);$$

(2) *the product defined as the product in $\mathfrak{M}$ on the set*

$$\mathcal{G}(\mathfrak{M})^{(2)} := \{(x, y) \in \mathcal{G}(\mathfrak{M}) \times \mathcal{G}(\mathfrak{M});\ \ \boldsymbol{s}(x) = \boldsymbol{t}(y)\};$$

(3) *the identity section $\varepsilon : \mathcal{L}(\mathfrak{M}) \hookrightarrow \mathcal{G}(\mathfrak{M})$ as the inclusion;*

(4) *the inverse map $\iota : \mathcal{G}(\mathfrak{M}) \to \mathcal{G}(\mathfrak{M})$ defined by*

$$\iota(x) := |x|^{-1}u^*;$$

is a groupoid over $\mathcal{L}(\mathfrak{M})$.

Proposition 2. *The $\mathcal{U}(\mathfrak{M}) \rightrightarrows \mathcal{L}(\mathfrak{M})$ is a wide subgroupoid of the groupoid $\mathcal{G}(\mathfrak{M}) \rightrightarrows \mathcal{L}(\mathfrak{M})$.*

Remark: The description of $\mathcal{U}(\mathfrak{M})$-orbits on $\mathcal{L}(\mathfrak{M})$ leads to the Murray-von Neumann classification of W^*-algebras.

Proposition 3. (1) *The groupoid $\mathcal{U}(\mathfrak{M})$ of partial isometries is a topological groupoid with respect to the $s^*(\mathcal{U}(\mathfrak{M}), \mathfrak{M}_*)$-topology and the uniform topology in $\mathfrak{M}$.*

(2) *The groupoid $\mathcal{G}(\mathfrak{M})$ is not a topological groupoid with respect to any natural topology in $\mathfrak{M}$.*

Proposition 4. *The groupoid $\mathcal{G}(\mathfrak{M})$ is the Banach-Lie groupoid with the complex Banach manifold structure modeled on the set of Banach spaces*

$$(1 - \tilde{p})\mathfrak{M}\tilde{p} \oplus \tilde{p}\mathfrak{M}p \oplus (1 - p)\mathfrak{M}p$$

indexed by the pairs $(\tilde{p}, p)$ of the equivalent projections of $\mathcal{L}(\mathfrak{M})$.

By $\mathcal{G}_{p_0}(\mathfrak{M}) \rightrightarrows \mathcal{L}_{p_0}(\mathfrak{M})$ we denote the transitive subgroupoid of $\mathcal{G}(\mathfrak{M}) \rightrightarrows \mathcal{L}(\mathfrak{M})$ defined as follows

$$\mathcal{L}_{p_0}(\mathfrak{M}) := \{l(x) : \quad x \in \mathcal{G}(\mathfrak{M}), \quad r(x) = p_0\},$$

$$\mathcal{G}_{p_0}(\mathfrak{M}) := l^{-1}(\mathcal{L}_{p_0}(\mathfrak{M})) \cap r^{-1}(\mathcal{L}_{p_0}(\mathfrak{M})).$$

The open subset $P_0 := \mathcal{G}_{p_0}(\mathfrak{M}) \cap \mathfrak{M}p_0 \subset \mathfrak{M}p_0$ is the total space of the principal bundle $P_0\,(\mathcal{L}_{p_0}(\mathfrak{M}),\ mathcalG_0,\ l)$, where $G_0 := G(p_0\mathfrak{M}p_0)$. The group G_0 is an open subset of the Banach space $p_0\mathfrak{M}p_0$. So, it is a Banach-Lie group which Lie algebra is $p_0\mathfrak{M}p_0$. Thus, the tangent bundle TP_0 can be identified with the trivial bundle $\mathfrak{M}p_0 \times P_0$. The groupoid $\mathcal{G}_0(\mathfrak{M}) \rightrightarrows \mathcal{L}_{p_0}(\mathfrak{M})$ is isomorphic to the gauge groupoid $\frac{P_0 \times P_0}{G_0}$ associated to the principal bundle $P_0\,(\mathcal{L}_{p_0}(\mathfrak{M}),\ G_0,\ l)$. This isomorphism is given by

$$I : \frac{P_0 \times P_0}{G_0} \ni \langle \eta, \xi \rangle \tilde{\mapsto} \eta\xi^{-1} = \eta(\xi^+\xi)^{-1}\xi^+ \ \in\ \mathcal{G}_{p_0}(\mathfrak{M}),$$

where $\langle \eta, \xi \rangle$ denotes the orbit of the G_0-action:

$$P_0 \times P_0 \times G_0 \ \ni\ (\eta, \xi, g) \mapsto (\eta g, \xi g)\ \in\ P_0 \times P_0$$

on the product $P_0 \times P_0$.

Using the above isomorphism one can identify the Banach Lie algebroid $\mathcal{A}_{p_0}(\mathfrak{M}) \to \mathcal{L}_{p_0}(\mathfrak{M})$ of the groupoid $\mathcal{G}_{p_0}(\mathfrak{M}) \rightrightarrows \mathcal{L}_{p_0}(\mathfrak{M})$ with the quotient vector bundle

$$\mathfrak{M}p_0 \times_{G_0} P_0 \cong TP_0/G_0 \to \mathcal{L}_{p_0}(\mathfrak{M}).$$

Let us consider the vector bundle

$$\mathcal{M}^L_{p_0}(\mathfrak{M}) := \{(x, p) \in \mathfrak{M} \times \mathcal{L}_p(\mathfrak{M}) :\ x \in \mathfrak{M}p_0\} \stackrel{pr_2}{\longrightarrow} \mathcal{L}_{p_0}(\mathfrak{M})$$

of the left $\mathfrak{M}$-ideals of the W^*-algebra $\mathfrak{M}$ over $\mathcal{L}_{p_0}(\mathfrak{M})$.

Proposition 5. *One has an isomorphism of vector bundles $\mathcal{A}_{p_0}(\mathfrak{M}) \stackrel{\pi}{\longrightarrow} \mathcal{L}_{p_0}(\mathfrak{M})$ and $\mathcal{M}^L_{p_0}(\mathfrak{M}) \stackrel{pr_2}{\longrightarrow} \mathcal{L}_{p_0}(\mathfrak{M})$, where π is the canonical projection of the algebroid $\mathcal{A}_{p_0}(\mathfrak{M})$ on the set of units $\mathcal{L}_{p_0}(\mathfrak{M})$. Hence one can consider $\pi : \mathcal{A}_{p_0}(\mathfrak{M}) \to \mathcal{L}_{p_0}(\mathfrak{M})$ as a bundle of left $\mathfrak{M}$-modules.*

The vector bundle $\mathcal{A}_{p_0 *}(\mathfrak{M}) \to \mathcal{L}_{p_0}(\mathfrak{M})$ predual to the algebroid bundle $\mathcal{A}_{p_0}(\mathfrak{M}) \to \mathcal{L}_{p_0}(\mathfrak{M})$ one can identify with $(\mathfrak{M}p_0)_* \times_{G_0} P_0 \cong p_0\mathfrak{M}_* \times_{G_0} P_0 \to \mathcal{L}_{p_0}(\mathfrak{M})$.

For $F, G \in C^\infty(T_*P)$ one has the canonical Poisson bracket

$$\{F, G\}_{T_*P_0}(b, \eta) := \left\langle \frac{\partial G}{\partial \eta}(b, \eta), \frac{\partial F}{\partial b}(b, \eta) \right\rangle - \left\langle \frac{\partial F}{\partial \eta}(b, \eta), \frac{\partial G}{\partial b}(b, \eta) \right\rangle,$$

where we identify the pre-cotangent bundle T_*P_0 with $p_0\mathfrak{M}_* \times P_0$ and $(b, \eta) \in p_0\mathfrak{M}_* \times P_0$. Note here that $\frac{\partial F}{\partial \eta}(b, \eta) \in \mathfrak{M}^*$ and $\frac{\partial F}{\partial b}(b, \eta) \in \mathfrak{M}$.

Proposition 6. *Let $q : T_*P_0 \to \mathcal{A}_{p_0 *}(\mathfrak{M})$ be the quotient projection. Then for $f, g \in C^\infty(\mathcal{A}_{p_0 *}(\mathfrak{M}))$ there exists only one function $\{f, g\}_{\mathcal{A}_{p_0 *}(\mathfrak{M})} \in C^\infty(\mathcal{A}_{p_0 *}(\mathfrak{M}))$ such that*

$$\{f \circ q, g \circ q\}_{T^*P_0} = \{f, g\}_{\mathcal{A}_{p_0 *}(\mathfrak{M})} \circ q.$$

We conclude from the above proposition that $(\mathcal{A}_{p_0 *}(\mathfrak{M}), \{\cdot, \cdot\}_{\mathcal{A}_{p_0 *}(\mathfrak{M})})$ is a Banach Poisson manifold. One can check that this is the linear Poisson structure related to the algebroid structure of $\mathcal{A}_{p_0}(\mathfrak{M}) \to \mathcal{L}_{p_0}(\mathfrak{M})$. So, it is the algebroid version of Lie-Poisson structure on $\mathfrak{M}_*$ investigated in [5].

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Moderate growth representations of Lie supergroups

Alexander Alldridge

In this talk, we reported upon work in progress on the theory of smooth supergroup representations and the use of certain convolution superalgebras. We propose this as a framework for the study of infinite-dimensional supergroup representations. This allows us to transpose with ease, several basic results from the theory of infinite-dimensional representations of real reductive Lie groups to the super world—whilst the algebraic theory of finite-dimensional Lie superalgebra representations differs sharply from the Lie algebra case. In particular, we discussed the globalisation of Harish-Chandra supermodules.

Smooth supergroup representations. Let $\mathbb{K}$ be $\mathbb{R}$ or $\mathbb{C}$. A $\mathbb{K}$-superspace is by definition a pair $X = (X_0, \mathcal{O}_X)$ of a topological space and a sheaf of supercommutative $\mathbb{K}$-superalgebras; a morphism $f : X \to Y$ is a pair $(f_0, f^\sharp)$ consisting of a continuous map and a morphism of superalgebra sheaves. Supermanifolds form the full subcategory of $\mathbb{K}$-superspaces whose underlying topological space is Hausdorff, and which are locally isomorphic to the model spaces $\mathbb{A}^{p|q} = (\mathbb{R}^p, \mathcal{C}^\infty_{\mathbb{R}^p} \otimes \bigwedge(\mathbb{K}^q)^*)$. A Lie supergroup G is a group object in this category; equivalently, these can be described in terms of pairs $(\mathfrak{g}, G_0)$ of a Lie superalgebra over $\mathbb{K}$ and real Lie group with an action of G_0 on $\mathfrak{g}$ by Lie superalgebra automorphisms satisfying some straightforward requirements.

Given a finite-dimensional super-vector space $V = V_{\bar 0} \oplus V_{\bar 1}$ over $\mathbb{C}$, an representation of G on V is an action $\pi : G \times V \to V$, which is linear in its second argument. Equivalently, it can be described by a the data of a smooth G_0-representation π_0 on V, together with a G_0-equivariant $\mathfrak{g}$-module structure extending the derived representation of π_0.

This equivalence can be generalised to infinite dimensions, at least if V carries a reasonable locally convex topology (for instance, V is bornological and Mackey-complete). Indeed, one may define for any such V a vector-space valued functor $V(-)$ on the category of supermanifolds, and this gives a notion of representations of the group-valued functor $G(-)$ associated with G. The category of representations thus obtained, called *smooth* G-representations, is equivalent to the category of smooth representations of the pair $(\mathfrak{g}, G_0)$, which may defined as in the finite-dimensional case.

Traditionally, one is interested in *unitary* representations, a notion which makes sense for supergroups in case $\mathbb{K} = \mathbb{R}$. However, unitarity is a very strong condition in the super case [5] and for some applications (such as Plancherel formulæ), it is necessary to consider more general representations [3], for instance such representations, which are unitary upon restriction to G_0. Our intention in the research presented here is to create a general framework, in which more special classes of representations can be investigated.

Convolution algebras. At least in the Fréchet case, an alternative approach to smooth representations, based on convolution superalgebras, is viable. Indeed, given a Lie supergroup G, the space $\mathcal{E}'(G)$ of continuous linear functionals on the space of all global sections of the structure sheaf $\mathcal{O}_G$ of G, inherits from the supergroup multiplication a natural convolution product. As a super-vector space, $\mathcal{E}'(G)$ isomorphic to $\mathcal{E}'(G_0) \otimes_{\mathfrak{U}(\mathfrak{g}_{\bar{0}})} \mathfrak{U}(\mathfrak{g})$. The categories of smooth Fréchet G-representations and of Fréchet $\mathcal{E}'(G)$-modules are equivalent.

The superalgebra $\mathcal{E}'(G)$ contains the space $\mathcal{C}_c^\infty(G)$ of compactly supported sections of the sheaf of Berezinian densities as a subalgebra. In particular, the latter superalgebra acts on any smooth Fréchet G-representation.

If V is a continuous Fréchet G_0-representation, then any extension of the G_0-representation on the space V_∞ of smooth vectors to a smooth Fréchet G-representation is called a *continuous* Fréchet G-representation on V. If V is in addition reflexive, then $\mathcal{C}_c^\infty(G)$ acts on V, and the following super Dixmier–Malliavin theorem holds: We have $V_\infty = \mathcal{C}_c^\infty(G)V = \mathcal{C}_c^\infty(G)V_\infty$. Thus, the categories of reflexive continuous Fréchet G-representations and of non-degenerate reflexive Fréchet $\mathcal{C}_c^\infty(G)$-modules are equivalent.

Moderate growth representations. From now on, we assume that G_0 is endowed with a scale [4]. One can then define a Schwartz convolution superalgebra $\mathcal{S}(G)$ of Berezinian densities, similar to case of compact supports considered above. As a super-vector space, one has $\mathcal{S}(G) = \mathcal{S}(G_0) \otimes_{\mathfrak{U}(\mathfrak{g}_{\bar{0}})} \mathfrak{U}(\mathfrak{g})$, where $\mathcal{S}(G_0)$ is the algebra of Schwartz densities associated with the scale on G_0. Moreover, one can define the notion of *moderate growth* for smooth Fréchet G-representations. If V is a continuous Banach G-representation, then V_∞ has moderate growth. Moreover, $\mathcal{S}(G)$ acts on smooth Fréchet representations of moderate growth, and the corresponding category is thus equivalent to the category of non-degenerate Fréchet $\mathcal{S}(G)$-modules.

A pair (U, V) of continuous G-representations is called *contragredient* if there exists a G_0-invariant perfect pairing of U and V, which restricts to a G-invariant perfect pairing of U_∞ and V_∞. Given a contragredient pair of continuous Hilbert G-representations, the matrix coefficient map $U_\infty \times V_\infty \to \mathcal{T}(G)$ (denoting by $\mathcal{T}(G)$ the space of tempered superfunctions) extends uniquely to a $G\times G$-equivariant map from the distribution vectors to the tempered generalised superfunctions $\mathcal{S}'(G)$. In particular, continuous Hilbert G-representations admit a distributional character.

Globalisation of Harish-Chandra supermodules. In the following, we assume that $\mathfrak{g}_{\bar{0}}$ is reductive and G_0 admits a finite-dimensional linear representation with closed image and finite kernel. Let $K_0 \subseteq G_0$ be a maximal compact subgroup. In analogy with the even case, we call a super-vector space V a $(\mathfrak{g}, K_0)$-module, if it is endowed with a locally finite K_0-module structure and with a K_0-equivariant $\mathfrak{g}$-module structure, which extends the derived representation of K_0. A $(\mathfrak{g}, K_0)$-module is called *Harish-Chandra* if it has finite K_0-multiplicities and is finitely generated as a $\mathfrak{g}$-module.

A smooth Fréchet G-representation is called *Casselmann–Wallach* if it has moderate growth and its $(\mathfrak{g}, K_0)$-module of K_0-finite vectors is Harish-Chandra. Given $V \in \mathrm{HC}(\mathfrak{g}, K_0)$, a *CW globalisation* of V is a Casselmann–Wallach G-representation E, together with the choice of an isomorphism of V with the $(\mathfrak{g}, K_0)$-module of K_0-finite vectors of E.

A basic fact is that $\mathfrak{U}(\mathfrak{g})$ is a Harish-Chandra $(\mathfrak{g}_{\bar{0}}, K_0)$-module. Hence, if $V \in \mathrm{HC}(\mathfrak{g}, K_0)$, then its restriction to $(\mathfrak{g}_{\bar{0}}, K_0)$ is again Harish-Chandra. Moreover, we have an isomorphism $\mathcal{S}(G) \cong \mathcal{S}(G_0) \otimes_{\mathfrak{U}(\mathfrak{g}_{\bar{0}})} \mathfrak{U}(\mathfrak{g})$ of super-vector spaces. (Here, we fix the algebraic scale structure on G_0.) Combined with the classical Casselmann-Wallach globalisation theorem, we obtain the following super version thereof: Any Harish-Chandra $(\mathfrak{g}, K_0)$-module has a unique CW globalisation. The proof makes use of the methods and results of Bernstein–Krötz [4].

In particular, the categories of smooth Casselmann–Wallach G-representations and of Harish-Chandra $(\mathfrak{g}, K_0)$-modules are equivalent. As another corollary, for any G-equivariant continuous linear map $\phi : U \to V$ of Casselmann–Wallach G-representations, the induced map $U/\ker\phi \to V$ is an isomorphism onto its closed image. Moreover, any Casselmann–Wallach G-representation is the space of smooth vectors of a Hilbert G_0-representation.

Gelfand–Kazhdan criterion for Casselmann–Wallach representations. As an application of the super Casselmann–Wallach theory sketched above, a generalisation of the Gelfand–Kazhdan criterion of Sun–Zhu [6] was given.

Indeed, let H_1, H_2 be closed subsupergroups of G, χ_j characters of H_j, and σ an anti-automorphism of G. Assume that these data satisfy the following property (GK): Any tempered generalised superfunction $T \in \mathcal{S}'(G)$, which is both relatively $H_1 \times H_2$-invariant for the character $\chi_1^{-1} \times \chi_2^{-1}$ and an a joint eigenvector for the even part of the centre of $\mathfrak{U}(\mathfrak{g})^G$, is fixed by σ. Then for any contragredient pair (U, V) of irreducible Casselmann–Wallach G-representations, we have

$$\dim \mathrm{Hom}_{H_1}(U, \chi_1) \cdot \dim \mathrm{Hom}_{H_2}(V, \chi_2) \leq 1.$$

In work in progress, we wish to apply this theorem to the study of multiplicity freeness for pairs of supergroups, analogous to the recent results by Aizenbud et al. [1], Aizenbud–Gourevich [2], and Sun–Zhu [7].

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Integration of vector fields on supermanifolds and the exponential morphism of a Lie supergroup

Tilmann Wurzbacher
(joint work with Stéphane Garnier)

In this text supermanifolds are always understood in the sense of Berezin-Kostant-Leites, i.e., a supermanifold M is a ringed space $(M_0, \mathcal{O}_M)$, where M_0 is a topological space, $\mathcal{O}_M$ is a sheaf of supercommutative unital superalgebras, and M is locally isomorphic, as a superringed space, to the standard model $\mathbf{R}^{m|n}$. The sheaf $\mathcal{O}_M$ modulo the nilpotent ideal $\mathcal{N}$ yields the structure of a classical, ungraded manifold on M_0. By a slight abuse of language we will denote this latter manifold simply by M_0. Morphisms $\Phi : M \to N$ are given by a pair $(\Phi_0, \Phi^\sharp)$ with $\Phi_0 : M_0 \to N_0$ a continuous map, and $\Phi^\sharp : \Phi_0^{-1}\mathcal{O}_N \to \mathcal{O}_M$ a morphism of sheaves of superrings over M_0. Furthermore, for $i \in \{\bar{0}, \bar{1}\}$,

$$\mathrm{Der}_\Phi(\mathcal{O}_N(N_0), \mathcal{O}_M(M_0))_i := \{D : (\mathcal{O}_N(N_0) \to \mathcal{O}_M(M_0) \,|\, D \text{ is linear and } \forall f, g \in \mathcal{O}_M(M_0) \text{ homogeneous, } D(f \cdot g) = D(f) \cdot \Phi^\sharp(g) + (-1)^{i \cdot |f|} \Phi^\sharp(f) \cdot D(g)\},$$

is called the "space of derivations of parity i along Φ". For $\Phi = \mathrm{id}_M$ we obtain the vector fields of parity i on M. General derivations (resp. vector fields) are sums of such homogeneous elements. We observe that an even vector field X on M can also be considered as a section σ_X of the (super) tangent bundle $TM \to M$. Furthermore, if X is a vector field on M, then it induces a unique vector field $\tilde{X}$ on the underlying classical manifold M_0.

In our article [1] we give a new, direct proof of the following result of J. Monterde and coworkers (compare [2]): given a vector field X on a supermanifold M, there exists a unique maximal flow domain Ω for X, an open sub supermanifold of $\mathbf{R}^{1|1} \times M$, containing $\{0\} \times M$, with a "flow morphism" $F : \Omega \to M$ fulfilling

$$F \circ \mathrm{inj}_{\{0\}\times M}^{\mathbf{R}^{1|1}\times M} = \mathrm{id}_M$$

and

$$\left(\mathrm{inj}_{\mathbf{R}\times M}^{\mathbf{R}^{1|1}\times M}\right)^\sharp \circ \left(\left(\frac{\partial}{\partial t} + \frac{\partial}{\partial \tau}\right) \circ F^\sharp - F^\sharp \circ X\right) = 0\,,$$

where t and τ are the natural coordinates on $\mathbf{R}^{1|1}$.

Our proof is based on the recursive solution of a finite number of systems of ordinary differential equations arising from the analysis of the coordinate expression for the unknown morphism F. The first step consists in integrating the vector field $\tilde{X}$ on the classical manifold M_0, the body of the supermanifold M. The following steps boil down to solving inhomogeneous linear systems. This procedure shows immediately why the maximal flow domain of X is the open sub supermanifold of $\mathbf{R}^{1|1}\times M$, whose body is the maximal flow domain of the underlying vector field $\tilde{X}$.

Our approach also yields the analogous conclusion in the case of a holomorphic vector field on a complex-analytic supermanifold, hitherto not treated in the literature.

As an application of the integration result to Lie theory we deduce the existence of an exponential morphism $\exp^G : T_eG \to G$ for every finite dimensional Lie supergroup G. More precisely, let $G = (G_0, \mathcal{O}_G)$ be a Lie supergroup, i.e. a group object in the category of supermanifolds. Then there is a canonical even vector field X on $G \times T_eG$ defined via the section

$$\sigma_X = (T\mu \circ (\sigma_0 \times \iota_e), 0)$$

of $T(G\times T_eG) = TG\times T(T_eG) \to G\times T_eG$, where $\mu : G\times G \to G$ is the multiplication morphism of G and $T\mu$ its differential, $\iota_e : T_eG \to TG$ the natural inclusion, $\sigma_0 : G \to TG$ the zero-section of TG and the last zero denotes the zero-section of $T(T_eG) \to T_eG$. The vector field X is complete by an easy application of the above result on the relation between the maximal flow domains of X and $\tilde{X}$. Thus the following makes sense:

Definition. Let G be a Lie supergroup with multiplication μ and neutral element e, and with the even vector field X and its flow morphism $F = F^X$. Then the "exponential morphism of G" is defined as

$$\exp^G := \mathrm{proj}_1 \circ F \circ \mathrm{inj}_{\{1\}\times\{e\}\times T_eG}^{\mathbf{R}\times G\times T_eG} : T_eG \to G,$$

where $\mathrm{proj}_1 : G \times T_eG \to G$ is the projection on the first factor.

Recall that, given a supermanifold M, we have a "hom functor", associating to every supermanifold S the set of S-points $M(S) = \mathrm{Hom}(S, M)$ of M, and to a morphism $\Phi : M \to N$ the map

$$\Phi(S) : M(S) \to N(S),\ \Psi \mapsto \Phi(S)(\Psi) = \Phi \circ \Psi\,.$$

Applying hom functors to superpoints, i.e. taking $S = \mathbf{R}^{0|k}$ for k in $\mathbf{N}_0$, we get a fundamental characterization of the exponential morphism:

Theorem. ([1]) The exponential morphism $\exp^G : T_eG \to G$ of a Lie supergroup G fulfills and is uniquely determined by the following condition: for all $k \geq 0$, $\exp^G(\mathbf{R}^{0|k}) : T_eG(\mathbf{R}^{0|k}) \to G(\mathbf{R}^{0|k})$ is the exponential map $\exp^{G(\mathbf{R}^{0|k})}$ of the finite-dimensional, ungraded Lie group $G(\mathbf{R}^{0|k})$.

The above constructed exponential morphism enjoys crucial properties similar to those of the exponential map of a classical, ungraded Lie group, as is shown by the next result.

Theorem. (W.) Let $G = (G_0, \mathcal{O}_G)$ be a Lie supergroup with exponential morphism $\exp^G$. Then

(1) given a Lie supergroup $H = (H_0, \mathcal{O}_H)$ with exponential morphism $\exp^H$ and a Lie supergroup morphism $\rho : G \to H$,
$$\exp^H \circ T_e\rho = \rho \circ \exp^G\,,$$
(2) $T_0\exp^G : T_0(T_eG) = T_eG \to T_eG$ equals id_{T_eG} and thus $\exp^G$ is a local diffeomorphism near 0 in T_eG,

(3) for $G = GL_{m|n}$ the morphism $\exp^G$ is given by the "usual exponential series".

We remark that the last property is to be interpreted as follows: denote, for an unital, supercommutative, associative superalgebra A, by $\mathrm{Mat}_{m|n}(A) = \mathrm{Hom}(A^{m|n}, A^{m|n})$ the set of A-module endomorphisms of $A^{m|n} = A^m \oplus (\Pi A)^n$, where Π is the parity change functor for A-modules. Then for all supermanifolds $S = (S_0, \mathcal{O}_S)$, the hom functors of the Lie superalgebra $T_eGL_{m|n}$ resp. the Lie supergroup $GL_{m|n}$ are given by $\mathrm{Hom}(S, T_eGL_{m|n}) = \mathrm{Mat}_{m|n}(\mathcal{O}_S(S_0))$ resp.

$$\mathrm{Hom}(S, GL_{m|n}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{Mat}_{m|n}(\mathcal{O}_S(S_0)) \,\middle|\, a, d \,\text{are invertible} \right\}\,.$$

The third point of the above theorem is now tantamount to the statement that the map $\exp^{GL_{m|n}}(S) : \mathrm{Hom}(S, T_eGL_{m|n}) = \mathrm{Mat}_{m|n}(\mathcal{O}_S(S_0)) \to \mathrm{Hom}(S, GL_{m|n}) \subset \mathrm{Mat}_{m|n}(\mathcal{O}_S(S_0))$ equals the (convergent) power series

$$\xi \mapsto \sum_{j \geq 0} \frac{1}{j!} \xi^j.$$

By Yoneda's lemma, this already specifies a unique morphism $\exp^{GL_{m|n}} : T_e GL_{m|n} \to GL_{m|n}$. Furthermore, using the first point, property (3) yields, of course, the exponential morphism for all linear Lie supergroups.

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Brownian and energy representation for path groups

Maria Gordina

(joint work with S.Albeverio, B.Driver, A.M.Vershik)

Let G be a connected compact semi-simple Lie group. We consider two representations of the infinite-dimensional group of paths from $[0, T]$ to G : one is in $L^2(W(G), \mu)$ where $W(G)$ is the space of continuous paths in G and μ is the Wiener measure, and the other is in $L^2(W(G), \nu)$ where ν is the Gaussian measure. The first representation comes from the quasi-invariance of μ with respect to the shifts by elements in $W(G)$, and the second representation is the energy representation studied before by Gelfand, Graev, Vershik, and Wallach. We prove that these two representations are unitarily equivalent, and then proceed to analyse their structures.

Unitary representations of gauge groups

Bas Janssens

(joint work with Karl-Hermann Neeb)

Let $P \to M$ be a smooth principal K_0-bundle, with K_0 a compact semisimple Lie group. Then the *gauge group* $\mathrm{Gau}(P)$ of vertical automorphisms of P is isomorphic to $\Gamma(K)$, the group of smooth sections of the adjoint bundle $K := P \times_{\mathrm{Ad}} K_0$. Let us denote $\mathfrak{k}_0 := \mathrm{Lie}(K_0)$, and $\mathfrak{g} := \mathfrak{k}_0 \otimes_{\mathbb{R}} \mathbb{C}$. Then the *gauge Lie algebra* of infinitesimal vertical automorphisms is isomorphic to $\Gamma(\mathfrak{K})$, with $\mathfrak{K} := P \times_{\mathrm{Ad}} \mathfrak{k}_0$ a bundle of Lie algebras. Both $\Gamma(\mathfrak{K})$ and its compactly supported version $\Gamma_c(\mathfrak{K})$ are locally convex topological Lie algebras. We classify their bounded unitary representations, i.e., the continuous homomorphisms $\pi\colon \Gamma(\mathfrak{K}) \to B(\mathcal{H})$ that satisfy $\pi(s)^\dagger = -\pi(s)$, (or alternatively, $\pi(s)^\dagger = -\pi(\overline{s})$ on the complexification $\Gamma(\mathfrak{K}_{\mathbb{C}})$).

The restriction of boundedness is quite severe, and will certainly exclude interesting examples. However, we expect the more inclusive class of *semibounded* representations, in which the Lie algebra has an open neighbourhood of elements with a spectrum bounded from one side, to be holomorphically induced from the bounded ones.

Consider, for example, the bundle of Lie algebras $\mathfrak{K} = (\mathbb{R} \times \mathfrak{k}_0)/\langle\sigma\rangle$ over S^1, where $\sigma \in \mathrm{Aut}(\mathfrak{k}_0)$ is a diagram automorphism of order 1, 2 or 3 acting by $\sigma(t, k) = (t + 2\pi, \sigma(k))$. The unitary irreducible highest weight representations of the associated twisted affine Kac-Moody algebra $\mathbb{C}c \oplus_\psi \Gamma(\mathfrak{K})_\mathbb{C} \rtimes \mathbb{C}\frac{d}{dt}$ are *semibounded*, but not bounded. However, one can show that every such semibounded irreducible unitary representation is holomorphically induced from a *bounded* representation of the centraliser $Z(\frac{d}{d\theta}) = \mathbb{C}c \oplus \mathfrak{g}^\sigma \oplus \mathbb{C}\frac{d}{dt}$.

Analogously, consider a Lie algebra bundle $\mathfrak{K} \to M$ and its pullback $p^*\mathfrak{K}$ along the 'spatial projection' $p\colon S^1 \times M \to M$. In this context, $\Gamma(\mathfrak{K}) < \Gamma(p^*\mathfrak{K})$ is to be thought of as the subalgebra of time-independent infinitesimal gauge transformations. We expect (and hope to prove in the near future) that the irreducible unitary *semibounded* representations of the Lie algebra $\mathbb{C}c \oplus_\psi \Gamma(p^*\mathfrak{K})_\mathbb{C} \rtimes \mathbb{C}\frac{d}{dt}$ (with ψ an appropriate cocycle) are holomorphically induced from irreducible unitary *bounded* representations of the centraliser $Z(\frac{d}{d\theta}) = \mathbb{C}c \oplus \Gamma(\mathfrak{K})_\mathbb{C} \rtimes \mathbb{C}\frac{d}{dt}$.

This motivates our investigation [1] into the bounded representations of $\Gamma(\mathfrak{K})$, where the more than casually interested reader may find proofs for the statements below. We proceed in two steps. First, we note that for M compact and $\mathfrak{K}$ trivial, we have $\Gamma(\mathfrak{K})_\mathbb{C} = \mathfrak{g} \otimes \mathcal{A}$, with $\mathcal{A} = C^\infty(M, \mathbb{C})$ a commutative involutive commutative inverse algebra (*cia*). The first step is now to classify bounded unitary irreducible representations in this setting. Examples of such representations are easily found; if χ is an involutive character of $\mathcal{A}$, and (ρ, V) a unitary irreducible representation of $\mathfrak{g}$, then the representation $\pi\colon \mathfrak{g} \otimes \mathcal{A} \to B(V)$ defined by $\pi(X \otimes a) = \rho(X)\chi(a)$ is called an *evaluation representation*. The following theorem will constitute the backbone of our classification:

Theorem 1. Every bounded irreducible unitary representation of $\mathfrak{g}\otimes\mathcal{A}$ is unitarily equivalent to a finite tensor product of evaluation representations.

These representations are, of course, finite dimensional. For the cias $C^\infty(M, \mathbb{C})$ (if M is compact) and $C_c^\infty(M, \mathbb{C}) \oplus \mathbb{C}\mathbf{1}$ (if M is noncompact), all characters are determined by points in M (resp. $M \cup \{\infty\}$) by $\chi(f) = f(x)$. Thus, for F in $C^\infty(M, \mathfrak{g})$ or $C_c^\infty(M, \mathfrak{g}) \rtimes \mathfrak{g}$, we have $\pi(F) = \rho(F(x))$.

For the second step, we wish to use the above 'local' result to attack the 'global' case, in which the bundle need not be trivial. Let $(\pi, \mathcal{H})$ be a bounded unitary irreducible (or factor) representation of $\Gamma_c(\mathfrak{K})$. Let U be an open subset of M such that $\mathfrak{K}$ trivialises in an open neighbourhood of $\overline{U}$. Then π restricts to a representation of the subalgebra $\Gamma_c(\mathfrak{K}|_U)_\mathbb{C} \simeq \mathfrak{g} \otimes C_c^\infty(U, \mathbb{C})$. We would like to apply the previous lemma to the algebra $\mathcal{A}_0 := C_c^\infty(U, \mathbb{C})$, but alas, $\mathcal{A}_0$ is not a cia, due to the lack of a unit. This inconvenience is easily circumvented by adjoining a unit, $\mathcal{A}_+ := \mathcal{A}_0 \oplus \mathbb{C}\mathbf{1}$, but then the algebra $\mathfrak{g} \otimes \mathcal{A}_+$ is no longer a subalgebra of $\Gamma_c(\mathfrak{K})$. The following 'localisation lemma' breaks the deadlock:

Lemma 2. The restriction of π to $\mathfrak{g} \otimes \mathcal{A}_0$ extends to a representation of $\mathfrak{g} \otimes \mathcal{A}_+$. Furthermore, there exist bounded irreducible (or factor) representations $(\pi_1, \mathcal{H}_1)$ and $(\pi_2, \mathcal{H}_2)$ of $\Gamma_c(\mathfrak{K})$ such that π_1 vanishes on $\Gamma_c(\mathfrak{K}|_U)$, π_2 is a finite tensor product of evaluation representations at points in U, $\mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2$, and $\pi = \pi_1 \otimes \pi_2$.

If M is a compact manifold, it can be covered by finitely many open sets U_i to which the above lemma can be successively applied, yielding:

Theorem 3. Every bounded irreducible unitary representation of $\Gamma(\mathfrak{K})$ is equivalent to a finite tensor product of evaluation representations.

This remains true even for noncompact manifolds.

For the Lie algebra $\Gamma_c(M)$, however, the situation is quite different. Indeed, we can construct *infinite* tensor products of evaluation representations as follows. Given a locally finite subset $\mathbf{x} \subseteq M$ and an irreducible unitary representation (ρ_x, V_x) for each $x \in \mathbf{x}$, we construct a Lie algebra homomorphism $\eta\colon \Gamma_c(\mathfrak{K}) \to \mathcal{A}_{\mathbf{x},\rho}$, where the UHF C*-algebra $\mathcal{A}_{\mathbf{x},\rho}$ is the (infinite) tensor product of C*-algebras $\bigotimes_{x\in\mathbf{x}} B(V_x)$. It is given by $\eta(s) = \sum_{x\in\mathbf{x}} \rho_x(s(x))$, where $\rho_x(s(x)) \in B(V_x)$ is considered as an element of $\mathcal{A}_{\mathbf{x},\rho}$. Any pure (factor) state ϕ of $\mathcal{A}_{\mathbf{x},\rho}$ now gives rise to an irreducible (factor) representation of $\Gamma_c(\mathfrak{K})$ by concatenating η with the GNS-representation $\pi_\phi\colon \mathcal{A}_{\mathbf{x},\rho} \to B(\mathcal{H}_\phi)$.

Theorem 4. Every bounded irreducible (or factor) unitary representation of $\Gamma(\mathfrak{K})$ is equivalent to such an infinite tensor product of evaluation representations.

If $\mathbf{x}$ is a finite set, then every pure state is given by a vector in $\bigotimes_{x\in\mathbf{x}} V_x$, so that up to unitary equivalence, there is a unique irreducible GNS representation. If $\mathbf{x}$ is infinite, however, the situation is radically different.

Although any two pure states are related by a $*$-automorphism of $\mathcal{A}_{\mathbf{x},\rho}$ (see [2]), this automorphism need not be inner, and there are many inequivalent irreducible representations associated to the same $\mathbf{x}$ and ρ. For example, if $\phi = \bigotimes_{x\in\mathbf{x}} \phi_x$ with ϕ_x the pure state on $B(V_x)$ given by the unit vector $\psi_x \in V_x$, then ϕ is pure, and $\mathcal{H}_\phi = \bigotimes_{x\in\mathbf{x}}(V_x, \psi_x)$. Two representations defined by $\{\psi_x\}$ and $\{\psi'_x\}$ are unitarily equivalent [3] if and only if the fidelity approaches 1 sufficiently fast, $\sum_{x\in\mathbf{x}} 1 - |\langle\psi_x, \psi'_x\rangle| < \infty$. Many non-equivalent irreducible representations, although certainly not all, are obtained in this way.

It is not hard to see that the von Neumann algebras $\pi(\Gamma_c(\mathfrak{K}))''$ for different states can be non-isomorphic factors of type I, II and III. For example, choose $\mathfrak{g} = \mathfrak{sl}_2(\mathbb{C})$, $\mathfrak{K} = M \times \mathfrak{k}_0$ trivial, $\mathbf{x} \subseteq M$ a locally finite countably infinite subset, and $V_x = \mathbb{C}^2$ the defining representation for all $x \in \mathbf{x}$. Then we have $\mathcal{A}_{\mathbf{x},\rho} = \widehat{\bigotimes}_{x\in\mathbf{x}} M_2(\mathbb{C})$, so that by choosing the factor state $\phi = \bigotimes_{x\in\mathbf{x}} \phi_x$ with

$$\phi_x(A) = \mathrm{Tr}\left(\begin{pmatrix} \frac{\lambda}{1+\lambda} & 0 \\ 0 & \frac{1}{1+\lambda} \end{pmatrix} A\right),$$

one obtains the type I_∞ factor for $\lambda = 0$ (when ϕ is pure), the hyperfinite type II_1 factor if $\lambda = 1$, and the hyperfinite type III_λ factors (the 'Powers factors') if $0 < \lambda < 1$.

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On the homomorphisms between generalized Verma modules arising from conformally invariant systems

Toshihisa Kubo

In this talk we studied systems of differential operators that are equivariant under an action of a Lie algebra. We call such systems of operators *conformally invariant.* To explain the meaning of the equivariance condition, suppose that $\mathcal{V} \to M$ is a vector bundle over a smooth manifold M and $\mathfrak{g}_0$ is a Lie algebra of first order differential operators that act on sections of $\mathcal{V}$. A system of linear differential operators $D_1, \ldots, D_n$ on sections of $\mathcal{V}$ is called a *conformally invariant system* if, for each $X \in \mathfrak{g}_0$, there are smooth functions $C_{ij}^X(m)$ on M so that, for all $1 \leq i \leq n$, and sections f of $\mathcal{V}$, we have

$$\big([X, D_i]f\big)(m) = \sum_{j=1}^{n} C_{ji}^X(m)(D_j f)(m), \tag{1}$$

where $[X, D_j] = XD_j - D_jX$. By extending $\mathbb{C}$-linearly, the identity (1) can be applied equally well to the complexified Lie algebra $\mathfrak{g} = \mathfrak{g}_0 \otimes_{\mathbb{R}} \mathbb{C}$.

A typical example for a conformally invariant system of one differential operator is the wave operator $\square = \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \frac{\partial^2}{\partial x_3^2} - \frac{\partial^2}{\partial x_4^2}$ on the Minkowski space $\mathbb{R}^{3,1}$. (See for example the introduction of [4].)

The notion of conformally invariant systems generalizes that of quasi-invariant differential operators introduced by Kostant in [3]. A systematic study of conformally invariant systems recently started with the work of Barchini-Kable-Zierau in [1] and [2].

Although the theory of conformally invariant systems can be viewed as an analytic-geometric theory, it is also closely related to algebraic objects such as generalized Verma modules. It has been shown in [2] that a conformally invariant system yields a homomorphism between certain generalized Verma modules, one of which is non-scalar. A homomorphism between generalized Verma modules is called *standard* if it is induced from a homomorphism between the corresponding (full) Verma modules.

In [5] we have built a number of systems of first and second order differential operators associated to a maximal parabolic subalgebra $\mathfrak{q}$ of quasi-Heisenberg type, that is, a maximal parabolic subalgebra $\mathfrak{q} = \mathfrak{l} \oplus \mathfrak{n}$ with nilpotent radical $\mathfrak{n}$ satisfying the conditions that $[\mathfrak{n}, [\mathfrak{n}, \mathfrak{n}]] = 0$ and $\dim([\mathfrak{n}, \mathfrak{n}]) > 1$. Then, in [6], we determined whether or not the homomorphisms between the generalized Verma modules arising from the systems of operators constructed in [5] are standard. The construction in [5] was uniform, but there were three open cases. Recently these gaps were filled in [7]. The standardness of the homomorphisms arising from the systems in the open cases was also determined.

To describe our work more precisely, we now briefly review the results of [5], [6], and [7]. Let G be a complex, simple, connected, simply-connected Lie group with Lie algebra $\mathfrak{g}$. Give a $\mathbb{Z}$-grading $\mathfrak{g} = \bigoplus_{j=-r}^{r} \mathfrak{g}(j)$ on $\mathfrak{g}$ so that $\mathfrak{q} = \mathfrak{g}(0) \oplus \bigoplus_{j>0} \mathfrak{g}(j) = \mathfrak{l} \oplus \mathfrak{n}$ is a maximal parabolic subalgebra. Let $Q = N_G(\mathfrak{q}) = LN$. For a real form $\mathfrak{g}_0$ of $\mathfrak{g}$ in which the parabolic subalgebra $\mathfrak{q}$ has a real form $\mathfrak{q}_0$, define G_0 to be an analytic subgroup of G with Lie algebra $\mathfrak{g}_0$. Set $Q_0 = N_{G_0}(\mathfrak{q})$. Our manifold is $M = G_0/Q_0$ and we consider a line bundle $\mathcal{L}_s \to G_0/Q_0$ for each $s \in \mathbb{C}$. By the Bruhat theory, the homogeneous space G_0/Q_0 admits an open dense submanifold $\bar{N}_0Q_0/Q_0$. We restrict our bundle to this submanifold. The systems that we construct act on smooth sections of the restricted bundle. By slight abuse of notation, we refer to the restricted bundle as $\mathcal{L}_s$.

We construct systems of kth order differential operators from L-irreducible constituents of $\mathfrak{g}(-r+k) \otimes \mathfrak{g}(r)$ for $1 \leq k \leq 2r$. The systems of operators are called Ω_k *systems*. (For the details see Section 3 of [5].) The conformal invariance of Ω_k systems depends on the complex parameter s for the restricted line bundle $\mathcal{L}_s \to \bar{N}_0$. We then say that an Ω_k system has *special value* s_k if the system is conformally invariant on the line bundle $\mathcal{L}_{s_k}$.

In [5] and [7], we find the special values of the Ω_1 system and Ω_2 systems associated to a maximal parabolic subalgebra $\mathfrak{q}$ of quasi-Heisenberg type. The special value s_1 of the Ω_1 system is $s_1 = 0$. To describe the results for the Ω_2 systems, let λ_i be the fundamental weight for the simple root α_i that determines the maximal parabolic subalgebra $\mathfrak{q}$. Tables 1 summarizes the line bundles $\mathcal{L}_s = \mathcal{L}(s\lambda_i)$ on which Ω_2 systems are conformally invariant. For the details of the table see Sections 4-7 of [5]. (Also see [7].)

Table 1. Line bundles with special values

Parabolic subalgebra	$\Omega_2\vert_{V(\mu+\epsilon_\gamma)^*}$	$\Omega_2\vert_{V(\mu+\epsilon_{n\gamma})^*}$
$B_n(i), 3 \leq i \leq n-2$	$\mathcal{L}\big((n-i-\frac{1}{2})\lambda_i\big)$	$\mathcal{L}(\lambda_i)$
$B_n(n-1)$	$\mathcal{L}\big(\frac{1}{2}\lambda_{n-1}\big)$	$\mathcal{L}(\lambda_{n-1})$
$B_n(n)$	$\mathcal{L}(-\lambda_n)$	–
$C_n(i), 2 \leq i \leq n-1$	$\mathcal{L}\big((n-i+1)\lambda_i\big)$	$\mathcal{L}(-\lambda_i)$
$D_n(i), 3 \leq i \leq n-3$	$\mathcal{L}\big((n-i-1)\lambda_i\big)$	$\mathcal{L}(\lambda_i)$
$E_6(3)$	$\mathcal{L}(\lambda_3)$	$\mathcal{L}(2\lambda_3)$
$E_6(5)$	$\mathcal{L}(\lambda_5)$	$\mathcal{L}(2\lambda_5)$
$E_7(2)$	$\mathcal{L}(2\lambda_2)$	–
$E_7(6)$	$\mathcal{L}(\lambda_6)$	$\mathcal{L}(3\lambda_6)$
$E_8(1)$	$\mathcal{L}(3\lambda_1)$	–
$F_4(4)$	$\mathcal{L}(-\lambda_4)$	–

In [6] and [7], for $k = 1, 2$, we classify the homomorphisms φ_{Ω_k} between the generalized Verma modules arising from the conformally invariant Ω_k system(s) as

Parabolic subalgebra	$\Omega_2\|_{V(\mu+\epsilon_\gamma)^*}$	$\Omega_2\|_{V(\mu+\epsilon^+_{n\gamma})^*}$	$\Omega_2\|_{V(\mu+\epsilon^-_{n\gamma})^*}$
$D_n(n-2)$	$\mathcal{L}(\lambda_{n-2})$	$\mathcal{L}(\lambda_{n-2})$	$\mathcal{L}(\lambda_{n-2})$

standard or non-standard. The map φ_{Ω_1} is standard for each parabolic subalgebra $\mathfrak{q}$ under consideration. Tables 2 exhibits the classification for φ_{Ω_2}.

TABLE 2. The classification of φ_{Ω_2}

Parabolic subalgebra	$\Omega_2\|_{V(\mu+\epsilon_\gamma)^*}$	$\Omega_2\|_{V(\mu+\epsilon_{n\gamma})^*}$
$B_n(i), 3 \le i \le n-2$	standard	non-standard
$B_n(n-1)$	standard	non-standard
$B_n(n)$	standard	–
$C_n(i), 2 \le i \le n-1$	non-standard	standard
$D_n(i), 3 \le i \le n-3$	non-standard	non-standard
$E_6(3)$	non-standard	non-standard
$E_6(5)$	non-standard	non-standard
$E_7(2)$	non-standard	–
$E_7(6)$	non-standard	non-standard
$E_8(1)$	non-standard	–
$F_4(4)$	standard	–

Parabolic subalgebra	$\Omega_2\|_{V(\mu+\epsilon_\gamma)^*}$	$\Omega_2\|_{V(\mu+\epsilon^+_{n\gamma})^*}$	$\Omega_2\|_{V(\mu+\epsilon^-_{n\gamma})^*}$
$D_n(n-2)$	non-standard	non-standard	non-standard

Now we have the following consequence:

Consequence. *Let $\mathfrak{q}$ be a maximal parabolic subalgebra of quasi-Heisenberg type. The map φ_{Ω_k} for $k = 1, 2$ is non-standard if and only if the special value s_k of an Ω_k system is a positive integer.*

Problem. Give a theoretical proof or understand mathematical significance of the consequence.

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Stepwise Square Integrable Representations of Nilpotent Lie Groups

Joseph A. Wolf

The talk started with a quick review of Kirillov's theory of unitary representations of nilpotent Lie groups [1], and my adaptation of this with Calvin Moore for square integrable (modulo the center) representations [2]. Then we described the conditions for a nilpotent Lie group to be foliated into subgroups that have square integrable (relative discrete series) unitary representations, that fit together to form a filtration by normal subgroups. We used that filtration to construct a class of "stepwise square integrable" representations on which Plancherel measure is concentrated and explicit. Further, we worked out the character formulae for those stepwise square integrable representations, and we gave an explicit polynomial Plancherel formula. Next, we used some structure theory to check that all these constructions and results apply to nilradicals of minimal parabolic subgroups of real reductive Lie groups. Finally, we developed multiplicity formulae, again explicit, for compact quotients N/Γ where Γ respects the filtration.

Very few real parabolic subgroups have nilradicals with square integrable representations [4], and those are generally maximal parabolics, so it was a surprise that all minimal parabolics have nilradicals with stepwise square integrable representations. Also, it is interesting that those nilradicals have such explicit Plancherel formulae, and that many of the technical results on lattice subgroups (see, for example, [3]) extend nicely to nilpotent groups with stepwise square integrable representations.

Commutative nilmanifolds are of the form $N \cdot K/K$ where K is compact and the normal subgroup N is 2–step nilpotent. Running through Vinberg's classification one sees that in all but a few cases N has square integrable representations. The application of stepwise square integrable representation theory may help to explain these few exceptions.

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Infinite-dimensional Schur–Weyl duality and the Coxeter–Laplace operator

Anatoly M. Vershik
(joint work with N.Tsilevich)

We extend the classical Schur–Weyl duality between representations of the groups $\mathrm{SL}(n,\mathbb{C})$ and $\mathfrak{S}_N$ to the case of $\mathrm{SL}(n,\mathbb{C})$ and the infinite symmetric group $\mathfrak{S}_{\mathbb{N}}$. Our construction is based on a "dynamic," or inductive, scheme of Schur–Weyl dualities. It leads to a new class of representations of the infinite symmetric group, which have not appeared earlier. We describe these representations and, in particular, find their spectral types with respect to the Gelfand–Tsetlin algebra. The main example of such a representation acts in an incomplete infinite tensor product. As an important application, we consider the weak limit of the so-called Coxeter–Laplace operator, which is essentially the Hamiltonian of the XXX Heisenberg model, in these representations.

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Koszul tensor categories of representations of Mackey Lie algebras and their dense subalgebras

Ivan Penkov
(joint work with Vera Serganova)

In [DPS] (see also Oberwolfach Report 51/2010, pp.3016 − 3018) we introduced the Koszul category of integrable representations $\mathbb{T}_{\mathfrak{g}}$ of the simple direct limit Lie algebras $\mathfrak{g} = \mathfrak{sl}(\infty) = \varinjlim \mathfrak{sl}(n), \mathfrak{g} = \mathfrak{o}(\infty) = \varinjlim \mathfrak{o}(n), \mathfrak{g} = \mathfrak{sp}(\infty) = \varinjlim \mathfrak{sp}(n)$. In this report we describe an extension of our results from [DPS] to a class of infinite-dimensional Lie algebras which we call Mackey Lie algebras. We also consider dense subalgebras of Mackey Lie algebras and draw corollaries about their respective categories of tensor representations.

The base field is $\mathbb{C}$. Let V and W be two vector spaces with a non-degenerate pairing $V \times W \to \mathbb{C}$. Then each of V and W can be considered as subspace of the dual of the other:

$$V \subset W^*, \ \ W \subset V^*.$$

Let $End_W\, V$ denote the algebra of endomorphisms $\phi : V \to V$ such that $\phi^*(W) \subset W$ where $\phi^* : V^* \to V^*$ is the dual endomorphism. Clearly, there is a canonical isomorphism of algebras

$$End_W\, V \xrightarrow{\sim} End_V\, W, \ \ \phi \longmapsto \phi^*_{|W}.$$

We call the Lie algebra associated with the associative algebra $End_W\, V$ a *Mackey Lie algebra* and denote it by $\mathfrak{m}_{V,W}$. G. Mackey calls the pair V, W a *linear system* and was the first to study linear systems in depth [M].

If V and W are countable dimensional, then up to isomorphism there is only one linear system [M]. In this case we set $V_* := W$ and write $\mathfrak{m}_{V,V_*}$. According to Mackey, there exists a basis $\{v_1, v_2, \dots\}$ of V such that $V_* = \text{span}\{v_1^*, v_2^*, \dots\}$, where $\{v_1^*, v_2^*, \dots\}$ is the set of linear functionals dual to $\{v_1, v_2, \dots\}$. Then it is easy to check that $\mathfrak{m}_{V,V_*}$ is identified with the Lie algebra of matrices $X = (x_{ij})_{i\geq 0, j\geq 0}$ such that each row and each column of X has finitely many non-zero entries. The Mackey Lie algebra $\mathfrak{m}_{V,V^*}$ (for a countable dimensional space V) is identified with the Lie algebra of matrices $X = (x_{ij})_{i\geq 0, j\geq 0}$ each column of which has finitely many non-zero entries. Alternatively, if a basis of V as above is enumerated by $\mathbb{Z}$ (i.e we consider a basis $\{v_j\}_{j\in\mathbb{Z}}$ such that $V_* = \text{span}\{v_j^*\}_{j\in\mathbb{Z}}$ where $v_j^*(v_i) = \delta_{i,j}$), then $\mathfrak{m}_{V,V_*}$ is identified with the Lie algebra of matrices $X = (x_{ij})_{i,j\in\mathbb{Z}}$ whose rows and columns have finitely many non-zero entries, and $\mathfrak{m}_{V,V^*}$ is identified with the Lie algebra of matrices $X = (x_{ij})_{i,j\in\mathbb{Z}}$ whose columns have finitely many non-zero entries.

The orthogonal and symplectic Mackey Lie algebras are introduced as follows. If V is a vector space endowed with a non-degenerate symmetric (respectively, antisymmetric) form, then $\mathfrak{om}_V$ (respectively, $\mathfrak{spm}_V$) is the Lie algebra

$$\{X \in \mathfrak{m}_{V,V} \,|\, (Xv, w) + (v, Xw) = 0 \ \forall\, v, w \in V\}.$$

If V is countable dimensional, there always is a basis $\{v_i, w_j\}_{i,j\in\mathbb{Z}}$ of V such that $\mathrm{span}\{v_i\}_{i\in\mathbb{Z}}$ and $\mathrm{span}\{w_j\}_{j\in\mathbb{Z}}$ are isotropic spaces and $(v_i, w_j) = \delta_{i,j}$ for $i \neq j$. The corresponding matrix form of $\mathfrak{om}_V$ consists of all block matrices

$$\left(\begin{array}{c|c} a_{ij} & b_{kl} \\ \hline c_{rs} & -a_{ji} \end{array}\right)$$

each row and column of which are finite and in addition $b_{kl} = -b_{lk}$, $c_{rs} = -c_{sr}$ where $i, j, k, l, r, s \in \mathbb{Z}$. The matrix form for $\mathfrak{spm}_V$ is similar: here $b_{kl} = b_{lk}$, $c_{rs} = c_{sr}$.

One verifies that there are the following exact sequences of Lie algebras:

$$0 \to V \otimes W \to \mathfrak{m}_{V,W} \to \mathfrak{m}_{V,W}/(V \otimes W) \to 0,$$
$$0 \to \Lambda^2 V \to \mathfrak{om}_V \to \mathfrak{om}_V/\Lambda^2 V \to 0,$$
$$0 \to S^2 V \to \mathfrak{spm}_V \to \mathfrak{spm}_V/S^2 V \to 0.$$

In what follows, $\tilde{\mathfrak{m}}$ stands for one of the Lie algebras $\mathfrak{m}_{V,W}, \mathfrak{om}_V, \mathfrak{sp}_V$, and $\mathfrak{m} \subset \tilde{\mathfrak{m}}$ is the respective subalgebra $V \otimes W, \Lambda^2 V$ or $S^2 V$. When V and W are countable dimensional we write $\tilde{\mathfrak{g}}$ instead of $\tilde{\mathfrak{m}}$, and $\mathfrak{g}$ instead of $\mathfrak{m}$. Note that in the letter case $V \otimes V_*$ is isomorphic to the Lie algebra $\mathrm{gl}(\infty)$ of finitary infinite matrices, while $\Lambda^2 V \simeq \mathfrak{o}(\infty)$ and $S^2 V \simeq \mathfrak{sp}(\infty)$ in the respective cases when V is endowed with a symmetric or simplectic non-degenerate form.

The following result describes the structure of the Mackey Lie algebras $\tilde{\mathfrak{g}}$.

Theorem 1. *a) Let V have basis $\{v_\alpha\}_{\alpha\in\mathbb{Z}}$ and $V_* := \mathrm{span}\{v^*_\alpha\}_{\alpha\in\mathbb{Z}}$ where $v^*_\alpha(v_\beta) = \delta_{\alpha,\beta}$. Set $C = \sum_\alpha v_\alpha \otimes v^*_\alpha$. Then $(V \otimes V_*) \oplus \mathbb{C}C$ is an ideal in $\mathfrak{m}_{V,V_*}$ and the quotient*

$$\mathfrak{m}_{V,V_*}/\left((V \otimes V_*) \oplus \mathbb{C}C\right)$$

is a simple Lie algebra.

b) If V is equipped with a non-degenerate symmetric (respectively, antisymmetric) bilinear form then $\mathfrak{om}_V/\Lambda^2 V$ (respectively $\mathfrak{spm}_V/S^2 V$) is a simple Lie algebra.

A Mackey Lie algebra $\mathfrak{m}_{V,W}$ has two *natural representations* V and W (which are non-isomorphic). For $\mathfrak{om}_V$ and $\mathfrak{spm}_V$, V is the (unique up to isomorphism) *natural representation.* We now define the category $\mathbb{T}_{\mathfrak{m}_{V,W}}$ (respectively $\mathbb{T}_{\mathfrak{om}_V}$ or $\mathbb{T}_{\mathfrak{spm}_V}$) of *finite-length tensor modules* as the full subcategory of the category $\mathfrak{m}_{V,W}-$mod (respectively, $\mathfrak{om}_V-$mod or $\mathfrak{spm}_V-$mod) whose objects are finite length subquotients of direct sums of copies of the tensor algebra $T(V \oplus W)$ (respectively $T(V)$). Note that $\mathbb{T}_{\mathfrak{m}_{V,W}}$ (respectively $\mathbb{T}_{\mathfrak{om}_V}$ and $\mathbb{T}_{\mathfrak{spm}_V}$) is a tensor category with respect to usual tensor product of $\mathfrak{m}_{V,W}-$modules (respectively $\mathfrak{om}_V-$ or $\mathfrak{spm}_V-$modules).

The following proposition provides equivalent characterizations of the category $\mathbb{T}_{\tilde{\mathfrak{g}}}$. We first need some definitions.

We call a subalgebra $\mathfrak{k}$ of a Mackey Lie algebra $\tilde{\mathfrak{g}}$ *large* if it contains the annihilator in $\tilde{\mathfrak{g}}$ of a pair of finite-dimensional subspaces $V' \subset V, W' \subset W$. A $\tilde{\mathfrak{g}}-$module M is $\mathfrak{g}-$*integrable* if any vector $m \in M$ generates a finite-dimensional $\mathfrak{g}_{fd}-$submodule of M where $\mathfrak{g}_{fd}$ is any finite dimensional subalgebra of $\mathfrak{g}$. A $\tilde{\mathfrak{g}}-$module M is an *absolute weight module* if for every splitting Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ (the definition see in [DPS]), M considered as an $\mathfrak{h}-$module is isomorphic to a direct sum of (not necessarily finite-dimensional) weight spaces.

Proposition 2. *The following conditions on a $\tilde{\mathfrak{g}}$-module M are equivalent:*
-M is an object of $\mathbb{T}_{\tilde{\mathfrak{g}}}$;
-M is a finite length $\mathfrak{g}-$integrable $\tilde{\mathfrak{g}}-$module and each vector $m \in M$ has a large annihilator in $\tilde{\mathfrak{g}}$;
-M is a finite length absolute weight-module.

Next we define dense subalgebras of Mackey Lie algebras. Let $\mathfrak{g}'$ be a Lie algebra and M be a $\mathfrak{g}'$-module. A Lie subalgebra $\mathfrak{g}'' \subset \mathfrak{g}'$ *acts densely* on M if for any finite set of vectors $m_1, \dots, m_n \in M$ and any $g' \in \mathfrak{g}'$ there is $g'' \in\gg''$ such that $g'' \cdot m_i = g' \cdot m_i$ for $i = 1, \dots, n$. We say that a Lie subalgebra $\mathfrak{a} \subset \mathfrak{m}_{V,W}$ (respectively $\mathfrak{a} \subset \mathfrak{om}_V$ or $\mathfrak{a} \subset \mathfrak{spm}_V$) is *dense in* $\mathfrak{m}_{V,W}$ if $\mathfrak{a}$ acts densely on $V \oplus W$ (respectively V).

Examples of dense subalgebras:

(1) For any Mackey Lie algebra $\tilde{\mathfrak{m}}$, the subalgebra $\mathfrak{m} \subset \tilde{\mathfrak{m}}$ is dense in $\tilde{\mathfrak{m}}$.
(2) Let $W = V_*$. Then the Jacobi Lie algebra $\mathfrak{j}_{V,V_*}$ consisting of matrices $J = (j_{kl})_{k,l\in\mathbb{Z}}$ such that $j_{kl} = 0$ when $|k-l| > m_J$ for some $m_J \in \mathbb{Z}_{>0}$, is dense in $\mathfrak{m}_{V,V_*}$.
(3) The Lie algebra $\mathfrak{sl}(2^\infty)$ (its definition see below) can be embedded into $\mathfrak{m}_{V,V_*}$ as a dense subalgebra. Recall that $\mathfrak{sl}(2^\infty)$ can be defined as the direct limit of the chain of inclusions
$$\mathfrak{sl}(2) \subset \mathfrak{sl}(4) \subset \dots \subset \mathfrak{sl}(2^n) \subset \mathfrak{sl}(2^{n+1}) \subset \dots$$
where $A \in \mathfrak{sl}(2^n)$ is being mapped to
$$\left(\begin{array}{c|c} A & 0 \\ \hline 0 & A \end{array}\right).$$
Notice that if $\phi_n : V_{2^n} \times V^*_{2^n} \to \mathbb{C}$ is the canonical pairing of the natural representation V_{2^n} of $\mathfrak{sl}(2^n)$ with its dual, then the mappings $\frac{\phi_n}{2^{n-1}}$ define a non-degenerate pairing between $\varinjlim V_{2^n}$ and $\varinjlim(V^*_{2^n})$, where V_{2^n} (respectively $V^*_{2^n}$) is embedded into $V_{2^{n+1}}$ (respectively $V^*_{2^{n+1}}$) via the formula $v \mapsto v \oplus v$. Since the pairing $(\varinjlim V_{2^n}) \times (\varinjlim V^*_{2^n}) \to \mathbb{C}$ is clearly $\mathfrak{sl}(2^\infty)-$invariant, we obtain an injective homomorphism
$$\mathfrak{sl}(2^n) \to \mathfrak{m}_{V,V_*}$$
for $V := \varinjlim V_{2^n}, V_* := \varinjlim(V^*_{2^n})$.

If $\mathfrak{a} \subset \tilde{\mathfrak{m}}$ is a dense subalgebra, a category $\mathbb{T}_{\mathfrak{a}}$ is defined in the same way as the category $\mathbb{T}_{\tilde{\mathfrak{m}}}$ via the tensor algebra $T(V \oplus W)$ (respectively $T(V)$ for $\tilde{\mathfrak{m}} = \mathfrak{om}_V, \mathfrak{spm}_V$).

The following is the main result of this report.

Theorem 3. *Let $\tilde{\mathfrak{m}}$ be a Mackey Lie algebra and $\mathfrak{a} \subset \tilde{\mathfrak{m}}$ be a dense subalgebra. Then the restriction functor*

$$R : \mathbb{T}_{\tilde{\mathfrak{m}}} \to \mathbb{T}_{\mathfrak{a}}$$

is an equivalence of tensor categories.

Corollary 4. *The subalgebra $\mathfrak{m} \subset \tilde{\mathfrak{m}}$ is dense in $\tilde{\mathfrak{m}}$, hence for any dense subalgebra $\mathfrak{a} \subset \tilde{\mathfrak{m}}$ the tensor categories $\mathbb{T}_{\mathfrak{m}}$ and $\mathbb{T}_{\mathfrak{a}}$ are equivalent.*

Corollary 5. *If V and $V_* = W$ are countable dimensional, then $\mathbb{T}_{\mathfrak{g}}$ is a Koszul tensor category in the sense of* [DPS] *and $T(V \oplus V_*)$ (respectively $T(V)$ for $\tilde{\mathfrak{g}} = \mathfrak{om}_V, \mathfrak{spm}_V$) is an injective cogenerator of $\mathbb{T}_{\mathfrak{g}}$. Hence the same applies to $\mathbb{T}_{\tilde{\mathfrak{g}}}$ and to $\mathbb{T}_{\mathfrak{a}}$.*

Corollary 6. *Under the assumption of Corollary 5, the description of irreducible objects of $\mathbb{T}_{\mathfrak{g}}$ from* [DPS]*, as well as the explicit computation of the dimensions of all* Ext*'s between irreducibles, applies to $\mathbb{T}_{\tilde{\mathfrak{g}}}$ and $\mathbb{T}_{\mathfrak{a}}$ for any dense subalgebra $\mathfrak{a} \subset \tilde{\mathfrak{g}}$.*

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Elliptic Lie Algebras and Gauge Invariant Functions

Kenji Iohara

(joint work with N. Suzuki, H. Terajima and H. Yamada)

A surface singularity $(X, *)$ is called a **simple singularity** if it is isomorphic to $(\mathbb{C}^2/\Gamma, \overline{0})$ for a finite subgroup Γ of SU_2. It is known that such singularities are classified in terms of simply laced Dynkin diagram, i.e., of type A_l, D_l and E_l. As one may expect, such a singularity admits a Lie theoretic construction.

Let G be a connected and simply connected simple Lie group over $\mathbb{C}$ of type A_l, D_l or E_l, $\mathfrak{g}$ be its Lie algebra, $\mathfrak{h} \subset \mathfrak{g}$ be a Cartan subalgebra and W be the Weyl group with respect to $\mathfrak{h}$. Recall that

$$\chi : \mathfrak{g} \longrightarrow \mathfrak{h}/W; \qquad x = x_s + x_n \longmapsto G.x_s \cap \mathfrak{h},$$

where $x = x_s + x_n$ signifies the Jordan-Chevalley decomposition with semi-simple x_s and nilpotent x_n, is called the **adjoint quotient map**. The next theorem was proved by E. Breiskorn [1]:

Theorem Let $x \in \mathfrak{g}$ be a **subregular** nilpotent element, i.e., a nilpotent element x such that $\dim Z_{\mathfrak{g}}(x) = \operatorname{rank} \mathfrak{g} + 2$. Let $S \subset \mathfrak{g}$ be a **transversal slice** of x to the G-orbit $G.x$, i.e., $S \subset \mathfrak{g}$ is a locally closed subvariety satisfying i) $x \in S$, ii) $G \times S \to \mathfrak{g}$; $(g, s) \mapsto \mathrm{Ad}(g)(s)$ is smooth, and iii) $\dim S = \operatorname{codim} G.x$. Then, $\chi|_S : (S, x) \to (\mathfrak{h}/W, \overline{0})$ is a semi-universal deformation of the corresponding singularity.

The proof of this theorem can be found in [4] and [5]. An explicit choice of a transversal slice, called a **Slodowy slice** nowadays, is also given.

Now, we consider a normal surface singularity $(X, *)$, called a **simple elliptic singularity** studied by K. Saito [3]. This is a singularity whose exceptional divisor of the minimal resolution consists of a single nonsingular elliptic curve. In particular, we are interested in the case when $(X, *)$ is an isolated hypersurface singularity; there are 3 such cases

$\widetilde{E_6}$	$x^3 + y^3 + z^3 + \lambda xyz = 0 \quad (\lambda \neq -3)$
$\widetilde{E_7}$	$x^4 + y^4 + z^2 + \lambda xyz = 0$
$\widetilde{E_8}$	$x^6 + y^3 + z^2 + \lambda xyz = 0$

The first aim of the work is to establish an analogous theorem to the above cited theorem of Brieskorn and Slodowy for these 3 singularities. For this purpose, we need

Step 1. the correct choice of a Lie algebra (and group),
Step 2. an analogue of the adjoint quotient map, and
Step 3. an analogue of a Slodowy slice.

The aim of this talk is to explain the **Step 1**. and the actual state of the **Step 2**. which is a *work in progress*.

Step 1. The geometry of principal G-bundles over $E_\tau := \mathbb{C}/\mathbb{Z} \oplus \tau\mathbb{Z}$ $(\mathrm{Im}\,\tau > 0)$ tells us what one should do.
Denote $z = x + \tau y \in \mathbb{C}$ $(x, y \in \mathbb{R})$, $\overline{\partial} = \tau\partial_x - \partial_y$, $w = e^{2\pi\sqrt{-1}z} \in \mathbb{C}^*$ and $q := e^{2\pi\sqrt{-1}\tau}$. Set

$$\mathcal{C}(\mathfrak{g})_\tau = \{\overline{\partial}\text{-connections on } G\text{-bundles over } E_\tau\},$$
$$L_{\mathrm{hol}}(G) := \{g : \mathbb{C}^* \to G;\ \text{holomorphic}\}.$$

Notice that $\mathcal{C}(\mathfrak{g})_\tau \cong \mathcal{E}(\mathfrak{g}) := C^\infty(E_\tau, \mathfrak{g})$ as affine space and the latter with point-wise Lie algebra structure is called an **elliptic Lie algebra**. It is known that the orbits of

(1) $L_{\mathrm{hol}}(G)$ by the q-twisted conjugation of $L_{\mathrm{hol}}(G)$: $(g.h)(w) := g(qw)h(w)g(w)^{-1}$, and of
(2) $\mathcal{C}(\mathfrak{g})_\tau$ by the gauge action of $\mathcal{E}(G)$: $(\overline{\partial}, A).g = (\overline{\partial}, \mathrm{Ad}(g^{-1})A + g^{-1}\overline{\partial}g)$,

both parametrize the isomorphism classes of holomorphic principal G-bundles over E_τ. See [2] and references therein, for detail.

This suggests that the right choice of a Lie algebra like object is $\mathcal{C}(\mathfrak{g})_\tau$ and of a Lie group is $L_{\mathrm{hol}}(G)$. But from view point of *invariant theory*, this is not a good choice; we need a $\mathbb{C}^*$-`extended` version of these objects, i.e., a **holomorphic affine Lie group** and a **$\mathbb{C}^*$-fibration** $\mathbb{C}^* \to \widetilde{\mathcal{C}}(\mathfrak{g})_\tau \to \mathcal{C}(\mathfrak{g})_\tau$!

Step 2. The gauge action of $\mathcal{E}(G)$ on $\mathcal{C}(\mathfrak{g})_\tau$ lifts to $\widetilde{\mathcal{C}}(\mathfrak{g})_\tau$ and an analogue of the adjoint quotient map should be an $\mathcal{E}(G)$-invariant map $\chi : \widetilde{\mathcal{C}}(\mathfrak{g})_\tau \to \mathbb{C}^{\mathrm{rank}\,\mathfrak{g}+1}$, *if it exists.* We observe that the restriction of χ on $(\overline{\partial}, \mathfrak{h}) \times \mathbb{C}^* \subset \widetilde{\mathcal{C}}(\mathfrak{g})_\tau$ can be given by the characters of fundamental representations over the corresponding affine Lie algebras. In our *work in progress*, we try showing the global existence of such *holomorphic* maps (in the Fréchet sense). Suppose that this has been proved. Then, we can deduce the following facts:

(1) The moduli space of holomorphic semi-stable principal G-bundles over E_τ is isomorphic to the weighted projective space $\mathbb{P}(a_0^\vee, a_1^\vee, \cdots, a_{\mathrm{rank}\,\mathfrak{g}}^\vee)$, where $a_i^\vee$ are the **co-labels** of the corresponding affine Lie algebra, and
(2) $\chi^{-1}(0)$ is the **unstable locus** of $\widetilde{\mathcal{C}}(\mathfrak{g})_\tau$.

Thus, the unstable locus of $\widetilde{\mathcal{C}}(\mathfrak{g})_\tau$ plays a role of the **nilpotent cone** of $\mathfrak{g}$ in the classical setting and the singularity in question should be living there.

Step 3. At least, we have a candidate...

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Classical algebras as graded commutative algebras

Valentin Ovsienko

The starting point of this work is the following observation, see [5]. The quaternion algebra $\mathbb{H}$ is $\mathbb{Z}_2^3$-commutative in the following sense. Associate to the elements of standard basis $\{1, i, j, k\}$ of $\mathbb{H}$ the following elements of $\mathbb{Z}_2^3$:

$$\bar{1} = (000), \quad \bar{i} = (011), \quad \bar{j} = (101), \quad \bar{k} = (110).$$

The usual product of quaternions then satisfies the condition of *graded commutative algebra*, i.e.,

$$b \cdot a = (-1)^{\langle \tilde{a}, \tilde{b} \rangle} a \cdot b, \tag{1}$$

where $\langle\,,\,\rangle$ is the usual *scalar product* of 3-vectors. This grading of $\mathbb{H}$ is different from the usual $\mathbb{Z}_2^2$-grading [4, 1].

It turns out that the above example is universal. The following result was obtained in [6].

Theorem 1. *If the abelian group $\mathcal{G}$ is finitely generated, then for an arbitrary $\mathcal{G}$-commutative algebra $\mathcal{A}$, there exists n such that $\mathcal{A}$ is $(\mathbb{Z}_2)^n$-commutative in the sense of (1), where the bilinear map $\langle\,,\,\rangle$ is the usual scalar product on $(\mathbb{Z}_2)^n$.*

The above grading can be easily extended to any Clifford algebra. We then show that every finite-dimensional simple associative $\mathcal{G}$-commutative algebra over $\mathbb{C}$ or over $\mathbb{R}$ is isomorphic to a Clifford algebra.

The problem of developing linear algebra with coefficients in the algebra of quaternions is a classical problem formulated by Cayley. This subject was intensively developed by many authors including Study, Dieudonné, Gelfand,... The commonly accepted version of quaternionic determinant is that of Dieudonné [3]. However, the Dieudonné determinant lacks almost all usual properties of the determinant, such as linearity in lines and columns, it is not polynomial. Therefore it is not related to any notion of trace.

A new approach using the notion of graded commutative algebras was recently developed in [2]. Every matrix with coefficients in $\mathbb{H}$ (or, more generally, in $\mathrm{Cl}_{p,q}$) corresponds to a homomorphism of graded modules and is written in the form:

$$A = \left(\begin{array}{c|c|c} A_{11} & \dots & A_{1p} \\ \hline \cdots & \cdots & \cdots \\ \hline A_{p1} & \dots & A_{pp} \end{array} \right), \tag{2}$$

One then obtains a natural notion of trace generalizing that of super trace.

Theorem 2. *There exists a unique (up to multiplication by a scalar of degree* 0*)* $\mathrm{Cl}_{p,q}$*-linear map* $\mathrm{gtr} : \mathrm{Mat}(N, \mathcal{A}) \to \mathcal{A}$, *such that* $\mathrm{gtr}\,([A, B]) = 0$. *It is defined for a homogeneous matrix* A *by*

$$\mathrm{gtr}(A) = \sum_{k=1}^{p} (-1)^{\langle x_k + \bar{A},\, x_k \rangle} \,\mathrm{tr}(A_{kk}) \,,$$

where x_k *are the degrees of the components of the module,* tr *is the usual trace and* A_{kk} *are diagonal blocks of* A*.*

Furthermore, one has the notion of graded determinant and Berezinian related to the draded trace by the usual Liouville formula. It is interesting that one recovers the Dieudonné determinant in the particular case of homogeneous matrices of degree 0.

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Spherical functions on compact Gelfand pairs of rank one

Maarten van Pruijssen
(joint work with Gert Heckman)

The K-biinvariant functions on a compact connected Riemannian symmetric space of rank one can be identified with families of Jacobi polynomials. The members of such a family are orthogonal, they satisfy a recurrence relation and they are eigenfunctions of a hypergeometric differential operator.

We generalize this construction. Let (G, K) be a pair of compact Lie groups with G simple and K connected and let P_G^+ and P_K^+ be two choices of sets of dominant integral weights. Given an irreducible representation π of G of highest weight $\lambda \in P_G^+$ and τ an irreducible representation of K of highest weight $\mu \in P_K^+$ we denote by $m_\lambda^{G,K}(\mu)$ the number of times that μ occurs in the decomposition of the restriction $\pi|_K$ of π to K.

A triple (G, K, τ) is called a multiplicity free triple if $m_\lambda^{G,K}(\mu) \leq 1$ for all $\lambda \in P_G^+$. If we can vary $\mu \in P_K^+$ in a certain face F of the P_K^+ then we call (G, K, F) a multiplicity free system. By a face $F \in P_K^+$ we mean the integral span of a number of fundamental weights of K in P_K^+.

If (G, K, F) is a multiplicity free system then (G, K) is automatically a Gelfand pair. We classified the multiplicity free systems with (G, K) a Gelfand pair of rank one in [6] and the forthcoming paper [5], the result is in Table 3. The spherical weight $\lambda_{\mathrm{sph}} \in P_G^+$ parametrizes all the irreducible G-representations that contain the trivial K-representation upon restriction.

For each item in Table 3 we determine the set

$$P_G^+(\mu) = \{\lambda \in P_G^+ | m_\lambda^{G,K}(\mu) = 1\}.$$

In particular, $P_G^+(0) = \mathbb{N}\lambda_{\mathrm{sph}}$. In the generality of Table 3 we need a bit of structure to determine the sets $P_G^+(\mu)$. Let $A \subset G$ be a circle outside K and let $M = Z_K(A)$ be the centralizer of A in K. Let $T_M \subset M$ be a maximal torus. The torus $T_M A \subset G$ is maximal and if we use this torus to describe the representations of G then we can decompose the highest weight $\lambda = \lambda_A + \lambda_M \in P_G^+$ of a G-type π into an A- and an M-part. We show that the M-part is the highest weight of an M-type that occurs in any K-representation τ that occurs in π.

On the other hand, if $\lambda \in P_G^+(\mu)$ then so is $\lambda + \lambda_{\mathrm{sph}}$. This shows that the set $P_G^+(\mu)$ is determined by the M-types that occur in μ and the spherical direction λ_{sph}. In the case $P_G^+(0)$ one considers the tensor products $\pi_\lambda \otimes \pi_{\lambda_{\mathrm{sph}}}$ of G-representations with weights $\lambda \in P_G^+(0)$ and $\pi_{\lambda_{\mathrm{sph}}}$ the irreducible G-representation of highest weight λ_{sph}. The decomposition of these tensor products lead to a three term recurrence relation for the involved spherical functions, see e.g. [7].

We show that we get a recurrence relation for the spherical functions of type μ if we multiply with the zonal spherical function that is associated to λ_{sph}. By analyzing the sets $P_G^+(\mu)$ we obtain information about the set of μ-spherical functions. It turns out that that this set can be described in terms of vector valued polynomials. By arranging the vector valued polynomials of degree d in a matrix

G	K	λ_{sph}	faces F
$\text{SU}(n+1)$	$\text{U}(n)$	$\varpi_1 + \varpi_n$	any
$\text{SO}(2n)$	$\text{SO}(2n-1)$	ϵ_1	any
$\text{SO}(2n+1)$	$\text{SO}(2n)$	ϖ_1	any
$\text{Sp}(2n)$	$\text{Sp}(2n-2) \times \text{Sp}(2)$	ϖ_2	$\dim F \leq 2$
F_4	$\text{Spin}(9)$	ϖ_1	$\dim F \leq 1$ or $F = \langle \omega_1, \omega_2 \rangle$
$\text{Spin}(7)$	G_2	ϖ_3	$\dim F \leq 1$
G_2	$\text{SU}(3)$	ϖ_1	$\dim F \leq 1$

Table 3. Classification of multiplicity free systems of rank one.

we obtain families of matrix valued polynomials. These families have properties that are similar to the properties of families of Jacobi polynomials: they satisfy orthogonality relations (deduced from Schur orthogonality), the satisfy a three-term-recurrence relation (deduced from the decomposition of the tensor products) and they are simultaneous eigenfunctions for a commutative algebra of differential operators.

The description of the families of spherical functions can become fairly explicit, see e.g. [1], [4], [2] and [3]. The construction that we described above depends heavily on the structure of the set $P_G^+(\mu)$. Ineed, the structure of $P_G^+(\mu)$ manages the degree of the involved polynomials. It would be desirable to have a better understanding of this structure, instead of the case by case analysis that we have used so far. This may then lead to a generalization of this construction to families of matrix valued polynomials in multiple variables, possessing nice properties that can be deduced from the Lie theory.

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Cartan decompositions for spherical spaces

Bernhard Krötz
(joint work with Thomas Danielson, Henrik Schlichtkrull)

Let G be a real reductive group, and H a reductive subgroup of G. We call the homogeneous space G/H *spherical* provided H admits an open orbit on G/P, where $P < G$ is a minimal parabolic subgroup. According to Wolf all symmetric spaces are spherical.

Examples of non-symmetric spaces are provided by triple spaces: $G = G_0 \times G_0 \times G_0$ and $H = \text{diag}(G_0)$ for

$$G_0 = \text{SL}(2,\mathbb{R}),\ \text{SL}(2,\mathbb{C}),\ \text{SO}_e(n,1) \quad (n = 2, 3, \dots) \tag{1}$$

It is interesting, in the non-symmetric setting, to explore properties which play an important role for the harmonic analysis on symmetric spaces.

One important structural result for symmetric spaces is the polar decomposition $G = KAH$. Here $K \subset G$ is a maximal compact subgroup, and $A \subset G$ is abelian. Polar decomposition for a Riemannian symmetric space G/K is due to Cartan, and it was generalized to reductive symmetric spaces in the form $G = KAH$ by Flensted-Jensen.

For triple spaces we show that indeed these spaces admit a polar decomposition as above, and we determine precisely for which maximal split abelian subgroups A the decomposition is valid. For the simplest choice of group A we describe the indeterminateness of the A-component for a given element in G, and we identify the invariant measure on G/H in these coordinates.

We conclude that there exist maximal split abelian subgroups A for which $G = KAH$, and for which PH is open for all minimal parabolic subgroups P with $P \supset A$, a property which plays an important role for estimating matrix coefficients on G/H [2].

An interesting observation (which surprised us) is that in some cases there are also maximal split abelian subgroups A for which PH is open for all minimal parabolic subgroups P with $P \subset A$, but for which the polar decomposition fails.

Further, we introduce an infinitesimal version of the polar decomposition, and show that in the current setting it is valid if and only if the global polar decomposition $G = KAH$ is valid.

For more general spherical spaces it is not clear whether polar decompositions exist. However, in case G is quasisplit and $H < G$ self-normalizing, then one can show that there exists a compact subset $\Omega \subset G$ and finitely many $A_1, \dots, A_n$ such that

$$\bigcup_{j=1}^{n} \Omega A_j H = G$$

and with $P_j A_j H$ open for all parabolics P_j above A_j, $j = 1, \dots, n$.

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A characterization of non-tube type Hermitian symmetric spaces by visible actions

Atsumu Sasaki

1. Introduction

Let us begin the talk with two facts on the relation between the multiplicity of a representation and the geometry.

The first fact is proved by T. Kobayashi and T. Oshima [9]. Let G be a reductive algebraic group and H its reductive algebraic subgroup. Then, for a pair (G,H), the homogeneous space G/H is *real spherical*, namely, there exists an open P-orbit in G/H where P is a minimal parabolic subgroup of G, if and only if the dimension of the intertwiners for any irreducible admissible representation $\pi \in \widehat{G}_{\mathrm{ad}}$ into the space $C^\infty(G/H)$ of smooth functions on G/H is finite, namely, $\dim \mathrm{Hom}(\pi, C^\infty(G/H)) < \infty$. Moreover, the complexified $G_{\mathbb{C}}/H_{\mathbb{C}}$ of G/H is *spherical*, namely, $G_{\mathbb{C}}/H_{\mathbb{C}}$ has an open Borel-orbit, if and only if $\sup_{\pi\in\widehat{G}_{\mathrm{ad}}} \dim \mathrm{Hom}(\pi, C^\infty(G/H)) < \infty$.

The second fact is concerned with the complex geometry as follows. Let H be a Lie group and $\mathcal{V}$ a H-equivariant Hermitian holomorphic vector bundle over a complex manifold D. Now, we consider a unitary representation $\mathcal{H}$ which is realized in the space $\mathcal{O}(D,\mathcal{V})$ of holomorphic sections of $\mathcal{V} \to D$. When is $\mathcal{H}$ multiplicity-free? For this, we consider a unitary representation of the isotropy subgroup H_x at $x \in D$ on the fiber $\mathcal{V}_x$. In general, the property of multiplicity-freeness of the unitary representation $\mathcal{H}$ is not fulfilled although $\mathcal{V}_x$ is multiplicity-free as a representation of H_x for any $x \in D$. However, this does hold if H acts on the base space D in a strongly visible fashion. This theory is established by T. Kobayashi (see [4, 5, 8]) and called *propagation theorem of multiplicity-freeness property* from fibers to $\mathcal{O}(D,\mathcal{V})$. This idea goes back to Gelfand–Kazhdan, S. Kobayashi [3], and Faraut–Thomas [1].

In summary, we have seen two different types of theorems, but both showing interaction between the multiplicity of representations and geometry. The common point is that one can expect some slice, or in other words, a kind of Cartan decomposition. In my talk, we discuss some family of examples which sit in the intersection of these two facts.

2. Visible actions on complex manifolds

A holomorphic action of a Lie group H on a connected complex manifold D is called *strongly visible* if there exists a *slice* S in D and an anti-holomorphic diffeomorphism σ on D satisfying the following conditions:

(V.1) $$S \text{ meets every } H\text{-orbit in } D,$$

(S.1) $$\sigma|_S = \mathrm{id}_S,$$

(S.2) $$\sigma \text{ preserves each } H\text{-orbit in } D.$$

We allow that S meets every H-orbit twice or more, i.e. S is not necessarily a complete representative of H-orbits in D.

In the Kobayashi's original definition [5, Definition 3.3.1], the concept of strongly visible actions is slightly broader, namely, he calls that this action is strongly visible if a complex manifold D contains an open set satisfying the conditions (V.1)–(S.2). For simplicity, we adopt this one in my talk.

3. A characterization of non-tube type Hermitian symmetric spaces by visible actions

Let G/K be a bounded symmetric domain. Since K is not semisimple, K has a non-trivial center. It is known that the dimension of its center is one. Then, G/K is a Hermitian symmetric space of non-compact type. We set $K^s := [K, K]$ which does not coincides with K. Then, we consider the complexification $G_{\mathbb{C}}/K^s_{\mathbb{C}}$ of G/K^s which is non-symmetric. By Matsushima's theorem, $G_{\mathbb{C}}/K^s_{\mathbb{C}}$ is a Stein manifold.

We take a compact real form G_u of $G_{\mathbb{C}}$.

Theorem 3.1 ([12, Theorem 1.1])**.** *The following are equivalent for non-compact irreducible Hermitian symmetric space G/K:*

(1) *The G_u-action on $G_{\mathbb{C}}/K^s_{\mathbb{C}}$ is strongly visible.*
(2) *G/K is of non-tube type.*

Let us illustrate our construction of a slice for the G_u-action on $G_{\mathbb{C}}/K^s_{\mathbb{C}}$ in case of non-tube type G/K. For this, we find a subset A satisfying the group decomposition $G_{\mathbb{C}} = G_u A K^s_{\mathbb{C}}$.

The key idea for finding such an A is as follows. Due to Flensted–Jensen [2], we have a Cartan decomposition for any symmetric pair (G, H) of a reductive Lie group G, namely, there exists an abelian B such that $G = KBH$. If $H = K$, then $G = KBK$ is nothing but the classical Cartan decomposition. Further, we use the herringbone stitch method which was initiated by Kobayashi in [6].

We return to the setting of Theorem 3.1. Since $(G_{\mathbb{C}}, K_{\mathbb{C}})$ is a symmetric pair, there exists an abelian A_1 whose dimension equals the rank of the symmetric space G/K such that $G_{\mathbb{C}} = G_u A_1 K_{\mathbb{C}}$. Next, we find one-dimensional subgroups $Z_{\mathbb{T}}, Z_{\mathbb{R}}$ of the centralizer $Z_{K_{\mathbb{C}}}(A_1)$ which are not contained in the semisimple part $K^s_{\mathbb{C}}$ such that $K_{\mathbb{C}} = Z_{\mathbb{T}} Z_{\mathbb{R}} K^s_{\mathbb{C}}$ and $Z_{\mathbb{T}} \subset G_u$. There exists such $Z_{\mathbb{T}}$ and $Z_{\mathbb{R}}$ if G/K is of non-tube type, and conversely. Using the herringbone stitch method, we get

$$G_{\mathbb{C}} = G_u A_1 K_{\mathbb{C}} = G_u A_1 (Z_{\mathbb{T}} Z_{\mathbb{R}} K_{\mathbb{C}}^s) = G_u Z_{\mathbb{T}} (A_1 Z_{\mathbb{R}}) K_{\mathbb{C}}^s = G_u (A_1 Z_{\mathbb{R}}) K_{\mathbb{C}}^s.$$

Hence, we take A as $A := A_1 Z_{\mathbb{R}} = Z_{\mathbb{R}} A_1$. It follows from our construction that A is an abelian. Therefore, we conclude:

Corollary 3.2 ([12, Corollary 4.1]). *For a non-tube type Hermitian symmetric space G/K, we can find an abelian A with dimension* $\operatorname{rank} G/K + \dim Z(K)$ *such that we have a generalization of a Cartan decomposition $G_{\mathbb{C}} = G_u A K_{\mathbb{C}}^s$. In particular, the A-orbit is a slice for the strongly visible G_u-action on $G_{\mathbb{C}}/K_{\mathbb{C}}^s$.*

Finally, we sketch the proof of the implication (2) $\Rightarrow$ (1), namely, $G_{\mathbb{C}}/K_{\mathbb{C}}^s$ is spherical if G/K is of non-tube type. Suppose that the G_u-action on $G_{\mathbb{C}}/K_{\mathbb{C}}^s$ is strongly visible. It follows from propagation theorem of multiplicity-freeness property (see Section 1) that the Hilbert space $L^2(G_u/K^s)$ of square integrable functions on the compact homogeneous space G_u/K^s is multiplicity-free as a representation of G_u. Thanks to Vinberg–Kimelfeld [14], $G_{\mathbb{C}}/K_{\mathbb{C}}^s$ is a spherical variety. In view of the classification of irreducible spherical varieties due to Krämer [10], G/K is of non-tube type.

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Semibounded non-Fock representations of generalized oscillator groups

Christoph Zellner

The aim of this talk is to report on a certain class of representations of (generalized) oscillator groups. These groups are defined as follows: Let (V, ω) be a locally convex symplectic vector space and $\gamma : \mathbb{R} \to \mathrm{Sp}(V)$ be a one-parameter group of symplectomorphisms defining a smooth action of $\mathbb{R}$ on V. The Lie group

$$G := \mathrm{Heis}(V, \omega) \rtimes_\gamma \mathbb{R}$$

is called an *oscillator group*, where $\mathrm{Heis}(V, \omega) = \mathbb{R} \times_\omega V$ is the Heisenberg group. We briefly explain the concept of semibounded representations as introduced in [5]. Let $\pi : G \to \mathrm{U}(\mathcal{H})$ be a smooth unitary representation and $\mathtt{d}\pi : \mathfrak{g} \to \mathrm{End}(\mathcal{H}^\infty)$ its derived representation. Then π is called *semibounded* if the self-adjoint operators $\overline{i\mathtt{d}\pi(x)}$ are uniformly bounded above for x in a non-empty open subset of the Lie algebra $\mathfrak{g}$. As a first result, it was shown in [6] that every oscillator group G, which has a semibounded representation π satisfying $[\mathfrak{g}, \mathfrak{g}] \not\subseteq \ker \mathtt{d}\pi$, embeds into a standard oscillator G_A. A *standard oscillator group* is of the form $G_A = \mathrm{Heis}(C^\infty(A), \omega_A) \rtimes_\gamma \mathbb{R}$, where γ is a unitary one-parameter group on a Hilbert space H, $C^\infty(A)$ is the space of γ-smooth vectors equipped with its natural C^∞-topology, $\omega_A(x, y) = \mathrm{Im}\langle Ax, y\rangle$ and A is the self-adjoint generator of γ satisfying $A \geq 0$ and $\ker A = \{0\}$. Further considerations show that, under some mild assumption on G, every semibounded representation π of G extends to a semibounded representation of G_A. Hence the standard oscillator groups are those which are relevant to get an understanding of semibounded representations. Moreover, one may assume that π is *normalized* which means that $\mathtt{d}\pi(1, 0, 0) = i\mathbf{1}$, and $\sup \mathrm{Spec}(i\mathtt{d}\pi(0, 0, 1)) = 0$.

The canonical example of a normalized irreducible semibounded representation of G_A is the Fock representation, and if $\inf \mathrm{Spec} A > 0$, this turns out to be the only one. If $\inf \mathrm{Spec} A = 0$, then twisting the Fock representation with certain automorphisms of G_A already yields examples which are not equivalent to the Fock representation. To obtain further examples of non-Fock representations, we shall consider holomorphic extensions of semibounded representations of G_A to certain complex semigroups.

In general, the group G_A does not have a complexification. However, by passing to the space $V^{\mathcal{O}} \subset V := C^\infty(A)$ of γ-holomorphic vectors equipped with its natural topology, we obtain the dense subgroup $G_A^{\mathcal{O}} := \mathrm{Heis}(V^{\mathcal{O}}, \omega_A) \rtimes_\gamma \mathbb{R}$ of G_A. The group $G_A^{\mathcal{O}}$ has a complexification

$$G_{A,\mathbb{C}} := \mathbb{C} \times_{(\omega_A)_\mathbb{C}} V_\mathbb{C}^{\mathcal{O}} \rtimes_{\gamma_\mathbb{C}} \mathbb{C},$$

which contains the open subsemigroup $S_A := \{(z, x, s) \in G_{A,\mathbb{C}} : \mathrm{Im}(s) > 0\}$. Consider the cone $W := \mathbb{R} \times V^{\mathcal{O}} \times]0, \infty[$ in the Lie algebra $\mathfrak{g}_A$ of G_A. Then we have the polar decomposition

$$\psi : G_A^{\mathcal{O}} \times W \to S_A, (g, w) \mapsto g \exp(iw)$$

of S_A, where ψ is a diffeomorphism and both ψ and ψ^{-1} are analytic. For a normalized semibounded representation π of G_A it turns out that $i\mathtt{d}\pi(x)$ is bounded above for all $x \in W$ and the following extension theorem holds:

Theorem 1. *For a normalized semibounded representation π of G_A the map*

$$\hat{\pi} : S_A \to B(\mathcal{H}), g\exp(iw) \to \pi(g)e^{i\mathtt{d}\pi(w)}$$

is a holomorphic representation of the complex semigroup S_A.

With the help of the preceding theorem, many semibounded non-Fock representations of standard oscillator groups can be obtained by extending continuous representations of countably infinite dimensional Heisenberg groups. This was described in the talk in more detail. Let us present the main result in the following:

Let (V, ω) be a countably infinite dimensional symplectic vector space. It is well known that V admits the structure of a pre-Hilbert space with $\omega(x, y) = \mathrm{Im}\langle x, y\rangle$. Moreover V contains an orthonormal basis $e_n, n \in \mathbb{N}$, and we equip $V = \varinjlim V_n$, where $V_n := \mathrm{span}\{e_1, \dots, e_n\}$, with the direct limit topology. We call a unitary representation $\pi : \mathrm{Heis}(V, \omega) \to \mathrm{U}(\mathcal{H})$ *regular* if it is continuous (w.r.t. the direct limit topology on V) and satisfies $\mathtt{d}\pi(1, 0, 0) = i\mathbf{1}$. Then the derived representation $\mathtt{d}\pi$ gives rise to the canonical commutation relations.

Theorem 2. *Let $\pi : \mathrm{Heis}(V, \omega) \to \mathrm{U}(\mathcal{H})$ be a regular representation. Then there exist a standard oscillator group $G_A = \mathrm{Heis}(C^\infty(A), \omega_A) \rtimes_\gamma \mathbb{R}$ and a dense inclusion $\iota : V \hookrightarrow C^\infty(A)$ such that the induced inclusion*

$$\mathrm{Heis}(V, \omega) \hookrightarrow G_A, (t, x) \mapsto (t, \iota(x), 0)$$

is a morphism of Lie groups. Moreover there exists a semibounded representation $\tilde{\pi} : G_A \to \mathrm{U}(\mathcal{H})$ such that $\tilde{\pi}|_{\mathrm{Heis}(V,\omega)} = \pi$ and the von Neumann-algebras generated by π and $\tilde{\pi}$ coincide.

There exist very many non-equivalent irreducible regular representations of $\mathrm{Heis}(V, \omega)$, cf. e.g. [2]. Moreover $\mathrm{Heis}(V, \omega)$ has regular factor representations of type II and III, cf. [3], [1], [4]. Hence the preceding theorem entails that there exist many non-equivalent semibounded irreducible representations as well as semibounded type II and III representations of certain oscillator groups.

The results presented in this talk can be found in [7].

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Regularity properties of infinite-dimensional Lie groups

Helge Glöckner

Let G be a Lie group modelled on a locally convex space, with neutral element e and Lie algebra $\mathfrak{g} := L(G) := T_e(G)$. To $g \in G$ and a tangent vector $v \in T_g(G)$, associate $v.g^{-1} := T_g\rho_{g^{-1}}(v) \in \mathfrak{g}$, using right translation $\rho_{g^{-1}}: G \to G$, $x \mapsto xg^{-1}$. Let $k \in \mathbb{N}_0 \cup \{\infty\}$. The Lie group G is called C^k*-regular* if the initial value problem

$$\eta'(t).\eta(t)^{-1} = \gamma(t), \quad \eta(0) = e$$

has a (necessarily unique) C^{k+1}-solution $\mathrm{Evol}_G(\gamma) := \eta\colon [0,1] \to G$ for each C^k-curve $\gamma\colon [0,1] \to \mathfrak{g}$, and the map

$$\mathrm{evol}_G\colon C^k([0,1],\mathfrak{g}) \to G, \quad \mathrm{evol}_G(\gamma) := \mathrm{Evol}_G(\gamma)(1) \tag{1}$$

is smooth (if $k = \infty$, then G is simply called *regular*). If G is regular and H a 1-connected Lie group with Lie algebra $\mathfrak{h}$, then every continuous Lie algebra homomorphism $\mathfrak{h} \to \mathfrak{g}$ is the tangent map of a smooth group homomorphism $H \to G$, by a famous theorem of John Milnor from 1984 – with many applications.

Besides basic facts, this note compiles examples of C^k-regular Lie groups, along with applications of regularity and C^k-regularity. It should be mentioned that the C^k-maps we use are so-called "Keller C^k_c-maps." For regularity in the inequivalent "convenient setting" of analysis, see works of Kriegl, Michor and Teichmann.

1. New regular Lie groups from given ones.

(a) *Let G be a C^k-regular Lie group and $H \subseteq G$ be a Lie subgroup* (*or initial Lie subgroup*) *of G such that $H = \{x \in G\colon (\forall j \in J)\ \phi_j(x) = \psi_j(x)\}$ for some families $(\phi_j)_{j\in J}$ and $(\psi_j)_{j\in J}$ of smooth homomorphisms from G to Lie groups. Then also H is C^k-regular.*

Applying this to complex conjugation σ, we get one half of:

(b) *If G is a real analytic Lie group which is a real Lie subgroup of a complex Lie group $G_{\mathbb{C}}$ with $L(G_{\mathbb{C}}) = L(G)_{\mathbb{C}}$ and admitting an anti-holomorphic involutive automorphism σ with fixed point set G, then $G_{\mathbb{C}}$ is C^k-regular if and only if G is C^k-regular and* evol_G (*as in* (1)) *is real analytic.*

The same conclusion holds if G is a real analytic *local* Lie group and $G_{\mathbb{C}}$ a complex analytic local Lie group with Lie algebra $L(G)_{\mathbb{C}}$. The definition of C^k-regularity for local Lie groups is as above, except that Evol_G only needs to be defined on an open 0-neighbourhood in $C^k([0,1],\mathfrak{g})$.

(c) *Let G be a Lie group G which is a projective limit of C^k-Lie groups G_j as a C^∞-manifold and a C^{k+1}-manifold, such that all bonding maps and all limit maps $q_j\colon G \to G_j$ are smooth homomorphisms and the maps $L(q_j)$ separate points on $L(G)$ (e.g., $L(G)$ might be the projective limit of the $L(G_j)$). Then G is C^k-regular.*

2. Some classical examples. The theory of ODEs in Banach spaces implies that every Banach-Lie group is C^0-regular. In particular, $C^\ell(M,H)$ is C^0-regular for each Banach-Lie group H, compact smooth manifold M and $\ell \in \mathbb{N}_0$. Using 1.c, we see that also the projective limit $C^\infty(M,H)$ of the $C^\ell(M,H)$ is C^0-regular (as shown first by Omori and co-workers, who also proved that $\mathrm{Diff}(M)$ is C^0-regular). More generally, if $r,k \in \mathbb{N}_0 \cup \{\infty\}$ and H is an arbitrary C^k-regular Lie group with Lie algebra $\mathfrak{h}$, one can use recent exponential laws [1] to identify $C^k([0,1],C^r(M,\mathfrak{h}))$ with $C^r(M,C^k([0,1],\mathfrak{h}))$ and finds that $G := C^r(M,H)$ is C^k-regular because evol_G can be identified with the smooth map $C^r(M,\mathrm{evol}_H)$.

3. Further examples.

(a) Direct limits $G = \bigcup_{n\in\mathbb{N}} G_n$ of finite-dimensional Lie groups $G_1 \subseteq G_2 \subseteq \cdots$ (such that all inclusions are smooth homomorphisms) are C^1-regular (as shown by the author in 2005). Dahmen gave criteria for C^1-regularity (resp, C^0-regularity) of G if the G_n are Banach-Lie groups [2]. Notably, $C^\omega(M,H)$ is C^0-regular for each real analytic compact manifold M and Banach-Lie group H (work in progress). Dahmen also showed that certain groups $\mathrm{GermDiff}(K)$ of germs of analytic diffeomorphisms around a compact subset $K \neq \emptyset$ of a complex Banach space X is C^0-regular. For $X = \mathbb{C}^d$, $K = \{0\}$, such groups were first studied by Pisanelli in 1976. The result solves an open problem in a 2006 survey by K.-H. Neeb.

(b) If $(G_j)_{j\in J}$ is a family of Lie groups, then the "weak direct product" $G := \bigoplus_{j\in J} G_j := \{(x_j)_{j\in J} \in \prod_{j\in J} G_j \colon x_j \neq e \text{ for only finitely many } j \in J\}$ is a Lie group [4]; its Lie algebra is the locally convex direct sum $\mathfrak{g} = \bigoplus_{j\in J} L(G_j)$. If J is countable and $k < \infty$, then the natural continuous isomorphism of vector spaces

$$\Phi\colon \bigoplus_{j\in J} C^k([0,1],L(G_j)) \to C^k([0,1],\mathfrak{g}) \tag{2}$$

is bicontinuous. If also each G_j is C^k-regular, then so is G, as $\mathrm{evol}_G = f \circ \Phi^{-1}$ with the C^∞-map $f\colon \bigoplus_{j\in J} C^k([0,1],L(G_j)) \to G$, $(\gamma_j)_{j\in J} \mapsto (\mathrm{evol}_{G_j}(\gamma_j))_{j\in J}$.

Remark. If each G_j is non-discrete, then Φ^{-1} is not continuous in the cases when J is uncountable (D. Vogt) or $k = \infty$. It is not known whether G is regular then.

(c) Let $\mathrm{Diff}(M)$ be the Lie group of all smooth diffeomorphisms of a paracompact finite-dimensional smooth manifold M, modelled on the space of compactly supported smooth vector fields, as discussed in P. W. Michor's book from 1980. As asserted by Milnor (without prooof) in a 1982 preprint, $\mathrm{Diff}(M)$ is regular (and indeed C^0-regular) if M is σ-compact, by an unpublished argument of the author which has now been extended to diffeomorphism groups of σ–compact orbifolds [6].

Remark. If M has uncountably many components M_j, then $\bigoplus_{j\in J} \mathrm{Diff}(M_j)$ is an open subgroup of $\mathrm{Diff}(M)$. It is not known whether $\mathrm{Diff}(M)$ is regular in this case, due to the problems described in the previous remark.

(d) If M is a σ-compact finite-dimensional smooth manifold, $r \in \mathbb{N}_0 \cup \{\infty\}$ and H a Lie group, then the compactly supported H-valued C^r-maps on M form a Lie group $G := C^r_c(M,H)$ (the author, 2002). If $(M_n)_{n\in\mathbb{N}}$ is a locally finite sequence of compact submanifolds with boundary whose interiors cover M, then

$$\rho\colon C^r_c(M,H) \to \bigoplus_{n\in\mathbb{N}} C^r(M_n,H), \quad \gamma \mapsto (\gamma|_{M_n})_{n\in\mathbb{N}}$$

is an isomorphism of Lie groups onto a Lie subgroup of the weak direct product, which is an intersection of equalizers of suitable point evaluations. Hence G is C^k-regular (for $k < \infty$) if H is C^k-regular, by 1.a, 2. and 3.b. An analogous reasoning shows that the gauge group $\mathrm{Gau}(P)$ (with $\mathrm{Gau}_c(P)$ as an open subgroup) is C^k-regular for finite k if $P \to M$ is a smooth principal bundle over a σ-compact finite-dimensional smooth manifold M whose structure group is a C^k-regular locally exponential Lie group. Then also the full Lie group $\mathrm{Aut}(P)$ of symmetries of the bundle (constructed in [7], cf. Wockel 2007 for compact M) is C^k-regular. In fact,

$$\mathbf{1} \to \mathrm{Gau}(P) \to \mathrm{Aut}(P) \to \mathrm{Diff}(M)_P \to \mathbf{1}$$

with a suitable open subgroup $\mathrm{Diff}(M)_P$ of $\mathrm{Diff}(M)$, and it is known that C^k-regularity is an extension property [5].

(e) The Lie group $\mathrm{Diff}_{\mathcal{S}}(\mathbb{R}^n)$ of diffeomorphisms ϕ of $\mathbb{R}^n$ such that $\phi - \mathrm{id}$ is in the Schwartz space $\mathcal{S}(\mathbb{R}^n,\mathbb{R}^n)$ of rapidly decreasing functions has been used in the physics literature (G. A. Goldin), and corresponding mapping groups $\mathcal{S}(\mathbb{R}^n,H)$ modelled on $\mathcal{S}(\mathbb{R}^n,L(H))$ have been considered in the book Boseck et al. 1981. My Ph.D.-student B. Walter published constructions of the Lie group structure and proofs of regularity for both groups (with H a Banach-Lie group) in 2012.

4. Some recent applications.

(a) A Lie group G has the Trotter property if the Trotter product formula for $\gamma_1,\gamma_2 \in \mathrm{Hom}(\mathbb{R},G)$ converges uniformly on compact sets to the one-parameter group γ with $\gamma'(0)=\gamma_1'(0)+\gamma_2'(0)$. The class of Lie groups which satisfy the Trotter property and are C^0-regular is closed under central extensions of Lie groups [5].

(b) Using Frobenius theorems for vector distributions on infinite-dimensional manifolds [3], one can make G/H a manifold with $G \to G/H$ a smooth H-principal bundle whenever H is a Lie subgroup of a Lie group G and H is a Banach-Lie group or H is regular and $L(H)$ is complemented in $L(G)$, with $L(G)/L(H)$ Banach.

5. L^∞-regularity. Using L^∞-maps to Fréchet spaces [4] and working with differentiability almost everywhere, one can define L^∞-regular Fréchet-Lie groups (replacing C^k-curves with L^∞-maps $\gamma\colon [0,1] \to \mathfrak{g}$ in the above definition of C^k-regularity). Using tools from [4], one can show that all Banach-Lie groups are L^∞-regular, and hence so is the projective limit $C^\infty(M,H)$ for all compact manifolds M and Banach-Lie groups H. Conjecture: $\mathrm{Diff}(M)$ is L^∞-regular.

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Reporter: Aprameyan Parthasarathy

Participants

PD. Dr. Alexander Alldridge
Mathematisches Institut
Universität zu Köln
Gebäude 318
Wilhelm-Waldeyer-Strasse
50937 Köln
GERMANY

Prof. Dr. Daniel Beltita
Institute of Mathematics
"Simion Stoilow" of the
Romanian Academy
P.O.Box 1-764
014 700 Bucharest
ROMANIA

Prof. Dr. John Arlo Caine
California Polytechnic University
3801 West Temple Avenue
Pomona CA 91768
UNITED STATES

Dr. Thomas Creutzig
Fachbereich Mathematik
TU Darmstadt
Schloßgartenstr. 7
64289 Darmstadt
GERMANY

Prof. Dr. Hendrik De Bie
Department of Mathematics
Ghent University
Krijgslaan 281
9000 Gent
BELGIUM

Prof. Dr. Michel Duflo
U.F.R. de Mathématiques
Université Paris 7 - Denis Diderot
Case Postale 7012
2, place Jussieu
75251 Paris Cedex 05
FRANCE

Prof. Dr. Jacques Faraut
Institut de Mathématiques
Equipe Analyse Algebrique - Case 246
Université Pierre et Marie Curie
4, place Jussieu
75252 Paris
FRANCE

Dr. Francois Gay-Balmaz
Laboratoire de meteorologie dynamique
École Normale Superieur
24, rue Lhomond
75005 Paris
FRANCE

Prof. Dr. Helge Glöckner
Institut für Mathematik
Universität Paderborn
Warburger Str. 100
33098 Paderborn
GERMANY

Prof. Dr. Gerald A. Goldin
Depts. of Mathematics and Physics
Rutgers University
SERC Bldg., Room 239
118 Frelinghuysen Road
Piscataway NJ 08854-8019
UNITED STATES

Prof. Dr. Masha Gordina
Department of Mathematics
University of Connecticut
196 Auditorium Road
Storrs, CT 06269-3009
UNITED STATES

Prof. Dr. Gert Heckman
Subfaculteit Wiskunde
Katholieke Universiteit Nijmegen
Postbus 9010
6500 GL Nijmegen
NETHERLANDS

Prof. Dr. Joachim Hilgert
Fakultät EIM - Elektrotechnik,
Informatik und Mathematik
Universität Paderborn
Warburger Str. 100
33098 Paderborn
GERMANY

Dr. Kenji Iohara
Institut Camille Jordan
Université Claude Bernard Lyon 1
43 blvd. du 11 novembre 1918
69622 Villeurbanne Cedex
FRANCE

Prof. Dr. Hideyuki Ishi
Graduate School of Mathematics
Nagoya University
Chikusa-ku, Furo-cho
Nagoya 464-8602
JAPAN

Dr. Bas Janssens
Department Mathematik
FAU Erlangen-Nürnberg
Cauerstr. 11
91058 Erlangen
GERMANY

Prof. Dr. Palle E. T. Jorgensen
Department of Mathematics
University of Iowa
Iowa City, IA 52242-1466
UNITED STATES

Prof. Dr. Toshiyuki Kobayashi
Grad. School of Mathematical Sciences
IPMU
University of Tokyo
3-8-1 Komaba, Meguro-ku
Tokyo 153-8914
JAPAN

Prof. Dr. Bernhard J. Krötz
Institut für Mathematik
Universität Paderborn
Warburger Str. 100
33098 Paderborn
GERMANY

Toshihisa Kubo
Grad. School of Mathematical Sciences
IPMU
University of Tokyo
3-8-1 Komaba, Meguro-ku
Tokyo 153-8914
JAPAN

Dr. Job Jacob Kuit
Dept. of Mathematical Sciences
University of Copenhagen
Universitetsparken 5
2100 Copenhagen
DENMARK

Dr. Gang Liu
Institut für Analysis
Leibniz Universität Hannover
Welfengarten 1
30167 Hannover
GERMANY

Prof. Dr. Manuel Medina
Laboratoire de Mathématiques
Université de Reims
Moulin de la Housse
BP 1039
51687 Reims Cedex 2
FRANCE

Dr. Stephane Merigon
Department Mathematik
FAU Erlangen-Nürnberg
Cauerstraße 11
91058 Erlangen
GERMANY

Prof. Dr. Jouko Mickelsson
Department of Theoretical Physics
Royal Institute of Technology
SCFAB
10691 Stockholm
SWEDEN

Dr. Jan Moellers
Matematisk Institut
Aarhus Universitet
Ny Munkegade 118
8000 Aarhus C
DENMARK

Prof. Dr. Karl-Hermann Neeb
Department Mathematik
FAU Erlangen-Nürnberg
Cauerstraße 11
91058 Erlangen
GERMANY

Prof. Dr. Anatol Odzijewicz
Institute of Mathematics
University of Bialystok
Lipowa 41
15-424 Bialystok
POLAND

Prof. Dr. Gestur Olafsson
Department of Mathematics
Louisiana State University
Baton Rouge LA 70803-4918
UNITED STATES

Prof. Dr. Bent Orsted
Matematisk Institut
Aarhus Universitet
Ny Munkegade 118
8000 Aarhus C
DENMARK

Dr. Yoshiki Oshima
Grad. School of Mathematical Sciences
IPMU
University of Tokyo
3-8-1 Komaba, Meguro-ku
Tokyo 153-8914
JAPAN

Prof. Dr. Valentin Ovsienko
Institut Camille Jordan
UMR 5208 du CNRS
Université Claude Bernard Lyon 1
21, bd. Claude Bernard
69622 Villeurbanne Cedex
FRANCE

Dr. Aprameyan Parthasarathy
FB 17 - Math./Inf. (D 344)
Universität Paderborn
Warburger Str. 100
33098 Paderborn
GERMANY

Prof. Dr. Angela Pasquale
Département de Mathématiques
Université de Lorraine
Batiment A
Ile Du Saulcy
57045 Metz Cedex 1
FRANCE

Prof. Dr. Ivan Penkov
School of Engineering and Science
Jacobs University Bremen
Postfach 750561
28725 Bremen
GERMANY

Prof. Dr. Michael Pevzner
Département de Mathématiques
Université de Reims
Moulin de la Housse
BP 1039
51687 Reims Cedex 2
FRANCE

Prof. Dr. Tomasz Przebinda
Department of Mathematics
University of Oklahoma
601 Elm Avenue
Norman, OK 73019-0315
UNITED STATES

Prof. Dr. Tudor S. Ratiu
Section de Mathématiques
Station 8
École Polytechnique Fédérale de
Lausanne
1015 Lausanne
SWITZERLAND

Prof. Dr. Hadi Salmasian
Department of Mathematics & Statistics
University of Ottawa
585 King Edward Avenue
Ottawa, Ont. K1N 6N5
CANADA

Dr. Atsumu Sasaki
Department of Mathematics
Faculty of Sciences
Tokai University
4-1-1 Kitakaname, Hiratsuka
Kanagawa 259-1292
JAPAN

Dr. Yurii Savchuk
Max-Planck-Institut für Mathematik
in den Naturwissenschaften
Inselstr. 22 - 26
04103 Leipzig
GERMANY

Jun.-Prof. Dr. Henrik Seppänen
Mathematisches Institut
Georg-August-Universität Göttingen
Bunsenstr. 3-5
37073 Göttingen
GERMANY

Prof. Dr. Erik P. van den Ban
Mathematisch Instituut
Universiteit Utrecht
Budapestlaan 6
P. O. Box 80.010
3508 TA Utrecht
NETHERLANDS

Maarten van Pruijssen
Institut für Mathematik
Universität Paderborn
Warburger Str. 100
33098 Paderborn
GERMANY

Prof. Dr. Anatoli M. Vershik
Steklov Mathematical Institute
PDMI
Fontanka 27
St. Petersburg 191 023
RUSSIAN FEDERATION

Stefan Wagner
Institut for Matematiske Fag
Kobenhavns Universitet
Universitetsparken 5
2100 Kobenhavn
DENMARK

Prof. Dr. Joseph Albert Wolf
Department of Mathematics
University of California
Berkeley CA 94720-3840
UNITED STATES

Prof. Dr. Tilmann Wurzbacher
Fakultät für Mathematik
Ruhr-Universität Bochum
Universitätsstr. 150
44801 Bochum
GERMANY

Prof. Dr. Oksana Yakimova
Mathematisches Institut
Friedrich-Schiller-Universität Jena
07737 Jena
GERMANY

Dr. Christoph Zellner
Department Mathematik
Universität Erlangen-Nürnberg
Cauerstr. 11
91058 Erlangen
GERMANY

Prof. Dr. Martin Zirnbauer
Institut für Theoretische Physik
Universität Köln
Zülpicher Str. 77
50937 Köln
GERMANY