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Extremes in Branching Random Walk and Branching Brownian Motion

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Abstract. Branching random walk (BRW) and branching Brownian motion (BBM) are mathematical models for population growth and spatial displacement. When resources are plentiful, population sizes grow exponentially in time. In such a situation, exceptional (or *extreme*) individuals will be found far from the bulk of the population. The study of such individuals, and their ancestral lineages, was the subject of the workshop. On one hand, this is a classical topic, with well-known connections to the KPP-equation and to search algorithms. On the other hand, substantial recent developments have recently been obtained via new approaches to the subject (stopping lines and spines, the view from the tip, multivariate analytic combinatorics), or from researchers working in seemingly distinct areas (from stochastic partial differential equations to theoretical physics).

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Introduction by the Organisers

A *branching random walk* (BRW) is a system of particles in some physical space such as $\mathbb{R}^d$, where, on the one hand, particles reproduce independently of one another according to some fixed distribution, and on the other, the displacement of particles with respect to their parent's position are also independent from one another, and are distributed according to some fixed law. *Branching Brownian motion* is also a process which involves branching and spatial displacement, but with particle trajectories given by Brownian motions.

Branching random walks and branching Brownian motion are fundamental objects of interest in probability, and have been studied in some depth at least since the 1950s. Yet many basic aspects of their behaviour remain poorly understood to this date – this is particularly true for questions regarding the extremal behaviour of such processes, which were the main focus of this workshop. While these are natural and intrinsic questions from a mathematical standpoint, it has appeared in recent years that they are also of fundamental importance in other scientific fields, from the analysis of search algorithms in computer science to the understanding of energy landscapes in random energy and spin glass models from theoretical physics, and including the description of the effect of natural selection on the genealogy of populations in theoretical biology.

There has been recently a surge of new ideas and breakthroughs which are moving the subject closer to the resolution of some of its longstanding problems. Strikingly, some of these key advances have been made simultaneously and independently by different groups of researchers, usually through entirely different methods. We outline a few themes which featured prominently.

– *Minimal position.* Bramson, Ding and Zeitouni; Addario-Berry and Reed; and Aïdékon have all proved related results on the position of the minimum and on its convergence in distribution; these results build on existing work by Biggins, Devroye, and McDiarmid, among others. These recent results are established using very different techniques. For instance, Aïdékon relies on spine methods and derivative martingales. Bramson, Ding and Zeitouni relied on robust versions of second moment arguments in a way which allowed them to treat the case of the discrete two-dimensional Gaussian Free Field. Addario-Berry and Reed use a rather combinatorial argument based on the second moment method. Some recent developments in this direction include the works of Arguin, Bovier and Kistler; and Aidékon, Berestycki, Brunet and Shi who show the existence of a limit for the point process of particles near the minimum for a branching Brownian motion. Moreover, this process is ergodic if correctly recentered by the derivative martingale (in order to take into account the early fluctuations of the process).

– *Aldous' conjecture.* This conjecture concerns the following situation: assume that all particles that reach a certain subset of $\mathbb{R}^d$ (say the negative half-line on $\mathbb{R}$) are immediately killed and removed from the system. Then there is a critical value β_c for the branching rate such that if $\beta \leq \beta_c$, the system dies out with probability 1, while it survives with positive probability if $\beta > \beta_c$. At $\beta = \beta_c$, Aldous conjectured that (in one dimension) the total number of individuals Z satisfies $\mathbf{E}(Z) < \infty$ but $\mathbf{E}(Z \log Z) = \infty$. Addario-Berry and Broutin made an even stronger conjecture that $\mathbf{P}(Z > x) \sim c/(x \log^2 x)$. Variants of this conjecture have been independently established by several researchers: by Maillard for branching Brownian motion, who introduced new ideas and substantially developed the singularity analysis approach; by Aidékon, Hu and Zindy, based on a trajectorial decomposition; and by Aidékon via spine methods.

– *Genealogies.* When a population evolves by branching and with a selection mechanism that maintains a fixed population size, only letting the fittest (extreme) particles survive, physicists Brunet and Derrida have made striking, non-rigorous predictions for the genealogy of the resulting population. Most notably, they predicted a characteristic “genealogical timescale” of $(\log N)^3$ generations if N is the population size, and a genealogy described by the Bolthausen-Sznitman coalescent. Aspects of this conjecture have recently been established by Bérard and Gouéré and by Berestycki, Berestycki and Schweinsberg, all matching perfectly the predictions. Relation to noisy travelling waves and in particular the stochastic KPP equation (which underpinned the non-rigorous approach of Brunet and Derrida) remain mysterious, even though separate but related predictions which they made for the propagation of the wavefront in this equation were recently proved by Mueller, Mytnik and Quastel. The Bolthausen-Sznitman coalescent is also the conjectured limiting object for the energy landscape in random energy models and related spin glass models. This is not a coincidence, but most aspects of this connection remain to be understood.

Workshop: Extremes in Branching Random Walk and Branching Brownian Motion

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Abstracts

The Branching Brownian Motion and lags in the Fisher-KPP equation

Éric Brunet

(joint work with B. Derrida, E. Aïdékon, J. Berestycki, Z. Shi)

The Branching Brownian Motion (BBM) is intimately related to the Fisher-KPP equation through the Mc-Kean representation: for any reasonable function ϕ, the quantity

$$H_\phi(x,t) = \Big\langle \prod_i \phi\big[x - X_i(t)\big] \Big\rangle$$

(where the product is over the particles present at time t and where $X_i(t)$ is the position of the i-th particle) is solution to the Fisher-KPP equation:

$$\partial_t H_\phi = \partial_x^2 H_\phi - H_\phi + H_\phi^2$$

with $H_\phi(x,0) = \phi(x)$.

If $\phi(x) = \mathbb{1}_{x>0}$, then $H_\phi(x,t)$ is the probability that the rightmost particle at time t is on the left of x; if $\phi(x) = \mathbb{1}_{x>0} + \lambda \mathbb{1}_{x<0}$, then $H_\phi(x,t)$ is the generating function of the number of particles on the right of x, etc.

By carefully choosing $\phi(x)$, one has access to any property of the distribution of the rightmost particles in the BBM at any time t. These properties can be measured numerically by integrating the Fisher-KPP equation, which is of course much simpler than simulating the BBM.

The Fisher-KPP equation has the property that for all the initial conditions $\phi(x)$ that we are interested in, the solution $H_\phi(x,t)$ converges to *the same* travelling wave $\omega(z)$ in the sense that there exists a m_t such that

$$H_\phi(m_t + z, t) \xrightarrow[t\to\infty]{} \omega(z).$$

Here, m_t is the position of the front at time t and, for a certain choice of the branching rate and diffusion coefficient, it is asymptotically given by

$$m_t = 2t - \frac{3}{2}\ln t + a_0(\phi) - \frac{3\sqrt{\pi}}{\sqrt{t}} + \frac{a_1(\phi)}{t} + \frac{a_{3/2}(\phi)}{t^{3/2}} + \cdots$$

It is worth emphasizing that the velocity 2, the 3/2 and the $3\sqrt{\pi}$ do not depend on the initial condition ϕ, while all the other coefficients in the expansion do. This means that two fronts started with different initial conditions ϕ will eventually converge to the same shape $\omega(z)$ travelling at velocity 2, with one lagging behind the other by an amount which is the difference of the two $a_0(\phi)$ corresponding to the two initial conditions ϕ. Because of the link between the Fisher-KPP fronts and the BBM, one can show from this convergence that the distribution of the rightmost particles in the BBM centered around the position $2t - (3/2)\ln t$ converges as $t \to \infty$ to a stationary distribution characterized by the set of all the lags $\{a_0(\phi)\}_\phi$. Furthermore, any property of this stationary distribution can be numerically measured.

The previous result can be refined by extending a famous result by Lalley and Sellke [4] stating that when $t \to \infty$ the position of the rightmost particle in the BBM can be written as

$$X_1(t) = 2t - \frac{3}{2}\ln t + \ln Z + \text{Cste} + \eta_1$$

where Z is the limit of the derivative Martingale

$$Z = \lim_{t\to\infty} \sum_i (2t - X_i(t)) e^{X_i(t)-2t}$$

and where η_1 is a Gumble distributed random variable. One can show by using the same method as Lalley and Sellke that the distribution of all the rightmost particles in the BBM centered around the random position $2t - \frac{3}{2}\ln t + \ln Z + \text{Cste}$ also converges to a stationary distribution which is independent of Z.

This distribution can be described as a *decorated exponential Poisson point process*: it is obtained by first selecting *leaders* according to a Poisson point process with density e^{-x} then, independently for each leader, particles are added according to an auxiliary distribution (the *decoration*) shifted by the position of the leader and such that all the particles in the decoration fall on the left of the leader. Thus, the rightmost particle of the BBM is the rightmost leader, i.e. the rightmost point in the exponential Poisson point process, and one recovers the Gumble distribution of Lalley and Sellke. The second rightmost particle in the BBM is either the rightmost point in the decoration of the rightmost leader, or the second rightmost leader, etc.

The rightmost particles in the BBM are therefore a collection of independent families (the leader and its decoration). As shown by Arguin, Bovier and Kistler [5], two particles belong to the same family if their lineage branched recently, a time of order one ago. They belong to two different families if their lineage branched near the origin of times. These are the only cases; two particles amongst the rightmost cannot have branched at an intermediary time which is neither very recent or very ancient.

Some beautiful descriptions of the auxiliary distribution forming the decoration of each leader exists, but it remains difficult to extract properties of this decoration.

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Convergence in law of the minimum of a branching random walk

Elie Aïdékon

We consider a branching random walk on the real line in discrete time. The process starts with one particle located at 0. At time 1, the particle dies and gives birth to a point process $\mathcal{L}$. Then, at each time $n \in \mathbb{N}$, the particles of generation n die and give birth to independent copies of the point process $\mathcal{L}$, translated to their position. Under some fairly general conditions, we can renormalize the branching random walk so that

$$\mathbf{E}\left[\sum_{|x|=1} \mathrm{e}^{-V(x)}\right] = 1, \qquad \mathbf{E}\left[\sum_{|x|=1} V(x)\mathrm{e}^{-V(x)}\right] = 0.$$

We denote by M_n the minimal position of the particles at time n. In this setting, it is known that $M_n - \frac{3}{2}\ln n$ is a tight sequence, see [1]. Writing $y_+ := \max(y, 0)$, we introduce the random variables

$$X := \sum_{|x|=1} \mathrm{e}^{-V(x)}, \qquad \tilde{X} := \sum_{|x|=1} V(x)_+ \mathrm{e}^{-V(x)}.$$

We assume

- the distribution of $\mathcal{L}$ is non-lattice,
- we have

$$\mathbf{E}\left[\sum_{|x|=1} V(x)^2 \mathrm{e}^{-V(x)}\right] < \infty,$$

$$\mathbf{E}\left[X(\ln_+ X)^2\right] < \infty, \qquad \mathbf{E}\left[\tilde{X} \ln_+ \tilde{X}\right] < \infty.$$

We prove

Theorem *The random variable $M_n - \frac{3}{2}\ln n$ converges in law as $n \to \infty$.*

In the branching Brownian case, this was known since the work of Bramson [6]. Moreover, we can characterize the limit law. We introduce the derivative martingale

$$D_n := \sum_{|x|=n} V(x)\mathrm{e}^{-V(x)}.$$

In [5], Biggins and Kyprianou prove this martingale converges almost surely to some D_∞. We prove that there exists a constant $C > 0$ such that, for any real x,

$$\lim_{n\to\infty} \mathbf{P}(M_n - \frac{3}{2}\ln n \geq x) = \mathbf{E}\left[\mathrm{e}^{-C\mathrm{e}^{x} D_\infty}\right].$$

Again, this was known in the branching Brownian motion setting, and proved by Lalley and Sellke [8].

Following this work, Madaule [9] proved that the point process of leftmost particles seen from the minimum converges in law. In the branching Brownian motion setting, the convergence of the extremal process was proved by Arguin, Bovier and Kistler [4] and Aïdékon, Berestycki, Brunet, Shi [3]. These works confirmed the prediction of physicists Brunet and Derrida [7], who conjectured that the limit point process was a *decorated* Poisson point process. Further conjectures on this extremal process are still open though: one can cite the asymptotic density in the tail of the extremal process, or the tail distribution of the gap between the leftmost and second leftmost particles.

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The almost sure limits of the minimal position in a branching random walk

Yueyun Hu

Consider a discrete-time branching random walk $\{V(u), u \in \mathbb{T}\}$ on the real line $\mathbb{R}$. The law of the branching random walk is determined by the point process $\Theta \equiv \sum_{|u|=1} \delta_{\{V(u)\}}$. The underlying Galton-Watson tree is denoted by $\mathbb{T}$ and is assumed to be supercritical (namely $\mathbb{E}[\Theta(\mathbb{R})] \in (1, \infty)$). Assuming that

$$\mathbb{E}\left[\int e^{-x}\Theta(dx)\right] = 1, \qquad \mathbb{E}\left[\int xe^{-x}\Theta(dx)\right] = 0. \tag{1}$$

When the hypothesis (1) is fulfilled, the branching random walk is called in the boundary case in the literature (see e.g. Biggins and Kyprianou [5], Aïdékon and

Shi [3]). Under some integrability conditions, a general branching random walk can be reduced to the boundary case after a linear transformation, see Jaffuel [12] for detailed discussions.

Denote by $\mathbb{M}_n := \min_{|u|=n} V(u)$ the minimal position of the branching random walk at generation n (with convention $\inf \emptyset \equiv \infty$). Hammersley [8], Kingman [13] and Biggins [4] established the law of large numbers for $\mathbb{M}_n$ (for any general branching random walk), whereas the second order limits have recently attracted many attentions, see [1, 11, 6, 2] and the references therein. In particular, Aïdékon [2] proved the convergence in law of $\mathbb{M}_n - \frac{3}{2}\log n$ under (1) and some mild conditions.

Concerning the almost sure limits of $\mathbb{M}_n$, there is a phenomena of fluctuation at the logarithmic scale ([11]): Under (1) and some extra integrability assumption, the following almost sure limits hold:

$$\limsup_{n\to\infty} \frac{\mathbb{M}_n}{\log n} = \frac{3}{2}, \qquad \mathbb{P}^*\text{-a.s.},$$
$$\liminf_{n\to\infty} \frac{\mathbb{M}_n}{\log n} = \frac{1}{2}, \qquad \mathbb{P}^*\text{-a.s.},$$

where $\mathbb{P}^*(\cdot) := \mathbb{P}(\cdot|\mathbb{S})$, and $\mathbb{S}$ denotes the event that the whole system survives.

We discuss here how $\mathbb{M}_n$ can approach its upper limit $\frac{3}{2}\log n$ and its lower limit $\frac{1}{2}\log n$:

Aïdékon and Shi [3] proved that under (1) and some $L(\log L)^2$-type condition,

$$\liminf_{n\to\infty} \left(\mathbb{M}_n - \frac{1}{2}\log n\right) = -\infty, \qquad \mathbb{P}^*\text{-a.s.}$$

Using their methods, we characterize the lower limits of $\mathbb{M}_n$ by an integral test:

Theorem 1 [9]**:** Assuming (1) and some $L(\log L)^2$-type integrability condition. For any function $f \uparrow \infty$, $\mathbb{P}^*$-almost surely,

$$\mathbb{P}^*\left(\mathbb{M}_n - \frac{1}{2}\log n < -f(n), \quad \text{i.o.}\right) = \begin{cases} 0 \\ 1 \end{cases} \iff \int^{\infty} \frac{dt}{t\exp(f(t))} \begin{cases} < \infty \\ = \infty \end{cases},$$

where i.o. means infinitely often as the relevant index $n \to \infty$.

On the upper limits of $\mathbb{M}_n$, we present a law of iterated logarithm:

Theorem 2 [10]**:** Assuming (1) and some $L(\log L)^2$-type integrability condition, we have

$$\limsup_{n\to\infty} \frac{1}{\log\log\log n}(\mathbb{M}_n - \frac{3}{2}\log n) = 1, \qquad \mathbb{P}^*\text{-a.s.}$$

We also discuss the related moderate deviation problem on $\mathbb{M}_n$, by distinguishing the Schröder case or the Böttcher case as for the limit of a Galton-Watson process (cf. [7]).

Theorem 3 [10]**:** Assuming (1) and the boundedness of the positive jumps in Θ, we have

$$\log \mathbb{P}^* \left(\mathbb{M}_n > \frac{3}{2} \log n + \lambda \right) = \begin{cases} -(\gamma + o(1))\lambda, & \text{in the Schröder case,} \\ -e^{(\beta+o(1))\lambda}, & \text{in the Böttcher case,} \end{cases}$$

where γ and β are two positive parameters which are determined by the law of Θ.

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The Aldous conjecture on a killed branching random walk

Olivier Zindy

(joint work with Elie Aïdékon, Yueyun Hu)

We consider a one-dimensional discrete-time branching random walk V on the real line $\mathbb{R}$. At the beginning, there is a single particle located at the origin 0. Its children, who form the first generation, are positioned according to a certain point process $\mathscr{L}$ on $\mathbb{R}$. Each of the particles in the first generation independently gives birth to new particles that are positioned (with respect to their birth places) according to a point process with the same law as $\mathscr{L}$; they form the second generation. And so on. For any $n \geq 1$, each particle at generation n produces new particles independently of each other and of everything up to the n-th generation.

Clearly, the particles of the branching random walk V form a Galton–Watson tree, which we denote by $\mathcal{T}$. Call $\varnothing$ the root. For every vertex $u \in \mathcal{T}$, we denote by $|u|$ its generation (then $|\varnothing| = 0$) and by $(V(u), |u| = n)$ the positions of the particles in the n-th generation. Then $\mathscr{L} = \sum_{|u|=1} \delta_{\{V(u)\}}$. The tree $\mathcal{T}$ will encode the genealogy of our branching random walk.

It will be more convenient to consider a branching random walk V starting from an arbitrary $x \in \mathbb{R}$ [namely, $V(\varnothing) = x$], whose law is denoted by $\mathbf{P}_x$ and the corresponding expectation by $\mathbf{E}_x$. For simplification, we write $\mathbf{P} \equiv \mathbf{P}_0$ and $\mathbf{E} \equiv \mathbf{E}_0$. Let $\nu := \sum_{|u|=1} 1$ be the number of particles in the first generation and denote by $\nu(u)$ the number of children of $u \in \mathcal{T}$.

Assume that $\mathbf{E}[\nu] > 1$, namely the Galton–Watson tree $\mathcal{T}$ is supercritical, then the system survives with positive probability $\mathbf{P}\big(\mathcal{T} = \infty\big) > 0$. Let us define the logarithmic generating function for the branching walk:

$$\psi(t) := \log \mathbf{E}\Big[\sum_{|u|=1} e^{tV(u)}\Big] \in (-\infty, +\infty], \qquad t \in \mathbb{R}.$$

We shall assume that ψ is finite on an open interval containing 0 and that $\mathrm{supp}\mathscr{L} \cap (0,\infty) \neq \emptyset$ [the later condition is to ensure that V can visit $(0,\infty)$ with positive probability, otherwise the problem that we shall consider becomes trivial]. Assume that there exists $\varrho^* > 0$ such that

$$\psi(\varrho^*) = \varrho^* \psi'(\varrho^*). \tag{1}$$

We also assume that ψ is finite on an open set containing $[0, \varrho^*]$. The condition (1) is rather mild, roughly saying, if we denote by $m^* = \mathrm{esssup}\,\mathrm{supp}\mathscr{L}$, then (1) is satisfied if either $m^* = \infty$ or $m^* < \infty$ and $\mathbf{E}\sum_{|u|=1} 1_{(V(u)=m^*)} < 1$.

Recall that (Kingman [8], Hammersley [6], Biggins [5]) conditioned on the survival of the system,

$$\lim_{n\to\infty} \frac{1}{n} \max_{|u|=n} V(u) = \psi'(\varrho^*), \qquad \text{a.s.}, \tag{2}$$

where ϱ^* is defined in (1). According to $\psi'(\varrho^*) = 0$ or $\psi'(\varrho^*) < 0$, we call the critical case or the subcritical case. Conditioned on $\{\mathcal{T} = \infty\}$, the rightmost particle in the branching random walk without killing has a negative speed in the subcritical case, while in the critical case it converges almost surely to $-\infty$ in the logarithmical scale (see Addario-Berry and Reed [2] and Hu and Shi [7] for the precise statement of the rate of almost sure convergence).

We now place a killing barrier at zero, hence in every generation $n \geq 0$, survive only the particles that always stayed nonnegative up to time n. Denote by $\mathscr{Z}$ the set of all living particles of the killed branching walk:

$$\mathscr{Z} := \Big\{u \in \mathcal{T} \,:\, V(v) \geq 0, \qquad \forall\, v \in [\varnothing, u]\Big\},$$

where $[\varnothing, u]$ denotes the shortest path relating u from the root $\varnothing$. We are interested in the total progeny

$$Z := \#\mathscr{Z},$$

on which David Aldous made the following conjecture:
Conjecture (D.Aldous [4]):

(i) (critical case): If $\psi'(\varrho^*) = 0$, then $\mathbf{E}[Z] < \infty$ and $\mathbf{E}[Z \log Z] = \infty$.
(ii) (subcritical case): If $\psi'(\varrho^*) < 0$, then there exists some constant $b > 1$ such that $\mathbf{P}(Z > n) = n^{-b+o(1)}$ as $n \to \infty$.

Let us call iid case if $\mathscr{L}$ is of form: $\mathscr{L} = \sum_{i=1}^{\nu} \delta_{\{X_i\}}$ with $(X_i)_{i\geq 1}$ a sequence of i.i.d. real-valued variables, independent of ν. There are several previous works on the critical and iid case: when (X_i) are Bernoulli random variables, Pemantle [9] obtained the precise asymptotic of $\mathbf{P}(Z = n)$ as $n \to \infty$, where the key ingredient of his proof is the recursive structure of the system inherited from the Bernoulli variables (X_i). For general random variables (X_i), Addario-Berry and Broutin [1] recently confirmed Aldous' conjecture (i) under some integrability hypothesis; This was improved later by Aïdékon [3] who proved that for a regular tree $\mathcal{T}$ (namely when ν equals some integer), for any fixed $x \geq 0$,

$$n(\log n)^2 \mathbf{P}_x(Z > n) \approx R(x)e^x,$$

where $R(x)$ is a renewal function.

We aim at the exact tail behavior of Z both in critical and subcritical cases and for a general point process $\mathscr{L}$.

Before the statement of our result, we remark that in the subcritical case ($\psi'(\varrho^*) < 0$), there are two real numbers ϱ_- and ϱ_+ such that $0 < \varrho_- < \varrho^* < \varrho_+$ and

$$\psi(\varrho_-) = \psi(\varrho_+) = 0,$$

[the existence of ϱ_+ follows from the assumption that $\mathrm{supp}\mathscr{L} \cap (0, \infty) \neq \emptyset$].

Assume that

$$\mathbf{E}[\nu^\alpha] < \infty, \qquad \text{for some } \begin{cases} \alpha > 2, & \text{in the critical case;} \\ \alpha > 2\frac{\varrho_+}{\varrho_-}, & \text{in the subcritical case.} \end{cases} \tag{3}$$

In the critical case, we suppose that

$$\mathbf{E}\left[\nu^{1+\delta^*}\right] < \infty, \qquad \sup_{\theta \in [-\delta^*, \varrho^* + \delta^*]} \psi(\theta) < \infty, \qquad \text{for some } \delta^* > 0. \tag{4}$$

In the subcritical case, we suppose that

$$\mathbf{E}\left[\sum_{|u|=1} (1 + e^{\varrho_- V(u)})\right]^{\frac{\varrho_+}{\varrho_-} + \delta^*} < \infty, \qquad \sup_{\theta \in [-\delta^*, \varrho_+ + \delta^*]} \psi(\theta) < \infty, \tag{5}$$

for some $\delta^* > 0$. In both cases, we always assume that there is no lattice that supports $\sum_{|u|=1} \delta_{V(u)}$ almost surely.

Our main result reads as follows.

Theorem 1 (Tail of the total progeny)**.** *Assume* (1)*,* (3)*.*

(i) (Critical case) If $\psi'(\varrho^*) = 0$ and (4) holds, then there exists a constant $c_{crit} > 0$ such that for any $x \geq 0$,

$$\mathbf{P}_x\Big(Z > n\Big) \sim c_{crit}\, R(x)\, e^{\varrho^* x} \frac{1}{n(\log n)^2}, \qquad n \to \infty,$$

where $R(x)$ is a renewal function.

(ii) (Subcritical case) If $\psi'(\varrho^*) < 0$ and (5) holds, then there exists a constant $c_{sub} > 0$ such that for any $x \geq 0$,

$$\mathbf{P}_x\Big(Z > n\Big) \sim c_{sub} R(x) e^{\varrho_+ x} n^{-\frac{\varrho_+}{\varrho_-}}, \qquad n \to \infty,$$

where $R(x)$ is a renewal function.

The values of c_{crit} and c_{sub} are given in Lemma 1. To explain the strategy of the proof of Theorem 1, we first introduce some notation: for any vertex $u \in \mathcal{T}$ and $a \in \mathbb{R}$, we define

$$\tau_a^+(u) := \inf\{0 \leq k \leq |u| : V(u_k) > a\}, \tag{6}$$

$$\tau_a^-(u) := \inf\{0 \leq k \leq |u| : V(u_k) < a\}, \tag{7}$$

with convention $\inf \emptyset := \infty$ and for $n \geq 1$ and for any $|u| = n$, we write $\{u_0 = \varnothing, u_1, ..., u_n\} = [\varnothing, u]$ the shortest path relating u from the root $\varnothing$ (u_k is the ancestor of k-th generation of u).

By using these notations, the living set $\mathscr{Z}$ of the killed branching random walk can be represented as follows:

$$\mathscr{Z} = \{u \in \mathcal{T} : \tau_0^-(u) > |u|\}.$$

For $a \leq x$, we define $\mathcal{L}[a]$ as the set of individuals which lives below a for its first time:

$$\mathcal{L}[a] := \{u \in \mathcal{T} : |u| = \tau_a^-(u)\}, \qquad a \leq x, \tag{8}$$

Since the whole system goes to $-\infty$, $\mathcal{L}[a]$ is well defined. In particular, $\mathcal{L}[0]$ is the set of leaves of the killed branching walk. As an application of a general fact for a wide class of graphs, we can compare the set of leaves $\mathcal{L}[0]$ with $\mathscr{Z}$. Then it is enough to investigate the tail asymptotics of $\#\mathcal{L}[0]$.

To state the result for $\#\mathcal{L}[0]$, we shall need an auxiliary random walk S, under a probability $\mathbf{Q}$, which depend on the parameter $\varrho = \varrho^*$ in the critical case, and $\varrho = \varrho^+$ in the subcritical case. We mention that under $\mathbf{Q}$, S is recurrent in the critical case and transient in the subcritical case. Let us also consider the renewal function $R(x)$ associated to S and τ_0^- the first time when S becomes negative.

Theorem 2 (Tail of the set of leaves)**.** *Assume (1).*

(i) Critical case : if $\psi'(1) = 0$ and (4) holds, then for any $x \geq 0$, we have when $n \to \infty$

$$\mathbf{P}_x(\#\mathcal{L}[0] > n) \sim c'_{crit} R(x) e^{\varrho^* x} \frac{1}{n(\log n)^2},$$

where $c'_{crit} := (\mathbf{Q}[e^{-S_{\tau_0^-}}] - 1)$.

(ii) Subcritical case : If $\psi'(1) < 0$ and (5) holds, then we have for any $x \geq 0$ when $n \to \infty$,

$$\mathbf{P}_x(\#\mathcal{L}[0] > n) \sim c'_{sub} R(x) e^{\varrho_+ x} n^{-\frac{\varrho_+}{\varrho_-}},$$

for some constant $c'_{sub} > 0$.

If $\sum_{|u|=1}(1+e^{\varrho - V(u)})$ has some larger moments, then we can give, as in the critical case (i), a probabilistic interpretation of the constant c'_{sub} for the subcritical case.

The next lemma establishes the relation between $\#\mathcal{L}[0]$ and the total progeny $Z = \#\mathscr{Z}$. Recall that $\mathbf{E}(\nu) > 1$.

Lemma 1. *Assume (3). Then Theorem 2 implies Theorem 1 with*
(i) in the critical case: $c_{crit} = (\mathbf{E}(\nu)-1)^{-1} c'_{crit}$,
(ii) in the subcritical case: $c_{sub} = (\mathbf{E}(\nu)-1)^{-\varrho_+/\varrho_-} c'_{sub}$.

The proof of Theorem 2 relies on an analysis of the maximum of the killed branching random walk and its progeny. We need to establish some Yaglom-type results. The main tool will be a spinal decomposition for the killed branching random walk.

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Consistent maximal displacement of branching Brownian motion

Matthew I. Roberts

A standard branching Brownian motion begins with one particle at the origin. This particle moves as a Brownian motion, until an independent exponentially distributed time of parameter 1, at which point it is instantaneously replaced by two new particles. These particles independently repeat the stochastic behaviour of their parent relative to their start position, each moving like a Brownian motion and splitting into two at an independent exponentially distributed time of parameter 1.

Let $N(t)$ be the set of all particles alive at time t, and for a particle $v \in N(t)$ let $X_v(s)$ represent its position at time $s \le t$ (or if v was not yet alive at time s, then the position of the unique ancestor of v that was alive at time s). If we define

$$M(t) = \max_{v \in N(t)} X_v(t)$$

then it is well known that

$$\frac{M(t)}{t} \to \sqrt{2} \quad \text{as } t \to \infty.$$

The problem of interest to us is that of consistent maximal displacements: how close can particles stay to the critical line $(\sqrt{2}t, t \geq 0)$? There are at least two ways of making this question precise, each of which has been considered before for the related model of branching random walks. The first is to ask for which curves $f : [0, \infty) \to \mathbb{R}$ it is possible for particles to stay above $f(t)$ for all times $t \geq 0$. That is, when is

$$\nu(f) := \mathbb{P}(\forall t \geq 0, \exists v \in N(t) : X_v(u) > f(u) \;\; \forall u \leq t)$$

non-zero? This was first considered by Jaffuel [3] (for branching random walks), who proved that there is a critical value $A_c = 3^{4/3}\pi^{2/3}2^{-7/6}$ such that if we set $f_a(t) = \sqrt{2}t - at^{1/3} - 1$ then $\nu(f_a) > 0$ if $a > A_c$, and $\nu(f_a) = 0$ if $a < A_c$.

The second approach is to look at recentered paths, specifically the value of

$$\lambda(v, t) = \sup_{s \in [0,t]} \{\sqrt{2}s - X_v(s)\},$$

and ask for the asymptotic behaviour of the minimum

$$\Lambda(t) = \min_{v \in N(t)} \lambda(v, t)$$

as $t \to \infty$. This quantity (or rather, again, its analogue for branching random walks) was studied by Fang and Zeitouni [1] and by Faraud, Hu and Shi [2], who showed that there is a critical value $a_c = 3^{1/3}\pi^{2/3}2^{-1/2}$ such that almost surely

$$\lim_{t \to \infty} \frac{\Lambda(t)}{t^{1/3}} = a_c.$$

To summarise, the two approaches to the question give similar results: in each case there appears to be a critical line on the $t^{1/3}$ scale above which particles cannot remain. We shall see, however, that if one peers more closely then the two situations are really quite different. Our first result is that not only is $\nu(f_{A_c}) > 0$ (which was previously unknown), but in fact particles may stay above $\sqrt{2}t - A_c t^{1/3} + t^{\gamma} - 1$ for any $\gamma < 1/3$. Secondly, we are able to give asymptotics on the log scale for $\Lambda(t)$: we have

$$\liminf_{t \to \infty} \frac{\Lambda(t) - a_c t^{1/3}}{\log t} = -\frac{1}{\sqrt{2}}$$

almost surely, and

$$\limsup_{t \to \infty} \frac{\Lambda(t) - a_c t^{1/3}}{\log t} = 0$$

almost surely.

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Radius of the support of super-Brownian motion

Andreas Kyprianou
(joint work with Marion Hesse)

Suppose that $X = \{X_t, t \geq 0\}$ is a Super-Brownian motion in $\mathbb{R}^d$, $d \geq 3$, with general branching mechanism ψ of the form

$$\psi(\lambda) = -\alpha\lambda + \beta\lambda^2 + \int_{(0,\infty)} (e^{-\lambda x} - 1 + \lambda x)\Pi(dx), \quad \lambda \geq 0,$$

where $\alpha = -\psi'(0+) \in (-\infty, \infty)$, $\beta \geq 0$ and Π is a measure concentrated on $(0,\infty)$ which satisfies $\int_{(0,\infty)}(x \wedge x^2)\Pi(dx) < \infty$. Assume $\psi(\infty) = \infty$ (the growth of the total mass of X is not monotone). Denote the largest root of ψ by $\lambda^* := \inf\{\lambda \geq 0 : \psi(\lambda) > 0\}$.

Using PDE theory, Sheu [1] introduces an unusual integral condition which offers a dichotomy regarding the existence of compact support of the range of X on the event that it does not survive. Specifically, he shows that there is compact support on the event that the process does not survive (irrespective of the value of $\psi'(0+)$) if and only if

$$\int^{\infty} \frac{1}{\sqrt{\int_{\lambda^*}^{\lambda} \psi(\theta)\mathrm{d}\theta}} \mathrm{d}\lambda < \infty. \tag{1}$$

In this talk, we give a probabilistic interpretation of where this condition comes from. We do this by looking at the total mass of the super-Brownian motion X upon its first exit from an increasing sequence of spheres. Let $D_t := \{x \in \mathbb{R}^d : ||x|| < t\}$ be the sphere of radius $t > 0$ around the origin. According to Dynkin's theory of exit measures we can describe the mass of X as it first exits the growing sequence of spheres $\{D_t, t > 0\}$ as a sequence of random measures on $\mathbb{R}^d$ known as branching Markov exit measures. Fix $r > 0$ and denote by $\{X_{D_t}, t > r\}$ this sequence of branching Markov exit measures. Then X_{D_t} is a measure supported on ∂D_t which consists of the configuration of mass of X as it first exits the sphere D_t.

Consider the total mass process $Z_t := ||X_{D_t}||$, $t > r$. We show that Z is a time-varying continuous-state branching process. By looking at its semi-group equations we can show that it can be assigned a time-dependent branching mechanism,

$\Psi(t, \lambda)$ and, moreover, with the help of an important martingale, we can show that, as $t \to \infty$, this sequence of branching mechanisms converges to one of a regular continuous-state branching process, say $\Psi(\infty, \lambda)$. It now appears that Sheu's condition (1) corresponds precisely to Grey's condition,

$$\int^{\infty} \frac{1}{\Psi(\infty, \lambda)} \mathrm{d}\lambda < \infty,$$

which describes the dichotomy of extinction (all mass has disappeared after a sufficiently large time) vs extinguishing (mass never disappears but limits to zero).

Intuitively speaking, on the event that X does not survive, there is compact support if and only if all mass can be contained in a sufficiently large sphere. This is equivalent to the mass on X_{D_t} becoming zero for all sufficiently large t. As the process X_{D_t} behaves 'asymptotically' as a continuous-state branching process, Grey's condition for extinction for this 'limiting process' is precisely what is needed for the compact support dichotomy. From this reasoning Sheu's condition emerges.

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Branching Brownian motion with absorption

Jason Schweinsberg

(joint work with Julien Berestycki, Nathanael Berestycki)

Brunet, Derrida, Mueller, and Munier [5, 6] proposed the following model of a population undergoing selection. The population has fixed size N. Each individual has $k \geq 2$ offspring. The fitness of each offspring is the parent's fitness plus an independent random variable with distribution ν, and out of the kN offspring, the N offspring with the highest fitness survive to form the next generation.

Brunet et al. conjectured that if two individuals are chosen at random in some generation, then the number of generations back to their most recent common ancestor is of order $(\log N)^3$. Second, they conjectured that if one takes a sample of n individuals from the population and follows their ancestral lines backwards in time, then the merging of these lineages is governed by a process called the Bolthausen-Sznitman coalescent [4]. This means that, unlike for population models without selection, more than two ancestral lines can merge at once. More precisely, when there are b lineages, the rate at which k particular lineages simultaneously merge into one is

$$\int_0^1 x^{k-2}(1-x)^{b-k} \, dx.$$

These conjectures remain open. However, in [1], we establish these results for a process known as branching Brownian motion, which has the same key features as the model of Brunet et al. but is simpler to analyze. We assume that at time zero, there is some configuration of particles to the right of the origin. Each particle moves according to Brownian motion with a drift of $-\mu$, and each particle splits into two at rate one. Particles are killed upon reaching the origin. In our application, particles represent individuals in a population, their positions correspond to their fitnesses, and branching events represent births.

Kesten [7] showed that if $\mu \geq \sqrt{2}$, then this process dies out almost surely, while if $\mu < \sqrt{2}$, then the process survives forever with positive probability. To model a population of approximately constant size of order N, we take the drift parameter to be

$$\mu_N = \sqrt{2 - \frac{2\pi^2}{(\log N + 3\log\log N)^2}}. \tag{1}$$

Theorem: Let $M_N(t)$ be the number of particles at time t. Denote the positions of the particles at time t by $X_{1,N}(t) \geq \cdots \geq X_{M_N(t),N}(t)$. Let $L_N = (\log N + 3\log\log N)/\sqrt{2}$. Let

$$Y_N(t) = \sum_{i=1}^{M_N(t)} e^{\mu_N X_{i,N}(t)}, \quad Z_N(t) = \sum_{i=1}^{M_N(t)} e^{\mu X_{i,N}(t)} \sin\left(\frac{\pi X_{i,N}(t)}{L_N}\right) \mathbf{1}_{\{X_{i,N}(t) \leq L_N\}}.$$

Suppose $Z_N(0)/[N(\log N)^2] \Rightarrow \nu$ as $N \to \infty$ for some nonzero probability measure ν, and $Y_N(0)/[N(\log N)^3] \Rightarrow 0$. Fix $t > 0$ and sample n particles uniformly at random at time $t(\log N)^3$. Let $\Pi_N(s)$ be the partition of $\{1, \ldots, n\}$ such that i and j are in the same block of $\Pi_N(s)$ if and only if the ith and jth sampled particles come from the same ancestor at time $(t - s/2\pi)(\log N)^3$. Then the finite-dimensional distributions of $(\Pi_N(s), 0 \leq s \leq 2\pi t)$ converge to those of the Bolthausen-Sznitman coalescent run for time $2\pi t$.

While the initial conditions appear complicated, $Z_N(t)$ is a natural measure of the "size" of the process at time t, and the initial conditions hold if $O(N)$ particles are placed in a relatively stable configuration. The $(\log N)^3$ time scale and the appearance of the Bolthausen-Sznitman coalescent match predictions in [5, 6].

The reason that multiple mergers arise is that on rare occasions, one particle drifts unusually far to the right. When this happens, many of its offspring survive, as they are able to avoid being killed at zero. Eventually, a significant fraction of the population can be descended from the particle that drifted far to the right. Consequently, when ancestral lines are followed backwards in time, many lineages get traced back to that particle. Because this happens on a time scale that is much faster than $(\log N)^3$, in the limit these ancestral lines merge simultaneously.

A particle must reach approximately L_N to have a large enough effect on the population to produce a multiple merger event. The number of particles to the left of L_N stays close to its expectation. Consequently, if $B \subset (0, L_N)$ and initially there is a single particle at x, then the number of particles in the set B at time t is approximately $\int_B p_t(x,y)\,dy$, where for $t \gg L_N^2$, we have

$$p_t(x,y) \approx \frac{2}{L_N} e^{\mu x} \sin\left(\frac{\pi x}{L_N}\right) e^{-\mu y} \sin\left(\frac{\pi y}{L_N}\right). \tag{2}$$

Note that (2) is proportional to $e^{\mu x}\sin(\pi x/L_N)$, which is why $Z_N(t)$ is the natural measure of the "size" of the process at time t. Also, one can see from (2) that the particles quickly settle into a configuration in which the "density" of particles near y is proportional to $e^{-\mu y}\sin(\pi y/L_N)$. Using this information, one can calculate that particles reach L_N at a rate that is roughly proportional to $Z_N(t)$, and that the time that it takes for a particle to reach L_N is $O((\log N)^3)$.

To understand what happens after a particle reaches L_N, consider branching Brownian motion with the critical drift of $-\sqrt{2}$ started with a single particle at L_N. For $y > 0$, let $M(y)$ denote the number of particles that would have reached the level $L_N - y$, had particles been killed upon reaching that level. Neveu [8] showed that there exists a random variable W such that

$$\lim_{y\to\infty} ye^{-\sqrt{2}y}M(y) = W \quad \text{a.s.} \tag{3}$$

We show in [1] that $P(W > x) \sim C/x$ as $x \to \infty$, where $C = 1/\sqrt{2}$. Because the contribution of the descendants of the particle that reaches L_N is approximately proportional to $M(y)$ for large y, the rate of jumps greater than x in the population size is proportional to $1/x$, which is the behavior needed for the genealogy of the population to be described by the Bolthausen-Sznitman coalescent.

The tools discussed above can also be used to obtain some new results about critical branching Brownian motion, in which the process starts with a single particle at $x > 0$ and the drift parameter is $\mu = \sqrt{2}$. Let ζ be the time at which the process goes extinct. We show in [2] that there exist positive constants C_1 and C_2 such that for any fixed $x > 0$, we have

$$C_1 xe^{\sqrt{2}x}e^{-(3\pi^2 t)^{1/3}} \le P(\zeta > t) \le C_2 xe^{\sqrt{2}x}e^{-(3\pi^2 t)^{1/3}}$$

for sufficiently large t. This improves upon a result on Kesten [7].

One can also consider the asymptotic behavior of the process as the position x of the initial particle tends to infinity. Let $\tau = 2\sqrt{2}/(3\pi^2)$. We show in [2] that for all $\varepsilon > 0$, there exists $\beta > 0$ such that for sufficiently large x,

$$P(|\zeta - \tau x^3| > \beta x^2) < \varepsilon.$$

Suppose $s = ux^3$, where $0 < u < \tau$, and $t = \tau x^3$ is the approximate extinction time. Let $N(s)$ be the number of particles at time s. We show in [3] that there exist positive constants C_3 and C_4 such that for sufficiently large x,

$$P(C_3 x^{-3}e^{\sqrt{2}(1-s/t)^{1/3}x} \le N(s) \le C_4 x^{-3}e^{\sqrt{2}(1-s/t)^{1/3}x}) > 1 - \varepsilon.$$

Also, let $R(s)$ be the position of the right-most particle at time s, and define $L(s) = (1 - s/t)^{1/3}x$. We show in [3] that there exists a constant $D > 0$ such that for sufficiently large x,

$$P\left(L(s) - \frac{3}{\sqrt{2}}\log x - D < R(s) < L(s) - \frac{3}{\sqrt{2}}\log x + D\right) > 1 - \varepsilon.$$

Finally, we have some information about the configuration of particles at time s. Roughly speaking, the "density" of particles near $y \in (0, L(s))$ is proportional to

$$e^{-\sqrt{2}y} \sin\left(\frac{\pi y}{L(s)}\right).$$

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Fluctuations of the front for a one-dimensional microscopic model for the spread of an infection

Jean Bérard

(joint work with Alejandro Ramírez)

In this talk, I shall discuss a microscopic stochastic model for the spread of an infection introduced by H. Kesten and V. Sidoravicius. The model consists of two kinds of individuals, safe and infected, with each site of the one-dimensional integer lattice bearing a certain number of individuals. Individuals move according to simple symmetric random walks, with two possibly distinct jump rates for safe and infected individuals. In the case where both jump rates are positive and equal, and one starts with a homogeneous Poisson initial condition, Kesten and Sidoravicius [2, 3] have proved a law of large number for the infection front (in fact, a shape theorem valid for all dimensions $d \geq 1$). In a joint work with A. Ramírez [1], we have obtained a central limit theorem for the position of the front, under the same assumptions. Our approach is based on the definition of an appropriate renewal structure for the model. It also provides results when the jump rate of infected particles is larger than the jump rate of safe particles, up to a slight modification of the model making the infection remanent.

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Ergodic Theorems at the edge of branching Brownian motion

Louis-Pierre Arguin
(joint work with Anton Bovier, Nicola Kistler)

We prove a conjecture of Lalley and Sellke [4] asserting that the empirical (time-averaged) distribution function of the maximum of branching Brownian motion converges almost surely to a double exponential, or Gumbel, distribution with a random shift. The result is also extended to prove that the empirical joint distribution of the positions of the particles at the edge converges to a Poisson cluster process. The method of proof is based on the decorrelation of the maximal displacements for appropriate time scales. A crucial input is the localization of the paths of particles close to the maximum that was previously established by the authors [3]. In [2], this is used to obtain a new result on genealogies of the extremal particles at different times.

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1. A note on stable point processes arising in branching Brownian motion. 2. The limiting process of N-particle branching random walk with polynomial tails

Pascal Maillard
(joint work with Jean Bérard (second part))

In the first part of my talk (based on [4]) I presented a characterization of *exponentially stable point processes* which arise in the study of the extremal processes of branching Brownian motion. A point process Z on $\mathbb{R}$ is called *exponentially 1-stable* or *exp-1-stable* if for every $\alpha, \beta \in \mathbb{R}$ with $e^\alpha + e^\beta = 1$, Z is equal in law to $T_\alpha Z + T_\beta Z'$, where Z' is an independent copy of Z, T_x is the translation by x and $+$ is the addition of measures. Such processes arise in the study of the extremal particles of branching Brownian motion and branching random walk and in this setting, several authors have proven the existence of a point process D on $\mathbb{R}$ such that Z is equal in law to $\sum_{i=1}^{\infty} T_{\xi_i} D_i$, where $(\xi_i)_{i\geq 1}$ are the atoms of a Poisson process of intensity $e^{-x}\,dx$ on $\mathbb{R}$ and $(D_i)_{i\geq 1}$ are independent copies of D and independent of $(\xi_i)_{i\geq 1}$. This is also called the *LePage decomposition* of a stable point process. In my talk, I showed how this decomposition holds in general for exponentially stable point processes and follows from a simple disintegration of the Lévy measure of the point process Z. The proof also extends to the general case of random measures on $\mathbb{R}$. Note that the LePage decomposition holds in much more general settings including point processes, as shown in [1], who rely on the theory of harmonic analysis on semigroups.

In the second part of my talk, unrelated to the first, I presented work in progress (joint with Jean Bérard) on N-particle branching random walk with jumps of polynomial tails. To be precise, let X be a random variable taking values in $[0,\infty)$ with $P(X > x) = 1/h(x)$, $h(x)$ regularly varying of index $\alpha > 1$. Consider the following particle system: Initially, there are N particles located at the origin of the real line (N is a large integer). At each time step, every particle duplicates, both copies jump according to the law of X (all independently), and then the N particles at the highest position are retained; the others are removed from the system. We show that the speed v_N of this system satisfies the asymptotic relation $v_N \sim C_\alpha h^{-1}(2N \log_2 N)/\log_2 N$ as $N \to \infty$, where h^{-1} is the generalized inverse of h and C_α is a constant depending on α only. The constant C_α turns out to be the speed of the process $R(t)$ defined as follows and dubbed the *stairs process* (see Figure 1 for a graphical representation).

- $R(t) = 0$ for all $t \leq 0$.
- For $t \in (n, n+1]$, $R(t)$ is the record process of the x-values of the atoms of the Poisson point process on $(n, n+1] \times \mathbb{R}_+$ with intensity $\mu_\alpha(dt, dx) = \alpha(x - R(t-1))^{\alpha-1} 1_{x > R(t-1)}\,dt\,dx$

Note that branching random walks with heavy tails have been considered earlier in the literature, see [2] for the case of regularly varying tails and [3] for the case of stretched exponential tails).

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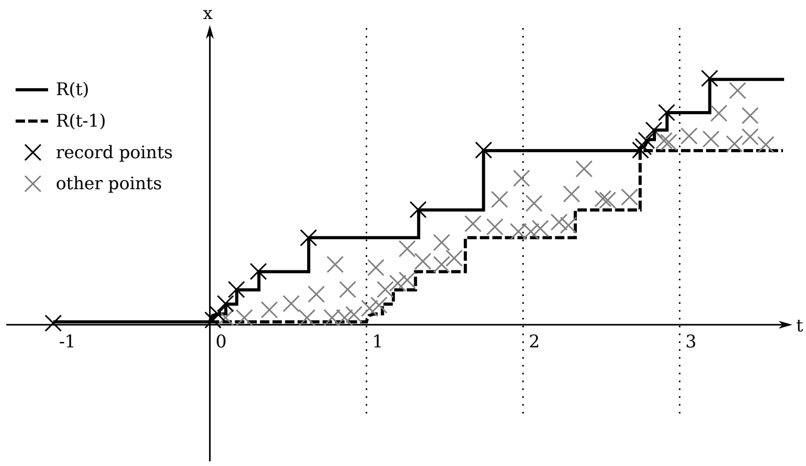

FIGURE 1. Graphical representation of the stairs process.

Convergence of the law of the maximum of two dimensional Gaussian free field

OFER ZEITOUNI
(joint work with Maury Bramson, Jian Ding)

The discrete Gaussian free field (GFF) $\{\eta_{v,N} : v \in V_N\}$, on a box $V_N \subset \mathbb{Z}^2$ of side length N with Dirichlet boundary data, is the mean zero Gaussian process that takes the value 0 on ∂V_N and satisfies the following Markov field condition for all $v \in V_N \setminus \partial V_N$: $\eta_{v,N}$ is distributed as a Gaussian random variable with variance 1, and mean equal to the average over its immediate neighbors given the GFF on $V_N \setminus \{v\}$. One aspect of the GFF that has received intense attention recently is the behavior of its maximum $\eta_N^* = \max_{v \in V_N} \eta_{v,N}$. Set

$$m_N = 2\sqrt{2/\pi}\big(\log N - \tfrac{3}{8}\log\log N\big). \tag{1}$$

The following is the main result presented in the talk:
Theorem: *The law of the random variable $\eta_N^* - m_N$ converges in distribution to a law μ_∞ as $N \to \infty$.*

A description of μ_∞ is a by-product of the proof. The technique of proof (which is contained in [2]) involves comparison with branching random walks (or their modifications), and is of interest even in the study of limit laws for maxima of branching random walks; in the latter case, convergence was established in a general setup in [1].

The talk also described some results on inhomogeneous branching Brownian motion obtained in collaboration with Ming Fang, and with Pascal Maillard.

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Quasi-stationary distributions and Yaglom limits of self-similar Markov processes

Bénédicte Haas

(joint work with Victor Rivero)

Consider X a continuous time $\mathbb{R}_+$-valued strong Markov process. We denote by $\mathbb{P}_x$ its distribution started at $x > 0$. This process is assumed to be self-similar, which means that there exists some $\alpha > 0$ such that for all $x > 0$,

$$\text{the distribution of } \{xX_{tx^{-\alpha}}, t \geq 0\} \text{ under } \mathbb{P}_1 \text{ is } \mathbb{P}_x.$$

We assume furthermore that $T_0 := \inf\{t > 0 : X_t = 0\} < \infty \quad \mathbb{P}_1$-a.s..

Our goal is to investigate the existence and characterization of quasi-stationary distributions and a Yaglom limit for this self-similar Markov process. By Yaglom limit, we mean the existence of a deterministic function g and a non-trivial probability measure ν such that the process rescaled by g and conditioned on non-extinction converges in distribution towards ν. If the study of quasi-stationary distributions is easy and follows mainly from a previous result by Bertoin and Yor [2], that of Yaglom limits is more challenging.

Using the well-known Lamperti representation of self-similar Markov processes in terms of Lévy processes [1], we prove that a Yaglom limit exits if and only if the extinction time at 0 of the process is in the domain of attraction of an extreme law. We then treat separately three cases, according whether the extinction time is in the domain of attraction of a Gumbel law, a Weibull law or a Fréchet law. In each of these cases, necessary and sufficient conditions on the parameters of the underlying Lévy process are given for the extinction time T_0 to be in the required domain of attraction. The limit of the process conditioned to be positive is then characterized by a multiplicative equation which is connected to a factorization of the exponential distribution in the Gumbel case, a factorization of a Beta distribution in the Weibull case and a factorization of a Pareto distribution in the Fréchet case. We emphasize that the existence of a Yaglom limit, the asymptotic behavior of the function g and the structure of the measure ν (when they exist) are strongly different for monotone processes X and non-monotone processes X. Applications to stable CSPB, stable Lévy trees and self-similar diffusions are then discussed. See also [3] for a related work and applications to the asymptotic behavior of solutions to the fragmentation equation.

Our proofs rely partly on results on the tail distribution of the extinction time T_0, which is known to be distributed as the exponential integral of a Lévy process.

In that aim, new results on such tail distributions are given, which may be of independent interest. Details can be found in the paper [4].

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Kardar-Parisi-Zhang equation and universality class

Jeremy Quastel

The KPZ equation in $d = 1$ is,

$$\partial_t h = -\tfrac{1}{2}(\partial_x h)^2 + \tfrac{1}{2}\partial_x^2 h + \xi \qquad (1)$$

where ξ denotes space-time white noise. It is an equation for a randomly evolving height function $h \in \mathbb{R}$ which depends on position $x \in \mathbb{R}$ and time $t \in \mathbb{R}_+$. The derivative $u = \partial_x h$ should satisfy the stochastic Burgers equation $\partial_t u = -\frac{1}{2}\partial_x u^2 + \frac{1}{2}\partial_x^2 u + \partial_x \xi$ which had been studied earlier: Forster, Nelson and Stephen predicted a dynamic scaling exponent $z = 3/2$ which roughly means we expect to see something interesting on the large scale $u_\epsilon(t,x) = \epsilon^{-1/2}u(\epsilon^{-3/2}t, \epsilon^{-1}x)$ as $\epsilon \searrow 0$.

Kardar, Parisi and Zhang reinterpreted (1) as a canonical model for random interface growth. The idea is to identify three key mechanisms of growth (in real situations there are of course many others): 1. Slope dependent, or lateral growth; 2. Relaxation; 3. Random forcing. The $\partial_x^2 h$ term represents the simplest possible form of relaxation/smoothing/diffusion. The simplest model for random forcing is that it is independent at different positions and different times, so we take it to be Gaussian space-time white noise. The key term is the lateral growth. It should depend on the slope only, and in a symmetric way, hence $F(\partial_x h)$ where F is the flux function. Expanding $F(s) = F(0) + F'(0)s + \frac{1}{2}F''(0)s^2 + \cdots$ one checks that the first and second term can be removed by simple changes of coordinates. Hence we choose (1) as the simplest and therefore canonical model. It is remarkable that through such a naive derivation, one arrives at what appears to be the only non-trivial model of this type (see [7], [6], [2]).

Two sided Brownian motion $B(x)$, $x \in \mathbb{R}$ normalized to have $E[(B(y)-B(x))^2] = |y-x|$ is invariant for (1). More precisely [4], the measure corresponding to $B(x) + N$, where N is given by Lebesgue measure (i.e. the product measure of Lebesgue measure for N and two-sided Brownian motion measure for $B(\cdot)$), is invariant for (1). For the stochastic Burgers equation, since the global height shift is killed by the derivative, the statement is simply that a spatial white noise is invariant [3]. For other initial data $h(0,x)$, for KPZ, no matter how smooth, $h(t,x)$ is locally Brownian in x, with the same local diffusivity, for any time $t > 0$. However, this means that the KPZ equation (1) is very ill-posed, as the non-linear term is clearly infinite. It needs some sort of infinite renormalization,

$$\partial_t h = -[\tfrac{1}{2}(\partial_x h)^2 - \infty] + \tfrac{1}{2}\partial_x^2 h + \xi. \tag{2}$$

The problem is that the nonlinear term is really being computed on a larger scale and is not supposed to be seeing the small scale fluctuations. Bertini and Giacomin [3], proposed that the solution of KPZ should simply be

$$h(t,x) = -\log z(t,x) \tag{3}$$

where $z(t,x)$ is the solution of the stochastic heat equation with multiplicative noise

$$\partial_t z = \tfrac{1}{2}\partial_x^2 z - z\xi. \tag{4}$$

It is to be interpreted in the Itô sense, in which case it is well posed. Recently it has been shown by M. Hairer [5] that (2) is well-posed with (3) as solutions.

If we look on the large scale $h_\epsilon(t,x) := \epsilon^{1/2} h(\epsilon^{-3/2}t, \epsilon^{-1}x)$ we get

$$\partial_t h_\epsilon = -\tfrac{1}{2}(\partial_x h_\epsilon)^2 + \tfrac{1}{2}\epsilon^{1/2}\partial_x^2 h_\epsilon + \epsilon^{1/4}\xi. \tag{5}$$

To see fluctuation on this scale is the roughest definition of the KPZ universality class, which is expected to include a wide variety of stochastically forced systems in $d = 1$ with non-linearities. On a finer scale one expects to see the random matrix distributions and Airy processes one obtains in the limit of (5) for special solvable models. An intriguing question which links KPZ to the topic of the meeting is

Conjecture 1. *Consider branching random walks on $\mathbb{Z}^2$ with a "cutoff", i.e. some rule which restricts the number of particles per site (for example, one could instantaneously kill any particle which jumped to, or was born at, at site with more than K particles). Then the boundary of the occupied set is in the KPZ universality class.*

Asymmetric simple exclusion process. In ASEP, particles on $\mathbb{Z}$ attempt to perform independent continuous time simple random walks jumping to the left at rate q and to the right at rate $p = 1 - q$. However, the jumps only take place if the target site is unoccupied. The height function $h^{ASEP}(x)$ is a random walk path that takes a jump up whenever there is a particle at that site, and a jump down whenever there is no particle at that site. The height function should be thought of as a special discretization of the KPZ equation, where the strength of the non-linearity is $q - p$. The weakly asymmetric limit of ASEP is:

Theorem 1. [3, 1] *Suppose that the initial data $\epsilon^{1/2}h^{\mathrm{ASEP}}(t = 0, \epsilon^{-1}x)$ are chosen to nicely approximate the initial conditions for the KPZ equation. Then there is a $C_\epsilon(t)$ such that $\epsilon^{1/2}h^{\mathrm{ASEP}}_{q-p=\epsilon^{1/2}}(\epsilon^{-2}t, \epsilon^{-1}x) - C_\epsilon(t) \to$ the Hopf-Cole solution.*

C. Tracy and H. Widom discovered exact formulas for ASEP.

Theorem 2. [9] *Let $q > p$ with $q + p = 1$, $\gamma = q - p$, $\tau = p/q$, $\alpha = (1-\rho)/\rho$. Let ASEP start with $\mathbb{Z}_-$ empty and Bernoulli product measure, density ρ on $\mathbb{Z}_+$. For $m = \lfloor \frac{1}{2}(s + x) \rfloor$, $t \geq 0$ and $x \in \mathbb{Z}$*

$$P(h^{ASEP}(t,x) \geq s) = \int_{S_{\tau+}} \frac{d\mu}{\mu} \prod_{k=0}^{\infty} (1 - \mu\tau^k) \det(I + \mu J\mu)_{L^2(\Gamma_\eta)} \tag{6}$$

where $S_{\tau+}$ is a circle centered at zero of radius strictly between τ and 1, and where $J_\mu(\eta, \eta') = \int_{\Gamma_\zeta} \exp\{\Lambda(\zeta) - \Lambda(\eta')\} \frac{f(\mu,\zeta/\eta')}{\eta'(\zeta-\eta)} \frac{g(\eta')}{g(\zeta)} d\zeta$ where $f(\mu, z) = \sum_{k=-\infty}^{\infty} \frac{\tau^k}{1-\tau^k\mu} z^k$, $\Lambda(\zeta) = -x\log(1-\zeta) + \frac{t\zeta}{1-\zeta} + m\log\zeta$, $g(\zeta) = \prod_{n=0}^{\infty}(1 + \tau^n\alpha\zeta)$.

By studying the weakly asymmetric limit using steepest descent, one obtains exact formulas for KPZ.

Theorem 3. [1], [8] *Let $z(t,x)$ be the solution of the stochastic heat equation (4) with initial data $z(0,x) = \delta_0(x)$ and $h(t,x) = -\log z(t,x)$.*

$$P(h(t,x) + \tfrac{x^2}{2t} + \log\sqrt{2\pi t} + \tfrac{t}{24} \geq -s) = \int_C \frac{d\mu}{\mu} e^{-\mu} \det(I - K_{\sigma_{t,\mu}})_{L^2(\kappa_t^{-1}s,\infty)}. \tag{7}$$

where $\kappa_t = 2^{-1/3}t^{1/3}$, C is a contour positively oriented and going from $+\infty + \epsilon i$ around $\mathbb{R}^+$ to $+\infty - i\epsilon$, and K_σ is an operator given by its integral kernel $K_\sigma(x,y) = \int_{-\infty}^{\infty} \sigma(\tau)\mathrm{Ai}(x+\tau)\mathrm{Ai}(y+\tau)d\tau$ and $\sigma_{t,\mu}(\tau) = \frac{\mu}{\mu - e^{-\kappa_t\tau}}$.

Since then, formulas for the one-point distribution have been obtained for other basic scaling invariant initial data (e.g. half-Brownian, and Brownian), in parallel by mathematicians through exact formulas for various microscopic modes, and by physicists using the (non-rigorous) replica methods. The most challenging case seems to be the flat and half-flat case. Proposed formulas were obtained by Calabrese and le Doussal. The following formula is a rigorously provable version of their divergent series for $E\big[e^{-Z(t,x)s}\big]$ in the half-flat case $Z(t=0,x) = \mathbf{1}_{x\geq 0}$, where $s = e^{-\frac{t}{24} - t^{1/3}r}$. It is joint work in progress with Janosch Ortmann and Daniel Remenik. Taking $0 < \delta_1 < \delta_2 << 1$ the formula is

$$\sum_{k=0}^{\infty} \frac{1}{(2\pi i)^{2k}k!} \int_{(\delta_1+i\mathbb{R})^k} d\vec{u} \int_{(\delta_2+i\mathbb{R})^k} d\vec{v} \prod_{a<b} \frac{\Gamma(u_a - u_b + v_a + v_b)\Gamma(u_b - u_a + v_a + v_b)}{\Gamma(-u_a - u_b + v_a + v_b)\Gamma(u_a + u_b + v_a + v_b)}$$
$$\times \prod_a \frac{2\pi}{\sin(-2\pi u_a)} \frac{\Gamma(2v_a)}{\Gamma(2(u_a+v_a))} e^{\frac{tu_a^3}{24} + \frac{u_a v_a x}{2} + \frac{tu_a v_a^2}{8} - 2rt^{1/3}u_a} \det\left[\frac{1}{u_a + u_b + v_a - v_b}\right].$$

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Last passage percolation and traveling fronts

Alejandro F. Ramírez
(joint work with Francis Comets, Jeremy Quastel)

We consider the following stochastic process introduced by Brunet and Derrida [3]. It consists of a fixed number $N \geq 1$ of particles on the real line, initially at the positions $X_1(0), \ldots, X_N(0)$. With $\{\xi_{j,i}(s) : 1 \leq i, j \leq N, s \geq 1\}$ an i.i.d. family of real random variables, the positions evolve as

$$X_i(t+1) = \max_{1 \leq j \leq N} \big\{ X_j(t) + \xi_{j,i}(t+1) \big\}. \tag{1}$$

The vector $X(t)$ describes the location after the t-th step of a population under reproduction, mutation and selection keeping the size constant. Given the current positions of the population, the next positions are a N-sample of the maximum of the full set of previous ones evolved by an independent step. It can be also viewed as long-range directed polymer in random medium with N sites in the transverse direction,

$$X_i(t) = \max \Big\{ X_{j_0}(0) + \sum_{s=1}^{t} \xi_{j_{s-1},j_s}(s); 1 \leq j_s \leq N \quad \forall s = 0, \ldots t-1, j_t = i \Big\}, \tag{2}$$

as can be checked by induction ($1 \leq i \leq N$). The model is long-range since the maximum in (1) ranges over all j's.

Traveling fronts appear in mean-field models for random growth. This was discovered by Derrida and Spohn [5] for directed polymers in random medium on the tree, and then extended to other problems [7, 8].

The present model was introduced by Brunet and Derrida in [3] to compute the corrections for large but finite system size to some continuous limit equations in front propagation. Corrections are due to finite size, quantization and stochastic effects. They predicted, for a large class of such models where the front is pulled by the farmost particles [3, 4], that the motion and the particle structure have universal features, depending on just a few parameters related to the upper tails. Some of these predictions have been rigorously proved in specific contexts, such as the corrections to the speed of the Branching Random Walk (BRW) under the effect of a selection [1], of the solution to KPP equation with a small stochastic noise [9], or the genealogy of branching Brownian motions with selection [2]. For the so-called N-BBM (branching Brownian motion with killing of leftmost particles to keep the population size constant and equal to N) the renormalized fluctuations for the position of the killing barrier converge to a Levy process as N diverges [6].

We now give a flavor of our results. The Gumbel law $\mathrm{G}(0,1)$ has distribution function $\mathbb{P}(\xi \le x) = \exp(-e^{-x}), x \in \mathbb{R}$. In [3] it is shown that an appropriate measure of the front location of a state $X \in \mathbb{R}^N$ in this case is

$$\Phi(X) = \ln \sum_{1 \le j \le N} e^{X_j}, \tag{3}$$

and that $\Phi(X(t))$ is a random walk, a feature which simplifies the analysis. For an arbitrary distribution of ξ, the speed of the front with N particles can be defined as the almost sure limit

$$v_N = \lim_{t \to \infty} t^{-1} \Phi(X(t)).$$

Our first result is the scaling limit as the number N of particles diverges.

Theorem 1. *Assume $\xi_{j,i}(t) \sim \mathrm{G}(0,1)$. Then, for all sequences $m_N \to \infty$ as $N \to \infty$,*

$$\frac{\Phi(X([m_N \tau])) - \beta_N m_N \tau}{m_N / \ln N} \xrightarrow{\text{law}} S(\tau)$$

in the Skorohod topology with $S(\cdot)$ a totally asymmetric Cauchy process with Lévy exponent

$$\psi_C := iCu - \frac{\pi}{2}|u|\left\{1 + i\frac{2}{\pi}\mathrm{sign}(u) \ln |u|\right\}$$

and where

$$\beta_N = \ln b_N + N b_N^{-1} \ln m_N,$$

with $\ln b_N = \ln N + \ln \ln N - \frac{\gamma}{\ln N} + \mathcal{O}(\frac{1}{\ln^2 N})$.

Fluctuations of the front location are Cauchy distributed in the large N limit. Keeping N fixed, the authors in [3] find that they are asymptotically Gaussian as $t \to \infty$. We prove here that, as N is sent to infinity, they are stable with index 1, a fact which has been overlooked in [3]. The Cauchy limit also holds true in the boundary case when time is not speeded-up ($m_N = 1$) and $N \to \infty$.

We next consider the case when ξ is a perturbation of the Gumbel law. Define $\varepsilon(x) \in [-\infty, 1]$ by

$$\varepsilon(x) = 1 + e^x \ln \mathbb{P}(\xi \le x). \tag{4}$$

Note that $\varepsilon \equiv 0$ is the case of $\xi \sim \mathrm{G}(0,1)$. The empirical distribution function (more precisely, its complement to 1) of the N-particle system (1) is the random function

$$U_N(t,x) = N^{-1} \sum_{i=1}^{N} \mathbf{1}_{X_i(t) > x}. \tag{5}$$

This is a non-increasing step function with jumps of size $1/N$ and limits $U_N(t,-\infty) = 1$, $U_N(t,+\infty) = 0$. It has the shape of a front wave, propagating at mean speed v_N, and it combines two interesting aspects: randomness and discrete values. We will call it the front profile, and we study in the next result its relevant part, around the front location.

Theorem 2. *Assume that*

$$\lim_{x\to+\infty} \varepsilon(x) = 0, \qquad \text{and} \quad \varepsilon(x) \in [-\delta^{-1}, 1-\delta], \tag{6}$$

for all x and some $\delta > 0$. Then, for all initial configurations $X(0) \in \mathbb{R}^N$, all $k \geq 1$, all $K_N \subset \{1,\dots,N\}$ with cardinality k, and all $t \geq 2$ we have

$$\Big(X_j(t) - \Phi(X(t-1)); j \in K_N\Big) \xrightarrow{\text{law}} G(0,1)^{\otimes k}, \quad N \to \infty, \tag{7}$$

with Φ from (3), and moreover,

$$U_N\big(t, \Phi(X(t-1)) + x\big) \longrightarrow u(x) = 1 - e^{-e^{-x}} \tag{8}$$

uniformly in probability as $N \to \infty$.

Finally, we study the finite-size corrections to the front speed in a case when the distribution of ξ is quite different from the Gumbel law.

Theorem 3. *Let $b < a$ and $p \in (0,1)$, and assume that the $\xi_{j,i}(t)$'s are integrable and satisfy*

$$\mathbb{P}(\xi > a) = \mathbb{P}(\xi \in (b,a)) = 0, \qquad \mathbb{P}(\xi = a) = p, \qquad \mathbb{P}(\xi \in (b-\varepsilon, b]) > 0 \tag{9}$$

for all $\varepsilon > 0$. Then, as $N \to \infty$,

$$v_N = a - (a-b)(1-p)^{N^2} 2^N + o\big((1-p)^{N^2} 2^N\big).$$

We note that in such a case, in the leading order terms of the expansion as $N \to \infty$, the value of the speed depends only on a few features of the distribution of ξ: the largest value a, its probability mass p and the gap $a - b$ with second largest one. All these involve the top of the support of the distribution, the other details being irrelevant. Such a behavior is expected for pulled fronts.

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Behaviour near the extinction time in self-similar fragmentations with negative index

Christina Goldschmidt
(joint work with Bénédicte Haas)

This work is based on the paper [5]. We study a Markovian model for the random fragmentation of an object. The state at any time takes values in

$$\mathcal{S} = \left\{ \mathbf{s} = (s_1, s_2, \ldots) : s_1 \geq s_2 \geq \ldots \geq 0, \sum_{i=1}^{\infty} s_i < \infty \right\},$$

where $\mathbf{s} \in \mathcal{S}$ represents the masses or sizes of the blocks present in the system. The transition mechanism depends only on these sizes and is characterised by two parameters: $\alpha \in \mathbb{R}$ (the *index of self-similarity*) and a probability measure ν on $\mathcal{S}_1 \setminus \{\mathbf{1}\}$, where $\mathcal{S}_1 = \{\mathbf{s} \in \mathcal{S} : \sum_{i=1}^{\infty} s_i = 1\}$ and $\mathbf{1} = (1, 0, 0, \ldots)$ (ν is called the *dislocation measure*). Different blocks evolve independently. A block of size x fragments at rate x^α into sub-blocks of random sizes $(xS_1, xS_2, \ldots)$ where $(S_1, S_2, \ldots) \sim \nu$. We write $F(t) = (F_1(t), F_2(t), \ldots) \in \mathcal{S}$ for the state at time t and, by default, take $F(0) = \mathbf{1}$. This is a special case of a model introduced by Bertoin in [3] (in particular, he allowed the possibility of an infinite dislocation measure ν, which means roughly that splitting events can take place on a dense set of times). We note that F started from state $x\mathbf{1}$ has the same law as $xF(x^\alpha \cdot)$ with F started from $\mathbf{1}$.

The Markov process $(F(t), t \geq 0)$ is clearly transient and has an absorbing state at $\mathbf{0} = (0, 0, \ldots)$. Its behaviour, however, is heavily dependent on the sign of α; here we focus on the case $\alpha < 0$, where small blocks split faster than larger ones. Indeed, jumps of the fragmentation accumulate in such a way that the random time

$$\zeta = \inf\{t \geq 0 : F(t) = \mathbf{0}\}$$

is almost surely finite. We call ζ the *extinction time*.

We observe that F can be thought of as a *discounted branching random walk*, a term coined by Athreya [2] in the case where the dislocation measure is $\nu = \delta_{(1/2, 1/2, 0, \ldots)}$. Imagine that we start with a single particle, of unit mass, whose displacement from the origin is exponentially distributed with parameter 1. Particles give birth to offspring particles of relative masses $(S_1, S_2, \ldots)$ distributed according to ν. In general, a particle of size x gets a displacement away from its parent distributed as $x^{-\alpha}$ times an independent standard exponential random variable. The random variable ζ then corresponds to the limiting position of the rightmost particle in this model.

In our work, we investigate how the fragmentation process F behaves as it approaches its extinction time.

Theorem 1. Suppose that ν is non-geometric (i.e.

$$\nu(s_i \in r^{\mathbb{N}} \cup \{0\} \text{ for all } i \geq 1) = 0$$

for every $r \in (0,1)$) and such that $\int_{\mathcal{S}_1} s_1^{-1} \nu(d\mathbf{s}) < \infty$. Then there exists a non-trivial càdlàg $\mathcal{S}$-valued self-similar process $(C(t), t \geq 0)$ such that, as $\epsilon \to 0$,

$$\left(\epsilon^{1/\alpha} F((\zeta - \epsilon t)-), t \geq 0\right) \to (C(t), t \geq 0)$$

in distribution for the Skorohod topology, with the pointwise distance on $\mathcal{S}$.

We proved an analogous version of this result in our earlier paper [4] for the *stable fragmentations*, a particular class of self-similar fragmentations which have infinite dislocation measures. The stable fragmentations can be represented in terms of stable Lévy trees and the methods used in our earlier paper rely heavily on excursion theory for these trees. The methods used in the present work are quite different. A key tool is the *last fragment process*, $(F_*(t), t \geq 0)$, which represents the size of the block at time t which is the ancestor of the unique block which disappears at time ζ. (That this statement makes sense is something which requires proof!) Of course, F_* is not measurable with respect to the natural filtration of the fragmentation process. However, it turns out that we can use it to give us a *spine decomposition* of the fragmentation.

First, let $0 = T_0 < T_1 < T_2 < \ldots$ be the successive split times of F_*, so that $T_n \to \zeta$ almost surely as $n \to \infty$. Then, if we think about the process from the perspective of the "natural timescale" for the last fragment (i.e. scaling the last fragment back to have size 1), at time T_n, our updated notion of the extinction time is given by $Z_n := (F_*(T_n))^\alpha(\zeta - T_n)$. It turns out that $(Z_n)_{n\geq 0}$ is a time-homogeneous $\mathbb{R}_+$-valued Markov chain in its own filtration which, by standard Foster-Lyapunov criteria, can be shown to possess a unique stationary distribution to which it converges. Moreover, the Markov chain $(Z_n)_{n\geq 0}$ drives a bigger Markov chain $(Z_n, \Theta_n, \mathbf{\Delta}_n)_{n\geq 0}$ which additionally tracks the relative sizes of the fragments produced by the split at time T_n (Θ_n corresponds to the last fragment and $\mathbf{\Delta}_n$ to the others). The whole state of the fragmentation at time $\zeta - \epsilon$ can be described as a complicated functional of these quantities. As $\epsilon \to 0$, the picture becomes stationary, and this gives rise to the limit in distribution stated in the theorem. Our proof makes crucial use of Markov renewal theory, as presented, for example, in Alsmeyer [1].

An interesting consequence of Theorem 1 is that it gives us access to a stationary measure for the fragmentation process F (since F is transient, this is necessarily an infinite measure).

Theorem 2. Suppose that ν is non-geometric and that $\int_{\mathcal{S}_1} s^{-1-\eta}\nu(d\mathbf{s}) < \infty$ for some $\eta > 0$. Then the σ-finite measure λ on $(\mathcal{S}, \mathcal{B}(\mathcal{S}))$ defined by

$$\lambda(A) = \int_0^\infty \mathbb{P}(C(t) \in A)dt$$

for $A \in \mathcal{B}(\mathcal{S})$ is invariant for the transition kernel of the fragmentation process.

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The size of the giant component in preferential attachment networks

Maren Eckhoff

(joint work with Peter Mörters)

We study the preferential attachment model introduced in [1]. Fix a concave function $f : \mathbb{N}_0 \to (0, \infty)$, called *attachment rule*, with $f(0) \le 1$ and $f(k+1) - f(k) < 1$ for all $k \in \mathbb{N}_0$. A growing sequence $(G_n : n \in \mathbb{N})$ of random graphs is defined by the following dynamics:

- Start with one vertex labelled 1 and no edges;
- Given the graph G_n, we construct G_{n+1} from G_n by adding a new vertex labelled $n+1$ and, for each $m \le n$ independently, inserting the directed edge $(n+1, m)$ with probability

$$\frac{f(\text{indegree of } m \text{ at time } n)}{n}.$$

The network $(G_n)_n$ has a *giant component* if there exists a constant $\theta(f) > 0$ such that

$$\frac{\#\text{vertices in the largest component of } G_n}{n} \xrightarrow{\mathbb{P}} \theta(f) \quad \text{as } n \to \infty. \tag{1}$$

Dereich and Mörters [2] coupled the neighbourhood of a vertex in the graph to a certain multitype branching random walk (BRW) with absorbing barrier at the origin. Offspring correspond to older vertices the further left of the origin they are located. The coupling was heuristically explained in the talk. Every particle in the BRW gives birth to a Poisson number of offspring to its left and an infinite number of offspring to its right, where all but a finite number of right-offspring are immediately killed by the absorbing barrier. The type of a particle represents

its relative position to its parent.
Dereich and Mörters [2] proved that the convergence in (1) holds and that the limit $\theta(f)$ is equal to the survival probability of the BRW with absorption. If the asymptotic slope of the attachment rule $\gamma := \lim_{k\to\infty} f(k)/k$ is greater or equal than $1/2$, then $\theta(f) > 0$. If $\gamma < 1/2$, then there exists $\delta \in [0, f(0))$ such that the network sequence generated with attachment rule $g(k) = f(k) - \delta$ does not have a giant component.
In the case $\gamma < 1/2$, we study the behaviour of $\theta(f)$ when f is close to criticality.

Let

$$A_\alpha g(\tau_0) := E_{\tau_0}\Big[\sum_{|x|=1} e^{-\alpha V(x)} g(\tau(x))\Big],$$

the Laplace transform of the offspring distribution of a particle of type τ_0. Here, $V(x)$ denotes the displacement of particle x to its parent's position and $\tau(x)$ denotes its type. Let $\rho(\alpha)$ be the spectral radius of A_α on the space of continuous, bounded functions and α^* the minimizer of ρ.

Theorem 1. *Let $(f_t)_t$ be attachment rules with $\gamma_t < 1/2$ and $\theta(f_t) > 0$ for all t. If $(f_t)_t$ is pointwise descreasing with $f_t \downarrow f$ and $\theta(f) = 0$, then*

$$\limsup_{t\to\infty} \sqrt{\log \rho_t(\alpha_t^*)}\, \log \theta(f_t) \le -\sqrt{\frac{\pi^2 \rho''(\alpha^*)}{2}}\,\alpha^*.$$

Here, all quantities derived from f_t are marked by an index t.

The most important parameter choice is f affine, that is $f(k) = \gamma k + \beta$ for $\gamma \in [0, 1)$, $\beta \in (0, 1]$. Theorem 1 implies that in this case, the size of the giant component decays exponentially fast at criticality. For instance, denoting by $\beta_c = \beta_c(\gamma)$ the largest intercept β with $\theta(\gamma \cdot +\beta) = 0$, we obtain:

Corollary 2. *Let $0 \le \gamma < 1/2$. Then*

$$\limsup_{\beta\downarrow\beta_c} \sqrt{\beta - \beta_c}\, \log \theta(\gamma \cdot +\beta) \le -\frac{\pi}{2}\frac{1}{\sqrt{1-\gamma}}.$$

The proof generalizes the techniques used by Gantert, Hu and Shi [3] to investigate a fixed BRW without types killed at a wall close to the asymptotic speed of the BRW. The core of our analysis is then the study of the Markov chain associated to the BRW via Lyons' change of measure.
Similar questions have been studied with different methods by Riordan [4] for certain random graph models. In the case $f(k) = \beta$, we recover the upper bound found by Riordan for the so-called "uniformly grown random graph" or "Dubins' model".

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The dual tree of a recursive triangulation of the disk

Nicolas Broutin
(joint work with Henning Sulzbach)

We consider the model of random recursive *lamination* of the disk introduced by Curien and Le Gall [4]. The disk we consider is $\mathscr{D} = \{z : 2\pi|z| \le 1\}$, so that the perimeter is one and can be identified with $[0,1]$. We use $[\![x,y]\!]$ to denote the chord with endpoints corresponding to $x < y$, $x, y \in [0,1]$.

For us, a lamination is a collection of straight chords of the disk which are non-intersecting, except possibly at their end points. The model is iterative: At $n = 1$, two points are sampled independently with uniform distribution on the circle. They are connected by a chord which splits the disk into two fragments. Later on, at each step, two independent points are sampled uniformly at random on the circle and are connected by a chord if the latter does not intersect any of the previously inserted chords; in other words the two points are connected by a chord if they both fall in the same fragment.

At time n the lamination $\mathfrak{L}_n$ consists of the union of the chords inserted up to time n. As an increasing closed subset of the disk, $\mathfrak{L}_n$ converges, and it is proved in [4] that

$$\mathfrak{L}_\infty = \overline{\bigcup_{n\ge 1} \mathfrak{L}_n}$$

is a triangulation of the disk in the sense that any face of the complement is an open triangle whose vertices lie on the circumference of the circle (see [1]).

Curien and Le Gall [4] also describe L_∞ as the lamination encoded by a random continuous process. Their analysis relies on the observation [1] that the lamination L_n may be described by the height function H_n such that for $s \in [0,1]$, $H_n(s)$ is the number of chords of L_n which intersect the straight line joining the points 0 and s on the circle. Using arguments based on the theory of fragmentations [2], they prove that there exists a random continuous process $\mathscr{M}$ such that, for every $s \in [0,1]$,

$$n^{-\beta/2} H_n(s) \to \mathscr{M}(s)$$

in probability as $n \to \infty$, where $\beta = (\sqrt{17}-3)/2$. The random process $\mathscr{M}$ inherits the recursive structure of the lamination and satisfies the following distributional fixed-point equation: let $\mathscr{M}^{(0)}, \mathscr{M}^{(1)}$ denote independent copies of $\mathscr{M}$, let also (U,V) be independent of $(\mathscr{M}^{(0)}, \mathscr{M}^{(1)})$ with density $2\mathbf{1}_{\{0\le u\le v\le 1\}}$ on $[0,1]^2$. Then

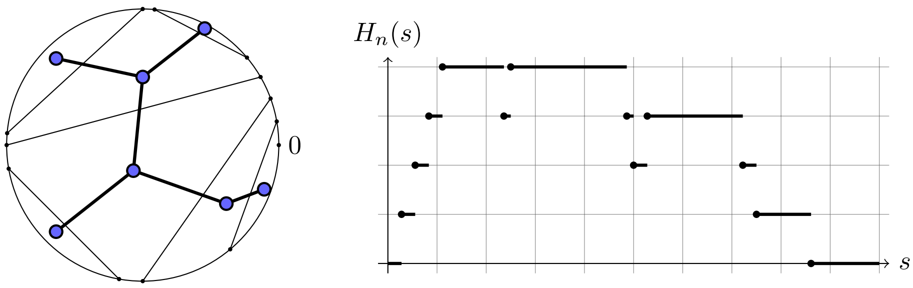

FIGURE 1. A lamination, its right-continuous height process and the corresponding rooted dual tree. Distances in the tree correspond to the number of chords separating fragments in the lamination.

the process defined by

$$\begin{cases} (1-V+U)^{\beta}\mathscr{M}^{(0)}\left(\dfrac{s}{1-(V-U)}\right) & \text{if } s<U \\ (1-V+U)^{\beta}\mathscr{M}^{(0)}\left(\dfrac{U}{1-(V-U)}\right)+(V-U)^{\beta}\mathscr{M}^{(1)}\left(\dfrac{s-U}{V-U}\right) & \text{if } U\le s<V \\ (1-V+U)^{\beta}\mathscr{M}^{(0)}\left(\dfrac{s-(V-U)}{1-(V-U)}\right) & \text{if } s\ge V, \end{cases}$$

is distributed as $\mathscr{M}$. Then, the limit triangulation $\mathfrak{L}_\infty$ is encoded by a random process $\mathscr{M} : [0,1] \to \mathbb{R}_+$ in the sense that $\mathfrak{L}_\infty$ is the set of all the chords $[\![x,y]\!]$ for which one has

$$\mathscr{M}(x) = \mathscr{M}(y) = \inf_{x\le s\le y} \mathscr{M}(s).$$

The lamination $\mathfrak{L}_n$ is naturally associated to its dual tree T_n: a node is associated to each connected component of $\mathscr{D} \setminus \mathfrak{L}_n$, and two nodes are connected by an edge if the two corresponding connected components share a chord as part of their boundary. Then, $H_n(s)$ is the distance in T_n between the root (the node associated to the fragment that has 0 on its boundary) and the node corresponding to the point $s \in [0,1]$. In other words, the function H_n encodes the metric of the tree T_n.

In the same way that a lamination of the disk is associated to a function, a tree-like metric space can be associated to continuous function $f : [0,1] \to \mathbb{R}_+$ such that $f(0) = f(1) = 0$. One first defines the pseudo-metric

$$d_f(x,y) = f(x) + f(y) - 2\inf\{f(s) : x \wedge y \le s \le x \vee y\}$$

and the equivalence relation $\sim_f$ by $x \sim_f y$ if $d_f(x,y) = 0$. Then $\mathcal{T}_f = ([0,1]/\sim_f, d_f)$ is a metric space which happens to be a compact real tree in the sense of [5, 6] (a geodesic metric space without loops).

Curien and Le Gall conjecture that, seen as a metric space, $n^{-\beta/2}T_n$ (the tree T_n equipped with the graph distance rescaled by $n^{-\beta/2}$) converges to the tree encoded by $\mathscr{M}$ in the Gromov–Hausdorff sense. We prove that this is indeed the case by showing:

Theorem [3]. *We have*

$$\|n^{-\beta/2}H_n - \mathscr{M}\|_\infty \to 0$$

almost surely and in every L^p*, for* $p \geq 1$*.*

Classical results about convergence of real trees [6] then imply convergence of the $n^{-\beta/2}T_n$ to $\mathcal{T}_{\mathscr{M}}$. The proof relies on a new construction of the limit process $\mathscr{M}$ which is functional from the start and allows coupling with the finite-n trees.

We also find the fractal dimension of the metric space $\mathcal{T}_{\mathscr{M}}$. For a compact metric space (X, d), let $N(X, r)$ denote the minimum number of balls of radius r required to cover X. Then, we prove:

Proposition. *We have, almost surely*

$$\frac{N(\mathcal{T}_{\mathscr{M}}, r)}{\log(1/r)} \xrightarrow[r\downarrow 0]{} \frac{1}{\beta}.$$

As a consequence, the box-counting dimension $\dim_M(\mathcal{T}_{\mathscr{M}}) = 1/\beta$*.*

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The shape of multidimensional Brunet–Derrida particle systems

Lee Zhuo Zhao

(joint work with Nathanael Berestycki)

We introduce particle systems in one or more dimensions in which particles perform branching Brownian motion and the population size is kept constant equal to $N > 1$, through the following selection mechanism: at all times only the N fittest particles survive, while all the other particles are removed. Fitness is measured with respect to some given fitness function $s : \mathbb{R}^d \to \mathbb{R}$.

This process can be seen as a multidimensional generalisation of the model of branching Brownian motion with selection in $\mathbb{R}$ introduced by Brunet, Derrida, Mueller and Munier [6, 7]. This is the model which arises as a particular case of the above description with $d = 1$ and $s(x) = x$.

The motivation for this process comes from the study of the effect of natural selection on the genealogy of a population. Using nonrigorous methods, Brunet et al. made several striking predictions, which we summarise below. Ordering the particles from right to left (so $X_1(t) \geq \ldots \geq X_N(t)$):

(i) For fixed N, $\lim_{t\to\infty}(X_1(t)/t) = \lim_{t\to\infty}(X_N(t)/t) = v_N$, almost surely, where v_N is a deterministic constant such that as $N \to \infty$, $v_N = v_\infty - c/(\log N)^2 + o((\log N)^{-2})$, where v_∞ is the speed of the rightmost particle in a free branching Brownian motion (or free branching random walk if time is discrete), and c is an explicit constant.

(ii) The genealogical time scale for this population is $(\log N)^3$. More precisely, the genealogy of an arbitrary sample of the population, rescaled by $(\log N)^3$, converges to the Bolthausen–Sznitman coalescent.

The arguments of Brunet et al. [6, 7] relied on a nonrigorous analogy with noisy Fisher–Kolmogorov–Petrovskii–Piskounov (FKPP) equation

$$\frac{\partial u}{\partial t} = \frac{1}{2}\frac{\partial^2 u}{\partial x^2} + u(1-u), \tag{1}$$

and relied strongly on ideas developed earlier by Brunet and Derrida [3, 4, 5] on the effect of noise on such an equation. For this reason this process is sometimes known as the Brunet–Derrida particle system. From a rigorous point of view, proofs of (i) can be found in the paper of Bérard and Gouéré [1], while a rigorous proof of (ii) for a closely related model can be found in [2]. However (ii) remains open for the original Brunet–Derrida process, though exciting progress in this direction has been achieved recently by Maillard [8].

We study geometric properties of the multidimensional system and show that for $s(x) = \|x\|_2$ (the Euclidean norm) the cloud of particles almost surely travels at speed converging to the same v_N as above and in some possibly random direction.

In the case where s is linear, we also obtain the same speed v_N for the cloud of particles in a deterministic direction depending only on s. Moreover, under some assumptions on the initial configuration, the shape of the cloud scales like $\log N$ in the direction parallel to the motion but at least $(\log N)^{3/2}$ in the orthogonal direction. This result is equivalent to the following result of independent interest: in one-dimensional systems, the genealogical time is lower bounded by $(\log N)^3$, thereby contributing a step towards resolving (ii) in the case $d = 1$.

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Conditioned regimes of the branching Brownian motion killed on a boundary

Damien Simon

The branching Brownian motion on the half-line with a negative drift and killing at 0 exhibits a phase transition between almost-sure extinction (if the drift is too small) and non-zero survival probability with explosion. An initial motivation of the study of this model was a comparison to branching Brownian motion with fixed size selection (the so-called N-BBM). To this purpose, it is interesting to study our model at time t conditioned to produce a *finite* number of individuals at a very large time T. Such events are rare but have non-zero probability and lead to interesting conditioned dynamics, which can be described through spine technique and careful studies of the F-KPP equation.

When the drift is too small, i.e. there is a non-zero survival probability, the conditioned process is defined by sending first $T \to \infty$ and then taking t large. There is a limit law, initially called *quasi-stationary* in [1, 2], that exhibits universal properties as the drift is close to the critical drift. In particular, the divergence of the size of the population is the same as the relation between size and velocity in the N-BBM and Bolthausen-Sznitman coalescent is expected to appear for the genealogies. Moreover, it is conjectured (and it is still open) in [3] that the renormalized size of the population should be an exponential law.

A major part of the talk consisted of explaining how the properties and conjectures can be guessed directly from the behaviour of solutions of the KPP equation. Rigorous proofs of partial results have been obtained since then in particular in [4], by defining suitable martingales inspired by our physical approach.

The second keypoint of the talk was an overview of what happens when the drift is too large. In this case, there is no such quasi-stationary conditioned regime but some other non-trivial scalings occur. Since [1, 3], it should be possible to obtain rigorous results based on similar arguments as the ones in [4] but it has not been done yet.

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Branching Brownian motion in a strip: Survival near criticality

Marion Hesse

(joint work with Simon C. Harris and Andreas E. Kyprianou)

We consider a branching Brownian motion X in which each particle performs a standard Brownian motion and is killed on hitting 0 or K. All living particles undergo branching at constant rate β to be replaced by two offspring particles. Once born, offspring particles move off independently from their birth position, repeating the stochastic behaviour of their parent.

We are interested in the following

- What is the critical size of the strip below which survival is no longer possible?
- How does the survival probability behave near criticality?
- What does the process look like near criticality when conditioned on survival?

Denote by P_x^K the law of the process started from an initial particle at $x \in (0, K)$. We can identify the critical width K_0 below which survival is no longer possible as follows.

Theorem: The survival probability $p_K(x) := P_x^K(\text{BBM in (0,K) survives})$ is positive if and only if $\beta - \pi^2/2K^2 > 0$ i.e. if and only if $K > K_0 := \frac{\pi}{\sqrt{2\beta}}$.

The proof uses a spine argument, decomposing X into a Brownian motion conditioned to stay in $(0, K)$ dressed with independent copies of (X, P^K) which immigrate along its path.

Let us turn to the second question concerning the asymptotics of the survival probability. We have the following result.

Theorem: Uniformly for all $x \in (0, K_0)$,

$$p_K(x) \sim C_K \sin(\pi x/K_0), \quad \text{as } K \downarrow K_0,$$

where C_K is independent of x and can explicitly be determined as

$$C_K = (K - K_0)\frac{3\pi^3}{8\beta K_0^3}, \quad \text{as } K \downarrow K_0,$$

and in particular $C_K \downarrow 0$ as $K \downarrow K_0$.

The first part of this result can be shown using spine arguments. However, to identify the exact form of C_K given above, we need yet another decomposition of the P^K-BBM. Loosely speaking, we identify the particles with infinite genealogical lines of descent, that is, particles which produce a family of descendants which

survives forever. To illustrate this, in a realisation of X, let us colour blue all particles with an infinite line of descent and colour red all remaining particles. Thus, on the event of survival, the resulting picture consists of a blue tree 'dressed' with red trees whereas, on the event of extinction, we see a red tree only.
Note that a particle at position x has probability $p_K(x)$ of being blue. We can then characterise the corresponding blue and red branching diffusions as follows.
In the blue branching diffusion, each particle

- branches at rate $\beta p_K(x)$,
- moves according to a diffusion with drift $\frac{p_K'}{p_K}$.

In the red branching diffusion, each particle

- branches at rate $\beta(1 - p_K(x))$,
- moves according to a diffusion with drift $-\frac{p_K'}{1-p_K}$.

Red branching diffusions immigrate along the trajectories of the blue particles at rate $2\beta(1 - p_K(x))$.

We can then construct a coloured tree starting from x by flipping a coin with probability $1 - p_K(x)$ of 'heads' and if it lands 'heads' we grow a red tree with initial particle at x, while if it lands 'tails' we grow a blue tree at x and dress its branches with red trees.

Theorem: Let $K > K_0$ and $x \in (0, K)$. The P^K-BBM X is equal in law to (the colour-blind view) of the coloured tree.

The explicit form of the constant C_K can then be derived from a careful study of the expected growth rate of the blue branching diffusion.
The decomposition above also allows us to answer our last question as it tells us that the BBM in $(0, K)$ conditioned on survival has the same law as a blue tree 'dressed' with red trees. Then, as $K \downarrow K_0$, we see that the blue branching rate βp_K drops down to 0, at the same time the red branching rate $\beta(1-p_K)$ increases to β and the rate of immigration $2\beta(1 - p_K)$ rises to 2β at criticality. Taking into account the change in the particle motion, the results reads as follows.

Theorem: For any finite time-horizon, the law of the BBM in $(0, K)$ under $\lim_{K\downarrow K_0} P^K_x(\cdot|\text{survival})$ is equal to the law of a particle system consisting of

- a spine performing Brownian motion conditioned to stay in $(0, K_0)$,
- immigration of P^{K_0}-BBM at rate 2β.

This work was motivated by recent work on survival of near critical branching Brownian motion with absorption at the origin by Aïdékon and Harris [1] as well as Berestycki et al. [2] .

References

[1] E. Aïdékon, S. Harris, *Near-critical survival probability of branching brownian motion with an absorbing barrier*, in preparation.
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Near critical survival probabilities for branching Brownian motion with killing

Simon Harris
(joint work with Elie Aïdékon, Andreas Kyprianou, Marion Hesse)

We will consider a branching Brownian motion where particles have a drift $-\rho$, binary branch at rate β and are killed if they hit the origin. This process is supercritical if and only if $\beta > \rho^2/2$ and we will discuss the behaviour of the survival probability in the regime as criticality is approached. This is joint work with Elie Aïdékon (Paris VI).

We will present some of the key ideas and techniques used in our proofs; namely, product martingales, additive martingale, the many-to-few lemma and stopping lines. Our approach gives alternative proofs to those found in [1, 2] in which the genealogy of the particles is also investigated.

We also briefly mention the closely related subsequent joint work with Marion Hesse and Andreas Kyprianou [3] for drifting BBM where there is killing at both 0 and K.

References

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