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Geometric Structures in Group Theory

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Abstract. The overall theme of the conference was geometric group theory, interpreted quite broadly. In general, geometric group theory seeks to understand algebraic properties of groups by studying their actions on spaces with various topological and geometric properties; in particular these spaces must have enough structure-preserving symmetry to admit interesting group actions. Although traditionally geometric group theorists have focused on finitely generated (and even finitely presented) countable discrete groups, the techniques that have been developed are now applied to more general groups, such as Lie groups and Kac-Moody groups, and although metric properties of the spaces have played a key role in geometric group theory, other structure such as complex or projective structures and measure-theoretic structures are being used more and more frequently.

Mathematics Subject Classification (2010): 20Fxx, 57Mxx.

Introduction by the Organisers

In addition to discussing the most recent developments within geometric group theory, the meeting also highlighted several dramatic contributions of geometric group theory to other fields. A particular emphasis within the field was studying several classes of groups which exhibit properties of classical examples such as arithmetic groups but are not themselves arithmetic.

The idea that a group can be thought of as a geometric object with non-positive or negative curvature is one of the most fundamental ideas in geometric group theory. Curvature conditions have helped us to understand both the general, randomly defined group and specific families of groups arising from topological of differential-geometric considerations. The focus has recently shifted to variants on these curvature conditions, both those which were defined long ago but not intensively studied and newly introduced notions. For example, Gromov introduced "relative hyperbolicity" at the same time as he defined hyperbolicity, but this was not studied deeply until at least a decade later. Relative hyperbolicity captures behavior similar to that of non-uniform lattices in real hyperbolic spaces in a more general, non-smooth framework. Other variants of hyperbolicity focus on properties of a particular group action rather on the group itself, and generalize classical small cancellation theory. This has led to the construction of quotient groups with prescribed properties, starting from a suitable action of a group on a space, and has had applications to groups arising from unexpected quarters, such as proving that the Cremona group is not simple.

Several talks during the week dealt with new techniques and questions. For example, in some talks the use of an auxiliary space with a group action is less central, such as in investigating the possible growth rates of finitely generated groups, or in attempts to establish a general theory of totally disconnected locally compact groups. In others, the structures on spaces preserved by the group action are more of an analytic nature than a geometric one, for example there are some exciting connections with measure theory and with operator algebras, some of which lead to deep topological questions.

Specific families of groups that were considered in the talks included mapping class groups $\mathrm{MCG}(\Sigma)$ of surfaces, groups of outer automorphisms $\mathrm{Out}(F_n)$ of non-abelian free groups and isometry groups of buildings. These are of particular interest because of their connections with many other areas of mathematics, and because each in its way generalizes the classical examples of linear groups acting on symmetric spaces. The construction of suitable substitutes for the symmetric spaces and the investigation of even the most basic properties are often very difficult.

Certainly one of the most exciting developments in the field was the recent use of geometric group theory to solve the last open conjecture on W. Thurston's famous list of problems on the structure of 3-manifolds. Two speakers gave talks explaining both the geometric group theory and its application to 3-manifolds during the official schedule, and informal sessions were held in the evenings for those who wanted to hear more details. Progress is currently being made on simplifying some of the proofs, and there are many further potential applications of the technology to geometric group theory.

We had 52 participants from a wide range of countries, and 26 official lectures. The staff in Oberwolfach was—as always—extremely supportive and helpful.

We are very grateful for the additional funding for 5 young PhD students and recent postdocs through Oberwolfach-Leibniz-Fellowships. In addition, there was one young student funded through the DMV Student's Conference. We think that this provided a great opportunity for these students.

We feel that the meeting was exciting and highly successful. The quality of all lectures was outstanding, and outside of lectures there was a constant buzz of intense mathematical conversations. We are confident that this conference will lead to both new and exciting mathematical results and to new collaborations.

Workshop: Geometric Structures in Group Theory

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Abstracts

Growth of groups

Laurent Bartholdi
(joint work with Anna Erschler)

Let G be a finitely generated group. Given a generating set S, the *growth function* $v_{G,S}$ of G is defined by

$$v_{G,S}(n) = \#\{g \in G \mid g = s_1 \dots s_m,\ s_i \in S^{\pm 1},\ m \leq n\}.$$

This function depends on the choice of S, but only mildly: if for functions v, w one defines $v \precsim w$ when there is a constant C with $v(n) \leq w(Cn)$ for all n, and $v \sim w$ means $v \precsim w \precsim v$, then the equivalence class of $v_{G,S}$ is independent of S, is written v_G, and is called the *growth type* of G.

Question 1. Which functions are equivalent to the growth type of a group?

Clearly $v_{G,S}(n) \precsim (2\#S)^n$, so $v_G \precsim \exp(n)$ for all G; and, if G contains a non-abelian free subsemigroup, then $v_G \sim \exp(n)$. Milnor [10] and Wolf [14] proved that virtually soluble groups have free subsemigroups if and only if they are not virtually nilpotent; in which case they have polynomial growth. The same conclusion holds for linear groups, by the Tits alternative. Bass [3] and Guivarc'h [8] determined the degree of polynomial growth in that case.

Milnor asked in [9] whether there are groups with growth type strictly between polynomial and exponential, and conjectured that $v_G \precsim n^d$ for some d if and only if G is virtually nilpotent. This was proven by Gromov in [6]. On the other hand, Grigorchuk [5] gave in 1983 examples of groups with growth type strictly between polynomial and exponential.

Question 2. Define the *infimal growth function* by

$$v_G^{\inf}(n) = \min_{S\colon \langle S\rangle = G} v_{G,S}(n).$$

How far can v_G and $v_G^{\inf}$ be?

Note that $v_G^{\inf}$ is by construction independent of any generating set. It does not appear to have been considered before, but a related invariant,

$$\lambda_G = \inf_{S\colon \langle S\rangle = G} \lim_{n\to\infty} \sqrt[n]{v_{G,S}(n)}$$

called the *infimal growth rate*, has been considered by Gromov [7], who asked whether $\lambda_G > 1$ whenever G has exponential growth type.

Eskin, Mozes, Oh [4] shew that for linear groups in characteristic 0 one has $\lambda_G > 1$ whenever G has exponential growth; and Osin [11] obtained the same conclusion for virtually soluble groups. In both cases, the stronger conclusion $v_G \sim v_G^{\inf}$ holds. On the other hand, Wilson [13] gave in 2003 examples of groups G of exponential growth type with $\lambda_G = 1$.

New results

Theorem 3. *Let* $\eta \cong 2.46$ *be the real root of* $\eta^3 - \eta^2 - 2\eta - 4 = 0$*. Let* $f : \mathbb{R} \to \mathbb{N}$ *be any function satisfying*

$$f(2n) \leq f(n)^2 \leq f(\eta n) \text{ for all } n \gg 0.$$

Then there exists a group G *with* $v_G \sim f$*.*

The only known restriction on growth types, due to Shalom and Tao [12], is that if $v_G \precsim n^{(\log\log n)^{1/100}}$ then $v_G \sim n^d$ for an integer d. Therefore, there remains "unchartered territory" between $n^{(\log\log n)^{1/100}}$ and $\exp(n^{\log 2/\log \eta})$. If there existed a group with growth strictly between polynomial and $\exp(n^{1/2})$, then it could not be residually nilpotent.

Theorem 4. *Let* H *be a countable group. Then there exists a finitely generated group* G *containing* H *as a subgroup, with*

$$v_G \sim \exp(n) \text{ and } v_G^{\inf} = \exp(n^{\log 2/\log \eta}).$$

Sketch of proofs

We start with a construction, due to Grigorchuk, of uncountably many groups of intermediate growth. Let V_4 denote the elementary 2-group with 4 elements, and let $\langle a \rangle$ denote the cyclic group with 2 elements. Consider then groups $G_\omega = \langle V_4, a \rangle$ depending on a parameter $\omega = \omega_0\omega_1 \dots$ with each ω_i an epimorphism $V_4 \twoheadrightarrow \langle a \rangle$, and defined by their action on $\mathbb{N} = \{0, 1, \dots\}$ by permutations. The action of the generator a is independent of ω, and is given by

$$a(2n) = 2n + 1, \qquad a(2n + 1) = 2n.$$

Elements $x \in V_4$ act by

$$x(0) = 0, \qquad x(2^k(2n + 1)) = 2^k(2\omega_k(n) + 1).$$

For ω as above, let $\sigma\omega$ denote the shifted sequence $\omega_1\omega_2 \dots$. There are then homomorphisms

$$\Phi_\omega : G_\omega \to (G_{\sigma\omega} \times G_{\sigma\omega}) \rtimes \langle a \rangle$$

given by restriction of the action to $2\mathbb{N}$ and $2\mathbb{N}+1$ and remembering whether these last two are permuted or not.

We construct simultaneously a metric $\|\cdot\|_\omega$ on each group G_ω, at bounded distortion from the word metric, and numbers $\eta_\omega \in [2, 3]$, such that if $\Phi_\omega(g) = (g_0, g_1)a^\varepsilon$ then

$$\|g_0\|_{\sigma\omega} + \|g_0\|_{\sigma\omega} \leq \tfrac{2}{\eta_\omega}\|g\|_\omega \tag{1}$$

holds, up to an additive constant.

If furthermore we replace G_ω by the restricted wreath product $F \wr G_\omega = F^{(\mathbb{N})} \rtimes G_\omega$, for a finite group F, then we may make the inequality in (1) sharp. It then follows that $v_{G_\omega,\|\cdot\|_\omega}(\eta_\omega n) \approx v_{G_{\sigma\omega},\|\cdot\|_{\sigma\omega}}(n)^2$.

Now constructing the sequence ω appropriately, controlling the partial products $\eta_\omega\eta_{\sigma\omega}\cdots\eta_{\sigma^k\omega}$ in terms of the function f, yields Theorem 3. In particular, $\omega =$

$(\pi_0\pi_1\pi_2)^\infty$, for π_0, π_1, π_2 the three epimorphisms $V_4 \to \langle a\rangle$, yields the minimal possible growth $\exp(n^{\log 2/\log\eta})$.

To prove Theorem 4, first embed H in a finitely generated group $\widehat{H}$ of exponential growth and generated by two elements x, y of finite order (say orders p, q respectively). Set $F = C_p \times C_q$. Consider then the group $G = \widehat{H} \wr G_\omega$. For $h \in \widehat{H}$, denote by h_i the function $\mathbb{N} \to \widehat{H}$ that takes value h at i and takes value 1 elsewhere. Consider the generating set $S_N = V_4 \cup \cup\{a, x_0, y_N\}$ of G. Then, as long as n is less than N, the ball of radius n in G is isomorphic to the ball of radius n in $F \wr G_\omega$, because x_0^g and $y_N^{g'}$ commute as long as g, g' are short enough. Therefore, $v_G^{\inf} = v_{F\wr G_\omega}^{\inf}$.

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Coarse median spaces

Brian H. Bowditch

We introduce the notion of a "coarse median space". This is space with a ternary operation satisfying the axioms of a median algebra up to bounded distance. This can be applied to a broad class of groups, via their Cayley graphs. Many results about such groups can be viewed in these terms. The idea was inspired by work of Behrstock and Minsky, and other people, on the mapping class group.

Recall that a "median algebra" is a set, M, together with a a ternary operation, $\mu : M^3 \longrightarrow M$, such that, for all $a, b, c, d, e \in M$,

(M1): $\mu(a, b, c) = \mu(b, c, a) = \mu(b, a, c)$,

(M2): $\mu(a, a, b) = a$,

(M3): $\mu(a, b, \mu(c, d, e)) = \mu(\mu(a, b, c), \mu(a, b, d), e)$.

Any finite median algebra can be identified as the vertex set of finite CAT(0) cube complex. Moreover, any finite subset of a median algebra lies inside a finite subalgebra. In view of this, we make the following definition [3].

Let (Λ, ρ) be a geodesic metric space and $\mu : \Lambda^3 \longrightarrow \Lambda$ be a ternary operation. We say that μ is a "coarse median" if it satisfies the following:

(C1): There are constants, k, $h(0)$, such that for all $a, b, c, a', b', c' \in \Lambda$ we have

$$\rho(\mu(a, b, c), \mu(a', b', c')) \leq k(\rho(a, a') + \rho(b, b') + \rho(c, c')) + h(0).$$

(C2): There is a function, $h : \mathbb{N} \longrightarrow [0, \infty)$, with the following property. Suppose that $A \subseteq \Lambda$ with $1 \leq |A| \leq p < \infty$, then there is a finite median algebra, (Π, μ_Π) and maps $\pi : A \longrightarrow \Pi$ and $\lambda : \Pi \longrightarrow \Lambda$ such that for all $x, y, z \in \Pi$ we have:

$$\rho(\lambda\mu_\Pi(x, y, z), \mu(\lambda x, \lambda y, \lambda z)) \leq h(p)$$

and

$$\rho(a, \lambda\pi a) \leq h(p)$$

for all $a \in A$.

The existence of a coarse median on a geodesic space is a quasi-isometry invariant, so we can apply this to finitely generated groups via their Cayley graphs. We can thus define a "coarse median group" as a finitely generated group whose Cayley graph is coarse median. For example, a hyperbolic group is a coarse median group of rank 1. Also, it follows using work of Behrstock and Minsky [2] that a mapping class group is coarse median of finite rank.

From this one can recover various facts [3, 4]. For example the asymptotic cone embeds into a finite product of $\mathbb{R}$-trees [1]. As a result, we recover the rank theorem of Behrstock and Minsky and Hamenstädt, as well as rapid decay, etc.

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Random groups contain surface subgroups

Danny Calegari
(joint work with Alden Walker)

The executive summary of our results is as follows:

(1) Most groups contain surface subgroups;
(2) These surfaces are usually $(1+\epsilon)$-quasiisometrically embedded; and
(3) These surfaces can usually be found and certified easily.

More precisely, by *most groups* we mean random groups in Gromov's density model.

If we fix a free group F_k of rank $k \geq 2$ and fix a free generating set, there are roughly $(2k-1)^n$ reduced words of length n. If we fix a *density* $0 \leq D \leq 1$, then a *random group at density* D *and length* 1 is generated by F_k, and has $(2k-1)^{Dn}$ relators of length n, chosen independently and at random with the uniform distribution. One is interested in statistical properties of such groups for fixed D that hold with probability going to 1 as $n \to \infty$.

Gromov showed that for $D > 1/2$ such groups are almost surely trivial or isomorphic to $\mathbb{Z}/2\mathbb{Z}$, but for $D < 1/2$, such groups are infinite, nonelementary, hyperbolic, and 2-dimensional (in fact, the obvious 2-complex coming from the presentation is almost surely aspherical).

Our main theorem is that at any density $0 \leq D < 1/2$, such random groups almost surely contain (many) closed surface subgroups (here $D = 0$ is suggestive notation for a random group in the few relators model, where one fixes some $N > 0$ and defines a group generated by F_k and with N relators of length n, chosen independently and at random with the uniform distribution, and then lets $n \to \infty$).

It is worth making several additional remarks.

(1) One can choose different probability distributions on the set of words of length n, with similar results. For example, suppose we generate reduced words by a (finite state, stationary) Markov process of entropy λ (so that some words are much more likely than others) but such that every reduced word σ has a positive probability of appearing as a subword (note that we do *not* assume the Markov process is symmetric). Define a random group at density D by adding λ^{Dn} relators of length n, chosen independently and at random with this distribution. Then for any $0 \leq D < 1/2$, such random groups almost surely contain (many) closed surface subgroups.
(2) It is overwhelmingly likely that a random N-relator group for $N \leq k$ has trivial H_2. But for $N > k$ in the few relators model, or for any $0 < D < 1/2$, we can insist that our surfaces are homologically essential.

(3) The surfaces we construct have genus $O(n)$. However, we conjecture that the simplest essential surfaces have genus $O(n/\log(n))$. Note that we can produce *homologically essential* maps from surfaces of genus $O(n/\log(n))$ to our random groups, and this order of estimate *is* a lower bound for the homologically essential surfaces.
(4) For $D < 1/6$, random groups are known to be virtually special, by Ollivier-Wise [3] and Agol [1]. Thus such groups virtually retract onto closed surface subgroups.
(5) There is tremendous flexibility in the construction, which depends on the (so-called) *Thin Fatgraph Theorem*, a combinatorial theorem which says that sufficiently random homologically trivial 1-chains in a free group bound trivalent fatgraphs with every edge very long. Such fatgraphs can be used to build many interesting injective 2-complexes.

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Locally normal subgroups of simple locally compact groups

Pierre-Emmanuel Caprace
(joint work with Colin D. Reid, George A. Willis)

The purpose of this talk was to describe some aspects of an ongoing joint research project with Colin Reid and George Willis, whose goal is to explore the structure of simple locally compact groups. It is inspired by earlier work due to J. Wilson [5] on just-infinite groups, and work by Barnea–Ershov–Weigel [1] on abstract commensurators of profinite groups. A more comprehensive research announcement is available in [2], so we will keep the present note brief. The details of the material elaborated in this project will be exposed in a series of papers, the first of which can be consulted in [3].

Since the identity component of a Hausdorff topological group is a closed normal subgroup, a topologically simple locally compact group is either connected or totally disconnected. A consequence of the solution to Hilbert's fifth problem is that all connected simple locally compact groups are Lie groups, and hence exhaustively understood. We focus on the complementary case of totally disconnected groups. The specific class, denoted by $\mathscr{S}$, which we focus on is that of compactly generated, non-discrete, topologically simple, totally disconnected locally compact (t.d.l.c.) groups. This class still includes all simple algebraic groups over non-Archimedean local fields, complete irreducible Kac–Moody groups over finite fields, certain groups of automorphisms of trees or higher dimensional non-positively curved cube complexes, and a few more exotic avatars of the latter groups, including Neretin's group of tree spheromorphisms.

The central concept of our study is that of a **locally normal subgroup**, which is defined as a compact subgroup whose normalised is open. Given a t.d.l.c. group G, we denote by $\mathcal{LN}(G)$ the set of commensurability classes of locally normal subgroups, endowed with the partial ordering induced by the relation of inclusion. Then $\mathcal{LN}(G)$ is a modular lattice, which we call the **structure lattice** of G, a term borrowed to J. Wilson [5]. The conjugation action of G on its locally normal subgroups induces a canonical action of G on $\mathcal{LN}(G)$ by automorphisms. We show that if G belongs to the class $\mathscr{S}$, then this action highlights an interesting dynamics, which is particularly rich when G contains two non-trivial commuting locally normal subgroups. One of our main results is that, under the latter condition, a group $G \in \mathscr{S}$ cannot be amenable. This fact lies in sharp contrast with the recent groundbreaking results by Juschenko–Monod [4] showing the existence of infinite finitely generated simple amenable groups.

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Contracting Boundaries of CAT(0) Spaces

Ruth Charney

(joint work with Harold Sultan)

Let X be a geodesic metric space which is either δ-hyperbolic or CAT(0). The visual boundary, ∂X, is the set of equivalence classes of geodesic rays in X, where two rays are equivalent if they have bounded Hausdorff distance. The (visual) topology on ∂X has as basis sets $N(\alpha, \epsilon, r)$ consisting of rays β which stay ϵ-close to the ray α for time $t \in [0, r]$.

If X is hyperbolic, this boundary has many nice properties. For example,

(1) Compactness: If X is a proper metric space, then ∂X is compact.
(2) Visibility: given any two points on ∂X, there is a geodesic in X joining them.
(3) QI-invariance: a quasi-isometry $f : X \to Y$ between two hyperbolic spaces induces a homeomorphism $\partial f : \partial X \to \partial Y$.

It follows from (3) that the boundary of a hyperbolic group G is well-defined.

In the case of a CAT(0) space, the boundary is not as well-behaved. Visibility fails on the boundary of any flat and, as shown by Croke and Kleiner [1], boundaries of CAT(0) spaces are not quasi-isometry invariant. In particular, there is no good notion of the boundary of a CAT(0) group.

To address this problem, we introduce a new notion of boundary, called the *contracting boundary* of X, defined as follows. Fix a base point $x_0 \in X$. A ray α based at x_0 is said to be D-contracting if for every ball B not intersecting α, the projection of B on α has diameter at most D. Let $\partial_c^D X$ denote the subspace of ∂X consisting of points represented by a D-contracting ray at x_0. Define the contracting boundary to be the union over $D \in \mathbb{N}$ of these subspaces, with the direct limit topology,

$$\partial_c X = \bigcup \partial_c^D X$$

One can show that this space is independent of choice of base point. We prove that it is a visibility space and it is quasi-isometry invariant.

The proof of quasi-isometry invariance depends on giving alternate characterizations of the contracting property. Let $M : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$. We say that a ray α is *M-Morse*, if for any (λ, ϵ)-quasi-geodesic β with endpoints on α, the quasi-geodesic β stays in the $M(\lambda, \epsilon)$–neighborhood of α. It is well-known that D-contracting geodesics are M-Morse where the function M depends only on D. We prove that the converse is also true: M-Morse geodesics are D-contracting where D depends only on M. The Morse property behaves well under quasi-isometry and allows us to control the contracting constants. Namely, if $f : X \to Y$ is a quasi-isometry, we prove that for any D, there exists D' such that f induces a continuous map $\partial_c^D X \to \partial_c^{D'} Y$.

Finally, we introduce a new notion of divergence, called the *lower divergence* of a ray and prove that α is contracting if and only if its lower divergence is super-linear. We use this to characterize which rays in a CAT(0) cube complex are contracting and apply this characterization to show that certain right-angled Coxeter groups have non-homeomorphic contracting boundaries and hence are not quasi-isometric.

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Suspensions and conjugacy of hyperbolic automorphisms

François Dahmani

Let F be a finitely presented group. A way to consider the conjugacy problem in $Aut(F)$, or $Out(F)$, is to relate it to an isomorphism problem on semi-direct products of F with $\mathbb{Z}$.

Given two semi-direct products, $F \rtimes_\alpha \langle t \rangle$ and $F \rtimes_\beta \langle t' \rangle$, their structural automorphisms α and β are conjugated in $Aut(F)$ if and only if there is an isomorphism

$F \rtimes_\alpha \langle t \rangle \to F \rtimes_\beta \langle t' \rangle$ sending F on F, and t on t'. They are conjugated in $Out(F)$ if and only if there is an isomorphism sending F on F, and t in $t'F$. The conjugacy problem in $Out(F)$ can be expressed as the problem of determining whether suspensions are fiber-and-orientation-preserving isomorphic.

Consider for instance $F = F_n$ a free group of finite rank n. In that case, a solution to the conjugacy problem in $Out(F_n)$ was announced by Lustig [Lu1, Lu2]. However, it might still be desirable to find short (in the sense that the exposition is conceptual and short; we ostensibly ignore complexity) complete solutions for specific classes of elements. Consider the class of atoroidal automorphisms: those that do not preserve any conjugacy class beside $\{1\}$. Since Brinkmann proved in [Br1] that an automorphism produces an hyperbolic suspension if and only if it is atoroidal, there is a conceptually simple (slightly brutal) way to algorithmically check whether two given automorphisms are indeed atoroidal (look for a preserved conjugacy class, and simultaneously look for a certificate of hyperbolicity of the semi-direct product), and if they are, our main result will allow to decide whether they are conjugate in $OutF_n$.

For hyperbolic groups, the isomorphism problem is solved [Sel, DGr, DGu2]. In several examples, the solution available can settle the conjugacy problem. Take two pseudo-Anosov diffeomorphisms of a hyperbolic surface. The mapping tori are closed hyperbolic 3-manifold, hence hyperbolic and rigid. Sela's solution to the isomorphism problem of their fundamental groups provides all conjugacy classes of isomorphisms (there are finitely many), and from that point, it is possible to check whether one of them preserves the fiber. For automorphisms of a free group, the analogous situation is when the two automorphisms are atoroidal, fully irreducible (with irreducible powers). However, there are atoroidal automorphisms for which the suspension, though hyperbolic, is not rigid. In [Br2] Brinkmann gave several examples with different behaviors. In particular, the solution to the isomorphism problem of hyperbolic groups will not reveal all isomorphisms between suspensions, and since the fibers are exponentially distorted in the suspensions, the usual rational tools do not work for solving the isomorphism problem with such a preservation constraint.

One can thus merely detect the existence of one isomorphism (say ι), but for investigating the existence of an isomorphism with the aformentionned properties, one is led to consider an orbit problem of the automorphism group of $F \rtimes \langle t \rangle$: decide whether an automorphism sends $\iota(F)$ on F and $\iota(t)$ on t' (or in $t'F$).

Orbits problems are not necessarily easier, especially if the group acting is large and complicated. For instance, Bogopolski, Martino and Ventura propose a subgroup of $GL(4, \mathbb{Z})$ whose orbit problem on $\mathbb{Z}^4$ is undecidable.

In this talk we prove that, if F is finitely generated and $F \rtimes \langle t \rangle$ hyperbolic, then $Out(F \rtimes \langle t \rangle)$ contains a finite index abelian subgroup, whose action on $H_1(F \rtimes \langle t \rangle)$ is generated by transvections. This allows us to prove that the specific orbit problem above is solvable in that case, by reducing it to a system of linear Diophantine equations, read in $H_1(F \rtimes \langle t \rangle)$.

These are thus the key steps to produce what we see as a picturesque way of solving the conjugacy problem for automorphisms of finitely presented groups with hyperbolic suspension.

The proof that $Out(F \rtimes \langle t \rangle)$ is virtually abelian is done by considering the canonical JSJ decomposition of the hyperbolic group $F \rtimes \langle t \rangle$. It suffices to show that this graph-of-groups decomposition does not contain any surface vertex group. Let us sketch the proof that takes roots in the way Brinkmann produces his examples in [Br2]. Consider the tree of the JSJ decomposition T, and $\mathbb{X}$ the graph of group quotient of T by $G = F \rtimes \langle t \rangle$. Since F is normal in G, T is a minimal tree for F, and $\mathbb{Y} = F \backslash T$ is a graph that is its own core, and since its genus is bounded by the rank of F, it is finite. It is a finite graph of groups decompisition of F. It follows by finiteness of $\mathbb{Y}$ that every vertex group (resp. edge group) in $\mathbb{X}$ is the suspension of a vertex group (resp. edge group) in $\mathbb{Y}$: lift the vertex in T, where its $\langle t \rangle$-orbit passes twice on a pre-image of a vertex in $\mathbb{Y}$, thus yielding the suspension. Since edge groups in $\mathbb{X}$ are cyclic, they are suspensions of the trivial group, and edge groups in $\mathbb{Y}$ must be trivial. Therefore $\mathbb{Y}$ is a free decomposition of F, and its vertex groups are of finite type. Going back to $\mathbb{X}$ again, vertex groups of $\mathbb{X}$ are suspensions of infinite groups of finite type, but finite type normal subgroups of free groups (or surface groups) are of finite index (or trivial).

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Subsurface projection in the $Out(F_n)$-setting

Mark Feighn

(joint work with Mladen Bestvina)

The curve complex $\mathcal{C}(S)$ of a compact surface S has as its vertices isotopy classes of essential simple closed curves in S. Its simplices are determined by pairwise disjoint curves. Masur and Minsky [1, 2] prove $\mathcal{C}(S)$ is hyperbolic and, for "general position" subsurfaces S_1 and S_2 of S, define a projection $\pi_{S_2}(S_1) \in \mathcal{C}(S_2)$. Bestvina-Bromberg-Fujiwara [3] finitely color the subsurfaces of S so that $\pi_{S_2}(S_1)$ is defined whenever S_1 and S_2 have the same color. They use properties of these projections to organize the curve complexes of subsurfaces of a common color into a hyperbolic space and prove:

Theorem 1 (Bestvina-Bromberg-Fujiwara [3])**.** *The mapping class group of S acts isometrically on a finite product of hyperbolic spaces such that orbits are quasi-isometrically embedded.*

The goal is to mimic this as much as possible for the outer automorphism group $Out(F_n)$ of a rank n free group F_n. The free splitting complex $\mathcal{S}(F_n)$ has for its k-simplices equivariant isomorphism classes of $(k+1)$-edge free splittings, i.e. minimal, simplicial F_n-trees with trivial edge stabilizers and $k+1$ orbits of edges. $\mathcal{S}(F_n)$ admits a natural $Out(F_n)$-action and plays the role ot the curve complex. Given distinct proper free factors A, B of F_n, $\pi_B(A)$ is defined as follows. Suppose $T \in \mathcal{S}(F_n)$ has trivial vertex stabilizers and $(A|T)/A \to T/F_n$ is an embedding where $A|T$ is the induced splitting of A. The projection $\pi_B(A) \in \mathcal{S}(B)$ is obtained by collapsing all but one orbit of edges of $B|T$. We show that the proper free factors of F_n can be finitely colored so that $\pi_B(A)$ is coarsely well-defined (choices result in splittings within uniform distance) whenever A and B have the same color and prove:

Theorem 2 (Bestvina-Feighn [4])**.** *$Out(F_n)$ acts isometrically on a finite product of hyperbolic spaces such that the translation lengths of automorphisms of exponential growth are positive.*

The main tool is a study of certain geodesics called *folding lines* in Culler-Vogtmann's Outer space of F_n and the resulting induced paths in Outer spaces of free factors of F_n. We also use the result of Handel-Mosher [5] that $\mathcal{S}(F_n)$ is hyperbolic and their characterization of which elements of $Out(F_n)$ have positive translation length when acting on $\mathcal{S}(F_n)$.

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Group actions on quasi-trees and strongly contracting orbits

Koji Fujiwara

(joint work with Mladen Bestvina, Ken Bromberg)

A *quasi-tree* is a graph which is quasi-isometric to a simplicial tree. In [1] we studied group actions on quasi-trees and found interesting applications. For example we showed that mapping class groups have finite asymptotic dimension.

In [1] we obtained a set of conditions (or Axioms) from which we can produce quasi-trees and group actions on them. It gives many natural examples for hyperbolic groups, mapping class groups and the outer automorphism groups of free groups.

In this talk I formulate a geometric property which can be used to construct quasi-trees. This property is called *strongly contracting* property, which is defined for a hyperbolic isometry of a geodesic space, [2]. For example, a hyperbolic isometry of a Gromov hyperbolic space and a rank-1 isometry of a CAT(0) space are strongly contracting.

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Higher dimensional cost and deficiency-gradient for Mapping Class Groups, $\mathbf{SL}(d, \mathbf{Z})$ and limit groups

Damien Gaboriau

(joint work with Miklos Abert)

If Γ is a finitely presented group, its **deficiency** is defined as

$$\operatorname{def}(\Gamma) := \max\{\#\text{generators} - \#\text{relators}\}$$

where the maximum is taken over all finite presentations of Γ.

If Γ is moreover residually finite, consider a **chain** $(\Gamma_n)_n$: a decreasing sequence of finite index normal subgroups with trivial intersection

$$\Gamma = \Gamma_0 > \Gamma_1 > \cdots > \Gamma_n > \cdots \qquad \Gamma \triangleright \Gamma_n \qquad [\Gamma : \Gamma_n] < \infty \qquad \cap_n \Gamma_n = \{id\}$$

and define the **deficiency-gradient** along the chain as

$$\operatorname{def} - \operatorname{grad}(\Gamma; (\Gamma_n)_n) := \lim_{n\to\infty} \frac{\operatorname{def}(\Gamma_n)}{[\Gamma : \Gamma_n]}$$

This is the analogue of the rank-gradient introduced by Lackenby with the rank (= minimum number of generators) replaced by the deficiency.

The goal of my talk is to explain how we compute the deficiency gradient along any chain for

$$\begin{array}{ll} \Gamma = \mathrm{MCG}(\Sigma_{g,p}), \quad g > 2 & \operatorname{def} - \operatorname{grad}(\Gamma; (\Gamma_n)_n) = 0 \\ \Gamma = \mathrm{SL}(d, \mathbb{Z}), \quad d > 3 & \operatorname{def} - \operatorname{grad}(\Gamma; (\Gamma_n)_n) = 0 \\ \Gamma \text{ a limit groups} & \operatorname{def} - \operatorname{grad}(\Gamma; (\Gamma_n)_n) = \beta_1(\Gamma) \end{array}$$

where β_1 is the first ℓ^2-Betti number.

Indeed, we identify the deficiency gradient as a **higher dimensional** $\mathbf{2}$**-cost** defined as the optimum deficiency of "measured leaf-simply-connected laminations" spanning the "action of Γ on the projectiv limit of the equiprobability preserving multiplication actions $\Gamma \curvearrowright \Gamma/\Gamma_n$".

We have some technics allowing us to compute the 2-cost in another way by using more geometric methods, by looking for instance at the action of Γ on its Cayley complex defined using Magnus-Nielsen presentation for $\mathrm{SL}(d,\mathbb{Z})$ and the Gervais presentation for the Mapping Class Group $\mathrm{MCG}(\Sigma_{g,p})$. As for the limit groups, they have **measured dimension** at most 2 and using of previous theorem of myself, we get an optimal measured leaf-simply-connected lamination for which the 2-cost coincides with the alternated sum of the ℓ^2-Betti numbers, which is easily seen to be a bound.

Obs. Bridson and Kochloukova have an alternative proof that $\mathrm{def} - \mathrm{grad}(\text{ limit group }) = \beta_1(\Gamma)$.

The Malnormal Special Quotient Theorem

Daniel Groves

(joint work with Ian Agol, Jason Manning)

The Malnormal Special Quotient Theorem is a key tool in Wise's work on groups with a quasi-convex hierarchy [2], and in Agol's recent proof of the Virtual Haken Conjecture [1]. Its statement is as follows:

Theorem 1. *Let G be a hyperbolic group which acts properly and cocompactly on a $CAT(0)$ cube complex with quotient admitting a finite special (orbi-)cover. Let $\{P_1, \ldots, P_m\}$ be an almost malnormal family of quasi-convex subgroups of G. There are finite-index $\dot{P}_i \trianglelefteq P_i$ so that for any finite-index $P_i' \leq \dot{P}_i$ the quotient:*

$$\bar{G} = G/\langle\{P_i'\}\rangle$$

is hyperbolic and acts on a cube complex with quotient admitting a finite special (orbi-)cover.

This allows one to pass to quotients while staying in the category of virtually special hyperbolic groups.

In this talk, I discussed the role of the MSQT in this theory, and did not have time to discuss a new proof due to Agol-G.-Manning.

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Thurston compactification of the Torelli space

Thomas Haettel

The content of this report comes from the article [7].

1. The Torelli group and the Torelli space

Let S be a closed orientable surface of genus $g \geqslant 1$. The Torelli group $I(S)$ of S is the kernel of the action of the mapping class group of S on the homology of S, equipped with the symplectic intersection form, i.e.

$$I(S) = \ker\left(\mathrm{MCG}(S) \twoheadrightarrow \mathrm{Sp}_{2g}(\mathbb{Z})\right).$$

The Torelli group is much less understood than the mapping class group itself, for instance it is not known if $I(S)$ is finitely presented if $g \geqslant 3$, see [2] for instance.

The Torelli space $\mathrm{Tor}(S)$ of S is the quotient of the Teichmüller space of S by the Torelli group, it comes with a natural action of $\mathrm{Sp}_{2g}(\mathbb{Z})$. It is also the space of equivalence classes of hyperbolic surfaces marked by S, where $f : S \to X$ is equivalent to $f' : S \to X'$ if there exists an isometry $i : X \to X'$ such that $(i \circ f)_* = f'_*$ in homology. In other words, $\mathrm{Tor}(S)$ is the moduli space of hyperbolic surfaces whose homology is marked by the homology of S.

Fix a marked hyperbolic surface $f : S \to X$, and let us define a Euclidean norm on $H_1(X, \mathbb{R})$. The Hodge theorem identifies the vector space of harmonic 1-forms with the first cohomology group $H^1(X, \mathbb{R})$, so the L^2 product of harmonic forms defines an inner product on $H^1(X, \mathbb{R})$, and hence an inner product on $H_1(X, \mathbb{R})$ (see [3] for a comparison between this Euclidean norm and the stable norm).

We will adapt the construction of the Thurston compactification of the Teichmüller space (see [1] for instance) to the Torelli space. Consider the $\mathrm{Sp}_{2g}(\mathbb{Z})$-equivariant mapping

$$\begin{array}{rcl} \psi : \mathrm{Tor}(S) & \to & \mathbb{P}({\mathbb{R}_+}^{H_1(S,\mathbb{Z})}) \\ {[f : S \to X]} & \mapsto & [c \in H_1(S, \mathbb{Z}) \mapsto \|f_*(c)\|]. \end{array}$$

Let us define the closure of the image of this map to be the Thurston compactification $\overline{\mathrm{Tor}(S)}$ of the Torelli space.

2. Relationship with the Siegel upper half-space

Fix b the standard symplectic form on $\mathbb{R}^{2g}$, and the standard Euclidean norm $\|\cdot\|$. If $\Lambda \subset \mathbb{R}^{2g}$ is a lattice, it is said to be symplectic if

$$\Lambda^* := \{x \in \mathbb{R}^{2g} \,:\, \forall y \in \Lambda, b(x, y) \in \mathbb{Z}\} = \Lambda.$$

Consider the space of isometry classes of marked symplectic lattices of $\mathbb{R}^{2g}$:

$$\mathcal{E} = \{f : \mathbb{Z}^{2g} \to \mathbb{R}^{2g} \,:\, f(\mathbb{Z}^{2g}) \text{ is a symplectic lattice of } \mathbb{R}^{2g}\}/\text{equivariant isometry}.$$

It is the symmetric space $\mathcal{E} = \mathrm{Sp}_{2g}(\mathbb{R})/U(g)$ of $\mathrm{Sp}_{2g}(\mathbb{R})$, also known as the Siegel upper half-plane, and comes with a natural action of $\mathrm{Sp}_{2g}(\mathbb{Z})$.

As a space of marked objects, it also admits a Thurston compactification $\overline{\mathcal{E}}$, defined by the $\mathrm{Sp}_{2g}(\mathbb{Z})$-equivariant embedding

$$\begin{array}{rcl}\phi : \mathcal{E} & \to & \mathbb{P}({\mathbb{R}_+}^{\mathbb{Z}^{2g}}) \\ {[f]} & \mapsto & [u \in \mathbb{Z}^{2g} \mapsto \|f(u)\|].\end{array}$$

Theorem 1. *The Thurston compactification $\overline{\mathcal{E}}$ is $\mathrm{Sp}_{2g}(\mathbb{Z})$-equivariantly isomorphic to the Satake compactification of $\mathcal{E}$ associated to the tautological representation of $\mathrm{Sp}_{2g}(\mathbb{R})$ on $\mathbb{R}^{2g}$.*

This isomorphism extends at least to all classical symmetric spaces of non-compact type, and to the symmetric space of non-compact type of the exceptional Lie group $E_{6(-26)}$.

We will compare the Thurston compactification $\overline{\mathrm{Tor}(S)}$ to the Satake compactification $\overline{\mathcal{E}}$ of the Siegel upper half-plane $\mathcal{E}$: consider the mapping

$$\begin{array}{rcl}p : \mathrm{Tor}(S) & \to & \mathcal{E} \\ {[f : S \to X]} & \mapsto & [f_* : H_1(S,\mathbb{Z}) \to H_1(X,\mathbb{R})],\end{array}$$

it is well-defined since the lattice $f_*(H_1(S,\mathbb{Z}))$ is symplectic with respect to the intersection form on $H_1(X,\mathbb{R})$.

Theorem 2. *The map p is the classical period map (see* [2], [4], [5], [6]*).*

Remark that $\phi = \psi \circ p : \mathrm{Tor}(S) \to \mathbb{P}({\mathbb{R}_+}^{\mathbb{Z}^{2g}})$, so $\overline{\mathrm{Tor}(S)}$ is $\mathrm{Sp}_{2g}(\mathbb{Z})$-equivariantly isomorphic to the Satake compactification of $\mathrm{Tor}(S)$ defined by $p : \mathrm{Tor}(S) \to \overline{\mathcal{E}}$.

3. Partial stratification of the boundary

We now describe a subset of the boundary of the compactification, namely the closure of the image of the map

$$\begin{array}{rcl}\widetilde{\psi} : \mathrm{Tor}(S) & \to & {\mathbb{R}_+}^{H_1(S,\mathbb{Z})} \\ {[f : S \to X]} & \mapsto & (c \in H_1(S,\mathbb{Z}) \mapsto \|f_*(c)\|).\end{array}$$

Let K^{sep} denote the complex of separating simple closed curves of S, and let $\sigma = \{\gamma_1, \dots, \gamma_k\}$ be a $(k-1)$-simplex. Topologically, $S \backslash \sigma$ is the disjoint union of $k+1$ surfaces with punctures, $(S_i \backslash P_i)_{0 \leqslant i \leqslant k}$. Consider the application

$$\begin{array}{rcl}\widetilde{\psi_\sigma} : \mathrm{Tor}_\sigma(S) = \displaystyle\prod_{i=0}^{k} \mathrm{Tor}(S_i \backslash P_i) & \to & {\mathbb{R}_+}^{H_1(S,\mathbb{Z})} \\ ([f_j : S_j \to X_j])_{j \in [\![0,k]\!]} & \mapsto & \left\{ c \mapsto \sqrt{\displaystyle\sum_{j=0}^{k} \|(f_j)_*(c_j)\|^2} \right\},\end{array}$$

where

$$c = \sum_{j=0}^{k} c_j \in H_1(S,\mathbb{Z}) = \bigoplus_{j=0}^{k} H_1(S_j,\mathbb{Z})$$

.

Theorem 3. *We have the following stratification*

$$\overline{\widetilde{\psi}(\mathrm{Tor}(S))} = \widetilde{\psi}(\mathrm{Tor}(S)) \sqcup \bigsqcup_{\sigma \in K^{sep}} \widetilde{\psi_\sigma}(\mathrm{Tor}_\sigma(S)).$$

This describe the subset of $\partial\overline{\mathrm{Tor}(S)}$ consisting of limits of hyperbolic surfaces whose nonseparating systole is bounded below. We would like to understand the full boundary of this compactification, and the relationship with the Thurston compactification of the Teichmüller space.

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Handlebody groups and Out(F_n) are siblings

Ursula Hamenstädt

A *handlbody* of genus $g \geq 2$ is a compact three-manifold H whose boundary ∂H is a closed oriented surface of genus g and which is homotopy equivalent to a rose with g petals. The *handlebody group* $\mathrm{Map}(H)$ is the group of all isotopy classes of orientation preserving diffeomorphisms of H.

Each diffeomorphism of H restricts to a diffeomorphism of ∂H. By a result of Laudenbach, two diffeomorphisms of H are isotopic if and only if their restrictions to ∂H are isotopic. Thus $\mathrm{Map}(H)$ embeds into the *mapping class group* $\mathrm{Mod}(\partial H)$ of ∂H of all isotopy classes of diffeomorphisms of ∂H.

On the other hand, a diffeomorphism of H induces an outer automorphism of the fundamental group of H which is a free group of rank g. In other words, there is a homomorphism $\mathrm{Map}(H) \to \mathrm{Out}(F_g)$ which is surjective, with infinitely generated kernel.

The mapping class group $\mathrm{Mod}(\partial H)$ and the group $\mathrm{Out}(F_g)$ are known to have some common properties, but there are also differences. Both groups are finitely presented residually finite. They both contain a torsion free subgroup of finite index, and every solvable subgroup is virtually abelian.

The group $\mathrm{Out}(F_2)$ simply is the group $GL(2,\mathbb{Z})$, in particular its Dehn function is linear. For $g \geq 3$ the Dehn function of $\mathrm{Out}(F_g)$ is of exponential growth type

[5, 1]. However, by a result of Mosher, the mapping class group $\mathrm{Mod}(S)$ of any closed surface S is automatic. In particular, its Dehn function is quadratic.

The handlebody group $\mathrm{Map}(H)$ is an exponentially distorted subgroup of the mapping class group $\mathrm{Mod}(\partial H)$ [3] and hence knowing the Dehn function of the mapping class group does not yield information on the Dehn function of the handlebody group. The goal of the lecture is to explain

Theorem 1. *For $g = 2$ the Dehn function of the handlebody group is quadratic. If $g \geq 3$ then the Dehn function of the handlebody group is exponential.*

The exponential upper bound for the Dehn function was established in [4]. For the lower bound in the case $g \geq 3$ one uses the strategy developed in [1] and shows that the so-called Gersten cycles which were constructed by Gersten to show that the Dehn function of $SL(3, \mathbb{Z})$ is exponential can be lifted to the handlebody group with geometric control.

The case $g = 2$ uses tools which are similar to tools developed recently for the investigation of $\mathrm{Out}(F_n)$. This includes the use of the *disc graph* which is a hyperbolic graph whose vertices are essential discs in H and where two such discs are connected by an edge of length one if they can be realized disjointly.

The disc graph of a handlebody is hyperbolic. To use disc graphs to understand the handlebody group it is however necessary to also use discs graphs of a handlebody with two spots on the boundary obtained by cutting a handlebody open along a non-separating disc. The crucial difference between the case $g = 2$ and $g \geq 3$ is that the disc graph of a solid torus with two spots on the boundary is a tree, while the disc graph of a handlebody of genus $g \geq 2$ with two spots on the boundary is not hyperbolic and has infinite asymptotic dimension [2]

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Hyperbolic elements for the action of $Out(F_n)$ on the free splitting complex

Michael Handel
(joint work with Lee Mosher)

A free splitting of the free group F_n of rank n is a minimal simplicial action of F_n on a simplicial tree with trivial edge stabilizers. Two free splittings are equivalent if there is an equivariant homeomorphism between the trees. We assume that all vertices have valence at least three and say that T is a k-edge splitting if there are k orbits of edges.

The free splitting complex $\mathcal{FS}$ of F_n is a simplicial complex whose k-simplices correspond bijectively to equivalence classes of $(k+1)$-edge splittings. The simplex corresponding to the free splitting T is a face of the simplex corresponding to the free splitting T' if T can be obtained from T' by collapsing orbits of edges. $\mathcal{FS}$ is a hyperbolic complex by [1].

The outer automorphism group of F_n, denoted $Out(F_n)$, acts on on $\mathcal{FS}$ and it is natural to ask which elements of $Out(F_n)$ have positive translation length and so are hyperbolic.

A set X of F_n-orbits of lines in F_n *fills* if it is not carried by any proper free factor system or equivalently if the realization of X in every marked graph G covers every edge of G. Associated to each $\phi \in Out(F_n)$ is a finite set $\mathcal{L}(\phi)$ of attracting laminations, each of which is a closed set of F_n-orbits of lines.

Theorem 1. *The following hold for all $\phi \in Out(F_n)$.*

(1) *The action of ϕ on $\mathcal{FS}$ is hyperbolic if and only if some element of $\mathcal{L}(\phi)$ fills.*
(2) *If the action of ϕ on $\mathcal{FS}$ is not hyperbolic then the action of ϕ on $\mathcal{FS}$ has bounded orbits.*
(3) *The action of some iterate of ϕ on $\mathcal{FS}$ fixes a vertex if and only if $\mathcal{L}(\phi)$ does not fill.*

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The hyperbolicity of the sphere complex via surgery paths

Camille Horbez
(joint work with Arnaud Hilion)

Historically, the study of the group $\mathrm{Out}(F_N)$ of outer automorphisms of a finitely-generated free group has found inspiration in analogies with mapping class groups of surfaces. During the past few years, research has focused on finding a hyperbolic $\mathrm{Out}(F_N)$-analogue of Harvey's curve complex. Bestvina and Feighn's proof of the hyperbolicity of the complex of free factors of F_N [1], and Handel and Mosher's proof of the hyperbolicity of the complex of free splittings of F_N [2], thus appeared as deep and sought-after results. My talk aimed at giving an alternative proof of Handel and Mosher's theorem, based on more geometric arguments in a dual model of the free splitting complex, Hatcher's sphere complex [3].

Let $M_N = \#_N S^1 \times S^2$ be the connected sum of N copies of $S^1 \times S^2$, whose fundamental group is free of rank N. A *system of spheres* in M_N is a collection of disjoint embedded 2-spheres in M_N, none of which bounds a ball, and no two of which are isotopic. The *sphere complex* $\mathcal{S}_N$ (or rather its first barycentric subdivision) is the graph whose vertices are isotopy classes of sphere systems in M_N, two vertices S and S' being joined by an edge whenever $S \subsetneq S'$ or $S' \subsetneq S$. It is equipped with a natural action of the *mapping class group* of M_N, defined as the quotient $MCG(M_N) := \text{Homeo}^+(M_N)/\text{Homeo}_0(M_N)$ of the group of orientation-preserving homeomorphisms of M_N by the subgroup consisting of those homeomorphisms that are isotopic to the identity. Every homeomorphism of M_N induces an automorphism at the level of fundamental groups, thus yielding a morphism from $MCG(M_N)$ to $\text{Out}(F_N)$. It follows from work by Laudenbach [5] that this morphism is surjective, and its kernel is finite, generated by Dehn twists along N disjoint spheres, and thus acts trivially on the sphere complex. Hence $\mathcal{S}_N$ comes equipped with an action of $\text{Out}(F_N)$.

Our proof of the hyperbolicity of the sphere complex relies on the following criterion due to Masur and Minsky. Basically, the idea is to check that a collection of paths in the complex satisfies certain axioms that describe the behaviour of geodesics in hyperbolic spaces. In the following statement, a *path* is thought of as a finite sequence $\gamma(0), \dots, \gamma(K)$ of vertices of $\mathcal{X}$ such that $d(\gamma(i), \gamma(i+1)) \leq 2$ for all $i \in \{0, \dots, K-1\}$. A set Γ of paths is said to be *transitive* if for all vertices $v, w \in \mathcal{X}$, there exists $\gamma \in \Gamma$ such that $\gamma(0) = v$ and $\gamma(K) = w$.

Theorem 1. *(Masur-Minsky* [6, Theorem 2.3]*) Let $\mathcal{X}$ be a connected simplicial complex equipped with the simplicial metric. Assume that there exist constants $A \geq 0$, $B > 0$, $C \geq 0$, a transitive set of paths Γ in $\mathcal{X}$ and for each path $\gamma \in \Gamma$ of length K, a map $\pi_\gamma : \mathcal{X} \to \{0, \dots, K\}$, such that*

- *(Coarse retraction) For all $k \in \{0, \dots, K\}$, the diameter of the set $\gamma([k, \pi(\gamma(k))])$ is less than C.*
- *(Coarse Lipschitz) For all vertices $v, w \in \mathcal{X}$ satisfying $d(v, w) \leq 1$, the diameter of the set $\gamma([\pi(v), \pi(w)])$ is less than C.*
- *(Strong contraction) For all vertices $v, w \in \mathcal{X}$ which satisfy $d(v, \gamma([0, K])) \geq A$ and $d(v, w) \leq B \cdot d(v, \gamma([0, K]))$, the diameter of the set $\gamma([\pi(v), \pi(w)])$ is less than C.*

Then $\mathcal{X}$ is Gromov hyperbolic, and there exist constants $K, L > 0$ only depending on A, B, C such that all the paths $\gamma \in \Gamma$ are (K, L)-unparameterized quasi-geodesics.

The paths we work with are defined using a surgery procedure on sphere systems, defined as follows – see Figure 1. Let $S, \Sigma \in \mathcal{S}_n$, which we assume have been isotoped so as to minimize their number of intersection circles. The intersection circles between S and Σ define a pattern of circles on Σ, each of which bounds two disks on Σ. Choose an innermost disk D in this collection, i.e. the disk D contains no other disk in this pattern, and let C be its boundary circle. The sphere $s \in S$ containing C is thus the union of two disks D_1 and D_2 which intersect along C. Performing surgery on S along D consists of replacing the sphere s by two disjoint spheres s_1 and s_2 that do not intersect s, the sphere s_1 being the union of a parallel copy of D_1 and a parallel copy of D, and s_2 being the union of a parallel copy of D_2 and a parallel copy of D. We then identify parallel spheres in $S - \{s\} \cup \{s_1, s_2\}$ to get a new sphere system S'. Given two sphere systems S and Σ, a *surgery path* from S to Σ is a finite sequence $S = S_0, \dots, S_K = \Sigma$ such that for all $i \in \{0, \dots, K-2\}$, the sphere system S_{i+1} is obtained from S_i by performing a single surgery step on S_i with respect to Σ, and K is the smallest integer such that S_{K-1} does not intersect $\Sigma = S_K$. Note in particular that for all $i \in \{0, \dots, K-1\}$, the sphere systems S_i and S_{i+1} do not intersect, so $d(S_i, S_{i+1}) \leq 2$ (as they are both contained in their union $S_i \cup S_{i+1}$).

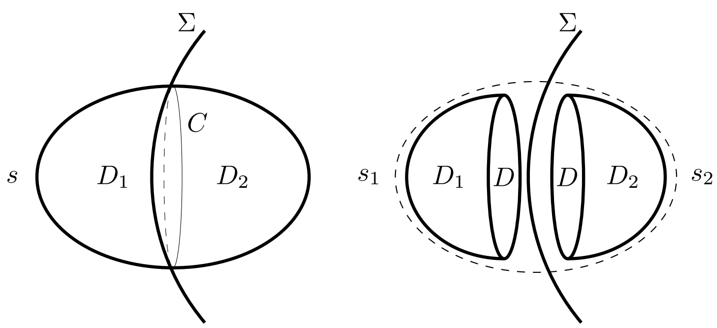

FIGURE 1. A surgery step.

Having in mind that surgery paths should behave like geodesics of a hyperbolic space, we define the projection map to a surgery path in the following way. Let S and Σ be two sphere systems, and let γ be a surgery path from S to Σ. Let S' be a sphere system. The projection $\pi(S')$ of S' to γ is the smallest integer k such that there exists a surgery path $S' = S'_0, \dots, S'_{K'} = \Sigma$ from S' to Σ (to the parameterization of which we allow to add "waiting times"), a surgery path $S = S_0, \dots, S_K = \Sigma$ from S to Σ obtained from γ by adding "waiting times" to its parameterization, and an integer t such that $S_t = \gamma(k)$ and S'_i shares a sphere with S_i for all $i \geq t$.

In my talk, I sketched the proof of the strong contracting property of the projection to a surgery path. Given a surgery path γ joining two sphere systems S

and Σ, and two sphere systems S^0 and S^{2^k} lying in a large ball which is disjoint from γ, we want to prove that S^0 and S^{2^k} have close projections to γ. Starting from a surgery path γ_0 from S^0 to Σ, we construct a surgery path from S^{2^k} to Σ which fellow travels γ_0 before it leaves the ball. The construction is roughly as follows :

- Join S^0 to S^{2^k} by a geodesic segment$(S^i)_{i\in\{0,\dots,2^k\}}$ in $\mathcal{S}_N$ (represented as a zig-zag path on the picture). Construct surgery paths joining each sphere system S^i to Σ in a compatible way.
- Prove that whenever a surgery path makes definite progress in $\mathcal{S}_N$, any two subsystems of S^i must "quickly" have a common descendant. This enables us to "contract" the diagram, as shown on Figure 2.

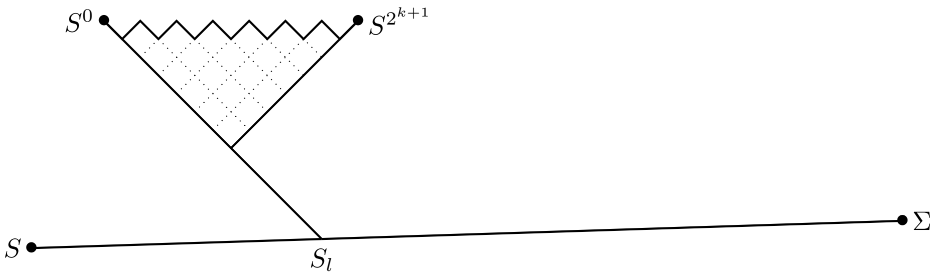

FIGURE 2. Schematic representation of the layout.

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Homological dimension of soluble groups

Peter H. Kropholler
(joint work with Martin Bridson, Karl Lorensen)

Let k be a non-zero commutative ring. The homological dimension $\mathrm{hd}_k(G)$ of a group G over k is defined by the assertion

$$\mathrm{hd}_k(G) \geq m \iff \text{there exists a } kG\text{-module } M \text{ such that } H_m(G, M) \neq 0.$$

Cohomological dimension works the same way:

$$\mathrm{cd}_k(G) \geq m \iff \text{there exists a } kG\text{-module } M \text{ such that } H^m(G, M) \neq 0.$$

These invariants are well understood for abelian groups. The infinite cyclic group has dimension one, both in homology and in cohomology, and independently of the choice of k.

If K is a normal subgroup of G then a Serre spectral sequence shows that

$$\mathrm{hd}_k(G) \leq \mathrm{hd}_k(K) + \mathrm{hd}_k(G/K).$$

The same method works and the same conclusion holds for cohomology:

$$\mathrm{cd}_k(G) \leq \mathrm{cd}_k(K) + \mathrm{cd}_k(G/K).$$

Each soluble group is built up from abelian groups by means of extensions. Therefore we can obtain upper bounds for the dimensions of a soluble group G in terms of its Hirsch length $h(G)$. For homology the statement goes like this:

if G is a soluble group with $\mathrm{hd}_k(G) < \infty$ then $\mathrm{hd}_k(G) \leq h(G) < \infty$.

The corresponding statement for cohomology is more complicated to state and moreover it is only conjectural:

If G is a soluble group with $\mathrm{cd}_k(G) < \infty$ then G has cardinality at most $\aleph_n$ for some $n < \omega$ and $\mathrm{cd}_k(G) \leq h(G) + n + 1 < \infty$.

For cohomology there are further complications as it is known that there is a distinction between the case when G satisfies Bieri–Eckmann duality and all other cases. A survey of known results is included in [3].

There is a natural conjecture for homological dimension that $\mathrm{hd}_k(G) = h(G)$ whenever hd_k is finite. This conjecture is explored in [1]. Additional progress is made in [2]. An early result of Urs Stammbach establishes the the natural homological dimension conjecture in the case $k = \mathbb{Q}$. It is then straightforward to deduce equality in the case $k = \mathbb{Z}$. Stammbach's method applies to soluble groups that are π-minimax provided k is a field whose characteristic does not belong to π. A π-minimax group is a group G with a series

$$1 = G_0 \lhd G_1 \lhd G_2 \lhd \cdots \lhd G_n = G$$

in which the factors G_i/G_{i-1} are cyclic, quasicyclic Prüfer p-groups ($p \in \pi$), or finite. Another case can be covered using an early result of Feld'man which says that the Serre spectral sequence inequalities become equalities if the normal subgroup K is an inverse duality group in the sense of Bieri and Eckmann.

In [1] the homological dimension conjecture is solved for abelian-by-polycyclic groups. Using splitting theorems developed in [2] we have the following most general case in which the conjecture is now known:

Let π be a set of primes. Let k be a finite field whose characteristic does not belong to π. Let G be an abelian-by-(π-minimax) group with finite homological dimension over k. Then $\mathrm{hd}_k(G) = h(G)$.

This is proved by first reducing to the finitely generated case and then to the case when A is torsion-free abelian. Assuming that G is the split extension of A by $Q := G/A$ we can then embed G in an ascending HNN extension $\widetilde{G} := G*_{G,t}$ in which the subgroup B generated by the stable letter together with A is a normal inverse duality group. Now Feld'man's equations may be brought into play and we deduce that $\mathrm{hd}_k(\widetilde{G}) = h(\widetilde{G})$. The ascending HNN extension has Hirsch length exactly one greater than that of G and a standard Mayer–Vietoris sequence shows that it has homological dimension at most one more than that of G. This information together with the more elementary inequalites already known now yields the desired conclusion.

There is one assumption here namely that the extension $A \to G \to Q$ is split. In general this is not the case. What is needed is some information about the second cohomology group $H^2(Q, A)$. This information is provided by the case $n = 2$ of the following result which may be found as Theorem A of [2]:

Let π be a set of primes. Let Q be a π-minimax group and let A be a $\mathbb{Z}Q$-module whose underlying additive group is minimax. Assume that there is no non-trivial torsion-free Q-module section of A with π-minimax underlying additive group. Then for all n, $H^n(Q, A)$ is finite.

From here, conclusions can be drawn using classical near splitting results and theory of Derek J. S. Robinson.

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Tame automorphisms of affine 3-folds acting on 2-dimensional complexes

Stéphane Lamy

I discuss various transformation groups arising in algebraic geometry, such as the group $\mathrm{Aut}(\mathbb{C}^2)$ of polynomial automorphisms of the plane, the Cremona group $\mathrm{Bir}(\mathbb{P}^2)$ of birational selfmaps of the projective plane, or the groups of tame automorphisms of $\mathbb{C}^3$ or of an affine quadric 3-fold.

An ubiquitous property in dimension 2 seems to be the existence of spaces with non positive curvature on which these groups act, leading to results such as the Tits' alternative or the non-simplicity.

We try to extend these results in higher dimension, in particular in an ongoing project with C. Bisi and J.-P. Furter we construct a CAT(0) hyperbolic square complex on which the tame group of an affine quadric acts, and we propose a similar candidate for the case of $\mathbb{C}^3$.

Vertex and extension finiteness for relatively hyperbolic groups

Gilbert Levitt
(joint work with Vincent Guirardel)

I discussed two finiteness properties of finitely generated groups:

Vertex finiteness: given G and a family $\mathcal{A}$ of subgroups, only finitely many vertex groups occur in splittings of G over groups in $\mathcal{A}$, up to isomorphism.

Extension finiteness: given any integer d, there are (up to isomorphism) only finitely many groups $\hat{G}$ containing G as a subgroup of index $\leq d$.

Extension finiteness holds if G is abelian, or free, or a surface group. It fails for $\mathbb{Z}/2\mathbb{Z} * (\mathbb{Z}/2\mathbb{Z} \wr \mathbb{Z})$, with $\mathbb{Z}/2\mathbb{Z} \wr \mathbb{Z}$ the lamplighter group.

Theorem 1 (Guirardel-L.). *Extension finiteness holds if G is toral relatively hyperbolic (and probably if G is hyperbolic relative to virtually polycyclic groups).*

As shown by $\mathbb{Z}/2\mathbb{Z} * (\mathbb{Z}/2\mathbb{Z} \wr \mathbb{Z})$, it may fail if G is hyperbolic relative to solvable subgroups.

Via standard extension theory, Theorem 1 follows from:

Theorem 2 (Guirardel-L.). *If G is as in Theorem 1, then $Out(G)$ has the following property (P): it only contains finitely many conjugacy classes of finite subgroups.*

I proved this in 2004 for cyclic finite subgroups when G is torsion-free hyperbolic. The extension to arbitrary finite subgroups uses a lemma by Ian Leary. A key difficulty in the proof of Theorem 2 is the fact that, if A is contained in B with finite index and A has (P), it does not always follow that B has (P).

Vertex finiteness holds for splittings over finite groups of bounded order, but not in general for splittings over finite groups, even for G finitely presented. It holds for cyclic splittings of one-ended hyperbolic groups (Sela, Delzant), but not in general, even for acylindrical cyclic splittings.

Theorem 3 (Guirardel-L.). *Vertex finiteness holds for:*

- *virtually cyclic splittings if G is hyperbolic relative to virtually polycyclic groups.*
- *abelian splittings of toral relatively hyperbolic groups.*

It does not always hold for cyclic splittings of groups hyperbolic relative to solvable groups, or for abelian splittings of groups hyperbolic relative to nilpotent groups.

The proof when G is one-ended uses a JSJ decomposition Γ_J and the following fact: given any splitting Γ, there exists a splitting $\hat{\Gamma}$ which collapses onto both Γ_J and Γ.

We use vertex finiteness to generalize Shor's theorem about finiteness of isomorphism types of fixed subgroups of automorphisms, to prove a uniform chain condition for fixed subgroups, and to study the action of $\mathrm{Out}(G)$ on spaces of $\mathbf{R}$-trees (such as the boundary of outer space).

From the classifications of Cuntz-Li-C^*-algebras to the cohomology of crystallographic groups

Wolfgang Lück

We want to discuss the following on the first glance unlinked problems:

Problem A. Classify all Cuntz-Li-C^*-algebras.

Problem B. Compute the group cohomology of crystallographic groups.

Problem C. Classify up to homeomorphism all closed manifolds occurring as total spaces of torus bundles over lens spaces.

Problem D. For which crystallographic groups is the unstable Gromov-Lawson-Rosenberg Conjecture true?

Given a number field K with ring of integers R, Cuntz and Li [2] associate to it a C^*-algebra $\mathcal{U}(R)$ which has some interesting connections to the number theory of K. The main result of Li-Lück [5] is an explicit calculation of the topological K-theory of $\mathcal{U}(R)$. These algebras turn out to be Kirchberg algebras and hence are classified by their K-theory. This has the surprising consequence

Theorem 1. *$\mathcal{U}(R)$ is up to isomorphism of C^*-algebras independent of R.*

The computation of the topological K-theory is reduced in Langer-Lück [4] to the computation of the group homology of the group $R \rtimes \mu$, where μ is the finite cyclic group of roots of unity in K and μ acts on R, which is as abelian group isomorphic to $\mathbb{Z}^n$, by multiplication in R. This boils down to

Conjecture 2 (Adem-Ge-Pan-Petrosyan)**.** *The Lyndon-Hochschild-Serre spectral sequence associated to the semi-direct product $\Gamma := \mathbb{Z}^n \rtimes_\rho \mathbb{Z}/m$ collapses in the strongest sense, i.e., all differentials in the E_r-term for $r \geq 2$ are trivial and all extension problems at the E_∞-level are trivial. In particular we get for all $k \geq 0$*

$$H^k(\Gamma;\mathbb{Z}) \cong \bigoplus_{i+j=k} H^i(\mathbb{Z}/m; H^j(\mathbb{Z}^n)).$$

This conjecture is known to be true if m is squarefree (see Adem-Ge-Pan-Petrosyan [1, Corollary 4.2]). Langer-Lück [3] prove the following positive result

Theorem 3. *Conjecture 2 is true, provided that the $\mathbb{Z}/m$-action on $\mathbb{Z}^n$ is free outside the origin.*

Notice that μ does acts freely on R so that the theorem above applies to $R \times \mu$. Hence Conjecture 2 is true in the case needed for the paper [5] by Li and Lück. However, it is false in general by the following result taken from [3].

Theorem 4 (Conjecture 2 is not true in general)**.** *Consider the special case $n = 6$ and $m = 4$, where ρ is given by the matrix*

$$\begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 1 \\ 0 & 0 & 0 & 0 & -1 & 0 \end{pmatrix}$$

Then the second differential in the Lyndon-Hochschild-Serre spectral sequence associated to the semi-direct product $\Gamma := \mathbb{Z}^n \rtimes_\rho \mathbb{Z}/m$ is non-trivial. In particular Conjecture 2 is not true.

Moreover, the following result is proved in [3].

Theorem 5.
(1) *If m is divisible by four, we can find n and ρ such that the second differential in the Lyndon-Hochschild-Serre spectral sequence associated to the semi-direct product $\Gamma := \mathbb{Z}^n \rtimes_\rho \mathbb{Z}/m$ is non-trivial;*
(2) *If m is not divisible by four, then for all n and ρ the second differential in the Lyndon-Hochschild-Serre spectral sequence associated to the semi-direct product $\Gamma := \mathbb{Z}^n \rtimes_\rho \mathbb{Z}/m$ is trivial.*

Finally we explain some partial results for Problems C and D which is work in progress with Jim Davis.

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Abstract simplicity of locally compact Kac–Moody groups

Timothée Marquis

In this talk, we report on the following theorem:

Theorem 1 ([Mar12])**.** *Let G be a complete Kac–Moody group over a finite field. Assume that G is of irreducible indefinite type. Then G is abstractly simple.*

Complete Kac–Moody groups over finite fields (which we also call locally compact Kac–Moody groups) are totally disconnected locally compact groups, obtained by completing a minimal Kac–Moody group $\mathfrak{G}(\mathbb{F}_q)$ (where $\mathfrak{G}$ is a Tits functor and $\mathbb{F}_q$ a finitel field) with respect to some natural filtration. Several (hopefully equivalent) constructions of these groups have appeared in the literature, from very different points of view. The construction we use is of algebraic nature and is due to O. Mathieu ([Mat88]) and G. Rousseau ([Rou12]).

The question whether an irreducible complete Kac–Moody groups $G(k)$ over an arbitrary field k is abstractly simple is very natural and was explicitly addressed by J. Tits [Tit89]. Abstract simplicity results for $G(k)$ over fields k of characteristic 0 were first obtained in an unpublished note by R. Moody ([Moo82]). Moody's proof has been recently generalised by G. Rousseau ([Rou12, Thm.6.19]) who extended Moody's result to fields k of positive characteristic p that are not algebraic over $\mathbb{F}_p$. The abstract simplicity of $G(k)$ when k is a finite field was shown in [CER08] in some important special cases, including groups of 2-spherical type over fields of order at least 4, as well as some other hyperbolic types under additional restrictions on the order of the ground field. Very recently, I. Capdeboscq and B. Rémy ([CR13]) managed to extend this result (with a similar approach) to all complete Kac–Moody groups $G(k)$ over finite fields k of order at least 4 and of characteristic p in case p is greater than the maximum (in absolute value) of the off-diagonal entries of the generalised Cartan matrix of $G(k)$. Their methods are of algebraic nature and rely on pro-p groups theory. In [Mar12], we establish the abstract simplicity of $G(k)$ over arbitrary finite fields, without any further restriction. Our proof relies on a dynamical approach (using contraction groups), and is very different from the one used in [CER08] and [CR13].

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Lattices in products of trees and a theorem of H.C. Wang

Shahar Mozes
(joint work with Marc Burger)

In 1967 H.C. Wang [Wa67] showed that for a connected semisimple group G without compact factors, a lattice $\Gamma < G$ is contained in only finitely many discrete subgroups. Shortly after, Kazhdan and Margulis [KM68] proved the stronger result that under the same hypothesis on G there is a constant $c_G > 0$ such that $\mathrm{Vol}(\Gamma\backslash G) \geq c_G$ for every lattice $\Gamma < G$.

In the meantime the scope of the study of lattices in locally compact groups has been greatly extended, including families of locally compact groups, like automorphism groups of trees [BL01], products of trees [BMZ09], and topological Kac-Moody groups [Re09].

In this context it was observed by Bass and Kulkarni that for the automorphism group of a d-regular tree $\mathcal{T}_d$ the analog of Kazhdan-Margulis as well as Wang's theorem, fail. They constructed for every $d \geq 3$ an infinite ascending chain $\Gamma_0 \lneq \Gamma_1 \lneq \cdots$ of discrete subgroups such that $\Gamma_\ell\backslash\mathcal{T}_d$ is a geometric loop; when d is composite they exhibited examples where $\Gamma_\ell\backslash\mathcal{T}_d$ is a geometric edge [BK90]. On the other hand, when $d \geq 3$ is a prime number, a deep conjecture of Goldschmidt-Sims in finite group theory implies that there are, up to conjugacy, only finitely many discrete subgroups $\Gamma < \mathrm{Aut}\,\mathcal{T}_d$ such that $\Gamma\backslash\mathcal{T}_d$ is an edge; this conjecture has been established for $d = 3$ [Go80]. For a product $\mathcal{T}_p \times \mathcal{T}_q$ of trees of prime degrees, Y. Glasner [Gl03] proved the remarkable result that up to conjugacy there are only finitely many $\Gamma < \mathrm{Aut}\,\mathcal{T}_p \times \mathrm{Aut}\,\mathcal{T}_q$ with non-discrete projections and such that $\Gamma\backslash(\mathcal{T}_p \times \mathcal{T}_q)$ is a geometric square.

We propose to study this finiteness problem in the framework of the theory of lattices in products of trees developed in [BM00a], [BM00b]. Our aim is to establish Wang's theorem for co-compact lattices in a product $\mathrm{Aut}\,(T_1) \times \mathrm{Aut}\,(T_2)$ of automorphism groups of regular trees by imposing non-properness and certain local transitivity properties for their actions on the individual factors T_1 and T_2. Our main result is:

Theorem 1. *Let $\Gamma < \mathrm{Aut}\,T_1 \times \mathrm{Aut}\,T_2$ be a co-compact lattice and $G_i = \overline{\mathrm{pr}_i(\Gamma)}$ the closures of its projections. Assume that each G_i is vertex transitive, non-discrete and locally quasi-primitive of almost simple type. Then Γ is contained in only finitely many discrete subgroups Λ with $\Lambda < G_1 \times G_2$.*

Recall that G_i is called locally quasi-primitive if for every vertex x of T_i the finite permutation group $\underline{G}_i(x)$ induced by the action of $G_i(x)$, the stabilizer of x, on $E(x)$, the set of edges based at x, is quasi-primitive. Such a permutation group has one or two minimal normal subgroups and is called of almost simple type if there is a unique minimal normal subgroup which is simple non-abelian [Pr97]. The theorem applies for instance when G_1 and G_2 are locally 2-transitive and the 2-transitive permutation groups have non-abelian socle.

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Acylindrically hyperbolic groups

Denis Osin

A group is acylindrically hyperbolic if it admits a non-elementary acylindrical action on a hyperbolic space. The class of acylindrically hyperbolic groups includes many examples of interest: hyperbolic and relatively hyperbolic groups, $Out(F_n)$ for $n > 1$, all but finitely many mapping class groups of closed surfaces, "most" fundamental groups of 3-manifolds, groups acting properly on proper $CAT(0)$ spaces and containing rank 1 elements, 1-relator groups with at least 3 generators, etc. On the other hand, many non-trivial results known for hyperbolic groups can be generalized to acylindrically hyperbolic groups. The purpose of my talk is to survey some of the recent progress in this direction.

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Telescopic actions

Anton Petrunin

(joint work with Dmitry Panov)

An isometric co-compact properly discontinuous group action H on X is called *telescopic* if for any finitely presented group G, there exists a subgroup H' of finite index in H such that G is isomorphic to the fundamental group of X/H'.

We construct examples of telescopic actions on some CAT[-1] spaces, in particular on 3 and 4-dimensional hyperbolic spaces. As applications we give new proofs of the following statements.

Aitchison's theorem. *Every finitely presented group G can appear as the fundamental group of M/J, where M is a closed 3-manifold and J is an involution which has only isolated fixed points.*

Taubes' theorem. *Every finitely presented group G can appear as the fundamental group of a compact complex 3-manifold.*

Separability of embedded surfaces in 3–manifolds

Piotr Przytycki

(joint work with Daniel T. Wise)

Here is a brief description of our recent joint work with Daniel T. Wise on separability of embedded surfaces in 3–manifolds.

A subgroup $H \subset G$ is *separable* if H equals the intersection of finite index subgroups of G containing H. Scott proved that if $G = \pi_1 M$ for a manifold M with universal cover $\widetilde{M}$, then H is separable if and only if each compact subset of $H\backslash\widetilde{M}$ embeds in an intermediate finite cover of M [5, Lem 1.4]. Thus, if $H = \pi_1 S$ for a compact surface $S \subset H\backslash\widetilde{M}$, then separability of H implies that S embeds in a finite cover of M. Rubinstein–Wang found a properly immersed π_1–injective surface $S \looparrowright M$ in a graph manifold such that S does not lift to an embedding in a finite cover of M, and they deduced that $\pi_1 S \subset \pi_1 M$ is not separable [4, Ex 2.6].

Our main result is:

Theorem 1. *Let M be a compact connected 3–manifold and let $S \subset M$ be a properly embedded connected π_1–injective surface. Then $\pi_1 S$ is separable in $\pi_1 M$.*

The problem of separability of an embedded surface subgroup was raised for instance by Silver–Williams — see [6] and the references therein to their earlier works. The Silver–Williams conjecture was resolved recently by Friedl–Vidussi in [2], who proved that $\pi_1 S$ can be separated from some element in $[\pi_1 M, \pi_1 M] - \pi_1 S$ whenever $\pi_1 S$ is not a fiber.

We proved Theorem 1 when M is a graph manifold in [3, Thm 1.1]. Theorem 1 was also proven when M is hyperbolic [7]. In fact, every finitely generated subgroup of $\pi_1 M$ is separable for hyperbolic M, by [7] in the case $\partial M \neq \emptyset$ and by Agol's theorem [1] for M closed.

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All finite groups are involved in the Mapping Class Group

Alan W. Reid

(joint work with Gregor Masbaum)

Let Σ_g be a closed orientable surface of genus $g \geq 1$, and Γ_g its Mapping Class Group.

A group H is involved in a group G if there is a finite index subgroup $K < G$ so that K subjects onto H. The question as to whether every finite group is involved in a fixed Γ_g was raised by U. Hamenstädt in her talk at the 2009 Georgia Topology Conference. This is easily seen to hold for the case $g = 1$ (since $\Gamma_1 = \mathrm{SL}(2, \mathbf{Z})$ is virtually free) and for $g = 2$ (since Γ_2 is large, see [5]). The main result of our joint work [8] is the following:

Theorem 1. *For all $g \geq 1$, every finite group is involved in Γ_g.*

From the remarks above, it suffices to deal with the case of $g \geq 3$. Although Γ_g is well-known to be residually finite [4], and therefore has a rich supply of finite quotients, apart from those finite quotients obtained from

$$\Gamma_g \to \mathrm{Sp}(2g, \mathbf{Z}) \to \mathrm{Sp}(2g, \mathbf{Z}/N\mathbf{Z})$$

very little seems known about what finite groups can arise as quotients of Γ_g (or subgroups of finite index). In particular, we emphasize that one cannot expect to prove Theorem 1 simply using the subgroup structure of the groups $\mathrm{Sp}(2g, \mathbf{Z}/N\mathbf{Z})$. The reason for this is that since $\mathrm{Sp}(2g, \mathbf{Z})$ has the Congruence Subgroup Property ([1]), it is well-known that not all finite groups are involved in $\mathrm{Sp}(2g, \mathbf{Z})$ (see [7] Chapter 4.0 for example).

The main new idea in the proof of Theorem 1 is to exploit the unitary representations arising in Topological Quantum Field Theory (TQFT) first constructed by Reshetikhin and Turaev [10]. We actually use the so-called SO(3)-TQFT following the skein-theoretical approach of [2] and the Integral TQFT refinement [3]. Theorem 1 easily follows from the next result (see [8]) which gives many new finite simple groups of Lie type as quotients of Γ_g. Let $\mathbf{F}_q$ denote a finite field of order q, and $\mathrm{SL}(N, q)$ (resp. $\mathrm{PSL}(N, q)$) will denote the finite group $\mathrm{SL}(N, \mathbf{F}_q)$ (resp. $\mathrm{PSL}(N, \mathbf{F}_q)$).

Theorem 2. *For each $g \geq 3$, there exists infinitely many N such that for each such N, there exists infinitely many primes q such that Γ_g surjects* $\mathrm{PSL}(N, q)$.

In addition we show that Theorem 2 holds for the Torelli group (with $g \geq 2$).

We briefly indicate the strategy of the proof of Theorem 2. The unitary representations that we consider are indexed by primes p congruent to 3 modulo 4. For each such p we exhibit a group Δ_g which is the image of a certain central extension $\widetilde{\Gamma}_g$ of Γ_g and satisfies

$$\Delta_g \subset \mathrm{SL}(N_p, \mathbf{Z}[\zeta_p]) ,$$

where ζ_p is a primitive p-th root of unity, and $\mathbf{Z}[\zeta_p]$ is the ring of integers in $\mathbf{Q}(\zeta_p)$. Moreover, the dimension $N_p \to \infty$ as we vary p.

The key part of the proof is the following. We use Strong Approximation in the form proved by Weisfeiler [11] (see also [9]) and a density result for these TQFT representations proved by Larsen and Wang [6] to exhibit infinitely many rational primes q, and prime ideals $\tilde{q} \subset \mathbf{Z}[\zeta_p]$ satisfying $\mathbf{Z}[\zeta_p]/\tilde{q} \simeq \mathbf{F}_q$, for which the reduction homomorphism $\pi_{\tilde{q}}$ from $\mathrm{SL}(N_p, \mathbf{Z}[\zeta_p])$ to $\mathrm{SL}(N_p, q)$ (induced by the isomorphism $\mathbf{Z}[\zeta_p]/\tilde{q} \simeq \mathbf{F}_q$) restricts to a surjection $\Delta_g \twoheadrightarrow \mathrm{SL}(N_p, q)$.

From this, it is then easy to get surjections $\Gamma_g \twoheadrightarrow \mathrm{PSL}(N_p, q)$, which will complete the proof. For more details on how all of this is achieved, see [8].

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CAT(0) braid groups

Petra Schwer
(joint work with Thomas Haettel, Dawid Kielak)

We address the question whether braid groups are CAT(0). In 2001 Thomas Brady [2] introduced a simplicial $K(B_n, 1)$ space for the braid groups whose universal cover is locally isomorphic to the order complex of the lattice NCP_n of non-crossing partitions on n points equidistributed on a circle.

Using this complex Tom Brady and Jon McCammond [3] showed in 2009 that the fact that the n-strand braid group is CAT(0) does follow from the diagonal link of X being CAT(1) for what they call the orthoscheme metric on NCP_n. They conjecture this to be true for all n and show it for $n \leq 5$ using the algorithmic methods of Elder-McCammond [4].

We improve their result to the following theorem and give a new proof for $n \leq 5$ avoiding any computational methods:

Theorem 1. *The n-strand braid group is CAT(0) for all $n \leq 6$.*

We use spherical buildings, a criterion for a space to be CAT(1) due to Brian Bowditch [1] and knowledge about turning points to show the desired curvature properties. Details can be found here [5].

Consider a subspace Y of a CAT(1) space B and equip Y with the induced length metric. Then a *turning point* is a point where a local geodesic in Y fails to be a local geodesic in B.

Why do partitions play a role?

Let U_n be the set of n^{th} roots of unity in the plane and let P_n denote the set of partitions of the set U_n into disjoint subsets. An element of a partition is called a *block*. One can show that P_n forms a bounded graded lattice of rank $n-1$, where the order is given by: $p \leq p'$ if and only if every block of p is contained in a block of p' and where joins and meets are defined via unions and intersections of blocks.

A partition is *non-crossing* if for every two distinct blocks of the partition, the convex hulls of these blocks in the plane do not intersect. The non-crossing partitions form a subposet NCP_n of P_n which is itself a bounded graded lattice of rank $n-1$.

Given an element of the braid group B_n one can associate to it an element of P_n as follows: First consider the map $\phi : B_n \to S_n$ which maps a braid to the induced permutation on strings. Now a permutation σ gives us a partition where we take the blocks to be the orbits of σ.

One can further introduce a partial order on S_n such that the elements smaller than a fixed n-cycle are poset-equivalent to NCP_n. Using this fact Brady [2] gave a presentation of the braid group in terms of non-crossing partitions and introduced a $K(B_n, 1)$ which is a quotient of the order complex of NCP_n.

Building up on Brady's work Brady-McCammond [3] defined an *orthoscheme metric* on the order complex of NCP_n as follows: A *standard n-orthoscheme* is a simplex in the barycentric subdivision of an n-cube of side length two. Now each simplex corresponding to a maximal chain in NCP_n is given the metric of a standard n-orthoscheme. Since NCP_n is a lattice all maximal simplices share an edge called the *diagonal*. They show:

Theorem 2 ([3]). *If NCP_n is CAT(0) then B_n is CAT(0). And if $\forall k \leq n$ the link of the diagonal in NCP_k is CAT(1) then NCP_n is CAT(0).*

Hence we are left to prove that the diagonal link is CAT(1). Each vertex in this link may be identified with and elements of NCP_n different from the minimal and maximal element.

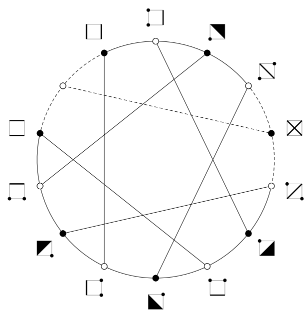

FIGURE 1. The diagonal link of NCP_4 inside the incidence graph of the Fano plane.

HOW DOES THE BUILDING COME INTO PLAY?

The diagonal link of the orthoscheme complex of the lattice NCP_n linearly embeds into a (in fact any thick) spherical building of type A_{n-2}.

Fix $n \geq 2$. If V is a $(n-1)$-dimensional vector space over a division algebra let $S(V)$ denote the rank $(n-1)$ lattice consisting of all the vector subspaces of V, with the order given by inclusion. We call $S(V)$ the *linear lattice* of V. It is not hard to see that the diagonal link of $S(V)$ is a spherical building of type A_{n-2}.

Now the partition and non-crossing partition lattices, P_n and NCP_n, are isomorphic to subposets of $S(V)$. To see this fix a field F, and let $V = \{(y_i) \in F^n \mid \sum_{i=1}^n y_i = 0\}$. For a partition $x \in P_n$ let then $f(x)$ be the sub-vector space of V defined by

$$f(x) = \big\{(y_i) \in V \mid \forall \text{ blocks } Q \in x : \sum_{i \in Q} y_i = 0\big\}.$$

One can show that f is an injective rank-preserving poset map from P_n to $S(V)$ which clearly restricts to NCP_n.

Figure 1 shows the diagonal link of NCP_4 (solid lines) inside the incidence graph of the Fano plane, i.e. the smallest possible spherical building in which NCP_4 embeds. The dotted lines are not in the image of NCP_4.

Why are turning points so important?

Our approach is based on investigating the relationship between the geometry of the diagonal link of NCP_n and the ambient building. We show inductively that the diagonal link is locally CAT(1) (in the sense of Bowditch [1]), and hence in order to verify the CAT(1) property we only need to show that it does not contain any locally geodesic loop of length smaller than 2π. In other word: if such a loop exists that it is shrinkable, that is may be homotoped to the trivial loop while not increasing its length along the way.

We show that a short loop has to pass through at least three turning points. Hence understanding positions and behaviour of turning points does allow us to gain control over these loops. The following three facts directly imply the theorem in case of $n = 5$. For the 6-string braid group more work remains to be done.

Facts.

- *If $n = 5$ then turning points are* universal vertices, *i.e. they correspond to a partition with only one non-singleton block which consists of consecutive numbers only.*
- *Each short loop through a universal vertex is shrinkable.*
- *Each short loop in NCP_n contains at least three (and a finite number of) turning points.*

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Varieties and geometric structures in semigroups.

Zlil Sela

In 1946 Quine proved that arithmetic can be interpreted in the theory of a free semigroup. Durnev and others proved that fragments of that theory (including the AE theory) are undecidable.

In a different direction, Makanin showed in 1977 that it is possible to decide if a system of equations over a free semigroup has a solution. This work preceeded his own work on the similar question for free groups. In 1987 Razborov managed to use Makanin's work on groups, and encoded the set of solutions to a system of equations over a free group in some combinatorial structures. No analogue of Razborov's work is known for varieties over a free semigroup.

We suggest a geometric approach to study varieties over a free semigroup. We manage to find analogues of (geometric) structures that were known to exist over groups to encode the points in a variety over a free semigroup.

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