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Hyperbolic Techniques for Phase Dynamics

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Abstract. The progress in the theory of hyperbolic conservation laws has always been and still is driven strongly by new fields of applications. The workshop addressed aspects of modelling, analysis and numerics for fundamental problems at the interface between hyperbolic evolution and the emerging mathematical theories of complex multiphasic materials. This includes problems in fluid and solid mechanics but also very recent applications in areas like swarm and traffic modelling.

Mathematics Subject Classification (2010): 35L65, 35L67, 65Mxx, 74N20, 76Txx, 80A17.

Introduction by the Organisers

This workshop welcomed 47 participants (thereof 9 women) from 11 different countries. The speakers included young PhD students as well as researchers well-established at the international level. The schedule of the workshop allowed the former to deliver 10 minutes presentations including open questions they face, while the latter introduced the audience to the most recent results. In the tradition of Oberwolfach workshops, discussions played a central role throughout the week. Often, they were explicitely initiated by the speakers, or motivated by the personal interests in the audience.

The hyperbolic modeling of phase dynamics turned out to be the underpinning cornerstone around which a large variety of subjects departed.

First of all, the classical topic of well-posedness, as well as the analytical techniques to achieve this property, was central in the presentations by De Lellis, Modena, and Nordli. In particular, a communication was devoted to the very interesting counter-example to uniqueness in the two-dimensional, isentropic fluid

system. The techniques based on relative entropies were discussed by Lattanzio and Seguin. On the other hand, Andreianov, Boutin, and Gwiazda presented recent results on multi-dimensional hyperbolic systems with various kinds of discontinuous flux.

Euler equations for fluid flows still keep their classical role of being a paradigm for the shock wave theory initiated by Peter Lax in the 60's and were central in several talks. Kinetic relations were addressed by Dreyer, who presented physically consistent relations to be imposed on subsonic phase transitions, and by Bedjaoui in joint work with LeFloch who derived kinetic relations based on traveling waves. Likewise, Corli, Fan, Freistühler, Giesselmann, Kraus, Kotschote, and Müller investigated liquid–vapor phase transitions from the modeling standpoint as well as the analytical and numerical points of view. Schleper presented new results on the compressible vs. incompressible limit for non-smooth solutions to the one-dimensional isentropic Euler equations. The case of elastodynamics was tackled by Tzavaras in his presentation devoted to the formation of cavitation, and by Luckhaus who approached it using the Hamiltonian formalism. Frid and Pop described recent results based on hyperbolic techniques and porous media flows.

The closely related topic of the dynamics of granular materials and its connection with the modeling of avalanches and erosion-deposition phenomena, was central in the communications by Guerra and Swierczewska-Gwiazda.

A less traditional field of applications is provided by the modeling of vehicular traffic and crowd dynamics. The issues related to the analytical study and as well as ad hoc numerical methods were considered by Amorim, Goatin, Marcellini, Rosini and Rossi.

Descriptions of the interplay among different models, both at different scales and devoted to different phenomena, motivate the communications by Garavello and Godlewski. In this context, Gasser drove the attention of the audience to models not yet much considered in the hyperbolic literature, while Borsche opened the discussion on the effectiveness and reliability of one-dimensional models in the description of three-dimensional phenomena.

Finally, several communications focused on the numerics. Mishra proposed a non deterministic approach to the lack of strong convergence of the numerical solutions to the two-dimensional Euler system. Amadori described well-balanced schemes for conservation laws with source terms, while Chalons and Turpault presented asymptotically-preserving schemes,. Pares discussed the approximation of non-conservative systems, and Helluy gave an overview of the use of graphic cards as an effective tool for large-scale computations. A general entropy dissipative approach was presented by Hiltebrand.

Workshop: Hyperbolic Techniques for Phase Dynamics

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Abstracts

The problem of dynamic cavitation in nonlinear elasticity

Athanasios Tzavaras
(joint work with Jan Giesselmann)

In this work we re-assess an example of non-uniqueness of entropy weak solutions for multi-dimensional systems of hyperbolic conservation laws constructed in [4, 5] and associated with the onset of cavitation in homogeneous and isotropic elastic media. The equations of nonlinear elasticity is the system

$$\mathbf{y}_{tt} - \operatorname{div} \frac{\partial W}{\partial \mathbf{F}}(\nabla \mathbf{y}) = 0\,, \tag{1}$$

where $\mathbf{y} : \mathbb{R}^d \times \mathbb{R}_+ \to \mathbb{R}$ stands for the motion, $\mathbf{F} = \nabla \mathbf{y}$ is the deformation gradient, and $W(\mathbf{F})$ stands for the stored elastic energy. The homogeneous deformation $\bar{\mathbf{y}}(\mathbf{x}, t) = \lambda \mathbf{x}$ is a particular solution of (1) with $\lambda > 0$ a given stretching.

For homogeneous and isotropic elastic materials W takes the simplified form $W(\mathbf{F}) = \Phi(\lambda_1, ..., \lambda_d)$, where Φ is a symmetric function of the eigenvalues $\lambda_1, ..., \lambda_d$ of $\sqrt{\mathbf{F}\mathbf{F}^T}$. Then (1) admits radially symmetric solutions, $\mathbf{y}(\mathbf{x}, t) = w(|\mathbf{x}|, t)\frac{\mathbf{x}}{|\mathbf{x}|}$, $R = |\mathbf{x}|$, generated by solving for the amplitude $w : \mathbb{R}_+ \times \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ the equations of isotropic radial elastodynamics,

$$w_{tt} = \frac{1}{R^{d-1}} \partial_R \left(R^{d-1} \frac{\partial \Phi}{\partial \lambda_1}(w_R, \frac{w}{R}, \ldots, \frac{w}{R}) \right) - \frac{1}{R}(d-1) \frac{\partial \Phi}{\partial \lambda_2}(w_R, \frac{w}{R}, \ldots, \frac{w}{R}). \tag{2}$$

$$w(R, 0) = \lambda R\,. \tag{3}$$

The special solution $\bar{w}(R, t) = \lambda R$ corresponds to a state of homogeneous deformation. The question arises if additional solutions of the initial value problem for (1) may be constructed by solving the problem (2)-(3).

This idea was pursued by Ball [1] first in the context of the static problem by using methods from the calculus of variations, who showed that there exists a critical stretching λ_{cr} so that: (i) for $\lambda < \lambda_{cr}$ the only equilibrium solution is the homogeneously deformed state; (ii) for $\lambda > \lambda_{cr}$ there exist non-trivial equilibria (corresponding to a cavity) with energy *less* than the energy of the homogeneous deformation [1]. In [4, 5], K.A. Pericak-Spector and S. Spector use the ansatz

$$w(R, t) = t\, r\Big(\frac{R}{t}\Big)\,, \tag{4}$$

to construct a self-similar weak solution for the dynamic problem (2)-(3) that corresponds to a spherical cavity emerging at time $t = 0$ from a homogeneously deformed state. The solution in [4] is constructed in dimension $d \geq 3$ for polyconvex stored energies of the special form

$$W(\mathbf{F}) = \frac{1}{2} \sum_{i=1}^{d} \lambda_i^2 + h\left(\prod_{i=1}^{d} \lambda_i \right) \tag{H_1}$$

where $h : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ satisfies the hypotheses

$$(\mathrm{H}_2) \qquad h'' > 0\,, \quad h''' < 0 \quad \lim_{v\to 0} h(v) = \lim_{v\to\infty} h(v) = \infty\,.$$

The hypothesis $h'' > 0$ refers to polyconvexity, while $h''' < 0$ indicates elasticity of softening type; more general stored energies were treated in [5]. It is proved in [4, 5] that the self-similar solution has smaller mechanical energy than the associated homogeneously deformed state from where it emerges, and thus provides an example of nonuniqueness of entropy weak solutions (at least for polyconvex energies). As already noted in [4], the paradox arises that by opening a cavity the energy of the material decreases, what induces an autocatalytic mechanism for failure. Cavitating solutions also exist in dimension $d = 2$ under different growth hypotheses on the stored energy function and it is shown that in dimensions $d = 2, 3$ the existence of a cavity entails that there exists a non-degenerate precursor shock [3].

There is a class of problems in solid mechanics, such as fracture, cavitation or shear bands, where discontinuous motions emerge from smooth motions via a mechanism of material instability. Any attempt to study solutions that lie at the limits of continuum modeling needs to reckon with the problem of giving a proper definition for such solutions. Once the material breaks or a shear band forms the motion can no longer be described at the level of continuum modeling and microscopic modeling or higher-order regularizing mechanisms have to be taken into account. Still, as such structures develop there is expected an intermediate time scale where both types of modeling apply.

The premise of this work is to introduce a meaning for solutions at this intermediate scale and to explore its ramifications. The idea is the following, presented here at the level of (1). Given a possibly discontinuous motion $\mathbf{y}(\mathbf{x}, t)$ we introduce the averaged motions $\mathbf{y}^n = \phi_n \star \mathbf{y}$, where ϕ is a mollifier, and define $\mathbf{y}$ to be a singular limiting induced from continuum solution (in short a *slic*-solution) if for every choice of the mollifier the smooth approximating family in the limit of small-scale averaging gives

$$(5) \qquad \mathbf{y}^n_{tt} - \operatorname{div} \mathbf{S}(\nabla \mathbf{y}^n) =: \mathbf{f^n} \to 0 \quad \text{in } \mathcal{D}'.$$

In this notion the precise form of regularizing mechanisms is not taken into account, instead it is enforced that they act in a stable way amounting to averaging of the tested solution. Moreover, the energy equation for (5),

$$(6) \qquad \partial_t \left(\frac{1}{2} |\mathbf{y}^n_t|^2 + W(\nabla \mathbf{y}^n) \right) - \operatorname{div} \left(\mathbf{y}^n_t \cdot \mathbf{S}(\nabla \mathbf{y}^n) \right) = \mathbf{y}^n_t \cdot \mathbf{f^n}\,,$$

suggests that, even though $\mathbf{f^n} \to 0$ in $\mathcal{D}'$, the power of the "microscopic forces" $\mathbf{y}^n_t \cdot \mathbf{f^n}$ may well have a non-trivial contribution in the limit, which needs to be calculated. The definition of slic-solutions exploits the fact that the momentum equation (1) is a second-order evolution. Slic-solutions are generated via averagings of a candidate discontinuous solution, and are natural in a context of discontinuous solutions for second order evolution equations. The latter property provides a mechanical intuition for the definition and suggests its name.

In the talk, we examine the ramifications of this definition in two examples. First, for the equations of one-dimensional elasticity

$$y_{tt} - (\tau(y_x))_x = 0 \tag{7}$$

we test a specific example of a motion,

$$y(x,t) = \begin{cases} \lambda x \mathbb{1}_{x<-\sigma t} + \big(t(\operatorname{sign} x)Y(0) + \alpha x\big)\mathbb{1}_{-\sigma t<x<\sigma t} + \lambda x \mathbb{1}_{\sigma t<x} & t>0 \\ \lambda x & t<0 \end{cases}, \tag{8}$$

towards being a slic-solution. The example (8) is a counterpart (for $d = 1$) of the cavitating solutions (for $d \geq 2$) and corresponds to a crack forming out of a homogeneously deformed state at the location $x = 0$ at time $t = 0$, in conjunction with two outgoing Lax-shocks propagating at $x = \pm\sigma t$. It is more singular than the dynamic cavitating solution in [4] as y_x has a delta-mass at the origin. We present a definition of the notion of slic-solution and provide conditions under which (8) is a slic-solution for (7). Then the energy balance for the approximate solution of the crack is computed and it turns out that there is an energetic cost for creating the crack that is projected in the limiting energy balance equation.

Then we consider the dynamically cavitating solution (4) with $r(s)$ the weak cavitating self-similar solution constructed in [4]. We introduce a counterpart of the notion of slic-solution adapted to this example and test the natural definition for the energy. The analysis is more cumbersome and is based on detailed estimations of the layers of the approximate solution, but the results parallel those of the one-dimensional example. Namely, conditions are given under which the cavitating solutions provide a slic-solution and the energy balance of the approximate solutions can be precisely computed [2] and shows that there is an energetic cost (corresponding to the surface energy) for creating the cavity..

Regarding the issue of uniqueness the results of [4] and [2] indicate : if the solution is construed as an entropy weak solution, then there is non-uniqueness. By contrast, if the solution is construed in the slic-sense, then there is a contribution to the total energy in the process of forming the cavity which results to the energy after the cavity formation being larger than before the cavitation. This indicates that the notion of entropy weak solutions is inadequate when dealing with discontinuous solutions and in particular it cannot account for the work needed to create the cavity. A more discriminating concept of solution has to be employed on strong singularities, and the slic-solution concept is such a possibility.

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Discrete Models for Nonlinear Elasticity

Stephan Luckhaus
(joint work with Roman Kotecky)

One way to derive the stored energy function of nonlinear elasticity from discrete particle models is to start from the equilibrium measure of a discrete lattice based Hamiltonian and show then that its infinite volume limit large deviations functional is a stored energy functional.That means the equilibrium measure with macroscopic boundary values is concentrating exponentially on minimizers of the stored energy functional with said boundary conditions. The discrete Hamiltonians we start from are lattice based finite range , i.e. the configurations are viewed as deformations of a ground state lattice interaction depends on the particle position in the ground state lattice and is finite range there.Furthermore the potential is invariant under rigid motions in phase space and under lattice translations. We have to assume growth conditions but try to keep them as general as possible. If the bound on the growth from above and below is with the same power p of the (discrete) gradient then the existence of the large deviations functional depending only on the gradient of deformation, the elastic free energy, can be proved.If the bound from below is p and the bound from above is a power less than the corresponding Sobolev exponent ,then we can prove the result provided the elastic energy has no Lawrentiev gap.The case with Lawrentiev gap is open.The interest of this case is that cavitation may occur under these conditions.

Ill-posedness for the isentropic system of gas dynamics

Camillo De Lellis
(joint work with Elisabetta Chiodaroli, Ondřej Kreml)

Consider the isentropic compressible Euler equations of gas dynamics in n space dimensions. This system consists of $n+1$ scalar equations, which state the conservation of mass and linear momentum. The unknowns are the density ρ and the velocity v and the system takes the the form:

$$\begin{cases} \partial_t \rho + \operatorname{div}_x(\rho v) \;=\; 0 \\ \partial_t(\rho v) + \operatorname{div}_x\left(\rho v \otimes v\right) + \nabla_x[p(\rho)] \;=\; 0 \end{cases} \tag{1}$$

The pressure p is a function of ρ determined from the constitutive thermodynamic relations of the gas under consideration and it is assumed to satisfy $p' > 0$ (this hypothesis guarantees also the hyperbolicity of the system on the regions where ρ is positive). A common choice is the polytropic pressure law $p(\rho) = \kappa\rho^\gamma$ with constants $\kappa > 0$ and $\gamma > 1$. The classical kinetic theory of gases predicts exponents $\gamma = 1 + \frac{2}{d}$, where d is the degree of freedom of the molecule of the gas.

A lot of attention has been devoted in the literature to the *Cauchy problem* which consists of solving (1) on a domain of the form $\mathbb{R}^2 \times [0, T[$ (where T might also be infinite), subject to an initial condition of type

$$\left\{ \begin{array}{l} \rho(\cdot, 0) \ = \ \rho^0 \\ v(\cdot, 0) \ = \ v^0 \, . \end{array} \right. \tag{2}$$

It is well known that, even starting from extremely regular initial data, the solutions of the Cauchy problem for the system (1) develops singularities in finite time. It is also well-known that after the appearance of the first singularity weak solutions (i.e. solutions in the usual distributional sense) are not unique: the standard example is provided by "non-physical" shocks, which can however be ruled out imposing that the weak solutions satisfy some further admissibility condition. Much effort has been put in understanding how this approach can give well-posedness results after the appearance of the first singularity, leading to a quite mature and successful theory in one space dimension (we refer the reader to the monographs [1],[4] and [9]).

Here we consider the case of two space dimensions and restrict our attention to bounded weak solutions of (1) which satisfy the following additional inequality in the sense of distributions (called usually *entropy inequality*, although for the specific system (1) this is rather a weak form of the energy balance):

$$\partial_t \left(\rho \varepsilon(\rho) + \rho \frac{|v|^2}{2} \right) + \mathrm{div}_x \left[\left(\rho \varepsilon(\rho) + \rho \frac{|v|^2}{2} + p(\rho) \right) v \right] \ \leq \ 0 \tag{3}$$

where the internal energy $\varepsilon : \mathbb{R}^+ \to \mathbb{R}$ is given through the law $p(r) = r^2 \varepsilon'(r)$. Indeed, admissible solutions are required to satisfy a slightly stronger condition, i.e. a form of (3) which involves also the initial data (see for instance [4] for the precise definition).

Starting from the work [6] it was observed that (3) is in this case not enough to restore uniqueness of admissible *bounded* solutions. The methods used in [6], inspired by techniques developed in the theory of differential inclusions, show a rather surprising abundance of admissible solutions to the Cauchy problem with certain particular initial data. However those specific data were rather irregular, leaving open the question whether this fact alone was responsible for such behavior. The investigations of [6] have been pushed further in [2] and in [3]: in the latter paper we have shown that the same nonuniqueness result holds even for *Lipschitz* initial data, therefore leading to the following theorem.

Theorem 1. *Let $p(\rho) = \rho^2$. Then there are* Lipschitz *initial data ρ^0 and v^0, with $\rho^0 \geq c_0 > 0$ for which there are* infinitely many *admissible bounded weak solutions (ρ, v) of the Cauchy problem* (1)-(2)*, with* $\inf \rho > 0$*. All these solutions coincide with the classical one as long as it exists and differ immediately after the formation of the first singularity.*

The proof of Theorem 1 relies heavily on the works of the author and László Székelyhidi, who in the paper [5] introduced methods from the theory of differential inclusions to explain the existence of compactly supported nontrivial weak

solutions of the *incompressible* Euler equations (discovered in the pioneering work of Scheffer [8]; see also [9]). The proof draws also heavily from the work [10] where Székelyhidi coupled the methods introduced in [5]-[6] with a clever construction to produce rather surprising irregular solutions of the incompressible Euler equations with vortex-sheet initial data.

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Some applications of relative entropy to hyperbolic PDE's

Nicolas Seguin

(joint work with Clément Cancès, Bruno Després, Hélène Mathis, Frédéric Lagoutière)

The notion of entropy plays a central role in the physical characterization of weak solutions of hyperbolic partial differential equations. It may provide uniqueness but also stability of the solutions with respect to the data of the problem. A key point in general is the ability of comparing two solutions using the entropy, which must fit with the topology associated with the well-posedness theory. To do so, one may use of what is called in the current vocabulary the *relative entropy*. Moreover, the entropy weak solutions can be directly defined as weak solutions which are entropy stable with respect to a convenient class of elementary solutions. Let us go into a more detailed presentation and consider a system of conservation laws (we restrict ourselves to the one-dimensional setting for sake of simplicity)

$$\partial_t u + \partial_x f(u) = 0 \tag{1}$$

which has to be understood in the weak sense. In order to select physically admissible solutions, we assume the existence of a Lax entropy pair (η, F) (i.e. $\eta'' > 0$

and $\eta' \cdot f' = F'$) and impose the inequality

$$\partial_t \eta(u) + \partial_x F(u) \leq 0 \tag{2}$$

which is an equality if u is a smooth solution. The so-called entropy relative and the associated flux are

$$\begin{aligned} H(u,v) &= \eta(u) - \eta(v) - \eta'(v) \cdot (u-v), \\ Q(u,v) &= F(u) - F(v) - \eta'(v) \cdot (f(u) - f(v)). \end{aligned} \tag{3}$$

It is worth noting that H is a positive function which vanishes only if $u = v$ and has a quadratic behavior. It is natural to define the *entropy weak solution* of (1) as a function u which satisfies for any constant vector $\bar{u}$ the inequality

$$\partial_t H(u,\bar{u}) + \partial_x Q(u,\bar{u}) \leq 0 \tag{4}$$

in the sense of distributions. Let us now provide two applications of the relative entropy.

1. Friedrichs systems with constraint

Let us focus on the case of Friedrichs systems, that is to say $f(u) = Au$ where A is a symmetric matrix [7]. The natural entropy is $\eta(u) = |u|^2$, which gives $H(u,v) = |u-v|^2$ and $Q(u,v) = (u-v) \cdot (A(u-v))$. Since in this case H and Q are symmetric, one can reproduce the Kruzhkov's proof to obtain uniqueness and stability [10]: for all $R > 0$ and $t > 0$,

$$\int_{|x|<R} H(u,v)(t,x)\ dx \leq \int_{|x|<R+Lt} H(u,v)(0,x)\ dx, \tag{5}$$

where u and v are two (entropy) weak solutions and L is the spectral radius of the matrix A. Now, motivated by some applications in elastoplasticity, let us add to (1) (still with $f(u) = Au$) the constraint

$$u(t,x) \in K \tag{6}$$

for all t and x, where K is a closed convex subset of the set of admissible states. We assume that (6) is not an involution. We propose in [4] to use (4) with (6) and for any constant vector $\bar{u}$ in K to define *constrained weak solutions* of (1)–(6). Once again, we can follow the Kruzhkov's proof to exactly obtain (5), leading to the uniqueness and the stability of constrained weak solutions. The computations are rather straightforward thanks to the adaptation of definition (4). Moreover, our result extends the classical ones obtained for instance in [11].

2. The nonlinear case and finite volume methods

We consider now the case of a nonlinear flux f. Here, H and Q are no longer symmetric functions so that the Kruzhkov's doubling variable cannot be used. However, let us recall the result obtained by Dafermos [3] and DiPerna [5], which is the uniqueness of smooth solutions to the Cauchy problem for (1) in the class

of entropy weak solutions. Indeed, consider a smooth solution v and an entropy weak solution u and after some computations, obtain

$$\partial_t H(u,v) + \partial_x Q(u,v) \leq -\partial_x v \cdot Z(u,v) \tag{7}$$

where $Z(u,v) = \eta''(v)(f(u) - f(v) - f'(v)(u-v))$. Therefore, using the quadratic behavior of Q and Z and that there exists $L > 0$ such that $|Q(u,v)| \leq L|H(u,v)|$, the Gronwall lemma enables to have the stability estimate

$$\int_{|x|<R} H(u,v)(t,x)\ dx \leq C \int_{|x|<R+Lt} H(u,v)(0,x)\ dx \tag{8}$$

where C involves a term of the form e^{Mt}, M being the Lipschitz constant of v.

In order to approximate system (1), a very classical method is based on the finite volume numerical schemes. Let Δx and Δt be respectively the space step and the time step. The mesh is defined by $x_{i+1/2} = (i+1/2)\Delta x$ and the successive times of computations are $t^n = n\Delta t$. We define the finite volume approximation u_i^n in the cell $(x_{i-1/2}, x_{i+1/2})$ at time t^n by

$$u_i^{n+1} = u_i^n - \frac{\Delta t}{\Delta x}(g(u_i^n, u_{i+1}^n) - g(u_{i-1}^n, u_i^n)) \tag{9}$$

where g is a consistent numerical flux (see [1] for more details). Let us assume that under the CFL condition

$$\lambda \Delta t \leq \Delta x \tag{10}$$

(λ being positive and depending on g), there exists a numerical entropy flux G consistent with F such that

$$\eta(u_i^{n+1}) \leq \eta(u_i^n) - \frac{\Delta t}{\Delta x}(G(u_i^n, u_{i+1}^n) - G(u_{i-1}^n, u_i^n)) \tag{11}$$

(see for instance [8, 12, 1] for more details). In [2], we propose to follow the formalism proposed in [6] to estimate the error of the numerical approximation

$$u_\Delta(t,x) = \sum_{i,n} u_i^n \mathbf{1}_{(t^n,t^{n+1})\times(x_{i-1/2},x_{i+1/2})}(t,x)$$

defined by (9). To do so, we first prove that, under some L^∞ and BV bounds (see [9] for more details on these assumptions), u_Δ satisfies in the weak sense

$$\partial_t u_\Delta + f(u_\Delta) = \mu_\Delta \quad \text{and} \quad \partial_t \eta(u)_\Delta + F(u_\Delta) \leq \bar{\mu}_\Delta, \tag{12}$$

where μ_Δ and $\bar{\mu}_\Delta$ are local Radon measures such that

$$\mu_\Delta(B(0,R)\times(0,T)) \leq C\Delta x \quad \text{and} \quad \bar{\mu}_\Delta(B(0,R)\times(0,T)) \leq \bar{C}\Delta x.$$

Using (12), we can compare u_h with any smooth solution v. Indeed, we have

$$\partial_t H(u_\Delta, v) + \partial_x Q(u_\Delta, v) \leq -\partial_x v \cdot Z(u,v) + \bar{\mu}_\Delta - \eta'(v) \cdot \mu_\Delta \tag{13}$$

which leads, after some careful computations, to the following error estimate

$$\int_0^T \int_{B(0,R)} H(u_\Delta, v)\ dx\ dt \leq C\Delta x. \tag{14}$$

Note that this estimate also holds in the multidimensional setting with unstructured grids in [2].

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Conservative scheme for two-fluid compressible flows without pressure oscillations

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(joint work with Jonathan Jung)

Compressible two-fluid flows are difficult to numerically simulate. Indeed, classic conservative finite volume schemes do not preserve the velocity and pressure equilibrium at the two-fluid interface. This leads to oscillations, lack of precision and even, in some liquid-gas simulations, to the crash of the computation. Several cures have been proposed to obtain better schemes (see [1] and included references). The resulting schemes are generally not conservative. Based on ideas of [2], we propose a new Lagrange-Projection scheme. The projection step is based on a random sampling strategy at the interface. The scheme has the following properties: it preserves constant velocity and pressure at the two-fluid interface, it preserves a perfectly sharp interface and it is fully conservative (in a statistical sense). The scheme can be extended to higher space dimensions through Strang

dimensional splitting. Finally, it is very simple to implement and thus well adapted to massively parallel GPU computations.

We are interested in the numerical resolution of the following system of partial differential equations, modeling a liquid-gas compressible flow

$$\partial_t W + \partial_x F(W) = 0, \tag{1}$$

where

$$W = (\rho, \rho u, \rho v, \rho E, \rho\varphi)^T, F(W) = (\rho u, \rho u^2 + p, \rho uv, (\rho E + p)u, \rho u\varphi)^T.$$

The unknowns are the density ρ, the two components of the velocity u, v, the internal energy e and the mass fraction of gas φ. The unknowns depend on the space variables x and on the time variable t. The total energy E is the sum of the internal energy and the kinetic energy $E = e+(u^2+v^2)/2$.The pressure p of the two-fluid medium is a function of the other thermodynamical parameters $p = p(\rho, e, \varphi)$. We consider a stiffened gas pressure law $p(\rho, e, \varphi) = (\gamma(\varphi)-1)\rho e-\gamma(\varphi)\pi(\varphi)$,where γ and π are given functions of the mass fraction φ, and $\gamma(\varphi) > 1$.

At the initial time, the mass fraction $\varphi(x, y, 0) = 1$ if the point (x, y) is in the gas region and $\varphi(x, y, 0) = 0$ if the point (x, y) is in the liquid region. The mass fraction is transported with the flow, which implies that for any time $t > 0$, $\varphi(x, y, t)$ can take only the two values 0 or 1. However, classic numerical schemes generally produce an artificial diffusion of the mass fraction, and in the numerical approximation we may observe $1 > \varphi > 0$. In classic conservative schemes, the artificial mixing zone implies a loss of the velocity and pressure equilibrium at the interface.

We construct a better numerical scheme for solving (1). We consider a sequence of time t_n, $n \in \mathbb{N}$ such that the time step $\tau_n = t_{n+1} - t_n > 0$. We consider also a space step h. We define the cell centers by $x_i = ih$. The cell C_i is the interval $]x_{i-1/2}, x_{i+1/2}[$. We look for an approximation$W_i^n \simeq W(x_i, t_n)$. Each time step of the scheme is made of two stages: an Arbitrary Lagrangian Eulerian (ALE) step and a Projection step.

ALE stage. In the first stage, we allow the cell boundaries $x_{i+1/2}$ to move at a velocity $\xi_{i+1/2}^n$. At the end of the first stage, the cell boundary is

$$x_{i+1/2}^{n+1,-} = x_{i+1/2} + \tau_n \xi_{i+1/2}^n.$$

Integrating the conservation law (1) on the moving cells, we obtain the following finite volume approximation

$$h_i^{n+1,-} W_i^{n+1,-} - hW_i^n + \tau_n(F_{i+1/2}^n - F_{i-1/2}^n) = 0.$$

The new size of cell i is given by

$$h_i^{n+1,-} = x_{i+1/2}^{n+1,-} - x_{i\text{-}1/2}^{n+1,-} = h + \tau_n(\xi_{i+1/2}^n - \xi_{i-1/2}^n).$$

The numerical flux is of the form

$$F_{i+1/2}^n = F(W_{i+1/2}^n) - \xi_{i+1/2}^n W_{i+1/2}^n.$$

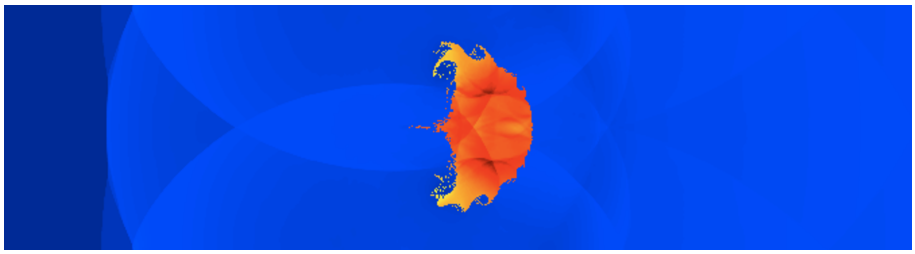

FIGURE 1. Shock-droplet simulation. Density plot.

The intermediate state $W^n_{i+1/2}$ is obtained by the resolution of a Riemann problem. More precisely, we consider the entropy solution of

$$\begin{aligned} \partial_t V + \partial_x F(V) &= 0, \\ V(x,0) &= \begin{cases} V_L & \text{if} \quad x < 0, \\ V_R & \text{if} \quad x > 0, \end{cases} \end{aligned}$$

which is denoted by $R(V_L, V_R, x/t) = V(x,t)$. The intermediate state is then $W^n_{i+1/2} = R(W^n_i, W^n_{i+1}, \xi^n_{i+1/2})$. In practice, R can also be an approximate Riemann solver.

Finally, the interface velocity is defined by

$$\xi^n_{i+1/2} = \begin{cases} u^n_{i+1/2} & \text{if} \quad (\varphi^n_i - 1/2)(\varphi^n_{i+1} - 1/2) < 0, \\ 0 & \text{else.} \end{cases} \tag{2}$$

The numerical flux is thus a classic Godunov flux in the pure fluid. It is a Lagrangian numerical flux at the two-fluid interface.

Projection step. The second stage of the time step is needed for returning to the initial mesh. We have to compute on the cells C_i of the initial mesh the averages of $W^{n+1,-}_i$, defined on the moved cells $C^{n+1,-}_i =]x^{n+1,-}_{i-1/2}, x^{n+1,-}_{i+1/2}[$. Instead of a standard integral averaging method, we rather consider a random sampling averaging process. We consider a pseudo random sequence $\omega_n \in [0,1[$ and we perform the following sampling

$$W^{n+1}_i = \begin{cases} W^{n+1,-}_{i-1}, \text{ if } \omega_n < \frac{\xi^n_{i-1/2}\tau_n}{h}, \\ W^{n+1,-}_i, \text{ if } \frac{\xi^n_{i-1/2}\tau_n}{h} \le \omega_n \le 1 + \frac{\xi^n_{i+1/2}\tau_n}{h}, \\ W^{n+1,-}_{i+1}, \text{ if } \omega_n > 1 + \frac{\xi^n_{i+1/2}\tau_n}{h}. \end{cases} \tag{3}$$

A good choice for the pseudo-random sequence ω_n is the (k_1, k_2) van der Corput sequence. In practice, we consider the $(5,3)$ van der Corput sequence.

We can extend the scheme to higher dimensions with dimensional splitting (more details in [3]). It is remarkable that the same random number can be used for one time step in the x and y directions. We present in Figure 1 the results of a two-dimensional shock-droplet GPU simulation. We observe that we are able to capture a sharp interface and small Kelvin-Helmholtz vortices. The numerical noise is moderate, despite the random nature of the scheme.

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A model for fluid flow in porous media with phase-dependent damping

Andrea Corli
(joint work with Haitao Fan)

We propose and discuss the following model for the isothermal and inviscid fluid flow through a porous medium, in presence of liquid-vapor phase changes:

$$\begin{cases} v_t - u_x & = 0\,, \\ u_t + p(v,\lambda)_x & = -\alpha(\lambda)u\,, \\ \lambda_t & = \frac{1}{\tau}\big(p(v,\lambda) - p_e\big)\lambda(\lambda-1)\,. \end{cases} \tag{1}$$

Here $t > 0$, $x \in \mathbb{R}$, $v > 0$ denotes the specific volume, u the velocity, $\lambda \in [0,1]$ the mass-density fraction of the vapor in the fluid, p_e an equilibrium pressure and $\tau > 0$ a reaction time. The pressure function $p(v,\lambda)$ satisfies

$$p_v < 0, \quad p_\lambda > 0, \quad p_{vv} > 0, \quad p_{v\lambda} < 0, \tag{2}$$

so that (1) is a hyperbolic system of balance laws. The flow is hosted in a medium that induces a friction force $-\alpha(\lambda)u$. We assume that $\alpha(\lambda) > 0$ for $\lambda \in [0,1)$ and $\alpha(1) \geq 0$; from a physical point of view, the function α is a decreasing function of λ.

The case without damping in the second equation has been studied in [6], even in a more general framework, as far as traveling waves are concerned; the Riemann problem in the limit case $\tau = \infty$ has been considered in [2] while in [1] the global existence of weak solutions and the vanishing relaxation limit has been proved. Here, we are interested in *traveling waves* to (1). The case when α is a positive constant was studied in [5]; analogous results are valid if α depends on λ but is bounded away from zero [4]. The case when $\alpha(1) = 0$ has been recently studied in [3]; such an assumption applies to vapor flows in materials with very high porosity. Now, we briefly introduce the dynamical system under consideration for the analysis of traveling waves, thus accounting for the results in [5, 4, 3].

We denote $U = (v,u,\lambda)$ and $\Omega = (0,+\infty) \times \mathbb{R} \times [0,1]$. A traveling wave to (1) with constant speed c is a solution to (1) of the form $U(\xi) = U\left(\frac{x-ct}{\tau}\right)$ that satisfies then the system

$$\begin{cases} -cv' - u' = 0\,, \\ -cu' + p' = -A(\lambda)u\,, \\ -c\lambda' = (p - p_e)\lambda(\lambda-1)\,, \end{cases} \tag{3}$$

together with

$$\begin{cases} (v,u,\lambda)(\pm\infty) = (v_\pm, u_\pm, \lambda_\pm)\,, \\ (v',u',\lambda')(\pm\infty) = 0\,, \end{cases} \tag{4}$$

for $(v_\pm, u_\pm, \lambda_\pm) \in \Omega$. Here above we denoted $A(\lambda) = \alpha(\lambda)\tau$. By integrating from $\xi = -\infty$ to $\xi = +\infty$ the first equation in (3) we get the necessary condition

$$u_- + cv_- = u_+ + cv_+. \tag{5}$$

We denote

$$\bar{v} = v_\pm + \frac{1}{c}u_\pm.$$

The first equation in (3) fully determines u through v:

$$u(\xi) = u_\pm - c\,(v(\xi) - v_\pm)\,.$$

Having eliminated u, system (3) can be written as follows, including the conditions at $\pm\infty$:

$$\begin{cases} (c^2 + p_v)v' = Ac(v-\bar{v}) + \frac{1}{c}p_\lambda(p-p_e)\lambda(\lambda-1), \\ \lambda' = -\frac{1}{c}(p-p_e)\lambda(\lambda-1), \\ (v,\lambda)(\pm\infty) = (v_\pm, \lambda_\pm). \end{cases} \tag{6}$$

If we introduce the notation

$$\begin{aligned} s(v,\lambda) &:= c^2 + p_v, \\ g(v,\lambda) &:= Ac(v-\bar{v}) + \frac{1}{c}p_\lambda(p-p_e)\lambda(\lambda-1), \end{aligned}$$

then, in turn, system (6) can be written as

$$\begin{cases} sv' = g, \\ \lambda' = -\frac{1}{c}(p-p_e)\lambda(\lambda-1), \\ (v,\lambda)(\pm\infty) = (v_\pm, \lambda_\pm). \end{cases} \tag{7}$$

The equilibrium points of (7) are deduced by those of (3) and depend on the choice of the end states $(v_\pm, u_\pm, \lambda_\pm)$.

Both in the case $\alpha(\lambda) > 0$ and in the case $\alpha(\lambda) > 0$ for $\lambda \in [0,1)$ and $\alpha(1) = 0$, we prove the existence and uniqueness of traveling waves for a large class of end states. The analysis is rather subtle because of the interplay of three important curves in the phase plane (v,λ): the *equilibrium curve* $\mathcal{G}$ where g vanishes, the *sonic curve* $\mathcal{S}$, defined by $p_v + c^2 = 0$, and the *equilibrium pressure curve* $p = p_e$, denoted by $\mathcal{P}$. In particular, the dynamical system is singular on the sonic curve $\mathcal{S}$. Further, the case $\alpha(1) = 0$ gives rise to several interesting patterns of solutions, since in that case any state $(v, \lambda = 1)$ is an equilibrium for the dynamical system.

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Multi-dimensional scalar conservation laws with fluxes discontinuous in the unknown and the spatial variable

Piotr Gwiazda

(joint work with Miroslav Bulíček, Josef Málek, Agnieszka Świerczewska-Gwiazda)

The studies on scalar hyperbolic conservation laws, including the topic of discontinuous fluxes, are analytically interesting and often undertaken. The research in this direction is a preface to studies on phase transition problems (discontinuity of the flux in the unknown function) and spatially inhomogeneous materials (discontinuity of the flux in the space variable). The presented framework captures implicit relations between the flux function and the unknown quantity, and can be viewed as an introduction to considering interesting phenomena in elasticity theory, namely the $p-$systems with an implicit relation between the stress tensor and the deformation.

We consider the conservation (or rather balance) law as follows

$$\frac{\partial u}{\partial t} + \operatorname{div}_x \boldsymbol{F}(x,u) = f(t,x,u) \tag{1}$$

$$u(0,\cdot) = u_0. \tag{2}$$

The problem (1) was considered in case of the unbounded domain $\mathbb{R}^d$, where d denotes an arbitrary spatial dimension, $u : \mathbb{R}_+ \times \mathbb{R}^d \to \mathbb{R}$ is an unknown and $\boldsymbol{F} : \mathbb{R}^d \times \mathbb{R} \to \mathbb{R}^d$ is a given flux of the quantity u, see [3, 4]. We were interested in identifying under which assumptions for a flux function $\boldsymbol{F}$ one can show well-posedness of the problem (1). We recall that for smooth $\boldsymbol{F}$ a weak solution to (1) is called the Kružkov entropy solution if it satisfies for all $k \in \mathbb{R}$ the following entropy inequality in the distributional sense in $\mathbb{R}_+ \times \mathbb{R}^d$

$$\begin{aligned} |u-k|_{,t} + \operatorname{div}_x \left(\operatorname{sgn}(u-k)(\boldsymbol{F}(x,u) - \boldsymbol{F}(x,k))\right) \\ + \operatorname{sgn}(u-k)\operatorname{div}_x \boldsymbol{F}(x,k) \leq 0. \end{aligned} \tag{3}$$

Hence an important aim was that a class of these fluxes is equivalent to Kružkov entropy solution in case that $\boldsymbol{F}$ is a sufficiently smooth function.

We provide a brief description of methodology in working with conservation laws with discontinuous fluxes, for brevity assume for the moment that $\boldsymbol{F}$ depends on u only. In the first step we identify a given discontinuous $\boldsymbol{F}$ with a continuous curve that consists of the graph of $\boldsymbol{F}$ and the vertical parts of the graph, which fill the jumps. Consequently, instead of a discontinuous $\boldsymbol{F}$ of the variable u we are working with an implicit relation

$$G(\boldsymbol{F}, u) = 0\,. \tag{4}$$

Equation (4) represents a curve in $\mathbb{R}^{d+1}$ and we have one degree of freedom to set up the "optimal" unknown (independent variable).

This leads us to the observation that it might be better in such cases to reformulate (1) in terms of the unknown $\boldsymbol{F}$ instead of in terms of the unknown u. The simple one-dimensional procedure is not transferable to higher dimensions, however the idea to transform the original problem into the new problem that has better properties (continuous flux) is the key motivation. In fact, in the multidimensional case we construct an appropriate parametrization of a discontinuous curve in $\mathbb{R}^{d+1}$ so that the composition of $\boldsymbol{F}$ with this parametrization has a natural continuous extension.

We partially follow the idea of entropy measure valued solutions tools and the method of doubling the variables, but on the level of measure valued solutions, see [3, 4, 6, 8]. The starting point in this framework is the definition of entropy measure valued solutions and the so-called contraction principle, which is satisfied by entropy measure valued solutions. To provide the essence of this concept, without introducing additional notions and notation, we recall them here, following DiPerna, for the case of continuous fluxes, namely for the following problem in $\mathbb{R}^d \times \mathbb{R}_+$

$$\begin{aligned} \frac{\partial u}{\partial t} + \operatorname{div}_x \boldsymbol{F}(u) &= 0, \\ u(\cdot, 0) &= u_0, \end{aligned} \tag{5}$$

with sufficiently regular $\boldsymbol{F}$ and u_0.

Definition 1. *A Young measure $\nu : \mathbb{R}^d \times \mathbb{R}_+ \to \operatorname{Prob}(\mathbb{R})$ is called an entropy measure valued solution to* (5) *if*

$$\frac{\partial}{\partial t}\langle \nu_{(\cdot)}, |\lambda - k| \rangle + \operatorname{div}_x \langle \nu_{(\cdot)}, \operatorname{sgn}(\lambda - k)(\boldsymbol{F}(\lambda) - \boldsymbol{F}(k)) \rangle \leq 0 \tag{6}$$

in distributional sense for all $k \in \mathbb{R}$ and if for all compact sets $K \subset \mathbb{R}^d$

$$\lim_{T \to 0} \frac{1}{T} \int_0^T \int_K \langle \nu_{(x,t)}, |\lambda - u_0(x)| \rangle \, dx \, dt = 0. \tag{7}$$

In the above definition by $\operatorname{Prob}(\mathbb{R})$ we mean the space of probability measures.

Theorem 1. *Let ν and σ be two entropy measure valued solutions to* (5)*. Then*

$$\frac{\partial}{\partial t}\langle \nu \otimes \sigma, |\lambda - \mu| \rangle + \operatorname{div}_x \langle \nu \otimes \sigma, \operatorname{sgn}(\lambda - \mu)(\boldsymbol{F}(\lambda) - \boldsymbol{F}(\mu)) \rangle \leq 0 \tag{8}$$

in distributional sense, where $\nu \otimes \sigma$ denotes the product measure on $\mathbb{R} \times \mathbb{R}$.

To generalize this notion to a class of fluxes $\boldsymbol{F}(x,u)$ discontinuous with respect both to x and to u we follow several recent results and combine them in a proper way to develop a unified theory. An important approach to problems with $x-$discontinuous fluxes appears for $d = 1$ in [2] and later in [1]. The authors generalized the entropy inequality (3) in a way that instead of a constant k they considered a stationary solution, namely a function $k(x)$ solving the equation

$$\partial_x \boldsymbol{F}(x, k(x)) = 0. \tag{9}$$

This idea of extending the definition of Kružkov solutions to a discontinuous case consisted in introducing adapted entropies $E(x,u) = |u - k(x)|$. For such choice of entropies we observe that the last term in (3), which is not well-defined in case of non-smooth $\boldsymbol{F}$ vanishes. For a sufficiently large class of k's satisfying (9) the uniqueness of solutions to (1)–(3) can be proved.

$$\boldsymbol{F}(x,u) \qquad \text{is Carathéodory,} \tag{10}$$

$$f(u) \leq |\boldsymbol{F}(x,u)| \leq g(u), \tag{11}$$

$$\boldsymbol{F}(x,u) \qquad \text{is for a.a. } x \text{ one to one locally Lipschitz .} \tag{12}$$

Later Panov [7] generalized the method developed in [1] by assuming that $\boldsymbol{F}$ is of the form

$$\boldsymbol{F}(x,u) = \mathsf{G}(\theta(x,u)), \tag{13}$$

where $\mathsf{G} \in \mathcal{C}(\mathbb{R}; \mathbb{R}^d)$ and $\theta(x,u) : \mathbb{R}^d \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function that is for almost all x strictly increasing with respect to u and for which there exist continuous functions f and g fulfilling $f(u) \to \infty$ as $|u| \to \infty$ such that

$$f(u) \leq |\theta(x,u)| \leq g(u). \tag{14}$$

In this setting he showed that it is natural to formulate the problem differently. Let $\eta(x,v)$ be the inverse to θ, i.e., $\theta(x, \eta(x,v)) = v$. Then assuming that G, θ, η and u are smooth one easily shows that u solves (1)–(3) if and only if $u(t,x) := \eta(x, v(t,x))$ (that is again smooth) satisfies for all $k \in \mathbb{R}$

$$|\eta(x,v) - \eta(x,k)|_{,t} + \operatorname{div}(\operatorname{sgn}(v-k)(\mathsf{G}(v) - \mathsf{G}(k))) \leq 0, \qquad \text{in } \mathbb{R}^{d+1}_+. \tag{15}$$

in the sense of distribution. It is evident that for introducing a notion of a weak solution satisfying (15) we do not need to require any smoothness of θ with respect to x, which is the case if one assumes (3).

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Kinetic Relations for Moving Phase Boundaries

Wolfgang Dreyer

Introduction. We describe a liquid-vapor phase transition by the Euler equations for the variables ρ - mass density, $\mathbf{v}$ - velocity and T - temperature. We consider both the isothermal and the adiabatic case. The interface between adjacent phases is represented by a moving interface. Across the interface we may have discontinuities of some fields. In the pure phases discontinuities may also develop, but in this case the discontinuities are classical shocks. The evolution of the interface are determined by interfacial balance equations and by a so called kinetic relation, which relates the mass flux across the interface to a corresponding thermodynamically consistent driving force. The main objective of this study is the construction of a scheme that generates kinetic relations.

The interfacial equations of balance for mass, momentum and energy. We consider an interface I that either separates two adjacent phases $\Omega_\pm$ or it indicates a classical shock. The interfacial normal and its normal speed are denoted by ν and w_ν, respectively. The jump of a generic quantity χ across the interface is denoted by $[[\chi]]$. The classical Rankine-Hugoniot equations relies on the balance equations for mass and momentum

$$[[\rho(v_\nu - w_\nu)]] = 0, \quad -\rho(v_\nu - w_\nu)[[\mathbf{v}]] + [[p]]\nu = 2\gamma k_{\mathrm{M}}\nu. \tag{1}$$

and on the energy balance

$$\rho(v_\nu - w_\nu)[[u + \frac{p}{\rho} + \frac{1}{2}(\mathbf{v} - \mathbf{w})^2]] + [[\mathbf{q}]] \cdot \nu = 0. \tag{2}$$

The newly introduced quantities are p - pressure, γ - surface tension, k_{M} - mean curvature, u - specific internal energy and $\mathbf{q}$ - heat flux.

Multiplication of the momentum balance by an interfacial tangent vector τ yields that $[[\mathbf{v}_\tau]] = 0$, thus the tangential velocity is continuous and we can set $\mathbf{v}_\tau = \mathbf{w}_\tau$ implying $(\mathbf{v} - \mathbf{w})^2 = (v_\nu - w_\nu)^2$.

Constitutive model for pressure, internal energy, surface tension, entropy and Gibbs energy. Pressure and the internal energy are given by constitutive functions with respect to the variables T and ρ. In particular, the modelling of phase transitions requires a non-monotone pressure - mass density dependence. This property is encoded in the specific (Helmholtz) free energy, which is represented by the function $\psi = \hat{\psi}(T, \rho)$. There two further constitutive quantities that will appear in the kinetic relation, namely s - specific entropy and g - specific Gibbs free energy. All these functions can be derived from ψ, because, according to the 2nd law of thermodynamics, we have

$$p = \rho^2 \frac{\partial \psi}{\partial \rho}, \quad u = -T^2 \frac{\partial}{\partial T}\left(\frac{\psi}{T}\right), \quad s = -\frac{\partial \psi}{\partial T}, \quad g = \frac{\partial \rho \psi}{\partial \rho}. \tag{3}$$

The surface tension is also given by a constitutive function, and it can likewise be derived from a free energy function. For illustration we assume now that the interface I is equipped with own thermodynamic properties, that are described by ρ_{I} - interfacial mass density, T_{I} - interfacial temperature and ψ_{I} - interfacial (Helmholtz) free energy density. Obviously the equations of balance (1) and (2) must be changed if we have $\psi_{\mathrm{I}} = \psi_{\mathrm{I}}(T_{\mathrm{I}}, \rho_{\mathrm{I}})$, but this will be discussed later. Here it is only important that thermodynamics implies

$$\gamma = \psi_{\mathrm{I}} - \rho_{\mathrm{I}} g_{\mathrm{I}} \quad \text{with} \quad g_{\mathrm{I}} = \frac{\partial \psi_{\mathrm{I}}}{\partial \rho_{\mathrm{I}}}, \tag{4}$$

where g_{I} denotes the interfacial Gibbs free energy density.

On the role of the heat flux. In the isothermal case the variables are ρ, $\mathbf{v}$ and w_ν. Then we only need the balance equations for mass and momentum for their determination. The jump of the heat flux is needed so that the energy balance is capable to allow $[[T]] = 0$. In the adiabatic case we have $\mathbf{q} = 0$ and the jump $[[T]] \neq 0$ is determined by the energy balance.

On the need of kinetic relations. Let us consider a Riemmann problem with two initial states in the liquid and the vapor phase, respectively, as it is indicated in Figure 1, see [1] for details. There are two possibilities to connect the two states at later times. In the left Figure the initial states are connected by a discontinuity and a rarefaction wave. However, it is also possible to connect the initial states exclusively by a single discontinuity as in the right Figure. The obvious question is now: Which solution is selected by *nature*?

Interfacial entropy production. The answer to this question relies on the interfacial entropy production $\zeta_{\mathrm{I}} \geq 0$. From the 2nd law of thermodynamics we

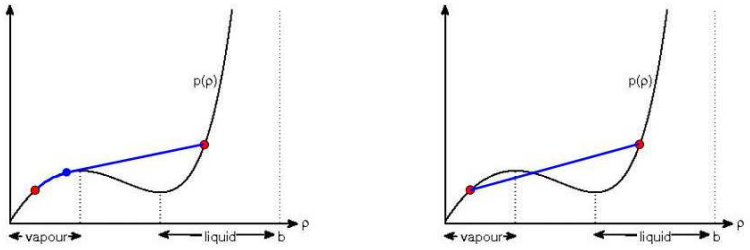

FIGURE 1. The non-uniqueness of the Riemann problem.

can deduce

$$\begin{aligned}\zeta_{\mathrm{I}} &= -[[(\frac{1}{T_{\mathrm{I}}}-\frac{1}{T})q_\nu]] \\ &\quad + [[(\frac{g_{\mathrm{I}}}{T_{\mathrm{I}}}-\frac{g}{T}-\frac{1}{2TI}(v_\nu-w_\nu)^2-(\frac{1}{T_{\mathrm{I}}}-\frac{1}{T})(u+\frac{p}{\rho}))\rho(v_\nu-w_\nu)]] \geq 0.\end{aligned} \tag{5}$$

We proceed with special cases. In the isothermal case with $\psi_{\mathrm{I}} =$ constant, we obtain

$$T\zeta_{\mathrm{I}} = -\rho(v_\nu-w_\nu)[[g+\frac{1}{2}(v_\nu-w_\nu)^2]] \geq 0. \tag{6}$$

In the adiabatic case with $\psi_{\mathrm{I}} =$ constant, the inequality reduces to

$$\zeta_{\mathrm{I}} = \rho(v_\nu-w_\nu)[[s]] \geq 0. \tag{7}$$

Kinetic relation in the isothermal case. We decompose the mass flux into the two mechanisms of the phase transitions, namely evaporation with rate γ_{E} and condensation with rate γ_{C}. We write $\rho(v_\nu - w_\nu) = m(\gamma_{\mathrm{E}} - \gamma_{\mathrm{C}})$, where m is the mass of an atom. A simple possibility to guarantee the sign of ζ_{I} is the choice

$$\ln(\frac{\gamma_{\mathrm{C}}}{\gamma_{\mathrm{E}}}) = \frac{m}{kT}[[g+\frac{1}{2}(v_\nu-w_\nu)^2]] \quad \text{implying} \quad \ln(\frac{\gamma_{\mathrm{C}}}{\gamma_{\mathrm{E}}})(\gamma_{\mathrm{E}}-\gamma_{\mathrm{C}}) \leq 0. \tag{8}$$

Thus in the isothermal case the kinetic relation reads

$$\rho(v_\nu-w_\nu) = m\gamma_{\mathrm{C}}(\exp(-\frac{m}{kT}[[g+\frac{1}{2}(v_\nu-w_\nu)^2]])-1). \tag{9}$$

We observe that thermodynamics restricts the ratio of condensation and evaporation rates. Only one of the two rates remains as an adjustable parameter. For example, we may assume $\gamma_{\mathrm{C}} =$ constant. Another choice results from the kinetic gas theory:

$$\gamma_{\mathrm{C}} = \sqrt{\frac{kT}{2\pi m}}\frac{p_{\mathrm{V}}}{kT}, \tag{10}$$

where p_{V} is the pressure on the vapor side of the interface and k denotes the Boltzmann constant.

Kinetic relation in the adiabatic case. Here we use the same decomposition of the mass flux and obtain in an analogous manner a kinetic relation for the adiabatic case,

$$\rho(v_\nu - w_\nu) = m\gamma_{\mathrm{C}}(\exp(\frac{m}{kT}[[s]]) - 1). \tag{11}$$

Scheme for construction of kinetic relations. We conclude that two steps are necessary to obtain a thermodynamic consistent kinetic relation.

(i): Calculate the interfacial entropy production as a bilinear form.

(ii): Relate the two factors to each other so that the sign of ζ_{I} is guaranteed

Discussion of more general cases. The presented scheme is the same in more involved cases. However, the exploitation of the interfacial entropy production might be very much more intricate. For illustration we go back to the entropy inequality (5). Note that the mass flux is only outside the jump brackets if it is continuous across the interface. If there is an interfacial mass density, the mass flux is not continuous. Consequently, the binary product in the interfacial entropy production is not $\rho(v_\nu - w_\nu)[[A]]$, instead $[[\rho(v_\nu - w_\nu)A]] = \rho^+(v_\nu^+ - w_\nu)A^+ - \rho^-(v_\nu^- - w_\nu)A^-$. Thus we have different binary products on each side of the interface whose further treatment must separately be carried out.

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Locally constrained conservation laws in traffic modeling

Paola Goatin

(joint work with Boris Andreianov, Christophe Chalons, Rinaldo M. Colombo, Maria L. Delle Monache, Mauro Garavello, Massimiliano D. Rosini, Nicolas Seguin)

Several phenomena displayed by vehicular traffic can be modeled using conservation laws in one space-dimension, see for example [9] for a survey of available models. Here we are interested in the following Cauchy problem with flux constraint

$$\partial_t \rho + \partial_x f(\rho) = 0, \qquad t > 0,\ x \in \mathbb{R}, \tag{1}$$

$$\rho(0, x) = \rho_0(x), \qquad x \in \mathbb{R}, \tag{2}$$

$$f(\rho(t, 0)) \leq F(t), \qquad t > 0. \tag{3}$$

This problem was originally proposed in [3, 4, 5] to model the presence of toll gates, but can be applied to other situations arising in traffic flow, such as traffic lights and construction sites [1, 5] or pedestrians exiting a corridor through a narrow

exit [2, 4]. The model can also be extended to systems [8] and moving bottlenecks caused by slow moving large vehicles [7].

For simplicity, here we assume that the flux function $f : [0,1] \to \mathbb{R}$ is Lipschitz continuous and

$$f(\rho) \geq 0,\ f(0) = f(1) = 0,\ f'(\rho)(\bar{\rho} - \rho) > 0 \quad \text{a. e. } \rho \in [0,1] \setminus \{\bar{\rho}\},$$

for some $\bar{\rho} \in (0,1)$. Moreover, we take $F \in L^\infty(\mathbb{R}^+; [0, f(\bar{\rho})])$ and $\rho_0 \in L^\infty(\mathbb{R}; [0,1])$.

We now explain how the Riemann solver for (1)-(3) is constructed. We denote by $\mathcal{R}$ the standard Riemann solver for (1), i.e. $(t,x) \mapsto \mathcal{R}(\rho^l, \rho^r)(x/t)$ is the standard weak entropy solution of (1)-(2), corresponding to the Riemann initial datum

$$\rho_0(x) = \begin{cases} \rho^l \text{ si } x < 0, \\ \rho^r \text{ si } x > 0. \end{cases}$$

Taking $F(t) \equiv F$ constant, we denote by $\check{\rho}_F \leq \hat{\rho}_F$ the solutions of the equation $f(\rho) = F$.

Definition (Riemann Solver). *The Riemann solver $\mathcal{R}^F : (\rho^l, \rho^r) \mapsto \mathcal{R}^F(\rho^l, \rho^r)$ for (1)-(3) is defined as follows.*
If $f\left(\mathcal{R}(\rho^l, \rho^r))(0)\right) \leq F$, then $\mathcal{R}^F(\rho^l, \rho^r) = \mathcal{R}(\rho^l, \rho^r)$.
Otherwise, $\mathcal{R}^F(\rho^l, \rho^r)(\lambda) = \begin{cases} \mathcal{R}(\rho^l, \hat{\rho}_F)(\lambda) \text{ if } \lambda < 0\,, \\ \mathcal{R}(\check{\rho}_F, \rho^r)(\lambda) \text{ if } \lambda > 0\,. \end{cases}$

Remark that, when the constraint is enforced, a discontinuity violating the Lax entropy condition arises at $x = 0$. The non-classical problem (1)-(3) can be interpreted as the singular limit of a classical Cauchy problem with discontinuous flux. Fix $\varepsilon > 0$ and consider the problem

$$(4) \qquad \begin{cases} \partial_t \rho^\varepsilon + \partial_x \left(k_\varepsilon(t,x)\, f(\rho^\varepsilon)\right) = 0, \\ \rho^\varepsilon(0,x) = \rho_0(x), \end{cases} \qquad k_\varepsilon(t,x) = \begin{cases} 1 & |x| > \varepsilon, \\ \dfrac{F(t)}{f(\bar{\rho})} & |x| \leq \varepsilon. \end{cases}$$

(We refer the interested reader to [3] for a rigorous justification of the limiting procedure.) Problem (4) fit the framework of [10, Theorem 4.5, 5.5 and 6.5], and the associated entropy condition is given by

$$\begin{aligned} \int_0^{+\infty} \int_{\mathbb{R}} \left(|\rho^\varepsilon(t,x) - \kappa|\ \partial_t + \Phi(\rho^\varepsilon(t,x), \kappa)\partial_x\right)\ \phi(t,x)\ dx\ dt \\ + \int_{\mathbb{R}} |\rho_0(x) - \kappa|\ \phi(0,x)\ dx \\ + \int_0^{+\infty} \left(1 - \frac{F(t)}{f(\bar{\rho})}\right) f(\kappa)\ \left(\phi(t,-\varepsilon) + \phi(t,\varepsilon)\right)\ dt \geq 0, \end{aligned}$$

for all $\kappa \in [0,1]$ and $\phi \in C_c^1(\mathbb{R}^+ \times \mathbb{R}; \mathbb{R}^+)$, where we have noted $\Phi(a,b) = \operatorname{sgn}(a-b)(f(a) - f(b))$. Remark that ρ^ε satisfy the constraint $f(\rho^\varepsilon(t,x)) \leq F(t)$ for a. e. $|x| < \varepsilon$.
Therefore, we give the following definition for (1)-(3).

Definition (Entropy Weak Solution). *A function $\rho \in L^\infty(\mathbb{R}^+ \times \mathbb{R}; [0,1])$ is an entropy weak solution of* (1)-(3) *if*
(i) entropy inequalities are satisfied: for all $\phi \in C^1_c(\mathbb{R}^+ \times \mathbb{R}; \mathbb{R}^+)$ and every $\kappa \in [0,1]$,

$$\int_0^{+\infty} \int_{\mathbb{R}} (|\rho(t,x) - \kappa| \partial_t + \Phi(\rho(t,x),\kappa)\partial_x)\ \phi(t,x)\ dx\ dt + \int_{\mathbb{R}} |\rho_0(x) - \kappa|\ \phi(0,x)\ dx + 2\int_0^{+\infty} \left(1 - \frac{F(t)}{f(\bar\rho)}\right) f(\kappa)\ \phi(t,0)\ dt \geq 0; \tag{5}$$

(ii) the constraint is satisfied:

$$f((\gamma^l \rho)(t)) = f((\gamma^r \rho)(t)) \leq F(t) \quad \text{a. e. } t > 0,$$

where $\gamma^{l,r}$ are the left and right strong trace operators at $\{x = 0\}$.

Remark. This definition selects the *maximal* solution, since all non-classical stationary discontinuities at $x = 0$ between two states ρ_2 and ρ_1 with $\rho_1 < \rho_2$ and $f(\rho_1) = f(\rho_2) < F(t)$ are ruled out by (5).

Existence and stability results for entropy weak solutions of (1)-(3) are detailed in [3], and in [2] for general fluxes with several points of local maximum/minimum. The construction of corresponding finite volume schemes, and the proof of their convergence, are given in [1].

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Energy consistent DG schemes for compressible two–phase flows

Jan Giesselmann
(joint work with Charalambos Makridakis, Tristan Pryer)

We describe a new class of discontinuous Galerkin (dG) finite element methods for the Navier-Stokes-Korteweg system which are *by design consistent with the energy dissipation structure of the problem.* The methods are of arbitrary high order of accuracy in space and second order in time. They provide stable approximations free of numerical artifacts.

Let $\Omega \subset \mathbb{R}^2$ be a bounded, open, connected set with Lipschitz boundary. Then the Navier-Stokes Korteweg model describing the behaviour of the density ρ and the velocity $\mathbf{u}$ in space and time reads

$$\begin{aligned} \rho_t + \operatorname{div}(\rho \mathbf{u}) &= 0 \\ (\rho \mathbf{u})_t + \operatorname{div}\left(\rho \mathbf{u} \otimes \mathbf{u} + p(\rho)\mathrm{Id}\right) &= \operatorname{div}(\mathbf{S}) + \gamma \rho \nabla \Delta \rho \end{aligned} \quad \text{in } \Omega \times \mathbb{R}_{>0}, \tag{1}$$

where

$$\mathbf{S} = \lambda(\operatorname{div} \mathbf{u})\mathrm{Id} + \mu(\nabla \mathbf{u} + (\nabla \mathbf{u})^T),$$

is the Navier-Stokes stress tensor modeling viscous effects, and λ and μ are viscosity coefficients satisfying $\mu > 0$ and $\lambda + 2\mu \geq 0$. Moreover, $\gamma > 0$ is a small number determining the strength of capillary effects and surface tension. This system is complemented by appropriate initial data $\rho_0, \mathbf{u}_0$ and the following boundary conditions

$$\mathbf{u} = 0, \ \nabla \rho \cdot \boldsymbol{\nu} = 0 \quad \text{in } \partial\Omega \times \mathbb{R}_{>0}. \tag{2}$$

There are two special features of this model which are intimately linked to the two phase character of the problem and surface tension effects: the third order terms in the momentum balance and the non-monotone pressure law, see Figure 1. In particular, to have two phases, we need a non-convex Helmholtz energy density function and it holds

$$p(\rho) := \rho W'(\rho) - W(\rho) \implies p'(\rho) = \rho W''(\rho).$$

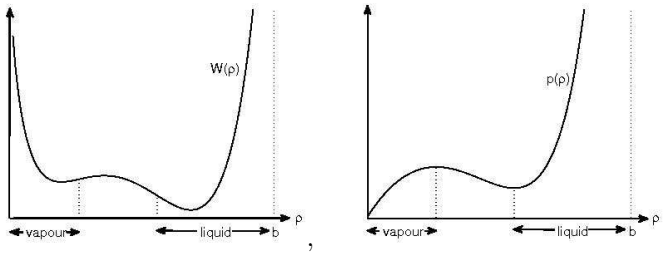

Figure 1. Helmholtz energy density and pressure.

It is classical to verify that smooth solutions of (1) satisfying (2) fulfill the *energy dissipation equality*

$$\frac{d}{dt}\int_\Omega W(\rho) + \frac{\rho}{2}|\mathbf{u}|^2 + \frac{\gamma}{2}|\nabla\rho|^2 \,\mathrm{dx} = -\int_\Omega \mathbf{S} : \nabla\mathbf{u} \,\mathrm{dx} \leq 0. \tag{3}$$

The inequality in (3) reflects the second law of thermodynamics while the equality shows that all energy dissipation in this model is due to viscous effects. The Ginzburg-Landau part of the energy leads to diffuse 'smeared out' interfaces of non-zero thickness as well as to surface tension effects as the perimeter of the interface is penalized energetically.

The Korteweg type third order term and the non-monotonicity of the pressure function cause several issues in the numerical treatment of this problem. In previous numerical studies [4, 2, 1] it has been observed that "classical" explicit-in-time finite volume (fv) and dG schemes using standard fluxes used in computational (compressible) fluid dynamics introduce several numerical artifacts.

The first artifact is non-monotonicity of the energy, i.e., an analogue of (3) is not true on the discrete level for classical fv and dG methods. This is mainly due to the fact that these "classical" schemes introduce standard diffusion in the mass conservation equation as a stabilising mechanism. While for convex energies standard diffusion, in fact, leads to energy dissipation, it may cause an increase in energy for multiphase flows [2]. Standard diffusion is also present in the (continuous) finite element method proposed in [1].

The second artifact are so called *parasitic currents*, i.e., the schemes are not well-balanced, as they do not preserve the correct (two phase) equilibria. Parasitic currents occur when equilibrium is approached and the numerical velocity field does not vanish uniformly, but in the interfacial layer large velocities whose magnitude is dependent on the gridsize and inversely dependent on the width of interfacial layer appear [2, §5]. As the interfacial layer is extremely thin this effect cannot be neglected in practical computations, although it is compatible with convergence of the scheme.

Both the non-monotone behaviour of the energy and the parasitic currents are due to numerical regularisation terms which are not adapted to the variational structure of the problem

Our approach is based on the idea that in case of an implicit-in-time discretisation no artificial regularisation in the space discretisation is needed. The main step in the construction of our scheme is a careful choice of the primary and auxiliary variables in the following first order system which is equivalent to (1) for smooth solutions. The mixed formulation is then to seek $(\rho, \mathbf{u}, \tau, \mathbf{q})$ such that

$$\begin{aligned}
\rho_t + \operatorname{div}(\rho\mathbf{u}) &= 0\\
\rho(\mathbf{u})_t + \operatorname{div}(\rho\mathbf{u}\otimes\mathbf{u}) - \operatorname{div}(\rho\mathbf{u})\mathbf{u} + \rho\nabla\tau - \frac{1}{2}\rho\nabla|\mathbf{u}|^2 - \mu\Delta\mathbf{u} &= 0\\
\tau - W'(\rho) + \gamma\operatorname{div}(\mathbf{q}) - \frac{1}{2}|\mathbf{u}|^2 &= 0\\
\mathbf{q} - \nabla\rho &= 0
\end{aligned} \tag{4}$$

where we have replaced $\operatorname{div}\mathbf{S}$ by $\mu\Delta\mathbf{u}$ for ease of exposition. The main feature of this mixed formulation is that all test functions needed in deriving (3) are linearly dependent on the primiary variables. Therefore, when we devise a dG scheme based on (4), they are elements of the appropriate ansatz and test function spaces.

We develop a dG scheme employing central fluxes and a Crank-Nicholson-like time discretisation based on the mixed formulation (4) . The details of this scheme and the proof of equation (5) can be found in [3]. Provided the numerical solutions are constructed in ansatz spaces enforcing the correct boundary conditions this scheme is mass conserving while its main property is that the numerical approximations $(\rho_h^n, \mathbf{u}_h^n, \tau_h^n, \mathbf{q}_h^n)$ and $(\rho_h^{n+1}, \mathbf{u}_h^{n+1}, \tau_h^{n+1}, \mathbf{q}_h^{n+1})$ at times t_n and t_{n+1} satisfy

$$\begin{aligned}(5)\quad &\int_\Omega W(\rho_h^{n+1}) + \frac{\rho_h^{n+1}}{2}|\mathbf{u}_h^{n+1}|^2 + \frac{\gamma}{2}|\nabla\rho_h^{n+1}|^2 \,\mathrm{dx} \\ &\qquad - \int_\Omega W(\rho_h^n) + \frac{\rho_h^n}{2}|\mathbf{u}_h^n|^2 + \frac{\gamma}{2}|\nabla\rho_h^n|^2 \,\mathrm{dx} = -B_h(\mathbf{u}_h^{n+\frac{1}{2}}, \mathbf{u}_h^{n+\frac{1}{2}}) \le 0\end{aligned}$$

where $B_h(\mathbf{u}_h^{n+\frac{1}{2}}, \mathbf{u}_h^{n+\frac{1}{2}})$ is a coercive discretisation of

$$\int_\Omega \Big|\nabla\Big(\frac{\mathbf{u}_h^n + \mathbf{u}_h^{n+1}}{2}\Big)\Big|^2 \,\mathrm{dx}\,.$$

Equation (5) shows nonlinear stability of the scheme as well as the fact that entropy dissipation is isolated to viscous effects.

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Nonconservative coupling and selection criteria

Benjamin Boutin

(joint work with Frédéric Coquel, Philippe G. LeFloch)

We consider the coupling of different nonlinear hyperbolic systems across a fixed interface, say at $x = 0$ considering for convenience only the one-space variable case, see [2], with unknown $u(x,t) \in \mathbb{R}^d$: $\partial_t u + \partial_x f^{\pm}(u) = 0$, where $t > 0$ and $\pm x > 0$ respectively. An augmented nonconservative PDE system is introduced by the authors in [1] that allows us, via Dafermos' self-similar viscosity method

to obtain the existence of self-similar solutions to the coupled Riemann problem, see also [3] for a similar approach. We then analyze and investigate the internal structure of the interfacial layer and recover selection criteria associated with the regularization mechanism. We then conclude by given evidence that solutions can be non-unique, even in some apparently simple situations.

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Stationary Evaporation Waves in a Spherical Symmetric Nozzle

Haitao Fan

(joint work with XiaoBiao Lin)

We study the possibility of making liquids evaporate as it flowing through a nozzle. We start with a model for flows involving liquid/vapor phase transitions:

$$
\begin{aligned}
&\rho_t + \nabla\cdot(\rho\mathbf{u}) = 0,\\
&(\rho\mathbf{u})_t + \nabla\cdot(\rho(\mathbf{uu}) + \mathbf{P}) = 0,\\
&(\lambda\rho)_t + \nabla\cdot(\lambda\rho\mathbf{u}) = \frac{w}{\gamma} + \nabla\cdot(\mu\rho\nabla\lambda),\\
&E_t + \nabla\cdot(\mathbf{u}E + \mathbf{u}\cdot\mathbf{P}) = \kappa\Delta T + L(T)\nabla\cdot(\mu\rho\nabla\lambda),\\
&\mathbf{P} = (p + (\frac{2}{3}\epsilon_1 - \epsilon_2)(\nabla\cdot\mathbf{u}))I - \epsilon_1(\nabla\mathbf{u} + \nabla\mathbf{u}^T),\\
&p_\rho > 0, \quad p_\lambda > 0.
\end{aligned}
\tag{1}
$$

The major symbols in the equations are as follows The major symbols in the equations are as follows

ρ	$\mathbf{u}$	λ	E	P
density	velocity	mass fraction of vapor	energy density	stress-strain tensor

The vapor production rate is

$$\frac{w}{\gamma} = \frac{1}{\gamma}(p - p_e)\lambda(\lambda - 1)\rho,$$

where γ is the typical reaction time and p_e the equilibrium pressure. The constants μ is the diffusion coefficient, κ the heat conduction coefficient, ϵ's the viscosity, and $L(T)$ is the latent heat. we shall assume that

$$\gamma = \epsilon/a, \quad \mu = \epsilon b.$$

To mimic the fuel being injected into a cylinder of a internal combustion engine through a cone shaped nozzle, we consider the spherical symmetric version of the isothermal case

$$
\begin{aligned}
&\rho_t + (\rho u)_r + \frac{2\rho u}{r} = 0,\\
&(\rho u)_t + (\rho u^2 + p)_r + \frac{2\rho u^2}{r} = \epsilon(u_r + \frac{2u}{r})_r, \qquad (2)\\
&(\lambda\rho)_t + (\lambda\rho u)_r + \frac{2\lambda\rho u}{r} = \frac{1}{\gamma}(p - p_e)\lambda(\lambda - 1)\rho + \mu((\rho\lambda_r)_r + \frac{2\rho\lambda_r}{r}).
\end{aligned}
$$

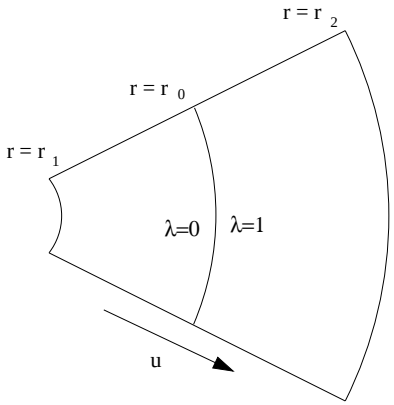

FIGURE 1

To achieve a steady evaporation inside the nozzle, we look for stationary solutions of (2) of the shape shown in Figure 1. Such solutions satisfy

$$
\begin{aligned}
&(\rho u)_r + \frac{2\rho u}{r} = 0,\\
&(\rho u^2 + p)_r + \frac{2\rho u^2}{r} = \epsilon(u_r + \frac{2u}{r})_r, \qquad (3)\\
&(\lambda\rho u)_r + \frac{2\lambda\rho u}{r} = \frac{a}{\epsilon}(p - p_e)\lambda(\lambda - 1)\rho + \epsilon b((\rho\lambda_r)_r + \frac{2\rho\lambda_r}{r}).
\end{aligned}
$$

In the $\epsilon \to 0+$ limit, the system has two types of such solutions, where the internal layers at $r = r_0$ are supersonic and subsonic respectively, as shown in Figure 2.

The existence of internal layers at $r = r_0$ is given by the following two theorems. Let $E_\pm$ denote the states at $r = r_0\pm$.

Theorem 1. *Given $E_\pm$, there is a subsonic evaporation internal layer solution if (a) $u^2 < p_\rho$ at $E_\pm$ and (b) $p_- < p_e$ and (c) $u^2 > 4ab(p_e - p)$ on the curve $m_0(u - u_-) + p(\lambda, m_0/u) - p_- = 0$ where the momentum $m = Cr^{-2}$.*

Furthermore, the solution is monontone: $\lambda' > 0, u' > 0$.

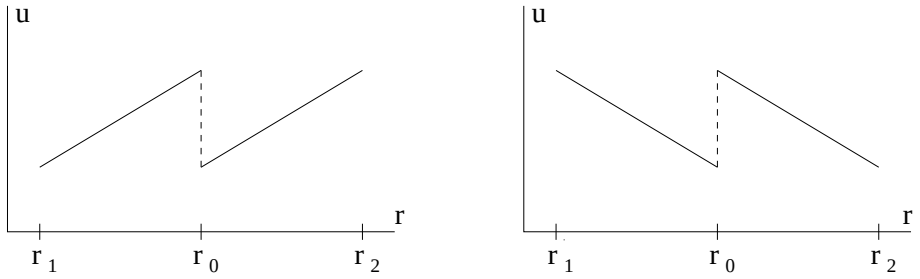

FIGURE 2. Left: Supersonic solution. Right: Subsonic solution

Theorem 2. *Given $E_\pm$, there is an unique supersonic evaporation internal layer solution if $u^2 > p_\rho$ and $p < p_e$ at E_+ and if $u^2 > 4ab[p_e - p(\lambda = 0, m_0/u)]$ for $u_+ < u < u_-$.*

Furthermore, the solution is monotone: $\lambda' > 0, u' < 0$.

The solution of the depicted in the smooth region of Figure 2 exists if $r_0 - r_1$ is small enough or $|u^2 - p_\rho|$ at r_0 is large enough.

The existence of solutions to the limiting system is then extended to the existence of similarly shaped solutions for (3). This is done using geometric theory of singular perturbations.

Theorem 3. *Assume $|ab|$ is sufficiently small, and that $r_0 - r_1$ is small enough or $|u^2 - p_\rho|$ at r_0 is large enough. Then the system* (3) *has solutions of the shape in Figure 1.*

The details of these results are given in [1].

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Entropy-Stable Path-Conservative Numerical Schemes

CARLOS PARÉS

(joint work with M.J.Castro, U.S. Fjordholm, and S. Mishra)

In [4] Tadmor introduced a sufficient condition for the numerical flux of a conservative method to be entropy-preserving. The goal of this work is to generalize this theory to strictly hyperbolic nonconservative systems of the form

$$u_t + A(u)u_x = 0, \quad x \in \mathbb{R},\ t > 0, \tag{1}$$

equipped with an entropy pair, i.e. a pair of functions (η, q), η being convex, such that

$$\nabla q(u) = \nabla \eta(u) \cdot A(u), \quad \forall\ u \in \mathbb{R}^n.$$

More precisely, the goal is to design semi-discrete path-conservative numerical schemes

$$\frac{d}{dt}u_i + \frac{1}{\Delta x}(D^+_{i-1/2} + D^-_{i+1/2}) = 0$$

(see [3]) that are entropy-preserving in the following sense: there exists a consistent numerical entropy flux $Q_{i+1/2}$ such that the numerical solutions also satisfy the equation:

$$\frac{d}{dt}\eta(u_i) + \frac{1}{\Delta x}(Q_{i+1/2} - Q_{i-1/2}) = 0. \tag{2}$$

An entropy-preserving scheme is not expected to be stable in presence of shocks and thus some numerical viscosity has to be added. What we propose here is to stabilize entropy-preserving path-conservative numerical schemes by using the physical viscosity of the problem. The resulting methods are expected to overcome, at least partially, the difficulty of convergence of the numerical solutions to the physical one discussed in [2]: see [1].

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A rigorous result on the discrete–continuum limit in traffic flow

Elena Rossi

Our aim is to investigate the connection between a macroscopic, or continuum, model and a microscopic, or discrete, model for traffic flow. In particular, we consider the Lighthill–Whitham and Richards model and a first order Follow the Leader model respectively: the first consists in a conservation law closed by a speed–density relation, the second in a system of ordinary differential equations.

The connection between these two descriptions is formalized and, through *ad hoc* operators, it is established a relation between the corresponding variables. Then, the discrete model is proved to converge to the continuum one in a sort of *kinetic limit*, i.e. as the number of vehicles increases to infinity while the total mass is kept fixed, see [2]. From the modelling point of view, this result justifies the Lighthill–Whitham and Richards model as the limit of a first order follow the leader model as the number of vehicles tends to infinity.

Numerical algorithms were developed to integrate both the continuum and the discrete models. The limiting procedure suggests the use of the ordinary differential system as a tool for the numerical integration of the corresponding partial differential equation. However, such a numerical algorithm to compute the continuum solutions hardly competes with an *ad hoc* method.

From the numerical point of view, the limiting result is extended to the case of two populations, referring to the macroscopic model in [1] and to the natural multi–population analogue of the microscopic one.

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Bifurcation and stability of liquid-vapor phase boundaries

Heinrich Freistühler

(joint work with M. Kotschote, S. Benzoni-Gavage)

After recalling results and techniques for viscous and inviscid classical shock-waves (Evans formulation, Majda determinant) the talk reported two results:

(a) The bifurcation of small-a,plitude traviling wave phase boundaries in appropriate temperature-parameterized families of Navier-Stokes Allen-Cahn systems

$$\begin{aligned}
\partial_t \rho + \nabla \cdot (\rho u) &= 0 \\
\partial_t(\rho u) + \nabla \cdot (\rho u \otimes u + p(\rho, c) I) &= \nabla \cdot (\mu(\nabla u)^{sym} + (\lambda \nabla \cdot u) I - \delta \rho \nabla c \otimes \nabla c) \\
\partial_t(\rho c) + \nabla \cdot (\rho c u) &= \delta^{-\frac{1}{2}} (\rho q(\rho, c) + \nabla \cdot (\delta \rho \nabla c)) \\
\bar{U}(\tau, c, |\nabla c|) &= U(\tau, c) + \frac{1}{2} \delta |\nabla c|^2 \\
U(\tau, c) &= \hat{U}(\tau, c) + W(c, \theta).
\end{aligned}$$

with

$$\hat{U}(\tau, c) = (1 - c) f\left(\frac{\tau - c\tau_1}{1 - c}\right),$$

$f' < 0 < f''$ and $W., \theta)$ undergoing a transition from convex (for $\theta > \theta_*$) to convex-concave-convex (for $\theta < \theta_*$) as an example. The justification for this form of $\hat{U}$ is

$$\hat{U} = cU_1(\tau_1) + cU_2(\tau_2), \quad \tau = c\tau_1 + (1 - c)\tau_2$$

with $\tau_1 = const$, $U_1(\tau_1) = 0$, $\tau_2 = \frac{1-c\tau_1}{1-c}$, i.e. one phase ("1") is strictly incompressible.

(b) The uniform Lopatinski stability of a phase front satisfying Euler in the bulk,

$$\partial_t \rho + \nabla \cdot (\rho u) = 0$$
$$\partial_t (\rho u) + \nabla \cdot (\rho u \otimes u + p(\rho) I) = 0$$

with a convex-concave-convex $\tilde{p}\left(\frac{1}{\rho}\right)$ and the jump conditions

$$[\rho V \cdot N] = 0$$
$$[\rho V (V \cdot N - s) + p(\rho) N] = \sigma H N$$
$$[g + \frac{1}{2}(V \cdot N - s)^2] = 0$$

accross the front with mean curvature H.

Mixed Systems in the Description of Traffic Flow

Francesca Marcellini
(joint work with Rinaldo M. Colombo and Michel Rascle)

We present mixed systems in the description of dynamics of traffic flow. In particular, we consider two different frameworks, both consisting in the coupling of systems of different types and both displaying 2 phases.

The first one is the Free–Congested model, see [9], where a scalar conservation law is coupled with a 2×2 system. The result is a macroscopic model based on a non-smooth 2×2 system of conservation laws and displaying 2 distinct phases: *Free* and *Congested*. Then, we present the coupling of a micro- and a macroscopic models, the former consisting in a system of ordinary differential equations and the latter in a scalar conservation law, see [8].

We recall at first the classical Lighthill-Whitham [14] and Richards [16] (LWR) traffic model

$$\partial_t \rho + \partial_x (\rho V) = 0 \tag{1}$$

which is a scalar conservation law, where $\rho = \rho(t, x)$ is the (mean) traffic density and $V = V(\rho)$ is the (mean) traffic speed.

Concerning the Free–Congested model, we consider the LWR model and than we extend this model with two different assumptions on the speed V. At first, we assume that, at a given density, different drivers may differ in their *maximal* speed w, so that $V = w\,\psi(\rho)$, with $w \in [\check{w}, \hat{w}]\,, \check{w} > 0$. The function ψ describes the attitude of drivers to choose their speed depending on the traffic density at their location and the maximal speed w is a specific feature of every single driver. Thus we are lead to study the system:

$$\begin{cases} \partial_t \rho + \partial_x (\rho v) = 0 \\ \partial_t w + v\,\partial_x w = 0 \end{cases} \qquad \text{with} \qquad v = w\,\psi(\rho)\,. \tag{2}$$

The role of the second equation above is to let the maximal velocity w be propagated with the traffic speed. We identify the different behaviors of the different drivers by means of their maximal speed, see also [4, 5].
The second assumption on the speed is the introduction of a uniform bound, a constant $V_{\max}$ that the drivers do not exceed. We obtain the following model:

$$\text{(3)} \qquad \begin{cases} \partial_t \rho + \partial_x(\rho v) = 0 \\ \partial_t w + v\,\partial_x w = 0 \end{cases} \qquad \text{with} \qquad v = \min\left\{V_{\max},\, w\,\psi(\rho)\right\}.$$

As a consequence of the introduction of this speed bound we have the formation of the two distinct phases, *Free* and *Congested*. The phases are presented in Figure 1; see [9, Section 2] for the notations.

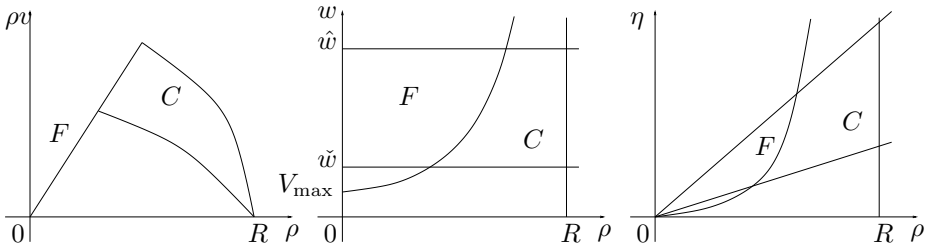

FIGURE 1. The phases F and C in the coordinates $(\rho, \rho v)$, (ρ, w) and (ρ, η).

The system in (3) is not in conservation form for the second equation, so similarly to [2, formula (3.1)], [3, formula (2.2)], [13, formula (1)], [15], we choose to reformulate (3) in conservation form, as follows:

$$\text{(4)} \qquad \begin{cases} \partial_t \rho + \partial_x\left(\rho\, v(\rho,\eta)\right) = 0 \\ \partial_t \eta + \partial_x\left(\eta\, v(\rho,\eta)\right) = 0 \end{cases} \qquad \text{with} \qquad v(\rho,\eta) = \min\left\{V_{\max}, \frac{\eta}{\rho}\,\psi(\rho)\right\}.$$

This model consists of a 2×2 system of conservation laws with a $\mathbf{C^{0,1}}$ but not $\mathbf{C^1}$ flow. Note in fact that $\frac{\eta}{\rho} = w \in [\check{w}, \hat{w}]$.

We study the Riemann Problem for (4), see [9, Section 2]. This model is also compared with other models of the same type in the current literature [2, 7, 17], as well as with a kinetic one [5].

The second traffic flow model that we present is the Micro–Macro model, see [8], consisting of a macroscopic and a microscopic descriptions glued together. The macroscopic part is described again through the LWR traffic model, as in (1), and the microscopic part through a Follow–the–Leader (FtL) model, see [1].

Microscopic models for vehicular traffic consist of a finite set of ordinary differential equations, describing the motion of each vehicle in the traffic flow. Below we consider a first order FtL model, where each driver adjusts his/her velocity to the vehicle in front, that is

$$\text{(5)} \qquad \dot{p}_i = v\left(\frac{\ell}{p_{i+1} - p_i}\right).$$

Here, $p_i = p_i(t)$ is the position of the i-th driver, for $i = 1, \dots, n$, and $p_{i+1} - p_i \geq \ell$ for all $i = 1, \dots, n-1$, the fixed parameter ℓ denoting the (mean) vehicles' length. Here, $\ell/(p_{i+1} - p_i)$ is the local traffic density seen by the driver p_i. Equation (5) needs to be closed with the trajectory of the first driver p_n. Throughout, we carefully select assumptions allowing us to prove that all speeds are bounded.

Our aim is to consider a general situation in which the two descriptions (1) and (5) are alternatively used in different segments of the real line. A similar approach to traffic modeling is in [12], where the interface between the micro- and macro description is kept fixed and the model in [2, 17] plays the role here played by the LWR one. See also [10] for the case $n = 1$.

Some numerical results complete the study of the model and prove the reasonableness of it's solutions: in particular they expain how the two micro- and macroscopic descriptions coexist in a single model, although being separated.

The Free–Congested and the Micro–Macro descriptions, both consist in the coupling of systems of different types and both display 2 phases. Moreover both the two descriptions display "free boundaries" models to be determined. Another analogy is the study of phase transitions, see also [6, 7, 9, 11]. In the Free–Congested model a vehicle can enter in or exit from one of the two distinct phases, depending on the traffic conditions. This does not occur in the Micro-Macro model: there is a backward propagating exchange of information between the different phases, although there is no exchange of mass.

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Entropy-stable discontinuous Galerkin finite element method with streamline diffusion and shock-capturing

Andreas Hiltebrand
(joint work with Siddhartha Mishra)

We propose and analyse entropy-stable discontinuous Galerkin finite element methods for hyperbolic systems of conservation laws in several space dimensions. To ensure entropy stability the discretisation is done in entropy variables and entropy-stable numerical fluxes are used. This leads to a formally arbitrarily high order accurate scheme. As we are interested in problems with shocks, a streamline diffusion and a shock-capturing term (compare [2, 3]) are added to control spurious oscillations at shocks. The resulting approximate solutions converge (under the assumption of a uniform L^∞ bound) to an entropy measure valued solution [1]. Various numerical experiments are shown to demonstrate the robustness and high resolution property of the scheme.

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Numerical schemes that converge to entropy measure valued solutions of systems of conservation laws

Siddhartha Mishra

Many interesting problems in physics and engineering are modeled using systems of hyperbolic conservation laws:

$$\partial_t u + \nabla \cdot f(u) = 0. \tag{1}$$

Here, $u : \mathbb{R}^n \times \mathbb{R}_+ \mapsto \mathbb{R}^m$ is the vector of unknowns and $f : \mathbb{R}^m \mapsto \mathbb{R}^m$ is the flux vector. The system has to be augmented with suitable initial and boundary conditions.

It is well known that solutions to (1), even for smooth initial data, do develop discontinuities in the form of shock waves in finite time. Hence, *weak solutions* of (1) are sought by interpreting the equation in the sense of distributions. Additional admissibility criteria or *entropy conditions* are imposed to ensure uniqueness of weak solutions.

Although entropy solutions have been accepted as the standard mathematical framework for solutions of systems of conservation laws, wellposedness results are only available in the scalar case and for systems in one space dimension and with small initial data. As of writing, there are no generic wellposedness results for systems of conservation laws in several space dimensions.

Numerical schemes to approximate entropy solutions of (1) have undergone extensive development over the last three decades or so. Finite volume and finite difference schemes based on (approximate) Riemann solvers, non oscillatory reconstruction procedures such as TVD, ENO, WENO and strongly stability preserving (SSP) Runge kutta time stepping methods, are very popular. Similarly, discontinuous Galerkin finite element methods and spectral viscosity have also been widely used.

Again, rigorous convergence results for numerical schemes approximating systems of conservation laws are only available for scalar problems (and only for first and second order schemes). No rigorous convergence results are currently available for system of conservation laws, particularly in several space dimensions.

The issues of wellposedness for the continuous problem and convergence of numerical approximations is intimately linked as numerical stability can be derived only if the solutions of the PDE have appropriate stability properties. Furthermore, the convergence of numerical approximations provides a constructive proof of existence for the underlying solutions of the PDE in question.

Given the lack of wellposedness results for multi-dimensional systems of conservation laws as well as the lack of rigorous convergence results for numerical approximations, it is natural to question whether entropy solutions of systems of conservation laws are well posed ? The question is investigated in a forthcoming paper [1].

Extensive numerical experiments in [1] show that many standard numerical schemes may not converge to any underlying function as the mesh is refined. Mesh refinement yields structures at finer and finer scale and strongly suggests that the

underlying entropy solutions are very unstable. Furthermore, a close examination of ensembles of solutions hint at a different, (more wider) notion of solutions might be appropriate in this context.

This more general concept of solutions is identified as *entropy measure valued solutions* i.e, the solutions of (1) are no longer functions, but rather space-time parametrized probability measures or *young measures*. In order words, the system of conservation laws (1) is interpreted as a measure valued cauchy problem:

$$\begin{aligned} \partial_t \langle \nu, \mathrm{id} \rangle + \nabla \cdot \langle \nu, f \rangle &= 0 \\ \nu_{(x,0)} &= \sigma_x, \end{aligned} \tag{2}$$

where $\sigma \in \mathbf{Y}(\mathbb{R}^n, \mathbb{R}^m)$ is the *initial measure-valued data* and $\nu \in \mathbf{Y}(\mathbb{R}^n \times \mathbb{R}_+, \mathbb{R}^m)$ is the sought after measure valued solution. Here $\mathbf{Y}(A, B)$ is the set of young measures mapping A to B. The above measure valued problem is augmented with entropy conditions to define entropy measure valued solutions.

The appropriate notion of solutions is defined in [1]. In particular, the sense in which the initial data is enforced is specified. The rest of [1] is focussed on showing existence of entropy measure valued solutions to a generic system of conservation laws by proving convergence of numerical approximations to them. A set of abstract stability criteria are proposed that amount to requiring that the approximate solutions (generated by the numerical scheme) are uniformly bounded in L^∞, satisfy a *weak BV* estimate and an entropy inequality. Given these assumptions, the authors of [1] prove that the corresponding approximate solutions generate a sequence of young measures that converges (sub sequentially) narrowly to a entropy measure valued solution of (2).

Two sets of numerical schemes that satisfy the above criteria are arbitrarily high-order TeCNO finite difference schemes of [2] and the space-time discontinuous Galerkin schemes of [3]. Thus, these schemes are rigorously shown to converge to entropy measure valued solutions of a generic system of conservation laws in several space dimensions. This result should be contrasted with the failure to obtain a rigorous convergence result for entropy solutions over the last three decades.

The rest of [1] examines different aspect of entropy measure valued solutions. Suitable notion of stability are defined and explored using the convergent numerical methods. In particular, *strong stability* i.e, if the initial measure valued data converging (narrowly) to a atomic measure implies that entropy measure valued solutions converge to a atomic measure, is precisely the classical notion of entropy solutions. Numerical experiments indicate that many interesting systems of conservation laws, particularly the Euler equations of gas dynamics, are *not strongly stable*. However, they also reveal that the solutions might be stable in more general metrics.

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Sharp interface limit of a diffuse interface model of Navier-Stokes-Allen-Cahn type for mixtures

Christiane Kraus

The mathematical main approaches to describe interfacial dynamics are diffuse interface (DI) models and sharp interface (SI) models. In this contribution, we consider the connection between these two kinds of models for the flow of compressible mixtures. We study the sharp interface limit of the following diffuse interface model for mixtures which is of Navier-Stokes-Allen-Cahn type:

$$\partial_t \rho + \operatorname{div}(\rho \mathbf{v}) = 0,$$
$$\partial_t \rho_\alpha + \operatorname{div}(\rho_\alpha \mathbf{v}) = \sum_{\beta=1}^{N-1} M_{\alpha\beta} \triangle \mu_\beta, \qquad \alpha = 1, \dots, N-1,$$
$$\partial_t(\rho \mathbf{v}) + \operatorname{div}(\rho \mathbf{v} \otimes \mathbf{v}) + \nabla \left(\sum_{\alpha=1}^{N-1} \rho_\alpha \mu_\alpha + \rho \mu_\rho - \Psi(\rho, \rho_\alpha) \right) - \frac{1}{\varepsilon} \nabla W(\varphi)$$
$$= \operatorname{div}(\boldsymbol{\sigma}_{NS}(\nabla \mathbf{v})) - \varepsilon \operatorname{div} \left(\nabla \varphi \otimes \nabla \varphi - |\nabla \varphi|^2 \mathbf{I} \right),$$
$$\rho \partial_t \varphi + \rho \mathbf{v} \cdot \nabla \varphi = -\mu_\varphi, \quad \mu_\varphi = \frac{1}{\varepsilon} W'(\varphi) - \varepsilon \Delta \varphi,$$

where $\boldsymbol{\sigma}_{NS}$ is the classical Navier-Stokes stress tensor, $\mu_\rho = \frac{\partial \Psi}{\partial \rho}$, $\mu_\alpha = \frac{\partial \Psi}{\partial \rho_\alpha} - \frac{\partial \Psi}{\partial \rho}$, $\alpha = 1, \dots, N-1$, and Ψ is the free energy given by

$$\Psi(\rho, \rho_\alpha) = h(\chi)\, \psi_1(\rho, \rho_\alpha) + (1 - h(\chi))\, \psi_2(\rho, \rho_\alpha).$$

Boundary conditions $(\mathrm{BC})_{\mathrm{DI}}$:

$$\mathbf{v} = 0, \quad \nabla \varphi \cdot \nu = 0, \quad \nabla \mu_\alpha \cdot \nu = 0, \quad \alpha = 1, \dots, N-1, \quad \text{on } \partial\Omega \times (0, T).$$

Initial conditions $(\mathrm{IC})_{\mathrm{DI}}$:

$$\mathbf{v}(0) = \mathbf{v}^0 \in H_0^1(\Omega; \mathbb{R}^n),\ \rho(0) = \rho^0 \in L^4(\Omega),\ \rho_\alpha(0) = \rho_\alpha^0 \in L^4(\Omega),\ \varphi(0) = \varphi^0 \in L^4(\Omega)$$

with $0 < d < \rho^0$ for some constant d, $0 \le \rho_\alpha^0$, $\alpha = 1, \dots, N-1$, and $\sum_{\alpha=1}^{N-1} \rho_\alpha^0 \le \rho^0$.

In the sharp interface limit $\varepsilon \to 0$, we can show rigorously that under suitable assumptions on Ω, W, h, ψ_1 and ψ_2 weak solutions of the DI-model converge to weak solutions of the following SI-model, where $\Omega^-(t) := \{x \in \Omega : \varphi(x,t) = -1\}$, $\Omega^+(t) := \{x \in \Omega : \varphi(x,t) = +1\}$ and $\Gamma(t) := \partial\Omega^-(t) \cap \Omega(t)$, $t \in (0,T)$.

In $\Omega^{\pm}(t)$, $t \in (0,T)$:

$$\partial_t \rho + \operatorname{div}(\rho \mathbf{v}) = 0,$$

$$\partial_t \rho_\alpha + \operatorname{div}(\rho_\alpha \mathbf{v}) = \sum_{\beta=1}^{N-1} M_{\alpha\beta} \triangle \mu_\beta, \qquad \alpha = 1, \dots, N-1,$$

$$\partial_t(\rho \mathbf{v}) + \operatorname{div}(\rho \mathbf{v} \otimes \mathbf{v}) + \nabla \left(\sum_{\alpha=1}^{N-1} \rho_\alpha \mu_\alpha + \rho \mu_\rho - \Psi(\rho, \rho_\alpha) \right) = \operatorname{div}(\boldsymbol{\sigma}_{NS}(\nabla \mathbf{v})),$$

$$\mu_\varphi = 0$$

On $\Gamma(t)$, $t \in (0,T)$:

$$[[\mathbf{v}]] = 0, \quad \mathbf{v}^+ \cdot \nu = \mathbf{v}^- \cdot \nu = w_\nu, \quad w_\nu : \text{normal velocity of } \Gamma,$$

$$[[\mu_\alpha]] = 0, \qquad \alpha = 1, \dots, N-1,$$

$$[[\rho_\alpha(\mathbf{v} \cdot \nu - w_\nu)]] = [[\nabla \mu_\alpha \cdot \nu]], \qquad \alpha = 1, \dots, N-1,$$

$$\left[\left[\sum_{\alpha=1}^{N-1} \rho_\alpha \mu_\alpha + \rho \mu_\rho - \Psi(\rho, \rho_\alpha) \right] \right] = \frac{1}{2} \sigma \kappa, \qquad \sigma = \int_{-1}^{+1} \sqrt{2W(s)}\, \mathrm{d}s,$$

$$[[\mu_\varphi]] = 0$$

Boundary conditions $(\mathrm{BC})_{\mathrm{SI}}$:
$\mathbf{v} = 0, \quad \nabla \mu_\alpha \cdot \nu = 0, \quad \alpha = 1, \dots, N-1, \quad$ on $\partial\Omega \times (0,T)$.

Initial conditions $(\mathrm{IC})_{\mathrm{SI}}$:

$$\mathbf{v}(0) = \mathbf{v}^0 \in H_0^1(\Omega; \mathbb{R}^n), \quad \rho(0) = \rho^0 \in L^4(\Omega), \quad \rho_\alpha(0) = \rho_\alpha^0 \in L^4(\Omega)$$

with $0 < d < \rho^0$ for some constant d, $0 \le \rho_\alpha^0$, $\alpha = 1, \dots, N-1$, and $\sum_{\alpha=1}^{N-1} \rho_\alpha^0 \le \rho^0$; $\Omega^-(0) = \Omega^{0,-}$ with $\partial\Omega^{0,-} \cap \partial\Omega = \emptyset$, $\Omega^{0,-} \subset \Omega$ and $\int_\Omega |\nabla \chi_{\Omega^{0,-}}| < \infty$, where $\chi_{\Omega^{0,-}}$ is the characteristic function of $\Omega^{0,-}$.

For the notion of weak solutions of the sharp interface model we use the varifold concept of weak solutions introduced by Chen 96.

Definition 1. *(Weak solutions of the sharp interface model)*
A tuple $(\mathbf{v}, \rho, \rho_\alpha, \Omega^-, V)$, $\alpha = 1, \dots, N-1$, *is called a weak solution of the SI-model with* $(BC)_{SI}$ *and* $(IC)_{SI}$ *if the following properties are satisfied:*

(i) *Space and set properties:*
$\mathbf{v} \in L^\infty(0,T; L^2(\Omega)) \cap L^2(0,T; H^1(\Omega))$,
$\rho, \rho_\alpha \in L^\infty(0,T; L^4(\Omega)), \alpha = 1, \dots, N-1$,
$0 < d_1 \le \rho$ *and* $\|\rho\|_{L^\infty(0,T;L^4(\Omega))} < d_2$ *for some constants* $0 < d_1, d_2$,
$0 \le \rho_\alpha, \quad \sum_{\alpha=1}^{N-1} \rho_\alpha \le \rho$,
$\Omega_T^- := \cup_{0 \le t < T} \Omega^-(t) \cup \{t\}$ *is a measurable subset of* $\Omega \times [0,T)$ *and* $\chi_{\Omega_T^-} \in L^\infty(0,T; BV(\Omega))$.

(ii) *Varifold: V is a Radon measure on $\overline{\Omega}\times G\times(0,T)$, $G:=S^{n-1}/\{\nu,-\nu\}$, such that for a.e. $t\in(0,\infty)$*

$$V=V^t\,\mathrm{d}t,$$

where V^t is a Radon measure on $\overline{\Omega}\times G$, called varifold.
In particular, for a.e. $t\in(0,T)$ and for all $\zeta\in C(\overline{\Omega}\times G)$:

$$\int_{\overline{\Omega}\times G}\zeta(x,\mathbf{p})\,\mathrm{d}V^t(x,\mathbf{p})=\sum_{k=1}^{n}\int_{\overline{\Omega}}h_k^t(x)\zeta\big(x,\mathbf{p}_k^t(x)\big)\mathrm{d}\lambda^t(x),$$

where λ^t is a Radon measure on $\overline{\Omega}$ such that $2\sigma|\nabla\chi_{\Omega^-(t)}|(\Omega)<\lambda^t(\Omega)$ and $h_k^t,\mathbf{p}_k^t$ are λ^t-measurable $\mathbb{R}$- and G-valued functions with

$$0\le h_k^t\le 1,\qquad \sum_{k=1}^{n}h_k^t\ge 1,\qquad \sum_{k=1}^{n}\mathbf{p}_k^t\otimes\mathbf{p}_k^t=\mathbf{I},\qquad \lambda^t-a.e..$$

The first variation is given by
$\langle\delta V^t,\boldsymbol{\zeta}\rangle=\int_{\overline{\Omega}\times G}(I-\mathbf{p}\times\mathbf{p}):\nabla\boldsymbol{\zeta}\mathrm{d}V(x,\mathbf{p})$ *for all* $\boldsymbol{\zeta}\in C^\infty(\overline{\Omega},\mathbb{R}^n)$.

(iii) *Mass balances: For all $\zeta\in C_c^\infty([0,T)\times\overline{\Omega})$*

$$\int_{\Omega_T}\big(\rho\partial_t\zeta+\rho\mathbf{v}\cdot\nabla\zeta\big)\,\mathrm{d}x\,\mathrm{d}t=-\int_\Omega\rho^0\zeta(0,\cdot)\,\mathrm{d}x,$$

$$\int_{\Omega_T}\Big(\rho_\alpha\partial_t\zeta+\rho_\alpha\mathbf{v}\cdot\nabla\zeta-\sum_{\beta=1}^{N-1}M_{\alpha\beta}\nabla\mu_\beta\cdot\nabla\zeta\Big)\,\mathrm{d}x\,\mathrm{d}t=-\int_\Omega\rho_\alpha^0\zeta(0,\cdot)\,\mathrm{d}x,$$

where $\mu_\alpha\in L^2(0,T;H^1(\Omega))$, $\alpha=1,\dots,N-1$.

(iv) *Momentum balance: For all $\zeta\in C_c^\infty([0,T)\times\overline{\Omega};\mathbb{R}^n)$:*

$$\int_{\Omega_T}\Big(\rho\mathbf{v}\cdot\partial_t\zeta+\rho\mathbf{v}\otimes\mathbf{v}:\nabla\zeta+\Big(\sum_{\alpha=1}^{N-1}\rho_\alpha\mu_\alpha+\rho\mu_\rho-\Psi(\rho,\rho_\alpha)\Big)\nabla\cdot\zeta\Big)\mathrm{d}x\,\mathrm{d}t$$
$$-\int_{\Omega_T}\boldsymbol{\sigma}_{NS}(\nabla\mathbf{v}):\nabla\zeta\,\mathrm{d}x\,\mathrm{d}t-\int_0^T\langle\delta V^t,\zeta\rangle\,\mathrm{d}t=-\int_\Omega\rho^0\mathbf{v}^0\cdot\zeta(0,\cdot)\,\mathrm{d}x$$

(v) *Reduced Allen-Cahn type law: For all $\zeta\in C_c^\infty([0,T)\times\overline{\Omega})$*

$$-2\int_{\Omega^-}\Big(\rho\partial_t\zeta+\rho\mathbf{v}\cdot\nabla\zeta\Big)\,\mathrm{d}x\,\mathrm{d}t=\int_{\Omega_T}\mu_\varphi\zeta\,\mathrm{d}x\,\mathrm{d}t+2\int_{\Omega^{0,-}}\rho^0\zeta(0,\cdot)\,\mathrm{d}x,$$

where $\mu_\varphi\in L^2(0,T;L^2(\Omega))$.

(vi) *Energy estimate: For a.e. $0<s\le t<T$*

$$\lambda^t(\overline{\Omega})+\int_\Omega\big(\Psi(\rho(t),\rho_\alpha(t))+\frac{\rho(t)}{2}|\mathbf{v}(t)|^2\big)\,\mathrm{d}x+\int_{\Omega_T}\Big(|\nabla\mathbf{v}|^2+\sum_{\alpha=1}^{N-1}|\nabla\mu_\alpha|^2$$
$$+|\mu_\varphi|^2\Big)\,\mathrm{d}x\mathrm{d}\tau\le\lambda^s(\overline{\Omega})+\int_\Omega\big(\Psi(\rho(s),\rho_\alpha(s))+\frac{\rho(s)}{2}|\mathbf{v}(s)|^2\big)\,\mathrm{d}x$$

Theorem (K. 13) Convergence and Existence of weak solutions
For all $\varepsilon \in (0,1]$, let $(\mathbf{v}_\varepsilon^0, \rho_\varepsilon^0, \rho_{\alpha,\varepsilon}^0, \varphi_\varepsilon^0) \in H_0^1(\Omega;\mathbb{R}^N) \times L^4(\Omega) \times (L^4(\Omega))^{N-1} \times H^1(\Omega)$ be given with $\int_\Omega \rho_\varepsilon^0 \,\mathrm{d}x = m$. Furthermore, let

$$E_{D,\varepsilon}(\varphi_\varepsilon^0, \rho_\varepsilon^0, \rho_{\alpha,\varepsilon}^0) + \int_\Omega \rho_\varepsilon^0 \frac{|\mathbf{v}^0|^2}{2} \,\mathrm{d}x \le C, \qquad C > 0 \text{ constant},$$

and $(\mathbf{v}_\varepsilon, \rho_\varepsilon, \rho_{\alpha,\varepsilon}, \varphi_\varepsilon)$ be a weak solution of the DI-model. Then there exists a sequence $\{\varepsilon_k\}_{k\in\mathbb{N}}$ such that weak solutions $(\mathbf{v}_{\varepsilon_k}, \rho_{\varepsilon_k}, \rho_{\alpha,\varepsilon_k}, \varphi_{\varepsilon_k})$ of the DI-model converge to a weak solution $(\mathbf{v}, \rho, \rho_\alpha, \Omega^-, V)$ of the SI-model as $k \to \infty$.

On the Cauchy problem for the Hunter–Saxton equation

Anders Nordli

The Hunter–Saxton equation given by

$$(u_t + uu_x)_x = \frac{1}{2}u_x^2, \tag{1}$$

was proposed by Hunter and Saxton as an asymptotic model of a one dimensional nematic liquid crystal [3]. In this talk we consider the Cauchy problem of a generalization of (1) on the real line,

$$(u_t + uu_x)_x = \frac{1}{2}u_x^2 + \frac{1}{2}\rho^2, \tag{2a}$$

$$\rho_t + (u\rho)_x = 0. \tag{2b}$$

We show the existence of a Lipschitz continuous semigroup of conservative solutions. This is done by mapping the problem to Lagrangian coordinates, solve it and construct a metric, and then map the solution back to Eulerian coordinates. The transformations between Eulerian and Lagrangian coordinates can be found in [2]. A similar result for (1) was obtained in [1].

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On a simplified fluid dynamic description of some (exotic) renewable power plants

Ingenuin Gasser
(joint work with Maria Bauer, Elisabetta Felaco)

In this talk we focus on a certain class of gas dynamics applications, which have the following common features

- The main features of the flows are one dimensional or the application is accessible by a one dimensional model.
- The flows are characterised by a low Mach number, i.e. the ratio between a typical fluid velocity and the speed of sound is small.
- The temperature variations in the flows cannot be negleted. In some examples the buoyancy force is the main driving force.
- A complete treatment of the application requires optimistion with respect to certain data or parameters.

Let us mention a few examples which fullfill the above mentioned examples:

- chimneys [FG]
- solar updraft towers [SS, G]
- energy towers [Z, BG]
- gas pipelines [BGH]
- receivers in parabolic trough power station
- exhaust tubes [GR]
- road and railway tunnel fires [GS1, G1].

The underlying equations are obtained by balancing mass, momentum and energy using the 1d Euler equations of gas dynamics (for variable cross section) with source terms. The various examples differ mainly in the source terms in the momentum and in the energy balance. Typically viscosity and diffuison can be neglected. In some cases – like in the case of an Energy Tower – we have a gas mixture.

After an appropriate scaling small parameters can be identified. Important parameters are the Mach number and the Froude number. Here the Mach number is everywhere small and remains always small. Typically this requires increased numerical effort when solving the fully compressible equations. The reason for that is that one resolves all the acustics even though it is not important for the above mentioned applications.

We use the smallness of the Mach number to perform an asymptotics and to further simplify the models. In combination with a possible small Froude number various asymptotic regimes are possible: small Mach number, Boussinesq, quasistatic approximations etc. We analyse the resulting asymptotic equations. Finally we perform numerical simulations. The asymptotic models result to recover the main features of the flows and to be numerically reasonalbe and robust.

In fact, one aim is to obtain simple models which allow fast robust numerical simulation tools. This is important and necessary when applying control or optimisation procedures or when considering applications on networks.

In the talk we focus on three applications: the chimney, the Solar Updraft Tower and the Energy tower. For all we show the modelling approach, numerical simulations, some optimisation with respect to parameters.

For the Solar Updraft Tower there are data from a prototype available [SS] and therefore we can validate our model. We obtain not only very good qualitative agreement but also surprisingly good quantitative agreement [G].

In case of the Energy Tower the model is used to confirm that such a power plant could produce a positive net power output [BG]. In addition we can estimate the net power as a function of temperature and relative humidity. Finally we can show - as predicted - that there is an optimal spray rate.

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Multivalued equations motivated by granular flow model

Agnieszka Świerczewska-Gwiazda
(joint work with Piotr Gwiazda)

We study the system describing flow of granular avalanches. The derivation of considered continuum flow models essentially bases on the fact that the characteristic length in the flowing direction is in general much larger than the thickness of an avalanche.

Such an approach resulted in depth-averaged equation governed by generalized system of shallow water equations (Saint-Venant equations). The evolution of granular avalanches along an inclined slope is described by the mass and momentum conservation laws. Among the variety of models capturing the dynamics of granular flow, the Savage-Hutter model, cf. [8], is one of the most commonly

used frameworks. The model covers the process of fast moving avalanche, where the contribution of kinetic energy is significant. The equations are obtained as a multi-scale limit of the three-dimensional free surface incompressible Navier-Stokes equations through depth-averaging process. Originally the model was proposed in one-dimension setting and later extended to two-dimensional topography in [1].

The system proposed by Savage and Hutter, cf. [8] consists of conservation laws (conservation of mass and balance of momentum) describing the motion of a one-dimensional avalanche flow down a smoothly varying slope (hard surface)

$$\begin{aligned} \frac{\partial}{\partial t}h + \frac{\partial}{\partial x}(hv) &= 0 \\ \frac{\partial}{\partial t}(hv) + \frac{\partial}{\partial x}\left(hv^2 + \frac{1}{2}\beta h^2\right) &= hg, \end{aligned} \tag{1}$$

where the dependent variables are the height $h : \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}$, and the velocity $v : \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}$. The height h is the thickness of the layer measured perpendicular to the given surface, and v is the mean velocity obtained by averaging over the thickness of the layer.

The coefficient β defines the geometry of the hard surface. We assume $\beta = \cos\gamma(x)$ where $\gamma(x)$ is the angle between down normal vector of the surface and the gravitation vector. The right-hand side results from gravitation and friction, i.e. internal friction (viscosity) and friction between granular layer and the given surface, in particular $g(v) := \sin\gamma - \operatorname{sgn}(v)\cos\gamma$ with the following definition of the function signum

$$\operatorname{sgn}(v) := \begin{cases} v/|v| & \text{for} \quad v \neq 0 \\ 0 & \text{for} \quad v = 0 \end{cases}$$

Such defined function g is discontinuous what provides mathematical difficulties. However, the crucial problem appears in physical description. Considering static and flat problem ($v = \gamma(x) = 0$) one obtains that the only solutions to such problem are constant solutions, what absolutely does not coincide with experiments conducted on granular materials. To include this observation the function g is extended to a multifunction as follows

$$\tilde{g}(v) = \begin{cases} \sin(\gamma) + [-\cos(\gamma), +\cos(\gamma)] & \text{for} \quad v = 0, \\ \sin(\gamma) - \frac{v}{|v|}\cos(\gamma) & \text{for} \quad v \neq 0 \end{cases}$$

and the second equation chagnes to the inclusion, namely

$$\frac{\partial}{\partial t}(hv) + \frac{\partial}{\partial x}\left(hv^2 + \frac{1}{2}\beta h^2\right) \quad \in \quad h\tilde{g}. \tag{2}$$

For simplicity the dependence of $\tilde{g}$ and β on x will be ignored, what does not influence the problem. The constant β is defined by $\beta = k\cos(\gamma)$, where $-\frac{\pi}{2} < \gamma < \frac{\pi}{2}$ is an angle between gravitational force and a constant slope ground and k is a positive constant.

We shall now in more detail discuss the concept of working with the system containing a differential inclusion (2). With some nonlinear transformations of

this system (and introducing the variables $u = (u_1, u_2) = (h, hv)$) we obtain

$$\frac{\partial}{\partial t}u + \frac{\partial}{\partial x}F(u) \in \tilde{G}(u), \tag{3}$$

where

$$F(u) = \begin{pmatrix} u_1 u_2 \\ \dfrac{u_2^2}{u_1} + \dfrac{u_1^2}{2} \end{pmatrix} \quad \text{and} \quad \tilde{G}(u) = \begin{pmatrix} 0 \\ \tilde{g}(u_2) \end{pmatrix}. \tag{4}$$

The proper class of solutions for the above system is the class of entropy weak solutions.

Definition 1. *We call $u \in L^\infty([0,T) \times \mathbb{R}; \mathbb{R}_+ \times \mathbb{R})$ a weak entropy solution to the system*

$$\frac{\partial}{\partial t}u + \frac{\partial}{\partial t}F(u) = G(t,x)$$

with the initial data $u^0 \in L^\infty(\mathbb{R}; \mathbb{R}_+ \times \mathbb{R})$ and source $G \in L^\infty([0,T) \times \mathbb{R}; \mathbb{R}^2)$ iff

1. u is a weak solution, i.e. for all test functions $\psi \in C_c^1([0,T) \times \mathbb{R}; \mathbb{R}^2)$ it holds

$$\begin{aligned} \int_{[0,T)\times\mathbb{R}} \big[u(t,x) \cdot \frac{\partial}{\partial t}\psi(t,x) + F(u(t,x)) \cdot \frac{\partial}{\partial x}\psi(t,x) \\ + G(t,x) \cdot \psi(t,x)\big]\, dtdx + \int_{\mathbb{R}} u^0(x) \cdot \psi(0,x) dx &= 0 \end{aligned}$$

2. The entropy inequality

$$\begin{aligned} \int_{[0,T)\times\mathbb{R}} [\eta(u(t,x))\frac{\partial}{\partial t}\phi(t,x) + q(u(t,x))\frac{\partial}{\partial x}\phi(t,x) \\ + \nabla_u \eta(u(t,x)) \cdot G(t,x)\phi(t,x)]\, dtdx + \int_{\mathbb{R}} \eta(u^0(x))\phi(0,x) dx &\geq 0 \end{aligned}$$

holds for all nonnegative test functions $\phi \in C_c^1([0,T) \times \mathbb{R}; \mathbb{R})$ and all convex entropy-entropy flux pairs (η, q).

The above definition is standard in the theory of conservation laws. However it cannot be used for system (3) because of the multifunction in a source term. Therefore we need the following extension.

Definition 2. *We call $u \in L^\infty([0,T) \times \mathbb{R}; \mathbb{R}_+ \times \mathbb{R})$ a weak entropy solution to the system (3) with the initial data $u^0 \in L^\infty(\mathbb{R}; \mathbb{R}_+ \times \mathbb{R})$ iff*

1. $\exists\, G(t,x) \in \tilde{G}(u(t,x))$ for a.a. $(t,x) \in [0,T) \times \mathbb{R}$.

2. u is an entropy weak solution according to Definition 1 to a system

$$\frac{\partial}{\partial t}u + \frac{\partial}{\partial x}F(u) = G(t,x).$$

Among various analytical results on this topic (cf. [2, 3, 4, 5]), we concentrated on L^1-stability of the Cauchy problem for 2×2 system coming out from the theory of granular media, namely the system of two independent inclusions coupled only by a right-hand side. The investigations are done in a class of weak entropy solutions. For solution $u \in L^\infty([0,T) \times \mathbb{R}; \mathbb{R}_+ \times \mathbb{R})$ in the sense of Definition 2 the result of uniqueness for the Cauchy problem cannot be obtained because of the possible occurrence of initial layer to such a solution.
Then it is natural to ask about possible class of solutions (and initial data) for the Cauchy problem in which we have global in time existence and uniqueness together. These are weak entropy solutions (in sense of Definition 2) with additional condition on the time regularity $C^0([0,T); L^1_{loc}(\mathbb{R}))$. Note that this is the typical time regularity for the uniqueness results for the scalar conservation law (cf. [6, 7],).

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Travelling waves and kinetic relations

Nabil Bedjaoui

We present some properties of the kinetic relations generated by traveling wave solutions of perturbed conservation laws.

We consider first the case of scalar conservation laws with concave-convex flux which are supplemented with *nonlinear, possibly singular,* diffusive and dispersive terms. The model takes the following form:

$$\partial_t u + \partial_x f(u) = \beta^{p+1} \left(b(u, \beta u_x) \, |u_x|^p u_x \right)_x + \delta \left(c_1(u) \, (c_2(u) \, u_x)_x \right)_x .$$

The case $p = 0$, i.e, *regular and linear* diffusion, was treated in [1]. We generalize these results in [2], for the case of singular diffusion ($p > 0$). The behavior of the

kinetic function near the origin is carefully investigated. We discover that $p = 1/3$ and $p = 1/2$ are somewhat unexpected critical values.

In the second part, we present a system of two conservation laws arising in elastodynamics and fluid dynamics in the case of concave-convex flux, with linear viscosity and capillarity terms, i.e, of the form:

$$\partial_t v - \partial_x u = 0,$$
$$\partial_t u + \partial_x \sigma(v) = \beta\, \partial_x(b(v)\ \partial_x u) - \delta \partial_x(a(v)\partial_x(a(v)\partial_x v)).$$

The kinetic relation obtained in the hyperbolic-elliptic case, for the $2-Shock$ wave is uniquely defined, but fails to be monotone near the Maxwell-line. This means that the kinetic relation for the $1-Shock$ wave is multivaluated in this region (see [3]).

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Homogenization of Degenerate Porous Medium Type Equations in Ergodic Algebras

Hermano Frid

(joint work with Jean Silva)

In this paper we consider the homogenization of a porous medium type equation of the general form

$$u_t = \Delta f(x, \frac{x}{\varepsilon}, u), \tag{1}$$

with $(x, y, t) \in \Omega \times \mathbb{R}^n \times (0, \infty)$, and $\Omega \subset \mathbb{R}^n$ is a, bounded or unbounded, open set . Here f is a continuous function of (x, y, u) and $f(x, y, \cdot)$ is locally Lipschitz continuous, uniformly in (x, y), and may be of two different types:

- In the type 1 case, $f(x, y, \cdot)$ is a general strictly increasing function; this is a mildly degenerate case.
- In the type 2 case, $f(x, y, u)$ has the form $h(x, y)F(u) + S(x, y)$, where $F(u)$ is just a nondecreasing function, which is *not* strictly increasing; this is a strongly degenerate case. Let us denote by G the strictly increasing right-continuous function such that $F(G(v)) = v$, for all $v \in \mathbb{R}$.

We consider the initial-boundary value problem where we prescribe an initial condition of the form

$$u(x,0) = u_0(x, \frac{x}{\varepsilon}), \tag{2}$$

and a boundary condition of the form

$$f(x, \frac{x}{\varepsilon}, u(x,t)) \,|\, \partial\Omega \times (0,\infty) = 0. \tag{3}$$

Our analysis applies equally well to a more general, non-homogeneous, boundary condition of the form

$$f(x, \frac{x}{\varepsilon}, u(x,t)) \,|\, \partial\Omega \times (0,\infty) = \beta(x),$$

for a function $\beta \in C(\bar{\Omega}) \cap H^1_{\rm loc}(\bar{\Omega})$, where $C(\bar{\Omega})$ denotes the space of bounded continuous functions on $\bar{\Omega}$, and $H^1_{\rm loc}(\bar{\Omega})$ denotes the space of functions defined on $\bar{\Omega}$, which multiplied by any function in $C_c^\infty(\mathbb{R}^n)$ gives a function in the usual Sobolev (Hilbert) space of first order $H^1(\bar{\Omega})$. We address the homogeneous case (3) just for convenience.

For fixed $(x,u) \in \Omega \times \mathbb{R}$, we will assume that $f(x,\cdot,u) \in \mathcal{A}(\mathbb{R}^n)$, where $\mathcal{A}(\mathbb{R}^n)$ is a general ergodic algebra, which means an algebra with mean value that is ergodic. An algebra with mean value (algebra w.m.v., for short) is an algebra of bounded uniformly continuous functions on $\mathbb{R}^n$, invariant by translations, each member of which possesses a mean value. It is said to be ergodic, roughly speaking, if, for any function $\varphi \in \mathcal{A}$, the averages of the translates $\varphi(\cdot + y)$, in balls of radius $R > 0$, converge as $R \to \infty$, in the norm of the mean value of the square of the absolute value, to the mean value $\bar{\varphi}$ of φ. The most elementary example of an ergodic algebra is the space of continuous functions in $\mathbb{R}^n$, which are periodic in each coordinate $\varphi(x + \tau_i e_i)$, $i = 1, \cdots, n$, for certain constants $\tau_i \in \mathbb{R}$, where e_i are the elements of the canonical basis of $\mathbb{R}^n$. Another well known example is the space of almost periodic functions which may be defined as the closure in the sup norm of the space spanned by the set $\{e^{i\lambda\cdot x} : \lambda \in \mathbb{R}^n\}$, in the complex case, or the real parts of the functions in such space, in the real case (*cf.* [5, 4]). Many other examples are known such as the space of Fourier-Stieltjes transforms, the weakly almost periodic functions, etc.; we will comment a bit on such examples in Section 2 of [14], below. We recall that the theory of algebras w.m.v. and ergodic algebras was first developed by Zhikov and Krivenko in [19] (see also [15]). Concerning the initial data in (2), we will, in general, assume that $u_0 \in L^\infty(\Omega; \mathcal{A}(\mathbb{R}^n))$.

We present two main results (see Theorems 6.1 and 7.1 in [14]) concerning the homogenization of the initial-boundary value problem (1), (2),(3).

Our first main result applies to an unbounded domain Ω and a general ergodic algebra $\mathcal{A}(\mathbb{R}^n)$, but we have to restrict ourselves to initial data that are "well-prepared", that is, of the form

$$u_0(x,y) = g(x,y,\phi_0(x)),$$

for some $\phi_0 \in L^\infty(\Omega)$, where, for all $(x,y) \in \Omega \times (0,\infty)$, $g(x,y,\cdot)$ is the strictly increasing right-continuous function satisfying $f(x,y,g(x,y,v)) = v$, for all $v \in \mathbb{R}$.

Our second main result applies to a general initial data $u_0 \in L^\infty(\Omega; \mathcal{A}(\mathbb{R}^n))$, but we have to compromise restricting ourselves to a bounded domain Ω and to an ergodic algebra $\mathcal{A}(\mathbb{R}^n)$ which is a regular algebra w.m.v., examples of the latter being provided by the periodic, almost periodic, and Fourier-Stieltjes transform functions, the precise definition is found in [14].

Both main results establish, under the mentioned assumptions, the weak star convergence in $L^\infty(\Omega \times (0,\infty))$ of the entropy solutions $u_\varepsilon(x,t)$ of (1),(2),(3) to the entropy solution $\bar{u}(x,t)$ of the problem

$$\begin{aligned} &u_t = \Delta \bar{f}(x,u), \\ &u(x,0) = \bar{u}_0(x), \\ &\bar{f}(x,u(x,t)) \,|\, \partial\Omega \times (0,\infty) = 0, \end{aligned} \tag{4}$$

where

$$\bar{u}_0(x) = \fint_{\mathbb{R}^n} u_0(x,y)\, dy,$$

and $\bar{f}(x,u)$ is defined by $\bar{f}(x, \bar{g}(x,v)) = v$, with

$$\bar{g}(x,v) := \fint_{\mathbb{R}^n} g(x,y,v)\, dy. \tag{5}$$

In the case where f is of type 2, $\bar{g}(x,\cdot)$, defined by (5), may, in general, be discontinuous, which is a bad situation for defining precisely $\bar{f}(x,\cdot)$, only from the knowledge of $\bar{g}$. In order to avoid such indetermination, we impose the additional assumption, concerning the functions $h(x,y)$ and $S(x,y)$ appearing in the definition of a pressure function of type 2:

$$\mathfrak{m}\left(\{z \in \mathcal{K} \,:\, \alpha h(x,z) + S(x,z) = v\}\right) = 0, \tag{6}$$

for all $(x,v) \in \Omega \times \mathbb{R}$, with α belonging to the discontinuity set of G, where $\mathcal{K}$ is the compact space associated with the algebra w.m.v. $\mathcal{A}(\mathbb{R}^n)$ and $\mathfrak{m}$ is the corresponding probability measure in $\mathcal{K}$ (*cf.*, [3]). Under the assumption (6) the function $\bar{g}(x,\cdot)$, defined in (5), turns out to be continuous and strictly increasing, and so is its inverse $\bar{f}(x,\cdot)$, which means that the limit problem (4) has, in any case, a pressure function of type 1. This has the additional advantage of making much easier to check the uniqueness of the solution of the limit problem, since, for pressure functions of type 1, the notions of entropy and weak solutions coincide (see details in [14]).

Moreover, both of our main theorems also give the existence of correctors, that is, both theorems assert that

$$u_\varepsilon(x,t) - g(x, \frac{x}{\varepsilon}, \bar{f}(\bar{u}(x,t))) \to 0, \quad \text{as } \varepsilon \to 0 \text{ in } L^1_{\mathrm{loc}}(\Omega \times (0,\infty)). \tag{7}$$

Again, to obtain (7) in the case where f is of type 2, we make essencial use of (6), which makes a study of sufficient conditions to guarantee this null measure property of great interest, and we obtain such conditions. For details see [14].

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Strong Solutions to the Navier-Stokes Allen-Cahn equations

Matthias Kotschote

In this talk we study two problems which are connected. First, I introduce a system of balance laws for two-phase mixtures of fluids undergoing phase transitions. In this model the interfaces between the phases are assumed to be of "diffuse" nature, that is, sharp interfaces are replaced by narrow transition layers. These regions as well as the two species are located by a phase field variable χ governed by the Allen-Cahn equation (AC), while the dynamics are described

by the Navier-Stokes equations (NS). The model was proposed by Blesgen [1], but recently I found out that Truskinowski [4] first developed this thermodynamically and mechanically consistent set of partial differential equations extending the Navier-Stokes equations to a compressible binary Allen-Cahn mixture.
The two-component (binary) viscous compressible fluid is characterised by its total density (of the mixture) $\rho : J \times \Omega \to \mathbb{R}_+$, velocity field $u : J \times \Omega \to \mathbb{R}^n$, temperature $\theta : J \times \Omega \to \mathbb{R}_+$ and phase field $\chi : J \times \Omega \to [0,1]$; the unknown functions ρ, u, θ and χ are governed by the so-called Navier-Stokes-Allen-Cahn (NSAC) system

(1)
$$\begin{aligned}
\partial_t \rho + \nabla \cdot (\rho u) &= 0, && (t,x) \in J \times \Omega,\\
\partial_t(\rho u) + \nabla \cdot (\rho u \otimes u) - \nabla \cdot (\mathcal{S} + \mathcal{P}) &= \rho f_{ext}, && (t,x) \in J \times \Omega,\\
\partial_t(\rho e) + \nabla \cdot (\rho e u) - \nabla \cdot (\alpha \nabla \theta) - \nabla \cdot [(\mathcal{S} + \mathcal{P}) \cdot u] &= \rho f_{ext} \cdot u, && (t,x) \in J \times \Omega,\\
\partial_t(\rho \chi) + \nabla \cdot (\rho \chi u) - \tau^{-1} \nabla \cdot (\rho \delta \nabla \chi) &= h, && (t,x) \in J \times \Omega,
\end{aligned}$$

with

(2)
$$\begin{aligned}
&\mathcal{S} = 2\mu \mathcal{D}(u) + \lambda \nabla \cdot u \mathcal{I}, \quad \mathcal{P} = -\rho^2 \partial_\rho \psi - \theta \rho \delta \nabla \chi \otimes \nabla \chi,\\
&\psi = \overline{\psi}(\theta, \rho, \chi) + \theta(\mathcal{W}(\theta, \chi) + \frac{\delta}{2}|\nabla \chi|^2), \quad e = \overline{e} + \frac{1}{2}|u|^2,\\
&\overline{e} = \psi - \theta \partial_\theta \psi = \overline{\psi} + \theta \partial_\theta \overline{\psi} - \theta^2 \partial_\theta \mathcal{W},\\
&h = -\tau^{-1} \theta^{-1} \partial_\chi(\rho \psi) = -\tau^{-1} \rho (\mathcal{W}'(\theta, \chi) + \frac{1}{\theta} \partial_\chi \overline{\psi}).
\end{aligned}$$

Here, the Helmholtz energy density ψ consists of two parts, namely $\overline{\psi}$ and the so-called mixing energy

$$\theta(\mathcal{W}(\theta, \chi) + \frac{\delta}{2}|\nabla \chi|^2)$$

which does not contribute to the internal energy e, since by using the Legendre transformation we obtain

$$\overline{e} = \psi - \theta \partial_\theta \psi = \overline{\psi} - \theta \partial_\theta \overline{\psi} - \theta^2 \partial_\theta \mathcal{W}.$$

Finally, these equations have to be complemented by initial conditions. One main interest was to handle non-constant coefficients. In fact, we are able to allow viscosity coefficients fulfilling the assumptions

$$\mu(\theta, \rho, \chi) > 0, \quad 2\mu(\theta, \rho, \chi) + \lambda(\theta, \rho, \chi) > 0, \quad \theta > 0, \rho > 0, \chi \in [0,1] \tag{3}$$

plus some regularity assumptions. Note that from the physical point of view it is more reasonable to consider viscosities depending on χ to model different viscosities for different phases.
Under some physically reasonable assumptions on coefficients and Helmholtz energy density the following well-posedness result is available (see [2]).

Theorem 1. *Let Ω be a bounded domain in $\mathbb{R}^n$, $n \geq 1$, with compact C^2-boundary Γ decomposing disjointly as $\Gamma = \Gamma_d \dot\cup \Gamma_s$, $J_0 = [0, T_0]$ with $T_0 \in (0, \infty)$ and $p \in (n+2, \infty)$. Further, assuming certain positive and regularity assumptions on the*

coefficients and the Helmholtz energy density and $f_{ext} \in \mathrm{L}_p(J_0;\mathrm{L}_p(\Omega;\mathbb{R}^n))$. *Then for each initial data* $(\rho_0,w_0)=(\rho_0,u_0,\theta_0,\chi_0)$ *in*

$$\begin{aligned}\mathcal{V} := \{(\rho,u,\theta,\chi) \in \mathrm{H}^1_p(\Omega)\times \mathrm{W}^{2-2/p}_p(\Omega;\mathbb{R}^n)\times \mathrm{W}^{2-2/p}_p(\Omega)\times \mathrm{W}^{2-2/p}_p(\Omega):\\ (u(y)|\nu(y)\geq 0,\ \forall y\in\Gamma,\ \rho(x)>0,\quad \theta(x)>0,\ \chi(x)\in[0,1],\ \forall x\in\overline{\Omega}\}\end{aligned}$$

and boundary data with enough regularity, there is a unique solution (ρ,w) *on a maximal time interval, which is* $J^*=[0,T^*)$, $T^*:=T^*(\rho_0,w_0)\in(0,T_0]$ *if the solution is not global. The solution* (ρ,w) *belongs to the class* $\mathbb{Z}_1(J)\times\mathbb{Z}_2(J)$,

$$\begin{aligned}&\mathbb{Z}_1(J) := \mathrm{H}^2_p(J;\mathrm{H}^{-1}_p(\Omega))\cap \mathrm{C}^1(J;\mathrm{L}_p(\Omega))\cap \mathrm{C}(J;\mathrm{H}^1_p(\Omega));\quad \mathrm{H}^{-1}_p := (\mathring{\mathrm{H}}^1_p(\Omega))',\\ &\mathbb{Z}_2(J) := Z_1(J)\times Z_2(J)\times Z_3(J),\\ &Z_1(J) := Z(J;\mathbb{R}^n),\quad Z_2(J) := Z(J;\mathbb{R}),\quad Z_3(J) := Z(J;\mathbb{R}),\\ &Z(J;E) := \mathrm{H}^1_p(J;\mathrm{L}_p(\Omega;E))\cap \mathrm{L}_p(J;\mathrm{H}^2_p(\Omega;E)),\quad E\in\{\mathbb{R}^n,\mathbb{R}\},\quad p\in(1,\infty),\end{aligned}$$

for each interval $J=[0,T]$, $T<T^*$, *or to the class* $\mathbb{Z}_1(J_0)\times\mathbb{Z}_2(J_0)$ *if the solution exists globally. The maximal time* T^* *is characterised by the property:*

$$\lim_{t\to T^*}(\rho,w)(t) \quad \textit{does not exist in } \mathcal{V}. \tag{4}$$

Moreover, χ *stays in* $(0,1)$ *and, if additionally* $\partial_\chi\overline{\psi}(\chi,\rho,0)=0$, *the temperature* θ *is nonnegative. The solution map* $(\rho_0,w_0)\to(\rho,w)(t)$ *generates a local semiflow on the phase space in the autonomous case.*

The second part of the talk is devoted to a stability result for the compressible Navier-Stokes equations. This problem is connected to the before-mentioned NSAC system by the following problem. Investigating the NSAC system regarding stability of non-constant steady states, e.g. due to considering large non-constant external forces, in the L_p framework, one immediately meets a regularity problem inherent in the linearization of the equation of conservation of mass. Therefore, to be able to handle the compressible Navier-Stokes equation regarding stability is a first step for considering the NSAC system regarding the stability issue, since the same problem arises due to the equation of conservation of mass being part of NSAC as well.

A global existence and uniqueness is presented for the non-isothermal compressible Navier-Stokes equations in bounded domains, where the initial data have to be near equilibria. Moreover, exponential stability of equilibria in the phase space is present and, above all, the approach in Eulerian coordinates is possible (see [3]).

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A New Quadratic Potential for Scalar Conservation Laws

Stefano Modena
(joint work with Stefano Bianchini)

We construct a new quadratic interaction potential $\mathfrak{Q}$ for the Cauchy problem associated to a scalar conservation law $u_t + f(u)_x = 0$. The fundamental property of $\mathfrak{Q}$ is the following: when an interaction between two wavefronts having the same sign occurs, $\mathfrak{Q}$ decreases and its decrease controls the change in speed times the strength of the shocks involved in the interaction. Moreover, although it can increase when a collision between shocks having opposite sign occurs, we are able to control its positive total variation with the total variation squared of the initial datum (see [2] for the details).

Thanks to this fundamental property, we think that this potential will be able to be used to prove useful results about the rate of convergence of the Glimm scheme, which is the approximation algorithm used to prove the existence of solutions of the Cauchy problem, about the stability of the solutions and about their structure.

Differently from other interaction potentials present in the literature (see for instance [1]), the form of this functional is the natural extension of the original Glimm functional [3], and coincides with it in the genuinely nonlinear case.

We also guess that our construction will be able to be extended to the vectorial case of hyperbolic systems of conservation laws.

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Compressible multi-component flow including phase transition

Siegfried Müller
(joint work with Maren Hantke, Ee Han)

Flows of compressible two-phase and multi-component mixtures mixtures have a wide range of applications. Difficulties in the modeling result from the interaction of the fluids, especially from the exchange of mass and energy across phase interfaces. So the treatment of the phase interfaces is in the focus of the modeling.

Several models are available in the literature, that are distinguished in sharp interface and diffuse interface models. A detailed survey of these models can be

found in Zein [13]. The model considered here was first proposed by Baer and Nunziato [1] for detonation waves in granular explosives. Later it was modified and generalized by several authors, see for example Saurel and Abgrall [9]. This model fits into the class of multi-component fluids considered in the book of Drew and Passman [2].

The original model by Baer and Nunziato is a full non-equilibrium model, which means, each component has its own pressure, velocity and temperature. The modified model of Saurel and Abgrall also includes relaxation terms for the pressure and the velocities of the components.

By instantaneous relaxation procedures equilibrium values for the pressure and the velocity can be found. Using further relaxation procedures to drive the temperatures and the chemical potentials into equilibrium mass transfer between the phases can be modeled, see Saurel et al. [10] or Zein et al. [14]. Typically the relaxation procedures for temperature and mass transfer are based on iterative algorithms that are very much time-consuming, see [15]. Thus multi-dimensional applications are only feasible in acceptable computational time on massive parallel architectures. Therefore our main objective is to improve the relaxation models and to design relaxation procedures that allow for efficient multi-dimensional computations. Efficiency is further improved by combining these with multiresolution-based grid adaptation techniques, see [6, 7].

Here we follow the ideas of Zein et al. [15] but improving the modeling in several aspects. First of all, we simplify the relaxation procedures for pressure, temperature and chemical potentials, both for two-phase models with two and three components, respectively. In contrast to [15] we avoid the calculation of some parameters and are able to find the relaxed pressures and temperatures directly *without* performing an iterative procedure. In addition, the modeling of mass transfer between two unpurified phases is improved. Instead of Gibbs free energies we relax the chemical potentials. This allows to take into account additional components in the phases by considering the mixture entropy and, thus, the model becomes more physically sound. Furthermore, we avoid the artificial definition of an interfacial region. This allows us to model *physical* cavitation, which means that we can start from a pure liquid phase. The vapor phase will be created by expansion. In previous work it was necessary to start with an appreciable amount of vapor, for instance 1% in [14]. Nevertheless in our modeling we avoid *unphysical* nucleation or *unphysical* cavitation.

For a physically relevant application we present simulations for laser induced-cavitation bubbles. These are motivated by experiments of Lauterborn et al.. An overview on this work can be found in the recent review article [5]. Recent experiments [11, 12] at elevated water temperatures indicate that the amount of non-condensable gas in the bubble might have a significant influence on the collapse and the rebound of the bubble. In the book of Müller [8], p. 301-312, it is explained that beside water vapor at least one further substance must be inside the bubble to guarantee a stable equilibrium state of a surviving bubble. In order

to investigate this numerically the two-component model had to be generalized to a multi-component model.

Quasi-one-dimensional numerical simulations of a collapsing bubble filled with non-condensable gas and condensable water vapor in liquid water are performed for varying amount of non-condensable gas. The computations show a strong effect on the rebound with increasing amount of non-condensable gas. However, the rebound is grid -dependent, i.e., the rebound increases under grid refinement. This indicates that some physical effect is still missing in the model. The asymptotic analysis of Guderley [4] for the collapse of a spherical shock wave in a single-phase fluid indicate that viscosity and heat conduction might be accounted for. This is in agreement with the findings in [3]. In addition, the phase transition might be modeled in non-equilibrium because of the very high speeds of the phase interface in the collapse.

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On the Zero Mach Limit of Compressible Two-Fluid Flow

Veronika Schleper
(joint work with Rinaldo M. Colombo)

The Zero Mach Limit of the compressible Euler equations is a classical problem and a variety of publications can found, discussing this topic in the framework of smooth solutions, see for example the classical results [2, 3, 5]. Contrary to the classical setting, we consider a two-fluid setting with one compressible and one weakly compressible phase and discuss the behavior of the system as the weakly compressible phase tends to be incompressible.

The basic model for the presentation will be the sharp-interface two-fluid model based on the isothermal Euler equations, given by

$$\text{(1a)} \qquad \begin{cases} \partial_t \rho_g + \partial_x (\rho_g v_g) = 0 \\ \partial_t (\rho_g v_g) + \partial_x \left(\rho_g v_g^2 + p_g(\rho_g)\right) = 0 \end{cases} \qquad \text{for } x \in \mathbb{R} \setminus [a(t), b(t)]$$

$$\text{(1b)} \qquad \begin{cases} \partial_t \rho_l + \partial_x (\rho_l v_l) = 0 \\ \partial_t (\rho_l v_l) + \partial_x \left(\rho_l v_l^2 + p_l(\rho_l)\right) = 0 \end{cases} \qquad \text{for } x \in [a(t), b(t)]$$

$$\text{(1c)} \qquad \begin{cases} v_g(t, a(t)-) = v_l(t, a(t)+) \\ p_g(\rho_g(t, a(t)-)) = p_l(\rho_l(t, a(t)+)) \\ v_g(t, b(t)+) = v_l(t, b(t)-) \\ p_g(\rho_g(t, b(t)+)) = p_l(\rho_l(t, b(t)-)) \end{cases} \qquad \text{interface conditions.}$$

where we allow for different pressure laws p_g and p_l in the different fluids. For analytical reasons, we exclude vacuum states and assume that the pressure law fulfills the standard requirements

$$\text{(2)} \qquad p \in \mathbf{C}^4(\mathbb{R}^+; \mathbb{R}^+), \quad p(\rho) > 0, \quad p'(\rho) > 0, \quad p''(\rho) > 0, \quad \text{for all } \rho \in \mathbb{R}^+.$$

In [1], a well-posedness result for system (1) was proven for initial data in $\mathbf{L}^1 \cap \mathbf{B}V$ with small total variation. This result provides the existence of a standard Riemann semigroup of solutions as well as the existence and regularity of the interface movement.

When considering the Riemann problem at the interface, we observe that the Lax curve associated with the liquid (and thus weakly compressible) fluid tends to a horizontal line, see Figure 1. Therefore, in the incompressible limit, the interface acts like a solid wall. This at first glance astonishing effect can be explained through the fact that the two-fluid Riemann problem considers fluids with infinite mass since they are present on the complete half space. Then it is obvious that a volume of incompressible fluid cannot be accelerated by a finite transfer of momentum.

It is therefore necessary to consider both interfaces and thus a Cauchy problem to circumvent the problem of infinite mass. For the discussion of the zero Mach

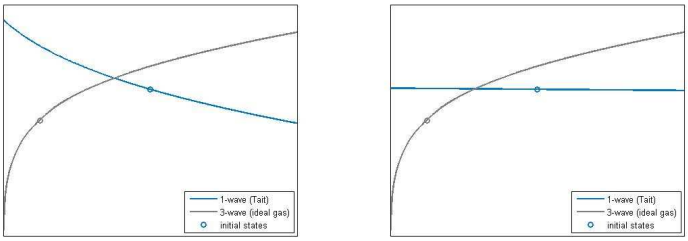

FIGURE 1. Illustration of the Lax-curves for gas (gray) and liquid (blue). The left figure shows a compressible liquid, while the figure on the right displays the case for a (very) weakly compressible fluid. The solution to the interface Riemann problem is given by the intersection point of the gray and blue curves.

limit, we restrict therefore to special initial data of the form

$$(\rho_0, v_0) = \begin{cases} (\bar\rho_g, \bar v_g) & \text{for } x < a(0) \\ (\rho_l^o, v_l^o) & \text{for } a(0) < x < b(0) \\ (\rho_g^o, v_g^o) & \text{for } x > b(0) \end{cases} \tag{3}$$

where we assume that (ρ_g^o, v_g^o) and (ρ_l^o, v_l^o) fulfill the coupling conditions $v_g^o = v_l^o$ and $p_g(\rho_g^o) = p_l(\rho_l^o)$ and $(\bar\rho_g, \bar v_g)$ can be connected to (ρ_g^o, v_g^o) by a *shock of the second family* of small size σ.

A detailed study of the wave interactions with the boundary shows that the initial acceleration of the liquid segment tends to the acceleration predicted by the incompressible Euler equations, i.e.

$$\dot a(0) = \dot b(0) = \frac{p_g(\rho_g(0, a(0)-)) - p_g(\rho_g(0, b(0)+))}{\rho_l^o \cdot (b(0) - a(0))}. \tag{4}$$

For more details on the derivation of the limit as well as on numerical examples, we refer the interested reader to [1, 4].

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Non-local constraints and crowd dynamics

Massimiliano D. Rosini
(joint work with B. Andreianov, C. Donadello)

Empirical studies [2] show that the capacity of the bottleneck can drop when high–density conditions occur upstream of the bottleneck. The corresponding model is

$$(1)\quad \partial_t \rho + \partial_x f(\rho) = 0, \quad f\left(\rho(t,0\pm)\right) \leq p\left(\int_{\mathbb{R}_-} w(x)\rho(t,x)\,\mathrm{d}x\right), \quad \rho(0,x) = \rho_0(x).$$

Above, $\rho \in [0,R]$ is the crowd density, $f \in [0,\bar{f}]$ is the flow, $p > 0$ is the capacity of the exit in $x = 0$ as a function of the weighted average density in a bounded left neighborhood of $x = 0$, $w \geq 0$ is the weight and ρ_0 is the initial density.

Theorem 1 ([1]). *(i) For any $\rho_0 \in \mathbf{L^i nfty}(\mathbb{R};[0,R])$, (1) admits a unique solution ρ. If $\tilde{\rho}$ is the solution corresponding to $\tilde{\rho}_0 \in \mathbf{L^i nfty}(\mathbb{R};[0,R])$, $T > 0$ and $L \gg 1$, then*

$$\|\rho(T) - \tilde{\rho}(T)\|_{\mathbf{L^1}([-L,L];\mathbb{R})} \leq e^{CT}\|\rho_0 - \tilde{\rho}_0\|_{\mathbf{L^1}([-(L+MT),(L+MT)];\mathbb{R})},$$

where $M = \mathrm{Lip}(f)$ and $C = 2\mathrm{Lip}(p)\|w\|_{\mathbf{L^i nfty}(\mathbb{R}_-;\mathbb{R})}$.
(ii) If $\rho_0 \in \mathcal{D} = \left\{\rho \in \mathbf{L^1}(\mathbb{R};[0,R]) : \Psi(\rho) \in \mathbf{BV}(\mathbb{R};\mathbb{R})\right\}$, $\Psi(a,b) = \mathrm{sgn}(a-b)\left(f(a)-f(b)\right)$, then for a.e. $t,s > 0$, $\rho(t,\cdot) \in \mathcal{D}$ and

$$\mathrm{TV}\left(\Psi\left(\rho(t)\right)\right) \leq C_t = \mathrm{TV}\left(\Psi\left(\rho_0\right)\right) + 4f(\bar{\rho}) + C\,t\,,$$
$$\|\Psi\left(\rho(t,\cdot)\right) - \Psi\left(\rho(s,\cdot)\right)\|_{\mathbf{L^1}(\mathbb{R};\mathbb{R})} \leq |t-s|\,\mathrm{Lip}(\Psi)\,C_T\,.$$

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Lipschitz semigroup and traveling waves for an integro–differential equation for slow erosion

Graziano Guerra
(joint work with Rinaldo M. Colombo, Wen Shen)

Introduction. Granular matter is being poured from an uphill location outside the interval of interest, and slides to the left down a hill. As it slides down, it interacts with the standing layer phase. In [9] the following 2×2 system of balance laws was proposed:

$$(1)\qquad \begin{cases} h_t - \left(\beta h u_x\right)_x = -\gamma\left(\alpha - |u_x|\right)h \\ u_t = \gamma\left(\alpha - |u_x|\right)h \end{cases}$$

where $u(t,x)$ is the height of the standing phase while $h(t,x)$ is the height of the moving layer phase, $\alpha,\ \beta,\ \gamma$ are positive constants. Time-dependent solutions for this model were first studied in [1, 11]. More complete models which include energy balance were proposed in [5, 6]. We will always suppose a strictly positive slope: $u_x \geq c > 0$. In [2] it is proved that, in the *slow erosion* limit ($h \to 0$, $t \to +\infty$), the limit standing profile depends only on the initial profile and on the total mass μ been poured from the uphill location outside the interval of interest. The limit profile $U(\mu, x)$ satisfies the following hyperbolic equation with a non local flux:

$$U_\mu(\mu,x) - \left(\exp \int_x^{+\infty} f\left(U_x(\mu,y)\right)\, dy\right)_x = 0, \qquad f(w) = \frac{w-1}{w}. \tag{2}$$

The *erosion function* f describes the interaction between the moving and the standing phase. It depends only on the slope and denotes the rate of mass being eroded or deposited per unit length and per unit mass passing through. There is a critical slope where no interaction happens and f vanishes. In a normalized model one could choose the critical slope to be 1. To adhere to the usual notation in the hyperbolic equations theory, we substitute μ with t in (2) and see what happens if we consider a general *erosion function* $f(w)$, not necessarily the one (2) which arises from the slow erosion limit of (1).

Existence and stability of solutions. With the following hypotheses on f

$$f(1) = 0,\ f' > 0,\ f'' < 0,\ \lim_{w\to 0^+} f(w) = -\infty,\quad \lim_{w\to+\infty} \frac{f(w)}{w} = 0 \tag{3}$$

D. Amadori and W. Shen in [3, 4] proved the existence and uniqueness of the solution to the Cauchy problem for (2). Allowing more erosion for large slopes i.e. substituting the last condition in (3) with a linear growth condition at $+\infty$: $\lim_{w\to+\infty}\left[f(w) - (\eta_o w + \beta)\right] = 0$, the solutions of (2) may develop various types of singularities, including jumps in the profile U, see [10] for a detailed discussion. Therefore, we expect $U(t)$ to attain values in $\mathbf{BV}$ and its space derivative U_x to be a measure. In order to give a meaning to (2), we introduce the inverse function $X = X(t,u)$ which is the graph completion of the inverse in space of U: $X(t,u) = x \iff u \in [U(t,x-), U(t,x+)]$. Whenever U has a jump, the inverse function X remains constant over the interval of jump in U. Under the further condition that there exists a positive κ such that

$$U(t,x_2) - U(t,x_1) \geq \kappa\,(x_2 - x_1) \quad \text{ for all } x_1, x_2 \in \mathbb{R}, x_1 \leq x_2\,, \tag{4}$$

the function X is Lipschitz continuous in u. Define the function $z(t,u)$ to be the u-derivative of $X(t,u)$: $z(t,u) \doteq X_u(t,u)$. This is a well defined function $z(t) \in \mathbf{L}^\infty(\mathbb{R};\mathbb{R}^+)$. In the case of a smooth function U, we can rewrite the integral in (2) as

$$\int_x^{+\infty} f\left(U_x(t,y)\right)\, dy = \int_{U(t,x)}^{+\infty} g\left(z(t,v)\right)\, dv, \quad g(s) = s\, f(1/s),\ s > 0. \tag{5}$$

Remark that the right hand side in (5) is well defined also if $U(t) \in \mathbf{BV}(\mathbb{R};\mathbb{R})$. From (2), thanks to (5) which allows to give a meaning to the nonlinear function f applied to the measure U_x, we are led to consider the conservation law

$$U_t(t,x) - \left(\exp \int_{U(t,x)}^{+\infty} g\left(z(t,v)\right)\, dv \right)_x = 0 \tag{6}$$

where we treat as unknown the function z. Moreover, to allow the reconstruction of U from z, we have to impose further constraints on z, namely that

$$z(t) \in \mathbf{L}^\infty(\mathbb{R};\mathbb{R}^+) \quad \text{and} \quad (z(t)-1) \in \mathbf{L}^1(\mathbb{R};\mathbb{R})\,. \tag{7}$$

Under these conditions, we reconstruct U from z as follows:

$$X(t,u) = u + \int_{-\infty}^{u} (z(t,v)-1)\, dv, \quad U(t,x) = \max\left\{v \in \mathbb{R}\colon X(t,v) \le x\right\}\,. \tag{8}$$

In [7] the following theorem was proved.

Theorem 1. *Fix $T > 0$ and consider the set*

$$\mathcal{Z} \doteq \left\{ z \in \mathbf{BV}\left(\mathbb{R};[0,+\infty)\right) : \begin{array}{l} z \text{ is right-continuous, and} \\ (z-1) \in \mathbf{L^1}\left(\mathbb{R};[0,+\infty)\right) \end{array} \right\}$$

For any f satisfying (3) with the slow growth condition substituted by the linear growth condition, let g be the corresponding erosion function defined in (5), then there exists a map $S^g\colon [0,T]\times\mathcal{Z}\to\mathcal{Z}$ with the following properties:

1. *(1) $S^g_0 = Id$ and for any $t_1, t_2 \in [0,T]$ with $t_1 + t_2 \in [0,T]$, the semigroup property holds: $S^g_{t_1} \circ S^g_{t_2} = S^g_{t_1+t_2}$.*
2. *(2) For any $z_o \in \mathcal{Z}$, the orbit $t \to S^g_t z_o$ solves (6)–(8) in the sense of distributions.*
3. *(3) There exists a constant $L > 0$, which depends on the system, on T and on the $\mathbf{L^1}$ norm and total variation of the initial data, such that*

$$\left\| S^g_t z - S^{\bar g}_{\bar t}\bar z \right\|_{\mathbf{L^1}} \le L\, \left(t\, \|g-\bar g\|_{\mathbf{W^{1,\infty}}} + e^{Lt}\, \|z-\bar z\|_{\mathbf{L^1}} + |t-\bar t| \right)\,.$$

Existence and stability of traveling waves. We consider "non decreasing" initial data $z_o(u) \in [0,1]$:

$$z_o(u) = \begin{cases} 1 & (u < u_a), \\ \tilde z_o(u) & (u \ge u_a), \end{cases} \qquad \tilde z_o(u_2) - \tilde z_o(u_1) \ge 0 \quad \text{for} \quad u_2 \ge u_1 \ge u_a\,, \tag{9}$$

define the "total drop" $D = \int_{\mathbb{R}} [1 - z_o(u)]\, du$ and consider the semigroup trajectory $z(t,u) = \left(S^g_t z_o\right)(u)$. Then we introduce the drop function

$$q(t,u) = \int_u^{+\infty} [z(t,v) - 1]\, dv \tag{10}$$

and the new unknown $\zeta(t,q)$ defined by $\zeta\left(t, q(t,u)\right) = z\left(t,u\right)$. In [8] it is shown that for smooth solution $z\left(t,u\right)$ the corresponding function $\zeta\left(t,q\right)$ satisfies the partial differential equation $\zeta_t - \left(1-\zeta\right)^2 \left(h'\left(\zeta\right)\zeta_q - h\left(\zeta\right)^2 \right) F\left(\zeta\right) = 0$, $(t,q) \in \mathbb{R}^+ \times [-D, 0]$ where $F\left(\zeta\right) = \exp \int_q^0 h\left(\zeta\left(t,s\right)\right) ds$ and $h\left(\zeta\right) = \frac{g(\zeta)}{1-\zeta}$. Traveling waves

in the unknown $z(t,u)$ turn out to correspond to stationary solutions in the unknown $\zeta(t,q)$. The following theorems are proved in [8].

Theorem 2. *For every value of total drop D, there exists exactly only one stationary traveling wave profile $\zeta(t,q) = Z_D(q)$, defined on $q \in [-D, 0]$.*

Theorem 3. *For every initial data (9), let D be its total drop, $z(t,u) = (S_t^g z_o)(u)$, $\zeta(t, q(t,u)) = z(t,u)$ where $q(t,u)$ is defined in (10), and $Z_D(q)$ be the unique stationary profile whose existence is guaranteed by Theorem 2, then one has the following convergence to the stationary traveling wave profile:*
$\lim_{t\to+\infty} \|\zeta(t,\cdot) - Z_D(\cdot)\|_{\mathbf{L^1}} = 0$.

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Asymptotic-preserving schemes for unusual long-time asymptotics ond 2D unstructured grids

Rodolphe Turpault

(joint work with C. Berthon, P. LeFloch, F. Marche, C. Sarazin)

We consider the long-time behaviour of hyperbolic systems supplemented by stiff source terms of the form:

$$\partial_t U + \mathrm{div} F(U) = -\sigma(U) R(U),$$

which satisfy the asumptions stated in [2]. In particular, they have a limit regime for $u = QU \in \mathbb{R}^n$ which writes:

$$\partial_t u - \mathrm{div}\Big(G(u, \nabla u)\Big) = 0. \tag{1}$$

Here, G is a nonlinear function.
The main purpose of this work concerns the numerical approximation of such systems through long-time asymptotic preserving schemes for both 1D and 2D geometries on unstructured grids. During the last decade, several schemes which can deal with scalar diffusive limits were proposed in the 1D case. However, none of these methods can be extended to the more difficult situations, even in 1D. Furthermore, the extension for unstructured grids in 2D is far from being straightforward in general. Indeed, only one such scheme has been designed for Friedrich-type systems (see [2]). On the contrary, we will show that a suitable extension of the scheme proposed in [1] naturally allows to tackle with these situations.

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Hyperbolic techniques for scalar degenerate parabolic problems

Boris Andreianov
(joint work with Mohamed Gazibo Karimou)

In the first part of the talk, we give a brief overview of the state of the art in the theory of scalar hyperbolic-parabolic-elliptic equations

$$(\mathbf{P}) \qquad b(u)_t + \mathrm{div}_x f(u) - \Delta_x \phi(u) = 0$$

with initial conditions and, when relevant, boundary conditions. In particular, the case $\mathbf{(H)}: \ b = \mathrm{Id},\ \phi \equiv 0$ is the hyperbolic scalar conservation law. Here b and ϕ are continuous non-strictly increasing nonlinearities; their regions on degeneracy sometimes correspond to different phases in the underlying physical models (flows in porous media, sedimentation, road traffic, Stefan and Hele-Shaw problems,...). The goal of the talk is to highlight hyperbolic techniques that remain applicable - and instrumental - for study of general degenerate parabolic equations **(P)**. We focus mainly in the hyperbolic-parabolic case ($b = \mathrm{Id}$).

Indeed, notions of solution and well-posedness techniques for **(P)** are inspired by the ones first established for the purely hyperbolic case **(H)**. Early works on the subject [19, 20] were carried out in the delicate BV setting; we refer to Evje and Karlsen [11] for a description of solutions and well-posedness results in this framework where existence is difficult to achieve. In the last fifteen years, the entropy formulation for L^∞ solutions [9], kinetic and renormalized formulations for L^1 solutions [10], [6] permitted to establish robust well-posedness and stability results. The important feature these formulations have in common is that they must keep track of parabolic dissipation associated with entropies (see, in particular, [20, 11, 9, 10, 6, 1]). Explicit representations of the dissipation measure (actually, different representations were chosen within the different approaches to

defining solutions) allow to go through the Kruzhkov doubling of variables method, establishing uniqueness and stability of solutions. Further, existence can be obtained by viscosity approximations or by convergent finite volume schemes (the latter ones are applicable when diffusion is approximated in a way compatible with the maximum principle, see [12]). At this stage, other "hyperbolic" techniques are useful, such as the measure-valued solutions or strong precompactness properties of sequences of solutions ([17]). Generalization to nonlinear diffusion operators can be handled within this approach and elliptic degeneracy can be taken into account following Alt and Luckhaus [2]. An example of well-posedness and structural stability study for a general triply degenerate parabolic problem in the framework of Carrillo entropy solutions can be found in [4]. Let us mention that recently, similar studies for nonlocal convection-diffusion problems have been developed ([1, 14]).

In the second part of this talk we concentrate on more recent trends related to taking into account boundary conditions for **(H)** and then for **(P)**: Dirichlet, zero-flux, obstacle... Some of the results can be found in the authors' work with K. Sbihi [3] and M. Gazibo Karimou [5], we also discuss the ongoing work with Gazibo and G. Vallet. Our goal is to set up techniques for treating initial-boundary value problems for **(P)** by comparing a general solution u to a "trace-regular" solution v. For trace-regular solutions, a strong interpretation of boundary condition can be given in terms of a suitable maximal monotone graph ([3]); uniqueness of such solutions is readily inferred (contrarily to the Otto's weak trace approach [15], see also [16, 18]). This program succeeded for the one-dimensional zero-flux problem

$$u_t+\Big(f(u)-\phi(u)_x\Big)_x=0 \ \text{ in } (0,T)\times(a,b), \ \ u|_{t=0}=u_0, \ \ \Big(f(u)-\phi(u)_x\Big)|_{x=a,b}=0$$

under the assumption $f(0)=0=f(u_{max})$, $0\le u_0\le u_{max}$ (see [5]), thus generalizing the result of Bürger, Frid and Karlsen [8] obtained for **(H)**. M. Gazibo also justified in [13] convergence of an implicit finite volume scheme (in the spirit of Eymard et al. [12]) for the zero-flux problem **(P)**. The key ingredients were:

- the obvious trace-regularity of solutions to the one-dimensional stationary problem $v+\Big(f(v)-\phi(v)_x\Big)_x=g$
- the notion of integral solution [7] to abstract evolution problems
- a new notion of "integral-process solution" to recast the nonlinear weak-* compactness idea exploited in [12] within the abstract framework of [7].

On-going works concern extension of this kind of techniques to various boundary conditions; they require obtention of wider classes of trace-regular solutions.

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Coupling, relaxation, adaptation of hyperbolic models

Edwige Godlewski
(joint work with C. Cancès, F. Coquel, H. Mathis, N. Seguin)

In the framework of the simulation of complex flows arising in industry, one may think for instance of coolant water circuits in pressurized water reactors (PWR), different models are used to treat the two-phase fluid flow in the different components. If a 3D model with turbulence is required in the core, simpler laminar 1D models are sufficient to describe the flow in most parts of the pipes in regular configuration, while in the case of loss-of-coolant accident, models should be able to describe these highly unsteady compressible flows with phase transition.

Generally speaking, the simulation of a complex system where one has to handle phenomena with different scales, both in time and space, involves several models, this means in turn several numerical codes, which need to be coupled. Moreover, this codes may already exist and have received some kind of validation, so they

cannot be changed in the coupling beyond the change in boundary conditions. Then, in order to reduce the computation time, and because the most refined models are too costly to implement everywhere, one would like to use them only when necessary. Which means the code should involve some indicator and adaptive procedure able to detect when some fine scale needs to be resolved.

Our aim is to understand on some simplified but relevant models the whole adaptation procedure, restricting ourselves to models of PDE of hyperbolic type which are discretized by finite volume methods. For that, we split the problem and address some key issues:

(1) identify suitable two-phase flow models
(2) define the interface coupling model
(3) derive numerical (finite volume) schemes with good stability properties
(4) describe the formal adaptation procedure,

together with some validation, either theoretical results for model problems or numerical experiments on some realistic situations.

In fact, the three first points have already received a lot of attention for the last few years, so we will mainly refer to some already published papers, only introducing a little more precisely the topics addressed in each paper. We restrict ourselves to a few selected papers in which many other references can be found.

Considering point (1), there exists a wide variety of models used for the simulation of two-phase flows in industrial thermo-hydraulic codes: mixture, drift (homogeneous or not), two-fluid or even multi-field models, in one or several dimensions. We are interested in exploiting some compatibility between models, either hierarchy which means that one model is obtained from the other by some known limiting process, such as relaxation to equilibrium, or asymptotic behavior when passing to the limit in some quantity (a flow entering a porous media, or large friction), or assuming some simplifying assumption: isentropic flow, slab symmetry, etc. For example [4] considers two Homogeneous Models (relaxation and equilibrium), while [3] proves a formal hierarchy between a two-fluid and a drift model; [14] contains theoretical results for a relaxation system approximating a *fluid system*; [11] studies the asymptotic behavior of solutions of Euler system with large friction.

Addressing point (3) is at the source of our interest in relaxation schemes [5], [2], [14], [11], [13], [1], [21]. Indeed, though the relaxation schemes we study result in very simple HLL type schemes, they can be understood as using a Godunov solver for a specific larger relaxation system. Then, these schemes inherit a good behavior directly from the PDE system together with the exact Godunov solver; the computations are explicit since the relaxation system has only linearly degenerate fields.

For point (2), since the codes should not be modified, we consider a thin interface and prescribe suitable boundary conditions on both sides of the coupling interface. This point raises some issues which have to be well understood:
• interface coupling in model cases and theoretical justification: scalar laws [15], [8]; linear systems [16]; the 2×2 Jin-Xin relaxation system and its scalar limit [9];
• coupling a 2D or 3D PDE system with its 1D version [17],[18];
• coupling of two systems of the same dimension with different fluxes [6];

• coupling of two systems of different dimensions.

In the second topic, one assumes some slab symmetry on one side, but the closure laws coincide. The third topic may illustrate the case a fluid flow where two different closure laws have been used in the modeling. The last one addresses the coupling of two *compatible* models, such as a relaxation system and its equilibrium limit as in [4], [9], or the full Euler system with energy and the isentropic Euler system [6]; and more recently an hyperbolic system with its diffusion limit [7].

We come now to point (4), which is quite challenging. The adaptation procedure computes an *adapted* solution at each time step t_n; it involves the coupling of a *coarse* model on a domain say D_c^n with a *fine* model in its complementary part, say D_f^n, at an interface $\overline{D}_c^n \cap \overline{D}_f^n$ which is found by some dynamical process. The ideas have been presented shortly in [20] for the 1D case, in the 'ideal' context of a relaxation system [12], coupled with its equilibrium limit. An indicator is found in the first order term of the Chapman-Enskog expansion, at least its discrete counterpart obtained by using a finite volume scheme. This *corrector* term naturally 'measures' the distance between the solution of the *fine* relaxation model with the *coarse* equilibrium limit. In [19], the adaption algorithm is detailed in the mutidimensional case, and illustrated by some convincing computations of compressible two-phase flows which do not strictly enter the 'ideal' setting.

We are also interested in the theoretical justification for a toy model, where the coarse model is a scalar law [10]. Though apparently simple, the situation is still tricky since the solution computed by the adaptation procedure necessarily involves discontinuity in the fluxes, and in order to use stability and error estimates, we need first to introduce some regularization, linked to a small parameter which has to be fitted wrt. the mesh size, the relaxation and the adaptation parameters.

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Dynamic capillarity models for porous media flows: travelling waves and non-classical shocks

Iuliu Sorin Pop

(joint work with Cornelis J. van Duijn, Yabin Fan, Lambertus A. Peletier)

1. Physical motivation

We investigate the existence of travelling waves for a mathematical model encountered in the flow of two fluid phases through a porous medium. In a simplified, dimensionless framework, given the total velocity of the flow, the model reads [8]

$$\partial_t u + \partial_x f(u) = \varepsilon\big(H(u)\partial_x P\big) \quad \text{for all } t > 0 \text{ and } x \in \mathbb{R}. \tag{1}$$

Here u is the saturation of one fluid (e.g. water) and P the dimensionless capillary pressure. The functions

$$f(u) = \frac{k_o(u)}{k_o(u) + Mk_w(u)}, \quad H(u) = k_w(u)f(u), \tag{2}$$

involve given monotone, bounded and continuous functions k_o, k_w. These are determined experimentally for $u \in [0, 1]$, and extended by constants outside this interval. Typically, the function f has a convex-concave profile.

Standard models assume that P depends only on u. Alternatively, one has [7]

$$P = P_e(u) + \tau \partial_t u. \tag{3}$$

P_e, determined experimentally, is the equilibrium component in the capillary pressure. The second term, involving a parameter $\tau \geq 0$, accounts for the dynamic effects. Common choices in the porous media literature [8] are

$$k_o(u) = u^{1+p}, k_w(u) = (1-u)^{1+q}, \text{ and } P_e(u) = (1-u)^{-\frac{1}{\lambda}},$$

where $p, q > 0$, $\lambda > 1$ and $M > 0$ are model specific.

Note the presence of two dimensionless numbers: the capillary number ε and the dynamic coefficient τ.

2. Trvelling waves and non-classical shocks

With the nonlinearities introduced in (2) and (3), (1) is a degenerate (pseudo) parabolic equation. In the limit $\varepsilon \searrow 0$, it becomes the Buckley-Leverett equation:

$$\partial_t u + \partial_x f(u) = 0 \text{ for all } t > 0 \text{ and } x \in \mathbb{R}, \tag{4}$$

a hyperbolic equation having (1) as physically motivated regularization. As in the classical theory of hyperbolic conservation laws, one can study the existence of travelling waves (TW) connecting two given states u_ℓ (the left state) and u_r (the right state) to define admissibility conditions for shock solutions to (4). Classical shocks are obtained for equilibrium models, when $\tau = 0$ in (3).

Motivated by the theory of non-classical shocks in [10], we study the existence of TWs for the non-equilibrium models, when $\tau > 0$. Clearly, these results will depend on τ. If for $\tau > 0$ the connectable states u_ℓ and u_r differ from those in the standard case $\tau = 0$, this justifies considering undercompressive shocks. Such shocks are motivated e.g. by dispersive waves [1]. Closely related is the liquid-vapour transition in porous media analyzed in [2]. Concerning (1), the case $H(u) \equiv 1$ and $P_e(u) = u$ is analyzed in [5]. A phase plane analysis based on a separation function is carried out in [12]. There saddle-to-saddle connections are identified for cases when classical (i.e. sufficiently smooth TW) exist.

Here we focus on the degenerate case, when at least one of the states is either 0 or 1. For these values, H in (1) is vanishing and the model degenerates. We restrict to the case $u_\ell > u_r$ and let α be the point where the tangent line through $(u_r, f(u_r))$ touches the graph of f. In this context, the following holds [4, 5]:

Theorem 1. *If $u_r < \alpha \leq u_\ell$, there exits $\tau_* > 0$ such that*

a. *If $0 \leq \tau \leq \tau_*$, TW connecting $u_\ell \leq \alpha$ to u_r exist and are monotone.*

b. *If $\tau > \tau_*$, there exists a unique $\bar{u}_\ell(\tau) > \alpha$ that can be connected to u_r through a monotone TW.*
c. *The $\bar{u} - \tau$ dependency is continuous and increasing for $\tau \geq \tau_*$.*

If $\tau > \tau_*$, let $u_\ell = \overline{u}(\tau)$. With $\underline{u}(\tau) \in (u_r, \alpha)$ being the intermediate intersection point of the graph of f with the chord $(u_\ell, f(u_\ell))$, $(u_r, f(u_r))$, then

d. *For each $u_\ell \in (u_r, \underline{u}(\tau)]$, there exists a TW connecting u_ℓ to u_r.*
e. *For each $u_\ell \in (\underline{u}(\tau), \bar{u}(\tau))$, no TW connecting u_ℓ to u_r exist.*
f. *For each $u_\ell \in (\underline{u}(\tau), \bar{u}(\tau))$, TW connecting u_ℓ to $\bar{u}(\tau)$ are possible.*

For fixed u_r, Fig. 1 left displays $\bar{u}(\tau)$ and $\underline{u}(\tau)$ for various τ. These are computed by a shooting method (see [4]).

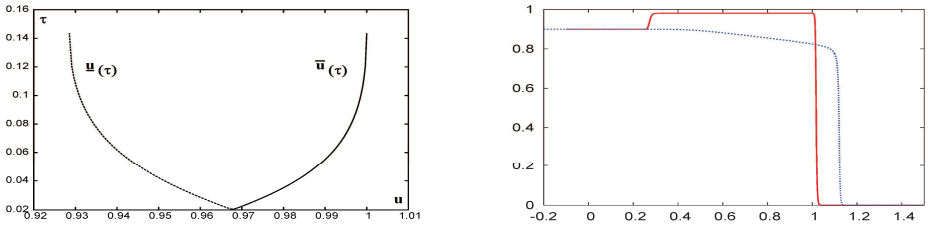

Fig. 1. Left: The $\bar{u}(\tau)$ and $\underline{u}(\tau)$ curves (left). Right: numerical solutions for the full model (1), $\tau > \tau_*$ (solid) and $\tau = 0$ (dotted).

As $\varepsilon \to 0$, a TW in Theorem 1, case b becomes a shock for (1) that violates the standard Oleinik entropy condition. For case f, a solution of the Riemann problem for (4) where the initial condition involves u_ℓ and u_r combines two shocks: one upwards from u_ℓ to $\bar{u}(\tau)\}$, and one downwards from $\bar{u}(\tau)$ to u_r. This is different from the classical construction, where the solution is a shock down to u_r, preceded by a rarefaction wave from u_ℓ to the tangent point α if $u_\ell > \alpha$. This behaviour is displayed in Fig. 1 right, presenting numerical solutions for the regularized models with $\tau > \tau_*$, respectively $\tau < \tau_*$ (see [3, 6] for details). For computing the limit shocks, a heterogeneous multiscale method is developed in [9].

A particular outcome of the degeneracy is observed in Fig. 1 left, where the branches are plotted up to a maximal value τ^*. As $\tau \nearrow \tau^*$, $\bar{u}(\tau) \nearrow 1$, where H vanishes. This prevents solutions to become larger than 1, and smooth TW connecting $u_\ell = 1$ to u_r do not exist anymore if $\tau > \tau^*$, [4]. Then non-smooth TW solutions (sharp waves) as defined [11] are considered. These waves remain 1 up to a certain argument η_1, where they start decreasing to u_r. At η_1, these waves have a discontinuous derivative. More special is the doubly degenerate case, when $u_r = 0$ and $\tau > \tau^*$ implying that $u_\ell = 1$. Then the waves may have another discontinuity in the derivative at $\eta_0 > \eta_1$, when u approaches 0. This situation is analyzed in detail in [4], and a selection criterion is introduced based on regularizing the states. The resulting waves have a kink at η_1 but approach 0 smoothly at some $\eta_0 \in (\eta_1, \infty)$. This situation is illustrated in Fig. 2, when the wave is computed for $\tau > \tau^*$ and for $u_\ell = 1$, $u_r = 0$. In the zoomed views, a kink appearing at η_1 and the smooth transition to 0 are observed.

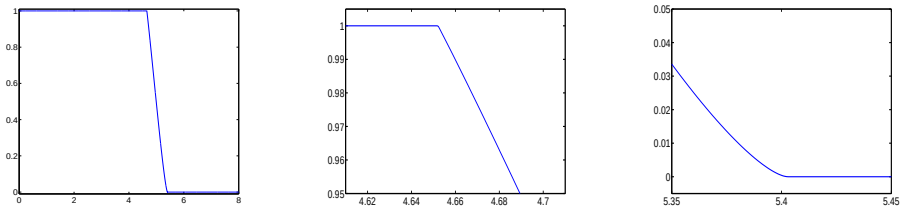

Fig. 2. The TW for $\tau > \tau^*$, $u_\ell = 1$ and $u_r = 0$ (left). Zoomed views: the transition from $u = 1$ to $u < 1$ (kink, middle) and from $u > 0$ to $u = 0$ (smooth, right).

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On two problems involving nonlocal hyperbolic conservation laws

Paulo Amorim

(joint work with Rinaldo Colombo, Andreia Teixeira)

In this work, we consider two problems involving nonlocal hyperbolic conservation laws. The first problem is about giving a sense to a hyperbolic conservation law with a constraint on the gradient of the solution. This is not straightforward since conservation laws naturally develop discontinuities. We give a sense to this problem by considering a related conservation law with a nonlocal term, and prove

existence, uniqueness and stability of solution of the Cauchy problem. This is then used to interpret the gradient constraint problem as a free boundary problem.

In the second part, we consider a class of nonlocal conservation laws used in recent years to study crowd dynamics. Extending results in [3], we present a convergence result for numerical approximations of this nonlocal model.

0.1. **A hyperbolic conservation law with a gradient constraint.** [1] In some models involving nonlinear conservation laws, physical mechanisms exist which prevent the formation of shocks, see [4]. This gives rise to conservation laws with a constraint on the gradient of the solution. We approach this problem by studying a related conservation law with a spatial nonlocal term.

More specifically, we consider the initial value problem for the scalar conservation law with an integral term

$$\begin{aligned} &\partial_t w(t,x) + \partial_x \Big(f'\Big(\int_{-\infty}^{x} w(t,z)dz \Big) g(w(t,x)) \Big) = 0, \\ &w(0,x) = w_0(x), \qquad x \in \mathbb{R}, \quad t \geq 0, \end{aligned} \tag{1}$$

with $f, g : \mathbb{R} \to \mathbb{R}$ given functions. The formulation (1) is motivated by the following problem: to find $u(x,t)$ verifying, in some appropriate sense, the following *conservation law with a gradient constraint,*

$$\begin{aligned} &\partial_t u + \partial_x (f(u)) = 0, \qquad u(0,x) = u_0(x), \\ &|\partial_x u(t,x)| \leq M, \qquad x \in \mathbb{R}, \quad t \geq 0, \end{aligned} \tag{2}$$

for some $M > 0$.

Problem (2) has no ready interpretation in the scope of conservation laws. Indeed, as is well known, a nonlinear conservation law will, even for smooth initial data, develop discontinuities in finite time, whose onset is preceded by a blowup of the spatial derivative. Therefore, it is hopeless to seek solutions verifying problem (2), without some additional information.

We now describe our interpretation of problem (2), which leads to the formulation (1). Setting $w := \partial_x u$, then spatial differentiation of (2) gives formally

$$\begin{aligned} &\partial_t w + \partial_x \Big(f'\Big(\int_{-\infty}^{x} w(t,z)dz \Big) w \Big) = 0, \\ &|w(t,x)| \leq M, \end{aligned} \tag{3}$$

so that the constraint on the gradient of u now acts on the function w itself. We provide a formulation for this problem by replacing the term w appearing inside the spatial derivative in (3) by a function *with compact support* $g(w)$. This limits the growth of $|w|$ (and, formally, of the gradient $|\partial_x u|$). Thus, we arrive at problem (1).

In [1], we establish existence, uniqueness, and continuous dependence results for problem (1) through a time-stepping (or layering) argument. In turn, this gives the following result for the problem with gradient constraint (2).

Theorem. Suppose that the function g in (1) has the form $g(w) = wh_\epsilon(w)$, where $h = h_\epsilon$ is some regularization of the characteristic function $\mathbb{1}_{[-M,M]}$, with $h = 1$ on $[-M+\epsilon, M-\epsilon]$, and let w be the solution of (1). Define the following sets contained in $(0,T)\times\mathbb{R}$,

$$I_\epsilon = \{(t,x) : |w| \le M-\epsilon\},$$
$$J_\epsilon = \{(t,x) : |w| = M\},$$
$$K_\epsilon = ((0,T)\times\mathbb{R}) \setminus (I_\epsilon \cup J_\epsilon).$$

Then, the function $u := \int_{-\infty}^{x} w(t,y)\,dy$ solves the conservation law with gradient constraint (2) in the sense that u is an entropy solution of

$$\begin{aligned} &\partial_t u + \partial_x(f(u)) = 0, \\ &|\partial_x u| < M, \end{aligned} \qquad \text{on} \quad I_\epsilon,$$

u verifies

$$|\partial_x u| = M \qquad \text{on} \quad J_\epsilon,$$

and u solves

$$\partial_t u + \partial_x(f(u))h(w) = 0$$

on the transition layer K_ϵ. Furthermore, u verifies the estimate

$$u \in L^\infty(0,T; W^{1,\infty}(\mathbb{R})).$$

In particular, u is continuous on $\mathbb{R}$ for each t.

0.2. **Convergence of a numerical scheme for nonlocal conservation laws.** This is joint work with Rinaldo Colombo (Brescia) and Andreia Teixeira (Lisboa).

We study in [2] a rather general class of 1D nonlocal conservation laws from a numerical point of view. First, following [3], we define an algorithm to numerically integrate them and prove its convergence. Then, we use this algorithm to investigate various analytical properties, obtaining evidence that usual properties of standard conservation laws fail in the nonlocal setting. Moreover, on the basis of our numerical integrations, we are led to conjecture the convergence of the nonlocal equation to the local ones, although no analytical results are, to our knowledge, available in this context.

Thus, we consider the Cauchy problem

$$\begin{cases} \partial_t \rho + \partial_x \left(f(t,x,\rho)\, v(\rho * \eta) \right) = 0 \\ \rho(0,x) = \rho^o(x) \end{cases} \qquad (t,x) \in \mathbb{R}^+ \times \mathbb{R} \tag{4}$$

and discretize it using an easy adaptation of the Lax–Friedrichs scheme.

Under an appropriate CFL condition, we obtain the following properties for the numerical scheme:

- Positivity
- L^1 estimate:
$$\|\rho^n\|_{L^1} \le \|\rho^o\|_{L^1}$$
- L^∞ estimate:
$$\|\rho^n\|_{L^\infty} \le e^{\mathcal{L}t}\|\rho^o\|_{L^\infty}$$

- Total variation estimate:

$$\sum_{j\in\mathbb{Z}} |\rho^n_{j+1} - \rho^n_j| \le \Big(\mathcal{K}_2\, t + \sum_{j\in\mathbb{Z}} |\rho^o_{j+1} - \rho^o_j|\Big) e^{\mathcal{K}_1 t}.$$

The constants $\mathcal{K}_{1,2}, \mathcal{L}$ depend on various norms of the data.

With an additional discrete entropy inequality, we obtain strong convergence of the numerical scheme towards the unique entropy solution of (4).

In the standard (local) case, when the flow is independent of t and x, the L^∞ norm and the total variation of the solution are well know to be non-increasing functions of time. Here, on the contrary, the total variation and the L^∞ norm of the solution to (4) may well sharply increase due to the nonlocal terms, even when the flow is independent from t and x.

Thus, we verify numerically in [2] that the usual properties of scalar conservation laws fail in this nonlocal setting. Moreover, these behaviors are consistent with the rigorous estimates obtained for the numerical solutions.

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Relative entropy methods for relaxation limits

Corrado Lattanzio

(joint work with A.E. Tzavaras)

Our aim is to analyze (singular) relaxation limits, in particular in the diffusive regimes, in connection with relative entropy methods. These techniques has been firstly utilized in the context of hyperbolic systems of conservation or balance laws equipped with dissipative convex entropies by Dafermos [2] and DiPerna [3], thus obtaining a remarkable stability formula. This property can be then utilized to obtain weak–strong uniqueness and stability properties for solutions of the systems under investigation. Lately, the same calculation has been carried out to compare weak entropy solutions of a relaxation systems with smooth solutions of its formal limit, in the case of hyperbolic relaxation (i.e. [1, 4, 6]), and, more recently, also in the case of a diffusive limits [5].

In this contribution, we shall generalize the results obtained in [5] to the framework of a large friction theory converging toward gradient flows:

$$(1)\qquad \begin{cases} \rho_t + \mathrm{div}_x(\rho u) = 0 \\ \\ (\rho u)_t + \mathrm{div}_x(\rho u \otimes u) = -\dfrac{1}{\varepsilon^2}\rho\left(u + \nabla_x \dfrac{\delta \mathcal{E}[\rho]}{\delta \rho}\right), \end{cases}$$

whose formal limit is given by

$$(2)\qquad \rho_t + \mathrm{div}_x(\rho u) = 0,\ u = -\nabla_x \frac{\delta \mathcal{E}[\rho]}{\delta \rho}$$

and $\mathcal{E}[\rho]$ is a convex functional. Starting from the energy dissipation for (1)

$$\frac{d}{dt}\left(\mathcal{E}[\rho] + \frac{\varepsilon^2}{2}\int \rho|u|^2 dx\right) + \int \rho|u|^2 dx = 0,$$

we obtain the following relative entropy estimate for any pairs of solutions $(\rho, \rho u)$ and $(\bar\rho, \bar\rho\bar u)$ of that system, the latter being smooth:

$$\begin{aligned} &\frac{d}{dt}\left(\mathcal{E}[\rho\,|\bar\rho] + \int \frac{\varepsilon^2}{2}|u-\bar u|^2 dx\right) + \int \rho|u-\bar u|^2 dx \\ &\quad = -\int \left(\varepsilon^2 \rho\big((u-\bar u)\otimes(u-\bar u)\big)\nabla_x \bar u + (\mathrm{div}_x\,\bar u)\mathcal{F}[\rho\,|\bar\rho]\right) dx, \end{aligned}$$

where $\mathcal{F}[\rho] := \langle \frac{\delta \mathcal{E}[\rho]}{\delta \rho}, \rho\rangle - \mathcal{E}[\rho]$ stands for the "pressure" functional. In addition, from that relation, we also obtain the following relation for solutions ρ, $\bar\rho$ ($\bar\rho$ smooth) of (2):

$$\frac{d}{dt}\mathcal{E}[\rho\,|\bar\rho] + \int \rho\left|\nabla_x \frac{\delta \mathcal{E}[\rho]}{\delta \rho} - \nabla_x \frac{\delta \mathcal{E}[\bar\rho]}{\delta \rho}\right|^2 dx = -\int (\mathrm{div}_x\,\bar u)\mathcal{F}[\rho\,|\bar\rho] dx,$$

which is the relative entropy estimate related to the usual energy dissipation for this gradient flow dynamics:

$$\frac{d}{dt}\mathcal{E}[\rho] + \int \rho\left|\nabla_x \frac{\delta \mathcal{E}[\rho]}{\delta \rho}\right|^2 dx = 0.$$

Finally, the stability of the diffusive relaxation limit, and in particular its rigorous justification, can also be achieved by using the following relative entropy estimate

$$\begin{aligned} &\frac{d}{dt}\left(\mathcal{E}[\rho\,|\bar\rho] + \int \frac{\varepsilon^2}{2}|u-\bar u|^2 dx\right) + \int \rho|u-\bar u|^2 dx \\ &\quad = -\int \left(\varepsilon^2 \rho\big((u-\bar u)\otimes(u-\bar u)\big)\nabla_x \bar u + (\mathrm{div}_x\,\bar u)\mathcal{F}[\rho\,|\bar\rho]\right) dx \\ &\qquad - \int\ \varepsilon^2 \rho(u-\bar u)\cdot\frac{\bar e}{\bar\rho} dx, \end{aligned}$$

where this time $(\rho, \rho u)$ is any solution of (1), $\bar\rho$, $\bar u$ is a smooth solution of (2), and the error term $\bar e$ is given by $\bar e = (\bar\rho\bar u)_t + \mathrm{div}_x(\bar\rho\bar u\otimes\bar u)$.

The above framework applies in particular to the case of the Euler equations with large friction converging toward the porous medium equation [5], and to the case of the diffusive relaxation limit of Keller–Segel type models. The latter consists in the study of the singular limit $\varepsilon \to 0$ of the following hyperbolic–elliptic system

$$\begin{cases} \rho_t + \dfrac{1}{\varepsilon} \operatorname{div}_x m = 0 \\ \\ m_t + \dfrac{1}{\varepsilon} \operatorname{div}_x \dfrac{m \otimes m}{\rho} + \dfrac{1}{\varepsilon} \nabla_x p(\rho) = -\dfrac{1}{\varepsilon^2} m + \dfrac{\mathcal{X}}{\varepsilon} \rho \nabla_x c \\ \\ -\triangle_x c + \beta c = \rho - \langle \rho \rangle, \end{cases}$$

where $t \in \mathbb{R}$, $x \in \mathbb{T}^n$, $\rho \geq 0$, $c \in \mathbb{R}$, $m \in \mathbb{R}^n$ and the pressure $p(\rho)$ satisfies $p'(\rho) \geq 0$. For the sake of simplicity, we can assume the usual γ–law case for the pressure, that is $p(\rho) = k\rho^\gamma$, $k > 0$ and $\gamma \geq 1$. Moreover, the screening coefficient β is nonnegative and the chemosensitive coefficient $\mathcal{X}$ is strictly positive. In addition, the elliptic equation above is equipped with periodic boundary conditions and with the constraint $\langle c \rangle = 0$ if $\beta = 0$; here the notation $\langle g \rangle$ stands for the mean of g. The formal limit of that system is given by

$$\begin{cases} \bar{\rho}_t + \operatorname{div}_x \left(\bar{\rho} \nabla_x (\mathcal{X} \bar{c} - h'(\bar{\rho})) \right) = 0 \\ -\triangle_x \bar{c} + \beta \bar{c} = \bar{\rho} - \langle \bar{\rho} \rangle, \end{cases}$$

where $h''(\rho) = \frac{p'(\rho)}{\rho}$. Then, the rigorous justification of this convergence by means of the relative entropy estimate can be obtained under suitable conditions on γ, $\beta \geq 0$, $\mathcal{X} > 0$ and on the mean $\langle \rho \rangle = \langle \bar{\rho} \rangle$.

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Error estimates for well-balanced Godunov schemes on scalar balance laws

Debora Amadori
(joint work with Laurent Gosse)

The main goal of this report is to compare two different approaches in the approximation of the scalar equation

$$\partial_t u + \partial_x f(u) = k(x)g(u)\,, \qquad k \in L^\infty_{loc}(\mathbb{R})\,, \tag{1}$$

namely the Time-Splitting (TS) and Well-Balanced (WB) approach.

In summary, the Time-Splitting approach consists in the approximation of (1) by treating in a separate way, as time evolves, the dynamic of the convective equation

$$\partial_t u + \partial_x f(u) = 0\,, \tag{2}$$

and the one given by the source term:

$$\partial_t u = k(x)g(u)\,. \tag{3}$$

On the other hand, in the Well Balanced approach one considers (2) together with the spatial dynamic given by the stationary equation:

$$\partial_x f(u) = k(x)g(u)\,. \tag{4}$$

Error estimates for time splitting operators were proved in [6, 5]:

$$\|u^{\Delta t}(t,\cdot) - u(t,\cdot)\|_{L^1(\mathbb{R})} \;\le\; C\,\exp(\sup[k(x)g'(u)]t)\,\sqrt{\Delta t}\,.$$

In these proofs, the Gronwall Lemma is applied and an exponential increase is possible when the source term is accretive, that is when

$$\sup_{x,u} k(x)g'(u) > 0 \tag{5}$$

for the solution under consideration. Numerical proofs show that the exponential amplification can actually occur, see Fig. 1.

In this note we provide rigorous estimates for a WB approximation of Godunov type. We will show that the error grows at most only **linearly** in time.

1. The main estimate

We provide (1) with initial data $u_0 \in BV_{loc}(\mathbb{R})$ for $t = 0$ and assume that the initial data are located in an invariant domain for the equation (see [1]). We assume that f, g are smooth and that

$$\inf_u f'(u) > 0\,. \tag{6}$$

Moreover we assume that the source term is accretive, see (5). Let L be an a-priori bound on f'. For $\Delta x > 0$, let a uniform Cartesian grid be given with mesh-width Δx and time-step Δt, that satisfy the relation $L\Delta t = \Delta x$. Then we define a WB

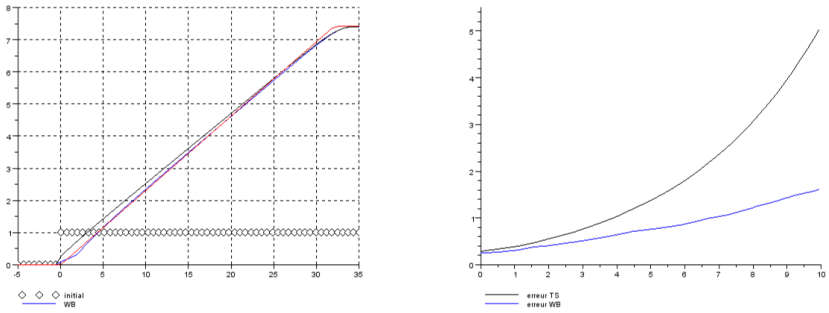

FIGURE 1. Time-evolution of the L^1 error for a rarefaction wave solution of Burgers equation.

scheme for equation (1), that basically consists in a Godunov approximation for the 2×2 non-conservative system

$$\partial_t u + \partial_x f(u) - g(u)\partial_x a = 0\,, \qquad \partial_t a = 0\,,$$

where a is an antiderivative of k:

$$\partial_x a(x) = k(x)\,.$$

Assumption (6) ensures that this system is strictly hyperbolic.

Let $u^{\Delta t}$ be this WB approximation of (1) (see [1] for precise definitions). For a specific Riemann invariant w, a local L^1 error holds for $u^{\Delta t}(t,\cdot)$: for all $t > 0$ and $x_1 < x_2$, one has

$$\int_{x_1}^{x_2} |u^{\Delta t}(t,x) - u(t,x)|dx \quad \leq \quad \min\{E_1, E_2\} \tag{7}$$

where

$$\begin{aligned} E_1(\Delta x, t) &= C\,\mathrm{e}^{\kappa\|k\|_{L^1}}\Big(\Delta x(\mathrm{TV}\{u_0\}+1) + \mathrm{TV}\{w_0\}\,L\,t\Big)\,,\\ E_2(\Delta x, t) &= \sqrt{\Delta x}\,\sqrt{A} + \Delta x\,B\,. \end{aligned}$$

Here C and κ are independent of time and Δx, while A and B depend on time as follows:

$$\begin{aligned} A(t) &= (\mathrm{TV}\{w_0\} + \|k\|_{L^1}) \cdot \Big((L+1)\,\mathrm{TV}\{u_0\}\,\mathrm{e}^{Nt} + \|k'g\|_\infty h(t)\Big) \cdot h(t)\,,\\ B(t) &= \mathrm{TV}\{u_0\}\,\mathrm{e}^{Nt} + \|k\|_\infty\,(\mathrm{TV}\{w_0\} + \|k\|_{L^1})\,h(t) \end{aligned}$$

being

$$\begin{aligned} h(t) &= \frac{\mathrm{e}^{Nt}-1}{N}\,,\\ N &= \sup\big\{k(x)g'(\xi)\,,\ x\in\mathbb{R}\,,\ |\xi| \leq \max\{\|u\|_\infty, \|u^{\Delta t}\|_\infty\}\big\} > 0\,. \end{aligned}$$

Here above, $w_0 = w_0(x)$ denotes the Riemann invariant at time $t = 0$. The L^1, L^∞ norms and TV above are referred to the interval $(x_1 - Lt, x_2)$.

The term E_1 grows linearly in time, while E_2 increases exponentially as time evolves:

$$E_2(\Delta x, t) \ \leq \ C_1 \sqrt{\Delta x}\, \mathrm{e}^{Nt}$$

for some constant C_1 depending on the coefficients and on the initial data. Hence the estimate (7) displays a linear growth in time beyond a certain time.

In terms of Δx, notice that E_1 does not vanish as $\Delta x \to 0$, differently from E_2. Therefore, the increased time accuracy is more effective when dealing with medium size grids, which is actually what occurs in practice.

While the error term E_2 is obtained through a classical Kuznetsov argument, [4], for the term E_1 a new approach is proposed: namely, the use of a decaying Lyapunov functional equivalent to the L^1 norm of two approximate (wave-front tracking) solutions ([2]).

The existence of such a functional for equation (1) was established in [3]. In reference [1], we employ successfully this tool to estimate the error between the Well-Balanced-Godunov solution and the exact one.

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Numerical comparison of coupling conditions for the isothermal Euler equations in tubes of varying cross sections

Raul Borsche

(joint work with Rinaldo M. Colombo, Mauro Garavello)

Consider the coupling of two circular tubes of different sizes. We compare numerically the description by the $3D$ isothermal Euler equations with descriptions by the $1D$ isothermal Euler equations equipped with different coupling consditions.

The governing equations are

$$\begin{cases} \partial_t \rho + \div(\rho\, v) = 0 \\ \partial_t(\rho\, v) + \div\,(\rho\, v \otimes v + p(\rho)\mathbf{I}) = 0 \end{cases} \quad \text{and} \quad \begin{cases} \partial_t \rho_i + \partial_x q_i & = & 0 \\ \partial_t q_i + \partial_x \left(\frac{{q_i}^2}{\rho_i} + p(\rho_i) \right) & = & 0 \end{cases},$$

where $i = l, r$ denotes the left and right tube. ρ is the fluid mass density, v is the fluid speed, $q = \rho\, v$ is its linear momentum density and $p = p(\rho)$ is its pressure. To complete the description in the $1D$ setting, coupling conditions of the form

$$\Phi\left(a_l, \rho_l(t, 0-), q_l(t, 0-); a_r, \rho_r(t, 0+), q_r(t, 0+)\right) = 0$$

have to be prescribed, where a_l and a_r denote the cross sectional areas of the left and right pipe. There exist various choices of the function Φ in the literature, see for instance [3, 1, 2, 4].

In the case of the trivial stationary state $q \equiv 0$ most of the $1D$ descriptions coincide with the $3D$ solutions. In the non-static cases we compare numerically the behavior of the different $1D$ descriptions with the solutions of the $3D$ model. As examples we study, starting from a stationary state, either shock or rarefaction waves passing the junction from both sides. In all considered test cases the $3D$ solutions show only in a small neighborhood of the jump in the area a spatially dependent pattern. Aside from the junction a $1D$ description of the flow seems appropriate. In each example there is always one candidate in set of considered coupling conditions, which captures quite accurately the global behavior, but there is none of them which is next to the $3D$ solutions in all cases.

Thus there is a possibility to describe the flow by $1D$ equations, but for a general description the design of the coupling conditions is currently under investigation.

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Large time step and asymptotic preserving numerical schemes for the gas dynamics equations with source terms

Christophe Chalons

(joint work with M. Girardin, S. Kokh)

We are interested in the simulation of subsonic compressible flows in a specific regime where the (main) driving phenomena are stiff source terms and material transport. More precisely, we consider the system of gas dynamics with external body forces and friction. Such flow configuration may be encountered in several industrial processes like the flows involved within the core of a nuclear power plant.

We propose here a method that fulfils our task by addressing three issues. First we require our method to enable the use of large time steps in order to avoid classic Courant-Friedrichs-Lewy (CFL) restriction based on the (fast) acoustic waves of the model. Second, we want our method to accurately approximate (slow) waves that account for material transport. Third, the discretization of the stiff source terms with large time steps may severely affect the accuracy of the method. We propose to overcome this difficulty by imposing an even stronger property on our numerical scheme. Indeed, when one considers the asymptotic regime obtained for both long time and large friction coefficients, the solution of the system is formally expected to behave like the solution of a typical parabolic system. We aim at deriving a scheme that preserves this property for the discrete approximation of the solution. Such property is referred to as an asymptotic preserving (AP) property. Since its introduction in the pioneer work of [5, 6], the notion of AP numerical schemes has been investigated and implemented in the past years in a wide range of context stemming from hydrodynamics with radiative transfer, multiscale kinetics, diffusive limit of the transport equation, to problems similar to our model. We refer the reader to the recent and abundant literature on this topic for more details.

Classic means to fulfil our first requirement for avoiding CFL based on the acoustic waves, consists in deriving an implicit in time discretization. Unfortunately this usually induces more numerical diffusion, including for the approximation of the material waves. In order to meet both first and second needs, we propose a mixed implicit-explicit strategy: the terms responsible for the acoustic waves receive a time implicit treatment while the ones responsible for the transport waves are treated by an explicit update. This task is achieved by means of a Lagrange-Projection algorithm as in [3]. This approach provides a natural decoupling of the acoustic waves and the material waves. An approximation based on a relaxation strategy (see for instance [1] but also the pioneering references therein) provides a simple mean to circumvent the nonlinearities involved with the equation of state of the fluid.

The effects of both gravity and friction source terms are then incorporated into the solver thanks to the concept of simple approximate Riemann solver and consistency with the integral form introduced by Gallice [4]. This powerful method allows to account for both source terms and convective fluxes at the same time. It has been applied in [2] for deriving an explicit scheme for the same flow model. We shall show that this approach can be followed within a mixed implicit-explicit framework for designing an AP scheme.

To conclude and summarize, we propose a *large time step* and *asymptotic preserving* scheme for the gas dynamics equations with external forces and friction terms. By asymptotic preserving, we mean that the numerical scheme is able to reproduce at the discrete level the parabolic-type asymptotic behaviour satisfied by the continuous equations. By large time-step, we mean that the scheme is stable under a CFL stability condition driven by the (slow) material waves, and not by the (fast) acoustic waves as it is customary in Godunov-type schemes. Numerical

evidences are proposed and show a gain of several orders of magnitude in both accuracy and efficiency.

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Balance laws coupled with ordinary differential equations

Mauro Garavello

(joint work with Raul Borsche, Rinaldo M. Colombo)

We consider a system of hyperbolic balance laws coupled, through the boundary data, with ordinary differential equations. More precisely, we study the following system

$$
\begin{cases}
\partial_t u + \partial_x f(u) = g(u) & t > 0,\, x > \gamma(t) \\
\dot w(t) = F\left(t, u(t, \gamma(t)), w(t)\right) & t > 0 \\
\dot\gamma(t) = \Pi(w(t)) & t > 0 \\
b(u(t, \gamma(t))) = B\left(t, w(t)\right) & t > 0 \\
u(0, x) = u_o(x) & x > x_o \\
w(0) = w_o & \\
\gamma(0) = x_o &
\end{cases} \tag{1}
$$

where $u \in \mathbb{R}^n$, $w \in \mathbb{R}^m$ and $\gamma \in \mathbb{R}$. The balance law $\partial_t u + \partial_x f(u) = g(u)$ is defined for $t > 0$ and $x > \gamma(t)$ while $\gamma(t)$ represents the position of a moving boundary at time t, which is not given a priori. Note that the solution to this partial differential equation depends on the position of the boundary and on the solution to the ordinary differential equation $\dot w(t) = F\left(t, u(t, \gamma(t)), w(t)\right)$ through the boundary data $b(u(t, \gamma(t))) = B\left(t, w(t)\right)$. Moreover the solution w to the ordinary differential equation depends on the trace $u(t, \gamma(t))$ of the solution of the balance laws. Here we present a result about the well posedness of solutions to (1) and also about the stability, i.e. the continuous dependence of the solution with respect to small variations of the source term g, of the vector fields F and Π and of the boundary term B.

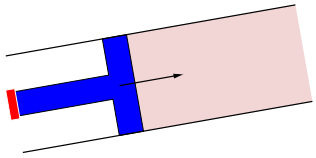

FIGURE 1. A fluid in a pipe with a piston at one end.

The present framework comprehends, for instance, the Eulerian description of a fluid in a pipe with a piston at one end, which leads to the following system (see Figure 1)

$$
(2)\qquad \begin{cases} \partial_t \rho + \partial_x q = 0 \\ \partial_t q + \partial_x \left(\frac{q^2}{\rho} + p(\rho) \right) = -\nu \frac{q|q|}{\rho} - g\,\rho\,\sin\alpha \\ V(t) = \dfrac{q\,(t, \gamma(t)+)}{\rho\,(t, \gamma(t)+)} \\ \dot V = p_{\mathrm{ext}}(t) - p\,(\rho\,(t, \gamma(t)+)) - g \sin\alpha \\ \dot\gamma(t) = V(t)\,. \end{cases}
$$

Here, ρ is the gas density, q is its linear momentum density, $p = p(\rho)$ is a pressure law playing the role of the gas equation of state, V is the piston speed and γ its position. Friction is described by the term $-\nu q\,|q|\,/\rho$, with ν being a suitable constant, while p_{ext} denotes the external pressure. The slope of the pipe is α, while g is gravity.

1. Analytical Results

Throughout, we denote $\mathbb{R}^+ = [0, +\infty[$. Let $\Omega \subseteq \mathbb{R}^n$ be an open set. With $B_r(w)$ we denote the open ball centered at w with radius r. Fix the reference states $\hat u \in \Omega$, $\hat w \in \mathbb{R}^m$ and a point $\hat x \in \mathbb{R}$. Define, for all positive δ, the sets

$$
(3)\quad \mathcal{U} = \left\{ u \in \hat u + (BV \cap L^1)(\mathbb{R}; \mathbb{R}^n) \colon u(\mathbb{R}) \subset \Omega \right\}, \quad \mathcal{U}_\delta = \{ u \in \mathcal{U} \colon TV(u) \le \delta \}\,.
$$

On system (1) we require the following conditions, where we refer to [5, 7] for the standard vocabulary about conservation laws.

- **(f):** $f \in C^4(\Omega; \mathbb{R}^n)$ is smooth and such that, for all $u \in \Omega$, $Df(u)$ is strictly hyperbolic and each characteristic field is either genuinely nonlinear or linearly degenerate. For $u \in \Omega$ and $i = 1, \dots, n$, call $\lambda_i(u)$ the i-th eigenvalue of $Df(u)$ and $r_i(u)$ the corresponding right eigenvector. We may assume that $\lambda_{i-1}(u) < \lambda_i(u)$ for all $u \in \Omega$ and $i = 2, \dots, n$.
- **(g):** For $\delta_o > 0$, $g \colon \mathcal{U}_{\delta_o} \to L^1(\mathbb{R}; \mathbb{R}^n)$ satisfies $\|g(u) - g(u')\|_{L^1} \le L_1 \|u - u'\|_{L^1}$ and $TV\,(g(u)) \le L_2$ for suitable $L_1, L_2 > 0$, and for every $u, u' \in \mathcal{U}_{\delta_o}$.
- **(Π):** $\Pi \in C^{0,1}(\mathbb{R}^m; \mathbb{R})$.
- **(NC):** There exist $c > 0$ and $\ell \in \{1, 2, \dots, n-1\}$ such that $\lambda_\ell(\hat u) < \Pi(\hat w) - c$ and $\lambda_{\ell+1}(\hat u) > \Pi(\hat w) + c$.
- **(b):** $b \in C^1(\Omega; \mathbb{R}^{n-\ell})$ is such that $\det\left(D_u b(\hat u)\,[r_{\ell+1}(\hat u) \quad \cdots \quad r_n(\hat u)]\right) \neq 0$.

(F): The map $F : \mathbb{R}^+ \times \Omega \times \mathbb{R}^m \longrightarrow \mathbb{R}^m$ is a Carathéodory function such that it is locally Lipschitz continuous with respect to u and w, uniformly in t and it has a sublinear growth with respect to w.

(B): $B \in C^1(\mathbb{R}^+ \times \mathbb{R}^m; \mathbb{R}^{n-\ell})$ is locally Lipschitz continuous.

Remark that condition **(NC)** is a Non Characteristic condition, while condition **(b)** is the usual assumption on the assignment of boundary data in a non characteristic problem for a conservation law, see for instance [1, 2, 3, 6].

Definition 1. *Let $T > 0$ and the state $\hat{u}$ be fixed. A triple (u, w, γ) with*

$$u \in C^0\left([0,T];\mathcal{U}\right) \qquad w \in W^{1,1}\left([0,T];\mathbb{R}^m\right) \qquad \gamma \in W^{1,\infty}\left([0,T];\mathbb{R}^m\right)$$

is a solution to (1) on $[0,T]$ with initial datum (u_o, w_o, x_o) such that $u_o \in \mathcal{U}$ with $u_o(x) = \hat{u}$ for $x < x_o$, $w_o \in \mathbb{R}^m$ and $x_o \in \mathbb{R}$, if

(1) u solves

$$\begin{cases} \partial_t u + \partial_x f(u) = g(u) & (t,x) \in \mathbb{R}^+ \times]\gamma_*(t), +\infty[\\ b\left(u\left(t, \gamma_*(t)+\right)\right) = B_*(t) & t \in \mathbb{R}^+ \\ u(0,x) = u_o(x) & x \in \mathbb{R}^+. \end{cases}$$

on $[0,T]$ with $B_*(t) = B\left(t, w(t)\right)$, $\gamma_*(t) = \gamma(t)$ and initial datum u_o;

(2) $w(t) = w_o + \int_0^t F_*\left(\tau, w(\tau)\right) d\tau$ for a.e. $t \in [0,T]$ with $F_*(t)$ given by $F\left(t, u\left(t, \gamma(t)+\right), w\right)$;

(3) $\gamma(t) = x_o + \int_0^t \Pi\left(w(\tau)\right) d\tau$ for a.e. $t \in [0,T]$.

We now present the main result of this work, which extends [4, Theorem 2.6] allowing variations of g, B, F and Π.

Theorem 1. *Assume we have functions f, g_1, g_2, b, B_1, B_2, F_1, F_2, Π_1 and Π_2 such that f satisfies **(f)**, g_1 and g_2 satisfy **(g)**, b satisfies **(b)**, B_1 and B_2 satisfy **(B)**, F_1 and F_2 satisfy **(F)**, and Π_1 and Π_2 satisfy **(Π)** and **(NC)**. Then, there exist positive δ, $\mathcal{L}$, T_δ, domains $\hat{\mathcal{D}}_t$ (for $t \in [0, T_\delta]$) and processes $\hat{P}_1(t,t_0), \hat{P}_2(t,t_0) \colon \hat{\mathcal{D}}_{t_0} \to \hat{\mathcal{D}}_{t_0+t}$ ($t_0, t_0+t \in [0,T_\delta]$) such that*

(1) for all t_0, t_1, t_2 with $t_0 \in [0, T_\delta[$, $t_1 \in [0, T_\delta - t_0[$ and $t_2 \in [0, T_\delta - t_0 - t_1]$, then $\hat{P}_l(t_2, t_0+t_1) \circ \hat{P}_l(t_1, t_0) = \hat{P}_l(t_1+t_2, t_0)$ and $\hat{P}_l(0,t_0) = Id$;

(2) for $t_0 \in [0, T_\delta[$, $t \in [0, T_\delta - t_0]$, and $(u,w,x), (\bar{u}, \bar{w}, \bar{x}) \in \hat{\mathcal{D}}_{t_0}$

$$\begin{aligned} &\left\| \hat{P}_1(t,t_0)(u,w,x) - \hat{P}_2(t,t_0)(\bar{u},\bar{w},\bar{x}) \right\|_{L^1 \times \mathbb{R}^m \times \mathbb{R}} \\ &\leq \mathcal{L}\left(\|u - \bar{u}\|_{L^1} + \|w - \bar{w}\|_{\mathbb{R}^m} + |x - \bar{x}|\right) + \mathcal{L}\left(\|g_1 - g_2\|_{L^1} + \|B_1 - B_2\|_{L^1}\right) \\ &+ \mathcal{L}\left(\|F_1 - F_2\|_\infty + \|\Pi_1 - \Pi_2\|_\infty\right) \end{aligned}$$

(3) for all $(u_0, w_0, x_0) \in \hat{\mathcal{D}}_0$, the maps $t \to \hat{P}_l(t,0)(u_0, w_0, x_0)$ ($l \in \{1,2\}$), defined for $t \in [0, T_\delta]$, solves (1) in the sense of Definition 1, respectively with $g = g_l$, $B = B_l$, $F = F_l$, $\Pi = \Pi_l$ ($l \in \{1,2\}$).

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