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Dynamische Systeme

Organised by
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Abstract. This workshop continued the biannual series at Oberwolfach on Dynamical Systems that started as the "Moser-Zehnder meeting" in 1981. The main themes of the workshop are the new results and developments in the area of dynamical systems, in particular in Hamiltonian systems and symplectic geometry related to Hamiltonian dynamics. Highlights were the solution of a fifty year old problem in Arnold diffusion and a KAM-result on quasi-linear perturbations of the KdV-equation.

Mathematics Subject Classification (2010): 37, 35, 53D.

Introduction by the Organisers

The workshop was organized by H. Eliasson (Paris), H. Hofer (Princeton) and J.-C. Yoccoz (Paris). It was attended by more than 50 participants from 11 countries and covered a large area of dynamical systems centered around classical Hamiltonian dynamics: KAM theory, Arnold diffusion, geodesic flows, periodic solutions of symplectic flows, Floer homology. Other subjects treated where dynamics of PDEs, magnetic fields, quasi-periodic co-cycles and Schrödinger operators, pseudo-rotations, Teichmuller dynamics, pentagram maps, discrete bicycle transformations, group actions and algebraic number theory.

C-Q Cheng and by V. Kaloshin presented their proofs of existence of Arnold diffusion in "generic" nearly integrable Hamiltonian systems in $2\frac{1}{2}$ degrees of freedom. This is a fifty year old problem to whose solution many mathematicians, in particular J. Mather, have contributed.

M. Berti presented a perturbation result of KAM-type for quasi-linear perturbations of the KdV equation. Quasi-linear perturbations are particularly important for the connection of KdV with water wave equations whose perturbation theory is one of the most challenging problem in the KAM-theory for PDE's. M. Guardia reported on growth of Sobolev norms for the non-linear cubic Schrödinger equation and L.-S. Young discussed a work on center manifolds and chaotic dynamics in infinite dimension with applications to certain PDE's.

A. Abbondandolo presented new results on the old problem of periodic solutions in magnetic fields. V. Ginzburg P. Albers and U. Hryniewicz discussed pseudo holomorphic curve and Floer homology methods in symplectic and contact dynamics. M.-C. Arnaud reported on a generalization of the (former) Hopf conjecture to Tonelli Hamiltonians.

Pseudo-rotations were discussed by B. Bramhan and P. Le Calvez and also in the talk of J. Franks on Tits alternative for symplectic surface diffeomorphisms. New results on reducibility and Lyapunov exponents for quasi-periodic co-cycles were presented by N. Karaliolios, K. Bjerklöv and J. You. Teichmuller dynamics was discussed in the talks of C. Matheus Silva Santos and C. Ulcigrai.

The meeting was held in an informal and stimulating atmosphere. The weather was very nice the whole week and the traditional walk to St. Roman, this year under the leadership of Sergei Tabachnikov, was even more pleasant than usual.

Workshop: Dynamische Systeme

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Abstracts

Local and global instability in nearly integrable Hamiltonian systems

Marian Gidea
(joint work with Rafael de la Llave, Tere Seara)

1. Introduction

A remarkable paradigm in Hamiltonian instability is the Arnold diffusion problem, asserting that generic, nearly integrable Hamiltonian systems possess trajectories that travel "wildly" and "arbitrarily far".

More precisely, one starts with a Liouville integrable Hamiltonian system, which determines a Liouville foliation of the phase space. Off the singular leaves of the Liouville foliations, such a system can be locally described via action-angle coordinates. Each trajectory of the system lying within an action-angle domain preserves the action coordinate indefinitely. Then one acts on the integrable system with a small perturbation of a generic type. One of the main question of the Arnold diffusion problem is whether there exist trajectories whose action coordinate changes by some positive constant independent of the size of the perturbation (diffusing orbits), as well as trajectories that move with prescribed frequencies for prescribed times (symbolic dynamics), for all sufficiently small perturbation. A survey on this problem can be found in [3].

We have developed a toolkit of geometric techniques that can be applied to prove the existence of diffusing orbits and of symbolic dynamics for large classes of nearly integrable Hamiltonian systems. We provide an outline of the method, as well as some relevant applications concerning both local and global aspects of the diffusion problem.

2. Shadowing lemma

A key ingredient in our approach is a very general shadowing lemma for normally hyperbolic invariant manifolds.

We consider a discrete dynamical system given by a C^r-smooth map f on a compact, C^r-smooth manifold M, where $r \geq 2$. We assume that there exists a normally hyperbolic invariant manifold Λ in M, as well as a a compact, $C^{\ell-1}$-smooth homoclinic manifold $\Gamma \subseteq W^s(\Lambda) \cap W^u(\Lambda)$, for some $2 \leq \ell \leq r$.

Under certain conditions on Γ, there exists a $C^{\ell-1}$-diffeomorphism $S : H^- \to H^+$, with H^-, H^+ open sets in Λ, defined as follows: for each $x \in \Gamma$, let $x^- \in \Lambda$ be the point uniquely defined by $x \in W^u(x^-)$, and let $x^+ \in \Lambda$ be the point uniquely defined by $x \in W^s(x^+)$; then $S(x^-) = x^+$. The mapping S is referred at as the scattering map associated to the homoclinic manifold Γ. In many examples, the scattering map can be computed explicitly via perturbation theory. The scattering map has been studied in [2].

In this context, we have the following shadowing lemma type of result, saying that for every pseudo-orbit obtained by alternately applying the scattering map and some power of the inner map $f_{|\Lambda}$, there exists a true orbit nearby.

Lemma 1. *Assume that $f : M \to M$, $\Lambda \subseteq M$ and $\Gamma \subseteq M$ are as above, and S is the scattering map associated to Γ. Then, for every $\delta > 0$ there exists $N > 0$ such that for every sequence of points $\{y_i\}_{i\geq 0}$ of the type $y_{i+1} = f^{m_i} \circ S \circ f^{n_i}(y_i)$, with $m_i, n_i \geq N$, there exists an orbit $\{z_i\}_{i\geq 0}$ of f, with $z_{i+1} = f^{n_i+m_i}(z_i)$, such that $\|z_i - y_i\| < \delta$ for all $i \geq 0$.*

The following result says that one can easily obtain pseudo-orbits as in Lemma 1 provided that f restricted to Λ satisfies some recurrence condition.

Theorem 1. *Assume that $f : M \to M$, $\Lambda \subseteq M$ and $\Gamma \subseteq M$ are as above, and S is the scattering map associated to Γ. Assume that S preserves a measure absolutely continuous with respect to Lebesgue measure on Λ. Let $\{x_i\}_{i=0,\dots,n}$ be an orbit segment of the scattering map, i.e., $x_{i+1} = S(x_i)$ for $i = 0, \dots, n-1$. Assume that each point x_i of the orbit has a neighborhood $U_i \subseteq \Lambda$ such that almost every point in U_i is recurrent. Then, for every $\delta > 0$ there exists an orbit $\{z_i\}_{i=0,\dots,n}$ of f in M, with $z_{i+1} = f^{k_i}(z_i)$ for some $k_i > 0$, such that $d(z_i, x_i) < \delta$ for all $i = 0, \dots, n$.*

We stress that, in the above statements, we do not require that Λ is a 2-dimensional annulus, or that f is a twist map.

The statements of the Lemma 1 and of the Theorem 1 remain true if one uses several scattering maps rather than one. In general, one has available an abundance of homoclinic orbits and corresponding scattering maps.

3. Local diffusion results

We illustrate the application of the above shadowing results with an example of a nearly integrable Hamiltonian system of the following type:

$$H_\varepsilon(p, q, I, \phi, t) = h_0(I) \pm \left(\frac{1}{2}p^2 + V(q)\right) + \varepsilon h(p, q, I, \phi, t; \varepsilon). \tag{3.1}$$

where $(p, q, I, \phi, t) \in \mathbb{R}^n \times \mathbb{T}^n \times \mathbb{R}^d \times \mathbb{T}^d \times \mathbb{T}^1$.

We make the following assumptions:

- (A1.) V, h_0 and h are uniformly C^r for some r sufficiently large.
- (A2.) The potential V is 1-periodic in q and has a unique non-degenerate global maximum.
- (A3.) The perturbation h is a trigonometric polynomial in (ϕ, t), and 1-periodic in q.
- (A4.) The perturbation h satisfies some explicit non-degeneracy condition (hypothesis H4 is [1]), which is satisfied by an open dense set of perturbations h.

Theorem 2. *Assuming the conditions A1-A4, there exists $\varepsilon_0 > 0$, and $\rho > 0$ such that, for each $\varepsilon \in (0, \varepsilon_0)$, there exists a trajectory $x(t)$ of the Hamiltonian flow and $T > 0$ such that*

$$\|I(x(T)) - I(x(0))\| > \rho.$$

A significant difference from other existing results is that we do not assume that h_0 has positive-definite normal torsion, i.e., $\partial^2 h_0/\partial I^2 \neq 0$. Hence, approaches based on KAM theory and Aubry-Mather theory do not apply in our context.

In the special case when Λ is 2-dimensional annulus and h_0 has positive-definite normal torsion we can also obtain quantitative estimates on the speed of the diffusion. More precisely, we show the existence of trajectories $x(t)$ which achieve a change of order $O(1)$ in the action coordinate during a time of order $O(\varepsilon^{-1}\ln(\varepsilon^{-1}))$. This diffusion time has been conjectured as optimal by Lochak [6]. Moreover, we can show the existence of diffusing orbits that visit any prescribed collection of Aubry-Mather sets.

Another example that we can treat with our methods is a geodesic flow on a compact manifold endowed with a generic Riemannian/Lorentz/Finsler metric, subject to a perturbation by a time-dependent potential. We show that, if the perturbation satisfies some mild recurrence condition, then there exist trajectories of the perturbed system along which the energy grows to infinity in time, at a linear rate. This energy growth rate is optimal. For details, see [5].

4. Global diffusion results

In the previous examples we focused on local diffusion, meaning that the diffusing trajectories lie within a single action-angle domain. In general, the geometry of an integrable system can be quite complicated, as the phase space is divided out by the singular leaves of the Liouville foliation into distinct action-angle domains. We now consider the question on whether, under generic conditions, there exist diffusing orbits that travel arbitrarily far across these different regions, and also follow prescribed frequencies of motion.

As an example, we consider a special class of a priori unstable Hamiltonian systems of three-degrees of freedom. The unperturbed Hamiltonian system is a product of a two-degree of freedom integrable Hamiltonian and a one-degree of freedom pendulum. We assume that, on some fixed energy level, there exists a normally hyperbolic invariant manifold diffeomorphic to a three-sphere, on which the Hamiltonian satisfies a strict convexity condition. The sphere is divided out by the singular leaves into disjoint, open domains, each completely described by one action and two angle coordinates. We assume that the stable and unstable invariant manifolds of the three-sphere coincide. We apply a small perturbation. The normally hyperbolic invariant manifold survives if the perturbation is small enough. Assuming some non-degeneracy conditions that are generic, the stable and unstable invariant manifolds intersect transversally along homoclinic manifolds. To each homoclinic manifold, one can associate a scattering map. Typically, there are many geometrically distinct homoclinic manifolds. For a generic perturbation, there exists a family of homoclinic manifolds with the property that the domains of the corresponding scattering maps cover all possible action level sets on each action-angle domain of the three-sphere.

Under these assumptions, we show that there exist trajectories that follow any prescribed sequence of frequencies of motion on the sphere. In this sense, we can say that the perturbed system exhibits global diffusion relative to the three-sphere. For details, see [4].

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KAM for quasi-linear KdV equations

Massimiliano Berti

The goal of this talk is to present recent existence results of Cantor families of quasi-periodic solutions of Hamiltonian *quasi-linear* KdV equations like

$$u_t + u_{xxx} - \partial_x u^2 + \mathcal{N}_4(x, u, u_x, u_{xx}, u_{xxx}) = 0\,, \quad x \in \mathbb{T}\,, \tag{0.1}$$

$$u_t + u_{xxx} \pm \partial_x u^3 + \mathcal{N}_4(x, u, u_x, u_{xx}, u_{xxx}) = 0\,, \quad x \in \mathbb{T}\,, \tag{0.2}$$

where the Hamiltonian quasi-linear term

$$\mathcal{N}_4(x, u, u_x, u_{xx}, u_{xxx}) := -\partial_x \big[(\partial_u f)(x, u, u_x) - \partial_x((\partial_{u_x} f)(x, u, u_x))\big] \tag{0.3}$$

vanishes of order 4 at the origin, i.e. $\mathcal{N}_4(x, \varepsilon u, \varepsilon u_x, \varepsilon u_{xx}, \varepsilon u_{xxx}) = O(\varepsilon^4)$ as $\varepsilon \to 0$, or, equivalently, the Hamiltonian density f vanishes of order 5, i.e. $f(x, \varepsilon u, \varepsilon u_x) = O(\varepsilon^5)$. Then (0.1)-(0.2) may be seen, close to the origin, as "small" perturbations of, respectively, the PDEs

$$(\mathbf{KdV})\ u_t + u_{xxx} - \partial_x u^2 = 0\,, \quad (\mathbf{mKdV})\ u_t + u_{xxx} \pm \partial_x u^3 = 0$$

both focusing/defocusing. Both KdV and mKdV are completely integrable equations. The natural question that we pose is whether their periodic, quasi-periodic or almost periodic solutions persist under small perturbations.

The first KAM results of quasi-periodic solutions have been proved by Kuksin [5] and Kappeler-Pöschel [4] for semilinear Hamiltonian perturbations $\varepsilon \partial_x (\partial_u f)(x, u)$, namely when the density f is independent of the first order derivatives u_x, i.e. (0.3) is a differential operator of order 1. This approach also works for Hamiltonian pseudo-differential perturbations of order 2 (in space), but *not* for a general quasi-linear perturbation as in (0.3), which is a nonlinear differential operator of the *same* order (i.e. 3) as the constant coefficient linear operator ∂_{xxx}. Such a strong perturbation term makes the answer quite delicate.

Another difficulty is that (0.1)-(0.2) are *completely resonant* PDEs, namely the linearized Airy-equation $u_t + u_{xxx} = 0$ (at the origin) possesses only the 2π-periodic in time solutions

$$u(t,x) = \sum_{j \in \mathbb{Z}} u_j e^{\mathrm{i} j^3 t} e^{\mathrm{i} j x} .$$

Hence the existence of quasi-periodic solutions of (0.1)-(0.2) is a purely nonlinear phenomenon. The nonlinearity will "twist" the frequencies of the trajectories according to the amplitude, producing quasi-periodic solutions.

Both (0.1) and (0.2) are Hamiltonian PDEs

$$u_t = \partial_x \nabla H(u) ,$$

where ∇H denotes the $L^2(\mathbb{T}_x)$ gradient of

$$H = \int_{\mathbb{T}} \frac{u_x^2}{2} + \frac{u^3}{3} + f(x,u,u_x)dx , \ H = \int_{\mathbb{T}} \frac{u_x^2}{2} \mp \frac{u^4}{4} + f(x,u,u_x)dx ,$$

and the phase space is $H_0^1(\mathbb{T}) := \{u(x) \in H^1(\mathbb{T},\mathbb{R}) \, : \, \int_{\mathbb{T}} u(x)dx = 0\}$.

Theorem 1 ('13, Baldi, Berti, Montalto [2]). *Given $\nu \in \mathbb{N}$, let $f \in C^q$ (with $q := q(\nu)$ large enough), $f(u,u_x) = O(|(u,u_x)|^5)$, and f reversible, namely $f(u,u_x) = f(u,-u_x)$. Then, for any choice of the "tangential sites"*

$$S := \{-\bar{\jmath}_\nu , \ldots , -\bar{\jmath}_1, \bar{\jmath}_1 , \ldots , \bar{\jmath}_\nu\} \subset \mathbb{Z} \setminus \{0\} ,$$

the mKdV equation (0.2) *possesses small amplitude quasi-periodic solutions with Sobolev regularity H^s, $s \leq q$, and zero Lyapunov exponents, of the form*

$$u = \sum_{j \in S} \sqrt{\xi_j}\, e^{\mathrm{i}\omega_j^\infty(\xi)t} e^{\mathrm{i}jx} + o(\sqrt{\xi}), \quad \omega_j^\infty(\xi) \overset{\xi \to 0}{\to} j^3 , \quad \xi_{-j} = \xi_j ,$$

for a "Cantor-like" set of small amplitudes $\xi \in \mathbb{R}_+^\nu$ with density 1 *at $\xi = 0$. The term $o(\sqrt{\xi})$ is small in the H^s-Sobolev norm. These quasi-periodic solutions are* LINEARLY STABLE.
For the KdV equation (0.1) *the same result holds for "generic" choices of the tangential sites S.*

Fixed the finite set of tangential sites S we perform a "very weak" Birkhoff normal form whose goal is only to find an invariant manifold of solutions of the third order approximate system, on which the dynamics is completely integrable. This Birkhoff map is close to the identity up to a finite rank operator. Hence it modifies $\mathcal{N}_4$ very mildly (for the more degenerate KdV equation (0.1) more Birkhoff normal forms steps are required). This procedure is different with respect to the usual Birkhoff normal forms. It is sufficient to find the first nonlinear approximation of the solutions so that the Nash-Moser iteration converges, and to extract the approximate "frequency-to-amplitude" modulation. We can not rely on the global "frequency-to-amplitude" map described in [4], because the Birkhoff coordinates are close to the Fourier transform only up to smoothing operators of order 1, and this does not seem enough to deal with quasi-linear perturbations such as (0.3).

Then the solution is obtained by a Nash-Moser iterative scheme in Sobolev scales. The key step is to construct an approximate inverse (*à la* Zehnder) of the linearized operator. For that we implement the abstract procedure in Berti-Bolle [3] developed to prove existence of quasi-periodic solutions for NLW (and NLS) with a multiplicative potential. This general method, based on symplectic techniques, decouples the "tangential" and the "normal" dynamics around the expected invariant torus, reducing the inversion of the full linearized operator to that of the linearized equation in the normal directions. This corresponds to a quasi-periodic linear Airy equation with variable coefficients which was investigated in [1] studying forced perturbations of the Airy equation

$$u_t + u_{xxx} + \varepsilon f(\omega t, x, u, u_x, u_{xx}, u_{xxx}) = 0 \tag{0.4}$$

where the nonlinearity is quasi-periodic in time with diophantine frequency

$$\omega = \lambda\bar{\omega} \in \mathbb{R}^\nu, \quad \lambda \in [1/2\,, 3/2], \quad |\bar{\omega}\cdot l| \geq \gamma |l|^{-\tau}, \ \forall l \in \mathbb{Z}^\nu \setminus \{0\}.$$

The only "external" parameter in (0.4) is λ. The perturbation may depend on u_{xxx} in a nonlinear way (fully nonlinear PDE).

In [1] we first prove, under some assumptions on f (without which it is easy to exhibit simple counterexamples), an existence theorem of quasi-periodic solutions of (0.4). Then we also prove their linear stability if the nonlinearity is *Hamiltonian* or *reversible*. In the Hamiltonian case, for example, we prove:

Theorem 2 ('12, Baldi, Berti, Montalto [1]). *There exist $s := s(\nu) > 0$, $q := q(\nu) \in \mathbb{N}$, such that for all $\varepsilon \in (0, \varepsilon_0)$, where $\varepsilon_0 := \varepsilon_0(f, \nu)$ is small enough, there exists a Cantor set $\mathcal{C}_\varepsilon \subset [1/2, 3/2]$ of asymptotically full Lebesgue measure, i.e. $|\mathcal{C}_\varepsilon| \to 1$ as $\varepsilon \to 0$, such that, $\forall \lambda \in \mathcal{C}_\varepsilon$ the equation*

$$u_t = \partial_x \nabla H, \quad H := \int_{\mathbb{T}} \frac{u_x^2}{2} + \varepsilon F(\omega t, x, u, u_x)\, dx, \tag{0.5}$$

has a quasi-periodic solution $u(\varepsilon, \lambda) \in H^s$ with frequency $\omega = \lambda\bar{\omega}$ satisfying $\|u(\varepsilon, \lambda)\|_s \to 0$ as $\varepsilon \to 0$. Moreover such solution is LINEARLY STABLE.

An essential ingredient in the proof is to conjugate the quasi-periodically forced linear PDE

$$h_t + (1 + a_3(\omega t, x))\partial_{xxx} h + a_2(\omega t, x)\partial_{xx} h + a_1(\omega t, x)\partial_x h + a_0(\omega t, x) h = 0, \tag{0.6}$$

obtained linearizing (0.5) at any approximate (or exact) solution u, to the dynamical system of infinitely many harmonic oscillators

$$\dot{v}_j + \mu_j v_j = 0, \quad \forall j \in \mathbb{Z}, \ \mu_j := \mathrm{i}(-m_3 j^3 + m_1 j) + r_j \in \mathrm{i}\mathbb{R}, \tag{0.7}$$

$$m_3, m_1 \in \mathbb{R}\,,\ m_3 = 1 + O(\varepsilon)\,,\ m_1 = O(\varepsilon)\,,\ \sup_j |r_j| \le C\varepsilon\,.$$

The main perturbative effect is clearly due to the term $a_3(\omega t, x)\partial_{xxx}$ and the standard reducibility KAM techniques do not work. The above conjugacy is obtained by changes of variables, like quasi-periodic time-dependent diffeomorphisms of the space variable x, multiplication operators and pseudo-differential operators, in order to reduce (0.6) to a constant coefficients equation $h_t + \tilde{m}_3\partial_{xxx}h + \tilde{m}_1\partial_x h = 0$, $\tilde{m}_1, \tilde{m}_3 \in \mathbb{R}$, plus a bounded remainder of order 0. These preliminary transformations, which are inspired to the works of Iooss-Plotnikov-Toland in water waves theory are very different from the usual KAM changes of variables. Then we perform a quadratic KAM reducibility scheme *à la* Eliasson-Kuksin, which completely diagonalizes the linearized operator, obtaining (0.7).

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Hyperbolic fixed points and periodic orbits of Hamiltonian systems

Viktor L. Ginzburg

(joint work with Başak Z. Gürel)

The central theme of this work is the existence problem for periodic orbits of Hamiltonian systems. We prove that for a certain class of closed monotone symplectic manifolds any Hamiltonian diffeomorphism with a hyperbolic fixed point must necessarily have infinitely many periodic orbits. Among the manifolds in this class are, for instance, the complex projective spaces $\mathbb{CP}^n$.

More specifically, let M be a closed symplectic manifold. Recall that the minimal Chern number N of M is defined by $\langle c_1(TM), \pi_2(M)\rangle = N\mathbb{Z}$, where $N \ge 0$, and that M is strictly monotone if $N > 0$ and $c_1(TM)|_{\pi_2(M)} = \lambda[\omega]|_{\pi_2(M)} \ne 0$ with $\lambda > 0$. Our main result is the following

Theorem 1 ([13]). *Let M be a closed strictly monotone symplectic manifold of dimension $2n$ such that $N \ge n/2 + 1$. Assume that*

$$\beta * \alpha = q[M]$$

in the quantum homology $\mathrm{HQ}_*(M)$ for some ordinary homology classes $\alpha \in \mathrm{H}_*(M)$ and $\beta \in \mathrm{H}_*(M)$ with $|\alpha| < 2n$ and $|\beta| < 2n$. Then any Hamiltonian diffeomorphism φ_H of M with a contractible hyperbolic periodic orbit γ has infinitely many periodic orbits.

Note that in our conventions $|q| = -2N$. Furthermore, the assumption that γ is contractible can be dropped when M is also toroidally monotone.

In addition to $\mathbb{CP}^n$, among the manifolds meeting the requirements of the theorem are the complex Grassmannians $\mathrm{Gr}(2, N)$, $\mathrm{Gr}(3, 6)$ and $\mathrm{Gr}(3, 7)$, the products $\mathbb{CP}^m \times P^{2d}$ and $\mathrm{Gr}(2, N) \times P^{2d}$, where P is symplectically aspherical and $d+2 \leq m$ in the former case and $d \leq 2$ in the latter, and monotone products $\mathbb{CP}^m \times \mathbb{CP}^m$.

To put Theorem 1 in perspective, recall that for many closed symplectic manifolds every Hamiltonian diffeomorphism has infinitely many periodic orbits and even periodic orbits of arbitrarily large (prime) period whenever the fixed points are isolated. This unconditional existence of infinitely many periodic orbits is usually referred to as the Conley conjecture. The Conley conjecture has been shown to hold for all symplectic manifolds M with $c_1(TM) |_{\pi_2(M)}= 0$ and also for negative monotone manifolds; see [3, 10, 17] and also [8, 9, 12, 18, 21, 23]. Ultimately, one can expect the Conley conjecture to hold for most of the symplectic manifolds.

There are, however, notable exceptions. The simplest one is S^2: an irrational rotation of S^2 about the z-axis has only two periodic orbits, which are also the fixed points; these are the poles. In fact, any manifold that admits a Hamiltonian torus action with isolated fixed points also admits a Hamiltonian diffeomorphism with finitely many periodic orbits. In particular, complex projective spaces, the Grassmannians, and, more generally, most of the coadjoint orbits of compact Lie groups as well as symplectic toric manifolds all admit Hamiltonian diffeomorphisms with finitely many periodic orbits.

An analogue of the Conley conjecture applicable to such manifolds is the conjecture that a Hamiltonian diffeomorphism with "more than necessary" fixed points has infinitely many periodic orbits; see [19, p. 263]. Here "more than necessary" is left deliberately vague although it is usually interpreted as a lower bound arising from some version of the Arnold conjecture. For $\mathbb{CP}^n$, the expected threshold is $n + 1$. This conjecture, referred to as the *HZ-conjecture* in [14, 15], is inspired by a celebrated theorem of Franks asserting that a Hamiltonian diffeomorphism of S^2 with at least three fixed points must have infinitely many periodic orbits, see [6, 7] and also [8, 21] for further refinements and [2, 4, 20] for symplectic topological proofs. (Franks' theorem holds for area preserving homeomorphisms. However, the discussion of possible generalizations of this stronger result to higher dimensions is far outside the scope of this work.) In fact, one can more generally conjecture that a Hamiltonian diffeomorphism has infinitely many periodic orbits whenever it has fixed points which are "homologically or geometrically unnecessary".

Theorem 1 provides some evidence supporting these conjectures beyond dimension two. (The HZ-conjecture, at least for non-degenerate Hamiltonian diffeomorphisms of $\mathbb{CP}^n$ and some other manifolds, would follow if we could replace a hyperbolic fixed point by a non-elliptic fixed point in the theorem.) Some further evidence is given by the results of [14], where a "local version" of the conjecture is considered, and also in [1, 15] concerning non-contractible periodic orbits.

Hyperbolicity is central to the proof of the theorem. (However, it can be replaced by a weaker requirement that γ is isolated as an invariant set.) The proof hinges on a result, of independent interest, asserting that the energy required for a Floer connecting trajectory of an iterated Hamiltonian to approach a hyperbolic orbit and cross its fixed neighborhood cannot be arbitrarily small: it is bounded away from zero by a constant independent of the order of iteration. (This is not true for, say, elliptic fixed points.) The proof of this lower bound relies on the variant of Gromov compactness theorem established in [5].

Finally note that the presence of one hyperbolic orbit implies, C^1-generically, the existence of transverse homoclinic points via the so-called connecting lemma; see [16, 24]. (The genericity assumption is essential here, although hypothetically this could be a C^∞-generic condition rather than C^1.) The existence of transverse homoclinic points has, in turn, rich dynamical consequences among which is the existence of infinitely many periodic orbits. Thus, under certain additional conditions on the ambient manifold, Theorem 1 recovers a fraction of this dynamics, but does this unconditionally rather than generically. Note also that the existence of infinitely many periodic orbits is a C^1-generic phenomenon, as follows from the closing lemma, [22], and in many instances even C^∞-generic (see [11]).

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Toward a smooth ergodic theory for infinite dimensional systems

Lai-Sang Young

The results I report on are part of a larger project the aim of which is to extend smooth ergodic theory, or ergodic theory of hyperbolic systems, to infinite dimensional dynamical systems satisfying technical conditions consistent with those defined by parabolic or dissipative evolutionary PDEs. Three sets of results are reported: (1) Absolute continuity of the strong stable foliation for systems with center manifolds; this result points to a notion of "almost everywhere" in Banach spaces that is natural in dynamical contexts [1]. (2) For semiflows in Hilbert spaces undergoing generic supercritical Hopf bifurcations, my co-authors and I proved that under suitable conditions, small time- periodic kicks will lead to the creation of strange attractors with SRB measures. These general results are applied to concrete equations such as the 1D Brusselator [2]. (3) Semiflows on Hilbert spaces admitting an invariant measure with only a single zero Lyapunov exponent are considered, and results extending Katok's earlier work on diffeomorphisms of finite dimensional manifolds, including the equivalence of and positive entropy with the presence of horseshoes and exponential growth of periodic orbits, are reported [3]. The results above are joint with three sets of co-authors; see below.

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Angular momentum and Horn's problem

Alain Chenciner

The *central* configurations of n point masses in the euclidean space E are those configurations

$$x = (\vec{r}_1, \cdots, \vec{r}_n) \in E^n$$

which, if released without initial velocity, homothetically collapse on their center of mass when submitted to Newtonian attraction. For example, Lagrange has proved that the only non collinear central configuration of 3 positive masses is the equilateral triangle.

Such configurations are known (see [3, 4]) to admit periodic rigid motions, which necessarily take place in an euclidean space E of even dimension $2p$ and are of the form

$$\vec{r}_i(t) = e^{\omega t J}\vec{r}_i(0)), \quad i = 1, \cdots n,$$

where J is a complex structure on E compatible with the euclidean structure, that is an isometry such that $J^2 = -Id$. The angular momentum bivector of such a motion defines, via the euclidean structure, a J-skew-hermitian endomorphism $\mathcal{C}$ of E of the form $\omega(S_0 J + JS_0)$, where the symmetric non negative $2p \times 2p$ matrix S_0 is the *inertia* matrix of the configuration $x(0)$ and ω is a real frequency. Replacing $\mathcal{C}$ by $\frac{1}{\omega}J^{-1}\mathcal{C}$, this leads to the following purely algebraic question :

Let S_0 be a symmetric non negative $2p \times 2p$ matrix; what is the image of the mapping $\mathcal{F}$ which, to each J, associates the ordered spectrum $\{\nu_1 \geq \nu_2 \geq \cdots \geq \nu_p\}$ of the J-hermitian matrix $S_0 + J^{-1}S_0 J$, considered as a complex $p \times p$ matrix ?

On the other hand, Horn's problem (now solved independantly by Klyashko and by Knutson and Tao) asks for the possible spectra of matrices of the form $C = A+B$, where A and B are complex hermitian (or real symmetric) with given spectra.

Introducing two Horn's problems, one in dimension p and one in dimension $2p$, one proves that the image of $\mathcal{F}$ is a convex polytope which can be described. The precise result was conjectured in [1] and proved there in case $p = 2$; the general case was proved in [2].

Moreover, to this polytope are associated subpolytopes whose faces correspond to the only values of the angular momentum for which bifurcations could occur to families of quasi-periodic relative equilibria with *balanced* configurations (see [4]; these are the *configurations équilibrées* of [3]).

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Orderability of contactomorphism groups

Peter Albers
(joint work with W. Merry, U. Fuchs, U. Frauenfelder)

We study the notion of orderability of contactomorphism groups as introduced by Eliashberg-Polterovich [EP00] and its connection to contact (non-)squeezing [EKP06] using Rabinowitz Floer homology. Eliashberg-Polterovich define a natural partial order on the group of contactomorphisms $\mathrm{Cont}_0(\Sigma,\xi)$ resp. its universal cover $\widetilde{\mathrm{Cont}}_0(\Sigma,\xi)$ where (Σ,ξ) is a closed contact manifold. The main feature of this partial order is that it is biinvariant. Eliashberg-Polterovich proved that the partial order on $\mathrm{Cont}_0(\Sigma,\xi)$ resp. $\widetilde{\mathrm{Cont}}_0(\Sigma,\xi)$ is trivial if and only if there exists a positive loop of contactomorphisms resp. a contractible positive loop. The questions whether this partial order is non-trivial is very subtle. For instance on S^{2n-1} a contractible positive loop exists whereas on its $\mathbb{Z}/2$-quotient, $\mathbb{R}\mathrm{P}^{2n-1}$, no contractible positive loop exists. Nonetheless a (non-contractible) positive loop does exist on $\mathbb{R}\mathrm{P}^{2n-1}$ e.g. the Reeb flow of the standard contact form.

If the contact manifold is the boundary of a compact exact symplectic manifold Cieliebak-Frauenfelder [CF09] constructed an invariant called Rabinowitz Floer homology. Examples are unit cotangent bundles as boundaries of unit codisk bundles. In this situation joint work with Frauenfelder [AF12] shows that perturbations of the underlying Rabinowitz action functional on can be used to relate a growth rate associated to a positive loops to growth rates of geodesics on the underlying manifold of the cotangent bundle. This implies that unless this underlying manifold is very simple, for instance spheres, projective spaces, there are no positive loops. Simple means that the cohomology ring has more than one generator.

Inspired by work of Sandon [San11] we studied jointly with Merry different perturbations of the Rabinowitz action functional. We prove [AM13] that non-vanishing of Rabinowitz Floer homology implies that there are no positive contractible loops of contactomorphisms and several other related results. Moreover, using spectral invariants we proved a general non-squeezing result. Both results subsume all previously know examples.

Finally, reporting joint work in progress with Merry and Fuchs we relate the orderability question to the famous Weinstein conjecture.

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Arnold diffusion in nearly integrable Hamiltonian systems

Chong-Qing Cheng

We consider nearly integrable Hamiltonian systems with 3 degrees of freedom:

$$H(x,y) = h(y) + \epsilon P(x,y), \qquad (x,y) \in \mathbb{T}^3 \times \mathbb{R}^3, \tag{0.1}$$

where h is assumed to strictly convex, namely, the Hessian matrix $\partial^2 h/\partial y^2$ is positive definite. It is also assumed that $\min h = 0$, both h and P are C^r-function with $r \geq 8$.

For $E > 0$, let $H^{-1}(E) = \{(x,y) : H(x,y) = E\}$ denote the energy level set, $B \subset \mathbb{R}^3$ denote a ball in $\mathbb{R}^3$ such that $\bigcup_{E' \leq E+1} h^{-1}(E') \subset B$. Let $\mathfrak{S}_a, \mathfrak{B}_a \subset C^r(\mathbb{T}^3 \times B)$ denote a sphere and a ball with radius $a > 0$ respectively: $F \in \mathfrak{S}_a$ if and only $\|F\|_{C^r} = a$ and $F \in \mathfrak{B}_a$ if and only $\|F\|_{C^r} \leq a$. They inherit the topology from $C^r(\mathbb{T}^3 \times B)$.

For perturbation P independent of y (classical mechanical system) we use the same notation $\mathfrak{S}_a, \mathfrak{B}_a \subset C^r(\mathbb{T}^3)$ to denote a sphere and a ball with radius $a > 0$

Let $\mathfrak{R}_a$ be a set residual in $\mathfrak{S}_a$, each $P \in \mathfrak{R}_a$ is associated with a set R_P residual in the interval $[0, a_P]$ with $a_P \leq a$. A set $\mathfrak{C}_a$ is said cusp-residual in $\mathfrak{B}_a$ if

$$\mathfrak{C}_a = \{\lambda P : P \in \mathfrak{R}_a, \lambda \in R_P\}.$$

Let Φ_H^t denote the Hamiltonian flow determined by H. Given an initial value (x,y), $\Phi_H^t(x,y)$ generates an orbit of the Hamiltonian flow $(x(t), y(t))$. An orbit $(x(t), y(t))$ is said to visit $B_\delta(y_0) \subset \mathbb{R}^3$ if there exists $t \in \mathbb{R}$ such that $y(t) \in B_\delta(y_0)$ a ball centered at y_0 with radius δ.

Theorem 1. *Given any two balls* $B_\delta(x_0, y_0), B_\delta(x_k, y_k) \subset \mathbb{T}^3 \times \mathbb{R}^3$ *and finitely many small balls* $B_\delta(y_i) \subset \mathbb{R}^3$ $(i = 0, 1, \cdots, k)$, *where* $y_i \in h^{-1}(E)$ *with* $E > 0$ *and* $\delta > 0$ *is small, there exists a cusp-residual set* $\mathfrak{C}_{\epsilon_0}$ *such that for each* $\epsilon P \in \mathfrak{C}_{\epsilon_0}$, *the Hamiltonian flow* Φ_H^t *admits orbits which, on the way between passing through* $B_\delta(x_0, y_0)$ *and* $B_\delta(x_k, y_k)$, *visit the balls* $B_\delta(y_i)$ *in any prescribed order.*

The result is generic not only in usual sense, but also in the sense of Mañé, namely, it is a typical phenomenon when the system is perturbed by potential. The details of the proof can be found in [1]. The results of the author for *a priori* unstable systems can be found in [2, 3, 4].

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A criterium for the simplicity of Lyapunov exponents of origamis

Carlos Matheus
(joint work with Martin Möller and Jean-Christophe Yoccoz)

The *Teichmüller flow* is a renormalization dynamics for interval exchange transformations, translation flows and billiards in rational polygons (cf. [17] and [19]). Among its recent applications, let us mention that the confirmation by Delecroix-Hubert-Lelièvre [3] of a conjecture of the physicists Hardy and Weber on abnormal rates of diffusion of typical trajectories in typical realizations of Ehrenfest's wind-tree model and the classification of commensurability classes of all known ball quotients by Kappes-Möller [13].

The phase space of Teichmüller flow is the *moduli space* $\mathcal{H}_g$ of unit area Abelian differentials on Riemann surfaces of genus $g \geq 1$, that is, the set of pairs (M, ω) where M is a Riemann surface of genus $g \geq 1$ and $\omega \neq 0$ is an Abelian differential (holomorphic 1-form) with unit total area (i.e., $(i/2)\int_M \omega \wedge \overline{\omega} = 1$) *modulo* biholomorphisms $f : (M, \omega) \to (M', \omega')$ respecting the mark ω (i.e., $f^*(\omega') = \omega$).

Given an Abelian differential (M, ω), let Σ be the (finite) set of zeroes of ω. By taking local primitives of ω, one obtains a *translation structure* on $M - \Sigma$, i.e., an atlas whose changes of coordinates are given by translations on the plane $\mathbb{R}^2 \equiv \mathbb{C}$. Conversely, from a translation structure, one can recover an Abelian differential by pulling back (via the corresponding atlas) the holomorphic 1-form dz on $\mathbb{C}$.

This correspondence between Abelian differentials and translation surfaces shows that there exists a natural action of $SL(2, \mathbb{R})$ on the moduli space $\mathcal{H}_g$ where an element $A \in SL(2, \mathbb{R})$ sends (M, ω) to the translation surface $A \cdot (M, \omega)$ obtained by post-composing the charts of the atlas of the translation structure (M, ω) with A. In this language, the Teichmüller flow is the action of the diagonal subgroup $g_t = \mathrm{diag}(e^t, e^{-t})$ of $SL(2, \mathbb{R})$.

The derivative Dg_t of the Teichmüller flow has the form $Dg_t = \mathrm{diag}(e^t, e^{-t}) \otimes G_t^{KZ}$ where G_t^{KZ} is the so-called *Kontsevich-Zorich (KZ) cocycle* over the Teichmüller flow g_t. In other words, the KZ cocycle G_t^{KZ} is the *interesting* part of the derivative Dg_t of Teichmüller flow.

As it turns out, KZ cocycle is a *symplectic* cocycle possessing $2g$ Lyapunov exponents of the form $1 = \lambda_1^\mu \geq \dots \lambda_g^\mu \geq -\lambda_g^\mu \geq \dots \geq -\lambda_1^\mu = -1$ with respect to any ergodic g_t-invariant probability measure μ. Furthermore, after the seminal works of Zorich [18] and Forni [6], we know that the Lyapunov exponents $\pm\lambda_i^\mu$, $i = 1, \dots, g$, with respect to $SL(2, \mathbb{R})$-invariant probability measures μ drive the deviations of ergodic averages of "most" interval exchange transformations and translation flows related to the support of μ.

In particular, the knowledge of the ergodic properties of $SL(2,\mathbb{R})$-invariant probability measures on the moduli space $\mathcal{H}_g$ are relevant for many applications and this (partly) explains the recent efforts to *classify* $SL(2,\mathbb{R})$-invariant ergodic probability measures (see, e.g., the works of Calta [2] and McMullen [16] for complete classification results in genus 2, and the recent preprint of Eskin and Mirzakhani [5] for a "Ratner-like" theorem in any genus $g \geq 2$).

Concerning the Lyapunov exponents of the KZ cocycle with respect to $SL(2,\mathbb{R})$-invariant probability measures, Forni [6] and Avila-Viana [1] confirmed a conjecture of Kontsevich and Zorich (based on some numerical experiments) on the *simplicity* of the Lyapunov exponents of a major class of (*absolutely continuous*) ergodic $SL(2,\mathbb{R})$-invariant probability measures on $\mathcal{H}_g$ known as *Masur-Veech measures* μ_{MV}, i.e., $\lambda_i^{\mu_{MV}} > \lambda_{i+1}^{\mu_{MV}}$ for each $i = 1, \ldots, g$. On the other hand, Forni and his coauthors (cf. [7] and [9]) gave examples of ergodic $SL(2,\mathbb{R})$-invariant probability measures μ_{EW} and μ_O (on $\mathcal{H}_3$ and $\mathcal{H}_4$, resp.) where $\lambda_2^{\mu_{EW}} = 0 = \lambda_2^{\mu_O}$, and, *a fortiori*, there is no simplicity of (non-tautological) Lyapunov exponents. Nevertheless, Forni [8] gave a criterion for the *non-uniform hyperbolicity* (i.e., $lambda_g^\mu > 0$) of ergodic $SL(2,\mathbb{R})$-invariant μ, but, as it is pointed out in his article, his conditions can *not* ensure simplicity in general.

This scenario motivates the following question: why a "naive" extension of Avila-Viana methods for the simplicity of Lyapunov exponents of Masur-Veech measures μ_{MV} does not work for general $SL(2,\mathbb{R})$-invariant ergodic probability measures? The answer comes from a close inspection of Forni's examples. In a nutshell, these examples come from *square-tiled surfaces* or *origamis*, i.e., translation surfaces (M,ω) obtained from finite branched covers of $(\mathbb{T}^2 = \mathbb{C}/(\mathbb{Z}\oplus i\mathbb{Z}), dz)$, say $\pi : (M,\omega) \to (\mathbb{T}^2, dz)$. It is know that the $SL(2,\mathbb{R})$-orbits of origamis are closed in $\mathcal{H}_g$ and they support an unique ergodic $SL(2,\mathbb{R})$-invariant probability measure. Also, still in the case of origamis, the first homology group $H_1(M,\mathbb{Q})$ decomposes into the space $H_1^{(0)}(M,\mathbb{Q})$ consisting of homology classes in M projecting to 0 in $\mathbb{T}^2$ and its symplectic orthogonal $H_1^{st}(M,\ mathbbQ) = \pi_*^{-1}(H_1(\mathbb{T}^2,\mathbb{Q}))$ and this decomposition is preserved by the KZ cocycle G_t^{KZ}. In the case of Forni's examples, Yoccoz and the author [15] showed that G_t^{KZ} acts on $H_1^{(0)}(M,\mathbb{Q})$ via some *finite* group of matrices, that is, the KZ cocycle is not sufficiently *rich* in the sense of Furstenberg [10], Guivarch-Raugi [12], Goldsheid-Margulis [11] and Avila-Viana [1] (for instance, its Zariski closure is far from the full symplectic group).

In this direction, Möller, Yoccoz and the author [14] showed that a "rich" KZ cocycle suffices to ensure simplicity of Lyapunov exponents of (ergodic $SL(2,\mathbb{R})$-invariant probability measures associated to) origamis. More precisely, the matrices of KZ cocycle G_t^{KZ} for origamis (M,ω) correspond to the action on homology of its *affine diffeomorphisms*, i.e., homeomorphisms preserving the zeroes of ω whose expressions in the charts of the translation structure (M,ω) are affine maps of $\mathbb{R}^2$. We denote by $\mathrm{Aff}(M,\omega)$ the group of affine diffeomorphisms of (M,ω), and, given $A \in \mathrm{Aff}(M,\omega)$, we let $DA \in SL(2,\mathbb{R})$ denotes its linear part. In this language, we proved that the following two results (the main theorems in [14]):

Theorem A. Let (M,ω) be an origami without automorphisms (i.e., without affine diffeomorphisms whose linear part is the identity matrix). Suppose that there are two affine diffeomorphisms $\phi,\psi\in\mathrm{Aff}(M,\omega)$ such that:

(a) $D\phi$ has trace >2 and ϕ is *Galois-pinching*, i.e., the action of ϕ on $H_1^{(0)}(M,\mathbb{Q})$ has real eigenvalues, irreducible characteristic polynomial and largest possible Galois group.

(b) ψ acts on $H_1^{(0)}(M,\mathbb{Q})$ via a non-trivial unipotent matrix $\mathrm{Id}+N_*$ so that $\mathrm{Im}(N_*)$ is not Lagrangian.

Then, the Lyapunov exponents of (M,ω) are simple.

Theorem B. Let (M,ω) be an origami without automorphisms and suppose that there are two affine diffeomorphisms $\phi,\psi\in\mathrm{Aff}(M,\omega)$ whose linear parts $D\phi$ and $D\psi$ are b-reduced (i.e., their entries are positive, the top leftmost entry is larger than the two anti-diagonal entries and the anti-diagonal entries are larger than the right bottomost entry), ϕ is Galois-pinching and the actions ϕ_*,ψ_* (resp.) of ϕ,ψ (resp.) on $H_1^{(0)}(M,\mathbb{Q})$ have disjoint splitting fields, the minimal polynomial of ψ_* has degree >2 and no irreducible factor of even degree. Then, the Lyapunov exponents of (M,ω) are simple.

Closing this report, let us make some comments on these theorems.

Firstly, we restricted our attention to origamis because the Teichmüller flow on $SL(2,\mathbb{R})$-orbits of origamis admit nice *codings* (countable Markov partitions with good distortion properties) thanks to the relationship between the geodesic flow on the modular curve $\mathcal{H}_1=SL(2,\mathbb{R})/SL(2,\mathbb{Z})$ and the continued fraction algorithm. It is worth to point out that the presence of codings is also useful in Avila-Viana arguments in [1] (in their context, the natural coding adapted to Masur-Veech measures μ_{MV} is the so-called *Rauzy-Veech algorithm*). Nevertheless, after the first versions of [14] were written, Eskin and the author [4] noticed that there is *no* need for codings in the context of *closed* $SL(2,\mathbb{R})$-orbits such as the ones of origamis. In other words, in this setting, the simplicity of Lyapunov exponents can be detected from the monoid generated by ϕ and ψ.

Secondly, the assumption in item (b) in Theorem A can be geometrically verified (without calculations) if the origami (M,ω) has a rational direction with homological dimension $1<k<g$ in the sense of [8]. In practice, this means that, for potential applications of Theorem A, once we have Galois-pinching, then the simplicity is *almost* automatic in the sense that the item (b) is satisfied unless your origami (M,ω) has a very special geometry (namely, no rational direction with homological dimension $1<k<g$). Interestingly enough, the fact expressed by Theorem A that simplicity is almost automatic from Galois-pinching fits nicely the philosophy behind some work in progress by I. Rivin and P. Sarnak (whose existence I learned from Eskin) where it is shown that a non-commutative monoid of matrices containing Galois-pinching elements has full Zariski closure.

Thirdly, in our joint work with Möller and Yoccoz [14], we combine Theorem A with Siegel's theorem on the finiteness of integral points on algebraic curves of genus >0 to show that, if a conjecture of Delecroix-Lelièvre on the classification of

$SL(2,\mathbb{R})$-orbits of origamis in the minimal stratum $\mathcal{H}(4)$ (i.e., the two connected components of $\mathcal{H}_3$ with the smallest dimension) is true, then, for all but finitely many origamis in $\mathcal{H}(4)$, the Lyapunov exponents are simple.

Finally, our Theorem B above was recently used by Eskin and the author [4] to show that the simplicity of Lyapunov exponents associated to *variations of Hodge structures* of mirror quintic Calabi-Yau 3-folds (a setting recently considered by Kontsevich as a natural generalization of the Teichmüller flow on the unit cotangent bundle of the moduli spaces of curves).

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A finite dimensional approach to Bramham's approximation theorem

Patrice Le Calvez

An *irrational pseudo rotation* is an area preserving homeomorphism of the Euclidean unit disk $\mathbf{D}$ that fixes 0 and that does not possess any other periodic point. To such a homeomorphism is associated an irrational number $\overline{\alpha} \notin \mathbf{Q}/\mathbf{Z}$ characterized by the following: every point admits $\overline{\alpha}$ as a rotation number. In particular the Poincaré rotation number of the restriction of f to the unit circle $\mathbf{S}$ is $\overline{\alpha}$. In the case where f is a C^k diffeomorphism, $1 \leq k \leq \infty$, we will say that f is a C^k irrational pseudo-rotation. Constructions of dynamically non trivial irrational pseudo-rotations are based on the method of fast periodic approximations, starting from the seminal paper of Anosov-Katok [1] to more recent developments (see Fayad-Saprykina [4] for example).

Barney Bramham has recently proved the following (see [2]):

Theorem 1: *Every C^∞ irrational pseudo-rotation f is the limit, for the C^0 topology, of a sequence of periodic smooth diffeomorphisms.*

The result is more precise. Let $(q_n)_{n\geq 0}$ be a sequence of positive integers such that the sequence $(q_n\overline{\alpha})_{n\geq 0}$ converges to 0 in $\mathbf{R}/\mathbf{Z}$. One can construct a sequence of homeomorphisms $(f_n)_{n\geq 0}$ fixing 0 and satisfying $(f_n)^{q_n} = \mathrm{Id}$ that converges to f in the C^0 topology. Such a map f_n is C^0 conjugate to a rotation of rational angle (mod. π). Approximating the conjugacy by a smooth diffeomorphism permits to approximate f_n by a smooth diffeomorphism of same period.

The proof uses pseudoholomorphic curves techniques from symplectic geometry. Trying to find a finite dimensional proof of this result is natural, as some results of symplectic geometry admit finite dimensional proofs by the use of generating families. A seminal example is Chaperon's proof of Conley-Zehnder's Theorem via broken geodesics method (see [3]): if F is the time one map of a Hamiltonian flow on the torus $\mathbf{R}^{2r}/\mathbf{Z}^{2r}$, a function can be constructed on a space $\mathbf{R}^{2r}/\mathbf{Z}^{2r} \times \mathbf{R}^{2n}$ whose critical points are in bijection with the contractible fixed points of F. Studying the dynamics of the gradient vector field ξ permits to minimize the number of critical points. Writing F as a composition of diffeomorphisms C^1 close to the identity is the way Chaperon constructs a generating family. Decomposing F in monotone twist maps alternatively positive or negative is another possible way. It is the fact that F is isotopic to the identity that is essential in the construction of the vector field ξ, but in the general case ξ has no reason to be a gradient vector field and its dynamics may be more complex. Nevertheless, if $r = 1$ it will satisfy some "canonical dissipative properties" and its dynamics can be surprisingly well understood (see [5] for the case where F is decomposed in monotone twist maps). Among the applications, one can note the following approximation result (see [6]): every minimal C^1 diffeomorphism F of $\mathbf{R}^2/\mathbf{Z}^2$ that is isotopic to the identity is a limit in the C^0 topology of a sequence of periodic diffeomorphisms. The proofs given in [2] and [6] share a thing in common: the construction of a foliation satisfying a certain "dynamical transverse property" on which a finite group acts, the approximating map being naturally related to this action. In [2] the foliation is defined on $\mathbf{R}\times\mathbf{D}\times\mathbf{R}/\mathbf{Z}$ and the leaves are either pseudoholomorphic cylinders or pseudoholomorphic half cylinders transverse to the boundary; in [6], the foliation is singular and naturally conjugate to the foliation by orbits of ξ on an invariant torus. Therefore it is natural to look for a proof of Bramham's theorems by a method close to the one given in [6]. The original proof of Theorem 1 is divided in two cases: the case where the restriction of f to $\mathbf{S}$ is smoothly conjugate to rotation, and the case where it is not. We succeeded to treat the first case, with some improvements due to the fact that we work in the C^1 category but unfortunately could not get the general case. Therefore we will prove:

THEOREM 1': *Every C^1 irrational pseudo-rotation f, whose restriction to $\mathbf{S}$ is C^1 conjugate to a rotation, is the limit, in the C^0 topology, of a sequence of periodic smooth diffeomorphisms.*

Observe that it is sufficient to prove Theorem 1' in the case where the restriction to $\mathbf{S}$ is a rotation. Indeed, every C^1 diffeomorphism of $\mathbf{S}$ can be extended to a C^1 area preserving diffeomorphism of $\mathbf{D}$). So, every C^1 irrational pseudo-rotation, whose restriction to $\mathbf{S}$ is C^1 conjugate to a rotation, is itself conjugate to a C^1 irrational pseudo rotation, whose restriction to $\mathbf{S}$ is a rotation.

Let us explain the idea of the proof. The first difficulty arises from the fact that f is defined on a surface with boundary. If one supposes that $f|_{\mathbf{S}}$ is a rotation, one can extend easily our map to the whole plane. Inside a small neighborhood of $\mathbf{D}$ we extend our map by an integrable polar twist map and outside by a rotation whose angle is irrational (mod. π) and close but different from $2\pi\alpha$. This implies that $\mathbf{S}$ is accumulated by invariant circles $S_{p/q}$ on which the map is periodic with a rotation number p/q that is a convergent of α, where $\alpha+\mathbf{Z}=\overline{\alpha}$. Our extended map is piecewise C^1 and one can construct a generating family of functions that are C^1 with Lipschitz derivatives. One knows that for every $q\geq 1$, the fixed point set of f^q corresponds to the singular points of a gradient vector field ξ_q defined on a space E_q depending on q. In particular each circle $S_{p/q}$ corresponds to a curve $\Sigma_{p/q}$ of singularities of ξ_q. There is a natural action of $\mathbf{Z}/q\mathbf{Z}$ that levaes ξ_q invariant. A crucial point is the fact that ξ_q is A Lipschitz with a constant A that does not depend on q. An important consequence is an uniform inequality between the L^2 norm of an orbit (the square root of the energy) and its L^0 norm. The fundamental result is the fact that $\Sigma_{p/q}$ bounds a disk $\Delta_{p/q}$ that contains the singular point corresponding to the fixed point 0 and that is invariant by the flow and by the $\mathbf{Z}/q\mathbf{Z}$ action. Moreover the dynamics on $\Delta_{p/q}$ is North-South and the non trivial orbits have the same energy. This energy can be explicitely computed and is small if p/q is a convergent of α. Consequently the vector field is uniformly small on $\Delta_{p/q}$. The approximation map will be related to $\xi_q|_{\Delta_{p/q}}$: one gets finite dimensional analogous of the arguments of [2]. The existence of the disk $\Delta_{p/q}$ is a consequence of the fact that ξ_q admits a *dominated structure*: there exists a canonical discrete Lyapounov function for the product flow on $E_q\times E_q$. The vector field would have been C^1, there would have exist a dynamically coherent dominated splitting with $\Delta_{p/q}$ as an integral manifold of the plane fields. In our Lipschitz situation, one can still use a graph transformation method to construct $\Delta_{p/q}$ and its invariance is a consequence of the following fact: the set $\{0\} \cup S_{p/q}$ is a *maximal unlinked fixed points set* of f^q, which means that there exists an isotopy from identity to f^q that fixes every point of $\{0\} \cup S_{p/q}$ and there is no larger subset of the fixed point set of f^q that satisfies this property.

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Growth of Sobolev norms for the cubic nonlinear Schrödinger equation

Marcel Guardia

(joint work with Vadim Kaloshin)

The purpose of this talk is to study the growth of Sobolev norms for the periodic cubic defocusing nonlinear Schrödinger equation with a convolution potential,

$$\begin{cases} -i\partial_t u + \Delta u + V(x) * u = |u|^2 u \\ u(0,x) = u_0(x), \end{cases} \tag{0.1}$$

where $x \in \mathbb{T}^2 = \mathbb{R}^2/(2\pi\mathbb{Z})^2$, $u : \mathbb{R} \times \mathbb{T}^2 \to \mathbb{C}$ and $V \in H^{s_0}(\mathbb{T}^2)$, $s_0 > 0$, with real Fourier coefficients. We also include the case $V = 0$. This equation is globally well posed in time in any Sobolev space H^s with $s \geq 1$.

If we write the Fourier series

$$u(t,x) = \sum_{n \in \mathbb{Z}^2} a_n(t) e^{inx} \qquad \text{and} \qquad V(x) = \sum_{n \in \mathbb{Z}^2} v_n e^{inx},$$

equation (0.1) becomes an infinite dimensional ordinary differential equation for the Fourier coefficients a_n,

$$-i\dot{a}_n = \left(|n|^2 + v_n\right) a_n + \sum_{\substack{n_1,n_2,n_3 \in \mathbb{Z}^2 \\ n_1 - n_2 + n_3 = n}} a_{n_1} \overline{a_{n_2}} a_{n_3}. \tag{0.2}$$

Note that the assumption that V has real Fourier coefficients implies that for this equation $a = 0$ is an elliptic critical point. Equation (0.2) is Hamiltonian.

The problem of growth of s-Sobolev norms in Hamiltonian Partial Differential Equations (PDE) has drawn a wide attention in the past decades and was considered by Bourgain one of the next century problems in Hamiltonian PDE [2]. The importance of this phenomenon is that solutions undergoing a large growth of s-Sobolev norm with $s > 1$ are solutions which, as time evolves, transfer energy to higher and higher modes. The s-Sobolev norm is defined by

$$\|u(t)\|_{H^s(\mathbb{T}^2)} := \|u(t,\cdot)\|_{H^s(\mathbb{T}^2)} := \left(\sum_{n\in\mathbb{Z}^2} \langle n\rangle^{2s} |a_n|^2\right)^{1/2},$$

where $\langle n\rangle = (1+|n|^2)^{1/2}$. It follows from conservation of energy that the H^1-norm of any solution of (0.1) is uniformly bounded. Therefore, if the H^s-norm of a solution grows indefinitely for some given $s > 1$ while the H^1-norm stays bounded, the energy of the solution of (0.1) must be transferred to higher modes.

In [1] (see also [7]), Bourgain considered equation (0.1) with $V = 0$ and obtained upper bounds for the possible growth of Sobolev norms. More concretely, he proved that

$$\|u(t)\|_{H^s} \le t^{C(s-1)} \|u(0)\|_{H^s} \quad \text{for} \quad t \to \infty.$$

In [2], Bourgain posed the following question, *Are there solutions of* (0.1) *with* $V = 0$ *with periodic boundary conditions in dimension* 2 *or higher with unbounded growth of* H^s*-norm for* $s > 1$*?*

Moreover, he conjectured, that the upper bound that he had obtained in [1] was not optimal and that the growth should be subpolynomial in time, that is,

$$\|u(t)\|_{H^s} \ll t^{\varepsilon} \|u(0)\|_{H^s} \quad \text{for} \quad t \to \infty, \text{ for all } \varepsilon > 0.$$

The question posed by Bourgain is still open. The first result for (0.1) with $V = 0$, is due to Kuksin. In [6], he proves the existence of solutions with any prescribed growth of s-Sobolev norm taking initial data large enough (depending on the prescribed growth). In this talk we are rather interested in showing growth of Sobolev norms for solutions with small initial data. That is, for solutions close (in some topology) to the solution $u = 0$. From the dynamical systems point of view, $u = 0$ is an elliptic critical point and therefore, showing growth of Sobolev norms for small initial data means that the critical point is unstable in the Sobolev spaces H^s, $s > 1$. The first paper dealing with such setting is [3].

Theorem 3 ([3])**.** *Fix* $s > 1$, $\mathcal{C} \gg 1$ *and* $\mu \ll 1$. *Then there exists a global solution* $u(t) = u(t,\cdot)$ *of* (0.1) *with* $V = 0$ *and* $T > 0$ *satisfying that*

$$\|u(0)\|_{H^s} \le \mu, \qquad \|u(T)\|_{H^s} \ge \mathcal{C}.$$

Note that the initial Sobolev norm gives bounds for the mass and energy of the solution u, which are constant as time evolves, and therefore are small for all time. The paper [3] does not give estimates for the time T with respect to the growth of the Sobolev norms, namely estimates of the speed of the growth. These estimates have been obtained recently in [4].

Theorem 4 ([4]). *Let $s > 1$. Then, there exists $c > 0$ with the following property: for any large $\mathcal{K} \gg 1$ there exists a a global solution $u(t) = u(t, \cdot)$ of (0.1) with $V = 0$ and a time T satisfying $0 < T \leq \mathcal{K}^c$ such that*

$$\|u(T)\|_{H^s} \geq \mathcal{K} \, \|u(0)\|_{H^s}.$$

Moreover, this solution can be chosen to satisfy

$$\|u(0)\|_{L^2} \leq \mathcal{K}^{-\sigma}.$$

for some $\sigma > 0$ independent of $\mathcal{K}$.

Note that this theorem does not give any information of the initial Sobolev norm but only on its growth. Nevertheless, it is dealing with small data since the L_2-norm of the solution is very small. One can impose also that the solution has small initial s-Sobolev norm at the expense of obtaining a slower growth.

Theorem 5 ([4]). *Let $s > 1$. Then there exists $c > 0$ with the following property: for any small $\mu \ll 1$ and large $\mathcal{C} \gg 1$ there exists a a global solution $u(t) = u(t, \cdot)$ of (0.1) with $V = 0$ and a time T satisfying*

$$0 < T \leq \left(\frac{\mathcal{C}}{\mu}\right)^{c \ln(\mathcal{C}/\mu)}$$

such that

$$\|u(T)\|_{H^s} \geq \mathcal{C} \quad \text{and} \quad \|u(0)\|_{H^s} \leq \mu.$$

In [5] it is shown that the instability mechanism developed in [3, 4] is also valid, with some modifications, if one considers equation (0.1) with a convolution potential. Therefore for this equation there also exist solutions with arbitrarily high, but finite, growth of Sobolev norm.

Theorem 6 ([5]). *Let $s_0 > 0$ and $s > 1$ and take $V \in H^{s_0}(\mathbb{T}^2)$ with real Fourier coefficients. Then, there exists $c > 0$ with the following property: for any large $\mathcal{K} \gg 1$ there exists a global solution $u(t) = u(t, \cdot)$ of (0.1) and a time T satisfying $0 < T \leq \mathcal{K}^c$ such that*

$$\|u(T)\|_{H^s} \geq \mathcal{K} \, \|u(0)\|_{H^s}.$$

Moreover, this solution can be chosen to satisfy

$$\|u(0)\|_{L^2} \leq \mathcal{K}^{-\sigma}.$$

for some $\sigma > 0$ independent of $\mathcal{K}$.

We can also impose initial small Sobolev norm as in Theorem 5. Nevertheless, then we obtain a slower growth, as happened in that theorem, and we need to add the extra hypothesis that the potential satisfies $V \in H^{s_0}$ with $s_0 > 70s/17$ instead of $s_0 > 0$. This hypothesis is certainly not optimal. Nevertheless, it is needed with our methods to impose the smallness of the initial Sobolev norms.

Theorem 7 ([5]). *Fix $s > 1$ and Let $s_0 > 70s/17$ and take $V \in H^{s_0}(\mathbb{T}^2)$ with real Fourier coefficients. Then, there exists $c > 0$ with the following property: for any* *small* $\mu \ll 1$ *and large* $\mathcal{C} \gg 1$ *there exists a global solution* $u(t) = u(t, \cdot)$ *of (0.1) and a time* T *satisfying*

$$0 < T \leq \left(\frac{\mathcal{C}}{\mu}\right)^{c \ln(\mathcal{C}/\mu)}$$

such that

$$\|u(T)\|_{H^s} \geq \mathcal{C} \quad \text{and} \quad \|u(0)\|_{H^s} \leq \mu.$$

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Reducibility of quasiperiodic cocycles in semi-simple compact Lie groups

Nikolaos Karaliolios

Let $\alpha \in \mathbb{T}$ be a topologically minimal translation. This property is, by Kronecker's theorem, equivalent α being an irrational.

If we also let $A(\cdot) \in C^s(\mathbb{T}, G)$, where G is a Lie group with Lie algebra g, and $s \in \mathbb{N} \cup \infty$, the couple $(\alpha, A(\cdot))$ acts on the fibered space $\mathbb{T} \times G \to \mathbb{T}$ defining a diffeomorphism by

$$(\alpha, A(\cdot)).(x, S) = (x + \alpha, A(x).S)$$

for any $(x, S) \in \mathbb{T} \times G$. We will call such a diffeomorphism a *quasiperiodic cocycle over* R_α (or simply a cocycle). The space of such actions is denoted by $SW^s_\alpha(\mathbb{T}, G) \subset Diff^s(\mathbb{T} \times G)$. We usually abbreviate the notation to SW^s_α. Cocycles are a class of fibered diffeomorphisms, since fibers of $\mathbb{T} \times G$ are mapped onto fibers, and the mapping from one fiber onto another in general depends on the base point. We restrict ourselves to the case of a semi-simple compact Lie group, and G will denote such a group from now on. We also remark that consideration of tori of higher dimension is equally relevant, but since the main theorem is not known to be true is these cases, we restrict ourselves to one-dimensional tori.

The n-th iterate of the action is given by

$$(\alpha, A(\cdot))^n.(x, S) = (n\alpha, A_n(\cdot)).(x, S) = (x + n\alpha, A_n(x).S)$$

where $A_n(\cdot)$ represents the *quasiperiodic product* of matrices equal to

$$A_n(\cdot) = A(\cdot + (n-1)\alpha)...A(\cdot)$$

for positive iterates. Negative iterates are found as inverses of positive ones:

$$\begin{aligned}(\alpha, A(\cdot))^{-n} &= ((\alpha, A(\cdot))^n)^{-1} \\ &= (-n\alpha, A^*(\cdot - n\alpha)...A^*(\cdot - \alpha))\end{aligned}$$

The cocycle $(\alpha, A(\cdot))$ is called a constant cocycle if $A(\cdot) = A \in G$ is a constant mapping. In that case, the quasiperiodic product reduces to a simple product of matrices

$$(\alpha, A)^n = (n\alpha, A^n)$$

and the dynamics become easy to describe. Another, more general, distinct class of cocycles having relatively simple dynamics is given by the applications $A(\cdot)$ taking values in an abelian subgroup of G. Such cocycles will be called *abelian*.

The relevant definition of dynamical conjugation in this case is the fibered conjugation, defined by

$$\begin{aligned}Conj_{B(\cdot)}.(\alpha, A(\cdot)) &= (\alpha, B(\cdot + \alpha).A(\cdot).B^{-1}(\cdot)) \\ &= (0, B(\cdot)) \circ (\alpha, A(\cdot)) \circ (0, B(\cdot))^{-1}\end{aligned}$$

where $B(\cdot) : \mathbb{T}^d \to G$. The dynamics of $Conj_{B(\cdot)}.(\alpha, A(\cdot))$ and $(\alpha, A(\cdot))$ are essentially the same, since

$$(Conj_{B(\cdot)}.(\alpha, A(\cdot)))^n = (n\alpha, B(\cdot + n\alpha).A_n(\cdot).B(\cdot))^{-1}$$

Naturally, we will say that two cocycles are *conjugate* iff there exists such a conjugation $B(\cdot)$.

The most interesting cases of cocycles are those who are conjugate to a constant one, called *reducible cocycles*, and those who are conjugate to an abelian cocycle, called *torus-reducible* since an abelian subroup of a compact group is isomorphic to a torus.

After having established the basic vocabulary, we can state the main theorem discussed in the talk.

Theorem 3. *Let $\alpha \in RDC$ and $(\alpha, A(\cdot)) \in SW^\infty(\mathbb{T}, G)$. Then, $(\alpha, A(\cdot))^{\chi_G}$ is accumulated by reducible cocycles in $SW^\infty_{\chi_G \alpha}(\chi_G \mathbb{T}, G)$.*

In other words, for such α, any cocycle $(\alpha, A(\cdot))$ in $\mathbb{T} \times G$ has an iterate which is accumulated by reducible cocycles and the maximal number χ_G of iterations needed in order to satisfy this property depends only on the group G. The constant χ_G is related with the homotopy of the group, and is equal to 1 for $G = SU(w+1)$, and 2 for $G = SO(3)$. For the remaining of this text we will suppose that $\chi_G = 1$ in order to simplify the exposition. The set $RDC \subset \mathbb{T}$ stands for the *Recurrent Diophantine Numbers*, and is a full Haar measure condition.

The proof of the theorem relies on the following results, which we will state informally. Let $(\alpha, A(\cdot)) \in SW^\infty_\alpha$. Then, for a.e. $x \in \mathbb{T}$, and for $n \in \mathbb{N}$ large enough, we can find a (maximal) subgroup $\tilde{G}(x) = G_0(x) \times G_+(x) \subset G$, satisfying the following properties. The subgroup G_0 is semi-simple, and for $S \in G_0$, the curve $t \mapsto (\alpha, A(\frac{\cdot - x}{n}))^n.(t, S)$ for $t \in [-1, 1]$ is close to a constant. The subgroup G_+ is abelian (and therefore a torus $\mathbb{T}^d$), and for $S \in G_+$, the curve $t \mapsto (\alpha, A(\frac{\cdot - x}{n}))^n.(t, S)$ for $t \in [-1, 1]$ is close to a mapping $t \mapsto \exp(t\phi(x))$, for some $\phi(x) \in \mathbb{R}^d \hookrightarrow g$. Here, $\mathbb{R}^d$ is the Lie algebra of G_+.

The groups $\tilde{G}(x)$ are isomorphic for a.e. x. The vectors $\phi(x)$ are well defined for a.e. x, depend measurably on x, and belong to the same class of vectors for conjugation in g (i.e. in a single orbit under $g \ni s \mapsto Ad(S).s \in g$, for $S \in G$ and a fixed $s \in g$). This fact allows us to suppose that $x = 0$ is a "good point", meaning that the subgroups G_0 and G_+, as well as the vector ϕ are well defined at this point.

For the identification of the vector $\phi(0)$ we use the renormalization scheme introduced in [6]. This scheme allows us to relate the dynamics of the given cocycle $(\alpha, A(\cdot))$ with those of $(\alpha_n, \hat{A}_n(\cdot))$, where $\alpha_n = G^n(\alpha)$ and G is the Gauss map $\alpha \mapsto [\alpha^{-1}]$ ($[\cdot]$ is the integer part). The cocycle $(\alpha_n, \hat{A}_n(\cdot))$ is obtained via continued fractions for α and conjugation in the fibers in an "almost canonical" way. By this, we mean that different cocycles $(\alpha_n, \hat{A}_n(\cdot))$ obtained in the same way are conjugate to each other, and that when the dynamics of $(\alpha_n, A_{q_n}(\cdot))$ (q_n is a denominator of a convergent of α) approach the limit object described previously, there is a natural way of obtaining $(\alpha_n, \hat{A}_n(\cdot))$ from the former. The proof of the convergence of the scheme shows that that for n big enough the derivative of such $\hat{A}_n(\cdot) \in C^\infty(\mathbb{T}, G)$ must be close to $\phi(0)$. Since $\hat{A}_n(\cdot)$ is one-periodic, $\phi(0)$, if it is non-zero, must be equal to the derivative of a periodic geodesic of G. Therefoere, its class modulo conjugations is quantized. If $\phi(0) = 0$, then $\hat{A}_n(\cdot)$ is close to a constant. In any case, we obtain

Theorem 4. *The models of the dynamics of cocycles in SW^∞_α are C^∞-small perturbations of abelian cocycles in $SW^\infty_{\alpha_n}$, for n big enough.*

In other terms, non-commutativity dies out (in a measurable way) due to ergodicity of $x \mapsto x + \alpha$. This observation is used in the proof of

Theorem 5. *Periodic geodesics are accumulated by cocycles whose limit object corresponds to a shorter geodesic.*

Here, by shorter we mean that the derivative within G_+ has smaller norm after perturbation, and eventually that the subgroup G_0 becomes bigger. The perturbation consists in a careful choice of terms not commuting with G_+.

This fact, together with

Theorem 6. *If $\alpha \in RDC$, then there exist n big enough such that $(\alpha_n, \hat{A}_n(\cdot))$ is accumulated by cocycles conjugate to the corresponding geodesic.*

allow us, after a finite number of arbitrarily small perturbations to obtain a cocycle $(\alpha, A'(\cdot))$, close to $(\alpha, A(\cdot))$, for which the corresponding subgroup $G_0 = G$. Therefore, there exists an $m \in \mathbb{N}$ ($m \gg n$) such that $(\alpha_m, \hat{A}'_m(\cdot))$ is close to a constant. Then, the local density theorem obtained in [5] can be applied.

Theorem 7. *If $\alpha \in RDC$, and $(\alpha, A'(\cdot))$ is as above, then there exist m big enough such that $(\alpha_m, \hat{A}'_m(\cdot))$ is accumulated reducible cocycles.*

The fact that "hatted" cocycles are obtained "almost canonically" by the "unhatted" ones and the fact that perturbations can be chosen to be arbitrarily small finishes the proof.

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First steps towards invariant circles using pseudoholomorphic curve methods

Barney Bramham

Let D be the closed unit disk in the plane. For this talk a pseudo-rotation was a C^∞-diffeomorphism $\varphi : D \to D$ having the following three properties in common with a rotation about the origin through an irrational angle: (1) φ is area and orientation preserving, (2) φ fixes the origin, and (3) φ has no other periodic points.

From work of Franks [6] one knows that the last property is equivalent to the following condition: that there exists an irrational number α such that every orbit having a well defined rotation number about the origin has rotation number α. The number α is then refered to as the *rotation number of φ*.

In 1970 Anosov and Katok [1] constructed "exotic" pseudo-rotations. So called, because they are ergodic (even weak mixing) and therefore cannot be simply in the conjugacy class of a rotation. Their examples were C^∞-limits of periodic diffeomorphisms. In [2] we showed that every pseudo-rotation is the C^0-limit of periodic diffeomorphisms.

Given the C^∞-convergence of the Anosov-Katok examples, and that these should include some of the "worst case scenarios", that is, where the limiting system is as far as possible from integrable, it is natural to ask whether the above C^0-convergence can be strengthened to some C^r-topology for $r > 0$. The following statement discussed in this talk is a small step in this direction.

Theorem 8. [1] *There exists a (non-empty, dense) subset $\mathcal{L}_0$ of the Liouville numbers such that the following holds. Let $\varphi : D \to D$ be a smooth irrational pseudo-rotation having rotation number α in $\mathcal{L}_0$. Suppose also that φ satisfies the following conditions on the boundary: for all $z \in \partial D$ and $v \in \mathbb{R}^2$,*

$$\varphi(z) = e^{2\pi i\alpha} z \qquad D\varphi(z)v = e^{2\pi i\alpha} v.$$

In other words, φ is a rotation up to first order on the boundary. Then there is a sequence of smooth diffeomorphims $\varphi_n : D \to D$, each periodic and fixing the origin, and converging to φ in the uniform C^0 topology, and moreover satisfying the uniform bounds

$$\sup_{n\in\mathbb{N}} \|D\varphi_n\|_{L^\infty(D)} < \infty.$$

In particular, for all $\gamma \in (0,1)$ we have convergence of $\varphi_n \to \varphi$ in the γ-Hölder topology.

Another reason to be interested in getting stronger convergence of the approximation maps is the following. The exotic pseudo-rotations of Anosov-Katok all had Liouvillean rotation number. Fayad and Saprykina [4] were able to significantly extend this statement and show that for every Liouville number α there exists an ergodic (again even weak mixing) pseudo-rotation with rotation number α. In particular such pseudo-rotations of course do not have interior invariant circles.

In contrast, for pseudo-rotations with Diophantine rotation number Herman showed that invariant circles always exist near the boundary. In [7] he raised the question whether moreover Diophantine pseudo-rotations are always globally conjugate to a rotation. In [5] Fayad and Krikorian prove that remarkably this is the case when the pseudo-rotation is sufficiently close to a rotation. In general the question remains open.

It seems natural to ask whether one could recover any of these existence results for invariant circles using the periodic approximation maps. (Especially as it might raise the possibility of moving away from the perturbative setting.) The naive idea would be to show that under the Diophantine condition one can control, e.g. get uniform Lipschitz bounds on, the invariant circles in the approximation maps and preserve them in the limit. For this it would suffice to bound certain directional derivatives of the approximation maps. Therefore we felt that the first small step in this direction would be to show that the approximation maps converge in a differentiable sense. This is the second motivation for theorem 8.

Of course, it is unfortunate that theorem 8 does not say anything for Diophantine rotation numbers. Moreover, even C^∞-convergence of the approximation maps would not in imply invariant circles for the limit as this would contradict the Anosov-Katok construction. So theorem 8 is very far from the statements one would like to prove. But it begins to make progress on understanding the approximation maps on an infinitesimal level, hence the, prematurely optimistic, allusion to invariant circles in the talk title.

[1] The proof is still being written up.

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Tonelli Hamiltonians with no conjugate points and C^0 integrability

Marie-Claude Arnaud
(joint work with Marc Arcostanzo, Philippe Bolle, Maxime Zavidovique)

This talk deals with C^0-integrable Tonelli Hamiltonians and Tonelli Hamiltonians without conjugate points of the cotangent bundle $T^*\mathbb{T}^n$ of the n-dimensional torus.

If the Tonelli Hamiltonian is a Riemannian metric, these properties coincide and have strong implications. Indeed, Heber showed (see [3]) in 1994 that for every Riemannian metric without conjugate points on the torus $\mathbb{T}^n$, there is a continuous foliation of the unit tangent bundle by tori which are Lipschitz, Lagrangian and invariant by the geodesic flow. The same year, this was improved by Burago and Ivanov who proved (see [2]) that such a metric has to be flat; as an immediate consequence, the continuous foliation in Heber's result is in fact smooth.

The notion of Tonelli Lagrangian is a vast extension of the concept of Riemannian metric, but we prove that Heber's result still holds:

Theorem 1 *Let H be a Tonelli Hamiltonian on $T^*\mathbb{T}^n$. Then H has no conjugate points if and only if there is a continuous foliation of $T^*\mathbb{T}^n$ by Lipschitz, Lagrangian, flow-invariant graphs.*

The proof uses ideas coming from weak KAM and Aubry-Mather theory. In fact, we establish that each leaf of the above foliation is the Aubry set corresponding to some cohomology class. More precisely, the first step of the proof is to see that some of those Aubry sets (later denoted by $\mathcal{G}_{T,r}$, with $T > 0$ and $r \in \mathbb{Z}^n$) are covered by periodic orbits of the Hamiltonian flow $(\phi_t^H)_{t\in\mathbb{R}}$, of a given period T (and a given homology class $r \in \mathbb{Z}^n$). In particular, the dynamics on the corresponding leaves is periodic.

The existence of those particular leaves is used again to prove a second theorem. Using a KAM theorem, we prove that such sets $\mathcal{G}_{T,r}$ are accumulated by KAM tori on which the dynamics is conjugated to an irrational rotation. We deduce:

Theorem 2 *Let (ϕ_t^H) be a C^∞ Tonelli flow of $T^*\mathbb{T}^n$ with no conjugate points and let $\mathfrak{F}$ be the continuous foliation in invariant Lagrangian tori that is given by theorem 1. Then there is a dense G_δ subset $\mathcal{G}$ of $\mathfrak{F}$ such that, for every $\mathcal{T} \in \mathcal{G}$, then $\phi^H_{1|\mathcal{T}}$ is strictly ergodic.*

We recall:

Definition A set $K \subset T\mathbb{T}^n$ that is invariant by a Tonelli flow (ϕ_t^H) is *strictly ergodic* if:

- the restricted flow $(\phi^H_{t|K})$ has a unique invariant Borel probability measure; this measure is denoted by μ;
- the support of μ is K.

The last section of this article is devoted to studying the entropy of Tonelli Hamiltonians without conjugate points. Indeed, it is not hard to see that a regular completely integrable Hamiltonian system has zero topological entropy. When singularities are allowed, the situation can become more complicated, as shown in the article [1] of Bolsinov and Taimanov.
Hence it seems to be natural to ask what can happen for a C^0-integrable Tonelli Hamiltonian. In this case, we don't know the restricted dynamics to all the invariant tori, hence it is not so obvious to decide if the topological entropy is zero or not. An answer to this question is provided by the following:

Theorem 3 *Let $H : T^*\mathbb{T}^n \to \mathbb{R}$ be a C^3 Tonelli Hamiltonian that is C^0-integrable. Then for every invariant Borel probability measure, the Lyapunov exponents are zero.*

This implies that both the metric, and topological entropies must also be 0. Observe that the conclusion of theorem 3 is true for a C^0-integrable Tonelli Hamiltonian defined on T^*M for any closed manifold M.

Some interesting questions remain open after this work, as:

Questions
1) Does a C^0 integrable Tonelli Hamiltonian exist that is not C^1 integrable?
2) Can an invariant torus of a C^0 integrable Tonelli Hamiltonian flow carry two invariant measures that have not the same rotation number (see the appendix for the definition of the rotation number)?

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Tire tracks geometry, continuous and discrete bicycle transformation, and the filament equation

Serge Tabachnikov

The goal of this talk is two-fold. The first part is an exposition of the joint work with R. Foote and M. Levi on the bicycle monodromy, see [5, 2]. The second part concerns the continuous and discrete bicycle (Darboux, Bäcklund) tyransformation, its complete integrability and its relation with the filament (smoke ring, binormal, local induction) equation. This is work in progress, see [7, 8].

The model of a bicycle is a directed segment of fixed length ℓ in the plane that can move so that the trajectory of the rear end is always tangent to the segment (since the rear wheel is fixed on the frame). Thus, given the rear wheel track γ and the direction, the front track Γ is uniquely determined: Γ is the locus of the end points of tangent segments to γ of length ℓ.

However, the front track Γ determines the rear track only if the initial position of the bicycle is given. The monodromy map $M_\ell(\Gamma)$ arises that assigns the terminal position to the initial one. If Γ is a closed curve, $M_\ell(\Gamma)$ is a self-map of a circle of radius ℓ. A similar construction can be made in any dimension resulting in a map of a sphere of radius ℓ.

The monodromy map $M_\ell(\Gamma)$ is a Möbius transformation (the sphere is considered as the sphere at infinity of the hyperbolic space, and the Möbius group as consisting of hyperbolic isometries). In the planar case, one identifies the circle with the real projective line via a stereographic projection, and the Möbius transformations with fractional-linear ones.

A Möbius transformation of a circle can be elliptic, parabolic, or hyperbolic, depending on the number of fixed points (0, 1, or 2). I outlined the proof of a hundred-year-old Menzin's conjecture: *if* Γ *is an oval of area greater than* $\pi\ell^2$ *then the monodromy is hyperbolic.* The proof makes use of a version of isoperimetric inequality for curves with cusps (wave fronts).

Similar results hold in the spherical and hyperbolic geometries, see [3].

Start with a closed cooriented rear track γ and consider the respective front track Γ. The monodromy $M_\ell(\Gamma)$ has a fixed point, hence there is another fixed point corresponding to another closed cooriented rear track, say γ'. The relation between γ and γ' is an involution. Change the coorientation of γ' to the opposite, and denote the resulting rear wheel track by γ^*. Let Γ^* be the respective front wheel track. The transformations $\gamma \mapsto \gamma^*$ and $\Gamma \mapsto \Gamma^*$ arise, called the bicycle transformations. Denote by T_ℓ the former.

In dimension three, Möbius transformations are complex fractional-linear, hence always have two fixed points. Thus T_ℓ is always defined in $\mathbf{R}^3$.

The following properties are valid in all dimensions:

(1) The bicycle transformations with different length parameters ℓ commute: $T_{\ell_1} \circ T_{\ell_2} = T_{\ell_2} \circ T_{\ell_1}$.

(2) If two curves are related by the bicycle transformation then their bicycle monodromies, for all values of the length parameter λ, are conjugated: $M_\lambda(T_\ell(\Gamma)) = M_\lambda(\Gamma)$.

(3) The following are integrals of the bicycle transformations:

$$\int_\Gamma \Gamma(t) \wedge \Gamma'(t)\ dt \quad \text{and} \quad \int_\Gamma (\Gamma(t) \cdot \Gamma'(t))\ \Gamma(t)\ dt$$

(the area bivector and the "center of mass").

Define two differential 2-forms on parametric closed curves in $\mathbf{R}^3$. Let $\Gamma(t)$ be a curve and $u(t), v(t)$ two vector fields along Γ. Then

$$\omega(u,v) = \int u'(t) \cdot v(t)\ dt, \quad \Omega(u,v) = \int \det(\Gamma'(t), u(t), v(t))\ dt.$$

Both forms are closed (in fact, exact).

One has the following result: *The bicycle transformation preserves both forms, ω and Ω.*

If $\Gamma(t)$ is an arc length parameterized curve in $\mathbf{R}^3$ then the evolution $\dot\Gamma = \Gamma' \times \Gamma''$ is called the binormal equation.

One has the following results: *the filament equation also preserves the forms ω and Ω. The two systems, the bicycle transformation and the filament equation, commute and share integrals.*

The sequence of integrals of the filament equation starts with

$$\int 1\ dx,\ \int \tau\ dx,\ \int \kappa^2\ dx,\ \int \kappa^2\tau\ dx,\ \int \left((\kappa')^2 + \kappa^2\tau^2 - \frac{1}{4}\kappa^4 \right)\ dx, \ldots$$

where τ is the torsion and κ is the curvature of a space curve, see, e.g., [4] .

A discrete version of the bicycle transformation is studied in [6, 8]. It is a discrete time dynamical system on polygons depending in a parameter ℓ (the length of the bicycle frame). In particular, the following construction of the *circumcenter of mass* defines an integral of the discrete bicycle transformation in the plane; see also [1].

Given a polygon, consider its triangulation and take the circumcenters of the triangles with the weights equal to their areas. One has the following result: *the center of mass of these points is independent of the triangulation (likewise, in the spherical and hyperbolic geometries, and for simplicial polyhedra too).*

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Existence of special finite-energy foliations on $SO(3)$ and applications to positively curved geodesic flows on the 2-sphere

Umberto L. Hryniewicz
(joint work with Joan E. Licata, Pedro A. S. Salomão and Kris Wysocki)

The concept of a (stable) finite energy foliation, introduced by Hofer, Wysocki and Zehnder in [6], is a powerful tool to study the global behavior of Hamiltonian dynamics on contact-type three-dimensional energy levels. Loosely speaking, it is a foliation of the energy level with a finite number of closed characteristics removed, satisfying special properties. In particular, each page is transverse to the Reeb flow, and is the projection of a finite-energy pseudo-holomorphic punctured sphere in the symplectization of the energy level. The applications are numerous, for instance, Hofer, Wysocki and Zehnder proved in [5] that strictly convex energy levels in $\mathbf{R}^4$ always have a finite energy foliation in the form of an open book decomposition with disk-like pages which are global surfaces of section for the flow. In particular, with the help of a deep result due to Franks [1], they are able to deduce the existence of two or infinitely many closed characteristics. In this talk we describe two applications of this method.

The first explores relationships between Reeb dynamics and the topology of three-dimensional contact-type energy levels. In order to investigate these relationships one may ask the following

Question I. Can the diffeomorphism type of a closed connected 3-manifold be determined in terms of dynamical properties of a flow defined on it?

A result in this direction would provide a dynamical characterization of the manifold. When one wishes to state and prove a characterization theorem it is a common procedure to endow the space with additional structure, hoping that this will provide additional tools. We propose to see the manifold as a contact-type energy level, so we end up looking for a characterization theorem not only of the diffeomorphism type but also of the induced contact structure. The tools that become available taking this particular viewpoint come from the theory of pseudo-holomorphic curves in symplectizations introduced by Hofer [2]. The study of this question was initiated by Hofer, Wysocki and Zehnder in [3, 4] where the tight three-sphere was first dynamically characterized.

Theorem A. (Hryniewicz, Licata and Salomão [7]**)** Let (M, ξ) be a closed connected tight contact 3-manifold such that $c_1(\xi)$ vanishes on $\pi_2(M)$. Then (M, ξ) is contactomorphic to the standard lens space $(L(p, q), \xi_{\text{std}})$ if, and only if, $\xi = \ker \lambda$ for a contact form λ admitting a closed Reeb trajectory K such that

i) K is p-unknotted, has self-linking number $-p$, monodromy $-q$, its p-th iterate has Conley-Zehnder index ≥ 3, and
ii) all other closed Reeb orbits which are contractible and have transverse rotation number 1 are linked with K.

The methods used to prove Theorem A can also be used to construct global surfaces of section for Reeb flows on $(L(p, q), \xi_{\text{std}})$, even for binding orbits with high Conley-Zehnder index. This has applications to the study of classical Hamiltonian systems, like geodesic flows on the two-sphere and the planar circular restricted three-body problem.

To describe the second application discussed in the talk we start by recalling an old theorem due to Birkhoff. Geodesic flows on the two-sphere always have simple closed geodesics, and to such a geodesic one can associate the so-called Birkhoff annulus: this is embedded in the unit sphere bundle and consists of the unit vectors along the geodesic pointing inside one of the hemispheres.

Theorem. (Birkhoff) If the Gaussian curvature is everywhere positive then a Birkhoff annulus is a global surface of section.

What is unsatisfactory about this statement is that positivity of Gaussian curvature is not a symplectically invariant condition, and one may ask

Question II. Is there a symplectically invariant condition which is sufficient for constructing annulus-like global surfaces of section for Reeb flows on $\mathbf{RP}^3$ with its standard contact structure?

The answer is provided by the following statement which is joint work with Pedro A. S. Salomão and Kris Wysocki.

Theorem B. (Hryniewicz, Salomão and Wysocki [8]**)** Assume that a tight Reeb flow on $\mathbf{RP}^3$ admits a pair of closed orbits P_0, P_1 forming a Hopf link. If all closed Reeb orbits have Conley-Zehnder index greater than or equal to 1 and all closed Reeb orbits with Conley-Zehnder index 2 are not contractible in $\mathbf{RP}^3 \setminus (P_0 \cup P_1)$, then there exists an annulus-like global surface of section bounded by $P_0 \cup P_1$.

By a Hopf link in $\mathbf{RP}^3$ we mean a transverse link that lifts to the transverse isotopy class of a standard Hopf link in the tight three-sphere. Theorem B implies the result of Birkhoff. In fact, if $\gamma(t)$ is a closed simple geodesic then $\dot{\gamma}(t)$ and

$-\dot{\gamma}(-t)$ form a Hopf link, moreover, positivity of the Gaussian curvature forces Conley-Zehnder indices to be strictly positive. The linking assumption is provided by the Gauss-Bonnet theorem. Hence Birkhoff's result gets generalized to a much broader class of Hamiltonian systems, and the symplectically invariant condition running in the background gets revealed.

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Arnold diffusion and weak quasiergodic hypothesis

Vadim Kaloshin

(joint work with Marcel Guardia, Ke Zhang)

1. Strong form of Arnold diffusion

Let $(\theta, p) \in \mathbb{T}^2 \times B^2$ be the phase space of an integrable Hamiltonian system $H_0(p)$ with $\mathbb{T}^2$ being 2-dimensional torus $\mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2 \ni \theta = (\theta_1, \theta_2)$ and B^2 being the unit ball around 0 in $\mathbb{R}^2$, $p = (p_1, p_2) \in B^2$. Assume that H_0 is strictly convex, i.e. Hessian $\partial^2_{p_i p_j} H_0$ is strictly positive definite.

Consider a smooth time periodic perturbation

$$H_\varepsilon(\theta, p, t) = H_0(p) + \varepsilon H_1(\theta, p, t), \quad t \in \mathbb{T} = \mathbb{R}/\mathbb{Z}.$$

We study a strong form of Arnold diffusion for this system, namely,

existence of orbits $\{(\theta_\varepsilon, p_\varepsilon)(t)\}_t$ going from one open set $p_\varepsilon(0) \in U$ to another $p_\varepsilon(t) \in U'$ for some $t = t_\varepsilon > 0$.

Arnold [1] proved existence of such orbits for an example and conjectured that they exist for a typical perturbation (see e.g. [2, 3, 4]).

Integer relations $\vec{k} \cdot (\partial_p H_0, 1) = 0$ with $\vec{k} = (\vec{k}_1, k_0) \in (\mathbb{Z}^2 \setminus 0) \times \mathbb{Z}$ and $\cdot$ being the inner product define *a resonant segment.* The condition that the Hessian of

H_0 is non-degenerate implies that $\partial_p H_0 : B^2 \to \mathbb{R}^2$ is a diffeomorphism and each resonant line defines a smooth curve embedded into action space

$$\Gamma_{\vec{k}} = \{p \in B^2 : \ \vec{k} \cdot (\partial_p H_0, 1) = 0\}.$$

If curves $\Gamma_{\vec{k}}$ and $\Gamma_{\vec{k}'}$ are given by two linearly independent resonances vectors $\{\vec{k}, \vec{k}'\}$, they either have no intersection or intersect at a single point in B^2.

We call a vector $\vec{k} = (\vec{k}_1, k_0) = (k_1^1, k_1^2, k_0) \in (\mathbb{Z}^2 \setminus 0) \times \mathbb{Z}$ and the corresponding resonance $\Gamma = \Gamma_{\vec{k}}$ *space irreducible* if either $(k_1^1, k_1^2) = (1, 0)$ or $(0, 1)$ or $\gcd(\vec{k}^1) = 1$, i.e. $|k_1^1|$ and $|k_1^2|$ are relatively prime.

Consider now two open sets $U, U' \subset B^2$. Select a finite collection of space irreducible resonant segments $\{\Gamma_j = \Gamma_{\vec{k}_j}\}_{j=1}^N$ for some collection of $\{\vec{k}_j\}_{j=1}^N$

- with neighbors $\vec{k}_j$ and $\vec{k}_{j+1}$ being linearly independent,
- $\Gamma_j \cap \Gamma_{j+1} \neq \emptyset$ for $j = 1, \dots, N-1$ and so that
- $\Gamma_1 \cap U \neq \emptyset$ and $\Gamma_N \cap U' \neq \emptyset$.

We would like to construct diffusing orbits along a connected path formed by segments inside Γ_j's, i.e. we select a connected piecewise smooth curve $\Gamma^* \subset \cup_{j=1}^N \Gamma_j$ so that $\Gamma^* \cap U \neq \emptyset$ and $\Gamma^* \cap U' \neq \emptyset$.

Consider the space of C^r perturbations $C^r(\mathbb{T}^2 \times B^2 \times \mathbb{T})$ with a natural C^r norm given by maximum of all partial derivatives of order up to r, here $r < +\infty$. Denote by $\mathcal{S}^r = \{H_1 \in C^r(\mathbb{T}^2 \times B^2 \times \mathbb{T}) : \ \|H_1\|_{C^r} = 1\}$ the unit sphere in this space. In [11] we prove the following

Theorem 8. *In the above notations fix the piecewise smooth segment Γ^* and $4 \leq r < +\infty$. Then there is an open and dense set $\mathcal{U} = \mathcal{U}_{\Gamma^*} \subset \mathcal{S}^r$ and a nonnegative function $\varepsilon_0 = \varepsilon_0(H_1)$ with $\varepsilon_0|_{\mathcal{U}} > 0$. Let $\mathcal{V} = \{\epsilon H_1 : H_1 \in \mathcal{U}, \quad 0 < \varepsilon < \varepsilon_0\}$, then for an open and dense set of $\epsilon H_1 \in \mathcal{W} \subset \mathcal{V}$, $\mathcal{W} \neq \mathcal{V}$ the Hamiltonian system $H_\epsilon = H_0 + \epsilon H_1$ has an orbit $\{(\theta_\varepsilon, p_\varepsilon)(t)\}_t$ whose action component satisfies*

$$p_\varepsilon(0) \in U, \qquad p_\varepsilon(t) \in U' \qquad \textit{for some} \ \ t = t_\varepsilon > 0$$

Moreover, for all $0 < t < t_\varepsilon$ the action component $p_\varepsilon(t)$ stays $O(\sqrt{\varepsilon})$-close to the union of resonances Γ^*.

Rational points on on Γ^* whose numerators and denominators are bounded by a certain constant K depending on H_0 and H_1, but independent of ε, are called strong double resonances. Diffusion away from strong double resonances is proven in [5]. Moreover, the result in [5] applies to the above setting with any number of degrees of freedom.

2. A weak form of quasiergodic hypothesis

A quasiergodic hypothesis states that a generic Hamiltonian system on a generic energy surface has a dense orbit (see e.g. [6, 7, 9]).

In [10] we construct an example of a nearly integrable system of the form $H(\theta, p) = \langle p, p \rangle /2 + \varepsilon H_1(\theta, p)$ with $p \in \mathbb{R}^3$, $\theta \in \mathbb{T}^3$ and $\langle \cdot, \cdot \rangle$ being the dot product. This system has an orbit dense in a set of Hausdorff dimension 5 inside of a

5-dimensional energy surface. We also construct examples of Lyapunov unstable KAM tori.

In [12] we modify this example and present an example of a nearly integrable system of the same form $H(\theta, p) = \langle p, p\rangle/2 + \varepsilon H_1(\theta, p)$ such that it has an orbit dense in a set of positive 5-dimensional measure. In particular, such an orbit accumulates to a positive measure set of KAM tori.

In [8] we considerably straighten both results. Let $\theta \in \mathbb{T}^2$ and $p \in B^2$. Suppose H_0 is sufficiently smooth and strictly convex. Consider a smooth time periodic perturbation

$$H_\varepsilon(\theta, p, t) = H_0(p) + \varepsilon H_1(\theta, p, t), \quad t \in \mathbb{T} = \mathbb{R}/\mathbb{Z}.$$

Fix $\gamma > 0$ and $\tau > 0$. We say that a vector $\omega \in \mathbb{R}^2$ is (τ, γ)-diophantine if $|(\omega, 1) \cdot (k_1, k_0)| \geq \gamma |(k_1, k_0)|^{-2-\tau}$ for any $(k_1, k_0) \in (\mathbb{Z}^2 \setminus 0) \times \mathbb{Z}$.

Theorem 9. [8] *With the above notations there is $r_0 > 0$ such that for any $\gamma > 0$ and $\tau > 0$ and any $r_0 \leq r < +\infty$ there is a dense set $\mathcal{D}$ in the unit sphere of perturbations $\mathcal{S}^r$ such that for any $H_1 \in \mathcal{D}$ there is ε with the property that H_ε has an orbit whose ω-limit set has Lebesgue measure at least $(1-\gamma)$ of Lebesgue measure of $\mathbb{T}^2 \times B^2 \times \mathbb{T}$. In particular, the ω-limit set contains all KAM tori with a (τ, γ)-diophantine rotation vector $(\omega, 1)$.*

By KAM theorem (see e.g. Pöschel [13]) for any sufficiently smooth small perturbation εH_1 the Hamiltonian $H_\varepsilon(\theta, p, t)$ has an invariant KAM torus for any (τ, γ)-diophantine rotation vector $(\omega, 1)$. Denote by $KAM_{\tau,\gamma}$ this union of KAM tori. Then one can define Whitney topology for pertubations vanishing on $KAM_{\tau,\gamma}$. Call such a Whitney topology *KAM Whitney topology*.

We also show that for each Hamiltonian H'_ε from this Theorem there is a KAM Whitney neighborhood such that conclusion holds for each element of this neighborhood.

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Lagrange spectra for translation surfaces

Corinna Ulcigrai
(joint work with Pascal Hubert and Luca Marchese)

The classical Lagrange Spectrum is a well studied object in Diophantine approximation and hyperbolic geometric. By Dirichlet theorem, for any irrational $\alpha \in \mathbb{R}$ there exists infinitely many integers p, q with $q \neq 0$ such that $|\alpha - p/q| < 1/q^2$. Let $L(\alpha) = \sup\{c > 0$ such that $|\alpha - p/q| < 1/(cq^2)$ for infinitely many $p, q\}$. One can show that $L(\alpha) = +\infty$ for almost every α and $L(\alpha)$ is finite if and only if α is of bounded type. The *Lagrange spectrum* $\mathcal{L} \subset \mathbb{R}$ is the set of finite values $L(\alpha)$ for $\alpha \in \mathbb{R}\backslash\mathbb{Q}$. An equivalent formulation of this Diophantine definition of $L(\alpha)$ is:

$$L(\alpha) = \limsup_{q,p\to\infty} \frac{1}{q|q\alpha - p|}. \tag{0.1}$$

The structure of $\mathcal{L}$ has been studied for more than a century. Some classical results, which, with the only exception of the recent (P6), can be found in the survey by Cusick and Flahive [1], are the following:

(P1) $\mathcal{L} \subset [\sqrt{5}, +\infty)$ and the infimum $\sqrt{5}$ is achieved (Hurwitz 1891).
(P2) $\mathcal{L}$ is a closed subset of the real line (Cusick 1975).
(P3) The values $L(\beta)$ for β quadratic irrational are dense in $\mathcal{L}$ (Cusick 1975).
(P4) $\mathcal{L}$ contains an *Hall's ray*, that is $[r, +\infty) \subset \mathcal{L}$ for some $r > 0$ (Hall 1947).
(P5) $\mathcal{L} \cap [\sqrt{5}, 3)$ is a discrete set (Markoff 1879).
(P6) $t \mapsto \text{Hdim}(\mathcal{L} \cap (-\infty, t])$ is a Cantor staircase. (Moreira 2009, see [4]).

In order to generalize Lagrange Spectra to the setting of translation surfaces and prove the analogous of properties (P1) to (P4), it is useful to first present two equivalent definitions of $\mathcal{L}$, one geometric and one dynamical. For the geometric interpretation, consider an unimodular lattice $\Lambda_\alpha = \mathbb{Z}v_1 + \mathbb{Z}v_2 \subset \mathbb{R}^2 \cong \mathbb{C}$ where v_1, v_2 are vectors with $\Re v_1 = -\alpha, \Re v_2 = 1 - \alpha$ and let $\mathbb{T}_\alpha = \mathbb{R}^2/\Lambda_\alpha$ be the associated flat torus. Remark that the Poincaré first return map of the linear vertical flow on $\mathbb{T}_\alpha$ is a rotation $R_\alpha(x) = x + \alpha \mod 1$. Then one can show that if $\mathrm{Area}(v) = |\Re v||\Im v|$ denotes the area of the rectangle which has v as diagonal,

$$a(\Lambda_\alpha) := \liminf_{\Im v \to +\infty} \{\mathrm{Area}(v), \quad v \in \Lambda_\alpha\} = \frac{1}{L(\alpha)}. \tag{0.2}$$

For the dynamical interpretation, let $\mathcal{M}_1 = SL(2,\mathbb{R})/SL(2,\mathbb{Z})$ be the moduli space of unimodular lattices (or flat tori). The linear action of the diagonal matrices $diag(e^t, e^{-t})$, $t \in \mathbb{R}$, on a lattice $\Lambda \subset \mathbb{R}^2$ defines the *geodesic flow* $(g_t)_{t\in\mathbb{R}}$ on $\mathcal{M}_1$. Let us recall that $\mathcal{M}_1$ is non compact and that the systole function given by $\mathrm{sys}(\Lambda) := \min\{\|v\|, v \in \Lambda\backslash\{0\}\}$ is a proper function. We also have

$$L(\alpha) = \frac{2}{s^2(\Lambda_\alpha)}, \qquad \text{where} \quad s(\Lambda) := \liminf_{t\to+\infty} \mathrm{sys}(g_t\Lambda). \tag{0.3}$$

In particular, $L(\alpha) < +\infty$ if and only if the geodesic ray $(g_t\Lambda_\alpha)_{t\geq 0}$ is bounded in $\mathcal{M}_1$ and $\mathcal{L}$ can be interpreted as the set of depths of penetration of bounded closed geodesics in the cusp of the modular surface $\mathbb{H}/SL(2,\mathbb{Z})$.

In order to generalize these three definitions (Diophantine, geometric and dynamical) of the Lagrange spectrum, we consider instead than a flat torus $\mathbb{T}_\alpha$ a higher genus translation surface X, that can be obtained for example by considering a polygon P (or more in general polygons) with pairs of parallel isometric sides identified by translations (see for example [5] for definitions). We then replace the moduli space of flat tori $\mathcal{M}_1$ by the moduli space of genus g and area one translation surfaces $\mathcal{M}_g$, see e.g. [5]. Poincaré first return maps of the linear flow on a translation surface to a transversal are interval exchange transformations (IETs for short), i.e. piecewise isometries of the unit interval which generalize rotations (see e.g. [5]). Let $\epsilon_q(T)$ be the shortest length of an interval exchanged by the q^{th} iterate T^q. We set

$$L(T) := \limsup_{q\to+\infty} \frac{1}{q\epsilon_q(T)}. \tag{0.4}$$

Remark that in the case of the rotation R_α, we have $\epsilon_q(R_\alpha) = \min_{0<p\leq q} |q\alpha - p|$., thus, definition (0.4) can be thought of as a generalization of (0.1).

The geometric quantity analogous to (0.1) associated to an area one translation surface X is defined considering all *saddle connections* (s.c. for short) of X or equivalently, if X is obtained from a polygon P, all *generalized diagonals* of P, that is all linear trajectories which connect a vertex of P to a vertex of P after possibly crossing some pairs of identified sides. Then we can set

$$a(X) := \liminf_{\Im v\to+\infty} \{\mathrm{Area}(v), \quad v \text{ saddle connection on } X\}. \tag{0.5}$$

Finally, for the dynamical definition that generalizes (0.3), consider the *Teichmüller geodesic flow* $(g_t)_{t\in\mathbb{R}}$ on $\mathcal{M}_g$ which is given by the linear action of the diagonal matrices $diag(e^t, e^{-t})$, $t \in \mathbb{R}$ on the polygons in $\mathbb{R}^2$ defining X. For the systole function given by $\mathrm{sys}(X) := \min\{|v|, \ v \text{ saddle connection on } X\}$, we set

$$s(X) := \liminf_{t\to+\infty} \mathrm{sys}(g_tX), \qquad X \in \mathcal{M}_g. \tag{0.6}$$

As first shown by Vorobets (see also [3]), if the IET T is a Poincaré map of the translation surface $X_T \in \mathcal{M}_g$, these quantities are related as in the case of the torus, that is $L(T) = 1/a(X_T) = 2/s^2(X_T)$.

Let $\mathcal{I} \subset \mathcal{M}_g$ be any closed subset of $\mathcal{M}_g$, invariant under the action of $SL(2,\mathbb{R})$ on $\mathcal{M}_g$. In [3], we define the (generalized) *Lagrange spectrum $\mathcal{L}_\mathcal{I}$ of the locus $\mathcal{I}$* by

$$\mathcal{L}_\mathcal{I} = \left\{ \frac{1}{a(X)}, \quad X \in \mathcal{I} \right\} \subset \mathbb{R} \cup \{+\infty\}.$$

In [3] we prove the following analogue of the properties (P1) to (P4) of the classical Lagrange spectrum. Let us recall that an origami is a translation surface X which is a branched cover of the standard torus $\mathbb{R}^2/\mathbb{Z}^2$ ramified over 0 (thus, X can be obtained by identifying parallel sides of squares).

Theorem. Let $\mathcal{I}$ be a closed $SL(2,\mathbb{R})$-invariant subset of the moduli space $\mathcal{M}_g$ consisting of translation surfaces with genus g and k singularities. Then

- (P1)' $\mathcal{L}_\mathcal{I} \subset [\pi \frac{2g+k-2}{2}, +\infty]$.
- (P2)' The finite Lagrange Spectrum $\mathcal{L}_\mathcal{I} \cap \mathbb{R}$ is a closed subset of the real line.
- (P3)' The values $1/a(X)$, where $X \in \mathcal{I}$ is a periodic point for the restriction of the Teichmuller flow to $\mathcal{I}$, are dense in $\mathcal{L}_\mathcal{I}$.
- (P4)' If $\mathcal{I}$ contains an origami, then $\mathcal{L}_\mathcal{I}$ contains an Hall's ray, that is there exist an explic

Property $(P1)'$ follows from a easy upper bound for the systole function and we do not claim that the bound stated is sharp. A natural and interesting open question is to compute the analogous of the *Hurwitz constant*, that is of the minimum of the spectrum $\mathcal{L}_\mathcal{I}$ in particular when $\mathcal{I}$ is a connected component of a stratum.

The *continued fraction algorithm* plays a crucial role in the proof of the results (P1) to (P6) on the classical Lagrange spectrum $\mathcal{L}$, thanks to the following beautiful formula for $L(\alpha)$. Let $\alpha = a_0 + [a_1, a_2, \dots]$ be the continued fraction expansion of α. Then we have

$$L(\alpha) = \limsup_{n\to\infty} \big([a_{n-1}, \dots, a_0] + a_n + [a_{n+1}, a_{n+2}, \dots]\big). \tag{0.7}$$

The main tools for the proof of the above results are two analogous explicit formulas to compute the values in $\mathcal{L}_\mathcal{I}$ via two generalizations of the continued fraction algorithm. The first, used to prove (P2)' and (P3)', is the so-called *Rauzy-Veech induction* for interval exchange transformations (see [5]). The second, used for (P4)' is an extension of the formula (0.7) to origamis which exploits a finite extension of the Farey map. The formulas are of independent interest and in particular the first one shows that Rauzy-Veech induction can be used to study invariant loci and the geometry of strata. Using these formulas and exploiting a symbolic coding of bounded Teichmüller geodesics by a subshift of finite type (provided by the Rauzy-Veech induction), we are able to adapt the strategy of the classical proofs by Hall and Cusick (see [1]) and hence prove (P2)', (P3)' and (P4)'.

We conclude by mentioning that it is not clear whether to expect or not that a generalization of the discreteness property (P5) to Lagrange spectra of translation surfaces hold. Some interesting results addressing this question for a similar generalization of the Lagrange spectrum in the context of IETs of 3 intervals were recently announced by Ferenczi [2]. In the special case of loci $\mathcal{I}$ that are $SL(2,\mathbb{R})$ orbits of an origami, in work in progress jointly with P. Hubert, L. Marchese and S. Lelièvre, we use a refinement of the formula for origamis to study the fine structure of the spectra of some explicit examples of origamis with 6 and 7 squares.

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The dynamics of a class of quasi-periodic Schrödinger cocycles

Kristian Bjerklöv

We are interested in the dynamics of the one-parameter family of quasi-periodic Schrödinger cocycle maps, parameterized by the real number E, and given by

$$(\omega, A_E) : \mathbb{T}\times\mathbb{R}^2 \to \mathbb{T}\times\mathbb{R}^2 \qquad (\theta, x) \mapsto (\theta+\omega, A_E(\theta)x). \tag{0.1}$$

Here $A_E : \mathbb{T} \to \mathrm{SL}(2,\mathbb{R})$ is defined by

$$A_E(\theta) = \begin{pmatrix} 0 & 1 \\ -1 & \lambda f(\theta) - E \end{pmatrix} \quad (E \in \mathbb{R}).$$

We shall assume that $f : \mathbb{T} \to \mathbb{R}$ is a C^2-function with exactly two, non-degenerate, critical points, and that the base frequency $\omega \in \mathbb{T}$ $(= \mathbb{R}/\mathbb{Z})$ satisfies the Diophantine condition

$$\inf_{p\in\mathbb{Z}} |q\omega - p| > \frac{\kappa}{|q|^\tau} \quad \text{for all } q \in \mathbb{Z}\setminus\{0\}$$

for some constants $\kappa > 0$ and $\tau \geq 1$. Furthermore, we shall assume that the coupling constant λ is large (depending on ω and f).

The cocycle (0.1) is closely related to the family of discrete Schrödinger operators

$$(H_\theta u)_n = -(u_{n+1} + u_{n-1}) + \lambda f(\theta + (n-1)\omega)u_n,$$

acting on $l^2(\mathbb{Z})$. The spectrum of H_θ is denoted by $\sigma(H)$. It is well-known that it is independent of $\theta \in \mathbb{T}$, since f is continuous. Moreover, the cocycle (0.1) is uniformly hyperbolic iff $E \notin \sigma(H)$.

Under the above assumptions on f and ω, Sinai [8] has shown that the operator H_θ has a pure-point spectrum with exponentially decaying eigenfunctions for a.e. $\theta \in \mathbb{T}$, provided that λ is sufficiently large. Moreover, the spectrum $\sigma(H)$ is a Cantor set. (Very similar results were also obtained by Fröhlich-Spencer-Wittwer [4].)

By $\gamma(E)$ we denote the (upper) Lyapunov exponent, i.e.,

$$\gamma(E) = \lim_{n\to\infty} \frac{1}{n} \int_{\mathbb{T}} \log \|A_E^n(\theta)\| d\theta \ (\geq 0)$$

where $A_E^n(\theta) := A_E(\theta + (n-1)\omega) \cdots A_E(\theta+\omega)A_E(\theta)$ $(n > 0)$.

Since $\mathrm{SL}(2,\mathbb{R})$ acts, in the natural way, on the real projective line $\mathbb{P}^1(\mathbb{R})$, the cocycle (ω, A_E) induces a "projective flow" on the space $\mathbb{T} \times \mathbb{P}^1(\mathbb{R})$. We let

$$\Phi_E : \mathbb{T} \times \mathbb{P}^1(\mathbb{R}) \to \mathbb{T} \times \mathbb{P}^1(\mathbb{R})$$

denote this map. Since $\mathbb{P}^1(\mathbb{R})$ is isomorphic to $\mathbb{T}$, we can view Φ_E as a map on the two-torus $\mathbb{T}^2$.

By works of M. Herman [5] and R. Johnson [6], we know that if $\gamma(E) > 0$ for some $E \in \mathbb{R}$, then the map Φ_E has either exactly two minimal sets (if the cocycle (0.1) is uniformly hyperbolic) or one unique minimal set (if (0.1) is not uniformly hyperbolic). Understanding the geometry of the unique minimal set is an interesting problem.

Our main results are the following:

Theorem. *Let f and ω be as above. There exists a $\lambda_0 = \lambda_0(\omega, f) > 0$ such that the following holds for all $\lambda \geq \lambda_0$:*

(1) *The Lyapunov exponent $\gamma(E)$ satisfies*

$$\gamma(E) \geq \frac{2\log\lambda}{3} \quad \textit{for all } E \in \mathbb{R}.$$

(2) *If E is on the edge of an open gap in the spectrum $\sigma(H)$, then there exists a phase $\theta \in \mathbb{T}$ and a vector $u \in l^2(\mathbb{Z})$, exponentially decaying at $\pm\infty$, such that $H_\theta u = Eu$.*

(3) *The map Φ_E has exactly two invariant and ergodic probability measures for all $E \in \mathbb{R}$, and, moreover,*

$$\Phi_E \textit{ is minimal } \iff E \in \sigma(H) \setminus \{\textit{edges of open gaps}\}.$$

Note that the theorem applies to the *almost Mathieu case*, that is, the case when $f(\theta) = \cos(\theta)$, which is by far the most studied case.

The proof of the Theorem is based on a detailed analysis of the map Φ_E. The analysis does not depend on the fact that this map comes from the linear system (0.1), and can be generalized to other classes of quasi-periodically forced circle maps.

It is an interesting problem to find conditions on $f : \mathbb{T} \to \mathbb{R}$ under which one has $\gamma(E) \geq c\log\lambda$ ($c > 0$ some constant) for all $E \in \mathbb{R}$, provided that λ is sufficiently large. This is true if f is (non-constant and) real-analytic [9]. It is also true for "many" functions f of Gevrey class [7] and of class C^3 [3], but it is not true if one only assumes f to be continuous [2].

Another open problem is to describe the dynamics of the map Φ_E (for all E) under more general assumptions on f.

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The Fried entropy for smooth group actions and connections with algebraic number theory

Svetlana Katok

(joint work with A. Katok and F. Rodriguez Hertz)

It is well-known that the standard notion of entropy for an action of locally compact topological group G by measure-preserving transformations assign value zero to the entropy of any smooth action unless the group G is virtually cyclic, i.e. a compact extension of $\mathbb{Z}$ or $\mathbb{R}$. We study numerical entropy-type invariants suitable for smooth actions of higher rank abelian groups on n-dimensional compact smooth manifolds by smooth transformations preserving a Borel probability measure μ. One such invariant, based on averaging approach, was introduced by D. Fried in 1983 [1] and for many years was essentially forgotten. We rediscovered it, but later found Fried's paper and now call this invariant the *Fried entropy.* For a $\mathbb{Z}^k$-action α it is given by the formula

$$h^*_\alpha = \frac{2^k}{k!vol(B(h^\alpha_\mu))},$$

where $B(h^\alpha_\mu)$ is the unit ball in the entropy norm/semi-norm of the action α. Arithmeticity of maximal rank smooth abelian actions ($k = n - 1$) proved by A. Katok and F. Rodriguez Hertz [2] implies that the Fried entropy for maximal rank positive entropy actions is closely related to regulators of totally real number fields. This leads to striking conclusions: (i) for maximal rank actions the Fried entropy can only take countably many values, (ii) in the weakly mixing case the Fried entropy is either equal to zero or is bounded away from zero by a positive function that depends only on the dimension n and grows exponentially with it. More precisely, the Fried entropy can be expressed as

$$h_\alpha^* = \frac{mR_K 2^{n-1}}{\binom{2n-2}{n-1}},$$

where m is a positive integer and R_K is the regulator of a totally real algebraic number field K associated to the algebraic model measurably isomorphic to α (by the Arithmeticity theorem mention above). We use Zimmert's analytic lower bound for regulators [3] for $s = 0.35$ to obtain $h_\alpha^* > 0.000752 \exp(0.244n)$.

Inspection of the number fields data at http://www.lmfdb.org/ identifies the quartic totally real number field of discriminant 725 as the field that conjecturally minimizes h_α^*. For it $h_\alpha^* = 0.330027....$

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A nightcap on magnetic dynamics

Karl Friedrich Siburg

(joint work with Andreas Knauf and Frank Schulz)

Consider a magnetic field in $\mathbb{R}^3$ whose field lines are perpendicular to the plane $\mathbb{R}^2 \times \{0\} \cong \mathbb{R}^2$. Then the motion of a particle of unit mass and unit charge in that plane is modelled by Newton's Second Law

$$\ddot{q} = B(q)J\dot{q} \tag{0.1}$$

where $B : \mathbb{R}^2 \to \mathbb{R}$ describes the field strength and the term on the right hand side is the Lorentz force corresponding to the magnetic field, with J being the symplectic matrix $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. The differential equation (0.1) can be written as the Hamiltonian system generated by the Hamiltonian $H : T^*\mathbb{R}^2 \to \mathbb{R}, H(q,p) = \frac{1}{2}\|p\|^2$ on $(T^*\mathbb{R}^2, \omega)$ with the twisted symplectic form

$$\omega = \omega_0 + B(q)dq_1 \wedge dq_2$$

where $\omega_0 = d\lambda = dp_1 \wedge dq_1 + dp_2 \wedge dq_2$ stands for the standard symplectic form on $T^*\mathbb{R}^2$.

In this talk, we study the dynamics of (0.1) from two perspectives. First of all, we assume that the magnetic field consists of $n \geq 2$ disjoint disk bumps, and explain why this system carries positive topological entropy $h_{\mathrm{top}} \geq \log n$. Secondly, we develop a scattering theory for magnetic fields vanishing at infinity, and show that under certain asymptotic assumptions on B there exists a well defined Møller transformation.

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Closed orbits for exact magnetic flows on surfaces below the Mañé critical value

Alberto Abbondandolo

(joint work with Leonardo Macarini and Gabriel P. Paternain)

Let M be an oriented closed surface equipped with a Riemannian metric g and a closed one-form θ. Denote by $\imath$ the almost complex structure induced by g and by $f : M \to \mathbb{R}$ the function defined by the identity $d\theta = f d\Omega_g$, where Ω_g is the area form induced by g. The magnetic flow equation for curves $x : \mathbb{R} \to M$ is

$$\nabla_t \dot{x} = f(x) \imath \dot{x}, \tag{0.1}$$

where ∇_t denotes the covariant derivatives. This second order equation defines a flow on TM which preserves the energy function

$$E : TM \to \mathbb{R}, \qquad E(x, v) := \frac{1}{2} g_x(v, v).$$

The equation (0.1) is the Euler-Lagrange equation associated to the Lagrangian

$$L : TM \to \mathbb{R}, \qquad L(x, v) := \frac{1}{2} g_x(v, v) + \theta_x(v).$$

Like the geodesic flow, the magnetic flow preserves the sphere-bundles $E^{-1}(\kappa)$, but, unlike the former flow, its behavior varies with the energy $\kappa \in [0, +\infty)$. Significant energy values are the Mañé critical values

$$c_u := \inf \{\kappa \mid \mathbb{S}_\kappa(\gamma) \geq 0 \text{ for every } \gamma \text{ contractible loop}\},$$

$$c_0 := \inf \{\kappa \mid \mathbb{S}_\kappa(\gamma) \geq 0 \text{ for every } \gamma \text{ null-homologous loop}\},$$

where $\mathbb{S}_\kappa$ is the action functional

$$\mathbb{S}_\kappa(\gamma) := \int_0^T \big(L(\gamma, \dot{\gamma}) + \kappa\big)\, dt, \quad \gamma : \mathbb{R}/T\mathbb{Z} \to M,$$

on loops of arbitrary period $T > 0$. In general, $0 \leq c_u \leq c_0$. The functional $\mathbb{S}_\kappa$ is bounded from below on a given connected component of the free loop space of M if and only if $\kappa \geq c_u$. When $\kappa > c_0$, the hypersurface $E^{-1}(\kappa)$ is of restricted contact type, and the magnetic flow on $E^{-1}(\kappa)$ is conjugated to a Finselr geodesic flow, up to a time reparametrization. In the latter energy range, multiplicity results for closed orbits can be thus deduced by corresponding statements about Finsler geodesics. In particular, if M is not the 2-sphere, then $E^{-1}(\kappa)$ has infinitely many closed orbits. If M is the 2-sphere, there are at least two closed orbits, as proved in [5], and Katok's example from [8] shows that there might be only two. The existence of infinitely many closed orbits holds also for $M \neq S^2$ and $\kappa > c_u$, because one can minimize $\mathbb{S}_\kappa$ on infinitely many mutually coprime connected components of the free loop space.

The energy range below c_u is less understood. In this energy range, there is always a closed orbit with negative action $\mathbb{S}_\kappa$ [9, 11], and there is a second closed orbit with positive action $\mathbb{S}_\kappa$ for almost every $\kappa \in (0, c_u)$ [6]. In this talk we shall discuss the following improvement of the above mentioned results:

Theorem 10. *For a.e. $\kappa \in (0, c_u)$ there are infinitely many periodic orbits on $E^{-1}(\kappa)$.*

Some preliminary discussion is needed, before presenting a sketch of the proof. The right functional setting for studying the functional $\mathbb{S}_\kappa$ is obtained by identifying the T-periodic loop $\gamma : \mathbb{R}/T\mathbb{Z} \to M$ with the pair (x, T), where $x : \mathbb{T} := \mathbb{R}/\mathbb{Z} \to M$, $x(s) := \gamma(Ts)$, and by seeing $\mathbb{S}_\kappa$ as a smooth functional on

$$\Lambda := W^{1,2}(\mathbb{T}, M) \times (0, +\infty).$$

Here $W^{1,2}(\mathbb{T}, M)$ denotes the Hilbert manifold of loops of Sobolev class $W^{1,2}$ on M. The starting point of our proof is the already mentioned existence of a close orbit with negative action from [9, 11], which can be upgraded to the following statement (see [7, 2]):

Theorem 11. *For every $\kappa \in (0, c_0)$ there exists a closed orbit α_κ of energy κ with $\mathbb{S}_\kappa(\alpha_\kappa) < 0$, all of whose iterates are local minimizers of $\mathbb{S}_\kappa$ on Λ.*

One would like to exploit the above result and the fact that $\mathbb{S}_\kappa$ is unbounded from below on each connected component of Λ for $\kappa < c_u$, in order to find other closed orbits as mountain pass critical points. The main difficulty is the fact that $\mathbb{S}_\kappa$ fails to satisfy the Palais-Smale (PS) condition when $\kappa \leq c_u$. Indeed:

Lemma 2 ([6, 1])**.** *Let $\gamma_n = (x_n, T_n)$ be a PS sequence for $\mathbb{S}_\kappa$. Then:*

(i) *If $0 < T_* \leq T_n \leq T^* < +\infty$, then (γ_n) is compact.*
(ii) *If $T_n \to 0$ then (x_n) converges to a constant loop and $\mathbb{S}_\kappa(\gamma_n) \to 0$.*
(iii) *If $\kappa > c_u$ then (T_n) is bounded from above (but not necessarily if $\kappa \leq c_u$).*

After these preliminaries, we sketch the proof of Theorem 1 under the following simplifying assumptions:

(a) α_κ is a strict local minimizer for every $\kappa \in (0, c_u)$;
(b) the map $\kappa \mapsto \alpha_\kappa$ is continuous;
(c) for a.e. $\kappa \in (0, c_u)$ all the closed orbits on $E^{-1}(\kappa)$ are non degenerate.

These assumptions are very strong and are introduced here only for the sake of simplicity. See [2] for the proof of Theorem 1 under the only assumption (c). The proof in the general case uses ideas from [4] and will appear elsewhere.

Fix a compact interval $I \subset (0, c_u)$ and find a loop β in the same component of the α_κ's such that

$$\mathbb{S}_\kappa(\beta) < \mathbb{S}_\kappa(\alpha_\kappa), \qquad \forall \kappa \in I.$$

Denoting by γ^n the n-th iterate of the loop γ, we define the minimax values

$$c_n(\kappa) := \inf_{\substack{u \in C([0,1],\Lambda) \\ u(0)=\alpha_\kappa^n,\, u(1)=\beta^n}} \max_{\sigma \in [0,1]} \mathbb{S}_\kappa(u(\sigma)), \qquad \forall n \in \mathbb{N},\ \forall \kappa \in I.$$

By (a) we have $c_n(\kappa) > \mathbb{S}_\kappa(\alpha_\kappa^n)$, and from (b) we deduce that for every $n \in \mathbb{N}$ the function $\kappa \mapsto c_n(\kappa)$ is increasing on I. Moreover, Bangert's homotopy argument from [3] allows to prove that

$$c_n(\kappa) \leq -an + b, \qquad \forall \kappa \in I,\ \forall n \in \mathbb{N},$$

for suitable positive constants a, b. In particular, $c_n(\kappa) \to -\infty$ for $n \to \infty$.

Struwe's monotonicity argument from [10] allows to prove that if c_n is differentiable at κ, then $c_n(\kappa)$ is a critical value of $\mathbb{S}_\kappa$. The following heuristic argument explains how the differentiability of c_n allows to bypass the lack of the PS condition: fix a small $\epsilon > 0$ and find a path u_κ connecting α_κ^n and β^n such that

$$c_n(\kappa) + \epsilon = \mathbb{S}_\kappa(u_\kappa(1/2)),$$

and $d\mathbb{S}_\kappa(u_\kappa(1/2))$ is small. By differentiating with respect to κ we obtain

$$c_n'(\kappa) = d\mathbb{S}_\kappa(x_\kappa, T_\kappa)[\partial_\kappa u_\kappa(1/2)] + T_\kappa, \quad \text{where } (x_\kappa, T_\kappa) = u_\kappa(1/2),$$

so T_κ has an upper bound. In this way we can construct a PS sequence with bounded periods, which converges to a critical point by Lemma 3, if n is so large that $c_n(\kappa) < 0$.

Since monotone functions are a.e. differentiable, we obtain a full measure subset $K \subset I$ such that $c_n(\kappa)$ is a critical value of $\mathbb{S}_\kappa$ for every $\kappa \in K$ and every $n \in \mathbb{N}$. For such a κ the functional $\mathbb{S}_\kappa$ has a sequence of critical points γ_n with action going to $-\infty$. By (c), up to considering a smaller K, we may assume that the mountain pass critical points γ_n's are non-degenerate and hence have Morse index one. An extension of Bott's index iteration theory to the free period action functional $\mathbb{S}_\kappa$ allows to prove that a non-degenerate critical point of $\mathbb{S}_\kappa$ with Morse index one has positive mean index. Together with the fact that $\mathbb{S}_\kappa(\gamma_n) \to -\infty$, this allows to exclude that the γ_n's are iterates of only finitely many closed orbits. Therefore, for every $\kappa \in K$ the level $E^{-1}(\kappa)$ has infinitely many periodic orbits. Since the compact subinterval $I \subset (0, c_u)$ is arbitrary, the thesis of Theorem 1 follows.

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Higher-dimensional pentagram maps and KdV flows

Boris Khesin
(joint work with Fedor Soloviev)

The pentagram map was originally defined by R. Schwartz in [6] as a map on plane convex polygons considered up to their projective equivalence, where a new polygon is spanned by the shortest diagonals of the initial one. This map is the identity for pentagons, it is an involution for hexagons, while for polygons with more vertices it was shown to exhibit quasi-periodic behaviour under iterations. The pentagram map was extended to the case of twisted polygons and its integrability in 2D was proved in [5], see also [7].

While this map is in a sense unique in 2D, its generalizations to higher dimensions seem to allow more freedom. A natural requirement for such generalizations, though, is their integrability. It turns out that that there is no natural generalization of this map to polyhedra, but one can suggest natural integrable generalizations of the pentagram map to the space of generic twisted polygons.

Define a *twisted n-gon* in a projective space $\mathbb{RP}^d$ with a monodromy $M \in SL_{d+1}(\mathbb{R})$ as a doubly-infinite sequence of points $v_k \in \mathbb{RP}^d$, $k \in \mathbb{Z}$ such that $v_{k+n} = M \circ v_k$ for each $k \in \mathbb{Z}$ and where M acts naturally on $\mathbb{RP}^d$. We assume that the vertices v_k are in general position (i.e., no $d+1$ consecutive vertices lie in the same hyperplane in $\mathbb{RP}^d$), and denote by $\mathcal{P}_n$ the space of generic twisted n-gons considered up to the projective equivalence. General pentagram maps are defined as follows.

Definition. We define two types of diagonal hyperplanes for a given twisted polygon (v_k) in $\mathbb{RP}^d$. The *higher diagonal hyperplane* P_k^{hi} is defined as the hyperplane passing through d vertices of the n-gon by taking every other vertex starting with v_k:

$$P_k^{hi} := (v_k, v_{k+2}, v_{k+4}, ..., v_{k+2(d-1)}) .$$

The *dented diagonal plane hyperplane* P_k^m for a fixed $m = 1, 2, ..., d-1$ is the hyperplane passing through all vertices but one from v_k to v_{k+d} by skipping only the vertex v_{k+m}:

$$P_k^m := (v_k, v_{k+1}, ..., v_{k+m-1}, v_{k+m+1}, v_{k+m+2}, ..., v_{k+d}) \,.$$

Now the corresponding *higher* or *dented pentagram maps* T^{hi} and T^m on twisted polygons (v_k) in $\mathbb{RP}^d$ are defined by intersecting d consecutive diagonal hyperplanes:

$$Tv_k := P_k \cap P_{k+1} \cap ... \cap P_{k+d-1} \,,$$

where for T^{hi} and T^m one uses the definition of the hyperplanes P_k^{hi} and P_k^m respectively. These pentagram maps are generically defined on the classes of projective equivalence of twisted polygon: $T : \mathcal{P}_n \to \mathcal{P}_n$.

Example. For $d = 2$ both definition coincide with the classical 2D pentagram map in [6]. For $d = 3$ the map T^{hi} uses the diagonal planes passing through $P_k^{hi} := (v_k, v_{k+2}, v_{k+4})$, while $P_k^1 = (v_k, v_{k+2}, v_{k+3})$ and $P_k^2 = (v_k, v_{k+1}, v_{k+3})$.

Theorem. *The higher T^{hi} and dented T_m pentagram map on both twisted and closed n-gons in any dimension d and any $m = 1, ..., d-1$ is an integrable system in the sense that it admits a Lax representation with a spectral parameter.*

Integrability for these maps in 2D (which coincide in this dimension) was proved in [5], while its Lax representation was found in [7]. For higher pentagram maps their Lax representation with a spectral parameter was found in [2]. It was based on a scale invariance of such maps proved in [2] for 3D and in [4] for higher d. For the dented pentagram maps their Lax representations and scale invariance in any dimension were described in [3]. The Lax representation provides first integrals (as the coefficients of the corresponding spectral curve) and allows one to use algebraic-geometric machinery to prove various integrability properties.

We also refer to [2, 3] for a detailed description of the algebraic-geometric integrability of these maps in 3D. One can show that in these cases the space of twisted n-gons in the complex space $\mathbb{CP}^3$ is generically fibered into (Zariski open subsets of) tori whose dimension is described in terms of n.

Remark. More generally, one can define generalized pentagram maps $T_{I,J}$ on (projective equivalence classes of) twisted polygons in $\mathbb{RP}^d$, associated with $(d-1)$-tuple of numbers I and J: the tuple I defines which vertices to take in the definition of the diagonal hyperplanes P_k, while the tuple J determines which of the hyperplanes to intersect in order to get the image point Tv_k. In general, their integrability is yet unknown, but there exists the following duality between such pentagram maps:

$$T_{I,J}^{-1} = T_{J^*,I^*} \circ Sh \,,$$

where I^* and J^* stand for the $(d-1)$-tuples taken in the opposite order and Sh is any shift in the indices of polygon vertices, see [3].

Remark. In [2, 3] it was also proved that the continuous limit of any higher or dented pentagram map (and more generally, of any generalized pentagram map) in $\mathbb{RP}^d$ is the $(2, d+1)$-KdV flow of the Adler-Gelfand-Dickey hierarchy on the circle. For 2D this is the classical Boussinesq equation on the circle: $u_{tt}+2(u^2)_{xx}+u_{xxxx} = 0$, which appears as the continuous limit of the 2D pentagram map [5].

A different integrable generalization to higher dimensions was proposed in [1], where the pentagram map was defined not on generic, but on the so-called corrugated polygons. These are twisted polygons in $\mathbb{RP}^d$, whose vertices v_{k-1}, v_k, v_{k+d-1}, and v_{k+d} span a projective two-dimensional plane for every $k \in \mathbb{Z}$. The pentagram map on corrugated polygons (denoted by T_{cor}) is integrable and admits an explicit description of the Poisson structure, a cluster algebra structure, and other interesting features [1]. It turns out that the pentagram map T_{cor} can be viewed as a particular case of the dented pentagram map:

Theorem [3]. *This pentagram map T_{cor} is a restriction of the dented pentagram map T_m for any $m = 1, ..., d-1$ from generic n-gons $\mathcal{P}_n$ in $\mathbb{RP}^d$ to corrugated ones (or differs from it by a shift in vertex indices). In particular, these restrictions for different m coincide modulo an index shift.*

Furthermore, by considering more general diagonal hyperplanes P_k, such as "deep-dented diagonals", i.e., those skipping more than one vertex, one can construct new integrable systems.

Theorem [3]. *The deep-dented pentagram maps in $\mathbb{RP}^d$ are restrictions of integrable systems to invariant submanifolds and have Lax representations with a spectral parameter.*

The main tool to prove integrability in this more general setting is an introduction of the corresponding notions of *partially corrugated polygons*, occupying intermediate positions between corrugated and generic ones. This approach brings about the following question, which manifests the change of perspective on generalized pentagram maps: Choose the diagonal hyperplane P_k so that the corresponding pentagram map turned out to be non-integrable.

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The group of symplectic surface diffeomorphisms

John Franks
(joint work with Michael Handel)

This talk focuses on some of the algebraic properties of symplectic diffeomorphisms of compact genus zero surfaces.

Definition. We will denote by $Cent^r(f)$, the centralizer of f, the subgroup of $\mathrm{Diff}^r(M)$ whose elements commute with f, and by $Cent^r_\mu(f)$ the subgroup of $\mathrm{Symp}^r_\mu(M)$ whose elements commute with f. We will denote by $Cent^r(f)$, the centralizer of f, the subgroup of $\mathrm{Diff}^r(M)$ whose elements commute with f, and by $Cent^r_\mu(f)$ the subgroup of $\mathrm{Symp}^r_\mu(M)$ whose elements commute with f.

A natural question in light of work of Farb and Shalen [1] on $\mathrm{Diff}^\omega(S^1)$ is the following: Suppose M is a closed surface and $f \in \mathrm{Diff}^\omega(M)$ has infinite order. Then is its centralizer, $Cent^\omega(f)$, always virtually abelian? We are able to answer this for the case of $\mathrm{Symp}^\omega_\mu(M)$ when M has genus zero.

Theorem 12. *Suppose M has genus zero and $f \in \mathrm{Symp}^\omega_\mu(M)$ has infinite order, then $Cent^\omega_\mu(f)$, the centralizer of f in $\mathrm{Symp}^\omega_\mu(M)$ is virtually abelian.*

In fact this is a special case of a more general result which is proved by the same techniques:

Theorem 13. *Suppose M is a compact oriented surface of genus zero and G is a subgroup of $\mathrm{Symp}^\omega_\mu(M)$. Suppose further that G has an infinite normal solvable subgroup. Then G is virtually abelian.*

An important corollary of Theorem 13 is the following.

Corollary 1. *Suppose M is a compact surface of genus zero and G is a solvable subgroup of $\mathrm{Symp}^\omega_\mu(M)$, then G is virtually abelian.*

The proof of these results is based on the observation that there are three types of structure for $f \in \mathrm{Symp}^\infty_\mu(M)$. Let M be a compact oriented surface with genus zero and let G be a subgroup of $\mathrm{Symp}^\infty_\mu(M)$.

- G contains an element of positive entropy
- G contains an element f which is multi-rotational, i.e. if $M = S^2$, then f has entropy 0 and at least three periodic points.
- G is a pseudo-rotation group.

and these exhaust the possibilities.

Definition 1. Suppose M is a compact genus zero surface and $f \in \mathrm{Symp}^\infty_\mu(M)$ and that the number of periodic points of f is greater than the Euler characteristic of M. If f has infinite order and entropy 0, we will call it a *multi-rotational diffeomorphism.*

Definition 2. An infinite order element $f \in \mathrm{Symp}^r_\mu(S^2)$ is a *pseudo-rotation* if for every $n > 0$ has exactly two fixed points, i.e. f has two fixed points and no other periodic points.

The case of Theorem 1 when G contains an element of positive entropy is due to Katok [3]. The that G contains an element f which is multi-rotational, relies on previous work of the author and Michael Handel providing a structure theorem for entropy zero area preserving diffeomorphisms of genus zero surfaces (see [2]). The last case is covered by the following:

Theorem 14. *Suppose f is an infinite order pseudo-rotation. Then $Cent^{\omega}_{\mu}(f)$, the centralizer of f in $\mathrm{Symp}^{\omega}_{\mu}(S^2)$ is virtually abelian.*

Recall that the *Tits alternative* is satisfied by a group G if every subgroup (or by some definitions, every finitely generated subgroup) of G is either virtually solvable or contains a non-abelian free group. This is a deep property known for finitely generated linear groups and some groups arising in geometric group theory. It is an important open question for $\mathrm{Diff}^{\omega}(S^1)$. (It is not true for $\mathrm{Diff}^{\infty}(S^1)$.)

Conjecture 15 (**Tits alternative**). If M is a compact surface then every finitely generated subgroup of $\mathrm{Symp}^{\omega}_{\mu}(M)$ is either virtually solvable or contains a non-abelian free group.

We are able to prove a special case of this conjecture.

Theorem 16. *Suppose M is a compact genus zero surface and G is a subgroup of $\mathrm{Symp}^{\omega}_{\mu}(M)$. If G contains at least one multi-rotational element then either G contains a subgroup isomorphic to F_2, the free group on two generators, or G has an abelian subgroup of finite index.*

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