Mathematisches Forschungsinstitut Oberwolfach

Mini-Workshop: New Crossroads between Mathematics and Field Theory

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ABSTRACT. In the last few years, it has been strongly emphasized the need to use new mathematical tools and structures which are not part of the traditional pool of expertise of the community working on the analysis of the mathematical and structural properties of classical and quantum field theory. Goal of the workshop has been to bring together some of the major experts in these topics to discuss the latest results and the new insights brought to field theory by techniques, such as microlocal analysis, infinite dimensional geometry and homological algebra.

Introduction by the Organisers

Goal of the workshop was, on the one hand, to bring together some of the major experts in the analysis of the mathematical aspects of classical and quantum field theory and, on the other hand, to discuss the key open problems as well as the main mathematical tools which are expected to play an active role in future researches. The hope was not only to look for opening new collaborations, but also to discuss new potential approaches to the various questions in this field, which are still unanswered. The event has been attended by 17 participants and it started with two introductory lectures, the first by Klaus Fredenhagen on the present status of algebraic quantum field theory and the second by Chris Fewster on the concept of Hadamard states and on the related quantum energy inequalities.

The most represented community in this event works on the algebraic approach to quantum field theory (AQFT). This is a well-established branch of mathematical physics which emphasizes the role of observables and their interplay with the notion of locality and causality. From a physical point of view, it has the net advantage of being essentially the only procedure which can be applied naturally when the underlying background is curved and thus it leads to concrete applications to cosmology. Overall it can be described as a two-step procedure. In the first one, a unital ∗-algebra of observables is assigned to a specific physical system. In this framework there have been several leaps forward during the past few years and, in particular, the formulation of all free field theories on arbitrary globally hyperbolic spacetimes is a topic fully understood. This particular aspect and, more precisely, the construction of the algebra of observables for a large class of free fields as well as their quantization via canonical commutation or anticommutation relations has been discussed in detail by Christian Bär during his talk.

The second step in the algebraic approach to quantum field theory is the assignment of a state, that is a normalized linear functional from the algebra of observables to the complex numbers which fulfills a positivity condition. Unfortunately not all states are physically acceptable and a precise mathematical characterization of those which are admissible has to be provided. It is universally accepted that the correct answer to this query are the so-called Hadamard states which are defined in terms of the wavefront set of their associated truncated two-point function. While the existence of such states is guaranteed by an old result by Wald, Fulling and Narcovich, their explicit construction is still an open issue which has been thoroughly investigated during the workshop. In his talk Christian Gérard has shown that a possible way to tackle this problem for a scalar field originates from a careful use of pseudo-differential calculus. Another approach has been discussed by Claudio Dappiaggi and it goes under the name of bulk-to-boundary correspondence. In spacetimes possessing a null (conformal) boundary it is possible to construct an injective ∗-homomorphism between the algebra of observables of the theory under analysis and an auxiliary one living intrinsically on the boundary. As a byproduct every state on the boundary induces a counterpart in the bulk and there exists a distinguished use of this procedure which allows to impose also the Hadamard condition. The biggest drawback of this method is the necessity to know in detail the fall-off condition of the solutions of the equation of motion for the field under analysis in the limit when they approach the null boundary and future timelike infinity. For this reason there is a close connection between these methods and the novel techniques aimed at the study of the stability of black holes. These were thoroughly discussed by Mihalis Dafermos in his talk.

Another open issue, which has been discussed extensively during this mini-workshop, is the analysis in the algebraic approach of gauge theories. Several frameworks have been proposed and outlined: For Abelian structure groups, one can exploit the affine character of the bundle of connections in order to quantize Yang-Mills theories adapting the procedure used for the usual standard linear

fields. Alexander Schenkel reported on this topic and he has also emphasized that an explicit characterization of the full gauge group is possible for the Abelian scenario. As a by-product one is able to identify and classify all observables which probe Aharonov-Bohm configurations. A connected problem consists of the construction of Hadamard states for such class of theories. In comparison with standard linear field theories, one has to cope with the additional difficulty of encoding gauge invariance. As reported by Alexander Strohmaier for the case of a $U(1)$ vector potential, it is possible to generalize also to curved backgrounds the Gupta-Bleuler mechanism, used in Minkowski spacetime. Hence existence of Hadamard states for a $U(1)$ Yang Mills theory is guaranteed.

For non-Abelian structure groups and more general scenarios, a possible approach consists of employing the Batalin-Vilkovisky formalism as reported by Katarzyna Rejzner. Although this approach has the net advantage of working in great generality, it leads naturally to deal with infinite dimensional spaces and differential calculus thereon. This is a topic, which is still intensively studied in pure mathematics and its applications to algebraic quantum field theory require careful analyses, particularly in connection with the microlocal properties inherited by the underlying Hadamard states. This issue has been discussed by Christian Brouder in this talk.

The use of infinite dimensional calculus within the framework of gauge theories comes parallel to an investigation on the possibility to discuss non linear classical field theories from an algebraic point view and in the spirit of the principle of general local covariance. This challenging investigation has been at the heart of the talks of Romeo Brunetti and of Pedro Laurdisen Ribeiro who introduced, most notably, the principle of spacetime descent, as a novel way to describe internal symmetries in classical field theories. The expectation is that a formulation also at the quantum level might be possible.

During the mini-workshop, concrete applications of the latest results in algebraic quantum field theory to physical phenomena and models have also been discussed. In particular Valter Moretti has shown how it is possible to translate into a rigorous framework the argument of Parikh and Wilczek which describes Hawking radiation as a tunneling process through the event horizon of a black hole. Most notably, in the talk, it has been shown how this procedure can be made intrinsically local and how the result holds true also for a polynomially interacting scalar field, at least at first order in the perturbation series. Applications to cosmology have been reported in the talks of Thomas-Paul Hack and Nicola Pinamonti. The former has discussed perturbations in inflationary models, seen as the linearization of a coupled Einstein-Klein-Gordon system. Thomas-Paul Hack has emphasized the role of gauge invariance in particular and he has discussed the related construction of the algebra of observables, also in view of the recent results in algebraic quantum field theory on the treatment of gauge systems. Nicola Pinamonti has discussed, instead, the use of semiclassical Einstein's equations in a cosmological framework under the further assumption that the matter content of the Universe can be described via a massive, conformally coupled scalar field. On

the one hand he has shown via a fixed point theorem argument that global solutions for this highly coupled system exist. On the other hand he has reinterpreted semiclassical Einstein's equations as an equation for the probability distributions of the curvature and of the the matter stress-energy tensor. In this procedure he has followed the same procedure according to which Brownian motion is treated as arising from a Langevin equation.

Mini-Workshop: New Crossroads between Mathematics and Field Theory

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Abstracts

CCR versus CAR quantization
Christian Bär
(joint work with Nicolas Ginoux)

This talk is based on [1]. A linear differential operator acting on sections of a vector bundles over a time-oriented Lorentzian manifold is called *Green hyperbolic* if its restriction to any globally hyperbolic subregion has advanced and retarded Green's operators. This class contains basically all matter fields of physical interest, namely wave operators (e.g. Klein-Gordon), the Proca equation, Dirac-type operators and the Rarita-Schwinger operator.

We show that to any globally hyperbolic spacetime equipped with such a Green-hyperbolic operator one can associate a symplectic vector space in a functorial manner. The composition of this functor with the Weyl functor yields a bosonic locally covariant quantum field theory in the sense of [2], i.e. is satisfies quantum causality and the time-slice axiom. This is CCR quantization. It is remarkable that this works even in cases when it is not expected from the side of physics; it is not reasonable to consider a bosonic quantum field theory for Dirac fields, for instance. This shows that there is no spin-statistics theorem on the level of observable algebras; one has to take states into account to obtain a spin statistics theorem as in [3].

In contrast, CAR quantization is much more restricted. We have to restrict to first order operators whose principal symbol satisfies a positivity condition. This holds for the classical Dirac operator (possibly twisted with a bundle carrying a postive definite metric) but fails for instance for Buchdahl operators. If the positivity condition holds, then one can functorially associate a pre-Hilbert space and thus construct a fermionic locally covariant quantum field theory.

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Functional properties of Hörmander's space of distributions
CHRISTIAN BROUDER
(joint work with Yoann Dabrowski)

The space $\mathcal{D}'_\Gamma(\Omega)$ of distributions defined on an open subset Ω of $\mathbb{R}^n$ and having their wavefront sets in a closed cone Γ was defined in 1971 by Lars Hörmander [1]. It has become important in physics because of its role in the formulation of renormalized quantum field theory in curved spacetimes since the ground-breaking articles by Radzikowski [2] and by Brunetti and Fredenhagen [3].

In this talk, which summarizes the results derived in our recent paper [4], the topological and bornological properties of $\mathcal{D}'_\Gamma(\Omega)$ are presented: when $\mathcal{D}'_\Gamma(\Omega)$ is equipped with a topology stronger than the one originallly defined by Hörmander, it is a complete, nuclear, semi-reflexive and semi-Montel normal space of distributions. A Banach-Steinhaus theorem provides several equivalent definitions of bounded sets in $\mathcal{D}'_\Gamma(\Omega)$.
Its topological dual is the space $\mathcal{E}'_\Lambda(\Omega)$ of compactly supported distributions whose wavefront sets are contained in the open cone $\Lambda = \{(x,k) \in T^*\Omega \backslash Z; (x,-k) \notin \Gamma\}$, where $Z = \{(x,0); x \in \Omega\}$ is the zero section of the cotangent bundle $T^*\Omega$. The space $\mathcal{E}'_\Lambda(\Omega)$ equipped with the strong topology is a nuclear, barrelled and bornological normal space of distributions. An explicit counter-example is constructed to show that, when Λ is not both open and closed, then $\mathcal{E}'_\Lambda(\Omega)$ is not sequentially complete.

Simple concrete rules are given to determine whether a distribution belongs to $\mathcal{D}'_\Gamma(\Omega)$, whether a sequence converges in $\mathcal{D}'_\Gamma(\Omega)$ and whether a set of distributions is bounded in $\mathcal{D}'_\Gamma(\Omega)$.

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On the algebraic structure of (quantum) field theories
ROMEO BRUNETTI
(joint work with K. Fredenhagen, P. L. Ribeiro)

The introduction of functional methods in classical field theory are reviewed [1]. The algebraic structure of the family of observables is discussed for the simplest example of a real scalar field and by means of the methods of micro-local analysis one proves that it gives a structure of a nuclear Poisson algebra after equipping it by the Peierls' bracket. All these rely only on the linearised dynamics, which is

introduced by defining Lagrangians as natural transformation between appropriate functors. The full dynamics is then studied via a kind of scattering procedure which amounts to prove the existence of certain diffeomorphisms in configuration space that intertwine, for instance, the free and the interacting Euler-Lagrange operators associated to a given first-order Lagrangian. The existence can be proven by the methods of Nash-Moser-Hörmander. Moreover, one proves that these maps represent canonical transformations by which an infinite dimensional analogue of the Darboux-Weinstein theorem can be discussed. One might consider all the results obtained as the first step into a novel kind of quantisation via a sort of Fedosov deformation quantization of the algebraic structures, these last results will be presented in a forthcoming joint paper with P. L. Ribeiro.

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Construction of Hadamard states from null boundaries

CLAUDIO DAPPIAGGI

(joint work with Valter Moretti and Nicola Pinamonti)

We review a procedure to construct explicitly states of Hadamard form on spacetimes possessing a null (conformal) boundary. Such method can be applied to a wide class of physically relevant models. The first one, which has been tackled, is the massless, conformally coupled, scalar field living on an asymptotically flat spacetime (M, g). As discussed in [1], one starts from the normally hyperbolic partial differential equation

$$\left(\Box_g - \frac{R}{6}\right)\phi = 0,$$

where $\phi : M \to \mathbb{R}$ and $\Box_g$ is the wave operator. If one considers smooth and compactly supported initial data, the resulting space of solutions $\mathcal{S}(M)$ is a symplectic space, if endowed with the following weakly non-degenerate bilinear form

$$\sigma_M(\phi, \phi') = \int_\Sigma d\mu(\Sigma)\left(\phi\nabla_n\phi' - \phi'\nabla_n\phi\right),$$

where Σ is an arbitrary Cauchy surface, with normal vector n and metric induced measure $d\mu(\Sigma)$. On the right hand side, the arguments $\phi, \phi' \in \mathcal{S}(M)$ are smooth functions, meant as being restricted on Σ. A well-known result in operator algebra guarantees that one can associate to $(\mathcal{S}(M), \sigma_M)$ a unique (up to $*$-isomorphisms) Weyl C*-algebra of observables $\mathcal{W}(M)$. The second ingredient of the so-called bulk-to-boundary mechanism is the assignment to the null boundary, $\Im$, of a symplectic space $\mathcal{S}(\Im)$ and, consequently of a second Weyl algebra $\mathcal{W}(\Im)$. The boundary symplectic space is not chosen out of the dynamics, as in the bulk, but rather by the requirement of the existence of an injective symplectomorphism

$\Gamma : \mathcal{S}(M) \to \mathcal{S}(\Im)$. The construction of the map Γ represents the key technical obstruction of this procedure, but tackling this issue is of paramount relevance since Γ yields in turn an injective $*$-homomorphism $\iota : \mathcal{W}(M) \to \mathcal{W}(\Im)$. Since a state is a normalized and positive linear functional from a C*-algebra into the complex numbers, it holds that, for every state $\omega : \mathcal{W}(\Im) \to \mathbb{C}$, it corresponds a counterpart $\omega' : \mathcal{W}(M) \to \mathbb{C}$ such that $\omega' \doteq \omega \circ \iota$. We show, moreover, that, on asymptotically flat spacetimes, there exists a distinguished choice for ω which yields for the bulk theory a counterpart ω' which is both Hadamard and invariant under all possible isometries. During the talk it is outlined how this bulk-to-boundary technique can be successfully applied in several other contexts, ranging from cosmology [2] to the construction of the Unruh state in Schwarzschild spacetime [3]. Noteworthy is the possibility to employ such procedure also for higher spin and gauge theories, as discussed recently, for example, in [4, 5].

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Hadamard states and Quantum Energy Inequalities

Christopher J. Fewster

In this introductory talk, I begin with a brief review of the framework of quantisation of a linear field in general globally hyperbolic spacetimes. I then turn to a discussion of physical states for the theory, beginning with the nonexistence of a single preferred choice of state in each spacetime, which is a special case of no-go theorem, due to Verch and myself [1]. Instead, it is now generally agreed that one must work with the class of Hadamard states. I review their original definition and the microlocal reformulation due to Radzikowski [2] and subsequently developed by other authors [3, 4]. I also discuss the recently proposed "S-J state" (or Sorkin-Johnston state) [5] and its failure to be Hadamard [6], and further recent results of Verch and myself which show that, among pure quasifree states only those that are Hadamard have finite fluctuations for all Wick polynomials (on "ultrastatic slab" spacetimes with compact Cauchy surface) [7]. (Note also that Brum and Fredenhagen have recently proposed a modification of the S-J procedure that does result in Hadamard states in certain circumstances [8].)

Finally, and briefly, I turn to the subject of Quantum Energy Inequalities (QEIs) (see, e.g., [9]) and describe recent results which establish QEIs for the massive Ising model [10].

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Quantum Field Theory and Mathematical Physics
KLAUS FREDENHAGEN

Quantum fields occur in physics and mathematics in several formalisms, which differ a lot in mathematical precision and applicability. An overview was given over an attempt to reformulate quantum field theory in such a way that it connects the axiomatic approaches (characterized by mathematical rigor, but typically quite remote from applications) with the practical applications. In addition this new formulation should allow to treat new situations, in particular theories on curved spacetimes (this was the main motivation for this new attempt), at finite temperature, in nonequilibrium situations etc.. This attempt relies on previous work on renormalization, including besides the well known work of Bogoliubov, Parasiuk, Hepp and Zimmermann less known papers by Epstein, Glaser, Steinmann, Stora, Scharf and Dütsch. These older works, however, restricted themselves essentially to theories on Minkowski space in the vacuum representation, there were unsolved infrared problems, and there were problems with the incorporation of gauge theories.

In the early 1990's, a new approach (the "modern appraoch") was started. It aimed at a formulation which is meaningful on generic Lorentzian spacetimes. It is based on ideas of algebraic quantum field theory by replacing algebras of operators on a given Hilbert space by algebras of functionals on the space of classical

field configurations. The crucial role of the spectrum condition in the older formulations, which is meaningful only under the condition of Poincaré symmetry, is replaced by the microlocal spectrum condition, a condition on the wave front set of functional derivatives of the functionals on field configurations. The condition of covariance under spacetime symmetries was replaced by a new principle, named the principle of local covariance, first formulated at an Oberwolfach Min-workshop 2000. This approach emphasizes the structural similarity with classical field theory; moreover it offers a unified view on different traditional frameworks, in particular the path integral formulation and canonical quantization.

Contributions to this new developments have been given by a number of authors, among them Brunetti, Dütsch, Fewster, Fredenhagen, Hollands, Kay, Radzikowski, Rejzner, Verch, Wald (see [1] for a review).

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Construction of Hadamard states by pseudodifferential calculus

Christian Gérard

(joint work with Michał Wrochna)

We give a new construction based on pseudo-differential calculus of quasi-free Hadamard states for Klein-Gordon equations on a class of space-times whose metric is well-behaved at spatial infinity. In particular on this class of space-times, we construct all pure Hadamard states whose two-point function (expressed in terms of Cauchy data on a Cauchy surface) is a matrix of pseudo-differential operators. We also study their covariance under symplectic transformations.

As an aside, we give a new construction of Hadamard states on arbitrary globally hyperbolic space-times which is an alternative to the classical construction by Fulling, Narcowich and Wald.

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Gauge-theoretic quantization of inflationary perturbations

Thomas-Paul Hack

Inflationary perturbations are the (quantum) field theory of the linearisation of the coupled Einstein-Klein-Gordon system

$$G_{ab}(g) = T_{ab}(g, \phi)$$

$$-\nabla_a \nabla^a \phi + \partial_\phi V(\phi) = 0 \,,$$

around any globally hyperbolic background solution (M, g, ϕ). While the full system is invariant under diffeomorphisms of the four-dimensional smooth manifold M, the linearised system is invariant under the linearisations of such diffeomorphisms, which constitute the gauge transformations of the theory. In the usual treatment of the subject, it is assumed that one linearises around a background solution of Robertson-Walker type, as this is intersting for physical applications in the framework of cosmology. Owing to high symmetry of the Cauchy surface of such a Lorentzian manifold, which is diffeomorphic to the three-dimensional Euclidean space, the linearised field variables (γ, φ) can be decomposed in irreducible representations w.r.t. the Euclidean group. In this way, gauge-invariant linear combinations of linearized field variables can be easily identified, including the so-called Bardeen potentials Φ and Ψ, the gauge-invariant scalar field perturbation χ, and the Mukhanov-Sasaki variable μ, which is a linear combination of Ψ and χ. While the partial differential equation satisfied by (γ, φ) does not possess a good Cauchy problem, the one satisfied by e.g. μ is of normally hyperbolic type. Thus, the usual approach consists in passing from the linearised field variables (γ, φ) to the gauge-invariant quantity μ, and to perform a CCR quantization of this induced field-theoretic system.

In the recent paper [1], Eltzner analysed the quantization of this system from first principles within algebraic quantum field theory on curved spacetimes. Starting from the fact, that the linearised Einstein-Klein-Gordon system can be rewritten as a constrained hyperbolic system for Ψ and χ, he first showed that it is impossible to quantize both Ψ and χ as local fields, i.e. as fields which commute at spacelike separations, and then demonstrated that the linear combination of these two fields given by μ is structurally preferred in that it is the only linear combination of Ψ and χ which satisfies a second order partial differential equation and whose canonical momentum in the sense of Lagrangian field theory is given by the derivative w.r.t. to (conformal) time. This can be considered as a structural justification of the usual approach to consider μ as a basic field variable for CCR quantization.

In this talk, we aim to consider a more general approach and to quantize the linearised Einstein-Klein-Gordon system as a proper gauge theory. Using the general framework to quantize linear gauge field theories on globally hyperbolic spacetimes developed in [2], we first provide a gauge-invariant quantization of the (γ, φ) system around any globally hyperbolic background solution of the Einstein-Klein-Gordon system. We obtain a local algebra of observables $\mathcal{A}$, which implies that all

observables commute at spacelike separations. Considering the situation in backgrounds of Robertson-Walker type, we find that there are certain relations between the general observable algebra $\mathcal{A}$ and the one obtained by CCR quantization of the Mukhanov-Sasaki variable μ, say, $\mathcal{A}_\mu$. We find that subalgebras of $\mathcal{A}$ and $\mathcal{A}_\mu$ are isomorphic, but that neither of this algebras can be considered as a subalgebra of the other, in particular, the field theories obtained by a gauge-theoretic quantization of the (γ, φ) field variables and by a usual quantization of the μ field variable as a scalar field on a globally hyperbolic spacetime don't coincide.

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Supersymmetry in curved spacetime

STEFAN HOLLANDS

(joint work with P. deMedeiros)

By conformally coupling vector and hyper multiplets in Minkowski space, we obtain a class of field theories with extended rigid conformal supersymmetry on any Lorentzian four-manifold admitting twistor spinors. We construct the conformal symmetry superalgebras which describe classical symmetries of these theories and derive an appropriate BRST operator in curved spacetime. In the process, we elucidate the general framework of cohomological algebra which underpins the construction. We then consider the corresponding perturbative quantum field theories. In particular, we examine the conditions necessary for conformal supersymmetries to be preserved at the quantum level, i.e. when the BRST operator commutes with the perturbatively defined S-matrix, which ensures superconformal invariance of amplitudes. To this end, we prescribe a renormalization scheme for time-ordered products that enter the perturbative S-matrix and show that such products obey certain Ward identities in curve d spacetime. These identities allow us to recast the problem in terms of the cohomology of the BRST operator. Through a careful analysis of this cohomology, and of the renormalization group in curved spacetime, we establish precise criteria which ensure that all conformal supersymmetries are preserved at the quantum level. As a by-product, we provide a rigorous proof that the beta-function for such theories is one-loop exact. We also briefly discuss the construction of chiral rings and the role of non-perturbative effects in curved spacetime.

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Black Hole radiation in the "local tunnelling" approach: The point of view of the Algebraic Quantum Field Theory

VALTER MORETTI

(joint work with N. Pinamonti and G. Collini)

Tunnelling processes through black hole horizons have recently been investigated by physicists in the framework of WKB theory discovering interesting interplay with the Hawking radiation [1, 2]. A more precise and general account of that phenomenon has been subsequently given within the framework of QFT in curved spacetime [3]. In particular, it has been shown that, in the limit of sharp localization on opposite sides of a Killing horizon, the quantum correlation functions of a scalar field appear to have thermal nature, and the tunnelling probability is proportional to $\exp\{-\beta_{Hawking}E\}$. This local result is valid in every spacetime including a local Killing horizon, no field equation is necessary, while a suitable choice for the quantum state is relevant. Indeed, the two point function has to verify a short-distance condition weaker than the Hadamard one. In this talk, after having presented the results obtained in [3], we discuss the more recent results obtained in [4]. As a matter of fact we focus on a massive scalar quantum field with a ϕ^3 self-interaction and we investigate the issue whether or not the black hole radiation can be handled at perturbative level, including the renormalisation contributions. We discuss how, for the simplest model of the Killing horizon generated by the boost in Minkowski spacetime, and referring to Minkowski vacuum, the tunnelling probability in the limit of sharp localization on opposite sides of the horizon preserves the thermal form proportional to $\exp\{-\beta_H E\}$ even taking the one-loop renormalisation corrections into account. A similar result is expected to hold for the Unruh state in the Kruskal manifold, since that state is Hadamard and looks like Minkowski vacuum close to the horizon.

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Semiclassical Einstein equations and their fluctuations in cosmology

Nicola Pinamonti

(joint work with Daniel Siemssen)

During this talk we shall consider the backreaction of quantum fields on gravity in a cosmological scenario. This will be done employing a very simple model for matter, namely a free massive quantum scalar field and using the Einstein equations in a semiclassical fashion

$$G_{ab} = \langle T_{ab} \rangle_\omega \ .$$

We shall see that the semiclassical Einstein equations become a well posed dynamical system and that uniqueness and existence of global solutions can be obtained.

More precisely, in the first part of the talk we assume that the spacetime is homogeneous, isotropic and spatially flat. We shall thus discuss the implication of this hypothesis on the form of the metric and on the form of the equation governing its evolution. In particular, in this scenario the metric possesses only one degree of freedom, the scale factor and, when an initial condition is fixed at some time, it will suffices to consider the trace of the Einstein equation to determine its dynamics

$$-R = \langle T \rangle_\omega \ .$$

We then proceed presenting the quantum matter we shall employ as source for gravity. For simplicity we consider a massive scalar field conformally coupled to gravity, namely

$$-\Box\varphi + \frac{1}{6}R\varphi + m^2\varphi = 0 \ .$$

The quantization of this system is performed within the framework of Algebraic Quantum Field Theory presenting the algebra generated by the identiy and the smeared abstract fields $\varphi(f)$. This procedure is functorial, hence we can assign to every cosmological spacetime M we are considering the algebra of fields $\mathcal{A}(M)$. We continue discussing in details the procedure of extending this algebra to encompass fields defined as point-like products of φ, like the stress tensor or $\varphi^2(f)$. After the extension, we can test our algebra on a smaller set of states (normalized linear functionals on the extended algebra), namely those enjoying the Hadamard condition.

The next preliminary element we have to discuss is the choice of states ω on which we can test T. Since we want to solve a dynamical system giving initial values at a fixed time, we have to do it employing a rule which uses only the form of the metric and its first time derivative at the initial time. We shall hence chose gaussian states which appear as close as possible to the vacuum at time zero. This kind of states are adiabatic states of zeroth order. Despite the lack of regularity, the expectation values of T do not diverge.

With all this preliminary work, we can now write the dynamical equation governing the evolution of the scale factor as a functional Volterra integral equation,

namely

$$a' = a'_0 + \int_0^t f(a, a')dt$$

where the functional f contains the expectation value $\omega(\varphi^2)$, which depends in a complicated functional manner on a. We will show that some nice estimate for this expectation value are possible, hence, we can employ Banach fixed point theorem to have local existence and uniqueness of the employed solution.

Thanks to the nice features of the obtained estimates, we show that extensions of local solutions can be always obtained provided the local solutions has not reached a singular point. Since the extended solution can be ordered, by Zorn lemma, a maximal solution exists and it can also be shown to be unique. This maximal solution is the sought global solution.

In the last part of the talk we shall also discuss the influence of the fluctuations of the quantum stress tensor on curvature treated stochastically. This is done reinterpreting the semiclassical Einstein equation as an equation for the probability distributions of the curvature and of the the matter stress tensor. This is the method assumed for example when the Brownian motion is treated as arising from a Langevin equation.

Finally we employ this idea to obtain the power spectrum of metric perturbation induced by matter fluctuations in an expanding universe. We see that it shows the characteristic scale free spectrum, even if the gravity field is not directly quantized.

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Local gauge symmetries in locally covariant field theory

Katarzyna Rejzner

In this talk I will show how the methods of infinite dimensional differential geometry and homological algebra can be applied to describe local symmetries in locally covariant classical and quantum field theory. These ideas allow to make precise the so called Batalin-Vilkovisky formalism in the context of infinite dimensional spaces. The framework which I propose works for theories on general globally hyperbolic specetimes and can be applied to the perturbative quantization of many physically interesting models, including QED, Yang-Mills theories and gravity.

The Principle of Space-Time Descent for Fiber Bundles
PEDRO LAURIDSEN RIBEIRO
(joint work with Romeo Brunetti and Klaus Fredenhagen)

The Principle of Local Covariance was introduced by Brunetti, Fredenhagen and Verch [1] as a synthesis of the axioms of algebraic QFT [2] which allows one to define field theories in the category of space-times in a functorial manner. In this framework, local covariant fields are viewed as suitable natural transformations. Recently, the Principle of Local Covariance was extended to classical field theory [3], formulated in terms of certain algebras of functionals over (real scalar) field configurations. Already at the classical level, however, one meets two different challenges to this principle: (I) the dynamics induced by hyperbolic (nonlinear) Euler-Lagrange equations makes necessary to restrict the domains of functionals, which means that we have to localize these algebras in space-time and field configuration space simultaneously [4, 5]; (II) if one tries to go beyond real scalar fields, the internal structure of the field configurations (internal symmetries, etc.) may pose obstructions to the Principle of Local Covariance [6]. We set a strategy to address both challenges, which consists in encoding both kinds of localization outlined by challenge (I) in another category (more precisely, a stack [7] over the category of space-times), in which subsets of field configurations over space-time coverings can be glued together in a unique way. This refinement of the Principle of Local Covariance, called the Principle of Space-Time Descent, provides a novel way to describe internal symmetries and may also be formulated at the quantum level.

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Quantized Abelian principal connections on Lorentzian manifolds

Alexander Schenkel

(joint work with Marco Benini, Claudio Dappiaggi and Thomas-Paul Hack)

In my talk I have reported on our recent work on the quantization of Abelian gauge theories in the framework of locally covariant quantum field theory. Our analysis is based on the description of principal connections in terms of sections of the bundle of connections. Since the bundle of connections is an affine bundle, we had to develop new techniques for the quantization of affine field theories [1]. We have performed a careful analysis of the gauge symmetries of theories of Abelian principal connections and we have found that compact and non-compact structure groups G lead to a structurally different class of gauge invariant observables [2]. For compact structure groups, such as $G = U(1)$ which is the case relevant in electromagnetism, we have obtained that the algebra of gauge invariant field polynomials (that is the classical Borchers-Uhlmann algebra of the vector dual bundle of the bundle of connections) is not separating on gauge equivalence classes of connections. We have extended this algebra in [3] by exponential observables, resembling regular versions of Wilson loop observables, and we could prove that these classical algebras separate gauge equivalence classes of connections. Our main theorem is:

Theorem 1 ([3]). *There exists a covariant functor $\mathfrak{A} : U(1)-\mathsf{PrBuGlobHyp} \to C^*\mathsf{Alg}$ describing suitable C^*-algebras of gauge invariant observables for quantized principal $U(1)$-connections. The covariant functor $\mathfrak{A}$ satisfies the quantum causality property and the quantum time-slice axiom. Furthermore, for each object Ξ in $U(1)-\mathsf{PrBuGlobHyp}$ the C^*-algebra $\mathfrak{A}(\Xi)$ is a quantization of an algebra of functionals on the affine space of connections that separates gauge equivalence classes of connections. However, $\mathfrak{A}$ does not satisfy the locality property and there exists no further quotient of $\mathfrak{A}$ such that the locality property is fulfilled.*

Regarding the violation of the locality property of our functor $\mathfrak{A}$, we have analyzed the restriction of the category $U(1) - \mathsf{PrBuGlobHyp}$ to a category of subbundles of a fixed principal $U(1)$-bundle over a globally hyperbolic Lorentzian manifold. This resembles the restriction from locally covariant quantum field theory to a Haag-Kastler-type approach. We have obtained the following

Theorem 2 ([3]). *For any fixed object Ξ in $U(1)-\mathsf{PrBuGlobHyp}$ our functor $\mathfrak{A}$ can be promoted to a Haag-Kastler quantum field theory on Ξ.*

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Gupta-Bleuler Quantization in Globally Hyperbolic Space-Times

Alexander Strohmaier

(joint work with Felix Finster)

This extended abstract is based on [1] where detailed proofs and references can be found. Suppose that (M, g) is a time-oriented oriented n-dimensional globally hyperbolic space-time. As usual denote the space of smooth p-forms by $\Omega^p(M)$. Let $\Omega_0^p(M)$ be the subspace of compactly supported forms. The space $\Omega_0^p(M)$ carries a natural possibly indefinite inner product $\langle f, g \rangle := \int_M f \wedge *g$, where $* : \Omega_0^p(M) \to \Omega_0^{n-p}(M)$ is the Hodge star operator. If d and δ denote differential and co-differential operator respectively, the wave operator $\square = d\delta + \delta d$ acting on the space of one forms is a normally hyperbolic operator and therefore possesses advanced and retarded Green's operators $G^{\pm} : \Omega_0^1(M) \to \Omega^1(M)$. Their difference $G = G^+ - G^-$ maps $\Omega_0^1(M)$ onto the space of solutions of the wave equation with compact spacelike support. By the Schwartz kernel theorem G has a distributional kernel and we write $G(f, g)$ for $\langle f, G(g) \rangle$.

The field algebra of the massless vector field in Lorentz gauge is defined to be the $*$-algebra $\mathcal{F}$ generated by symbols $A(f), f \in \Omega_0^1(M)$ and relations

$$f \mapsto A(f) \text{ is linear,}$$
$$A(f)A(g) - A(g)A(f) = -\mathrm{i}\, G(f, g),$$
$$A(\square f) = 0, \quad \text{for all } f \in \Omega_0^1(M),$$
$$(A(f))^* = A(f).$$

The subalgebra $\mathcal{A}$ of observables is generated by $A(f)$ with $\delta f = 0$.

Let $(\mathfrak{K}, \langle \cdot, \cdot \rangle)$ be a locally convex topological vector spaced endowed with an indefinite inner product and suppose π is a representation of $\mathcal{F}$ on $\mathfrak{K}$ and $\Omega \in \mathfrak{K}$ such that the following hold:

(a) $\overline{\pi(\mathcal{F})\,\Omega} = \mathfrak{K}$,   (cyclicity)

(b) $\pi(\mathcal{A})\,\Omega$ is a positive semi-definite subspace $\mathfrak{H}_0 \subset \mathfrak{K}$ and $\langle \Omega, \Omega \rangle = 1$.

(c) the representation is quasifree: the n-point distributions

$$\omega_n(f_1, \ldots, f_n) = \left\langle \Omega, \pi\big(A(f_1) \cdots A(f_n)\big)\, \Omega \right\rangle.$$

satisfy

$$\omega_m(f_1, \ldots, f_m) = \sum_P \prod_r \omega_2(f_{(r,1)}, f_{(r,2)}),$$

where P denotes a partition of the set $\{1, \ldots, m\}$ into subsets which are pairings of points labeled by r.

(d) For any $n \in \mathbb{N}$, the space $\mathfrak{K}_n$ defined by

$$\mathfrak{K}_n = \overline{\mathcal{F}_n \Omega} \subset \mathfrak{K}$$

is a Krein space (endowed with the inner product $\langle \cdot, \cdot \rangle$ and the locally convex topology induced by $\mathfrak{K}$). Here $\mathcal{F}_n$ is the span of products of the form $A(f_1) \cdots A(f_n)$.

(e) microlocal spectrum condition:

$$\mathrm{WF}\Big(A(\cdot)\Omega\Big) \subseteq J^+ = \{(x,\xi) \mid g_x(\xi,\xi) \geq 0 \text{ and } \xi_0 \geq 0\} \ ,$$

where $A(\cdot)\Omega$ is a Krein-space-valued distribution.

(f) Gupta-Bleuler condition:

$$\big\langle \phi, \pi\big(A(df)\big)\,\phi \big\rangle = 0 \qquad \text{for all } \phi \in \mathfrak{H}_0 \ .$$

Then $(\pi, \mathfrak{K}, \Omega)$ is called a **quasi free Gupta-Bleuler representation**. Dividing out the null subspace

$$\mathcal{N} = \{\psi \in \mathfrak{H}_0 \mid \langle \psi, \psi \rangle = 0\}$$

and forming the completion, one gets a Hilbert space, which in the usual Gupta-Bleuler formalism is interpreted as the physical Hilbert space.

We show that Gupta-Bleuler representations can be constructed on a large class of space-times employing the well known argument that globally hyperbolic space-times are past isometric to ultrastatic space-times. We then use canonical quantization on ultrastatic space-times $M = \mathbb{R} \times \Sigma$. This canonical quantization procedure relies on the existence of so-called one particle structures. The presence of zero modes creates a difficulty in the construction of such one particle structures. Whereas the discrete spectrum at zero (the L^2-cohomology of Σ) can be projected out and quantized separately, the presence of so-called zero-resonance states can not be easily overcome. One of our result states that the presence of such zero-resonance modes is of topological nature depending on the topology of the boundary at infinity for a large class of manifolds. In particular we could show that there exist one-particle structures with positive energy as long as Σ is a compact (metric or topological) perturbation of $\mathbb{R}^{2m+1}$. Our main result is the existence of these special states on these ultrastatic space-times which implies the existence of Gupta-Bleuler representations on a very large class of globally hyperbolic space-times.

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