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Multiscale and High-Dimensional Problems

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Abstract. High-dimensional problems appear naturally in various scientific areas, such as PDEs describing complex processes in computational chemistry and physics, or stochastic or parameter-dependent PDEs leading to deterministic problems with a large number of variables. Other highly visible examples are regression and classification with high-dimensional data as input and/or output in the context of learning theory. High dimensional problems cannot be solved by traditional numerical techniques, because of the so-called *curse of dimensionality*.

Such problems therefore amplify the need for novel theoretical and computational approaches, in order to make them, first of all, tractable and, second, offering finer and finer resolutions of relevant features. Paradoxically, increasing computational power serves to even heighten this demand. The wealth of available data itself becomes a major obstruction. Extracting essential information from complex structures and developing rigorous models to quantify the quality of information in a high dimensional context leads to tasks that are not tractable by existing methods.

The last decade has seen the emergence of several new computational methodologies to address the above obstacles. Their common features are the nonlinearity of the solution methods as well as the ability of separating solution characteristics living on different length scales. Perhaps the most prominent examples lie in adaptive grid solvers, tensor product, sparse grid and hyperbolic wavelet approximations and model reduction approaches. These have drastically advanced the frontiers of computability for certain problem classes in numerical analysis.

This workshop deepened the understanding of the underlying mathematical concepts that drive this new evolution of computation and promoted the exchange of ideas emerging in various disciplines about the handling of multiscale and high-dimensional problems.

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Introduction by the Organisers

Inherently high-dimensional problems appear naturally in various scienitific areas, such as the Fokker-Planck and the Schrödinger equations as examples of PDEs describing complex processes in computational chemistry and physics, or stochastic or parameter-dependent PDEs leading to deterministic problems with a large number of variables. Complex scientific models like climate models, of turbulence, fluid structure interaction, nanosciences and reliability control, demand finer and finer resolution in order to increase reliability. This demand is not simply solved by increasing computational power. Indeed, higher computability even contributes to the problem by generating wealthy data sets for which efficient organization principles are not available. Extracting essential information from complex structures and developing rigorous models for quantifying the quality of information is an increasingly important issue. This manifests itself through recent developments in various areas.

The mathematical methods emerging to address these problems have to exploit in a much more subtle way the structure of the problem in order to extract the necessary information. They have several common features including the question whether the underlying objects have a sufficiently small information content, how this content might be accessible through certain sparse representations, the nonlinearity of the solution methods as well as the ability of separating solution characteristics living on different length scales. Having to deal with the appearance and interaction of local features at different levels of resolution has, for instance, brought about spatially adaptive methods as a key methodology that has advanced the frontiers of computability for certain problem classes in numerical analysis.

A related but different concept for managing the interaction of different length scales centers on wavelet bases and multilevel decompositions. In the very spirit of harmonic analysis they allow one to decompose complex objects into simple building blocks that again support analyzing multiscale features. However, for high-dimensional problems, wavelets are not the only answer although they may serve as a reference for classical representations.

While this ability was exploited first primarily for treating *explicitly* given objects, like digital signals and images or data sets, the use of such concepts for recovering also *implicitly* given objects, like solutions of partial differential or boundary integral equations, has become a major recent focus of attention. The close marriage of discretization, analysis and the solution process based on *adaptive* wavelet methods has led to significant theoretical advances as well as new algorithmic paradigms for linear and nonlinear stationary variational problems. Through thresholding and best N-term approximation based on wavelet expansions, concepts from nonlinear approximation theory and harmonic analysis become practically manageable. In our opinion, these ideas have opened promising perspectives not only for signal and image processing but also for the numerical analysis of

differential and integral equations covering, in particular, such operator equations with high dimensional deterministic or stochastic parameter dependence. For the latter, smooth dependence of the solution on the parameters can be exploited for achieving highly efficient approximations.

These various concepts have developed relatively independently of one another. Our previous Oberwolfach Workshops "Wavelet and Multiscale Methods" held in July 2004, August 2007 and August 2010 sought to bring various disciplines utilizing multiscale techniques together by inviting leading experts and young emerging scientists in areas that rarely interact. Those workshops not only accelerated the advancement of nonlinear and multiscale methodologies but also provided beneficial cross–fertilizations to an array of diverse disciplines which participated in the workshop, see the Oberwolfach Reports 34/2004, 36/2007 and 33/2010. Among the several recognizable outcomes of the workshops were: (i) the emergence of compressed sensing as an exciting alternative to the traditional sensing-compression paradigm, (ii) fast online computational algorithms based on adaptive partition for mathematical learning, (iii) clarification of the role of coarsening in adaptive numerical methods for PDEs, (iv) injection of the notion of sparsity into stochastic models to identify computational paradigms that are more efficient than Monte Carlo techniques.

One of the main objectives of this workshop was to foster synergies by the interaction of scientists from different disciplines resulting in more rapid developments of new methodologies in these various domains. It also served to bridge theoretical foundations with applications, such as mathematical finance, quantum chemistry, signal and image processing, complex fluid flows. Examples of conceptual issues that were advanced by our workshop were:

- adaptive and nonlinear multilevel methods for high-dimensional PDEs;
- multilevel and high dimensional meshless methods;
- Convergence of low-rank tensor approximations to solutions of high-dimensional PDEs;
- adaptive treatment of nonlinear and time–dependent variational problems;
- interaction of different scales under nonlinear mappings;
- convergence theory and analysis for model reduction;
- extension of model reduction methods to unsymmetric, indefinite and singularly perturbed problems;
- polynomial interpolation and adaptive quadrature in high dimensions;
- uncertainty quantification and stochastic inversion;
- regularity of solutions to stochastic differential equations;
- high dimensional dynamical systems for modeling flocking and consensus formation;
- harmonic analysis and frames on general compact domains and manifolds.

The workshop had large success in bringing together researchers from diverse disciplines who rarely see one another but have common interest in high dimensional problems and their numerical treatment. This led to a lot of interesting discussions and new ideas that will surely be pursued over the next years.

In summary, we feel that the conceptual similarities that occur in these diverse application areas suggest a wealth of synergies and cross–fertilization. These concepts are in our opinion not only relevant for the development of efficient solution methods for large scale and inherently high-dimensional problems but also for the formulation of rigorous mathematical models for quantifying the extraction of essential information from complex objects in many dimensions.

As in the previous workshops, the participants are experts in areas like nonlinear approximation theory (e.g., Cohen, Dahmen, DeVore), statistical learning theory (e.g., Kerkyacharian, Picard), tensor approximations (Grasedyck, Hackbusch, Oseledets, Schneider, Yserentant), sparse grids (e.g., Harbrecht, Schwab), finite elements (e.g., Oswald, Stevenson), convergence of adaptive methods (e.g. Dahlke, Stevenson), spectral methods (e.g., Canuto), harmonic analysis and wavelets (e.g., Cohen, Dahmen, Petrushev, Schneider), strongly nonlinear DPEs in high dimensions (e.g., Süli), numerical fluid mechanics and conservation laws (e.g., Müller, Popov, Tadmor), inverse problems (e.g., de Mol), multiscale modeling (e.g., Müller, Ohlberger, Tadmor), parameter-dependent PDE-constrained control problems (e.g., Kunoth), model reduction and reduced basis functions (e.g., Dahmen, Grepl, Ohlberger, Urban, Wojtaszczyk), stochastic PDEs and regularity of their solutions (e.g., Dahlke, Larsson, Nobile, Schwab), modeling flocking and consensus formation processes (e.g. Fornasier, Tadmor) and tractability of multivariate problems (Wozniakowski).

Workshop: Multiscale and High-Dimensional Problems

Table of Contents

Abstracts

Space-time sparse tensor Petrov-Galerkin discretization of parabolic evolution equations

Roman Andreev

Introduction.

The main novel contribution of this work is the construction of practical algorithms for the stable solution of linear parabolic evolution equations in space-time sparse tensor product finite element spaces that are specified a priori. Furthermore, we propose efficient multilevel preconditioners. All results are reflected in numerical experiments.

Sparse-tensor discretizations of parabolic evolution equations were considered in [4], but were not shown to be stable; on the other hand, space-time adaptive wavelet discretizations of [5] are optimal in terms of work and accuracy, but require Riesz bases of wavelet type that are difficult to construct in practice, especially on non-trivial domains.

Main theoretical results are given in [1]. For details, as well as further references we refer to [2]. A concise modular Matlab implementation is described in [3].

Model problem. Let D be an open, connected, nonempty and bounded subset of $\mathbb{R}^d$ with a locally Lipschitz continuous boundary ∂D, and $J = (0, T)$ a bounded nonempty temporal interval. We will have $T := 2$. Consider the linear parabolic evolution equation $\partial_t u(t,x) - \nabla \cdot (a(t,x)\nabla u(t,x)) = f(t,x)$, $(t,x) \in J \times D$, with the initial condition $u(0,x) = g(x)$, $x \in D$, and boundary condition $u(t,x) = 0$, $(t,x) \in J \times \partial D$. Here and in the following, $\nabla\cdot$ and ∇ denote the divergence and the gradient with respect to the spatial variable x. To formalize the equation, we introduce the Lebesgue/Sobolev spaces $H := L^2(D)$ and $V := H^1_0(D)$, and define the Bochner spaces $X := L^2(J;V) \cap H^1(J;V')$ and $Y := H \times L^2(J;V)$. Recall that X embeds continuously into $C^0([0,T];H)$. We write $(\cdot,\cdot)$ for the scalar product on H and for the duality pairing on $V' \times V$. We define the family of symmetric linear operators $A(t) : V \to V'$, a.e. $t \in J$, by $(A(t)\chi)(\chi) := \int_D a(t,x)\nabla\chi(x) \cdot \nabla\chi(x)dx$, $\chi \in C^\infty_0(D)$. If $a \in L^\infty(J \times D)$ is uniformly positive, $f \in L^2(J;V')$ and $g \in H$, a well-posed space-time (variational) formulation of the parabolic evolution equation is given by: find $u \in X$ such that $Bu = F$ in Y', where $B : X \to Y'$ and $F \in Y'$ are defined by $(Bw)(v) := \int_J (\partial_t w + Aw, v_1)dt + (w(0), v_0)$, and $Fv := \int_J (f, v_1)dt + (g, v_0)$, for all $w \in X$ and $(v_0, v_1) \in Y$.

Discretization framework. We describe the stable approximate resolution of $Bu = F$. Assume that subspaces $X_L \subset X$ and $Y_L \subset Y$ satisfy the so-called discrete inf-sup condition $\gamma_L := \inf_{\|w_L\|_X=1} \sup_{\|v_L\|_Y=1} (Bw_L)(v_L) > 0$. In this context, w_L, resp. v_L, range in $X_L \setminus \{0\}$, resp. $Y_L \setminus \{0\}$. The minimal residual discrete solution is defined as $u_L := u_L(F) := \operatorname{argmin}_{w_L \in X_L} R_F(w_L)$, where $R_F(w_L) := \sup_{\|v_L\|_Y=1} |(F - Bw_L)(v_L)|$, $w_L \in X_L$. One can show that $F \mapsto u_L(F)$ is linear and continuous, the quasi-optimality estimate $\|u - u_L\|_X \le \gamma_L^{-1}\|B\|\|u - w_L\|_X$,

$w_L \in X_L$, holds. Given bases on $\Phi \subset X_L$ and $\Psi \subset Y_L$, define the Gram matrices $\mathbf{B} := (B\Phi)(\Psi)$, $\mathbf{M} := (\Phi, \Phi)_X$, $\mathbf{N} := (\Psi, \Psi)_Y$ and the vector $\mathbf{F} := F\Psi$. Expanding $u_L = \Phi^\top \mathbf{u}$ in terms of the basis Φ, the minimal residual discrete solution is obtained from $\mathbf{B}^\top \mathbf{N}^{-1} \mathbf{B}\mathbf{u} = \mathbf{B}^\top \mathbf{N}^{-1} \mathbf{F}$. The (symmetric positive definite) matrix $\mathbf{B}^\top \mathbf{N}^{-1} \mathbf{B}$ can be preconditioned with $\mathbf{M}^{-1}$, and the resulting condition number is bounded by $\gamma_L^{-2} \|B\|^2$, motivating the application of the conjugate gradient method for the iterative solution.

Main results. We now discuss how discrete trial and test spaces $X_L \subset X$ and $Y_L \subset Y$ satisfying the discrete inf-sup condition $\gamma_L > 0$ can be constructed. To that end, for each integer $L \geq 0$, let $\{0, T\} \subset T_L \subset [0, T]$ be a temporal mesh consisting of $2^{L+1} + 1$ equispaced nodes. Let $E_L \subset H^1(J)$, resp. $E'_L \subset L^2(J)$, be the space of piecewise affine continuous, reps. piecewise constant, functions on T_L. Let $V_L \subset V_{L+1} \subset V$ be a sequence of finite-dimensional nested subspaces whose union is dense in V. We assume that $\kappa := \inf_{L \geq 0} \left[\inf_{\|\chi'_L\|_{V'}=1} \sup_{\|\chi_L\|_V=1} (\chi'_L, \chi_L) \right]$, is positive. This is, e.g., the case for continuous piecewise affine functions on a quasi-uniform sequence of simplicial triangulations of a polygonal domain D.

For $\mathbf{M}$ and $\mathbf{N}$ we will substitute computationally accessible spectrally equivalent matrices. Let $P_k : L^2(J) \to E_k$, $k \geq 0$, and $Q_\ell : H \to V_\ell$, $\ell \geq 0$, be orthogonal projections. Then $\mathbf{M}$ is taken as the Gram matrix of the symmetric linear operator $M : X \to X'$ given by $M := \sum_{k,\ell \geq 0} (2^{2\ell} + 2^{2k-2\ell})(P_k \otimes Q_\ell)$. The matrix $\mathbf{N}$ is defined similarly.

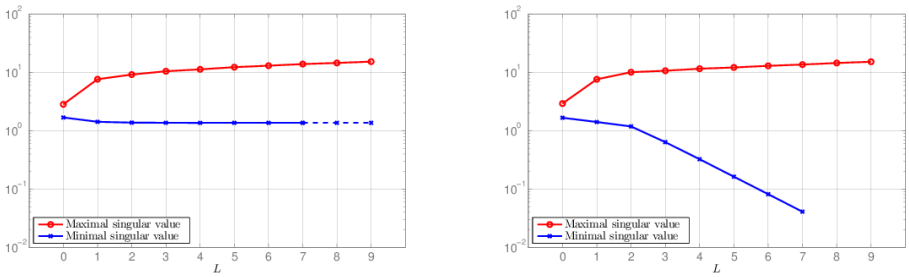
Let now $X_L := E_L \otimes V_L$. Then there exists $c > 0$ such that the following holds for any $L \geq 0$:

a) For $Y_L := V_L \times [E_{L+1} \otimes V_L]$ or $Y_L := V_L \times [E'_{L+1} \otimes V_L]$, one has $\gamma_L \geq c\kappa$.

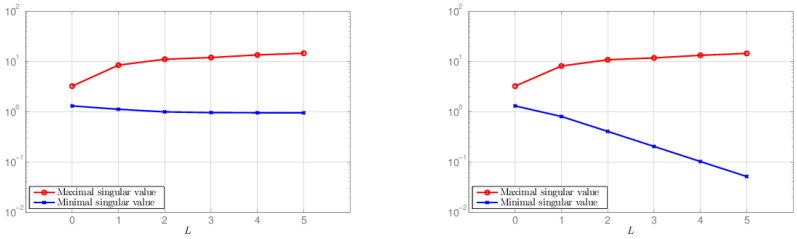
b) For $Y_L := V_L \times [E'_L \otimes V_L]$ one has only conditional stability $\gamma_L \geq c\mathrm{CFL}_L$ for $\mathrm{CFL}_L := 2^{-(L+1)} |t - s| \sup_{\|\chi'_L\|_{V'}=1} \|\chi_L\|_V$. Note that this discretization, with trapezoidal quadrature in time, exactly corresponds to the Crank-Nicholson time-stepping scheme.

The estimates are illustrated in the following plots. These show the minimal and the maximal singular values of the preconditioned system matrix for the case $a = 1$ with continuous piecewise affine functions V_L on a uniform partition of $D := (-1, 1)$ with $2^{L+1} - 1$ inner nodes.

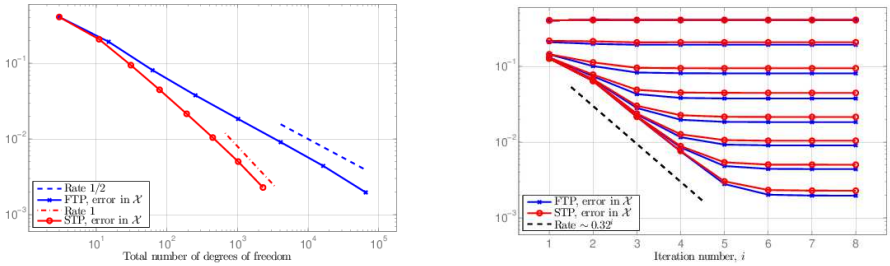
a) For $Y_L := V_L \times [E_{L+1} \otimes V_L]$, or $Y_L := V_L \times [E_L \otimes V_L]$:



b) For $X_L = V_L \otimes E_{2L}$ with $Y_L := V_L \times [E'_{2L} \otimes V_L]$, or $X_L = V_L \otimes E_L$ with $Y_L := V_L \times [E'_L \otimes V_L]$:



Space-time sparse tensor product. Another pair of discrete trial and test spaces that satisfy the discrete inf-sup condition $\gamma_L \geq c\kappa > 0$ are the space-time sparse tensor product subspaces $X_L := \sum_{k+\ell\leq L} E_k \otimes V_\ell$ and $Y_L := V_L \times \sum_{k+\ell\leq L} E_{k+1} \otimes V_\ell$. To illustrate the potential of the space-time sparse tensor product (STP) subspaces we consider the semi-linear parabolic evolution equation $\partial_t u(t,x) - \partial_{xx} u(t,x) + 10u(t,x)^3 = f(t,x)$ on the spatial domain $D := (-1,1)$ with V_L as above, for the non-separable source $f(t,x) = \sin(\pi t/2)^2 \cos(\cos(\pi t/2) + x)$ and with initial value $g = 0$. This equation can be cast into the operator form $Bu + G(u) = F$, and we solve it by the fixed point iteration $w \mapsto u_L(F - G(w))$, resulting in the sequence of discrete solutions for each $i = 0, 1, \ldots$. We compare the error in X with the corresponding solutions on the full tensor product (FTP) subspaces in terms of the total number of degrees of freedom 1) for fixed $i = 10$ with $L = 0, \ldots, 7$, and 2) for $L = 0, \ldots, 7$ as a function of the number of fixed point iterations i:



In particular, one observes the expected improved rate of the sparse-tensor product solution, and a discretization-independent contraction rate in the fixed point iteration.

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References

[1] R. Andreev, Stability of sparse space-time finite element discretizations of linear parabolic evolution equations, IMA J. Numer. Anal., 33, 2013

[2] R. Andreev, Stability of space-time Petrov-Galerkin discretizations for parabolic evolution equations, ETH Diss. No. 20842, ETH Zürich, 2012

[3] R. Andreev, Space-time discretization of the heat equation. A concise Matlab implementation, arXiv:1212.6037, 2012

[4] M. Griebel and D. Oeltz, A sparse grid space-time discretization scheme for parabolic problems, Computing, 81, 2007

[5] Ch. Schwab and R. Stevenson, Space-time adaptive wavelet methods for parabolic evolution problems, Math. Comp., 78, 2009

Adaptive near–optimal rank tensor approximation for high–dimensional operator equations

Markus Bachmayr
(joint work with Wolfgang Dahmen)

We consider a framework for the construction of iterative schemes for operator equations that combine adaptive approximation in a basis and low-rank approximation in tensor formats. Our starting point is an operator equation $Au = f$, where $A\colon \mathcal{H} \to \mathcal{H}'$ is a bounded and elliptic linear operator, where $\mathcal{H}$ is a function space on a product domain in potentially high dimension d. Assuming that a Riesz basis of $\mathcal{H}$ is available, the original problem can be rewritten equivalently as a bi-infinite linear system $\mathbf{Au} = \mathbf{f}$, where $\mathbf{u}, \mathbf{f} \in \ell_2$, and the infinite matrix $\mathbf{A}\colon \ell_2 \to \ell_2$ is bounded and continously invertible. In what follows, $\|\cdot\|$ always denotes the norm on ℓ_2.

Under the given assumptions, the Richardson iteration for the continuous problem $\mathbf{u}_{n+1} = \mathbf{u}_n - \omega(\mathbf{Au}_n - \mathbf{f})$ is convergent for sufficiently small $\omega > 0$. This is the starting point for adaptive wavelet methods [3] that dynamically approximate this ideal iteration by finite quantities. They exploit the approximate sparsity of the coefficient sequences $\mathbf{u}$, $\mathbf{f}$, and of the operator $\mathbf{A}$.

The new aspect here is that we make use of additional tensor product structure of the problem. For this discussion, we assume $\mathcal{H} = \mathcal{H}_1 \otimes \cdots \otimes \mathcal{H}_d$, and that we have a tensor product Riesz basis $\{\Psi_\nu := \psi_{\nu_1} \otimes \cdots \otimes \psi_{\nu_d}\}_{\nu \in \mathbb{N}^d}$ of $\mathcal{H}$. Note that these assumptions can be relaxed, which is necessary e.g. for problems on Sobolev spaces; the latter case, however, requires special care.

We now use a structured tensor format for coefficient sequences with respect to $\{\Psi_\nu\}$. An example of a suitable tensor structure is the *Tucker* or *subspace format*,

$$\mathbf{u} = \sum_{k_1=1}^{\infty} \cdots \sum_{k_d=1}^{\infty} \mathbf{a}_{k_1,\dots,k_d}\, \mathbf{U}^{(1)}_{k_1} \otimes \cdots \otimes \mathbf{U}^{(d)}_{k_d}\,, \tag{1}$$

with the order-d core tensor $\mathbf{a}$ and orthonormal mode frames $\mathbf{U}^{(i)}_k \in \ell_2$. An alternative that is also suitable for higher dimensions is the hierarchical Tucker format, which amounts to a further recursive decomposition of the core tensor in (1), and retains the important properties of the Tucker format. A common feature of both formats is of particular interest for our purposes: near-best approximations by lower-rank tensors, with controlled error in ℓ_2-norm, can be computed by procedures implementable by standard linear algebra routines. Concerning these and related tensor representations, we refer to [4] and the references therein.

In the case of the Tucker format, the resulting approximation of u is of the form

$$u = \sum_{k_1=1}^{r_1} \cdots \sum_{k_d=1}^{r_d} \mathbf{a}_{k_1,\dots,k_d} \bigotimes_{i=1}^{d} \Big(\sum_{\nu_i \in \Lambda_i \subset \mathbb{N}} \mathbf{U}^{(i)}_{k_i,\nu_i} \psi_{\nu_i} \Big)\,.$$

This is a highly nonlinear type of approximation: besides the multiplicative nonlinearity in the tensor representation, we aim to adaptively and simultaneously

determine suitable finite approximation ranks $r_1, \dots, r_d$ (with additional ranks for the decomposition of $\mathbf{a}$ in the hierarchical format), the active indices Λ_i for the mode frames, and corresponding coefficients.

This is accomplished by a perturbed iteration

$$\mathbf{u}_{n+1} = \mathrm{R}_{\varepsilon_1(n)}\mathrm{C}_{\varepsilon_2(n)}\big(\mathbf{u}_n - \omega(\mathbf{A}_{\varepsilon_3(n)}(\mathbf{u}_n) - \mathbf{f}_{\varepsilon_3(n)})\big)\,. \tag{2}$$

Here $\mathbf{f}_\varepsilon$ and $\mathbf{A}_\varepsilon$ approximate $\mathbf{f}$ and the action of $\mathbf{A}$, respectively, leading to an implicit adjustment of approximation ranks and active basis indices; R_ε yields a recompressed low-rank approximation with error bound ε; and C_ε is a coarsening operation that eliminates basis indices corresponding to negligibly small coefficients.

A crucial aspect is that the latter operation needs to be performed by inspecting only lower-dimensional quantities derived from the high-dimensional coefficient sequence. To this end, we consider the lower-dimensional sequences

$$(\pi^{(i)}(\mathbf{v}))_{\nu_i} := \Big(\sum_{\substack{\nu_1,\dots,\nu_{i-1},\\ \nu_{i+1},\dots,\nu_d}} |\mathbf{v}_{\nu_1,\dots,\nu_d}|^2\Big)^{\frac{1}{2}}, \quad i = 1,\dots,d,$$

whose entries, making use of orthonormality properties of the underlying tensor format, can be computed efficiently [2]. Deleting the basis indices corresponding to the smallest entries of the $\pi^{(i)}(\mathbf{v})$ yields a realization of $\mathrm{C}_\varepsilon\mathbf{v}$ with the desired properties.

Under the present quite general assumptions, we can give a choice of parameters for the iteration (2) that ensures its convergence. Under suitable further conditions, we can obtain estimates for its complexity in dependence on the target accuracy as well. For such a result, we need appropriate approximability assumptions. On the one hand, we assume that $\mathbf{u}$, $\mathbf{f}$, and $\mathbf{A}$ each have low-rank approximations for which the error in ℓ_2- and spectral norm, respectively, decays exponentially or almost exponentially with the maximal rank in the respective representation. On the other hand, we assume that the best N-term approximations for $\pi^{(i)}(\mathbf{f})$ and $\pi^{(i)}(\mathbf{u})$ converge as N^{-s} for some $s > 0$. This corresponds to approximate sparsity of mode frames. Furthermore, we assume that the lower-dimensional factors in the approximations of the operator have suitable wavelet compressibility properties. These exemplary assumptions hold in a number of applications, see e.g. [1].

As our main result [2], we find that under these conditions, (2) produces $\mathbf{u}_\varepsilon$ with $\|\mathbf{u}_\varepsilon - \mathbf{u}\| \le \varepsilon$ using a number of operations that can be bounded by

$$C|\ln\varepsilon|^K \varepsilon^{-\frac{1}{s}}\,.$$

Here C and K are independent of ε, with only algebraic and logarithmic explicit dependencies, respectively, on d; there may, however, be further dependencies on d via the approximability of $\mathbf{A}$, $\mathbf{f}$, and $\mathbf{u}$. In other words, up to a fixed polylogarithmic factor we recover the complexity of approximating the lower-dimensional tensor components of the solution. Moreover, assuming an appropriate dimension-dependence of the required approximability estimates, the complexity of the iteration has an algebraic dependence on d.

References

[1] M. Bachmayr. *Adaptive Low-Rank Wavelet Methods and Applications to Two-Electron Schrödinger Equations*. PhD thesis, RWTH Aachen, 2012.
[2] M. Bachmayr and W. Dahmen. Adaptive near-optimal rank tensor approximation for high-dimensional operator equations. arXiv:1304.7796 [math.NA], 2013.
[3] A. Cohen, W. Dahmen, and R. DeVore. Adaptive wavelet methods II – beyond the elliptic case. *Foundations of Computational Mathematics*, 2(3):203–245, 2002.
[4] W. Hackbusch. *Tensor Spaces and Numerical Tensor Calculus*, volume 42 of *Springer Series in Computational Mathematics*. Springer-Verlag Berlin Heidelberg, 2012.

Instance optimality for hp-type approximation

Peter Binev

Adaptive approximation by piecewise polynomials can be generalized in different ways. One of the most investigated forms of it, often used in finite element methods (FEM), is the *hp-approximation* in which the local size of the elements of the partition and the degree of the polynomials may vary but the total number of degrees of freedom is controlled.

Given a domain D, we partition it using a fixed binary refinement scheme and relate each partition to a binary tree $\mathcal{T}$ with root D and leaves $\mathcal{L}(\mathcal{T})$ that correspond to the elements of the partition. As usual, subdividing the element corresponding to the node $\Delta \in \mathcal{L}(\mathcal{T})$ into two is related to adding two nodes Δ' and Δ'' to the tree and connecting them to Δ. We referred to Δ as "parent" of the new nodes Δ' and Δ'', called "children" of Δ. Note that the binary tree $\mathcal{T}$ should be *admissible*, namely, with the exception of the root for each node Δ of $\mathcal{T}$ its sibling is also in $\mathcal{T}$.

We consider a sequence of polynomial spaces $\mathcal{P}_1 \subset \mathcal{P}_2 \subset \mathcal{P}_3 \subset ...$ of orders $p = 1, 2, 3, ...$ and assume that the orders correspond to the numbers of degrees of freedom introduced by the polynomial space. Evidently, in multidimensional settings the order cannot correspond to the degree of the polynomial space. There are two options to recommend in such cases: *(i)* to consider $\mathcal{P}_k$ to be a linear combination of k monomials; or *(ii)* to set $\mathcal{P}_k = \mathcal{P}_{k-1}$ in case k is not high enough to match the number of degrees of freedom of the next polynomial space of choice.

Given a tree $\mathcal{T}$, we define an hp-tree $\mathcal{T}^{\rm hp}$ and an approximating space $\mathcal{A}(\mathcal{T}^{\rm hp})$ by decorating each leave $\Delta \in \mathcal{L}(\mathcal{T})$ with a polynomial space $\mathcal{P}_{p(\Delta)}$ of a *node-specific* order $p(\Delta)$. The complexity $N = \mathcal{N}(\mathcal{T}^{\rm hp})$ of the approximation by functions from $\mathcal{A}(\mathcal{T}^{\rm hp})$ is set to be $\mathcal{N}(\mathcal{T}^{\rm hp}) := \sum_{\Delta\in\mathcal{L}(\mathcal{T})} p(\Delta)$.

Given a function f, we define its best hp-type approximation as

$$\sigma_n(f) := \inf_{\mathcal{N}(\mathcal{T}^{\rm hp})\le n} \inf_{f_n\in\mathcal{A}(\mathcal{T}^{\rm hp})} E_f(f_n) \ ,$$

where $E_f(f_n) = E(\mathcal{T}^{\rm hp})$ is the error of approximating f by f_n.

The goal of this research is to examine hp-adaptive approximation and to establish for it results similar to the ones from [3] used in [2] and several other papers

as an important ingredient in proving optimal rates of convergence for adaptive FEM.

We assume that for any node $\Delta \in \mathcal{T}$ we can calculate the local error of approximation $e_k(\Delta)$ by functions from $\mathcal{P}_k$ at the element of the partition corresponding to Δ. We also assume that these errors satisfy the subadditivity condition that in this setup is as follows:

$$e_k(\Delta) \geq e_{k+1}(\Delta) \text{ for } k \geq 1 \qquad \text{and} \qquad e_1(\Delta) \geq e_1(\Delta') + e_1(\Delta'')\,,$$

where Δ' and Δ'' are the children of Δ. A weaker assumption, similar to the one considered in [3], can be handled using the same general ideas.

The total error $E(\mathcal{T}^{\mathrm{hp}})$ is defined as

$$E(\mathcal{T}^{\mathrm{hp}}) := \sum_{\Delta \in \mathcal{L}(\mathcal{T}^{\mathrm{hp}})} e_{p(\Delta)}(\Delta)\,.$$

Note that because of the subadditivity condition often the total error is not exactly the norm of the difference $f - f_n$. For example, in L_2 we have to define $E_f(f_n) := \|f - f_n\|_{L_2}^2$.

We want to establish a coarse-to-fine algorithm that analyzes the errors at the current tree $\mathcal{T}_N^{\mathrm{hp}}$ and decides how to define the next tree $\mathcal{T}_{N+1}^{\mathrm{hp}}$ with the degrees of freedom increased by one. To this end, we introduce the admissible binary tree $\mathcal{T}_N^{\mathrm{h}}$ with root node D and number of leaves $\#\mathcal{L}(\mathcal{T}_N^{\mathrm{h}}) = N$. For each node $\Delta \in \mathcal{T}_N^{\mathrm{h}}$ define $T_N^{\mathrm{h}}(\Delta)$ to be the maximal subtree of $\mathcal{T}_N^{\mathrm{h}}$ with root Δ. The order $p(\Delta) := \#\mathcal{L}(T_N^{\mathrm{h}}(\Delta))$ of a node $\Delta \in \mathcal{T}_N^{\mathrm{h}}$ is defined as the number of leaves of the tree $T_N^{\mathrm{h}}(\Delta)$. Then for every subtree $\mathcal{T}$ of $\mathcal{T}_N^{\mathrm{h}}$ with root node D, the tree $\mathcal{T}_N^{\mathrm{hp}}$ is defined by assigning the polynomial orders $p(\Delta)$ for $\Delta \in \mathcal{L}(\mathcal{T})$.

We have to show now how to define the tree $\mathcal{T}_N^{\mathrm{h}}$ and then how to choose its subtree $\mathcal{T}$ in such a way that the error is as small as possible.

The first part of the algorithm follows the ideas from [3] and [1] and defines the modified error functionals

$$\tilde{e}^{\mathrm{h}}(D) := e_1(D) \qquad \text{and} \qquad \tilde{e}^{\mathrm{h}}(\Delta) := \left(\frac{1}{e_1(\Delta)} + \frac{1}{\tilde{e}^{\mathrm{h}}(\Delta^+)}\right)^{-1}\,,$$

where Δ^+ is the parent of the node $\Delta \neq D$. Although these quantities are good indicators how to grow the tree $\mathcal{T}_N^{\mathrm{h}}$, they have to be further adjusted to take into account the improvements of the approximation provided by the hp-option. The adjusted quantities $\tilde{e}(\Delta)$ are defined below. We then grow $\mathcal{T}_N^{\mathrm{h}}$ to $\mathcal{T}_{N+1}^{\mathrm{h}}$ by subdividing the leaf $\Delta_N \in \mathcal{L}(\mathcal{T}_N^{\mathrm{h}})$ with the largest $\tilde{e}(\Delta_N) = \tilde{e}_N := \max_{\Delta \in \mathcal{L}(\mathcal{T}_N^{\mathrm{h}})} \tilde{e}(\Delta)$.

In the second part of the algorithm we check in a fine-to-coarse manner which subtree $\mathcal{T}$ of $\mathcal{T}_N^{\mathrm{h}}$ gives the smallest total error. We start with $\mathcal{T} = \mathcal{T}_N^{\mathrm{h}}$ and set $\tilde{e}(\Delta) := \tilde{e}^{\mathrm{h}}(\Delta)$ for all leaves $\Delta \in \mathcal{L}(\mathcal{T}_N^{\mathrm{h}})$. In the course of the algorithm we observe the dynamic quantities

$$E^{\mathrm{hp}}(\Delta) := \sum_{\Delta' \in \mathcal{L}(\mathcal{T}_N^{\mathrm{h}}(\Delta) \cap \mathcal{T})} e_{p(\Delta')}(\Delta')\,,$$

where $\mathcal{T}$ is the current subtree. We examine level by level, starting from the finest, the internal nodes Δ of $\mathcal{T}$ and compare this quantity with $e_{p(\Delta)}(\Delta)$. If $e_{p(\Delta)}(\Delta) < E^{\rm hp}(\Delta)$, we make Δ a leaf node of $\mathcal{T}$ by trimming all its descendants and also modify $\tilde{e}(\Delta') := \tilde{e}(\Delta')\frac{e_{p(\Delta)}(\Delta)}{E^{\rm hp}(\Delta)}$ for all $\Delta' \in \mathcal{L}(T_N^{\rm h}(\Delta))$; otherwise we do nothing at this step. It should be clear that after examining all the internal nodes this algorithm will result in a subtree $\mathcal{T}$ and the corresponding decorated tree $\mathcal{T}^{\rm hp}$ with minimal total error $E(\mathcal{T}^{\rm hp})$ among all the subtrees $\mathcal{T}$ of $\mathcal{T}_N^{\rm h}$. We denote this optimal hp-tree by $\mathcal{T}_N^{\rm hp}$.

It is important to note that once the tree $\mathcal{T}_N^{\rm hp}$ is found, it is not necessary to repeat the second part of the algorithm in its entirety to obtain $\mathcal{T}_{N+1}^{\rm hp}$ but only to reexamine the quantities related to the node Δ, subdivided to receive $\mathcal{T}_{N+1}^{\rm h}$ from $\mathcal{T}_N^{\rm h}$, and all its ancestors. Therefore, the complexity of the algorithm varies between $\mathcal{O}(N \log N)$ for well balanced trees to $\mathcal{O}(N^2)$ for highly unbalanced ones.

The analysis of the performance of the algorithm is based on the comparison of the resulting tree $\mathcal{T}_N^{\rm hp}$ with the best possible hp-tree $T_n^\star$ with n degrees of freedom. We set $t := \min_{k\le N} \tilde{e}_k$ and estimate in terms of t the errors $E^{\rm hp}(\mathcal{T}_N^{\rm hp})$ from above and $E^{\rm hp}(T_n^\star)$ from below to obtain the following result.

Theorem. *Let $\mathcal{T}_N^{\rm hp}$ is the tree received by the algorithm. Then for $n \le N$ we have*

$$E^{\rm hp}(\mathcal{T}_N^{\rm hp}) \le \frac{N+n-1}{N-n+1}\, \sigma_n(f)\,.$$

Assuming that the calculation of each local error $e_k(\Delta)$ requires at most constant number of operations, the complexity of the algorithm cannot exceed $\mathcal{O}(N^2)$.

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References

[1] P. Binev. Adaptive Methods and Near-Best Tree Approximation. Mathematisches Forschungsinstitut Oberwolfach, Report No. 29, 2007, 1669–1673.
[2] P. Binev, W. Dahmen, and R. DeVore. Adaptive Finite Element Methods with Convergence Rates. Numer. Math., 97, 2004, 219–268.
[3] P. Binev and R. DeVore. Fast Computation in Adaptive Tree Approximation. Numer. Math., 97, 2004, 193–217.

Low-rank techniques applied to moment equations for the stochastic Darcy problem with lognormal permeability

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(joint work with Fabio Nobile, Daniel Kressner and Christine Tobler)

In many natural phenomena and engineering applications the problem data are either incompletely known or contain a certain level of uncertainty due to the material properties, boundary conditions, loading terms, domain geometry, etc. One way to treat and include this uncertainty in the model is to describe

the problem data as random variables or random fields, so that the deterministic problem turns into a stochastic differential equation.

The solution of a stochastic differential equation is itself a random field $u(\omega)$ with values in a suitable function space V. The description of the stochastic solution requires the knowledge of its statistical moments. The simplest approach is Monte Carlo Method, which however is very costly and performs a slow rate of convergence, even if independent of the probability space dimension. As an alternative, a generalized Polynomial Chaos Expansion of the stochastic solution, coupled with a projection or interpolation strategy, may be considered. It exploits the regularity of the solution in the random variable, but can not handle with high dimensional probability spaces. We propose to derive the *moment equations*, that is the deterministic equations solved by the probabilistic moments of the stochastic solution.

We are interested in studying the fluid flow in a heterogeneous porous domain. To model this phenomenon we consider the Darcy boundary value problem with randomly varying permeability:

$$-div_x\left(a(\omega,x)\nabla_x u(\omega,x)\right) = f(x) \tag{1}$$

where the forcing term is deterministic. A frequently used model in geophysical applications [9, 8] describes the permeability as a lognormal random field: $a(\omega,x) = e^{Y(\omega,x)}$, $Y(\omega,x)$ Gaussian random field with standard deviation σ. Recently it has appeared also in the mathematical community: see e.g. [4, 7, 6].

Under the assumption of small standard deviation σ, we expand the random solution $u(Y,x)$ in Taylor series in a neighborhood of $\mathbb{E}[Y]$, and approximate u using its Taylor polynomial $T^K u = \sum_{k=0}^{K} \frac{u^k}{k!}$. This approach is known as *perturbation technique*. We provide an a priori error bound which both predicts the divergence of the Taylor series for any positive σ, and the existence of an optimal degree K^{σ}_{opt} such that adding further terms to the Taylor polynomial will deteriorate the accuracy instead of improving it. Our theoretical findings are confirmed by some numerical tests in the simple case where a single random variable is considered. The divergence of the Taylor series is strictly linked to the model we have adopted. Indeed, if the permeability is described as a finite or infinite sum of bounded random variables, then the convergence of the Taylor series has been proved. [1, 5].

The Taylor polynomial is directly computable only in the finite-dimensional setting, that is when $Y(\omega,x)$ is parametrized by a finite number of random variables. In the infinite-dimensional setting the Taylor polynomial involves the Gateaux derivatives of u with respect to Y, which are not computable. However, it is possible to derive the deterministic equations solved by $\mathbb{E}[T^K u] = \sum_{k=0}^{K} \frac{\mathbb{E}[u^k]}{k!}$, with $K \leq K^{\sigma}_{opt}$.

Starting from the stochastic problem (1) we derive the problem solved by $\mathbb{E}[u^k]$ for $k = 0,\dots,K$ and state its well-posedness. The solution of this k-th order correction problem requires the solution of a recursion on the $(l+1)$-points correlations $\mathbb{E}[u^{k-l}\otimes Y^{\otimes l}]$, for $l = 1,\dots,k$. Each correlation $\mathbb{E}[u^{k-l}\otimes Y^{\otimes l}]$ is defined on

the tensorized domain $D^{\times(l+1)}$, and solves a high dimensional problem, which we prove to be well-posed and for which we show regularity results in mixed Hölder spaces.

In the discrete setting, each correlation $\mathbb{E}[u^{k-l} \otimes Y^{\otimes l}]$ is represented by a tensor of order $l+1$. The *curse of dimensionality* affects the recursion we are studying, since the number of entries of a tensor grows exponentially in its order. To overcome this problem, we propose to store and make computations between tensors in a data-sparse or *low-rank format*. Of particular interest is the *Tensor Train (TT) format*. A tensor in TT-format is represented as a linear combination of order three tensors whose dimensions are called TT-ranks. We represent all correlations $\mathbb{E}[u^{k-l} \otimes Y^{\otimes l}]$ in TT-format so that the curse of dimensionality is greatly reduced.

We develop an algorithm in TT-format able to compute $\mathbb{E}[T^K u]$, the K-th order approximation of $\mathbb{E}[u]$. In the simple one-dimensional case $D = [0, 1]$ we perform some numerical tests both to study the complexity of the algorithm and the accuracy of the TT-solution. We compare the TT-solution with a collocation or Monte Carlo solution.

A short version of this work can be found in [2], whereas all the details can be found in [3].

References

[1] I. Babuška and P. Chatzipantelidis. On solving elliptic stochastic partial differential equations. Comput. Methods Appl. Mech. Engrg., 191, 2002, 4093–4122.

[2] F. Bonizzoni and F. Nobile. Perturbation analysis for the stochastic Darcy problem. Proceeding in 6th European Congress on Computational Methods in Applied Sciences and Engineering, 2012.

[3] F. Bonizzoni. Analysis and approximation of moment equations for PDEs with stochastic data. PhD Thesis, Department of Mathematics, Politecnico di Milano, Italy, 9, 2013.

[4] J. Charrier and A. Debussche. Weak truncation error estimates for elliptic PDEs with lognormal coefficients. Stochastic Partial Differential Equations: Analysis and Computations, 1, 2013, 63-93.

[5] A. Cohen, R. Devore and C. Schwab. Analytic regularity and polynomial approximation of parametric and stochastic elliptic PDE's. Anal. Appl. (Singap.), 9, 2011, 11–47.

[6] J. Galvis and M. Sarkis. Approximating infinity-dimensional stochastic Darcy's equations without uniform ellipticity. SIAM J. Numer. Anal., 47, 2009, 3624–3651.

[7] C.J. Gittelson. Stochastic Galerkin discretization of the log-normal isotropic diffusion problem. Math. Models Methods Appl. Sci., 20, 2010, 237–263.

[8] A. Guadagnini and S.P. Neuman. Nonlocal and localized analyses of conditional mean steady state flow in bounded, randomly nonuniform domains: 1. Theory and computational approach. Water Resour. Res., 35, 1999, 2999–3018.

[9] D.M. Tartakovsky and S.P. Neuman. Transient flow in bounded randomly heterogeneous domains: 1. Exact conditional moment equations and recursive approximations. Water Resour. Res., 34, 1998, 1–12.

A multivariate approximation problem in statistics

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(joint work with Holger Dette)

The problem of constructing optimal designs for the discrimination of several regression models is considered. You are given k real-valued functions $f_j(x)$ and regression functions $v_j(x,\theta_j) \subset V_j$. Here f_j arises from a priori information on a model, and it is compared with the regression function of a different model. Some components of $f = (f_1,\dots,f_k)$ and some components of $v = (v_1,\dots,v_k)$ may coincide if they are involved in more than one comparison.

We consider designs that are defined as probability measures with finite support on a compact interval X. If the design ξ has masses $w_1,\dots,w_\nu$ at the distinct points $x_1,\dots,x_\nu$, then observations are taken at these points with the relative proportions given by the masses.

A design is considered as optimal if it is the solution of the max-min problem

$$\max_\xi \min_\theta \int_X \sum_j |f_j(x) - v_j(x,\theta_j)|^2 d\xi.$$

The min problem describes how good the functions f_j can be approximated by the regression function v_j (that refers to a different model). A good approximation is associated with a bad discrimination. Therefore we have the max-min problem.

Let $\langle\cdot,\cdot\rangle$ be the inner product in Euclidean k-space and $|\cdot|$ be the associated norm, and we write the max-min problem as

$$\max_\xi \min_\theta \int_X |f(x) - v(x,\theta)|^2 d\xi.$$

If the family $V = \otimes V_j$ is linear or appropriate conditions are satisfied in the nonlinear case, then there is no duality gap, i.e., the max-min problem and the min-max problem have the same solution

$$\max_\xi \min_\theta \int_X |f(x) - v(x,\theta)|^2 d\xi$$
$$\min_\theta \max_\xi \int_X |f(x) - v(x,\theta)|^2 d\xi = \min_\theta \|f(x) - v(x,\theta)\|^2.$$

Here $\|\cdot\|$ denotes the sup norm on X.

In the linear case there is a straight forward generalization of the classical theory of univariate Chebyshev approximation [2].

——— **Characterization Theorem** Let $f \in C(X)^k$ and V be an n-dimensional subspace of $C(X)^k$. Set $\varepsilon := f - u$. The function u is a best approximation to f in V if there exist $\nu \le n+1$ points $x_1, x_2, \dots, x_\nu$ with $|(f-u)(x_i)| = \|f-u\|$ and there are ν weights $w_1, w_2, \dots, w_\nu \ge 0$, $\sum_{i=1}^\nu w_i = 1$ such that the functional

$$\ell(g) := \frac{1}{\|\varepsilon\|} \sum_{i=1}^{\nu} w_i \,\langle \varepsilon(x_i), g(x_i)\rangle$$

satisfies

$$\ell(\varepsilon) = \|\varepsilon\|, \quad \|\ell\| = 1, \quad \text{and} \quad V \subset \ker(\ell).$$

The points x_i become the sample points and the w_i their weights.

The characterization theorem gives rise to the impression that the numerical determination of the design ξ is easy, but the existing algorithms were known to be successful only for $k = 1$ and $k = 2$. It is less known that this holds also for a famous algorithm that is based on the equivalence theorem by Atkinson and Fedorov [1]

A shortcoming of the characterization theorem is the fact that the cardinality ν of the support of the design ξ is bounded from above by $\dim V + 1$, but this bound may be far from the actual number of support points that is often close to $\max_j \dim V_j + 1$. A real-life problem cited in [2, 3] has an optimal design with 4 points although the dimension of V is 15. The consequences are severe.

The approach to the solution of the approximation problem is usually by an iteration. Each iteration step has two parts like the classical Remez algorithm for univariate functions. Let u_0 be a suboptimal approximation. In the first part the extreme points, say $x_1, x_2, \ldots, x_\nu$, of the error function

$$|\varepsilon_0| = |f - u_0|$$

are chosen as candidates for the support. The second part with the improvement of the parameters θ and $v = v(\cdot, \theta)$ is the difficult one.

We describe a procedure of Newton type for the linear case. It applies similarly to the gradient space in the nonlinear case. The minimization of the error at the points x_i by a correction v is given by

$$\min_{v \in V} \max_{1 \le i \le \nu} |(f - u_0 - v)(x_i)|^2 = \min_{v \in V} \max_{1 \le i \le \nu} |\varepsilon_0(x_i)|^2 - 2\,\langle \varepsilon_0(x_i), v(x_i)\rangle + |v(x_i)|^2\}.$$

Now we proceed in the spirit of Newton's method and drop the quadratic term to obtain a linear program,

$$\min_{v \in V} \max_{1 \le i \le \nu} |\varepsilon_0(x_i)|^2 - 2\,\langle \varepsilon_0(x_i), v(x_i)\rangle\}.$$

Unfortunately, the objective function is no longer bounden from below. The Newton corrections turn out to be useless. The difference between the number of restrictions and the number of variables is so great that even a damping of the correction does not help whenever $k \ge 3$.

A partial improvement is obtained by performing the minimization successively in the subspaces V_j instead of doing it at once in the entire space. The reduction of the dimension yields linear programs with lower bounds. This is only a preparation, but it is an important one. It makes that the restrictions $w_i \ge 0$ are satisfied in the following step without requiring them explicitly.

Since the original minimization problem is not suitable for the numerical solution, we return to the max-min problem and consider a linearization of the saddle point problem. The sample points x_i are considered as fixed in this part of the exchange algorithm, and we write the max-min problem with the correction

$v = \sum_j \theta_j v_j$ in matrix-vector notation

$$\max_w \min_\theta \{\theta' A(w)\theta - 2w'R\theta + b'w\},$$

subject to the normalization $e'w := \sum_i w_i = 1$. The matrices $A(w)$, R, and the vector b can be found in [2]. In particular, the (inner) minimization for given w gives rise to a linear equation. We compute the matrix $A(w)$ for a guess of w and obtain a linear saddle point problem

$$\begin{pmatrix} A & -R^T & \\ -R & & e \\ & e^T & \end{pmatrix} \begin{pmatrix} \theta \\ w \\ \lambda \end{pmatrix} = \begin{pmatrix} 0 \\ -\frac{1}{2}b \\ 1 \end{pmatrix}.$$

Here λ is the Lagrange multiplier associated to the normalization of the weights.

This iterative procedure yields designs with an efficiency of 99.9 % in 8 to 12 steps, i.e., we get optimal designs up to rounding errors. Moreover we get designs with the right number of support.

Concluding remark. We returned successfully from the approximation problem to the max-min problem but not to a procedure that was motivated by the equivalence theorem by Atkinson and Fedorov [1, 4].

References

[1] A. C. Atkinson and V. V. Fedorov, The designs of experiments for discriminating between two rival models, Biometrika, 62, 1975, 57–70.

[2] D. Braess and H. Dette, Optimal discriminating designs for several competing regression models. The Annals of Statistics, 12, 2013, 897–922.

[3] J. Pinheiro, F. Bretz, and M. Branson, Analysis of dose-response studies: Modeling approaches. In Dose Finding in Drug Development (N. Ting, ed.) 146–171. Springer, New York, 2006.

[4] C. Tommasi and J. López-Fidalgo, Bayesian optimum designs for discriminating between models with any distribution. Computational Statistics & Data Analysis 54, 2010, 143–150.

High-dimensional adaptive sparse polynomial interpolation and application for parametric and stochastic elliptic PDE's

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(joint work with Albert Cohen and Christoph Schwab)

The numerical approximation of parametric partial differential equations is a computational challenge, in particular when the number of involved parameter is large. We considers a model class of second order, linear, parametric, elliptic PDEs on a bounded domain D with diffusion coefficients depending on the parameters in an affine manner. For such models, it was shown in [2] that under very weak assumptions on the diffusion coefficients, the entire family of solutions to such equations can be simultaneously approximated in the Hilbert space $V = H_0^1(D)$ by multivariate sparse polynomials in the parameter vector y with a controlled

number N of terms. The convergence rate in terms of N does not depend on the number of parameters in V, which may be arbitrarily large or countably infinite, thereby breaking the curse of dimensionality. However, these approximation results do not describe the concrete construction of these polynomial expansions, and should therefore rather be viewed as benchmark for the convergence analysis of numerical methods. We present the polynomial interpolation process in high dimension proposed in [4]. We show that the interpolation operator can be recursively computed using a Newton-like interpolation formula in dimension 1. The interpolation points are picked in a multi-dimensional grid based on a tensorization of an infinite uni-dimensional sequence. We show through an analysis of the Lebesgue constant that the stability of the interpolation process is very tied to the choice of the infinite sequence. For instance, linear growth of the Lebesgue constants is insured when the sequence is of type of the so-called $\Re$-Leja sequence studied in [3]. Numerical experiments are presented in large parameter dimension and confirm the effectiveness of the process.

References

[1] A. Cohen, R. DeVore, and Ch. Schwab. Analytic regularity and polynomial approximation of parametric and stochastic PDE's. Analysis and Applications (Singapore) 9, 1-37 (2011).

[2] A. Chkifa, A. Cohen, R. DeVore, and Ch. Schwab. Sparse Adaptive Taylor Approximation Algorithms for Parametric and Stochastic Elliptic PDEs. M2AN, volume 47-1, pages 253-280, 2013.

[3] A. Chkifa. On the Lebesgue constant of Leja sequences for the complex unit disk and of their real projection. JAT volume 166, pages 176-200, 2013.

[4] A. Chkifa, A. Cohen, and Ch. Schwab. High-dimensional adaptive sparse polynomial interpolation and applications to parametric PDEs. JFoCM May 2013

[5] A. Chkifa, A. Cohen. Breaking the curse of dimensionality in sparse polynomial approximation of parametric PDEs. Submitted.

Adaptive wavelet methods for SPDEs: Theoretical analysis and practical realization

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(joint work with P.A. Cioica, N. Döhring, S. Kinzel, F. Lindner, T. Raasch, K. Ritter and R.L. Schilling)

We are interested in the numerical treatment of stochastic partial differential equations (SPDEs) of the form

$$\mathrm{d}u(t) = (A(u(t)) + F(t, u(t)))\mathrm{d}t + B(t, u(t))\mathrm{d}W_t, \tag{1}$$

on a bounded Lipschitz domain $\mathcal{O} \subseteq \mathbb{R}^d$, where $W = (W(t))_{t\in[0,T]}$ is a cylindrical Wiener process. These equations have important applications, e.g., in computational finance, epidemiology, population genetics and many others. We are in particular concerned with *adaptive* numerical schemes based on *wavelets*. The first step is the theoretical foundation of adaptive numerical methods for SPDEs. It is

well–known that the order of approximation that can be achieved by adaptive and other nonlinear methods is determined by the regularity of the exact solution in the specific scale $B^s_{\tau,\tau}(\mathcal{O})$, $1/\tau = s/d + 1/2$, of Besov spaces. In contrast, the approximation order of nonadaptive (uniform) schemes is determined by the Sobolev smoothness. Therefore, to justify the use of adaptive algorithms, sufficiently high Besov regularity (in above scale), compared to the Sobolev smoothness has to be established. It turns out that for linear SPDEs of the form:

$$\mathrm{d}u(t) = \sum_{\mu,\nu=1}^{d} a^{\mu\nu} u_{x_\mu x_\nu} \mathrm{d}t + \sum_{k=1}^{\infty} g^k(t) \mathrm{d}w_t^k, \tag{2}$$

this is indeed the case. To establish Besov regularity, the first step is to study these equations in weighted Sobolev spaces. We consider the spaces

$$H^m_{p,\theta}(\mathcal{O}) := \left\{ u \,:\, \rho^{|\alpha|} D^\alpha u \in L_p(\mathcal{O}, \rho^{\theta-d}\mathrm{d}x) \text{ for all } \alpha \in \mathbb{N}^d \text{ with } |\alpha| \le m \right\},$$

where ρ is some distance to the boundary, and their stochastic counterparts

$$\begin{aligned}
\mathbb{H}^\gamma_{p,\theta}(\mathcal{O},T) &:= L_p\left(\Omega_T, \mathcal{P}, \mathbb{P}\otimes\lambda;\, H^\gamma_{p,\theta}(\mathcal{O})\right),\\
\mathbb{H}^\gamma_{p,\theta}(\mathcal{O},T;\ell_2) &:= L_p\left(\Omega_T, \mathcal{P}, \mathbb{P}\otimes\lambda;\, H^\gamma_{p,\theta}(\mathcal{O};\ell_2)\right),\\
U^\gamma_{p,\theta}(\mathcal{O}) &:= L_p\big(\Omega, \mathcal{F}_0, \mathbb{P}; H^{\gamma-2/p}_{p,\theta+2-p}(\mathcal{O})\big).
\end{aligned}$$

Then we seek for solutions in the spaces:

$$\begin{aligned}
\mathfrak{H}^\gamma_{p,\theta}(\mathcal{O},T) := \Big\{ u \in \mathbb{H}^\gamma_{p,\theta-p}(\mathcal{O},T) \,:\, & u(0,\cdot\,) \in U^\gamma_{p,\theta}(\mathcal{O}) \ \text{ and}\\
& \mathrm{d}u = f\,\mathrm{d}t + \sum_{\kappa=1}^{\infty} g^\kappa\,\mathrm{d}w_t^\kappa \ \text{ for some}\\
& f \in \mathbb{H}^{\gamma-2}_{p,\theta+p}(\mathcal{O},T),\ g \in \mathbb{H}^{\gamma-1}_{p,\theta}(\mathcal{O},T;\ell_2)\Big\},
\end{aligned}$$

The following theorem has been proved in [5]:

Theorem 1. *Let $\gamma \in \mathbb{R}$. For $p \in [2,\infty)$, there exists a constant $\kappa_0 \in (0,1)$, depending only on d, p, $(a^{\mu\nu})_{1\le\mu,\nu\le d}$ and $\mathcal{O}$, such that for any $\theta \in (d+p-2-\kappa_0, d+p-2+\kappa_0)$, $g \in \mathbb{H}^{\gamma-1}_{p,\theta}(\mathcal{O},T;\ell_2)$ and $u_0 \in U^\gamma_{p,\theta}(\mathcal{O})$, Equation (2) has a unique solution u in the class $\mathfrak{H}^\gamma_{p,\theta}(\mathcal{O},T)$. For this solution*

$$\|u\|^p_{\mathfrak{H}^\gamma_{p,\theta}(\mathcal{O},T)} \le C\left(\|g\|^p_{\mathbb{H}^{\gamma-1}_{p,\theta}(\mathcal{O},T;\ell_2)} + \|u_0\|^p_{U^\gamma_{p,\theta}(\mathcal{O})}\right), \tag{3}$$

where the constant C depends only on d, p, γ, θ, $(a^{\mu\nu})_{1\le\mu,\nu\le d}$, T and $\mathcal{O}$.

Based on this result, the following fact has been shown in [2]:

Theorem 2. *Let $g \in \mathbb{H}^{\gamma-1}_{2,\theta}(\mathcal{O},T;\ell_2)$ and $u_0 \in U^\gamma_{2,\theta}(\mathcal{O})$ for some $\gamma \in \mathbb{N}$, where θ fulfils*

$$\theta \in (d-\kappa_0, d+\kappa_0),$$

with κ_0 from Theorem 1. Let u be the unique solution in the class $\mathfrak{H}^\gamma_{2,\theta}(\mathcal{O},T)$ of Equation (2) and assume furthermore that

$$u \in L_2\big(\Omega_T;\, W_2^s(\mathcal{O})\big) \quad \text{for some} \quad s \in \Big(0, \gamma \wedge \Big(1+\frac{d-\theta}{2}\Big)\Big]. \tag{4}$$

Then, we have

$$u \in L_\tau(\Omega_T; B^\alpha_{\tau,\tau}(\mathcal{O})), \quad \frac{1}{\tau} = \frac{\alpha}{d} + \frac{1}{2}, \quad \text{for all} \quad \alpha \in \Big(0, \gamma \wedge \frac{sd}{d-1}\Big),$$

and the following estimate holds

$$\|u\|_{L_\tau(\Omega_T;B^\alpha_{\tau,\tau}(\mathcal{O}))} \le C\left(\|g\|_{\mathbb{H}^{\gamma-1}_{2,\theta}(\mathcal{O},T;\ell_2)} + \|u_0\|_{U^\gamma_{2,\theta}(\mathcal{O})} + \|u\|_{L_2(\Omega_T;W^s_2(\mathcal{O}))}\right). \tag{5}$$

The constant C depends only on d, γ, α, s, θ, $(a^{\mu\nu})_{1\le\mu,\nu\le d}$, T and $\mathcal{O}$.

Generalizations to other kinds of noise such a multiplicative noise and to semilinear SPDEs also exist [1, 2].

Since the Sobolev smoothness is multiplied by $d/(d-1)$, the Besov smoothness of the solution is generically higher than the Sobolev regularity so that the use of adaptive wavelet algorithms is completely justified.

The next step is the practical realization of these schemes. To discretize the underlying SPDE, we use the horizontal method of lines, i.e., the Rothe method. We interpret the SPDE as an abstract Cauchy problem in a suitable function space and discretize first in time, then in space. Then, in each time step, an elliptic subproblem has to be solved which will be performed by the well-known adaptive wavelet algorithms that are guaranteed to converge with optimal order [4]. Although each subproblem can be solved up to any given tolerance, it remains to investigate how the errors in the different time steps accumulate. We present a strategy for the choice of the tolerances such that the overall Rothe scheme provides the same approximation order in time as the unperturbed one (where the elliptic subproblems are solved exactly). We need the following assumptions:

- $A : D(A) \subset U \to U$ is linear, densely defined, strictly negative definite, and self-adjoint. Zero belongs to the resolvent set of A and the inverse $A^{-1} : U \to U$ is compact. There exists an $\alpha > 0$ such that $(-A)^{-\alpha}$ is a trace class operator on U.
- For certain parameters $\rho \ge 0,\ \sigma < 1, \beta < (1-\alpha)/2$,

$$\begin{aligned} F : D((-A)^\rho) &\to D((-A)^{\rho-\sigma}), \\ B : D((-A)^\rho) &\to \mathcal{L}(\ell_2; D((-A)^{\rho-\beta})), \end{aligned}$$

  are both globally Lipschitz continuous.

Under these assumptions and for a uniform discretization in time with stepsize τ the following holds [3].

Theorem 3. *Let $(u(t))_{t\in[0,T]}$ be the unique mild solution to Eq. (1) and let*

$$\delta < \min\{1-\sigma, (1-\alpha)/2 - \beta\}.$$

Let $(u_k)_{k=1}^K$ be the time discretization by means of the linearly-implicit Euler scheme. Then

$$\Big(\mathbb{E}\|u(T)-u_K\|^2_{D((-A)^\rho)}\Big)^{1/2} \le C\tau^\delta$$

with $C = C(u_0, \delta, A, B, F, \alpha, \beta, \sigma, T)$.

The next theorem states the conditions under which the perturbed system realizes the same order of convergence in time [3].

Theorem 4. *Let $(u(t))_{t\in[0,T]}$ be the unique mild solution to Eq.* (1) *and let*

$$\delta < \min\{1-\sigma, (1-\alpha)/2-\beta\}.$$

If one chooses

$$\epsilon_k \le \tau^{1+\delta}$$

for all $k = 0, \dots, K-1$, $K \in \mathbb{N}$, then the output $\tilde{u}_K$ of the inexact linearly-implicit Euler scheme satisfies

$$\Big(\mathbb{E}\|u(T)-\tilde{u}_K\|^2_{D((-A)^\rho)}\Big)^{1/2} \le C\tau^\delta$$

with $C = C(u_0, \delta, A, B, F, \alpha, \beta, \sigma, T)$.

References

[1] P.A. Cioica, S. Dahlke. Spatial Besov Regularity for Semilinear Stochastic Partial Differential Equations on Bounded Lipschitz Domains. Int. J. Comput. Math., 89, 2012, 2443–2459.

[2] P.A. Cioica, S. Dahlke, S. Kinzel, F. Lindner, T. Raasch, K. Ritter, R.L. Schilling. Spatial Besov Regularity for SPDEs on Lipschitz Domains. Studia Math., 207, 2010, 197–234.

[3] P.A. Cioica, S. Dahlke, N. Döhring, S. Kinzel, F. Lindner, T. Raasch, K. Ritter, and R.L. Schilling. On the Convergence Analysis of Rothe's Method. Preprint No. 124, 2012, DFG Priority Program 1324 "Extraction of Quantifiable Information from Complex Systems".

[4] A. Cohen, W. Dahmen, and R. DeVore. Adaptive Wavelet Methods for Elliptic Operator Equations – Convergence Rates. Math. Comput., 70, 2001, 27–75.

[5] K.-H. Kim. Weighted Sobolev Space Theory of Parabolic Stochastic PDEs on Non-smooth Domains. J. Theor. Probab., 2012, DOI 10.1007/s10959-012-0459-7.

Multivariate polynomial interpolation on monotone sets

Nira Dyn

(joint work with Michael S. Floater)

This talk presents results from [1]. A monotone set of indices in $\mathbb{R}^d$ is a subset of $\mathbb{Z}^d_+$ such that

$$\alpha \in M \Rightarrow \{\beta \in \mathbb{Z}^d_+ \ : \ \beta \le \alpha\} \subset M.$$

In this talk we investigate interpolation from the space of polynomials related to M, $\mathrm{span}\{x^\alpha = x_1^{\alpha_1} \dots x_d^{\alpha_d} \ : \ \alpha \in M\}$, at the d-dimensional interpolation points $\{x_\alpha = (x^{[1]}_{\alpha_1}, \dots, x^{[d]}_{\alpha_d}) : \ \alpha \in M\}$, where $\{x_i^{[j]}\}_{i\in\mathbb{Z}_+}$ is a sequence of distinct real numbers for $j = 1, \dots, d$.

The focus of the talk is the expression of the interpolant defined by M, $P(M)$, in terms of tensor-product polynomial interpolants, corresponding to blocks of indices. For a monotone set which is a union of n blocks, $M_n = \cup_{i=1}^n B_i$, we obtain the combinatorial formula

$$P(M_n) = \sum_{k=1}^{n} (-1)^{k-1} \sum_{1 \le i_1 < i_2 < \ldots i_k \le n} P(B_{i_1} \cap B_{i_2} \ldots \cap B_{i_k})$$

and its simplification

$$P(M) = \sum_{\alpha \in M} c_\alpha P(B_\alpha), \quad c_\alpha = \sum_{\epsilon \in I_\alpha} (-1)^{|\epsilon|},$$

where $I_\alpha = \{\epsilon \in \{0,1\}^d \ : \ \alpha + \epsilon \in M\}$. A wide class of indices for which the corresponding coefficients c_α vanish is given. The simplified formula is then applied to the example of interpolation by total degree polynomials,

$$P(\{\alpha \ : \ |\alpha| \le m\}) = \sum_{j=0}^{\min\{m,d-1\}} \binom{d-1}{j} (-1)^j \sum_{|\alpha| = m-j} P(\{\beta \ : \ \beta \le \alpha\}).$$

Similar formulas were obtained for piecewise polynomial interpolants on sparse grids for the numerical solution of PDE. In particular the formula for the total degree interpolant given above is similar to the "combination technique" for sparse grids (see e.g. [2]).

References

[1] N. Dyn and M.S. Floater. Multivariate polynomial interpolation on monotone sets. J. Approx. Theory, to appear.

[2] M. Griebel, M. Schneider, and C. Zenger. A Combination technique for the solution of sparse grid problems. Iterative methods in linear algebra (Brussels, 1991), 263–281, North Holand, Amsterdam, 1992.

Reduced basis methods for parametrized optimal control problems

Martin A. Grepl

(joint work with Mark Kärcher)

Many problems in science and engineering can be modeled in terms of optimal control problems governed by parametrized partial differential equations (PDEs) [6]. While the PDE describes the underlying system or component behavior, the parameters often serve to identify a particular configuration of the component — such as boundary and initial conditions, material properties, and geometry. The solution of these problems using classical discretization techniques such as finite elements or finite volumes is sometimes computationally expensive and time-consuming. One way to decrease the computational burden is the surrogate model approach, where the original high-dimensional model is replaced by a reduced order approximation. These ideas have received a lot of attention in the past and various model order reduction techniques — e.g., proper orthogonal

decomposition (POD), reduction based on inertial manifolds, and reduced basis methods — have been used in this context. However, the solution of the reduced order optimal control problem is generally suboptimal and reliable error estimation is thus crucial.

A posteriori error bounds for reduced order solutions of optimal control problems have been proposed for proper orthogonal decomposition (POD) and reduced basis surrogate models in [9] and [2, 7], respectively. However, all of these results have slight deficiencies, i.e., evaluation of the bounds in [9] requires a forward-backward solution of the underlying high-dimensional state and adjoint equations and is thus computationally expensive, the error estimator in [2] is not a rigorous upper bound for the error, and the result in [7] only applies to optimal control problems without control constraints involving stationary (time-independent) PDEs.

In this talk, we employ the reduced basis method [8] as a surrogate model for the solution of optimal control problems. We consider the following simplified problem setting: Given $\mu \in \mathcal{D} \subset \mathbb{R}^P$, we want to solve the optimal control problem

$$(1) \quad \min J(y,u;\mu) \quad \text{s.t.} \quad (y,u) \in Y \times \mathcal{U} \ \text{ solves } \ a(y,v;\mu) = b(v;\mu)\,u, \ \forall v \in Y.$$

Here, μ and $\mathcal{D}$ are the parameter and parameter domain, respectively; Y is an appropriate Hilbert space with associated inner product $(w,v)_Y$ and norm $\|\cdot\|_Y$; $\Omega \subset \mathbb{R}^d$, $d = 1,2,3$, is our spatial domain, a point in which shall be denoted $(x_1,\ldots,x_d)$; $u \in \mathcal{U} \equiv \mathbb{R}$ is the (scalar) control; $y(\mu) \in Y$ is the state variable; b is a linear bounded functional on Y; and, for any $\mu \in \mathcal{D}$, $a(\cdot,\cdot;\mu) : Y \times Y \to \mathbb{R}$ is a coercive continuous bilinear form. Finally, the quadratic cost functional $J(\cdot,\cdot;\mu) : Y \times \mathcal{U} \to \mathbb{R}$ is given by $J(y,u;\mu) = \frac{1}{2}\|y - y_d\|^2_{L^2(D)} + \frac{\lambda}{2}\|u - u_d\|^2_{\mathcal{U}}$, where $y_d \in Y$ and $u_d \in \mathcal{U}$ are the desired state and control, respectively; $D \subset \Omega$ is a measurable set; and $\lambda > 0$ is the given regularization parameter. It follows from our assumptions that there exists a unique optimal solution to (1) [6].

In the reduced basis methodology, we next introduce a truth finite element approximation space $Y_{\mathcal{N}} \subset Y$ of very large dimension $\mathcal{N}$ and reduced basis space $Y_N \subset Y_{\mathcal{N}}$ of dimension N, where usually $N \ll \mathcal{N}$. We denote the truth and reduced basis solutions to the optimal control problem (1) — obtained through a Galerkin projection onto the respective spaces — by $(y^*_{\mathcal{N}}, u^*_{\mathcal{N}}) \in Y_{\mathcal{N}} \times \mathcal{U}$ and $(y^*_N, u^*_N) \in Y_N \times \mathcal{U}$, respectively.

We first show, for the simple problem setting (1), how to derive rigorous and efficiently evaluable *a posteriori* error bounds for the optimal control, $\|u^*_{\mathcal{N}} - u^*_N\|_{\mathcal{U}}$, and the associated cost functional, $|J(y^*_{\mathcal{N}}, u^*_{\mathcal{N}};\mu) - J(y^*_N, u^*_N;\mu)|$ [3, 4]. We start with the bound from [9], replace the required high-dimensional state and adjoint solution by the solution to the associated reduced basis approximation, and bound the error introduced by extending the standard reduced basis *a posteriori* error bounds. The error bound for the cost functional is based on the standard result in [1], again by extending standard reduced basis *a posteriori* error bounds. The offline-online decomposition directly applies to the reduced basis optimal control

problem and the associated *a posteriori* error bounds: the computational complexity in the online stage to evaluate (y_N^*, u_N^*) as well as the control and associated cost functional error bounds depends only on N and is independent of $\mathcal{N}$. Our approach thus allows not only the efficient real-time solution of the reduced optimal control problem, but also the efficient real-time evaluation of the quality of the suboptimal solution.

Finally, we consider various extensions of the problem setting (1) to elliptic optimal control problems with distributed controls and to parabolic problems with multiple (time-dependent) controls and control constraints [5]. We also present numerical results to confirm the validity of our approach. In the parabolic case, for example, we can guarantee — thanks to our *a posteriori* error bounds — a relative error in the control of less than 1% whilst obtaining an average (online) speed-up of approximately 580 for the solution of the reduced basis optimal control problem compared to the truth finite element optimal control problem and an average speed-up of approximately 400 for the solution of the optimal control problem *and* evaluation of the error bound for the control and cost functional.

References

[1] R. Becker, H. Kapp, and R. Rannacher. Adaptive finite element methods for optimal control of partial differential equations: Basic concept. SIAM J. Control Optim., 39:113–132, 2000.

[2] L. Dedè. Reduced basis method and a posteriori error estimation for parametrized linear-quadratic optimal control problems. SIAM J. Sci. Comput., 32(2):997–1019, 2010.

[3] M. A. Grepl and M. Kärcher. Reduced basis a posteriori error bounds for parametrized linear-quadratic elliptic optimal control problems. C. R. Acad. Sci. Paris, Ser. I, 349(15-16):873–877, 2011.

[4] M. Kärcher and M. A. Grepl. A certified reduced basis method for parametrized elliptic optimal control problems. ESAIM: COCV, 2013 (accepted for publication).

[5] M. Kärcher and M. A. Grepl. A posteriori error estimation for reduced order solutions of parametrized parabolic optimal control problems. submitted to ESAIM: M2AN, 2013.

[6] J. L. Lions. Optimal Control of Systems Governed by Partial Differential Equations. Springer, 1971.

[7] F. Negri, G. Rozza, A. Manzoni, and A. Quarteroni. Reduced basis method for parametrized elliptic optimal control problems. Technical report, Submitted, 2012.

[8] G. Rozza, D. B. P. Huynh, and A. T. Patera. Reduced basis approximation and a posteriori error estimation for affinely parametrized elliptic coercive partial differential equations. Archives of Computational Methods in Engineering, 15(3):229–275, 2008.

[9] F. Tröltzsch and S. Volkwein. POD a-posteriori error estimates for linear-quadratic optimal control problems. Comput. Optim. Appl., 44:83–115, 2009.

Numerical tensor calculus and tensorisation

Wolfgang Hackbusch

The numerical tensor calculus is concerned with the representation of tensors from

$$\mathbf{V} := \bigotimes_{j=1}^{d} V_j$$

and the treatment of the various operations between tensors. V_j are any vector spaces; e.g., function spaces like $L^2(\Omega_j)$, discrete analogues like $\mathbb{R}^{n_j}$, operators between tensor spaces, or their discrete analogues: matrix spaces $\mathbb{R}^{n_j \times m_j}$.

A well-suited method for a data-sparse representation of $\mathbf{v} \in \mathbf{V}$ is the hierarchical tensor format (cf. [11]). It is characterised by (a) a dimension splitting tree T_D and (b) subspaces $U_\alpha \subset V_\alpha := \bigotimes_{j\in\alpha} V_j$. More precisely, $D := \{1, \ldots, d\}$, T_D is a binary tree with vertices $\alpha \subset D$ such that $\alpha = \alpha_1 \cup \alpha_2$ is the disjoint union of the sons α_1, α_2 of α. The set of leaves $\mathcal{L}(T_D)$ consists of $\{j\}$, $j \in D$. The subspaces satisfy the nestedness condition

$$U_\alpha \subset U_{\alpha_1} \otimes U_{\alpha_2} \text{ for } \alpha \in T_D \backslash \mathcal{L}(T_D). \tag{1}$$

At the root, $\mathbf{v} \in U_D$ holds (cf. [9, Chap. 11]).

Given so-called representation ranks r_α ($\alpha \in T_D$) and the tuple $\mathfrak{r} = (r_\alpha)$, the set $\mathcal{H}_\mathfrak{r}$ of hierarchical tensors of representation rank $\mathfrak{r}$ is defined by

$$\mathcal{H}_\mathfrak{r} = \{\mathbf{v} \in \mathbf{V} : \exists U_\alpha \text{ with (1) and } \mathbf{v} \in U_D\}.$$

In fact, an α-rank $rank_\alpha(\mathbf{v})$ (cf. [12]) and subspaces $U_\alpha^{\min}(\mathbf{v})$ of minimal dimension can be defined such that $\mathcal{H}_\mathfrak{r} = \{\mathbf{v} \in \mathbf{V} : rank_\alpha(\mathbf{v}) \le r_\alpha\}$ and $U_\alpha = U_\alpha^{\min}(\mathbf{v})$ have dimension $rank_\alpha(\mathbf{v})$. For best approximation problems it is important that $\mathcal{H}_\mathfrak{r}$ is weakly closed (cf. [4]).

For the numerical implementation the subspaces U_α are spanned by bases $\{b_\ell^{(\alpha)} : 1 \le \ell \le r_\alpha\}$. However, only for $\alpha \in \mathcal{L}(T_D)$ the bases are stored explicitly. Otherwise, we make use of (1) and store only the matrices $C^{(\alpha,\ell)} \in \mathbb{R}^{r_{\alpha_1} \times r_{\alpha_2}}$ whose entries satisfy

$$b_\ell^{(\alpha)} = \sum_{i=1}^{r_{\alpha_1}} \sum_{j=1}^{r_{\alpha_2}} C_{ij}^{(\alpha,\ell)} b_i^{(\alpha_1)} \otimes b_j^{(\alpha_2)} \qquad (\alpha_1, \alpha_2 \text{ sons of } \alpha).$$

Finally, $\mathbf{v} \in U_D$ is characterised by $\mathbf{v} = c_1^{(D)} b_1^{(D)}$ (note that $r_D = 1$ holds). The overall storage of $\mathbf{v} \in \mathcal{H}_\mathfrak{r}$ is $(d-1)\, r^3 + rdn$, where $r := \max_\alpha r_\alpha$ and $n := \max_j \dim(V_j)$.

Basic operations in $\mathcal{H}_\mathfrak{r}$ are basis transformation, transformation to orthonormal bases, and in particular the computation of the HOSVD (higher order singular value decomposition, cf. [2], [5]), which allows an easy truncation of tensors to smaller representation ranks with full control of the error (even other norms than Hilbert norms can be obtained, cf. [10]). The typical cost of tensor operations is $O(dr^4 + r^2 dn)$ (cf. [9, Chap. 13])

The TT format corresponds to the hierarchical format with the linear tree $T_D = \{\{j\}, \{1,\dots,j\} : 1 \le j \le d\}$, where the sons of $\{1,\dots,j\}$ are $\{1,\dots,j-1\}$ and $\{j\}$, and where the subspaces $U_j = V_j$ are chosen with maximal dimension (cf. [14, 15, 16], [9, Chap. 12]).

Since the order d of the tensor appears linearly, there is no curse of dimensionality. One may even map standard vectors $v \in \mathbb{R}^n$ ($n = 2^d$) into tensors $\mathbf{v}$ of the isomorphic tensor space $\mathbf{V} := \bigotimes_{j=1}^{d} \mathbb{R}^2$. The concrete isomorphism is $\mathbf{v}[i_1,\dots,i_d] = v[i]$ ($0 \le i \le n-1$) with the binary integer representation $i = i_1 + 2i_2 + \dots$ ($0 \le i_j \le 1$). This procedure is called *tensorisation* (cf. [9, Chap. 14]). It can be considered as a multi-scale approach, since the different directions $j = 1,\dots,d$ correspond to different scales of the grid function $v \in \mathbb{R}^n$ (think of $v_i = \varphi(ih)$, $h = 1/n$, for a function φ on $[0,1]$).

The advantage of the tensorisation comes from the fact that the tensor truncation can be applied to $\mathbf{v}$ and yields $\mathbf{v}_\varepsilon$ with $\|\mathbf{v} - \mathbf{v}_\varepsilon\| \le \varepsilon$. For sufficiently smooth functions, one obtains a strong data compression up to $O(\log n)$. Furthermore, all operations like scalar products, Hadamard products, convolutions (cf. [8]) can be applied to the tensorised vectors with a cost corresponding to the compressed data size.

Upper estimates for the obtainable data size can be obtained from the data size of approximations by exponential sums (cf. [1]), trigonometric polynomial, hp methods (cf. [6]), or wavelets. Differently from the previous approaches, the tensor truncation is a blackbox method, which needs no a priori knowledge about the position of singularities and no adaptive approach. Therefore, the implementational overhead is minimal.

For a survey concerning low-rank tensor approximations see [9] and [7].

References

[1] D. Braess, W. Hackbusch. On the efficient computation of high-dimensional integrals and the approximation by exponential sums. In: DeVore and Kunoth [3], pp. 39–74

[2] L. De Lathauwer, B. De Moor, J. Vandewalle. A multilinear singular value decomposition. SIAM J. Matrix Anal. Appl. **21**, 2000, 1253–1278

[3] R.A. DeVore, A. Kunoth, (eds.). Multiscale, nonlinear and adaptive approximation. Springer, Berlin (2009)

[4] A. Falcó, W. Hackbusch.On minimal subspaces in tensor representations. Found. Comput. Math. **12**, 2012, 765–803

[5] L. Grasedyck. Hierarchical singular value decomposition of tensors. SIAM J. Matrix Anal. Appl. **31**, 2010, 2029–2054

[6] L. Grasedyck. Polynomial approximation in hierarchical Tucker format by vector-tensorization. DFG-SPP 1324 Preprint 43, Philipps-Universität Marburg (2010)

[7] L. Grasedyck, D. Kressner, C. Tobler: A literature survey of low-rank tensor approximation techniques. GAMM-Mitteilungen **36**, 2013

[8] W. Hackbusch. Tensorisation of vectors and their efficient convolution. Numer. Math. **119**, 2011, 465–488

[9] W. Hackbusch. Tensor spaces and numerical tensor calculus. Springer, Berlin (2012)

[10] W. Hackbusch. L^∞ estimation of tensor truncations. Numer. Math. (2013). Published online

[11] W. Hackbusch, S. Kühn. A new scheme for the tensor representation. J. Fourier Anal. Appl. **15**, 2009, 706–722
[12] F.L. Hitchcock. The expression of a tensor or a polyadic as a sum of products. Journal of Mathematics and Physics **6**, 1927, 164–189
[13] B. Khoromskij. $O(d \log N)$-quantics approximation of $N - d$ tensors in high-dimensional numerical modeling. Constr. Approx. **34**, 2011, 257–280
[14] I.V. Oseledets. A new tensor decomposition. Doklady Math. **80**, 2009, 495–496
[15] I.V. Oseledets. Approximation of matrices using tensor decomposition. SIAM J. Matrix Anal. Appl. **31**, 2010, 2130–2145
[16] I.V. Oseledets. Tensor-train decomposition. SIAM J. Sci. Comput. **33**, 2011, 2295–2317

N-term approximation and Besov regularity for parametric elliptic PDEs

Markus Hansen

We are interested in the Besov regularity for non-parametric and parametric problems. More specifically, in a first step we shall investigate the regularity of solutions to the problem

$$-\nabla\big(A(x)\cdot\nabla u(x)\big) = f \text{ in } D\,, \qquad u|_{\partial D} = 0\,, \tag{1}$$

where $A = (a_{i,j})_{i,j=1}^{d}$ is symmetric, with further properties to be specified later on, and $D \subset \mathbb{R}^d$, $d \in \{2,3\}$ is a bounded Lipschitz domain either of polygonal ($d = 2$) or polyhedral ($d = 3$) structure (see [4] for more details on this type of domains). The study of the regularity of this problem will consist of two parts: First we consider the regularity in unweighted and weighted Sobolev spaces, and subsequently we investigate embeddings of the weighted Sobolev spaces into Besov spaces.

N-term approximation of elliptic PDEs. It is well-known that Besov spaces are closely related to approximation classes for several approximation procedures, in particular also for n-term wavelet approximation and adaptive Finite element approximation. Apart from the classical result by DeVore, Jawerth and Popov [3], which identifies the approximation class $\mathcal{A}_\tau^{\alpha/d}\big(L_p(D)\big)$ as the Besov space $B^\alpha_{\tau,\tau}(D)$, $\frac{1}{\tau} = \frac{\alpha}{d} + \frac{1}{p}$, $1 < p < \infty$, there is the following more recent result: If D is a bounded Lipschitz domain and $s_0 - s_1 > d(\frac{1}{p_0} - \frac{1}{p_1})$, $0 < p_0 \le p_1 \le \infty$, then it holds

$$\sigma_N(f)_{L_{p_1}(D)} \lesssim N^{-s_0/d} \|f|B^{s_0}_{p_0,\infty}(D)\|\,.$$

This result can be found in [2]. In the sequel we hence focus on the Besov spaces $B^s_{\tau,\infty}(D)$, for suitable parameters s and τ.

Now lets turn back to the weighted Sobolev spaces and the regularity of elliptic problems. It is known that on non-smooth domains the solution for (1) even for smooth data will possess only limited Sobolev regularity. For instance, a famous result by Jerison and Kenig shows that on general Lipschitz domains the solution u to Poisson's problem (with homogeneous Dirichlet boundary condition) belongs only to $H^{3/2}(D)$, even for smooth right-hand side f. To circumvent this deficiency,

one approach for polygonal and polyhedral domains consists in studying regularity in the scale of weighted Babuška-Kondratiev spaces $\mathcal{K}_a^m$, defined via

$$\mathcal{K}_a^m(D) = \left\{ u \middle| \|u|\mathcal{K}_a^m(D)\|^2 = \int_D \sum_{|\alpha|\le m} \left|\rho(x)^{|\alpha|-a} D^\alpha u(x)\right|^2 dx < \infty \right\},$$

where $m \in \mathbb{N}_0$, $a \in \mathbb{R}$, and ρ is the regularized distance to the singular set $S \subset \partial D$ (see [6] for a construction of a smooth distance function). This singular set S consists exactly of those points on the boundary, where singularities may occur: For polygons S consists of its vertices, for polyhedra S generally consists of its vertices and edges. Within this scale of function spaces, a regularity result can be formulated as follows:

Proposition 1. *Let $D \subset \mathbb{R}^d$ be a bounded domain with polyhedral structure. Consider problem (1) with*

$$a_{i,j} \in \mathcal{W}_\infty^m = \left\{ v : D \longrightarrow \mathbb{R} \middle| \ \rho^{|\alpha|} D^\alpha v \in L_\infty(D), |\alpha| \le m \right\}.$$

for all $1 \le i,j \le d$. Assume that the associated bilinear form is bounded and coercive on $H^1(D)$. Then there exists some $\overline{a} > 0$ such that for any $m \in \mathbb{N}_0$, any $|a| < \overline{a}$ and any $f \in \mathcal{K}_{a-1}^{m-1}(D)$ the problem (1) admits a uniquely determined solution $u \in \mathcal{K}_{a+1}^{m+1}(D)$, and it holds

$$\|u|\mathcal{K}_{a+1}^{m+1}(D)\| \le C \, \|f|\mathcal{K}_{a-1}^{m-1}(D)\|$$

for some constant C independent of f.

Our first main result then can be summarized as follows. Therein we denote by δ the dimension of the singular set (for polygons, singularities possibly occur only in its vertices, for polyhedra in its vertices and edges).

Theorem 1. *Let $D \subset \mathbb{R}^d$ be some bounded polyhedral domain. Further suppose $\min(s,a) > \frac{\delta}{d} m$. Then there exists some $0 < \tau_0 \le 2$ such that we have an embedding*

$$\mathcal{K}_a^m(D) \cap H^s(D) \hookrightarrow B_{\tau,\infty}^m(D) \hookrightarrow\hookrightarrow L_2(D)$$

for all $\tau_* < \tau < \tau_0$, where $\frac{1}{\tau_*} = \frac{s}{d} + \frac{1}{2}$. Consequently,

$$\sigma_N(u)_{L_2(D)} \lesssim N^{-m/d} \max\left(\|u|\mathcal{K}_a^m(D)\|, \|u|H^s(D)\| \right), \quad u \in \mathcal{K}_a^m(D) \cap H^s(D).$$

In particular, under the assumptions of Proposition 1 the solution $u \in \mathcal{K}_1^{m+1}(D)$ of problem (1) for $f \in H^{m-1}(D) \hookrightarrow \mathcal{K}_{-1}^{m-1}(D)$ can be approximated at the rate

$$\sigma_N(u)_{H^1(D)} \lesssim N^{-m/d} \|f|H^{m-1}(D)\|.$$

The argument is based on corresponding results in [1] and mainly uses wavelet characterizations of Besov spaces. Noteworthy is a wavelet based splitting of $u \in \mathcal{K}_a^m(D) \cap H^s(D)$ into a regular part (interior wavelets), governed by the weighted regularity, and a singular part (wavelets whose support intersects the singular set), where the unweighted regularity comes into play.

Parametric elliptic PDEs. In a second part, we now thansfer the above results for a single elliptic PDE to a parametric PDE. More specifically, we now study in the problem

$$\begin{aligned} -\nabla\big(A(y)\nabla u(y)\big) &= f(y) \quad \text{in} \quad D\,, \\ u(y) &= 0 \qquad \text{on} \quad \partial_d D\,, \\ \nabla^A_\nu u(y) &= g(y) \quad \text{on} \quad \partial_n D\,, \end{aligned} \tag{2}$$

where $A = (a_{i,j})_{i,j=1}^d$ and $a_{i,j} : U \longrightarrow L^\infty(D)$ are given fixed mappings with parameter domain $U = [-1,1]^{\mathbb{N}}$, i.e. the countable cartesian product of intervals $[-1,1]$, either interpreted as the unit ball of $\ell_\infty(\mathbb{N})$ or as the compact subset of the Frechet space $\mathbb{R}^{\mathbb{N}}$ (i.e. equipped with the product topology).

For every fixed parameter, we are back in the setting of the first part. Here we are interested in regularity results for the parametric mapping $y \mapsto u(y)$ form the parametric domain into $B^m_{\tau,\infty}(D)$. It is known (see [5]) that Gateaux-differentiability of order k of the input data f, g and $a_{i,j}$ transfers to differentiability of u of the same order, where we consider u with values in $\mathcal{K}^{m+1}_{a+1}(D)$. Moreover, this remains true for $k = \infty$ and even for analytic functions.

Theorem 2. *Let the assumptions of Theorem 1 be fulfilled for every parameter $y \in U$. Then, under corresponding assumptions on the differentiablity (or analyticity, respectively) of f, g and $a_{i,j}$, the mapping $U \ni y \mapsto u(y) \in B^m_{\tau,\infty}(D)$ is Gateaux-differentiable of order k (or analytic, respectively).*

Finally, we turn to generalized polynomial chaos (gpc) expansions of the parametric mapping $u(y)$, e.g. into a series

$$u(y) = \sum_{\nu\in\mathcal{F}} w_\nu T_\nu(y)\,, \qquad \mathcal{F} = \{\nu \in \mathbb{N}_0^{\mathbb{N}} : \operatorname{supp}\nu < \infty\}\,,$$

of tensorized Chebyshev polynomials. Therein the coefficients w_ν itself again belong to $B^m_{\tau,\infty}(D)$. To obtain approximation results for the fully discretized problem, i.e. for a wavelet system $(\psi_i)_{i\in\mathcal{I}}$ on D we consider N-term approximation w.r.to the system $(\psi_I)_{i\in\mathcal{I}} \otimes (T_\nu)_{\nu\in\mathcal{F}}$, we have to equilibrate the approximation of the expansion by appropriate partial sums, and the approximation of the exact Chebyshev coefficients. For approximation results by partial sums we need to assume some specific structure for $A(y)$: For $A(y) = a(y)I_d$, I_d being the identity matrix, and $a(y) = a_0 + \sum_{j\geq 1} y_j\phi_j$ we obtain the following result:

Theorem 3. *Denote by $(\psi_i)_{i\in\mathcal{I}}$ some wavelet basis for $B^m_{\tau,\infty}(D)$. Assume*

$$\sum_{j\geq 1} \|\phi_j\|^p_{\mathcal{W}^m_\infty(D)} < \infty$$

for some $0 < p < 1$. Then the N-term approximation error with respect to the dictionary $(\phi_i)_{i\in\mathcal{I}} \otimes (T_\nu)_{\nu\in\mathcal{F}}$ can be estimated by

$$\sigma_N(u)_{H^1_0(D)} \leq c\, N^{-\min(m/d,r)}\,, \quad r = \frac{1}{p} - 1\,.$$

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References

[1] S. Dahlke and R. A. DeVore. Besov regularity for elliptic boundary value problems. Comm. Partial Differential Equations, 22(1-2), 1997, 1–16.
[2] S. Dahlke, E. Novak, W. Sickel. Optimal approximation of elliptic problems by linear and nonlinear mappings II. J. Complexity, 22, 2006, 549–603.
[3] R. A. DeVore, B. Jawerth, V. Popov. Compression of wavelet decompositions. Amer. J. Math., 114, 1992, 737–785.
[4] A. Mazzucato and V. Nistor. Well posedness and regularity for the elasticity equation with mixed boundary conditions on polyhedral domains and domains with cracks. Arch. Ration. Mech. Anal., 195, 2010, 25–73.
[5] V. Nistor and Ch. Schwab. High order Galerkin approximations for parametric second order elliptic partial differential equations. Mathematical Models and Methods in Applied Sciences, 2, 2013, World Scientific Publishing Company, DOI: 10.1142/S0218202513500218.
[6] E. M. Stein. Singular integrals and differentiability properties of functions. Princeton Mathematical series 30, Princeton University Press, Princeton, New Jersey, 1970.

Multilevel quadrature for elliptic stochastic partial differential equations

Helmut Harbrecht
(joint work with Michael Peters and Markus Siebenmorgen)

1. Introduction

This talk is concerned with elliptic second order boundary value problems with random diffusion. In parametrized form, such problems are of the form

$$\begin{aligned} &\text{find } u \in L^2_\rho\big(\square; H^1_0(D)\big) \text{ such that} \\ &\quad -\operatorname{div}\big(\alpha(\mathbf{y})\nabla u(\mathbf{y})\big) = f \text{ in } D \text{ for all } \mathbf{y} \in \square, \end{aligned} \tag{1}$$

where $D \subset \mathbb{R}^d$ is the physical domain, $\rho : \square \to \mathbb{R}_{\geq 0}$ is the joint density function, and $\square = (-1,1)^m$ (in the uniformly elliptic case) or $\square = \mathbb{R}^m$ (in the log-normal case) is the parameter domain of the stochastic variable. The quantities of interest are the solution's expectation

$$\mathbb{E}_u(\mathbf{x}) = \int_\square u(\mathbf{x},\mathbf{y})\rho(\mathbf{y})\,\mathrm{d}\mathbf{y}, \tag{2}$$

its variance, or even higher order moments.

A principal approach to compute (2) is the Monte Carlo method. However, it is extremely expensive to generate a large number of suitable samples and to solve the deterministic boundary value problem (1) on each sample. To overcome this obstruction, the multilevel Monte Carlo method (MLMC) has been developed in [1]. From the stochastic point of view, it is a variance reduction technique which considerably decreases the complexity. The idea is to combine the Monte Carlo

quadrature of the stochastic variable with a multilevel splitting of the Bochner space which contains the random solution. Then, to compute (2), most samples can be performed on coarse spatial discretizations while only a few samples must be performed on fine spatial discretizations. This proceeding is a sparse grid approximation of the expectation. If we replace the Monte Carlo quadrature by another quadrature rule for high-dimensional integrals, we obtain for example the multilevel quasi Monte Carlo method (MLQMC) or the multilevel Gaussian quadrature method (MLGQ).

2. Quadrature in the stochastic variable

To compute the integral (2), we have to provide a sequence of quadrature formulae $\{Q_\ell\}$ for the Bochner integral

$$I : L^2_\rho(\square; X) \to X, \quad Iv = \int_\square v(\cdot, \mathbf{y})\rho(\mathbf{y})\,\mathrm{d}\mathbf{y}$$

where $X \subset L^2(D)$ denotes a Banach space. The quadrature formula

$$Q_\ell : L^2_\rho(\square; X) \to X, \quad Q_\ell v = \sum_{i=1}^{N_\ell} \omega_{\ell,i} v(\cdot, \boldsymbol{\xi}_{\ell,i})\rho(\boldsymbol{\xi}_{\ell,i}) \tag{3}$$

is supposed to fulfill the error bound

$$\|(I - Q_\ell)v\|_X \lesssim 2^{-\ell}\|v\|_{\mathcal{H}(\square;X)}$$

uniformly in $\ell \in \mathbb{N}$, where $\mathcal{H}(\square; X) \subset L^2_\rho(\square, X)$ is a suitable Bochner space.

3. Finite element approximation in the spatial variable

In order to apply the quadrature formula (3), we shall calculate the solution $u(\mathbf{y}) \in H^1_0(D)$ of the diffusion problem (1) in certain points $\mathbf{y} \in \square$. To this end, consider a coarse grid triangulation/tetrahedralization $\mathcal{T}_0 = \{\tau_{0,k}\}$ of the domain D. Then, for $\ell \geq 1$, a uniform and shape regular triangulation/tetrahedralization $\mathcal{T}_\ell = \{\tau_{\ell,k}\}$ is recursively obtained by uniformly refining each triangle/tetrahedron $\tau_{\ell-1,k}$ into 2^n triangles/tetrahedrons with diameter $h_\ell \sim 2^{-\ell}$. Then, define the finite element spaces

$$\mathcal{S}_\ell(D) := \{v \in C(D) : v|_{\partial D} = 0 \text{ and } v|_\tau \text{ is linear for all } \tau \in \mathcal{T}_\ell\} \subset H^1_0(D)$$

and let

$$G_\ell(\mathbf{y}) : H^1_0(D) \to \mathcal{S}_\ell(D), \quad v \mapsto v_\ell$$

denote the Galerkin projection related with (1), given by Galerkin orthogonality

$$\int_D \alpha(\mathbf{x}, \mathbf{y})\nabla\big(v(\mathbf{x}) - v_\ell(\mathbf{x})\big)\nabla w_\ell(\mathbf{x})\,\mathrm{d}\mathbf{x} = 0 \text{ for all } w_j \in \mathcal{S}_\ell(D).$$

Then, the approximate solution $G_\ell(\mathbf{y})u(\mathbf{y}) \in \mathcal{S}_\ell(D)$ to (1) of a finite element method in the space $\mathcal{S}_\ell(D)$ satisfies the error estimate

$$\|u(\mathbf{y}) - G_\ell(\mathbf{y})u_\ell(\mathbf{y})\|_{H^1(D)} \lesssim 2^{-\ell}\sqrt{\frac{\alpha_{\max}(\mathbf{y})}{\alpha_{\min}(\mathbf{y})}}\|u(\mathbf{y})\|_{H^2(D)}$$

provided that the domain D is convex and $f \in L^2(D)$.

4. Multilevel quadrature

We now have to combine the quadrature method with the multilevel finite element discretization. To this end, we define the ansatz spaces

$$V_j^{(1)} := \Big\{ G_j(\mathbf{y}) v(\mathbf{x},\mathbf{y}) : v \in C\big(\square; H_0^1(D)\big) \text{ and } \mathbf{y} \in \square \Big\} \subset L^2_\rho\big(\square; \mathcal{S}_j(D)\big).$$

To compute the expectation (2), we shall apply the quadrature rule Q_j to the finite element solution in $\mathcal{S}_j(D)$ which yields

$$\mathbb{E}_u(\mathbf{x}) \approx Q_j\big(G_j(\mathbf{y}) u(\mathbf{x},\mathbf{y})\big) = \sum_{i=0}^{N_j} \omega_{j,i} G_j(\boldsymbol{\xi}_{j,i}) u(\mathbf{x},\boldsymbol{\xi}_{j,i}) \rho(\boldsymbol{\xi}_{j,i}). \tag{4}$$

This can be interpreted as the *full* tensor product approximation of the function $\mathbb{E}_u$ in the product space $V_j^{(1)} \otimes V_j^{(2)}$ where the quadrature rule Q_j serves as "space" $V_j^{(2)}$. It produces the error estimate

$$\Big\| \mathbb{E}_u(\mathbf{x}) - Q_j\big(G_j(\mathbf{y}) u(\mathbf{x},\mathbf{y})\big) \Big\|_{H^1(D)} \lesssim 2^{-j} j \|u\|_{\mathcal{H}(\square;H_1(D)) \cap L^2_\rho(\square;H_2(D))}.$$

In contrast to this, setting $G_{-1}(\mathbf{y}) := 0$ for all $\mathbf{y} \in \square$, the *sparse* tensor product of the spaces $V_j^{(1)}$ and $V_j^{(2)}$ is built with the help of the complement spaces

$$W_\ell^{(1)} := \Big\{ \big(G_\ell(\mathbf{y}) - G_{\ell-1}(\mathbf{y})\big) v(\mathbf{x},\mathbf{y}) : v \in C\big(\square; H_0^1(D)\big) \text{ and } \mathbf{y} \in \square \Big\} \subset V_\ell^{(1)}$$

in accordance with

$$\widehat{V_j^{(1)} \otimes V_j^{(2)}} = \bigoplus_{\ell=0}^{j} W_\ell^{(1)} \otimes \Big(\bigoplus_{\ell'=0}^{j-\ell} W_{\ell'}^{(2)} \Big) = \bigoplus_{\ell=0}^{j} W_\ell^{(1)} \otimes V_{j-\ell}^{(2)}.$$

This means, we consider the sparse tensor product approximation

$$\begin{aligned} \mathbb{E}_u(\mathbf{x}) &\approx \sum_{\ell=0}^{j} Q_{j-\ell}\big(G_\ell(\mathbf{y}) u(\mathbf{x},\mathbf{y}) - G_{\ell-1}(\mathbf{y}) u(\mathbf{x},\mathbf{y})\big) \\ &= \sum_{\ell=0}^{j} \sum_{i=0}^{N_{j-\ell}} \omega_{j-\ell,i} \big(G_\ell(\boldsymbol{\xi}_{j-\ell,i}) u(\mathbf{x},\boldsymbol{\xi}_{j-\ell,i}) - G_{\ell-1}(\boldsymbol{\xi}_{j-\ell,i}) u(\mathbf{x},\boldsymbol{\xi}_{j-\ell,i})\big) \rho(\boldsymbol{\xi}_{j-\ell,i}). \end{aligned}$$

Loosely speaking, the function $u \in L^2_\rho\big(\square; H_0^1(D)\big)$ is divided into j slices which are related to the modulus of its entity. Then, for every slice, the precision of the quadrature is properly chosen. We refer to Figure 1 for a graphical illustration.

Under the assumption that the random solution provides mixed regularity in terms of $u \in \mathcal{H}\big(\square; H_2(D)\big)$, the multilevel quadrature produces essentially the same accuracy as the standard tensor product quadrature (4):

$$\Big\| \mathbb{E}_u(\mathbf{x}) - \sum_{\ell=0}^{j} Q_{j-\ell}\big(G_\ell(\mathbf{y}) u(\mathbf{x},\mathbf{y}) - G_{\ell-1}(\mathbf{y}) u(\mathbf{x},\mathbf{y})\big) \Big\|_{H^1(D)} \lesssim 2^{-j} j \|u\|_{\mathcal{H}(\square;H_2(D))}.$$

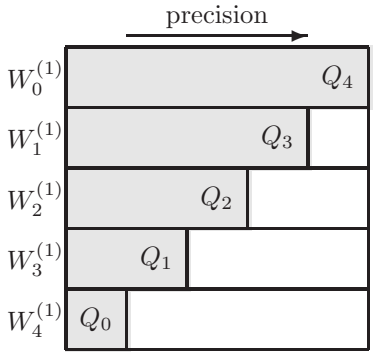


FIGURE 1. Visualization of the multilevel quadrature.

Notice that the multilevel quadrature idea can be generalized also to higher order moments or other output functionals, see [2, 3] for the details.

REFERENCES

[1] A. Barth, C. Schwab, and N. Zollinger. Multi-level Monte Carlo finite element method for elliptic PDEs with stochastic coefficients. Numer. Math., 119, 2011, 123–161.

[2] H. Harbrecht, M. Peters, and M. Siebenmorgen. On multilevel quadrature for elliptic stochastic partial differential equations. In J. Garcke and M. Griebel, editors, Sparse grids and applications, volume 88 of Lecture Notes in Computational Science and Engineering, pages 161–179, Berlin-Heidelberg, 2013. Springer.

[3] H. Harbrecht, M. Peters, and M. Siebenmorgen. Multilevel accelerated quadrature for PDEs with log-normal distributed random coefficient. Preprint 2013-18, Mathematisches Institut, Universität Basel, Switzerland, 2013.

Analysis on convex subset of a Riemannian manifold and classical polynomial approximation

GERARD KERKYACHARIAN

(joint work with P. Petrushev, Y. Xu)

It is a classical topic to look to orthonormal basis of polynomials on a compact set X of R^d, with respect to some Radon measure μ. For exemple : the one dimensional interval (Jacobi), the unit sphere (Spherical harmonics), the ball and the simplex (work of Petrushev, Xu, ...) In this framework, one can be interested in the best approximation of functions by polynomials of fixed degree, in $L_p(\mu)$, and to built a suitable frame for characterization of function spaces related to this approximation. This constructions have been carried using special functions estimates.

We will be interested by spaces where the polynomials give the spectral spaces of some positive selfadjoint operator. Under suitable conditions, a "natural" metric ρ could be defined on X such that (X, ρ, μ) is a homogeneous space, and if the

associated semi-group has a good "Gaussian" behavior, then we could apply the procedure developed in recent works by P. Petrushev, T. Coulhon and G.K., to built such frames, and such function spaces.

Actually we will show that analysis on "convex" open set in a Riemannian manifold can help to understand these classical topics.

References

[1] Th. Coulhon, G. Kerkyacharian, P. Petrushev, Heat kernel generated frames in the setting of Dirichlet spaces. Journal of Fourier Analysis and Applications 18 : 995-1066, 2012.
[2] G. Kerkyacharian, P. Petrushev, Heat kernel based decomposition of spaces of distributions in the framework of Dirichlet spaces, to appear in: Transaction of American Math. Soc.
[3] G. Kerkyacharian, P. Petrushev, Y. Xu, Gaussian bounds for the heat kernel on convex compact sets. Preprint.

Isotropic Gaussian random fields on the sphere

Annika Lang
(joint work with Christoph Schwab)

Sample regularity and fast simulation of isotropic Gaussian random fields on the sphere are for example of interest for the numerical analysis of stochastic partial differential equations and for the simulation of ice crystals or Saharan dust particles as lognormal random fields. In what follows we recall the results from [2], which include the approximation of isotropic Gaussian random fields with convergence rates as well as the regularity of the samples in relation to the smoothness of the covariance expressed in terms of the decay of the angular power spectrum. As example we construct isotropic Q-Wiener processes out of isotropic Gaussian random fields and discretize the stochastic heat equation with spectral methods.

Before we state the results, we start with a short review of the basics. Therefore, let $(\Omega, \mathcal{A}, (\mathcal{F}_t), P)$ be a filtered probability space and denote by $\mathbb{S}^2 \subset \mathbb{R}^3$ the unit sphere. A $\mathcal{A} \otimes \mathcal{B}(\mathbb{S}^2)$-measurable mapping $T : \Omega \times \mathbb{S}^2 \to \mathbb{R}$ is called an *isotropic Gaussian random field* if, for all $k \in \mathbb{N}$, $x_1, \dots, x_k \in \mathbb{S}^2$, $a_1, \dots, a_k \in \mathbb{R}$, the real-valued random variable $\sum_{i=1}^k a_i T(x_i)$ is Gaussian and the distribution of $(T(x_1), \dots, T(x_k))$ is invariant under rotations. By [3], the isotropic Gaussian random field T admits a Karhunen–Loève expansion $T = \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} a_{\ell m} Y_{\ell m}$, where $(Y_{\ell m}, \ell \in \mathbb{N}_0, m = -\ell, \dots, \ell)$ denotes the sequence of spherical harmonic functions and $(a_{\ell m}, \ell \in \mathbb{N}_0, m = -\ell, \dots, \ell)$ is a sequence of normally distributed random variables, whose properties are characterized by the angular power spectrum $(A_\ell, \ell \in \mathbb{N}_0)$. For $\ell \in \mathbb{N}$, $m = 1, \dots, \ell$, and $\vartheta \in [0, \pi]$, let

$$L_{\ell m}(\vartheta) := \sqrt{\frac{2\ell + 1}{4\pi} \frac{(\ell - m)!}{(\ell + m)!}} P_{\ell m}(\cos \vartheta)$$

be a weighted version of the associate Legendre polynomials ($P_{\ell m}, \ell \in \mathbb{N}_0, m = 0, \dots, \ell$). Then the random field generated by

$$\sum_{\ell=0}^{\infty}\Big(\sqrt{A_\ell}X^1_{\ell 0}L_{\ell 0}(\vartheta) + \sqrt{2A_\ell}\sum_{m=1}^{\ell} L_{\ell m}(\vartheta)(X^1_{\ell m}\cos(m\varphi) + X^2_{\ell m}\sin(m\varphi))\Big) + \mathbb{E}(T)$$

is equal in law to T, where $((X^1_{\ell m}, X^2_{\ell m}), \ell \in \mathbb{N}_0, m = 0, \dots, \ell)$ is a sequence of independent standard normally distributed random variables with $X^2_{\ell 0} = 0$ for all $\ell \in \mathbb{N}_0$. A truncation of the series expansion leads to the following convergence results which rely on the decay of the angular power spectrum.

Theorem 1. *Assume that $A_\ell \le C \cdot \ell^{-\alpha}$ for some $\alpha > 2$ and $C > 0$. Then for all $0 < p < +\infty$ there exists $\hat{C}_p > 0$ such that*

$$\|T - T^\kappa\|_{L^p(\Omega; L^2(\mathbb{S}^2))} \le \hat{C}_p \cdot \kappa^{-(\alpha-2)/2},$$

where the truncated series expansion T^κ is given by

$$\sum_{\ell=0}^{\kappa}\Big(\sqrt{A_\ell}X^1_{\ell 0}L_{\ell 0}(\vartheta) + \sqrt{2A_\ell}\sum_{m=1}^{\ell} L_{\ell m}(\vartheta)(X^1_{\ell m}\cos(m\varphi) + X^2_{\ell m}\sin(m\varphi))\Big) + \mathbb{E}(T).$$

Furthermore, for all $\beta < (\alpha - 2)/2$ it holds asymptotically $\|T - T^\kappa\|_{L^2(\mathbb{S}^2)} \le \kappa^{-\beta}$, P-a. s..

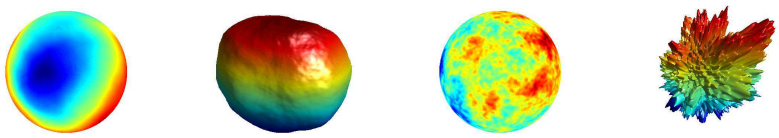
The decay of the angular power spectrum is linked to the regularity of the covariance kernel in the following proposition and can be extended to non-integers with fractional weighted Sobolev spaces.

Proposition 2. *For every $n \in \mathbb{N}_0$, it holds that the sequence $(\ell^{n+1/2}A_\ell, \ell \ge n)$ is in $\ell^2(\mathbb{N}_0)$ if and only if the covariance kernel $(1-\mu^2)^{n/2}\frac{\partial^n}{\partial \mu^n}\sum_{\ell=0}^{\infty} A_\ell \frac{2\ell+1}{4\pi}P_\ell(\mu)$ is in $L^2(-1,1)$, where $(P_\ell, \ell \in \mathbb{N}_0)$ denotes the sequence of Legendre polynomials.*

Furthermore, the decay of the angular power spectrum determines the sample regularity of the random field.

Theorem 3. *Assume that $\sum_{\ell=0}^{\infty} A_\ell \ell^{1+\beta} < +\infty$ for some $\beta > 0$. Then there exists a continuous modification of T which is Hölder continuous with exponent γ for all $\gamma < \min\{\beta/2, 1\}$. Furthermore, the modification is k-times continuously differentiable for all $k < \beta/2 - 1$. The corresponding* lognormal random field $\exp(T)$ *has the same regularity properties.*

The Hölder continuity in the previous theorem is proven using the following lemma and a version of the Kolmogorov–Chentsov theorem, which we state for completeness, while the differentiability is a direct consequence of Sobolev embeddings. The same regularity of the lognormal random field results from the properties of the exponential function. Samples of Gaussian and the corresponding lognormal random fields are shown in Figure 1.



(a) $\alpha = 5$ (b) $\alpha = 5$ (c) $\alpha = 3$ (d) $\alpha = 3$

FIGURE 1. Samples of isotropic Gaussian and the corresponding lognormal random fields as radius of the deformed sphere with $A_\ell = (\ell+1)^{-\alpha}$.

Lemma 4. *Assume that* $\sum_{\ell=0}^{\infty} A_\ell \ell^{1+\beta} < +\infty$ *for some* $\beta \in [0,2]$. *Then the corresponding kernel function* $k(r) = \sum_{\ell=0}^{\infty} A_\ell \frac{2\ell+1}{4\pi} P_\ell(\cos r)$ *satisfies that*

$$|k(0) - k(r)| \leq C_\beta r^\beta$$

for some $C_\beta > 0$, *which implies that for all* $0 < p < +\infty$ *there exists* $C_{\beta,p} > 0$ *such that*

$$\mathbb{E}(|T(x) - T(y)|^{2p}) \leq C_{\beta,p}\, d(x,y)^{\beta p}.$$

The second step in the proof is the Kolmogorov–Chentsov theorem for random fields on $\mathbb{S}^2$ which is proven by applying a version of the theorem for domains on six charts and patching the resulting random fields together with a partition of unity. This is extended to general manifolds and from Hölder continuity to Hölder differentiability in [1].

Theorem 5 (Kolmogorov–Chentsov theorem)**.** *Let* T *be a random field on* $\mathbb{S}^2$ *that satisfies*

$$\mathbb{E}(|T(x) - T(y)|^p) \leq C d(x,y)^{2+\epsilon p}$$

for some $p > 0$, $C > 0$, *and some* $\epsilon \in (0,1]$. *Then there exists a continuous modification of* T *that is locally Hölder continuous with exponent* γ *for all* $\gamma \in (0,\epsilon)$.

Besides the already mentioned application to lognormal random fields, isotropic Gaussian random fields can also be used to define a Q-Wiener process W taking values in $L^2(\mathbb{S}^2)$ by the Karhunen–Loève expansion

$$\sum_{\ell=0}^{\infty} \sqrt{A_\ell}\beta^1_{\ell 0}(t) L_{\ell 0}(\vartheta) + \sqrt{2A_\ell} \sum_{m=1}^{\ell} L_{\ell m}(\vartheta)(\beta^1_{\ell m}(t)\cos(m\varphi) + \beta^2_{\ell m}(t)\sin(m\varphi)),$$

where $((\beta^1_{\ell m}, \beta^2_{\ell m}), \ell \in \mathbb{N}_0, m = 0, \ldots, \ell)$ is a sequence of independent Brownian motions and $\beta^2_{\ell 0} = 0$ for $\ell \in \mathbb{N}_0$. Here, the covariance operator Q is characterized by $QY_{\ell m} = A_\ell Y_{\ell m}$. Let us observe that the Laplace–Beltrami operator $\Delta_{\mathbb{S}^2}$ on $\mathbb{S}^2$ satisfies that $\Delta_{\mathbb{S}^2} Y_{\ell m} = -\ell(\ell+1)Y_{\ell m}$. We want to simulate the stochastic heat equation on $\mathbb{S}^2$ driven by additive Q-Wiener noise on some finite time interval

$$dX(t) = \Delta_{\mathbb{S}^2} X(t)\, dt + dW(t)$$

with initial condition $X(0) = X_0 \in L^2(\Omega; L^2(\mathbb{S}^2))$, i. e., in mild form

$$X(t) = X_0 + \int_0^t \Delta_{\mathbb{S}^2} X(s)\, ds + \int_0^t dW(s) = X_0 + \int_0^t \Delta_{\mathbb{S}^2} X(s)\, ds + W(t).$$

This equation can be expanded with respect to the spherical harmonic functions and leads to the stochastic differential equations

$$(X(t), Y_{\ell m})_{L^2(\mathbb{S}^2)} = (X_0, Y_{\ell m})_{L^2(\mathbb{S}^2)} - \ell(\ell+1) \int_0^t (X(s), Y_{\ell m})_{L^2(\mathbb{S}^2)}\, ds + a_{\ell m}(t)$$

with scaled Brownian motions $a_{\ell m}$, which can be solved with the variations of constants formula. We are able to simulate the solution with the observation that the stochastic convolutions $\int_0^t e^{-\ell(\ell+1)(t-s)}\, d\beta_{\ell m}(s)$ are by the Itô formula normally distributed with mean zero and variance $(2\ell(\ell+1))^{-1}(1 - e^{-2\ell(\ell+1)t})$. In what follows, we obtain convergence results under weaker assumptions on $(A_\ell, \ell \in \mathbb{N}_0)$ than in Theorem 1 due to the smoothing of the heat kernel.

Theorem 6. *Assume that $A_\ell \leq C \cdot \ell^{-\alpha}$ for some $\alpha > 0$ and $C > 0$. Then, for all $0 < p < +\infty$, $t < +\infty$, and $\kappa \in \mathbb{N}$,*

$$\|X(t) - X^\kappa(t)\|_{L^p(\Omega; L^2(\mathbb{S}^2))} \leq \hat{C}_p \cdot \kappa^{-\alpha/2}$$

with $\hat{C}_p > 0$ independent of the time discretization, where X^κ denotes the truncated Karhunen–Loève expansion of the solution of the stochastic heat equation. Furthermore, for all $\beta < \alpha/2$ it holds asymptotically $\|X(t) - X^\kappa(t)\|_{L^2(\mathbb{S}^2)} \leq \kappa^{-\beta}$, P-a. s..

References

[1] Roman Andreev and Annika Lang. Kolmogorov–Chentsov theorem and differentiability of random fields on manifolds. arXiv:1307.4886 [math.PR], SAM-Report 2013-22, July 2013.

[2] Annika Lang and Christoph Schwab. Isotropic Gaussian random fields on the sphere: regularity, fast simulation, and stochastic partial differential equations. arXiv:1305.1170 [math.PR], SAM-Report 2013-15, May 2013.

[3] Domenico Marinucci and Giovanni Peccati. *Random Fields on the Sphere. Representation, Limit Theorems and Cosmological Applications*. Cambridge University Press, 2011.

Adaptive, hierarchical cones of convex functions

Jean-Marie Mirebeau

A number of mathematical problems are formulated, or can be formulated, as the minimization of a convex functional over the *cone of convex functions* over a convex domain Ω of $\mathbb{R}^d$. Among them the best known is Optimal Transport, with the standard quadratic transportation cost. Let us also mention Newton's problem of the convex body of least resistance [5] moving through a gas of particles. The monopolist problem, arising from economics and introduced in 1978 [7], is one of our main motivations. Despite a significant research effort [8, 2, 3, 1], numerical

studies of this problem, in dimension $d \geq 2$, remain expensive and limited to small resolutions.

While existence results for the solutions of these problems follow from standard convex analysis, obtaining numerical approximations raises unexpected issues, due to the difficulty of discretizing the cone of convex functions. For simplicity, we assume that the domain is bi-dimensional, $\Omega \subset \mathbb{R}^2$, and sampled on a grid $X := \Omega \cap \mathbb{Z}^2$ containing N points. Smooth convex functions $u : \Omega \to \mathbb{R}$ are *locally* characterized by the inequalities, in the sense of symmetric matrices

$$\forall x \in \Omega,\ d^2u(x) \succcurlyeq 0.$$

A discretization of the cone of convex functions, based on this characterization and using $\mathcal{O}(N)$ semi-definite (non-linear) constraints, was presented in [1]. However, a discrete map on X obeying these constraints *cannot* in general be extended into a convex continuous map on Ω, which is an issue for some applications. On the other hand, convex functions $u : \Omega \to \mathbb{R}$ are also characterized by the linear, *non-local* inequalities

$$\forall x, y \in \Omega, \forall\, 0 \leq \lambda \leq 1,\ u(\lambda x + (1-\lambda)y) \leq \lambda u(x) + (1-\lambda)u(y). \tag{1}$$

The cone of restrictions to X of convex maps

$$\mathrm{Conv}(X) := \{u_{|X};\ u : \Omega \to \mathbb{R}^d,\ \text{convex}\}, \tag{2}$$

has a minimal characterization [2] in terms of $\mathcal{O}(N^2)$ *non-local* linear inequalities of form similar to (1). Let us also mention a third discretization introduced in [3], also based on $\mathcal{O}(N^2)$ non-local, linear inequalities.

Our objective is to combine the strongpoints of these different methods - number of constraints growing (quasi-)linearly with $N := \#(X)$, simple linear constraints, discrete maps that are restrictions of convex maps - by the use of adaptive, multiscale constructions. For that purpose, we have introduced a hierarchical family of sub-cones $\mathrm{Conv}(\mathcal{V}) \subset \mathrm{Conv}(X)$, each defined by some linear inequalities associated to a *stencil*: the data $\mathcal{V} = (\mathcal{V}(x))_{x\in X}$ of a selected collection of neighbors $\mathcal{V}(x) \subset X$, for each $x \in X$ (see Figure 1, center). The number of inequalities defining $\mathrm{Conv}(\mathcal{V})$ is bounded by $\#(\mathcal{V}) := \sum_{x\in X} \#(\mathcal{V}(x))$, up to a fixed multiplicative constant. For each convex map $u : \Omega \to \mathbb{R}$, there exists a stencil $\mathcal{V}$ depending on u and X, such that $u_{|X} \in \mathrm{Conv}(\mathcal{V})$ and $\#(\mathcal{V}) \lesssim N \ln^3 N$ (to be precise, this estimate holds in average over random grid orientations). We use adaptive stencil refinement strategies, allowed by the hierarchy property $\mathrm{Conv}(\mathcal{V} \cap \mathcal{V}') = \mathrm{Conv}(\mathcal{V}) \cap \mathrm{Conv}(\mathcal{V}')$, to iteratively construct in applications a problem dependent stencil $\mathcal{V}$. (Ideally, this would be the stencil $\mathcal{V}$ of minimal cardinality such that the cone $\mathrm{Conv}(\mathcal{V})$ contains the restriction to X of the continuous problem solution.) The constructed local stencils $\mathcal{V}(x)$, $x \in X$, are generally sparse and highly anisotropic (Figure 1, center), and the corresponding linear constraints defining $\mathrm{Conv}(\mathcal{V})$ are in part short range, and in part domain wide. Our discretization of the cone of convex functions is thus neither local [1], nor global [2, 3], but adaptive and hierarchical, multi-scale and anisotropic.

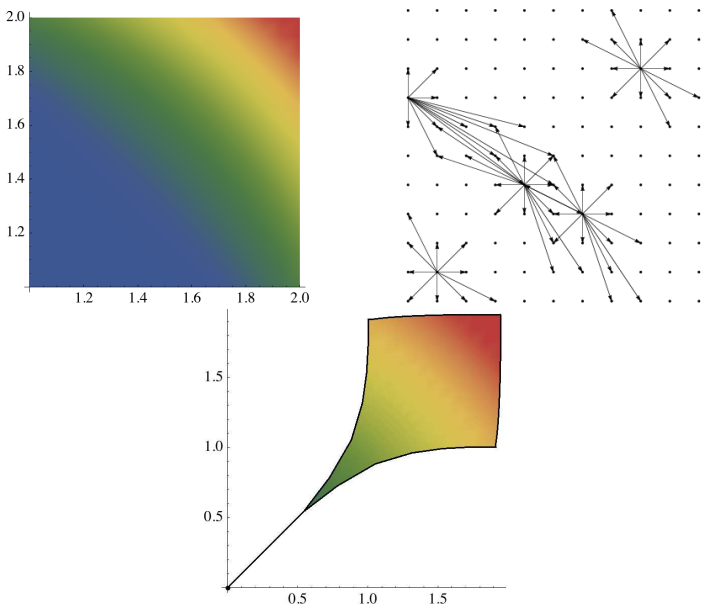


FIGURE 1. Left: solution u of (3). Center: stencil $\mathcal{V}(x)$ used in the computation, at a few points $x \in X$. Right: optimal product line Q (black contour), which is here the union of a "low end" one dimensional segment and of a "high end" two dimensional domain; optimal pricing strategy π (colors).

We applied the proposed method to an economical problem, where a monopolist [7] wants to design a product line Q, and a pricing strategy $\pi : Q \to \mathbb{R}$ so as to maximize his profit in a captive market ("take it or leave it" sales policy, rational customers, no second hand market). We illustrate here a prototypical example, see [3] for details, where products q are characterized by two traits: $q \in Q \subset \mathbb{R}^2_+$, and have production cost $\frac{1}{2}\|q\|^2$. The utility of a product q to a customer z is modeled by the scalar product $\langle z, q\rangle$; customers have have uniform density on $[1,2]^2$, they are rational and individually choose the product of maximal *net* utility (or no product at all): $u(z) := \max\{0, \max_{q\in Q}\langle z, q\rangle - \pi(q)\}$. This net utility u is thus convex by construction, and maximizing the monopolist's profit amounts to, see [3]:

$$\min\left\{\int_{[1,2]^2}\left(\frac{1}{2}\|\nabla u\|^2 - \langle u, z\rangle + u\right)dz;\ u \text{ convex},\ u \geq 0\right\}. \tag{3}$$

The monopolist's optimal product line Q is recovered from the minimizing net utility u as $Q := \{\nabla u(z);\ z \in [1,2]^2\}$, and the optimal pricing strategy π as the Legendre Fenchel dual of u. Numerical experiments reproduce an expected

qualitative model prediction: high-end products are more diverse than low-end products, see Figure 1 right.

In summary, we address variational problems posed on the cone of convex functions using a numerical method involving adaptive, anisotropic, sparse, long range stencils. The underlying domain needs to be a grid, since its arithmetic properties play an essential role. This is part of a larger program to apply similar strategies to stiff and strongly anisotropic Partial Differential Equations (PDEs), which has already seen some success with Anisotropic Diffusion [4], and Anisotropic static Hamilton-Jacobi PDEs [6].

References

[1] N. E. Aguilera, P. Morin. Approximating optimization problems over convex functions, Numerische Mathematik, vol. 111, no 1, p. 1-34, 2008.

[2] G. Carlier, T. Lachand-Robert, and B. Maury. A numerical approach to variational problems subject to convexity constraint, Numerische Mathematik, vol. 88, no 2, p. 299-318, 2001

[3] I. Ekeland, and S. Moreno-Bromberg. An algorithm for computing solutions of variational problems with global convexity constraints, Numerische Mathematik, vol. 115, no 1, p. 45-69, 2010.

[4] J. Fehrenbach, and J.-M. Mirebeau. Sparse Non-Negative Stencils for Anisotropic Diffusion, Journal of Mathematical Imaging and Vision, to appear, 2013

[5] T. Lachand-Robert, and M. A. Peletier Newton's Problem of the Body of Minimal Resistance in the Class of Convex Developable Functions, Mathematische Nachrichten, vol. 226, p. 153-176, 2000

[6] J.-M. Mirebeau. Efficient Fast Marching with Finsler Metrics, Numerische Mathematik, to appear, 2013

[7] M. Mussa, and S. Rosen, Monopoly and product quality, Journal of Economic Theory, vol. 18, p. 301-317, 1978

[8] J. C. Rochet, and P. Choné. Ironing, Sweeping, and Multidimensional Screening, Econometrica, p. 783-826, 1998.

Stability of Petrov-Galerkin discretizations: Application to the weak space-time formulation for parabolic PDEs

Christian Mollet

Suppose that we have a boundedly invertible operator $B \in \mathcal{L}(X, Y')$ mapping from a Hilbert space X into a dual Hilbert space Y'. We denote by $\mathcal{L}(X, Y')$ the set of all linear and bounded mappings form X into Y'. Consider the abstract operator equation

$$Bu = f, \qquad u \in X, \quad f \in Y'.$$

A Petrov-Galerkin solution is now given as the solution $u_j \in S_j$ of the discrete variational problem

$$\langle Bu_j, q_\ell \rangle = \langle f, q_\ell \rangle \qquad \text{for all } q_\ell \in Q_\ell,$$

for reasonably chosen discrete subspaces $S_j \subset X$ and $Q_\ell \subset Y$. The stability of such Petrov-Galerkin approaches is characterized by the discrete inf-sup condition

$$\inf_{v_j \in S_j \setminus \{0\}} \sup_{q_\ell \in Q_\ell \setminus \{0\}} \frac{|\langle B v_j, q_\ell \rangle|}{\|v_j\|_X \|q_\ell\|_Y} =: \beta_{j,\ell} > 0.$$

In order to obtain well-defined solutions which converge quasi-optimally to the exact solutions, it is crucial that this discrete inf-sup constants $\beta_{j,\ell}$ can be bounded from below by a constant β independently of the discretizations. Therefore, one needs to construct appropriate discretization spaces S_j and Q_ℓ. It turned out that one generally has to enrich the test space in order to ensure uniform stability. This in turn leads to a minimal residual Petrov-Galerkin approach [1]. We were able to prove stability for sufficiently regular operators under standard smoothness and approximation properties of the discrete spaces [5].

Moreover, we can explicitly state the number of extra layers L ensuring stability of the minimal residual Pertov-Galerkin approach when choosing $\ell \leq j + L$ and also the lower bound β of the discrete inf-sup condition is given explicitly. The ideas for the proof mainly stem form [3].

Next, this result will be applied to the full space-time weak formulation of parabolic problems, which serves as an important model example. Starting with a parabolic PDE in standard weak form with respect to space

$$\begin{aligned} \frac{du}{dt} + Au &= f \quad \text{in } \Omega \times [0,T] \\ u &= g \quad \text{on } \partial\Omega \times [0,T] \\ u(\cdot,0) &= u_0 \quad \text{in } \Omega, \end{aligned}$$

the space-time formulations reads as

$$\langle Bu, q \rangle = \ell(q) \qquad \forall q \in Y := L_2(0,T;V) \cap H^1_{\{T\}}(0,T;V'),$$

where

$$\langle Bu, q \rangle := \int_0^T -\left\langle u(t), \frac{dq(t)}{dt} \right\rangle \, dt + \int_0^T \langle A(t)u(t), q(t) \rangle \, dt,$$

$$\ell(q) := \int_0^T \langle f(t), q(t) \rangle \, dt + \langle u_0, q(0) \rangle,$$

with solution space $X := L_2(0,T;V)$ and $H^1_{\{T\}}(0,T;V') := \{\phi \in H^1(0,T;V') : \phi(T) = 0\}$, cf. [2, 6]. A detailed stability analysis for such problem classes using slightly different techniques can also be found in [1]. A criterion for the regularity of the operator B which only depends on the spatial operator A was already given in [2], so we restrict ourselves mainly to the choice of appropriate discrete subspaces. We choose the temporal and spatial hierarchy of subspaces

$$S^x_{j_0} \subset S^x_{j_0+1} \subset \cdots \subset H^m(\Omega), \qquad \operatorname{clos}_{H^m(\Omega)} \left(\bigcup_{j=j_0}^{\infty} S^x_j \right) = H^m(\Omega),$$

and similar S_j^t, Q_ℓ^x and Q_ℓ^t and define the corresponding tensor product spaces

$$\begin{aligned} S_j &:= S_j^t \otimes S_j^x \subset L_2(0,T;V) = X, \\ Q_\ell &:= Q_\ell^t \otimes Q_\ell^x \subset L_2(0,T;W) \cap H^1_{\{T\}}(0,T;H) = Y_+. \end{aligned}$$

Moreover, we arrange the temporal ('t') and spatial ('x'), primal(' ') and dual ('$\sim$') subspaces such that they are L_2-stable and satisfy Jackson and Bernstein estimates with given parameters γ_{F_k} respectively d_{F_k}. These conditions are satisfied e.g. for B-Splines on uniform grids with grid spacing 2^{-j} respectively $2^{-\ell}$, where the parameters d_{F_k} in the Jackson estimates are simply given by the polynomial order and the parameters γ_{F_k} in the Bernstein estimates by the global smoothness of the B-Splines. It is worth mentioning that these conditions are not only restricted to uniform grids, but also hold for certain sparse tensor grids for instance. When using wavelet bases respectively Riesz bases, one can easily show that the discrete inf-sup constants are equivalent to the smallest singular values of the corresponding system matrices, i.e.,

$$\beta_{j,\ell} = \inf_{v_j \in S_j \setminus \{0\}} \sup_{q_\ell \in Q_\ell \setminus \{0\}} \frac{|\langle B v_j, q_\ell \rangle|}{\|v_j\|_X \|q_\ell\|_Y} \quad \sim \quad \lambda_{\min}(\mathbf{B}_{j,\ell}^T \mathbf{B}_{j,\ell})^{\frac{1}{2}} =: \sigma_{\min}(\mathbf{B}_{j,\ell}).$$

That is, one can study the qualitative behavior of the discrete inf-sup constants via these singular values. Our predictions on the stability, especially in view of the dependence on the refinement levels ℓ and j with respect to the test and trial spaces, respectively, are underlined e.g. by the numerical results in Figure 1. In Figure 1 we considered the smallest and largest singular values of the system

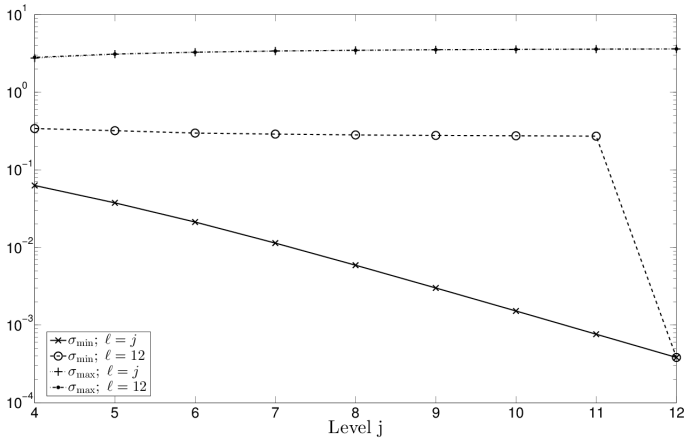


FIGURE 1. Plot of $\sigma_{\min}(\mathbf{B}_{j,\ell}) \sim \beta_{j,\ell}$ for same levels $j = \ell$ and fixed level $\ell = 12$.

matrices which stem from the ODE

$$\frac{du(t)}{dt} = f(t), \qquad t \in [0,1],$$

with zero initial condition. We observe that all theoretical predictions are already confirmed by this ODE example. Considering e.g. the Helmholz equation, i.e., a PDE with Bochner spaces respectively tensor product spaces involved, yields qualitatively similar results. These results for a parabolic PDE can be found in [5].

It can be expected, that the results can be improved by replacing S_j by $S_{(j_1,j_2)} := S^t_{j_1} \otimes S^x_{j_2}$ and also Q_ℓ by $Q_{\ell_1,\ell_2} := Q^t_{\ell_1} \otimes Q^x_{\ell_2}$, since the levels with respect to time and space can be handled differently. Moreover, one could try to apply the used techniques also to stochastic partial differential equations, as for instance to a space-time weak formulation of the stochastic heat equation [4].

References

[1] R. Andreev. Stability of sparse space-time discretizations of linear parabolic evolution equations. IMA J. Numer. Anal., 33, 2013, 242–260.

[2] N. Chegini and R. Stevenson. Adaptive Wavelet Schemes for Parabolic Problems: Sparse Matrices and Numerical Results. SIAM J. Numer. Anal, 49, 1, 2011, 182–212.

[3] W. Dahmen and A. Kunoth. Appending boundary conditions by Lagrange multipliers: General criteria for the LBB condition. Numer. Math, 88, 2001, 9–42.

[4] S. Larsson and M. Molteni. Weak space-time variational formulation of the stochastic heat equation. Work in progress.

[5] C. Mollet. Stability of Petrov-Galerkin discretizations: Application to the space-time weak formulation for parabolic evolution problems. Manuscript, Feb. 2013, in revision.

[6] C. Schwab and R. Stevenson. Space-Time Adaptive Wavelet Methods for Parabolic Evolution Problems. Math. Comp., 78(267), 2009, 1293–1318.

Effective boundary conditions for compressible flows over rough boundaries

Siegfried Müller

(joint work with Wolfgang Dahmen, Giulia Deolmi)

Domains with microscopic rough boundaries frequently arise in applications in engineering. For instance, space shuttles are often covered with tiles, while small air injecting nozzles are used over wings of aircrafts to reduce the drag [14]. Examples can also be found in nature, e.g. the skin of sharks [10], and in everyday life, e.g. golf balls.

The challenge inherent in the numerical simulation of such problems is the high resolution that is needed to resolve the roughness. In general, the computational costs will be prohibitively high and a direct numerical simulation will not be feasible although nowadays computer are becoming even more powerful. To deal with this type of problems we thus need concepts that allow to quantify the influence of small scale effects on the resolved large scale effects *without* resolving small scale structures. For this purpose concepts based on either *homogenization techniques*

[5, 13, 15] or *(heterogeneous) multiscale modeling* [8, 9] can be used. These concepts need always to be adapted to the problem at hand, i.e., for a given concrete application the main task is to derive an appropriate upscaling strategy.

Our particular interest is on the flow over a rough surface. A possible approach for the derivation of an appropriate upscaling strategy is to smooth artificially the boundary and solve the flow equations in the artificial smooth domain, cf. [1, 12]. Of course, this will introduce a significant error because micro-scale effects due to the roughness are discarded in this *zeroth order solution*. Therefore it has to be corrected by an appropriate correction term that depends on macro-scale *and* micro-scale variables. Plugging the modified solution into the original problem, another typically much simpler problem, the so-called *cell problem*, can be derived by means of an asymptotic expansion. In addition, we obtain also a correction for the boundary conditions on the artificial smooth surface where the solution of the cell problem enters. These are referred to as *effective boundary conditions* or *Navier wall law* [6] and can be considered the upscaling model. Finally the effective problem can be solved on the smooth domain with the effective boundary conditions.

Previous work in this regard has been done for laminar and *incompressible* fluid flow modeled by the incompressible Navier-Stokes equations for moderate Reynolds numbers: in [1] a Navier wall law is derived applying homogenization techniques starting from a Taylor expansion of velocity and pressure. In [2, 4, 3] these ideas are extended to the unsteady problem. In [12, 11] effective boundary conditions at the contact interface between a porous medium and a viscous incompressible fluid are derived. The corresponding asymptotic expansions have analogies with those in [1]. However, instead of applying a Taylor expansion, a zeroth order approximation is computed firstly on a smooth *subset* of the rough domain and then it is continuously extended to the boundary of effective domain *including* the roughness where it establishes the Navier wall law.

In our work we adapt the ideas of Achdou et al. [1] for incompressible low Reynolds number flow to the more complex mathematical model of the compressible Navier-Stokes equations, where transport effects are dominating dissipative effects due to viscosity and heat conduction and boundary layers are much thinner. As a consequence some simplifications adopted in [1] cannot be applied here. In particular, for compressible flows the solution of the zeroth order approximation enters the cell problem, so we must deal with a coupling between micro and macro scales. For this purpose we approximate the zeroth order problem by van Driest's similarity solution for the laminar compressible boundary layer of a flat plate. From this we compute the macro-scale parameters involved in the cell problem on the micro scale. Averaging the solution of the cell problem provides us with the effective constant that characterizes the Navier wall law in the effective problem on the smooth domain. The ill-conditioning of the effective problem is overcome using an a posteriori procedure based upon the multiscale-based adaptive algorithm.

For proof of concept we investigate the laminar flow over a flat plate with partially embedded periodic roughness. The roughness elements are characterized by

different heights and spacings. Of particular interest is the skin friction coefficient that serves as a measure for the quality of the effective model compared to direct numerical simulations performed on the rough domain. Details on the results can be found in [7].

In future work we will investigate laminar as well as turbulent flow over riblet structures in three dimensions. Opposite to the two-dimensional configuration considered in the present work the roughness will be aligned in spanwise direction instead of streamwise direction. This configuration is certainly more appropriate to reduce drag by influencing turbulence structures near to the surface. This technology might be useful to increase the efficiency of nowadays airplanes with regard to energy consumption where riblet structures are mounted to the airplane wings. Alternatively, one might think of high-frequency waves actuated in spanwise direction on the surface of an airplane.

References

[1] Y. Achdou, O. Pironneau, F. Valentin, *Effective boundary conditions for laminar flows over periodic rough boundaries*, *J. Comp. Phys.*, **147**, 187–218, 1998.

[2] Y. Achdou, O. Pironneau, F. Valentin, *New wall laws for unsteady incompressible Navier-Stokes equations*, *ECCOMAS 2000*, 2000.

[3] Y. Achdou, O. Pironneau, F. Valentin, *Comparison of wall laws for unsteady incompressible Navier-Stokes equations*, *First MIT Conference on Computational Fluid and Solid Mechanics, Boston, USA, June 12-15*, 2001.

[4] G.R. Barrenechea, P. Le Tallec, F. Valentin, *New wall laws for the unsteady incompressible Navier-Stokes equations on rough domains*', *Mathematical Modeling and Numerical Analysis*, **36**, 177–203, 2002.

[5] A. Bensoussan, J.L. Lions, G. Papanicolaou, *Asymptotic analysis for periodic structures*, North-Holland, Amsterdam, 1978.

[6] J. Casado-Diaz, E. Fernandez-Cara, J. Simon, *Why viscous fluids adhere to rugose walls: a mathematical explanation*, *J. Differential Equations*, **189**, 526–537, 2003.

[7] W. Dahmen, G. Deolmi, S. Müller, *Effective boundary conditions for Compressible Flows over Rough Boundaries*, in preparation.

[8] W. E, B. Engquist, *The heterogeneous multiscale methods*, *Commun. Math. Sci.* **1(1)**, 87–132, 2003.

[9] W. E, B. Engquist, *The heterogeneous multi-scale method for homogenization problems*, in Multiscale Methods in Science and Engineering, 89–110, Lecture Notes Comput. Sci. Eng., Vol. 44, Springer, Berlin, 2005.

[10] E. Friedmann, *The Optimal Shape of Riblets in the Viscous Sublayer*, *J. Math. Fluid Mech.*, **12**, 243–265, 2010.

[11] W. Jaeger, A. Mikelic, *On the interface boundary conditions of Beavers, Joseph and Saffman*, *SIAM J. Appl. Math.*, **60**, 1111–1127, 2000.

[12] W. Jaeger, A. Mikelic, *On the roughness-induced effective boundary conditions for an incompressible viscous flow*, *Journal of Differential Equations*, **170**, 96–122, 2001.

[13] V. Jikov, S. Kozlov, O. Oleinik, *Homogenization of differential operators and integral functionals*, Springer, Berlin, 1995.

[14] A. Madureira, F. Valentin, *Asymptotics of the Poisson problem in domains with curved rough boundaries*, *SIAM J. Math. Anal.*, **38**, 1450–1473, 2006/07.

[15] L. Tartar, *The general theory of homogenization. A personalized introduction*, Lecture Notes of the Unione Matematica Italiana, 7. Springer-Verlag, Berlin; UMI, Bologna, 2009.

Model reduction for multiscale problems

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(joint work with F. Albrecht, M. Drohmann, B. Haasdonk, P. Henning, S. Kaulmann, and B. Schweizer)

In this contribution we present efficient numerical multiscale methods for flow in heterogeneous porous media, in particular also in situations where the resulting equations are to be solved repeatedly for varying parameters, as e.g. in the context of uncertainty quantification, time dependent scenarios or optimal control problems. We discuss a posteriori based discretization methods and suggest a suitable conceptual approach for an efficient numerical treatment of parameterized variational multiscale problems where the parameters are either chosen from a low dimensional parameter space or consists of parameter functions from some compact low dimensional manifold that is embedded in some high dimensional or even infinite dimensional function space. Our general approach [14] covers a large class of numerical multi-scale schemes based on an additive splitting of function spaces into macroscopic and fine scale contributions combined with a tensor decomposition of function spaces in the context of multi query applications.

In detail, let U, V denote suitable function spaces over a domain $\Omega \subset \mathbb{R}^d$ and let us look at solutions $u_\mu^\epsilon \in U$ of parameterized variational problems of the form

$$R_\mu^\epsilon[u_\mu^\epsilon](v) = 0 \qquad \forall v \in V.$$

with an ϵ and μ-dependent mapping $R_\mu^\epsilon : U \to V'$ where ϵ denotes a parameter that indicates the multiscale character of the problem, and $\mu : \Omega \to \mathbb{R}^p, p \in \mathbb{N}$ denotes a vector of parameter functions that do not depend on ϵ.

Numerical multiscale methods make use of a possible separation of scales in the underlying problem. The macroscopic scale is defined by a priori chosen macroscopic approximation spaces $U_H \subset U, V_H \subset V$, typically chosen as piecewise polynomial functions on a uniform coarse partition $\mathcal{T}_H$ of Ω. The fine scale in the multiscale problem is usually defined by a priori chosen microscopic approximation spaces $U_h \subset U, V_h \subset V$, also typically chosen as piecewise polynomial functions on a uniform fine partition $\mathcal{T}_h$ of Ω. For suitable choices of polynomial degrees and meshes the spaces should satisfy $U_H \subset U_h \subset U$, and $V_H \subset V_h \subset V$, respectively. In this setting, let us denote with $\pi_{U_H} : U \to U_H$, $\pi_{V_H} : V \to V_H$ projections into the coarse spaces. We then define fine parts of U_h, or V_h through

$$U_{f,h} := \{u_h \in U_h : \pi_{U_H}(u_h) = 0\}, V_{f,h} := \{v_h \in V_h : \pi_{V_H}(v_h) = 0\}.$$

The discrete solution $u_{\mu,h}^\epsilon \in U_h$ is then defined through its decomposition $u_{\mu,h}^\epsilon = u_H + u_{f,h} \in U_H \oplus U_{f,h}$, satisfying

$$R_\mu^\epsilon[u_H + u_{f,h}](v_H) = 0 \qquad \forall v_H \in V_H, \tag{1}$$

$$R_\mu^\epsilon[u_H + u_{f,h}](v_{f,h}) = 0 \qquad \forall v_{f,h} \in V_{f,h}. \tag{2}$$

In a further step, a localization of the fine scale correction $u_{f,h}$ is obtained. Thus, let a coarse partition $\mathcal{T}_H$ of Ω and macroscopic discrete function spaces

$U_H(\mathcal{T}_H), V_H(\mathcal{T}_H)$ be given, e.g. by choosing globally continuous, piecewise polynomial finite element spaces on $\mathcal{T}_H$. Furthermore, we choose quadrature rules $(\omega_{T,q}, x_{T,q})_{q=1}^{Q}$ for $T \in \mathcal{T}_H$ and associate with each quadrature point $x_{T,q}$ a local function space $U^{\delta}_{f,x_{T,q}}$ which might e.g. be given as

$$U^{\delta}_{f,x_{T,q}} := \{u_{f,x_{T,q}} = u_{f,h}|_{Y^{\delta}(x_{T,q})} : u_{f,h} \in U_{f,h})\}$$

where $Y^{\delta}(x_{T,q})$ is an appropriate discrete δ-environment of $x_{T,q}$ that can be decomposed with elements from the fine mesh $\mathcal{T}_h$. Local function spaces $V^{\delta}_{f,x_{T,q}}$ are defined analogously.

Next, we define local corrector operators $\mathcal{Q}_{x_{T,q}} : U_H \to U^{\delta}_{f,x_{T,q}}$ through an appropriate localization of 2, e.g.

$$R^{\epsilon}_{\mu}[u_H + \mathcal{Q}_{x_{T,q}}(u_H)](v_{f,x_{T,q}}) \quad = \quad 0 \qquad \forall v_{f,x_{T,q}} \in V^{\delta}_{f,x_{T,q}}. \tag{3}$$

A corresponding local reconstruction operator $\mathcal{R}_{x_{T,q}}$ is then given as

$$\mathcal{R}_{x_{T,q}}(u_H) = u_H + \mathcal{Q}_{x_{T,q}}(u_H) \tag{4}$$

and we obtain the overall method using numerical quadrature in the coarse scale equation (1) and by replacing $u_H + u_{f,h}$ in (1) by the localized reconstruction $\mathcal{R}_{x_{T,q}}(u_H)$. Depending on the choice of trail and test functions, and on the choice of specific localizations of the function space for the fine scale correctors and by choosing corresponding localized corrector operators a variety of numerical multiscale methods can be recovered. For a detailed derivation of the multiscale finite element method (MsFEM), the variational multiscale method, and the heterogeneous multiscale method (HMM) in such a framework we refer to the expositions in [6] and [12]. We in particular focus on a posteriori error estimation and adaptivity for HMM approximations of elliptic problems [13, 7] and for approximation of immiscible two phase flow in porous media [9, 10]. We also refer to [9] for homogenization of degenerate two phase flow in porous media in a more complex situation where also jumps in the capillary pressure and relative permeability curves on the fine scale are allowed. Finally, we present an a posteriori error estimate for MsFEM that in particular is able to measure the error due to oversampling in heterogeneous scenarios [8].

To efficiently cope with two phase flow in porous media in multi-query scenarios, we introduce the reduced basis approach [5] with extensions for non-linear PDEs, based on the concept of empirical operator interpolation [3]. Numerical experiments are given for two phase flow in porous media [4]. Finally, we present a generalization of the classical projection based reduced basis approach to efficiently cope with multiscale problems in multi-query scenarios. Let thus suppose that in a first step we have computed snapshots, i.e. solutions $u^{\epsilon}_{\mu,h}$ with our favorite numerical multiscale method for suitable chosen parameters $\mu_i, i = 1, \ldots, N$. The choice of suitable parameters may for example be done by a Greedy algorithm based on efficient a posteriori error estimates. Let us denote Φ_N a orthogonalized Basis of $V_N := \text{span}(\mu_i, i = 1, \ldots, N)$. The classical reduced basis approach is based on approximating solutions by linear expansions of the form

$u^{\epsilon}_{\mu,N}(x) = \sum_{i=1}^{N} a_i\phi_i(x), x \in \Omega$. Hence, the spatial variation of the solution is represented by the globally defined basis functions only. Here, we apply a generalization of this approach (see [14]) by replacing the linear combination of reduced basis functions by the nonlinear combination $u^{\epsilon}_{\mu,N} = \sum_{i=1}^{N} a_i(x)\phi_i(x)$, with the hope to significantly reduce the number N of reduced basis functions needed to represent the solution manifold of the underlying parameterized problem. Here the coefficients a_i are now supposed to be macroscopic functions that are able to take care of the macroscopic spatial variation of the solution manifold. Let us for instance assume $a_i \in U_H$, while $\phi_i \in V_N \subset U_h$. The reduced multiscale solution space is then given as $U_{H,N} := \{u_{H,N}(x) = \sum_{i=1}^{N} a_i(x)\phi_i(x) | a_i \in U_H, \phi_i \in \Phi_N\}$ and a corresponding reduced scheme is obtained by suitable projection of the original problem onto such function spaces.

Particular realizations of this approach are the local reduced basis discontinuous Galerkin method [11] and the localized reduced basis multiscale method [2], but also other approaches such as the mixed multiscale finite element method using limited global information [1], the generalized finite element method or partition of unity methods fit into this framework.

Numerical experiments are given to demonstrate the efficiency of the new approach.

References

[1] J. E. Aarnes, Y. Efendiev, and L. Jiang. Mixed multiscale finite element methods using limited global information. *Multiscale Model. Simul.*, 7(2):655–676, 2008.

[2] F. Albrecht, B. Haasdonk, S. Kaulmann, and M. Ohlberger. The localized reduced basis multiscale method. In *Proceedings of Algoritmy 2012, Conference on Scientific Computing, Vysoke Tatry, Podbanske, September 9-14, 2012*, pages 393–403. Slovak University of Technology in Bratislava, Publishing House of STU, 2012.

[3] M. Drohmann, B. Haasdonk and M. Ohlberger. Reduced basis approximation for nonlinear parametrized evolution equations based on empirical operator interpolation. *SIAM J. Sci. Comput.*, 34(2):A937–A969, 2012.

[4] M. Drohmann, B. Haasdonk and M. Ohlberger. Reduced Basis Model Reduction of Parametrized Two-Phase Flow in Porous Media . Submitted to the Proceedings of MATHMOD 2012 - 7th Vienna International Conference on Mathematical Modelling, Vienna, February 15 - 17, 2012.

[5] B. Haasdonk and M. Ohlberger. Reduced basis method for finite volume approximations of parametrized linear evolution equations. *ESAIM: Mathematical Modelling and Numerical Analysis*, 42(02):277–302, 2008.

[6] P. Henning and M. Ohlberger. A Newton-scheme framework for multiscale methods for nonlinear elliptic homogenization problems. In *Proceedings of Algoritmy 2012, Conference on Scientific Computing, Vysoke Tatry, Podbanske, September 9-14, 2012*, pages 65–74. Slovak University of Technology in Bratislava, Publishing House of STU, 2012.

[7] P. Henning and M. Ohlberger. A-posteriori error estimation for a heterogeneous multiscale method for monotone operators and beyond a periodic setting. *DCDS-S. To appear*, 2013.

[8] P. Henning, M. Ohlberger, and B. Schweizer. An adaptive multiscale finite element method. Preprint 05/12 - N, FB 10 , Universität Münster, 2012.

[9] P. Henning, M. Ohlberger, and B. Schweizer. Homogenization of the degenerate two-phase flow equations. *M3AS, to appear*, 2013.

[10] P. Henning, M. Ohlberger, and B. Schweizer. Adaptive Heterogeneous Multiscale Methods for immiscible two-phase flow in porous media . FB 10, University of Muenster arXiv:1307.2123 [math.NA], Preprint, 2013.
[11] S. Kaulmann, B. Haasdonk, and M. Ohlberger. A new local reduced basis discontinuous galerkin approach for heterogeneous multiscale problems. *C. R. Math. Acad. Sci. Paris*, 349(23-24) :1233–1238, 2011.
[12] A. Målqvist. Multiscale methods for elliptic problems. *Multiscale Model. Simul.*, 9(3):1064–1086, 2011.
[13] M. Ohlberger. A posteriori error estimates for the heterogeneous multiscale finite element method for elliptic homogenization problems. *Multiscale Model. Simul.*, 4(1):88–114, 2005.
[14] M. Ohlberger. Error control based model reduction for multi-scale problems. In *Proceedings of Algoritmy 2012, Conference on Scientific Computing, Vysoke Tatry, Podbanske, September 9-14, 2012*, pages 1–10. Slovak University of Technology in Bratislava, Publishing House of STU, 2012.

Compactly supported frames for spaces of distributions in the framework of Dirichlet spaces

Pencho Petrushev

(joint work with Shai Dekel, Gerard Kerkyacharian, and George Kyriazis)

Compactly supported frames and bases are an important tool in Harmonic analysis and its applications in allowing to represent functions and distributions in terms of building blocks of small supports. Our goal is to construct frames with compactly supported frame elements of small shrinking supports in the general framework of Dirichlet spaces [1, 3], which we next describe briefly:

I. We assume that (M, ρ, μ) is a metric measure space satisfying the conditions: (M, ρ) is a locally compact metric space with distance $\rho(\cdot, \cdot)$ and μ is a positive Radon measure such that the following *volume doubling condition* is valid

$$0 < \mu(B(x, 2r)) \le c_0 \mu(B(x, r)) < \infty \quad \text{for all } x \in M \text{ and } r > 0,$$

where $B(x, r)$ is the open ball centered at x of radius r and $c_0 > 1$ is a constant.

II. The main assumption is that the local geometry of the space (M, ρ, μ) is related to an essentially self-adjoint positive operator L on $L^2(M, d\mu)$ such that the associated semigroup $P_t = e^{-tL}$ consists of integral operators with (heat) kernel $p_t(x, y)$ obeying the conditions:

• *Small time Gaussian upper bound*:

$$|p_t(x, y)| \le \frac{C^\star \exp\{-\frac{c^\star \rho^2(x,y)}{t}\}}{\sqrt{\mu(B(x, \sqrt{t}))\mu(B(y, \sqrt{t}))}} \quad \text{for } x, y \in M,\ 0 < t \le 1. \tag{1}$$

• *Hölder continuity*: There exists a constant $\alpha > 0$ such that

$$\big|p_t(x, y) - p_t(x, y')\big| \le C^\star \Big(\frac{\rho(y, y')}{\sqrt{t}}\Big)^\alpha \frac{\exp\{-\frac{c^\star \rho^2(x,y)}{t}\}}{\sqrt{\mu(B(x, \sqrt{t}))\mu(B(y, \sqrt{t}))}}$$

for $x, y, y' \in M$ and $0 < t \le 1$, whenever $\rho(y, y') \le \sqrt{t}$.

- *Markov property*:

$$\int_M p_t(x,y)d\mu(y) \equiv 1 \quad \text{for } t > 0.$$

Above $C^\star, c^\star > 0$ are structural constants.

A natural effective realization of the above setting appears in the general framework of strictly local regular Dirichlet spaces with a complete intrinsic metric, where it only suffices to verify the local scale-invariant Poincaré inequality and the global doubling condition on the measure and then the above general setting applies in full. A key observation is that situations, where our theory applies are quite common, in particular, this theory applies on Lie groups or homogeneous spaces with polynomial volume growth, complete Riemannian manifolds with Ricci curvature bounded from below and satisfying the volume doubling condition. Naturally, it covers the more classical cases on the sphere, interval, ball, and simplex with weights. For more details, see [1, 3, 4].

We build on results on functional calculus, frames and spaces of distributions developed in [1, 3]. To achieve our goals we first develop a general small perturbation scheme for construction of frames in a general quasi-Banach space $\mathcal{B}$ of distributions given a pair of dual frames $\{\psi_\xi\}$, $\{\tilde{\psi}_\xi\}$. In fact, this is the situation in [3]. Such a method has been developed in [7] in the more favorable situation when a single frame $\{\psi_\xi\}$ for $\mathcal{B}$ exists. The latter scheme can be applied directly in our setting in the spacial case when the spectral spaces have the polynomial property (see [3]) as on the sphere, interval, ball, and simplex. The idea of these schemes is rooted in the development of bases in [9], also in [5, 6].

The construction of compactly supported frames relies heavily on the *finite speed propagation property* of solutions of the wave equation associated with the operator L. This property follows from the Gaussian bound (1) on the heat kernel $p_t(x,y)$. The finite speed propagation property alone, however, is not sufficient. The other properties of the heat kernel and the doubling condition on the measure are also important for the development of a complete theory. In particular, they allowed to develop in [3] Besov and Triebel-Lizorkin spaces with full set of indices and their frame characterization, which play a critical role here.

Compactly supported frames have already been constructed on the sphere in [7] and on the ball with weight $w_\mu(x) = (1-|x|)^{\mu-1/2}$, where μ is a half integer and $\mu \geq 0$ in [8]. One of the strengths of our method is that although it is general it allows to obtain in particular settings better results than the existing ones. For example, combining results from this work and [4] enable us to improve the results on the ball from [8] by relaxing the condition on μ from a half integer and $\mu \geq 0$ to any $\mu > -1/2$. Another application of the current results and results from [4] is to the development of compactly supported frames on the interval with Jacobi weights and on the simplex with weights.

A key feature of the new frames is that they can be used for decomposition of the Besov and Triebel-Lizorkin spaces in the general framework of Dirichlet spaces developed in [3], and therefore, in many particular settings of interest.

An important application of the compactly supported frames from this work is to atomic Hardy spaces H^p_A, $0 < p \leq 1$. The compactly supported frames provide a vehicle in establishing Littlewood-Play characterization of the Hardy spaces H^p_A and their frame decomposition.

The results of this work are established with complete proofs in [2].

References

[1] T. Coulhon, G. Kerkyacharian, P. Petrushev, Heat kernel generated frames in the setting of Dirichlet spaces, J. Fourier Anal. Appl. 18 (2012), 995–1066.

[2] S. Dekel, G. Kerkyacharian, G. Kyriazis, and P. Petrushev, Compactly supported frames for spaces of distributions in the framework of Dirichlet spaces, preprint.

[3] G. Kerkyacharian, P. Petrushev, Heat kernel based decomposition of spaces of distributions in the framework of Dirichlet spaces, Trans. Amer. Math. Soc. (to appear).

[4] G. Kerkyacharian, P. Petrushev, and Yuan Xu, Gaussian bounds for the heat kernel on the ball and simplex, preprint.

[5] G. Kyriazis, P. Petrushev, New Bases for Treibel-Lizorkin and Besov spaces, Trans. Amer. Math. Soc. 354 (2002), 749–776.

[6] G. Kyriazis, P. Petrushev, On the construction of frames for Treibel-Lizorkin and Besov spaces, Proc. Amer. Math. Soc. 134 (2006), 1759–1770.

[7] G. Kyriazis, P. Petrushev, On the construction of frames for spaces of distributions, J. Funct. Anal. 257 (2009), 2159–2187.

[8] G. Kyriazis, P. Petrushev, "Compactly" supported frames for spaces of distributions on the ball, Monatsh. Math. 165 (2012), 365–391.

[9] P. Petrushev, Bases consisting of rational functions of uniformly bounded degrees or more general functions, J. Funct. Anal. 174 (2000), 18–75.

Maximum principle and entropy consistency for numerical approximations of nonlinear hyperbolic conservation laws

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(joint work with Jean-Luc Guermond and Orhan Mehmetoglu)

The scalar theory for nonlinear hyperbolic conservation laws is well developed. Namely, maximum principle, entropy stability and convergence of viscosity approximations has been established a long time ago. However, in the case of second or higher order schemes, there are few limited convergence results. We present here our recent stability and convergence result for the second order Nessyahu-Tadmor scheme, see [1, 2].

Theorem 1 (Mehmetoglu and P.)**.** *Let $f \in C^4(\mathbb{R})$ be strictly convex. Then, under a standard CFL condition the Nessyahu-Tadmor scheme based on the minmod limiter converges strongly on compact sets to the unique entropy solution of the conservation law*

$$u_t + f(u)_x = 0, \quad (x,t) \in \mathbb{R} \times (0,\infty)$$

for any initial data $u(x,0) = u^0(x) \in L^\infty(\mathbb{R})$.

The relation of the above result with entropy viscosity and entropy stable schemes will be explained. In the case of the Euler system of gas dynamics, we will present a class of viscosity approximations and prove that they have an invariant domain property (the analog of maximum principle for systems). Moreover, we will identify the subclass of these approximations which is consistent with all entropy inequalities, see [3]. The connections with the related works of Lax [4], Tadmor [5], and Harten et al. [6] will be discussed. Some numerical results obtained with entropy viscosity schemes based on this first order approximation of the Euler system will be presented.

References

[1] O. Mehmetoglu and B. Popov. Maximum principle and convergence of central schemes based on slope limiters. Math. Comp. 81, No. 277: 219–231, 2012.

[2] O. Mehmetoglu and B. Popov. Convergence of the NT scheme for bounded initial data. submitted.

[3] J.-L. Guermond and B. Popov. Viscous regularization of the Euler equations and entropy principles. submitted.

[4] P. Lax. Shock waves and entropy. In Contributions to nonlinear functional analysis (Proc. Sympos., Math. Res. Center, Univ. Wisconsin, Madison, Wis., 1971): 603–634. Academic Press, New York, 1971.

[5] E. Tadmor. A minimum entropy principle in the gas dynamics equations. Appl. Numer. Math., 2(3-5): 211–219, 1986.

[6] A. Harten, P. D. Lax, C. D. Levermore, and W. J. Morokoff. Convex entropies and hyperbolicity for general Euler equations. SIAM J. Numer. Anal., 35(6): 2117–2127, 1998.

Convergence of dynamical low rank approximation in hierarchical tensor formats

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(joint work with M. Bachmayr)

Let $\mathcal{V}_1, \dots, \mathcal{V}_d$ Hilbert spaces, where e.g. $\mathcal{V}_i = \mathbb{R}^{n_i}$ or $\mathcal{V}_i = L^2(\mathbb{R})$ may hold. An order-d tensor over these spaces is then given by any $U \in \otimes_{i=1}^d \mathcal{V}_i$. For sake of simplicity let us interpret such tensors as multivariate functions

$$U : \mathcal{I}_1 \times \cdots \times \mathcal{I}_d \to \mathbb{R},\ \underline{x} = (x_1, \dots, x_d) \mapsto U(x_1, \dots, x_d),$$

with index sets $\mathcal{I}_i$ to be either discrete, e.g. $\mathcal{I}_i = \{1, \dots, n_i\}$ in case that $\mathcal{V}_i = \mathbb{R}^{n_i}$, or continuous, e.g. $\mathcal{I}_i = \mathbb{R}$ in the case that, for instance, $\mathcal{V}_i = L^2(\mathbb{R})$. Such tensors play an important role in the description of many complex systems. While in some applications, they are given explicitly, they are often defined only implicitly as the solution partial differential or integral equations in high dimensions. e.g in $\mathbb{R}^d$. Examples are the Fokker-Planck equations or many body Schrödinger equations. At the latest after a discretization, tensor quantities take on n^d different values $(x_1, \dots, x_d)$, assuming $n = \max\{n_i; i = 1, \dots, d\}$. The complexity $\mathcal{O}(n^d)$ grows at least exponential in d, Even if n is small. e.g. $n = 2$, for large d it is impossible to handle the full tensor. Tensor product approximation aims to approximate these

tensors by products of univariate functions. In tensor product approximation, the *Hierarchical Tucker tensor format (HT)* (Hackbusch-Kühn) and *Tensor Trains (TT)* (Tyrtyshnikov-Oseledets) have been introduced recently offering stable and robust approximation by a low order cost, see [2]. The representation of these tensors can be described by a *tensor network* with tree structure. The appearing component tensors are of low order, e.g. for binary trees they are of order 3, independent of the original order d. For example, the TT format provides a special case of hierarchical tensor formats. Here $U(x_1,\dots,x_d)$ is represented in terms of d component matrices $U_1(x_1), U_2(x_2),\dots,U_d(x_d)$. A value of U at point $(x_1,\dots,x_d)$ can be computed by

$$U(x_1,\dots,x_d) = U_1(x_1)U_2(x_2)\cdots U_d(x_d)\ ,$$

or explicitely

$$= \sum_{k_1=1}^{r_1} \dots \sum_{k_{d-1}=1}^{r_{d-1}} U_1(x_1,k_1)U_2(k_1,x_2,k_2)\dots U_{d-1}(k_{d-2},x_{d-1},k_{d-1})U_d(k_{d-1},x_d).$$

The numbers r_i, define the rank numbers of the TT decomposition or the rank vector $\underline{r} = (r_1,\dots,r_{d-1})$ that mainly governs the complexity of the representation. Letting $r = \max r_i$, storage of the TT decomposition is $\mathcal{O}(r^2nd)$ and is thus formally free from the curse of dimensionality. For many problems of interest, which could not be handled so far, this approach has the potential to circumvent from the curse of dimensionality. The important observation is that the hierarchical tensors inherit major properties from low rank factorization of matrices. It is already known that, given a partition tree, then the optimal ranks are defined by the rank of the corresponding *matricisation* or *matrix unfolding* $\mathbf{A}_t$, see [2]. Moreover a quasi-optimal approximation can be found from the singular value decomposition of these matrices, the HOSVD see [2]. Best N-term approximation resultss has been derived from this observation in a recent paper [7]. Let $\mathbf{A}_t = \mathbf{U}^T\Sigma\mathbf{V}\ , (SVD)\ \Sigma = \mathrm{diag}(\sigma_i)$ be the singular value decomposition of the unfolding at a node t in the tree. For $0 < p \le 2$, we refer to the the nuclear norm $p = 1$ and van Schatten class quasi-norms

$$\|\mathbf{A}_t\|_{*,p} := \big(\sum_i \sigma_{t,i}^p\big)^{\frac{1}{p}}\ .$$

Then, by Stechkins lemma the best rank r_t, for $s := \frac{1}{p} - \frac{1}{2}$, best N-term approximation satisfies

$$\inf_{\mathrm{rank}\,\mathbf{V}\le k} \|\mathbf{A}_t - \mathbf{V}\|_2 \lesssim r_t^{-s}\|\mathbf{A}_t\|_{*,p}\ .$$

Furthermore this results is quite sharp. Altogether this implies the following result.

Theorem 1. *Assume that* $\|\mathbf{A}\|_{*,p} := \max_t \|\mathbf{A}_t\|_{*,p} < \infty$, *and* $|\underline{r}| := \max\{r_t\}$, *then*

$$\inf_{\{V:\textit{ranks of } V\le\underline{r}\}} \|U - V\|_2 \lesssim C(d)|\underline{r}|^{-s}\|\mathbf{A}\|_{*,p}$$

with a prefactor $C(d) \lesssim \sqrt{d}$ *, scaling mildly with* d*. Notice that the complexity scales* $\sharp U \lesssim d|\underline{r}|^3$ *(HT),* $(\lesssim nd|\underline{r}|^2$ *(TT)).*

It has been shown in [7] that *mixed Sobolev spaces* are embedded in these classes. Perhaps, due to the complexity scaling $\mathcal{O}(r^3)$, these results are only suboptimal for these classes. In a forthcoming paper. with M. Bachmayr, consider the following parabolic problem, see e.g. [5]

$$\dot{U} = (-\mathcal{A} + \mathcal{B})U + f \ , \ U_0 \in \mathcal{M}_{\underline{r}} \ ,$$

where $\mathcal{A}$ is the sum of univariate operators. We notice that the simple transformation $U \to \widetilde{U}_\gamma(t) = e^{\gamma t}U(t)$ does not change the rank structure. It provides the transformed equation

$$\dot{U}_\gamma = (-\mathcal{A} - \gamma + \mathcal{B})U_\gamma + f_{-\gamma} \ , \ U_0 \in \mathcal{M}_{\underline{r}} \ .$$

In this setting, we derive the following kind of regularity result, provide that the right hand side is sufficiently regular in the above setting i.e. $f_t \in L_{*,p}$, $\forall t$.

Theorem 2. *Let* $1 \leq p \leq 2$*, i.e. we consider only those van Schatten classes which are Banach spaces,*

$$\mathcal{B} = \sum_{k=1}^{\infty} B_{x,k} \otimes B_{y,k} \quad , \quad \sum_{k=0}^{\infty} \|B_{x,k}\|_{L_{*,p} \to L_{*,p}} \|B_{y,k}\|_{L_{*,p} \to L_{*,p}} \leq L < \infty \ , \ \textit{for all} \ \ t,$$

(1) *if* L *is sufficiently small, then*

$$\|U\|_{*,p} < \infty$$

(2) *there exist* $\gamma \in \mathbb{R}$ *such that*

$$\|\widetilde{U}_\gamma\|_{*,p} < \infty$$

The set of tensors of prescribed rank is neither a linear space nor convex. It has been shown, partly by the author [6], that given a tree, the set hierarchical tensor of optimal rank forms smooth (open) manifolds $\mathcal{M}_{\underline{r}}$. For numerical computations, we cast the computation of an approximate solution into an optimization problems constraint to this manifold. In particular, for approximation by elements from this highly nonlinear manifold , we apply the well known *Dirac Frenkel variational principle*, see e.g [3],

$$\dot{U}(t) = \operatorname{argmin}\{\|V - (-\mathcal{A} + \mathcal{B})\Psi(t) + f(t)\| : V \in \mathcal{T}_U\} \ .$$

By straightforward manipulations, this provides the equations of motion in weak form,

$$\langle \dot{U} + (\mathcal{A} - \mathcal{B})\Psi - f, V\rangle \ , \ \forall V \in \mathcal{T}_U, U(0) = \Psi_0 \in \mathcal{M}_{\underline{r}} \ .$$

or

$$\dot{U} = P_{\mathcal{T}_U}(-\mathcal{A} + \mathcal{B})\Psi + f, U(0) = \Psi_0 \in \mathcal{M}_{\underline{r}} \ .$$

In [4], we have analysed the (open) manifold of such tensors and its projection onto the tangent space $\mathcal{T}_U$ at point $U(t) \in \mathcal{M}_{\underline{r}}$. First convergence estimates in L_2 has been derived, providing quasi-optimal converge local in time, i.e. for $0 \leq t \leq T$, T sufficiently small.

The Dirac Frenkel principle is a Galerkin approximation, where the differential equation has to be satisfied in weak form on the tangent space $\mathcal{T}_U$ at each time $U(t)$. In a forthcoming paper with M. Bachmayr [1], we tried to extend some results from well established theory of spatial Galerkin approximation w.r.t. a fixed linear ansatz space, e.g. FEM space. Aside the fact that in our case the space $\mathcal{T}_U$, is not fixed, it moves by time, further obstructions has to be handled. Due to the richness of the present nonlinear parametrization, we cannot expect quasi-optimal convergence for long times without severe restrictions and further assumptions. Indeed some counter examples are provided. Our major assumption requires that the actual approximation, together with the a quasi best approximation, touches only those parts of the manifold, where the curvature is bounded by a uniform constant ρ.

References

[1] M. Bachmayr and R. Schneider, Convergence estimates for low rank approximation of parabolic problems in preparation

[2] W. Hackbusch, *Tensor Spaces and Numerical Tensor Calculus*, SSCM Vol. 42, Springer, 2012.

[3] C. Lubich, *From Quantum to Classical Molecular Dynamics: Reduced methods and Numerical Analysis*, Zürich Lectures in advanced mathematics, EMS, 2008.

[4] C. Lubich, T. Rohwedder, R. Schneider and B. Vandereycken, Dynamical approximation of hierarchical Tucker and tensor train tensors. SPP1324 Preprint (126/2012), to appear in SIMAX 2013

[5] B. Khoromskij, I. Oseledets and R. Schneider Efficient time-stepping scheme for dynamics on TT-manifolds, MIS Preprint 80/2011

[6] S. Holtz, T. Rohwedder and R. Schneider, On manifolds of tensors with fixed TT rank, Numer. Math. 120, no. 4, 701 – 731, 2012.

[7] R. Schneider and A. Uschmajew, Approximation rates for the hierarchical tensor formats in periodic Sobolev spaces submitted to Journal of Complexity

Sparse quadrature approach to Bayesian inverse problems

Christoph Schwab

(joint work with Claudia Schillings)

We consider the parametric deterministic formulation of Bayesian inverse problems with distributed parameter uncertainty from infinite dimensional, separable Banach spaces X, with uniform prior probability measure on space X of all uncertainties. Under the assumption of given observation data δ subject to additive observation noise $\eta \sim N(0, \Gamma)$ with positive covariance Γ, an infinite-dimensional version of Bayes' formula has been shown in [14].

For problems with uncertain, distributed parameters $u \in X$ (which could be a diffusion coefficient, elastic moduli in solid mechanics, shape of the domain D of definition of the physical problem [1], kinetic parameters in stoichiometric models of reaction-systems in biological systems [4, 7], permeability in porous media or optimal control of uncertain systems [9]), we develop a practical, adaptive computational algorithm for the efficient approximation of the infinite-dimensional

integrals with respect to the Bayesian posterior (conditional on given data δ) μ^δ which arise in Bayes' formula in [14].

The Bayesian posterior μ^δ is shown to admit a representation in terms of a (generalized) polynomial chaos expansion in the (countably many) coordinates y_j which parametrize the uncertainty. We prove that if the uncertain datum $u \in X$ admits the (norm-convergent in X) expansion

$$u = \langle u \rangle + \sum_{j \geq 1} y_j \psi_j(x)$$

with $|y_j| \leq 1$ and with $(\|\psi_j\|_X)_{j \geq 1} \in \ell^p(\mathbb{N})$ for some $0 < p < 1$, then the solution $q(u) = (A(u))^{-1} f$ of the parametric operator equation will depend holomorphically on the parameters y_j, with precise control of the domain of holomorphy [3, 8, 5, 1, 9]. In two-scale limits of homogenization theory, these domains are independent of physical scale parameters [8].

We prove, generalizing [13], that the holomorphic dependence on the parameters y_j of the forward solution of the above problems implies p-sparsity of the polynomial chaos expansion for the parametric forward solution $q(u) = (A(u))^{-1} f$ and also for the density function of derivative of the Bayesian posterior μ^δ with respect to the prior μ_0, conditional on given data δ.

The proof of the p-sparsity in [12, 10] is based on verification of holomorphic dependence of the polynomial chaos representation for the density of the Bayesian posterior with respect to the prior, following the proofs in the linear, elliptic diffusion problems in [3, 13].

Based on this sparsity result, dimension independent convergence rates of best N-term approximations of the parametric forward map as well as of the parametric density of the Bayesian posterior with respect to uniform prior follows from Stechkin's lemma.

We propose a deterministic, adaptive algorithm inspired by [6] and analogous to the adaptive interpolation methods in [2] and the references there. The proposed algorithm determines iteratively, and depending on the observation data δ and the observation noise variance Γ a sequence of quadrature dimensions and quadrature orders.

Convergence rates for the adaptive Smolyak quadrature approximation are shown, computationally, to coincide with the best N term approximation rates of the Bayesian posterior density which, in turn [12, 10] depend only on the sparsity class (characterized in turn by the summability exponent $p \in (0, 1)$ of the uncertain distributed parameter $u \in X$).

Convergence rates are obtained in [12, 10] via monotone L^∞ N-term approximations of the posterior density from [2]. The resulting rates $1/p - 1$ are larger than the rate $1/2$ afforded by Monte-Carlo methods and their variants (notably MCMC) for $p < 2/3$ when stated in terms of the number N of (numerical) solutions of the forward problems which are necessary in the quadrature algorithm.

Applications with verified holomorphic dependence include high-dimensional parametric initial value problems [4], semilinear elliptic equations [5, 1] with uncertain differential operators, parabolic evolution problems with uncertain operators [1], elliptic multiscale problems with uncertain coefficients [8] and from biological systems sciences [7], as well as optimal control of uncertain systems [9], and problems with uncertain shape [1].

Numerical examples are presented for diffusion problems with uncertain diffusion coefficient from [12], [10], and for large, parametric systems of initial value problems from stoichiometric models for biological systems with mass-action kinetics from [4, 7]. Computational savings with respect to adaptivity in the forward solver are indicated. Here, we present numerical experiments based on the parametric, parabolic initial boundary-value problem

$$\begin{aligned} \partial_t q(t,x) - \operatorname{div}(u(x)\nabla q(t,x)) = f(t,x) \quad (t,x) \in T \times D\,, \\ q(0,x) = 0 \quad x \in D\,, \\ q(t,0) = q(t,1) = 0 \quad t \in T\,, \end{aligned}$$

with $f(t,x) = 100 \cdot tx$, $D = (0,1)$ and $T = (0,1)$. The uncertain coefficient u is parametrized as $u(x,y) = \bar{a} + \sum_{j=1}^{128} y_j \psi_j$, where $\bar{a} = 1$ and $\psi_j = \alpha_j \chi_{D_j}$ with $D_j = [(j-1)\frac{1}{128}, j\frac{1}{128}]$, $\boldsymbol{y} = (y_j)_{j=1,\ldots,128}$ and $\alpha_j = \frac{0.6}{j^\zeta}$, $\zeta = 3$. Figure 1 shows the convergence behavior of the adaptive Smolyak algorithm for the approximation of the normalization constant considering a variation of the number of observation points as well as of the observational noise. A detailed discussion of the numerical

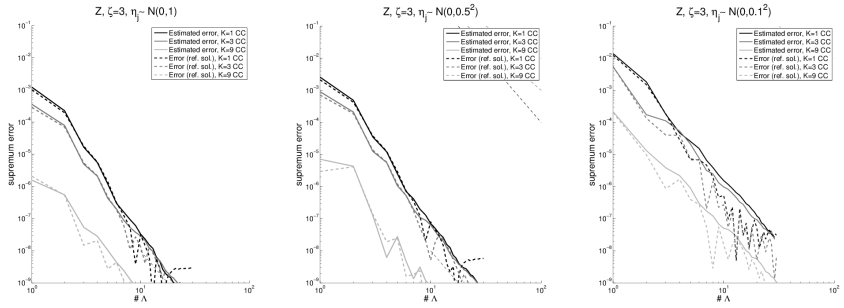

FIGURE 1. Comparison of the estimated error and actual error of the normalization constant $Z = \mathbb{E}^{\mu^\delta}[1]$ with respect to the cardinality $\#\Lambda$ of the index sets Λ_N (Clenshaw-Curtis quadrature) with $K = 1, 3, 9$ (number of observation points), $\eta \sim \mathcal{N}(0,1)$ (l.), $\eta \sim \mathcal{N}(0, 0.5^2)$ (m.), $\eta \sim \mathcal{N}(0, 0.1^2)$ (r.).

experiments for the parametric, parabolic evolution problem with random coefficients can be found in [10]. The numerical experiments indicate that, as the observation noise with variance $\Gamma \to 0$, growth of the constants in the Smolyak quadrature error estimates. In [11], we present an asymptotic analysis and prove

$C \sim \exp(b/\Gamma)$ for some constants $b, C > 0$. We also show in [11] that the Bayesian estimate admits a finite limit in the case $\Gamma \to 0$, and propose regularization of the integrand functions arising in the computation of the conditional expectation.

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References

[1] Albert Cohen and Abdellah Chkifa and Christoph Schwab, Breaking the curse of dimensionality in sparse polynomial approximation of parametric PDEs, Report 2013-25, Seminar for Applied Mathematics, ETH Zürich, 2013.

[2] Abdellah Chkifa and Albert Cohen and Christoph Schwab, High-dimensional adaptive sparse polynomial interpolation and applications to parametric PDEs, Journ. Found. Comp. Math., http://dx.doi.org/10.1007/s10208-013-9154-z, 2013.

[3] Albert Cohen and Ronald A. DeVore and Christoph Schwab, Analytic regularity and polynomial approximation of parametric and stochastic elliptic PDEs, Analysis and Applications **9**(1) 11–47, http://dx.doi.org/10.1142/S0219530511001728, 2010.

[4] Markus Hansen and Christoph Schwab, Sparse Adaptive Approximation of High Dimensional Parametric Initial Value Problems, Vietnam Journal of Mathematics**41** (2) 181–215, http://dx.doi.org/10.1007/s10013-013-0011-9, 2013.

[5] Markus Hansen and Christoph Schwab, Analytic regularity and nonlinear approximation of a class of parametric semilinear elliptic PDEs, Mathematische Nachrichten, http://dx.doi.org/10.1002/mana201100131, 2013.

[6] Thomas Gerstner and Michael Griebel, Dimension-adaptive tensor-product quadrature *Computing* **71** 65-87, 2003.

[7] Markus Hansen, Claudia Schillings and Christoph Schwab, Sparse Approximation Algorithms for High Dimensional Parametric Initial Value Problems, Report 2013-10, Seminar for Applied Mathematics, ETH Zürich (2013) (to appear in Proc. HPSC5, Hanoi, 2013 in Springer LNCSE).

[8] Viet H. Hoang and Christoph Schwab, Analytic regularity and polynomial approximation of stochastic, parametric elliptic multiscale PDEs, Analysis and Applications (Singapore) **11** (01), http://dx.doi.org/10.1142/S0219530513500012, 2013.

[9] Angela Kunoth and Christoph Schwab, Analytic regularity and GPC approximation for control problems constrained by linear parametric elliptic and parabolic PDEs, SIAM Journ. Control and Optimization **51**(3) pp. 2442 – 2471, 2013.

[10] Claudia Schillings and Christoph Schwab, Sparsity in Bayesian Inversion of Parametric Operator Equations, Report 2013-17, Seminar for Applied Mathematics, ETH Zürich, 2013.

[11] Claudia Schillings and Christoph Schwab, Scaling Limits in Computational Bayesian Inversion of Parametric and Stochastic Operator Equations (in preparation) , 2013.

[12] Claudia Schillings and Christoph Schwab, Sparse, adaptive Smolyak quadratures for Bayesian inverse problems, Inverse Problems, **29** (6), http://dx.doi.org/10.1088/0266-5611/29/6/065011, 2012.

[13] Christoph Schwab and Andrew M. Stuart, Sparse deterministic approximation of Bayesian inverse problems, Inverse Problems **28**(4) http://dx.doi.org/10.1088/0266-5611/28/4/045003, 2012.

[14] Andrew M. Stuart, Inverse problems: a Bayesian perspective, Acta Numerica, **19** (2010) pp. 451–559, http://dx.doi.org/10.1017/S0962492910000061.

Instance optimality of the adaptive maximum strategy

Rob Stevenson
(joint work with Christian Kreuzer and Lars Diening)

Adaptive algorithms for the solution of PDEs that have been proposed since the 70's are nowadays standard tools in science and engineering. In contrast to uniform refinements, adaptive mesh modifications do not guarantee that the maximal mesh size tends to zero. For this reason, even convergence of adaptive finite element methods (AFEM's) was unclear for a long time, though practical experiences often showed optimal convergence rates.

In one dimension, convergence of an AFEM for elliptic problems was proved by Babuška and Vogelius in [3] under some heuristic assumptions. Later, Dörfler introduced in [5] a bulk chasing marking strategy thereby proving linear convergence of an AFEM in two space dimensions for a sufficiently fine initial triangulation. This restriction was removed in [7, 8] by Morin, Nochetto, and Siebert.

In [1], Binev, Dahmen and DeVore extended the AFEM analysed in [7] by a a so-called *coarsening* routine, and showed that the resulting method is *instance optimal*, cf. also [2]. This means that the energy norm of the error in any approximation produced by the algorithm, with underlying triangulation denoted as $\mathcal{T}$, is less than some constant multiple of the error w.r.t. any *admissible* triangulation $\tilde{\mathcal{T}}$ satisfying $\#(\tilde{\mathcal{T}} \setminus \mathcal{T}_\perp) \leq \lambda \#(\mathcal{T} \setminus \mathcal{T}_\perp)$, for some fixed constant $\lambda \in (0,1)$. Here, an admissible triangulation is a conforming triangulation, which is created by finitely many newest vertex bisections (NVB) from a fixed initial triangulation $\mathcal{T}_\perp$.

In [10], it was shown that already without the addition of coarsening, the AFEM is *class optimal*: Whenever the solution can be approximated at some asymptotic (algebraic) convergence rate s by finite element approximations, then the AFEM produces a sequence of approximations, which converges with precisely this rate s. In [4], a similar result was shown with a refinement routine that is not required to produce "interior nodes", and with a different treatment of the approximation of the right-hand side. In that paper, the AFEM is considered as a procedure for reducing the *total error*, being the sum of the error in the energy norm and the so-called oscillation. This is also the point of view that will be taken in the present work.

In the last few years, in numerous works class optimality results for AFEMs have been derived for arbitrary space dimensions, finite elements of arbitrary orders, the error measured in L^2, right-hand sides in H^{-1}, nonconforming triangulations, discontinuous Galerkin methods, general diffusion tensors, (mildly) non-symmetric problems, nonlinear diffusion equations, and indefinite problems.

In all these works the marking strategy is *bulk chasing*, also called *Dörfler marking*. In [9], Morin, Siebert and Veeser considered also the maximum and equidistribution strategies, without proving any rates though.

In our recent work [6], we consider a standard AFEM, so without coarsening, in the model setting of Poisson's equations with homogeneous Dirichlet boundary conditions on a two-dimensional polygonal domain, the error measured in the

energy norm, square integrable right-hand side, linear finite elements, and conforming triangulations created by NVB. The refinement routine is not required to create interior nodes in refined triangles. Our method utilizes a (modified) *maximum marking strategy* for the standard residual error estimator organised by edges.

The maximum strategy marks all edges for bisection whose indicator is greater or equal to a constant $\mu \in (0, 1]$ times the largest indicator. This strategy is usually preferred by practitioners since, other than with Dörfler marking, it does not require the sorting of the error indicators, and in practise the results turn out to be very insensitive to the choice of the marking parameter $\mu \in (0, 1]$.

Roughly speaking, our modification of the maximum marking strategy replaces the role of the error indicator associated with an edge S by the sum of the error indicators over those edges that necessarily have to be bisected together with S in order to retain a conforming triangulation.

The main result of [6] states, that for any $\mu \in (0, 1]$, our AFEM is *instance optimal* for the *total error*. Clearly, instance optimality implies class optimality for any (algebraic) rate s, but not vice versa.

To prove instance optimality, we will show that the *total energy* associated with any triangulation $\mathcal{T}$ produced by our AFEM is not larger than the total energy associated with any conforming triangulation $\tilde{\mathcal{T}}$ created by NVB with $\#(\tilde{\mathcal{T}} \setminus \mathcal{T}_\perp) \leq \lambda \#(\mathcal{T} \setminus \mathcal{T}_\perp)$, for some fixed constant $\lambda \in (0, 1)$. Here the total energy is defined as the sum of the Dirichlet energy and the "element residual part of the a posteriori estimator".

References

[1] P. Binev, W. Dahmen, and R. DeVore, *Adaptive finite element methods with convergence rates*, Numer. Math. **97** (2004), no. 2, 219–268.

[2] P. Binev, *Adaptive methods and near-best tree approximation*, Oberwolfach Reports, vol. 29, 2007, pp. 1669–1673.

[3] I. Babuška and M. Vogelius, *Feedback and adaptive finite element solution of one-dimensional boundary value problems*, Numer. Math. **44** (1984), no. 1, 75–102.

[4] J. M. Cascón, C. Kreuzer, R. H. Nochetto, and K. G. Siebert, *Quasi-optimal convergence rate for an adaptive finite element method*, SIAM J. Numer. Anal. **46** (2008), no. 5, 2524–2550.

[5] W. Dörfler, *A convergent adaptive algorithm for Poisson's equation*, SIAM J. Numer. Anal. **33** (1996), 1106–1124.

[6] C. Kreuzer, L. Diening, R.P. Stevenson, *Instance optimality of the adaptive maximum strategy*, preprint, June 2013, arXiv:1306.0377 [math.NA]

[7] P. Morin, R. H. Nochetto, and K. G. Siebert, *Data oscillation and convergence of adaptive FEM*, SIAM J. Numer. Anal. **38** (2000), 466–488.

[8] ———, *Convergence of adaptive finite element methods*, SIAM Review **44** (2002), 631–658.

[9] P. Morin, K. G. Siebert, and A. Veeser, *A basic convergence result for conforming adaptive finite elements*, Math. Models Methods Appl. Sci. **18** (2008), no. 5, 707–737.

[10] R.P. Stevenson, *Optimality of a standard adaptive finite element method*, Found. Comput. Math. **7** (2007), no. 2, 245–269.

Analysis and approximation of Hamilton–Jacobi–Bellman equations with Cordès coefficients

Endre Süli
(joint work with Iain Smears)

Non-divergence form linear second-order elliptic equations with discontinuous coefficients do not generally possess a weak formulation in the Sobolev space H^1, thus presenting an obstacle to their numerical solution by classical finite element methods.

Consider, for example, the boundary-value problem

$$\begin{aligned} Lu &= f \qquad \text{in } \Omega, \\ u &= 0 \qquad \text{on } \partial\Omega, \end{aligned} \tag{1}$$

on a bounded open convex domain Ω in $\mathbb{R}^n$, where $f \in L^2(\Omega)$, and L is a second-order elliptic operator in non-divergence form, i.e., the leading term of L is of the form $\sum_{i,j=1}^n a_{ij}\, u_{x_i x_j}$, with coefficients $a_{ij} \in L^\infty(\Omega)$.

PDEs of this form appear in many areas, including probability theory, stochastic analysis and statistical physics. Such equations also arise as linearizations of fully nonlinear PDEs, as obtained for instance from the use of iterative solution algorithms. In such cases, it can rarely be expected that the coefficients of the operator be smooth or even continuous. For example, in applications to Hamilton–Jacobi–Bellman equations, the coefficients a_{ij} may be merely essentially bounded.

In the case of continuous but possibly non-differentiable coefficients in the differential operator, the Calderón–Zygmund theory of strong solutions establishes the well-posedness of the boundary-value problem (1) in sufficiently smooth domains. However, without additional hypotheses, well-posedness of (1) is generally lost in the case of discontinuous coefficients.

Despite these difficulties, well-posedness, for solutions in the space $H^2(\Omega) \cap H_0^1(\Omega)$, is recovered in convex domains for elliptic operators L with L^∞ coefficients, provided that the coefficients of L satisfy an additional condition, known as the *Cordès condition*; in the case when $Lu := \sum_{i,j=1}^n a_{ij}\, u_{x_i x_j}$, the Cordès condition demands the existence of an $\varepsilon \in (0, 1]$ such that, for a.e. $x \in \Omega$,

$$\frac{|a|^2}{(\mathrm{Tr}\,(a))^2} \leq \frac{1}{n - 1 + \varepsilon}, \tag{2}$$

where $|\cdot|$ represents the Frobenius norm of a matrix and $\mathrm{Tr}\,(\cdot)$ denotes the trace of a matrix. The Cordès condition encompasses a large range of applications. For example, in two spatial dimensions, the condition amounts to simply requiring uniform ellipticity of the diffusion coefficient.

Unlike elliptic equations in divergence form, the literature on the numerical analysis of non-divergence form equations is comparatively sparse. In view of the applications mentioned above, it is important to consider methods that do not assume *a priori* information about the location of the discontinuities of the

coefficients. Our main objective is the construction of stable and convergent high-order finite element methods, and their rigorous mathematical analysis, for non-divergence form linear second-order elliptic PDEs, and the (fully nonlinear) elliptic Hamilton–Jacobi–Bellman equation.

The application of a conforming finite element method to (1) would require at least H^2-regularity of the approximation, which, in turn, amounts to an, excessively restrictive, C^1-continuity condition on the finite element space. For this reason, we consider discontinuous Galerkin (DG) finite element methods, which allow the approximation to be discontinuous between elements in the computational grid, with the continuity conditions being enforced only weakly through the discretized problem. These methods have been analyzed and applied to a large range of PDEs. The ability of DG methods to handle *hp*-refinement, where one varies both mesh size and polynomial degree, is of significant interest here, in view of the potential loss of higher regularity of the solution near discontinuities of the coefficients. Indeed, *hp*-refinement has been used in the context of continuous and discontinuous Galerkin finite element methods to obtain exponential convergence for problems with non-smooth solutions.

We propose a new *hp*-version DG finite element method for linear second-order elliptic PDEs in non-divergence form, satisfying a Cordès condition. It is shown that the method exhibits a convergence rate that is optimal with respect to the mesh size h and suboptimal with respect to the polynomial degree p by only half an order. Numerical experiments demonstrate the accuracy of the method and illustrate the potential of exponential convergence under *hp*-refinement for problems with discontinuous coefficients and nonsmooth solutions. The Cordès condition (2) plays a central role in the numerical analysis of the proposed method (cf. [1]).

We further construct an *hp*-version DG finite element method for the numerical solution of, fully nonlinear, second-order elliptic Hamilton–Jacobi–Bellman (HJB) equations:

$$\sup_{\alpha\in\Lambda} [L^\alpha u - f^\alpha] = 0 \quad \text{in } \Omega, \tag{3}$$

where Ω is a bounded open convex domain in $\mathbb{R}^n$, $n \geq 2$, Λ is a compact metric space, and the L^α, $\alpha \in \Lambda$, are elliptic operators of the form

$$L^\alpha v = \sum_{i,j=1}^{n} a_{ij}^\alpha \, v_{x_i x_j} + \sum_{i=1}^{n} b_i^\alpha \, v_{x_i} - c^\alpha \, v. \tag{4}$$

The function c^α is supposed to be non-negative on $\overline{\Omega}$, for each $\alpha \in \Lambda$. We assume *the Cordès condition*: there exist $\lambda > 0$ and $\varepsilon \in (0, 1]$ such that, for each $\alpha \in \Lambda$,

$$\frac{|a^\alpha|^2 + |b^\alpha|^2/2\lambda + (c^\alpha/\lambda)^2}{(\mathrm{Tr}\,(a^\alpha) + c^\alpha/\lambda)^2} \leq \frac{1}{n+\varepsilon} \quad \text{in } \overline{\Omega}, \tag{5}$$

where $|\cdot|$ represents the Euclidian norm for vectors and the Frobenius norm for matrices. In the special case $b^\alpha \equiv 0$ and $c^\alpha \equiv 0$ for each $\alpha \in \Lambda$, we set $\lambda = 0$ and the Cordès condition (5) is replaced by (2).

HJB equations characterize the value functions of stochastic control problems, which arise from applications in engineering, physics, economics, and finance. The solution of (3) leads to the best choices of controls from the set Λ for steering a stochastic process towards optimizing the expected value of a functional.

We prove the existence and uniqueness of a strong solution (in $H^2(\Omega) \cap H_0^1(\Omega)$) of equation (3). We then construct a stable, consistent, convergent and high-order *hp*-version DG finite element method, for which we prove convergence rates in a discrete H^2-type norm that are optimal with respect to mesh size, and suboptimal in the polynomial degree by only half an order. In contrast with monotone finite difference schemes, the proposed method is consistent regardless of the choice of the mesh, thereby permitting *hp*-refinement on very general shape-regular sequences of meshes. Our numerical experiments show gains in computational efficiency, flexibility, and accuracy over existing monotone methods (cf. [2]).

References

[1] I. Smears and E. Süli. Discontinuous Galerkin finite element approximation of non-divergence form elliptic equations with Cordès coefficients. SIAM J. Numer. Anal. 51 (2013). pp. 2088–2106.

[2] I. Smears and E. Süli. Discontinuous Galerkin finite element approximation of Hamilton–Jacobi–Bellman equations with Cordès coefficients. SIAM J. Numer. Anal. (Submitted, 2013). Available from `http://eprints.maths.ox.ac.uk/1671/`

Clustering and consensus in collective dynamics

Eitan Tadmor

1. Self-organized dynamics. We discuss self-organized dynamics of agent-based models with focus on a prototype model driven by non-symmetric self-alignment. Models for self-alignment dynamics have appeared in a large variety of different contexts, including load balancing in computer networks, evolution of languages, gossiping, algorithms for sensor networks, emergence of flocks, herds, schools and other biological "clustering", pedestrian dynamics, ecological models, multi-agent robots, models for opinion dynamics, economic networks and more; consult [4, 5, 6, 9, 10, 11, 13, 14, 15, 16, 18, 23] and the references therein.

The starting point for our discussion is the evolution of N agents, each of which is identified by its "position" $\mathbf{p}_i(t) \in \mathbb{R}^d$. The position $\mathbf{p}_i(t)$ may account for opinion, velocity, or other attributes of agent "i" at time t. Each agent adjusts its position according to the position of his neighbors:

$$\frac{d}{dt}\mathbf{p}_i = \alpha \sum_{j \neq i} a_{ij}(\mathbf{p}_j - \mathbf{p}_i), \qquad a_{ij} \geq 0.$$

Here, $\alpha > 0$ is a scaling parameter and the coefficients a_{ij} quantify the strength of interaction between agents i and j: the larger a_{ij} is, the more weight is given to agent j to align itself with agent i, based on the difference of their positions $\mathbf{p}_i - \mathbf{p}_j$. The underlying fundamental assumption here is that agents do not react to the *position* of others but to their differences relative to other agents. In particular, the

a_{ij}'s themselves are allowed to depend on the relative differences, $\mathbf{p}_i - \mathbf{p}_j$. Indeed, we consider nonlinear models in the sense that the coefficients may depend on the position, $a_{ij} = a_{ij}(\mathcal{P}(t)),\ \mathcal{P}(t) := \{\mathbf{p}_k(t)\}_k$.

2. Unconditional consensus and flocking. We distinguish between two main classes of self-alignment models. In the global case, the rules of engagement are such that every agent is influenced by every other agent, $a_{ij} > \eta > 0$. The dynamics in this case is driven by *global* interactions. We have a fairly good understanding of the large time dynamics of such models, e.g., [3, 6, 8, 20, 21, 24]: global interactions which are sufficiently strong lead to *unconditional consensus* in the sense that *all* initial configurations of agents concentrate around an emerging limit state, the "consensus" $\mathbf{p}^\infty :\ \mathbf{p}_i(t) \overset{t\to\infty}{\longrightarrow} \mathbf{p}^\infty$.

In more realistic models, however, interactions between agents are limited to their local neighbors, [1, 2, 7, 22]. The behavior of *local* models where some of the a_{ij} may vanish, requires a more intricate analysis. In the general scenario for such local models, agents tend to concentrate into one or more separate *clusters*. The particular case in which agents concentrate into one cluster, that is the emergence of a consensus or a flock, depends on the propagation of *uniform connectivity* of the underling (weighted) graph associated with the adjacency matrix, $\{a_{ij}\}$. Thus, the question of consensus for local models is resolved here in terms of persistence of connectivity over time. However, even if the initial configuration is assumed connected, there is still a possibility of losing connectivity as the a_{ij}'s may vary in time together with the positions $\mathcal{P}(t)$. This leaves open the main question of tracing the propagation of connectivity in time.

3. Heterophilious interactions. Many standard models for self-organized dynamics in social, biological and physical science make the intuitive assumption that the intensity of alignment increases as agents get closer, reflecting a common tendency to align with those who think or act alike. "Birds of feather flock together" reflects our intuition that increasing intensity of alignment as the difference of positions decreases, is more likely to lead to a consensus. We argue in [21] that the converse is true: heterophily — the tendency to bond more with those who are different rather than with those who are similar, plays a decisive role in the process of clustering. In particular, sufficiently strong heterophilious interactions enhance consensus. We are not unaware that this phenomenon of enhanced consensus in the presence of heterophilious interactions, may have intriguing consequences in different areas other than social networks, e.g., global bonding in atomic scales, avoiding materials' fractures in mesoscopic scales, or "cloud" formations in macroscopic scales.

References

[1] I. Aoki, A simulation study on the schooling mechanism in fish *Bull. Japanese Society of Scientific Fisheries*, 48(8), 1081-1088, 1982.

[2] M. Ballerini, N. Cabibbo, R. Candelier, A. Cavagna, E. Cisbani, I. Giardina, V. Lecomte, A. Orlandi, G. Parisi, A. Procaccini, M. Viale and V. Zdravkovic, Interaction ruling

animal collective behavior depends on topological rather than metric distance *PNAS*, 105(4), 1232-1237, 2008.

[3] V. D. Blondel, J. M. Hendricks, and J. N. Tsitsiklis, On Krause's multi-agent consensus model with state-dependent connectivity. *Automatic Control, IEEE Transactions on*, 54(11):2586–2597, 2009.

[4] S. Camazine, J. L. Deneubourg, N. R Franks, J. Sneyd, G. Theraulaz, and E. Bonabeau, Self-organization in biological systems. *Princeton University Press; Princeton, NJ: 2001*, 2001.

[5] R. Carli, F. Fagnani, A. Speranzon, and S. Zampieri, Communication constraints in the average consensus problem. *Automatica*, 44, 671-684, 2008.

[6] I.D. Couzin, J. Krause, N.R. Franks, and S.A. Levin, Effective leadership and decision-making in animal groups on the move. *Nature*, 433(7025):513-516, 2005.

[7] I. D. Couzin, J. Krause, R. James, G. D. Ruxton and N. R. Franks, Collective Memory and Spatial Sorting in Animal Groups *J. of Theoretical Biology*, 218(1), 1-11, 2002.

[8] F. Cucker and S. Smale. Emergent behavior in flocks. *IEEE Transactions on automatic control*, 52(5):852, 2007.

[9] F. Cucker, S. Smale and D.X. Zhou, Modeling language evolution. *Found. Comput. Math.*, 4, 315-343, 2006.

[10] F. Fagnani and S. Zampieri, Randomized consensus algorithms over large scale networks. *IEEE J. Selected Areas of Communications* , 26, 634-649, 2008.

[11] V. Grimm and S. F Railsback, *Individual-based modeling and ecology*. Princeton Univ Pr, 2005.

[12] S. Y. Ha and E. Tadmor, From particle to kinetic and hydrodynamic descriptions of flocking. *Kinetic and Related Models*, 1(3):415–435, 2008.

[13] D. Helbing, Traffic and related self-driven many particle systems. *Reviews of Modern Physics*, 73, 1067-1141, 2001.

[14] C. K Hemelrijk and H. Hildenbrandt, Self-organized shape and frontal density of fish schools. *Ethology*, 114(3):245–254, 2008.

[15] M. O. Jackson, Social and Economic Networks. Princeton University Press, 2010.

[16] A. Jadbabaie, J. Lin, and A.S. Morse, Coordination of groups of mobile autonomous agents using nearest neighbor rules. *IEEE Trans. Automat. Control*, 48, 988-1001, 2003.

[17] P. F. Lazarsfeld, R. K. Merton, Friendship as a Social Process: A Substantive and Methodological Analysis. In "Freedom and Control in Modern Society" (M. Berger, T. Abel, and C. H. Page, eds.) New York, Van Nostrand, 18-66, 1954.

[18] J.Lorenz, Continuous opinion dynamics of multidimensional allocation problems under bounded confidence. A survey. *Internat. J. Modern Phys. C*, 18, 1819-1838, 2007.

[19] M. McPherson, L. Smith-Lovin, and J. M Cook, Birds of a Feather: Homophily in Social Networks. *Annual Review of Sociology*, Vol. 27: 415-444 2001.

[20] S. Motsch and E. Tadmor, A new model for self-organized dynamics and its flocking behavior. *Journal of Statistical Physics*, 144(5):923–947, August 2011.

[21] S. Motsch and E. Tadmor. Heterophilious dynamics enhances consensus arXiv:1301.4123, 2013.

[22] C. W. Reynolds, Flocks, herds and schools: A distributed behavioral model. In *ACM SIGGRAPH Computer Graphics*, 21, 25-34, 1987.

[23] J. Toner and Y. Tu, Flocks, herds, and schools. A quantitative theory of flocking. *Physical Review E*. 58, 4828-4858 (1998).

[24] T. Vicsek, A. Czirók, E. Ben-Jacob, I. Cohen, and O. Shochet, Novel type of phase transition in a system of self-driven particles. *Physical Review Letters*, 75(6), 1226–1229, 1995.

Reduced basis methods for transport dominated problems

Gerrit Welper
(joint work with Wolfgang Dahmen and Christian Plesken)

The reduced basis method has been successfully applied to a wide range of problems including elliptic and parabolic parametric PDEs. However, its application to transport dominated problems is still difficult and the main obstructions are examined in the first part of the talk. Especially, we wish to apply the known greedy theory from [2, 1, 7], to find reduced bases that achieve optimal convergence rates with respect to the Kolmogorov n-width. This amounts to finding new snapshots satisfying the weak greedy condition: Find a parameter μ_{n+1} such that

$$\inf_{\varphi\in X_n} \|u_{\mu_{n+1}} - \varphi\| \geq c \sup_{\mu} \inf_{\varphi\in X_n} \|u - \varphi\|$$

where X_n is the current reduced basis, u_μ is the solution of the parametric PDE for the parameter μ and $c \geq 0$ is a constant. We follow the typical approach to solve this optimization problem by providing two major ingredients for the transport dominated case:

(1) A numerical scheme that produces, up to constants, a best approximation $u_{\mu,n} \in X_n$, i.e.
$$\|u_\mu - u_{\mu,n}\| \leq C \inf_{\varphi\in X_n} \|u_\mu - \varphi\|,$$
to the solution u_μ of the parametric PDE for a fixed parameter μ, where $C > 0$ is a constant.
(2) An error estimate for $\|u_\mu - u_{\mu,n}\|$.

While these two ingredients are well understood in the elliptic case, they are typically not satisfied by standard methods for transport dominated problems, like SUPG, variational multiscale or DG schemes. However, they are provided by some more recent methods from [5, 6, 3, 4]. These schemes can be understood as a "stabilization on the infinite dimensional level" by first providing two, eventually parameter dependent Hilbert spaces such that the operator A_μ of the parametric PDE is an isomorphism satisfying a mapping property

$$\|\cdot\|_{X_\mu} \sim \|A_\mu \cdot\|_{Y'_\mu}.$$

For convection dominated convection-diffusion problems, it is ensured that the involved constants do not depend on the Péclet number, in fact the norms can be chosen such that the constants are one.

This mapping property allows the construction of residual based error estimators presented in the talk. In addition, it also implies that the infinite dimensional operator satisfies an inf-sup condition. It follows that a Petrov-Galerkin method yields a best approximation (up to constants) from the reduced basis if we can guarantee a discrete inf-sup condition. However, the corresponding test spaces are typically parameter dependent, whereas we wish to have only one single reduced test space for computational efficiency. To this end, the Petrov-Galerkin method is reformulated as a saddle point problem which requires the same inf-sup condition

but allows a test space which is larger than the trial space. This freedom is then used to find a test space that achieves a discrete inf-sup condition uniformly in the parameter.

To actually compute these test spaces, we use a greedy strategy (see [8]) by inductively adding a test function for the parameter and trial function with worst discrete inf-sup constant. For singularly perturbed convection-diffusion equations, it is shown that this "inner" greedy loop terminates after $\mathcal{O}(n)$ steps where n is the size of the reduced trial space.

These constructions provide all ingredients of the indicated ansatz to achieve the week greedy property. This yields the main result of the talk: The proposed method produces reduced bases for which the error of the reduced basis approximation has the same rate as the Kolmogorov n-width.

Finally, the theoretical statements are supplemented by numerical tests.

References

[1] P. Binev, A. Cohen, W. Dahmen, R. DeVore, G. Petrova, P. Wojtaszczyk, Convergence Rates for Greedy Algorithms in Reduced Basis Methods, SIAM J. Math. Anal., 43 (2011), 1457–1472.

[2] A. Buffa, Y. Maday, A.T. Patera, C. Prud'homme, and G. Turinici, A Priori convergence of the greedy algorithm for the parameterized reduced basis, ESAIM: Mathematical Modelling and Numerical Analysis, 46 (03) (2012), 595-603.

[3] A. Cohen, W. Dahmen, G. Welper, Adaptivity and Variational Stabilization for Convection-Diffusion Equations, ESAIM: Mathematical Modelling and Numerical Analysis, 46 (5) (2012), 1247 - 1273.

[4] W. Dahmen, C. Huang, C. Schwab, G. Welper, Adaptive Petrov-Galerkin methods for first order transport equations, SIAM J. Numer. Anal., 50(5) (2012), 2420–2445.

[5] L. Demkowicz and J. Gopalakrishnan, A class of discontinuous Petrov-Galerkin Methods I: The transport equation, Comput. Methods Appl. Mech. Engrg., 199 (No. 23-24) (2010), pp. 1558–1572.

[6] L. Demkowicz, J. Gopalakrishnan, A class of discontinuous Petrov-Galerkin methods. Part II: Optimal test functions, Numerical Methods for Partial Differential Equations, 27 (No.1) (2011), pp. 70 - 105.

[7] R. DeVore, G. Petrova, P. Wojtaszczyk, Greedy algorithms for reduced bases in Banach spaces, preprint, April 2012, to appear in Constructive Approximation.

[8] A. Gerner, K. Veroy-Grepl, Certified reduced basis methods for parametrized saddle point problems, preprint 2012, SIAM J. Sci. Comput. accepted for publication.

Properties of greedy algorithm used in reduced basis method

Przemek Wojtaszczyk

Reduced basis: Let $\Delta \subset R^d$ be a compact set e.g. $\Delta = [0,1]^d$ with d big and let

$$D_\mu f = g \quad \mu \in \Delta \tag{1}$$

be a family of problems. For a given $\mu \in \Delta$ solving (1) is time consuming.

However we want to prepare ourselves to do it fast – on line. The idea of *reduced basis method* is the following. We solve (1) for $\mu_1, \dots, \mu_n$ to get $f_{\mu_1}, \dots, f_{\mu_n}$ and

for given μ we approximate the solution

$$f_\mu \sim \sum_{j=1}^{n} a_j f_{\mu_j}.$$

The real question is how to find basis elements? Greedy selection was suggested by Maday-Patera-Turinici, 2002 and used many times in various problems.

Let $K =: \{f_\mu \ : \ \mu \in \Delta\}$ be a compact subset of certain Banach space X or a Hilbert space H.

We define $\mu_1, \dots \mu_n$ as follows ($f_j =: f_{\mu_j}$)

(1) $f_1 = \text{argmax}\{\|f\| \ : \ f \in K\}$
(2) Given $f_1, \dots, f_n$ we define $E_n = \text{span}\,\{f_1, \dots, f_n\}$ and put $f_{n+1} = \text{argmax}\{\text{dist}\,(f, E_n) \ : \ f \in K\}$

To estimate the performance of this procedure we define $\sigma_n(K) = \sup\limits_{f\in K} \text{dist}\,(f, E_n)$.

Recall the classical Kolmogorov width

$$d_n(K) = \inf_F \sup_{f\in K} \text{dist}(f, F)$$

for F a subspace of dimension $\le n$. Clearly always $d_n(K) \le \sigma_n(K)$ so we need to estimate $\sigma_n(K)$ from above by $d_m(K)$.

Among other things the following sample results will be discussed.

Note that $f_n \in K$, so E_n is spanned by a basis of functions from K. There is no such requirement in the definition of Kolmogorov widths. We define

$$\bar{d}_n(K) = \inf_F \sup_{f\in K} \text{dist}(f, F)$$

where F is a subspace of dimension $\le n$ spanned by elements from K.

Theorem [1] *The following holds:*

(i) *For any compact set $K \subset H$ and any $n \ge 0$, we have*

$$\bar{d}_n(K) \le (n+1) d_n(K).$$

(ii) *Given any $n > 0$ and $\epsilon > 0$, there is a set $K \subset H$ such that*

$$\bar{d}_n(K) \ge (n - 1 - \epsilon) d_n(K).$$

It is notable that in many cases this extra factor is not relevant.

Theorem [2] *For the greedy algorithm in a Hilbert space H and for any compact set K and $N \ge 1$, $N = km$ with $k \ge 1$ we have*

$$\sigma_{km}(K) \le \sqrt{2}(d_m(K))^{1-\frac{1}{k}}$$

in particular

$$\sigma_{2m}(K) \le \sqrt{2}\sqrt{d_m(K)},$$

$n = 1, 2 \dots$.

Theorem [1, 2] *If $d_n(K) \le C_0 n^{-\alpha}$, $n = 1, 2, \dots$, then $\sigma_n(K) \le C_1 n^{-\alpha}$, $n = 1, 2 \dots$, with $C_1 := 2^{5\alpha+1} C_0$.*

Similar results hold in the Banach space setting [2] but the right hand side has an extra factor of order $\sqrt{n}$ or somewhat bigger.

References

[1] Peter Binev, Albert Cohen, Wolfgang Dahmen, Ronald DeVore, Guergana Petrova, P.W., Convergence Rates for Greedy Algorithms in Reduced Basis Methods SIAM J. Math. Anal. vol.43 N.3 pp. 1457-1472 (2011)
[2] Ronald DeVore, Guergana Petrova, P.W., Greedy Algorithms for Reduced Basis in Banach Spaces Constructive Approximation 37N.3 pp.455-466 (2013)
[3] A. Buffa, Y. Maday, A.T. Patera, C. Prud'homme, and G. Turinici, A Priori convergence of the greedy algorithm for the parameterized reduced basis, M2AN Math. Model. Numer. Anal., **46**(2012), 595–603.

Tractability of multivariate problems

Henryk Woźniakowski

Many multivariate problems suffer from the curse of dimensionality. This is usually for multivariate problems defined in the worst case setting over isotropic classes of functions for which the role of each group of variables is the same. We illustrate the curse by two examples of multivariate integration defined for classes of smooth functions. We indicate three possible ways to vanquish the curse of dimensionality. They are:

- increasing smoothness with respect to successive variables,
- monitoring different groups of variables by decreasing weights,
- switching to a more lenient setting such as the randomized or the average case setting.

The thorough study of various tractability issues can be found in [1, 2, 3].

References

[1] E. Novak and H. Woźniakowski, Tractability of Multivariate Problems, Volume I: Linear Information, European Mathematical Society, Zürich, 2008.
[2] E. Novak and H. Woźniakowski, Tractability of Multivariate Problems, Volume II: Standard Information for Functionals, European Mathematical Society, Zürich, 2010.
[3] E. Novak and H. Woźniakowski, Tractability of Multivariate Problems, Volume III: Standard Information for Operators. European Mathematical Society, Zürich, 2010.

Regularity and approximability of the solutions to the chemical master equation

Harry Yserentant
(joint work with Ludwig Gauckler)

The chemical master equation is a fundamental equation in chemical kinetics. It is an adequate substitute for the classical reaction-rate equations whenever stochastic effects become relevant. In the present talk we gave a simple argument showing that the solutions of a large class of chemical master equations, including all those in which elementary reactions between two and more molecules do not generate a larger number of molecules than existed before, are bounded in polynomially weighted ℓ_1-spaces. As an illustration for the implications of this kind

of regularity we analyzed the effect of truncating the state space. This leads to an error analysis of the finite state projection of the chemical master equation, an approximation that underlies many numerical methods.

The chemical master equation can be regarded as a differential-difference equation, a differential equation in time and a difference equation in the rest of the variables, or in other words a possibly infinite system

$$\frac{\mathrm{d}}{\mathrm{d}t}\, p(\Omega;t,x) = \sum_{r=1}^{R} \Big(a_r(t,x-\nu_r)\, p(\Omega;t,x-\nu_r) - a_r(t,x)\, p(\Omega;t,x) \Big), \quad x \in \Omega,$$

of ordinary differential equations. The solution components are labeled by vectors $x \in \mathbb{Z}^d$ and take the value $p(\Omega;t,x) = 0$ for all x outside a given subset Ω of $\mathbb{Z}^d$. The ν_r are given elements in $\mathbb{Z}^d$ and the coefficient functions $a_r(t,x)$ are nonnegative and continuous in the variable t for each single x.

The existence and uniqueness of a minimal nonnegative solution for given summable, nonnegative initial values can be shown by means of a monotonicity argument due to Reuter and Ledermann [1]. We have shown a priori estimates

$$\sum_{x\in\Omega} \gamma(x)\, p(\Omega;t,x) \;\le\; c(t) \sum_{x\in\Omega} \gamma(x)\, p(\Omega;0,x)$$

for these solutions, with weight functions $\gamma : \mathbb{Z}^d \to \mathbb{R}_{\ge 0}$ satisfying an estimate

$$\sum_{r=1}^{R} a_r(t,x)\big(\gamma(x+\nu_r) - \gamma(x)\big) \;\le\; \kappa(t)\,\gamma(x)$$

for all x in the domains Ω under consideration. In particular, we studied the case of the set $\Omega = \mathbb{N}_0^d$ that is relevant for chemical kinetics. Assume that the coefficient functions $a_r(t,x)$ grow at most linearly in x for those indices r for which the components of ν_r sum up to a positive value. This condition is satisfied in chemical kinetics if one restricts oneself to systems in which reactions between two and more molecules are excluded that generate a larger number of molecules than existed before. For all multi-indices $\alpha > 0$ the moments

$$\sum_{x\in\Omega} x^{\alpha} p(\Omega;t,x)$$

then remain finite and can be estimated by the moments of the initial values. This fact can be used to study the convergence rate of finite state projections $p(\Omega_n;t,x)$ in which the solution on the whole set $\Omega = \mathbb{N}_0^d$ is replaced by solutions $p(\Omega_n;t,x)$ on finite subsets Ω_n of $\mathbb{N}_0^d$, with same initial values on Ω_n. Choosing

$$\Omega_n = \big\{ x \in \mathbb{N}_0^d \,\big|\, x_1 + \ldots + x_d < n \big\},$$

the ℓ_1-error tends super-algebraically to zero, corresponding initial values provided.

References

[1] Reuter, G. E. H., Ledermann, W: On the differential equations for the transition probabilities of Markov processes with enumerably many states. *Proc. Cambridge Philos. Soc.* 49:247–262, 1953.

[2] Gauckler, L., Yserentant, H.: Regularity and approximability of the solutions to the chemical master equation. Matheon-Preprint 1010 (2013), urn:nbn:de:0296-matheon-12146.

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