Mathematisches Forschungsinstitut Oberwolfach

Partial Differential Equations

Organised by
Alice Chang (Princeton)
Camillo De Lellis (Zürich)
Reiner Schätzle (Tübingen)

4 August – 10 August 2013

ABSTRACT. The workshop dealt with partial differential equations in geometry and technical applications. The main topics were the combination of nonlinear partial differential equations and geometric problems, and fourth order equations in conformal geometry.

Introduction by the Organisers

The workshop *Partial differential equations*, organised by Alice Chang (Princeton), Camillo DeLellis (Zürich), and Reiner Schätzle (Tübingen) was held August 4-10, 2013. This meeting was well attended by 51 participants, including 8 females, with broad geographic representation. The program consisted of 15 talks and 9 shorter contributions and left sufficient time for discussions.

There were several contributions to regularity of solutions of partial differential equations and to geometric flows, for example concerning the Monge-Ampere equation, minimizers of the Mumford-Shah functional, the structure of branch sets of stable minimal submanifolds, mean curvature flow and Ricci flow.

A major part of the leading experts of partial differential equations with conformal invariance attended the workshop. Here new results were presented in conformal geometry, for the Yamabe type problem, for uniformization and critical metrics.

The organisers and the participants are grateful to the Oberwolfach Institute for presenting the opportunity and the resources to arrange this interesting meeting.

Workshop: Partial Differential Equations

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Abstracts

A new conformally covariant operator in CR geometry

JEFFREY S. CASE

(joint work with Paul Yang)

In four-dimensional conformal geometry, the pair (P_4, Q_4) of the Paneitz operator and the Q-curvature generalize in many ways properties of the pair $(-\Delta, K)$ of the Laplacian and the Gauss curvature on surfaces. For example, P_4 is a fourth-order conformally covariant self-adjoint operator of the form Δ^2 plus lower-order terms, P_4 kills the constants, and if $\hat{g} = e^{2\sigma} g$ is a conformal change of metric, then

$$(0.1) \qquad e^{4\sigma} \hat{Q}_4 = Q_4 + P_4(\sigma).$$

In particular, the total Q-curvature of a compact manifold is a conformal invariant, and indeed one readily checks that the Euler characteristic can be expressed as a linear combination of the total Q-curvature and the integral of a pointwise conformal invariant. The pair (P_4, Q_4) also appear in the sharp Moser–Trudinger–Onofri inequality

$$(0.2) \qquad \int_{S^4} P_4(u)u + 2 \int_{S^4} Q_4 u - \frac{1}{2} \left(\int_{S^4} Q_4 \right) \log \fint e^{4u} \geq 0$$

proven by Beckner [1]. In the curved case, work of Chang and Yang [5] shows that minimizers of the left hand side of (0.2) exist as long P_4 is nonnegative with $\ker P_4 = \mathbb{R}$ and the total Q-curvature is strictly less than the corresponding value of the sphere. Gursky [8] showed that these assumptions can be checked using simpler conformal invariants. For example, he gave a characterization of the conformal four-sphere by showing that if (M^4, g) is a compact Riemannian manifold with nonnegative Yamabe constant, then

$$\int_M Q_4 \leq 8\pi^2$$

with equality if and only if (M^4, g) is conformally equivalent to the standard four-sphere.

A natural question is whether there exists a CR analogue of the pair (P_4, Q_4) for three-dimensional CR manifolds. It has been known for some time that there is a pair (P_4, Q_4) of a fourth-order conformally covariant operator and its associated Q-curvature which satisfy the analogue of the transformation formula (0.1). However, one does not expect this pair to be so closely related to Moser–Trudinger inequalities and topological questions for two reasons. First, the kernel of the CR Paneitz operator contains the CR pluriharmonic functions, which is an infinite-dimensional vector space. Second, the total Q-curvature of a compact three-dimensional CR manifold is always zero, and the Q-curvature itself vanishes identically for contact forms on boundaries of strictly pseudoconvex domains which arise by formally solving Fefferman's Monge-Ampère equation [7].

Recent work of Branson, Fontana and Morpurgo [2] constructed on the standard CR three-sphere a fourth-order conformally covariant operator acting *only* on the CR pluriharmonic functions for which an analogue of the Beckner–Onofri inequality (0.2) holds. This suggests that one should look for the abstract analogue of this operator as the "right" analogue of the Paneitz operator.

In my talk, I described joint work with Paul Yang where we construct on a general three-dimensional pseudohermitian manifold (M^3, J, θ) a fourth-order operator P_4' which, when restricted to the space $\mathcal{P}$ of CR pluriharmonic functions, is conformally-covariant and self-adjoint. Formally, this operator arises as the "Q-curvature operator" associated to the Paneitz operator, meaning that one can define $P_4' \colon \mathcal{P} \to \mathbb{R}$ as the limit

$$P_4' = \lim_{n \to 1} \frac{2}{n-1} P_{4,n}|_{\mathcal{P}}$$

of the Paneitz operator on a $(2n+1)$-dimensional CR manifold via Branson's argument of analytic continuation in the dimension. In particular, P_4' has the property that if $\hat{\theta} = e^\sigma \theta$, then

$$e^{2\sigma} P_4'(u) = P_4'(u) + P_4(u\sigma)$$

for all $u \in \mathcal{P}$. This implies that $e^{2\sigma} \hat{P}_4' = P_4'$ mod $\mathcal{P}^\perp$, giving the sense in which we regard P_4' as a Paneitz-type operator.

For a distinguished class of contact forms, we can also associate to P_4' a scalar invariant Q_4'. In general, $P_4'(1) = Q_4$, which need not vanish. However, if θ is a so-called pseudo-Einstein contact form, then Q_4 vanishes identically, and we can formally define the Q_4' curvature by

$$Q_4' = \lim_{n \to 1} \frac{4}{(n-1)^2} P_{4,n}(1).$$

If θ is pseudo-Einstein, then $\hat{\theta} = e^\sigma \theta$ is pseudo-Einstein if and only if $\sigma \in \mathcal{P}$, and in this case we see that the Q'-curvature transforms by

(0.3) $$e^{2\sigma} \hat{Q}_4' = Q_4' + P_4'(\sigma) + \frac{1}{2} P_4(\sigma^2).$$

This implies that $e^{2\sigma} \hat{Q}_4' = Q_4' + P_4'(\sigma)$ mod $\mathcal{P}^\perp$, giving the sense in which we regard Q_4' as the Q-curvature associated to P_4'. This also implies that for compact three-dimensional pseudo-Einstein manifolds, the total Q'-curvature is an invariant of the class of pseudo-Einstein contact forms. Additionally, on the standard CR three-sphere $Q_4' = 1$, so that the work of Branson, Fontana and Morpurgo [2] gives the sharp inequality

(0.4) $$\int_{S^3} P_4'(u)u + 2 \int_{S^3} Q_4' u - \left(\int_{S^3} Q_4' \right) \log \fint e^{2u} \geq 0$$

for all $u \in W^{2,2} \cap \mathcal{P}$.

We are also able to generalize to the CR setting most of the results pertaining to the Paneitz operator and Q-curvature of a four-dimensional manifold mentioned above. First, for pseudo-Einstein manifolds, the Burns–Epstein invariant [3, 6] is

a nonzero multiple of the total Q'-curvature. In particular, we have the following result.

Theorem 1. *Let (M^3, J) be the boundary of a strictly pseudoconvex domain X, then*

$$\int_X \left(c_2 - \frac{1}{3} c_1^2 \right) = \chi(X) - \frac{1}{16\pi^2} \int_M Q' \, \theta \wedge d\theta,$$

where c_1 and c_2 are the first and second Chern forms of the Kähler–Einstein metric in X obtained by solving Fefferman's equation and $\chi(X)$ is the Euler characteristic of X.

Generalizing Gursky's result [8], we can also give a characterization of the standard CR three-sphere via the total Q'-curvature.

Theorem 2. *Let (M^3, J, θ) be a compact three-dimensional pseudo-Einstein manifold with nonnegative Paneitz operator and nonnegative CR Yamabe constant. Then*

$$\int_M Q_4' \, \theta \wedge d\theta \leq 16\pi^2,$$

with equality if and only if (M^3, J) is CR equivalent to the standard CR three sphere.

Finally, we have a partial generalization of the result of Chang and Yang [5] concerning the minimization of the conformal II-functional.

Theorem 3. *Let (M^3, J, θ) be a compact three-dimensional pseudo-Einstein manifold such that $\int Q_4' < 16\pi^2$ and $P_4' \geq 0$ with $\ker P_4'$ consisting only of the constant functions. Define the functional $II \colon \mathcal{P} \to \mathbb{R}$ by*

$$II(u) = \int_M P_4'(u)u + 2 \int_M Q_4' u - \left(\int_M Q_4' \right) \log \fint e^{2u}.$$

Then there exists a $w \in W^{2,2} \cap \mathcal{P}$ such that

$$II(w) = \inf \left\{ II(u) \colon u \in W^{2,2} \cap \mathcal{P} \right\}.$$

In particular, for a natural class of boundaries of pseudoconvex domains, we can use the P'-operator to obtain estimates for the space of CR pluriharmonic functions. We expect to be able to prove that w is smooth and that the contact form $e^{2w}\theta$ has constant Q'-curvature.

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Higher integrability for the gradient of minimizers of the Mumford-Shah functional

Guido De Philippis

1. Introduction

Free discontinuity problems are a class of variational problems which involve pairs (u, K) where K is some closed set and u is a function which minimizes some energy outside K. One of the most famous examples is given by the Mumford-Shah energy functional, which arises in image segmentation [11]: given a open set $\Omega \subset \mathbb{R}^n$, for any $K \subset \Omega$ relatively closed and $u \in W^{1,2}(\Omega \setminus K)$, one defines the *Mumford-Shah energy* of (u, K) in Ω to be

$$MS(u, K)[\Omega] := \int_{\Omega \setminus K} |\nabla u|^2 + \mathcal{H}^{n-1}(K \cap \Omega).$$

We say that the pair (u, K) is a *local minimizer* for the Mumford Shah energy in Ω if, for every ball $B = B_\varrho(x) \Subset \Omega$,

$$MS(u, K)[B] \leq MS(v, H)[B]$$

for all pairs (v, H) such that $H \subset \Omega$ is relatively closed, $v \in W^{1,2}(\Omega \setminus H)$, $K \cap (\Omega \setminus B) = H \cap (\Omega \setminus B)$, and $u = v$ almost everywhere in $(\Omega \setminus B) \setminus K$. We denote the set of local minimizers in Ω by $\mathcal{M}(\Omega)$. We refer to [2, 4] for a general account on the theory.

In [6], De Giorgi formulated a series of conjectures on the properties of local minimizers. One of them states as follows [6, Conjecture 1]:

Conjecture (De Giorgi): *If (u, K) is a (local) minimizer of the Mumford-Shah energy inside Ω, then there exists $\gamma \in (1, 2)$ such that $|\nabla u|^2 \in L^\gamma(\Omega' \setminus K)$ for all $\Omega' \subset\subset \Omega$.*

A positive answer to the above conjecture was given in [8] when $n = 2$. The proof there strongly relies on the two-dimensional assumption, since it uses the description of minimal Caccioppoli partitions. Here I present a recent result obtained in collaboration with Alessio Figalli which provides a positive answer to the De Giorgi conjecture.

Theorem 1.1. [5] *There exist dimensional constants $\bar{C} > 0$ and $\bar{\gamma} = \bar{\gamma}(n) > 1$ such that, for all $(u, K) \in \mathcal{M}(B_2)$,*

$$\tag{1.1} \int_{B_{1/2} \setminus K} |\nabla u|^{2\bar{\gamma}} \leq \bar{C}.$$

2. Idea of the proof

The proof of the above Theorem is based on the following key property of the set K which follows combining the regularity theory of [1] with [9, 10, 12].

There exists a constant C such that for all $x \in K \cap B_1$ and all $\varrho < 1$ there exists a $y \in B_{\varrho/2}(x) \cap K$, a unit vector $\bar{\nu}$ and a $C^{1,1/4}$ function $f : \mathbb{R}^{n-1} \to \mathbb{R}$ satisfying

$$K \cap B_{2\varrho/C}(y) = \left[y + \operatorname{graph}_{\bar{\nu}}(f) \right] \cap B_{2\varrho/C}(y),$$

where

$$\operatorname{graph}_{\bar{\nu}}(f) := \left\{ z \in \mathbb{R}^n : \; z \cdot \bar{\nu} = f\left(z - (\bar{\nu} \cdot z)z \right) \right\}.$$

In other words the (closed) set Σ where K is not a C^1 manifold is *porous* in K. This immediately implies that $\dim_{\mathcal{H}}(\Sigma) < n - 1$. Relying on this result the proof of the higher integrability goes as follows:
Let $M \gg 1$ and for $h \in \mathbb{N}$ define the following set

$$\tag{2.1} A_h := \left\{ x \in B_2 \setminus K \text{ such that } |\nabla u(x)|^2 \geq M^{h+1} \right\}.$$

The goal is to show that $|A_h| \lesssim M^{-h(1+\varepsilon)}$ for some positive ε. Since u is harmonic outside K and the integral of $|\nabla u|^2$ over a ball of radius r is controlled by r^{n-1} it follows by elliptic regularity that A_h is contained in a M^{-h}-neighborhood of K. However, for the set K we have a porosity estimate. This together with classical estimates for the Neumann problem, tells us that inside every ball of radius ϱ there is a ball of comparable radius where $|\nabla u|^2 \leq C/\varrho$. Using this one can show that the size of A_h is smaller than what one would get by just using that A_h is contained in a M^{-h}-neighborhood of K. Indeed, by induction over h we can show that A_h is contained in the M^{-h}-neighborhood of a set K_h obtained from K_{h-1} by removing the "good balls" where the gradient is not too big. One can then shows that the $\mathcal{H}^{n-1}$ measure of K_h decays geometrically, this allows us to obtain a stronger estimate on the size of A_h which immediately implies the desired higher integrability.

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Entropy and differential Harnack type formulas for evolving domains
Klaus Ecker

1. Introduction

For bounded open subsets $\Omega \subset \mathbb{R}^N$ with smooth boundary, smooth functions $f : \overline{\Omega} \to \mathbb{R}$ and $\beta : \partial\Omega \to \mathbb{R}$ and for $\tau > 0$ the quantity

$$\mathcal{W}_\beta(\Omega, f, \tau) = \int_\Omega \left(\tau|\nabla f|^2 + f - (n+1)\right) u \, dx + 2\tau \int_{\partial\Omega} \beta u \, dS$$

with

$$u = \frac{e^{-f}}{(4\pi\tau)^{\frac{n+1}{2}}}$$

and the associated entropy

$$\mu_\beta(\Omega, \tau) = \inf\left\{\mathcal{W}_\beta(\Omega, f, \tau) , \int_\Omega u \, dx = 1\right\}$$

were studied in [E].

These generalize expressions introduced by Perelman in [P] where instead a compact Riemannian manifold is considered that is, in particular, no boundary integral appears and the integrand above has an extra summand which is the scalar curvature of that manifold. The metric additionally enters via the volume element and the squared length of the gradient. Perelman proved that for a family of metrics evolving by Ricciflow, a family of functions solving a suitable version of the backward heat equation and for a time-dependent parameter τ with time derivative equal to -1 his entropy is monotonically increasing in time. He then used this to establish an important local non-collapsing result for Ricciflow which

we will explain below in detail within our context.

In [E], we investigated an analogous situation to Perelman's. This can be described as follows: Consider a family of bounded open regions $(\Omega_t)_{t \in (0,T)}$ in $\mathbb{R}^N$ with smooth boundary hypersurfaces $M_t = \partial \Omega_t$ evolving with smooth normal speed $\beta = \beta_{M_t}$ given by

$$\beta_{M_t} = -\frac{\partial x}{\partial t} \cdot \nu.$$

Here x denotes the embedding map of M_t and ν is the normal to M_t pointing out of Ω_t. We then derived a formula which states that if $\tau = \tau(t) > 0$ satisfies $\frac{\partial \tau}{\partial t} = -1$, f solves the evolution equation

$$\frac{\partial f}{\partial t} + \Delta f = |\nabla f|^2 + \frac{n+1}{2\tau}$$

in Ω_t with Neumann boundary condition

$$\nabla f \cdot \nu = \beta$$

on $M_t = \partial \Omega_t$ and we introduce a family of diffeomorphisms $\varphi_t : \overline{\Omega} \to \overline{\Omega}_t$ with $x = \varphi_t(q)$, $q \in \overline{\Omega}$ satisfying

$$\frac{\partial x}{\partial t} = -\nabla f(x,t)$$

then

$$\frac{d}{dt} \mathcal{W}_\beta(\Omega_t, f(t), \tau(t)) = 2\tau(t) \int_{\Omega_t} \left| \nabla_i \nabla_j f(t) - \frac{\delta_{ij}}{2\tau(t)} \right|^2 u(t)\, dx$$
$$- \int_{M_t} \nabla W_{\tau(t)}(f(t)) \cdot \nu\, dS$$

where $u(t) = u(f(t))$ with u defined above and $W_\tau(f) \equiv \tau(2\Delta f - |\nabla f|^2) + f - (n+1)$. Note that the equations for f and τ are the same ones considered by Perelman except for a summand given by the scalar curvature of the metric evolving under Ricciflow. The Neumann condition above which does not appear in [P] arises naturally in our setting.

The main observation in [E] was that the above formula can be converted to

$$\frac{d}{dt} \mathcal{W}_\beta(\Omega_t, f(t), \tau(t)) = 2\tau(t) \int_{\Omega_t} \left| \nabla_i \nabla_j f(t) - \frac{\delta_{ij}}{2\tau(t)} \right|^2 u(t)\, dx$$
$$+ 2\tau(t) \int_{M_t} \left(\frac{\partial \beta}{\partial t} - 2\nabla^{M_t} \beta \cdot \nabla^{M_t} f(t) + A_{M_t}(\nabla^{M_t} f(t), \nabla^{M_t} f(t)) - \frac{\beta}{2\tau(t)} \right) u(t) dS$$

where A_{M_t} denotes the second fundamental form of M_t and $\beta = \beta_{M_t}$ is our speed function. If for each fixed t we write this for the time variable $s \in (t - \epsilon, t)$, $\epsilon > 0$ instead where $(f(s))_{s \in (t-\epsilon, t)}$ is a solution of the above Neumann boundary value problem satisfying the end condition $f(t) = f$ and f is a minimizer for $\mu_\beta(\Omega_t, \tau(t))$ then the integrals on the right hand side are continuous in s up to time t by

standard regularity theory (see for instance [L], Theorems 5.18 and 5.19). Hence we obtain, using the minimizing property of f,

(1.1)
$$\frac{d}{dt}\mu_\beta(\Omega_t, \tau(t)) \geq 2\tau(t) \int_{\Omega_t} \left| \nabla_i \nabla_j f - \frac{\delta_{ij}}{2\tau(t)} \right|^2 u\, dx$$
$$+ 2\tau(t) \int_{M_t} \left(\frac{\partial \beta}{\partial t} - 2\nabla^{M_t}\beta \cdot \nabla^{M_t} f + A_{M_t}(\nabla^{M_t} f, \nabla^{M_t} f) - \frac{\beta}{2\tau(t)} \right) u\, dS$$

for every $t \in (0, T)$. In this inequality the time derivative on the left hand side has to be understood in the sense of $\liminf_{s \nearrow t}$. Alternatively, we could write this as an inequality involving a difference quotient for the entropy and a space-time integral on the right hand side.

It is our goal to show that for certain natural speed functions β, for instance the mean curvature H_{M_t} of the evolving hypersurface M_t, the right hand side of (1.1) is non-negative that is the monotonicity inequality

(1.2)
$$\frac{d}{dt}\mu_\beta(\Omega_t, \tau(t)) \geq 0$$

holds or at least some suitable differential inequality. For a fixed bounded region Ω, that is when $\beta = 0$, Lei Ni [N] has proved this inequality if additionally the boundary is weakly convex. Note that an alternative statement of his result is as that $\mu_0(\Omega, \tau)$ is non-increasing in τ if $\partial\Omega$ is weakly convex.

As was proven in [E], inequality (1.2) implies, analogously as in [P], that if the evolving domains are bounded and $T < \infty$ then there is a constant $\kappa > 0$ depending only on $n, \Omega_0, T, \sup_{M_0} |\beta|$ and c_1 such that

$$\frac{|\Omega_t \cap B_r(x_0)|}{r^{n+1}} \geq \kappa$$

holds for all $t \in [0, T)$ and $r \in (0, \sqrt{T}]$ in balls $B_r(x_0) \subset \mathbb{R}^N$ satisfying the conditions $|\Omega_t \cap B_{r/2}(x_0)| > 0$ and

$$\frac{|\Omega_t \cap B_r(x_0)| + r^2 \int_{M_t \cap B_r(x_0)} |\beta|\, dS}{|\Omega_t \cap B_{r/2}(x_0)|} \leq c_1.$$

This lower bound on the volume ratio is termed *local non-collapsing of volume*.

Since the lower volume ratio bound is scaling invariant for certain homogeneous β it is also valid for any smooth limit of suitably rescaled solutions of the flow consisting of smooth, compact embedded hypersurfaces, but now for all radii $r > 0$ as long as the other conditions still hold for the balls $B_r(x_0)$ we consider. The latter information can be used to rule out certain degenerate rescaling limits which in turn provides invaluable information on the singularity structure of solutions for our evolution equation.

An important special case is mean curvature flow, that is where β is the mean curvature H_{M_t} of the hypersurfaces M_t. On the shrinking spheres $M_t = \partial B_{\sqrt{-2nt}}$ for $t < 0$ which form a particular solution of this flow the right hand side of (1.1) vanishes if we choose

$$f(x,t) = -\frac{|x|^2}{4t} + c$$

for a suitable constant c depending only on n.

In the mean curvature flow case we furthermore observe that the expression

$$Z(V) \equiv \frac{\partial H}{\partial t} + 2\,\nabla^M H \cdot V + A_{M_t}(V,V)$$

defined for tangent vectorfields V to M_t which features in (1.1) with $V = -\nabla^{M_t} f$ is the central quantity in Hamilton's differential Harnack inequality for convex solutions of mean curvature flow, see [Ha1]. In [E], it was therefore suggested to search for a generalized form of Hamilton's Harnack inequality, even an integrated (average) version, as a potential approach to establishing monotonicity of our entropy in the mean curvature flow case.

So far we have not succeeded in proving non-negativity of the right hand side of (1.1) but have realized that is is possible to derive a lower volume ratio bound also from a certain weakened version of (1.2), that is where the time derivative is allowed to become negative in a controlled fashion.

In this talk we first recall several properties of the entropy $\mu_\beta(\Omega, \tau)$ from [E] such as lower and upper bounds. A lower bound can be derived using the logarithmic Sobolev inequality for Ω. An upper bound for $\mu_\beta(\Omega, r^2)$ is the logarithm of the quantity

$$\frac{|\Omega \cap B_r(x_0)|}{r^{n+1}}$$

for arbitrary balls plus an additional term which usually can be easily controlled for many domains and functions β.

We mention the following property of the entropy which was established in [P], [KL] and [E]:

Proposition. *A smooth function $f : \overline{\Omega} \to \mathbb{R}$ is a minimizer for $\mu_\beta(\Omega, \tau)$ if and only if it satisfies the conditions*

(1.3) $W_\tau(f) \equiv \tau(2\Delta f - |\nabla f|^2) + f - (n+1) = \mu_\beta(\Omega, \tau)$ *in Ω,*

(1.4) $\nabla f \cdot \nu = \beta$ *on $\partial \Omega$*

and

(1.5) $\int_\Omega u = 1, \qquad u = \frac{e^{-f}}{(4\pi\tau)^{\frac{n+1}{2}}}.$

We also investigate the quantity $\gamma_\beta(\Omega) \equiv \inf_{\tau > 0} \mu_\beta(\Omega, \tau)$ which for the Ricci-flow was introduced in [P] (denoted by $\nu(g)$ there) and studied in detail in [CHI]. A lower bound on $\gamma_\beta(\Omega_t)$, $t \in [0, T)$ for $T < \infty$ only depending on n, T, Ω_0 and $\sup_{M_0} |\beta|$ also leads to a local non-collapsing result for Ω_t in the sense discussed above.

Our main results give expressions for the time-derivative of $\mu_\beta(\Omega_t, \tau(t))$ and $\gamma_\beta(\Omega_t)$ in terms of well-known Harnack type expressions:

Theorem. *Let $(M_t)_{t \in (0,T)} = (\partial \Omega_t)_{t \in (0,T)}$ be a solution of mean curvature flow consisting of compact hypersurfaces. Then we have the inequality*

$$\frac{d}{dt} \mu_{H_{M_t}}(\Omega_t, \tau(t)) \geq \int_{\Omega_t} \left(\frac{n+1}{2\tau(t)} - \Delta f \right) u \, dx$$
$$+ 2\tau(t) \int_{M_t} H_{M_t} \left(|A_{M_t}|^2 - \frac{1}{2\tau(t)} + \nabla^2 f(\nu, \nu) - \frac{1}{2\tau(t)} \right) u \, dS$$

where f is a minimizer for $\mu_{H_{M_t}}(\Omega_t, \tau(t))$.

Theorem. *Let $(M_t)_{t \in (0,T)} = (\partial \Omega_t)_{t \in (0,T)}$ be a solution of mean curvature flow consisting of compact hypersurfaces which satisfy $H > 0$ for all times. Then at those $t \in (0, T)$ where a minimizing pair (f, τ) for $\gamma_{H_{M_t}}(\Omega_t)$ satisfies $\tau > 0$ we have the inequality*

$$\frac{d}{dt} \gamma_{H_{M_t}}(\Omega_t) \geq 2\tau \int_{M_t} H_{M_t} \left(|A_{M_t}|^2 + \nabla^2 f(\nu, \nu) - \frac{1}{2\tau} \right) u \, dS.$$

In the above statements, $|A|^2$ denotes the second fundamental form of M_t.

These formulas provide an alternative to inequality (1.1) in the special case of mean curvature flow. Note that the functions f enter the right hand sides only via the quantity

$$\nabla_i \nabla_j f - \frac{\delta_{ij}}{2\tau}$$

which features in the differential Harnack inequalities of Li and Yau [LY] and of Hamilton [Ha2]. Our results therefore transform the problem of bounding the time derivatives of $\mu_{H_{M_t}}(\Omega_t, \tau(t))$ and $\gamma_{H_{M_t}}(\Omega_t)$ from below which would lead to a local non-collapsing result for Ω_t into the task of establishing Li-Yau-Hamilton type Harnack inequalities for solutions f of the Neumann problem associated with the Euler-Lagrange equation for $\mu_\beta(\Omega_t, \tau(t))$ or for functions f corresponding to a minimizing pair (f, τ) of $\gamma_{H_{M_t}}(\Omega_t)$. One can show that the latter functions satisfy an extra identity in addition to the three relations given by the Euler-Lagrange equation, its natural boundary condition and the normalization condition for the function u associated with f.

A closely related problem which is of independent interest is to investigate which new expressions arise in the differential Harnack inequalities in [LY] and [Ha2] for generalizations of the situations considered in these papers such as for instance positive solutions of the heat equation on evolving domains with Neumann boundary condition given by the speed function of the boundary. This should be particularly natural in the mean curvature flow case that is when $\beta = H_{M_t}$ as in this case the Codazzi equations could provide additional information. It may also prove to be an easier task than trying to derive an extension of Hamilton's Harnack inequality for the mean curvature function.

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On the inverse σ_k problem in Kähler geometry

HAO FANG

Let (M, ω) be a compact Kähler manifold. Let χ be another Kähler metric on M, and $[\chi]$ be its cohomology class. We define

$$\sigma_k(\chi) = \binom{n}{k} \frac{\chi^k \wedge \omega^{n-k}}{\omega^n},$$

for some integer $1 \leq k \leq n$. One readily finds that this quantity is just the k-th elementary symmetric polynomial of eigenvalues of χ with respect to ω. We fix ω^n as the volume form on M. Then the constant

$$c_k := \frac{\int_M \sigma_{n-k}(\chi)}{\int_M \sigma_n(\chi)} = \binom{n}{k} \frac{[\chi]^{n-k} \cdot [\omega]^k}{[\chi]^n},$$

is a topological constant depending only on classes $[\chi]$ and $[\omega]$.

In joint work with M. Lai and X. Ma [FLM], we have raised the following question

Question 0.1. *Let (M, ω), χ and c_k be given as above, does there exist a metric $\tilde{\chi} \in [\chi]$ such that*

$$(0.1) \qquad c_k \tilde{\chi}^n = \binom{n}{k} \tilde{\chi}^{n-k} \wedge \omega^k?$$

In [FLM], we consider the following geometric flow to study (0.1):

$$(0.2) \qquad \begin{cases} \frac{\partial}{\partial t} \varphi &= c_k^{1/k} - (\frac{\sigma_{n-k}(\chi_\varphi)}{\sigma_n(\chi_\varphi)})^{1/k}, \\ \varphi(0) &= 0 \end{cases}$$

in the space of Kähler potentials of χ:

$$\mathcal{P}_\chi := \{\varphi \in C^\infty(M) | \chi_\varphi := \chi + \frac{\sqrt{-1}}{2} \partial \bar{\partial} \varphi > 0\}.$$

When $k = 1$, the flow is the J-flow which appeared first in [D] in the setting of moment map and by Chen in [Ch1, Ch2], as the gradient flow of the J-functional, which appears as a term of the Mabuchi energy. In this case, Song and Weinkove [SW] provide a necessary and sufficient condition for the flow to converge to the critical metric. For general k, the question is solved in [FLM] with an analogous condition which we now describe.

For $k \neq n$, we define $\mathcal{C}_k = \mathcal{C}_k(\omega)$ as

$$(0.3) \qquad \mathcal{C}_k(\omega) = \{[\chi] > 0, \exists \chi' \in [\chi],$$

$$\text{such that } nc_k \chi'^{n-1} - \binom{n}{k}(n-k)\chi'^{n-k-1} \wedge \omega^k > 0\}.$$

For $k = n$, we take $\mathcal{C}_n(\omega)$ to be the entire Kähler cone of M.

Theorem 1. *Let (M, ω) be a compact Kähler manifolds. Let k be an integer $1 \leq k \leq n$. Assume χ is another Kähler form with its class $[\chi] \in \mathcal{C}_k(\omega)$, then the flow*

$$(0.4) \qquad \frac{\partial}{\partial t} \varphi = c_k^{1/k} - (\frac{\sigma_{n-k}(\chi_\varphi)}{\sigma_n(\chi_\varphi)})^{1/k},$$

with any initial value $\chi_0 \in [\chi]$ has long time existence and converges to a unique smooth metric $\tilde{\chi} \in [\chi]$ satisfying

$$(0.5) \qquad c_k \tilde{\chi}^n = \binom{n}{k} \tilde{\chi}^{n-k} \wedge \omega^k.$$

It would be interesting to discuss the same problem when the cone condition fails.

In joint work with M. Lai [FL], we discuss the problem when M processes some symmetry. In particular, we assume that the initial data (M, ω, χ_0) satisfy the Calabi Ansatz. We prove that χ_t converges to smooth conic metric on M away from a sub-varietry when $t \to \infty$ if the cone condition fails.

In joint work with M. Lai, J. Song and B. Weinkove [FLSW], we study the J-flow on Kähler surfaces when the Kähler class $[\chi]$ lies on the boundary of the open cone $\mathcal{C}_1(\omega)$, and satisfies a nonnegative condition. We obtain a C^0 estimate and show that the J-flow converges smoothly to a singular Kähler metric away from a finite number of curves of negative self-intersection on the surface. We discuss an application to the Mabuchi energy functional on Kähler surfaces with ample canonical bundle.

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Partial regularity for Monge-Ampère type equations

Alessio Figalli

The Monge-Ampère equation is a fully nonlinear degenerate elliptic equation which arises in several problems from analysis and geometry.

More recently, a more general class of Monge-Ampère type equations of the form

$$(0.1) \qquad \det\big(D^2 u - A(x, u, \nabla u)\big) = f(x, u, \nabla u),$$

has found applications in several other problems. In particular, this class of equations arises in the regularity theory for optimal transport maps. To explain this fact, we first introduce some notation.

We introduce first some conditions on the cost function $c : X \times Y \to \mathbb{R}$ $(X, Y \subset \mathbb{R}^n)$:

(C0) The cost function $c : X \times Y \to \mathbb{R}$ is of class C^2 with $\|c\|_{C^2(X \times Y)} < \infty$.

(C1) For any $x \in X$, the map $Y \ni y \mapsto -D_x c(x, y) \in \mathbb{R}^n$ is injective.

(C2) For any $y \in Y$, the map $X \ni x \mapsto -D_y c(x, y) \in \mathbb{R}^n$ is injective.

(C3) $\det(D_{xy} c)(x, y) \neq 0$ for all $(x, y) \in X \times Y$.

It is a classical result in optimal transport theory that, given two probability densities f and g supported respectively on X and Y, if c satisfies **(C0)** and **(C1)** then there exists a unique optimal transport T sending f onto g, and such a map satisfies

$$(0.2) \qquad D_x c(x, T(x)) + \nabla u(x) = 0$$

for some c-convex function $u : X \to \mathbb{R}$, that is, u can be written as

$$u(x) = \sup_{y \in Y} \{ -c(x,y) + \lambda_y \}$$

for some family of constants $\lambda_y \in \mathbb{R}$.

Since c satisfies **(C1)** we can define the *c-exponential map*:

for any $x \in X$, $y \in Y$, $p \in \mathbb{R}^n$,   c-$\exp_x(p) = y \iff p = -D_x c(x,y)$.

This allows us to rewrite (0.2) as $T(x) = $ c-$\exp_x(\nabla u(x))$.

We now observe that, whenever c satisfies **(C0)**, **(C1)**, and **(C3)**, then the transport condition $T_\sharp f = g$ gives

$$(0.3) \qquad\qquad |\det(DT(x))| = \frac{f(x)}{g(T(x))} \qquad \text{a.e.}$$

In addition, the c-convexity of u implies that, at every point x where u is twice differentiable,

$$(0.4) \qquad\qquad D^2 u(x) + D_{xx} c\big(x, \text{c-}\exp_x(\nabla u(x))\big) \geq 0.$$

Hence, differentiating (0.2) with respect to x, and using (0.3) and (0.4), we obtain

$$(0.5) \quad \det\Big(D^2 u(x) + D_{xx} c\big(x, \text{c-}\exp_x(\nabla u(x))\big)\Big)$$
$$= \big|\det\big(D_{xy}c\big(x, \text{c-}\exp_x(\nabla u(x))\big)\big)\big| \frac{f(x)}{g(\text{c-}\exp_x(\nabla u(x)))}$$

at every point x where u it is twice differentiable.

Hence u solves a Monge-Ampère type equation of the form (0.1) with

$$\mathcal{A}(x, \nabla u(x)) := -D_x^2 c\big(x, \text{c-}\exp_x\big(\nabla u(x)\big)\big).$$

The breakthrough for the regularity of solutions to this class of equations came with the paper of Ma, Trudinger and Wang [3], where the authors found a mysterious fourth-order condition on the cost functions, which turned out to be sufficient to prove the regularity of u. Such condition was then shown by Loeper [2] to be necessary for regularity. Unfortunately, this condition is very restrictive and it is satisfied only in very particular cases. Hence the need to develop a partial regularity theory: is it true that solutions are always smooth outside a "small" singular set?

In a joint paper with Guido De Philippis [1], we proved the following results:

Theorem 0.1. *Let $X, Y \subset \mathbb{R}^n$ be two bounded open sets, and let $f : X \to \mathbb{R}^+$ and $g : Y \to \mathbb{R}^+$ be two continuous probability densities, respectively bounded away from zero and infinity on X and Y. Assume that the cost $c : X \times Y \to \mathbb{R}$ satisfies* **(C0)**-**(C3)**, *and denote by $T : X \to Y$ the unique optimal transport map sending f onto g. Then there exist two relatively closed sets $\Sigma_X \subset X, \Sigma_Y \subset Y$ of measure zero such that $T : X \setminus \Sigma_X \to Y \setminus \Sigma_Y$ is a homeomorphism of class $C_{\text{loc}}^{0,\beta}$ for any $\beta < 1$. In addition, if $c \in C_{\text{loc}}^{k+2,\alpha}(X \times Y)$, $f \in C_{\text{loc}}^{k,\alpha}(X)$, and $g \in C_{\text{loc}}^{k,\alpha}(Y)$ for some $k \geq 0$ and $\alpha \in (0,1)$, then $T : X \setminus \Sigma_X \to Y \setminus \Sigma_Y$ is a diffeomorphism of class $C_{\text{loc}}^{k+1,\alpha}$.*

Theorem 0.2. *Let M be a smooth Riemannian manifold, and let $f, g : M \to \mathbb{R}^+$ be two continuous probability densities, locally bounded away from zero and infinity on M. Let $T : M \to M$ denote the optimal transport map for the cost $c = d^2/2$ sending f onto g, d being the Riemannian distance on M. Then there exist two closed sets $\Sigma_X, \Sigma_Y \subset M$ of measure zero such that $T : M \setminus \Sigma_X \to M \setminus \Sigma_Y$ is a homeomorphism of class $C^{0,\beta}_{\text{loc}}$ for any $\beta < 1$. In addition, if both f and g are of class $C^{k,\alpha}$, then $T : M \setminus \Sigma_X \to M \setminus \Sigma_Y$ is a diffeomorphism of class $C^{k+1,\alpha}_{\text{loc}}$.*

The rough idea of the proof is the following: if $\bar{x}$ is a point where the semiconvex function u is twice differentiable (these are almost every point, by Alexandrov's Theorem), then around that point u looks like a parabola. In addition, by looking close enough to $\bar{x}$, the cost function c will be very close to the linear one and the densities will be almost constant there. Hence u is close to a convex function v solving an optimal transport problem with linear cost and constant densities. In addition, since u is close to a parabola, so is v. Hence, by Caffarelli's regularity theory for the classical Monge-Ampère equation v is smooth, and we can use this information to deduce that u is even closer to a second parabola (given by the second order Taylor expansion of v at the origin) inside a small neighborhood around of origin. By rescaling back this neighborhood at scale 1 and iterating this construction, we obtain that u is $C^{1,\beta}$ at the origin for any $\beta \in (0, 1)$. Since this argument can be applied at every point in a neighborhood of the origin, we deduce that u is $C^{1,\beta}$ there.

Once this result is proved, we can show that u enjoys a comparison principle, and this allows us to use a second approximation argument with solutions of the classical Monge-Ampère equation to conclude that u is $C^{2,\sigma'}$ in a smaller neighborhood, for some $\sigma' > 0$. Then higher regularity follows from standard elliptic estimates.

These results imply that T is of class $C^{0,\beta}$ in neighborhood of $\bar{x}$ (resp. T is of class $C^{k+1,\alpha}$ if $c \in C^{k+2,\alpha}_{\text{loc}}$ and $f, g \in C^{k,\alpha}_{\text{loc}}$). Being our assumptions completely symmetric in x and y, we can apply the same argument to the optimal map T^* sending g onto f. Since $T^* = T^{-1}$, it follows that T is a global homeomorphism of class $C^{0,\beta}_{\text{loc}}$ (resp. T is a global diffeomorphism of class $C^{k+1,\alpha}_{\text{loc}}$) outside a closed set of measure zero.

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Layer solutions for the fractional Laplacian on hyperbolic space
Maria del Mar Gonzalez Nogueras

Abstract: We consider a semilinear equation for the fractional Laplacian on
hyperbolic space with a nonlinearity that comes from a double well potential. We
prove existence and uniqueness of layer solutions, together with some qualitative
and symmetry properties. One of the difficulties is to show a priori regularity
for the equation, for which we need to derive a singular integral formula for the
fractional Laplacian on hyperbolic space.

1. The fractional Laplacian on hyperbolic space

The Fourier transform on hyperbolic space may be defined as

$$\hat{w}(\lambda,\theta) = \int_{\mathbb{H}^n} w(\Omega) h_{\lambda,\theta}(\Omega) d\Omega,$$

where $h_{\lambda,\theta}(\Omega)$ are generalized eigenfunctions of the Laplace-Beltrami operator on
$\mathbb{H}^n$ associated to the eigenvalue $\lambda^2 + \frac{(n-1)^2}{4}$. This allows to define the fractional
Laplacian on hyperbolic space $(-\Delta_{\mathbb{H}^n})^\gamma w$ as the operator with principal symbol

$$\widehat{(-\Delta_{\mathbb{H}^n})^\gamma} w = \left(\frac{(n-1)^2 + \lambda^2}{4} \right)^\gamma \hat{w}.$$

Theorem ([1]). *The fractional Laplacian may be written as a integral formula
against a convolution kernel:*

$$(-\Delta_{\mathbb{H}^n})^\gamma w(x) = \int_{\mathbb{H}^n} (w(x') - w(x)) K_\gamma(\rho) \, dx',$$

where the kernel $K_\gamma(\rho)$ has the asymptotic behavior:

  i. *As $\rho \to 0$, $K_\gamma(\rho) \sim \frac{1}{\rho^{n+2\gamma}}$.*
  ii. *As $\rho \to \infty$, $K_\gamma(\rho) \sim \rho^{-1-\gamma} e^{-(n-1)\rho}$.*

Although the formula for the kernel in the previous theorem may be written
explicitly, for our purposes it is enough to consider its asymptotic behavior. Indeed,
one obtains as a corollary some $\mathcal{C}^\alpha$ a-priori regularity results for the equation
$(-\Delta_{\mathbb{H}^n})^\gamma w = v$.

Our second theorem deals with the Caffarelli-Silvestre [5] extension problem for
the calculation of the fractional Laplacian:

Theorem ([1]). *Let $\gamma \in (0,1)$, $a = 1 - 2\gamma$. Given w on $\mathbb{H}^n$, there exists a unique
solution of the extension problem*

$$\begin{cases} div_g(y^a \nabla_g u)(x,y) = 0 \text{ for } (x,y) \in \mathbb{H}^n \times \mathbb{R}_+, \\ u(x,0) = w(x) \qquad \text{for } x \in \mathbb{H}^n, \end{cases}$$

where g is the product metric on $\mathbb{H}^n \times \mathbb{R}_+$ given by $g = g_{\mathbb{H}^n} + dy^2$ Moreover,

$$(-\Delta_{\mathbb{H}^n})^\gamma w = -d_\gamma \lim_{y \to 0} y^a \partial_y u,$$

for a constant $d_\gamma = 2^{2\gamma-1} \frac{\Gamma(\gamma)}{\Gamma(1-\gamma)}$.

2. Layer solutions

We next investigate in [6] the equation

$$(-\Delta_{\mathbb{H}^n})^{\gamma} w = f(w) \quad \text{in } \mathbb{H}^n,$$

for $\gamma \in (0,1)$ and f a smooth nonlinearity that typically comes from a double well potential. We prove the existence of heteroclinic connections in the following sense; a so-called layer solution is a smooth solution of the previous equation converging to ± 1 at any point of the two hemispheres $S_{\pm} \subset \partial_{\infty}\mathbb{H}^n$ and which is strictly increasing with respect to the signed distance to a totally geodesic hyperplane Π. We prove that under additional conditions on the nonlinearity uniqueness holds up to isometry.

Note that layer solutions for the fractional Laplacian on Euclidean space were studied in [4, 2, 3]. Many of their arguments generalize to hyperbolic space, however, two main difficulties arise:

- All the a-priori estimates need to be reproven taking into account the geometry of the problem (which is the motivation for the results in Section 1).
- The lack of translation invariance, which is replaced by isometries on hyperbolic space

Finally, we consider the Hamiltonian energy

$$V(t) = \frac{1}{2} \int_0^{\infty} y^a \left((\partial_t u)^2 - (\partial_y u)^2 \right) \, dy - \frac{1}{d_{\gamma}} \left(F(u(t,0)) - F(1) \right),$$

where $F' = -f$. We show that V is decreasing to zero along the trajectories when $t \to +\infty$.

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A mean curvature type flow and isoperimetric inequality in warped product space

PENGFEI GUAN

Abstract

We report a joint work in progress with Junfang Li and Mutao Wang,

Suppose (M, g) is a hypersurface in a warped product space $(N, \bar{g})$, with $N = \mathbb{R}^+ \times \mathbb{S}^n$, $\bar{g} = d\rho^2 + \phi(\rho)g_{\mathbb{S}^n}$, where $g_{\mathbb{S}^n}$ is the standard metric on $\mathbb{S}^n$. Define

$$V = \phi(\rho)\frac{\partial}{\partial \rho}, \quad u = <V, \nu>,$$

where ν the normal of M. We assume $u > 0$, that is M is starshaped. We consider the following geometric flow involving mean curvature for starshaped hypersurfaces in the warped product space.

$$(0.1) \qquad\qquad\qquad X_t = (n\phi^{'} - uH)\nu,$$

where H the mean curvature of M. The flow is designed to preserve the volume of the enclosed domain by the hypersurfaces, via Minkowski type identities. Under the assumption that

$$(0.2) \qquad\qquad\qquad 1 \geq 1 - (\phi^{'})^2 + \phi\phi^{''} \geq 0,$$

we prove that the surface area is monotone decreasing along the flow and starshapedness is preserved. The longtime existence and exponential convergence of the flow to a slice is obtained without any additional curvature assumption, with the establishment of the key gradient estimate

$$(0.3) \qquad\qquad\qquad e^{\alpha t}|\nabla_{g_{\mathbb{S}^n}}\rho|^2 \leq C,$$

for some $\alpha > 0$ (depending only on n), $C > 0$ (depending only on n and initial data) and for all $t > 0$. A sharp isoperimetric inequality in the warped product space follows from the monotonicity and the convergence. The result extends a previous result (jointly with Junfang Li) in the space form.

Critical metrics on connected sums of Einstein four-manifolds

MATTEW J. GURSKY

Abstract: We develop a gluing procedure designed to obtain canonical metrics on connected sums of Einstein four-manifolds. These metrics are critical points of quadratic Riemannian functionals of the form

$$g \mapsto \int |W|^2 \ dvol + t \int R^2 \ dvol,$$

where W is the Weyl tensor, R is the scalar curvature, and $t < 0$ is a real parameter.

The main application is an existence result, using two well-known Einstein manifolds as building blocks: the Fubini-Study metric on $\mathbb{CP}^2$, and the product metric on $S^2 \times S^2$. Using these metrics in various gluing configurations, critical

metrics are found on connected sums for a specific value of $t < 0$ depending on the global geometry of the factors.

Some remarks on singularities of weak solutions to the Monge-Ampère equation

JUN KITAGAWA

(joint work with Young-Heon Kim)

Equations of Monge-Ampère type arise naturally in many contexts, such as geometric analysis and optimal transport problems. We consider a case where necessary conditions for the regularity theory fail and singularities may be unavoidable, and ask what can be said about the structure of such singularities.

First recall some notions of weak solutions to the (classical) Monge-Ampère equation,

$$(0.1) \qquad \lambda^{-1}\chi_\Omega \leq \det \nabla^2 u \leq \lambda \chi_\Omega$$

where $\lambda > 0$ and $\Omega \subset \mathbb{R}^n$ is open and bounded. First, we say that a convex function $u : \mathbb{R}^n \to \mathbb{R}$ is a *Brenier solution* (from the domain Ω to Ω') of (0.1) if it satisfies

$$\lambda^{-1}\chi_{\Omega'} \leq \nabla u_\# \chi_\Omega \leq \lambda \chi_{\Omega'},$$
$$\nabla u(\mathbb{R}^n) \subset \bar{\Omega}',$$

here ∇u is defined Lebesgue a.e. and $\nabla u_\#$ denotes the push-forward of a measure by the map ∇u. Brenier solutions originate from the optimal transport (Monge-Kantorovich) problem with a cost function $c(x,y) = -\langle x, y \rangle$. However, Brenier solutions are in some sense not strong enough for the study of regularity theory. Toward that end one says that a convex function $u : \mathbb{R}^n \to \mathbb{R}$ is an *Aleksandrov solution* of (0.1) if

$$\lambda^{-1}|E \cap \Omega| \leq |\partial u(E)| \leq \lambda |E \cap \Omega|$$

for any measurable set E, where here ∂u denotes the subdifferential image of a set under u. There is a well-known $C^{1,\alpha}$ theory for Aleksandrov solutions of (0.1), pioneered by Caffarelli:

Theorem 1 (Caffarelli [1]). *Suppose that u is a globally Lipschitz Aleksandrov solution of* (0.1). *Then u is strictly convex and $C^{1,\alpha}$ for some α in the interior of Ω. Moreover, if Ω' is convex, then a Brenier solution of* (0.1) *is an Aleksandrov solution of* (0.1).

However, if one removes the lower bound condition in the definition of Aleksandrov solution, the solution u may no longer be differentiable everywhere. At the same time, one can still obtain some amount of control on the "severity" of the singularities that can form, as illustrated by another result of Caffarelli:

Theorem 2 (Caffarelli [2]). *Suppose that a convex $u : \mathbb{R}^n \to \mathbb{R}$ satisfies $|\partial u(E)| \leq \lambda |E \cap \Omega|$ for any measurable set E. Then, at any point x such that $\partial u(x) \cap (\text{spt}\,(\nabla u_\# \chi_\Omega))^\circ \neq \emptyset$, the affine dimension of $\partial u(x)$ is strictly less than $n/2$.*

In our result, we have both found an alternative proof of Caffarelli's theorem above, and extended it to a more general class of optimal transport problems.

Theorem 3 (Kim-K. [3]). *The result of Theorem 2 holds for an optimal transport problem with cost function satisfying the conditions (Twist), (Nondeg), and the weak (MTW) condition, along with appropriate conditions on the domains involved.*

For a more detailed statement and definitions of the relevant conditions, the reader is directed to [3].

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A compactness theorem for a fully nonlinear Yamabe problem under a lower Ricci curvature bound

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(joint work with Luc Nguyen)

In this talk we present the proof of a result in [Li-Nguyen] on compactness of solutions of a fully nonlinear Yamabe problem satisfying a lower Ricci curvature bound, when the manifold is not conformally diffeomorphic to the standard sphere. The result allows us to prove the existence of solutions when the associated cone Γ satisfies $\mu_\Gamma^+ \leq 1$, which include the σ_k-Yamabe problem for k not smaller than half of the dimension of the manifold.

Let (M, g) be a compact smooth Riemannian manifold of dimension $n \geq 3$. Let A_g denote the Schouten tensor of g, i.e.

$$A_g = \frac{1}{n-2}\Big(\mathrm{Ric}_g - \frac{1}{2(n-1)}\,R_g\,g\Big),$$

where Ric_g and R_g are respectively the Ricci curvature and the scalar curvature of g. Let $\lambda(A_g) = (\lambda_1, \cdots, \lambda_n)$ denote the eigenvalues of A_g with respect to g, and

(0.1)

$\Gamma \subset \mathbb{R}^n$ be an open convex symmetric cone with vertex at the origin,

(0.2)

$\{\lambda \in \mathbb{R}^n | \lambda_i > 0, 1 \le i \le n\} \subset \Gamma \subset \{\lambda \in \mathbb{R}^n | \lambda_1 + \ldots + \lambda_n > 0\}$,

(0.3)

$f \in C^\infty(\Gamma) \cap C^0(\overline{\Gamma})$ be concave, homogeneous of degree one, symmetric in λ_i,

(0.4)

$f > 0$ in Γ, $\quad f = 0$ on $\partial\Gamma$; $\quad f_{\lambda_i} > 0$ in Γ $\forall 1 \le i \le n$.

For a positive function u, let g_u denote the metric $u^{\frac{4}{n-2}} g$. Note that the Schouten tensor of g_u is given by

$$A_{g_u} = -\frac{2}{n-2} u^{-1} \nabla_g^2 u + \frac{2n}{(n-2)^2} u^{-2} du \otimes du - \frac{2}{(n-2)^2} u^{-2} |du|_g^2 \, g + A_g.$$

Theorem 0.1. *([Li-Nguyem]) Let (f, Γ) satisfy (0.1)-(0.4), (M, g) be a compact, smooth Riemannian manifold of dimension $n \ge 3$. For any $\alpha \ge 0$, there exists $C = C(M, g, f, \Gamma, \alpha) > 0$ such that if $u \in C^2(M)$ is a positive solution of*

(0.5) $$f(\lambda(A_{g_u})) = 1, \quad \lambda(A_{g_u}) \in \Gamma, \quad on \ M$$

and satisfies $\text{Ric}_{g_u} \ge -(n-1)\alpha^2 \, g_u$, then

$$\max_M u \le C.$$

Equation (0.5) is a second order fully nonlinear elliptic equation of u. For $1 \le k \le n$, let $\sigma_k(\lambda) = \sum_{1 \le i_1 < \cdots < i_k \le n} \lambda_{i_1} \cdots \lambda_{i_k}, \lambda = (\lambda_1, \cdots, \lambda_n) \in \mathbb{R}^n$, denote the k-th elementary symmetric function, and let Γ_k denote the connected component of $\{\lambda \in \mathbb{R}^n | \sigma_k(\lambda) > 0\}$ containing the positive cone $\{\lambda \in \mathbb{R}^n | \lambda_1, \cdots, \lambda_n > 0\}$. Then $(f, \Gamma) = (\sigma_k^{1/k}, \Gamma_k)$ satisfies (0.1)-(0.4).

Let Γ be the cone above, and let μ_Γ^+ be a constant defined by the following:

$\mu_\Gamma^+ \in [0, n-1]$ is the unique number such that $(-\mu_\Gamma^+, 1, ...1) \in \partial\Gamma$.

For $\Gamma = \Gamma_k$, we have

$$\mu_{\Gamma_k}^+ = \frac{n-k}{k} \text{ for } 1 \le k \le n.$$

In particular,

$$\begin{cases} \mu_{\Gamma_k}^+ > 1 & \text{if } k < \frac{n}{2}, \\ \mu_{\Gamma_k}^+ = 1 & \text{if } k = \frac{n}{2}, \\ \mu_{\Gamma_k}^+ < 1 & \text{if } k > \frac{n}{2}. \end{cases}$$

Note that for $\mu_\Gamma^+ \le 1$, the lower Ricci bound assumption with $\alpha = 0$ is satisfied automatically.

As a consequence of Theorem 0.1, we have:

Theorem 0.2. *([Li-Nguyen]) Let (f, Γ) satisfy (0.1)-(0.4) and (M, g) be a compact, smooth Riemannian manifold of dimension $n \geq 3$ satisfying $\lambda(A_g) \in \Gamma$ on M. If $\mu_\Gamma^+ \leq 1$, then there is a smooth solution u of (0.5). Moreover, all solutions u of (0.5) satisfy $\|\ln u\|_{C^5(M,g)} \leq C$ for some constant C depending only on (f, Γ) and (M, g).*

Remark 0.3. In fact, the degree of all solutions in the above theorem is equal to -1.

Remark 0.4. Note that the above theorem covers the case $(f, \Gamma) = (\sigma_k^{1/k}, \Gamma_k)$ for $k \geq \frac{n}{2}$.

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Degree theory for oblique boundary problems

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(joint work with Yanyan Li and Luc Nguyen)

Consider a second order fully nonlinear elliptic operator with a nonlinear oblique boundary condition of the general form,

$$(0.1) \qquad\qquad F[u] = f(\cdot, u, Du, D^2 u), \quad \text{in } \Omega,$$

$$(0.2) \qquad\qquad G[u] = g(\cdot, u, Du), \quad \text{on } \partial\Omega,$$

where Ω is a bounded smooth domain in Euclidean n-space, $\mathbb{R}^n$, and $f \in C^{3,\alpha}(\overline{\Omega} \times \mathbb{R} \times \mathbb{R}^n \times \mathcal{S}^n)$ and $g \in C^{4,\alpha}(\overline{\Omega} \times \mathbb{R} \times \mathbb{R}^n)$ are real valued functions. Here $\mathcal{S}^n$ denotes the $n(n+1)/2$ dimensional linear space of $n \times n$ real symmetric matrices, and $Du = (D_i u)$ and $D^2 u = [D_{ij} u]$ denote the gradient vector and Hessian matrix of the real valued function u.

Let (x, z, p, r) denote points in $\overline{\Omega} \times \mathbb{R} \times \mathbb{R}^n \times \mathcal{S}^n$. The operator $F : C^{4,\alpha}(\overline{\Omega}) \to C^{2,\alpha}(\overline{\Omega})$ is *uniformly elliptic* on some bounded open subset $\mathcal{O}$ of $C^{4,\alpha}(\overline{\Omega})$, namely there exists a constant $\lambda > 0$ such that for all $u \in \mathcal{O}, x \in \overline{\Omega}$ and $\xi \in \mathbb{R}^n$ there holds

$$(0.3) \qquad\qquad \frac{\partial f}{\partial r_{ij}}(x, u, Du, D^2 u)\xi_i \xi_j \geq \lambda |\xi|^2.$$

The operator $G : C^{4,\alpha}(\overline{\Omega}) \to C^{3,\alpha}(\partial\Omega)$ is *uniformly oblique* on $\mathcal{O}$, namely there exists a constant $\chi > 0$ such that for all $u \in \mathcal{O}$ and $x \in \partial\Omega$

$$(0.4) \qquad\qquad \frac{\partial g}{\partial p}(x, u, Du) \cdot \gamma(x) \geq \chi,$$

where $\gamma(x)$ denotes the outer unit normal of $\partial\Omega$ at x.

Theorem 0.1 ([3]). *Let $\mathcal{O} \subset C^{4,\alpha}(\overline{\Omega})$ be a bounded open set with $\partial\mathcal{O} \cap (F,G)^{-1}(0) = \emptyset$, where F, G are as above. There exists a unique integer-valued degree for (F,G) on $\mathcal{O}$ at 0, which satisfies the following key properties:*

- *If $\deg((F,G),\mathcal{O},0) \neq 0$, then $\exists u \in \mathcal{O}$ s.t. $(F[u],G[u]) = 0$.*
- *If $\mathcal{U}_1, \mathcal{U}_2 \subset \mathcal{O}$ and $(\mathcal{O}\backslash\mathcal{U}_1 \cup \mathcal{U}_2) \cap (F,G)^{-1}(0) = \emptyset$, then $\deg((F,G),\mathcal{O},0) = \deg((F,G),\mathcal{U}_1,0) + \deg((F,G),\mathcal{U}_2,0)$.*
- *(Homotopy invariance) If $t \mapsto (f_t, g_t)$ is continuous from $[0,1]$ to $C^{3,\alpha}(\overline{\Omega} \times \mathbb{R}^n \times \mathbb{R}^n \times S^{n \times n}) \times C^{4,\alpha}(\overline{\Omega} \times \mathbb{R} \times \mathbb{R}^n)$ s.t. F_t is uniformly elliptic, G_t is uniformly oblique on $\mathcal{O}$, and $\partial\mathcal{O} \cap (F_t, G_t)^{-1}(0) = \emptyset$ for all $t \in [0,1]$, then $\deg((F_t, G_t), \mathcal{O}, 0)$ is independent of t.*

Moreover, we have the following immediate properties:

- Assume $(F,G)[u_0] = 0$ and the Fréchet derivative $(F', G')[u_0]$ is invertible. Then $\deg((F,G),\mathcal{O},0) = \deg((F',G'),\mathcal{O},0)$, where $\mathcal{O}$ is a neighborhood of u_0 in $C^{4,\alpha}(\overline{\Omega})$ which does not contain any other points of $(F,G)^{-1}(0)$.
- *Compatibility with Leray-Schauder degree in linear cases*: If F and G are linear, then $deg((F,G),\mathcal{O},0)$ "coincides" with the Leray-Schauder degree for linear operators.

Sketch of proof: Consider the operator $S : C^{2,\alpha}(\overline{\Omega}) \to C^{\alpha}(\overline{\Omega}) \times C^{1,\alpha}(\partial\Omega)$ such that

$$u \mapsto \left(\triangle u, \; (\gamma_i D_i u + u)|_{\partial\Omega} \right), \quad \gamma : \text{unit outer normal}$$

and the boundary operator $T : C^{3,\alpha}(\partial\Omega) \to C^{1,\alpha}(\partial\Omega)$ s.t.

$$u \mapsto \triangle_T u - u,$$

where $\triangle_T$ denotes the tangential Laplacian over $\partial\Omega$. Define

$$(0.5) \qquad \tilde{F} = \left(\tilde{F}_{(1)}, \tilde{F}_{(2)} \right) = S \circ F : C^{4,\alpha}(\overline{\Omega}) \to C^{\alpha}(\overline{\Omega}) \times C^{1,\alpha}(\partial\Omega)$$

$$(0.6) \qquad \tilde{G} = T \circ G : C^{4,\alpha}(\overline{\Omega}) \to C^{1,\alpha}(\partial\Omega).$$

It is well-known that S, T are bijections, $(F,G) = 0$ if and only if $(\tilde{F}, \tilde{G}) = 0$. We are going to define a degree for (F,G) by defining a degree for $(\tilde{F}, \tilde{G})$.

We write

$$\tilde{F}_{(1)}[u] = a_{st}(x, u, Du, D^2u)D_{iist}u + C_*(x, u, Du, D^2u, D^3u),$$

$$\tilde{F}_{(2)}[u] = \left(a_{st}(x, u, Du, D^2u)D_{sti}u\gamma_i + E_*(x, u, Du, D^2u) \right)\big|_{\partial\Omega},$$

$$\tilde{G}[u] = \left(b_i(x, u, Du)\triangle_T(D_i u) + H_*(x, u, Du, D^2u) \right)\big|_{\partial\Omega},$$

where $a_{st} = \frac{\partial f}{\partial r_{st}}(x, u, Du, D^2u)$ and $b_i = \frac{\partial g}{\partial p_i}(x, u, Du)$.

For a constant $N > 0$, define a linear operator

$$L^N \; : \; C^{4,\alpha}(\overline{\Omega}) \to C^{\alpha}(\overline{\Omega}) \times C^{1,\alpha}(\partial\Omega) \times C^{1,\alpha}(\partial\Omega)$$

$$w \mapsto \left(L_{(1)}^N w, \; L_{(2)}^N w, \; L_{(3)}^N w \right),$$

where

$$L_{(1)}^N w = a_{st} D_{iist} w - N a_{st} D_{st} w,$$
$$L_{(2)}^N w = (a_{st} D_{sti} w \gamma_i)|_{\partial \Omega},$$
$$L_{(3)}^N w = (b_i \triangle_T (D_i w) - N b_i D_i w - N w)|_{\partial \Omega}.$$

Then we split the operators $(\tilde{F}[u], \tilde{G}[u]) = L^{u,N}[u] + R^{u,N}[u]$, where $R^{u,N} = (R_{(1)}^{u,N}, R_{(2)}^{u,N}, R_{(3)}^{u,N})$ maps $C^{4,\alpha}(\overline{\Omega})$ into $C^{1,\alpha}(\overline{\Omega}) \times C^{2,\alpha}(\partial\Omega) \times C^{2,\alpha}(\partial\Omega)$ and

$$R_{(1)}^{u,N}[u] = N a_{st}(x, u, Du, D^2 u) D_{st} u + C_*(x, u, Du, D^2 u, D^3 u),$$
$$R_{(2)}^{u,N}[u] = E_*(x, u, Du, D^2 u)|_{\partial \Omega},$$
$$R_{(3)}^{u,N}[u] = N b_i(x, u, Du) D_i u + N u + H_*(x, u, Du, D^2 u)|_{\partial \Omega}.$$

A main technical estimate is the following

Theorem 0.2 ([3]). *Let $a_{st} \in C^{2,\alpha}(\overline{\Omega})$, symmetric, and there exists a constant $\lambda > 0$ such that $a_{st}(x)\xi_i\xi_j \geq \lambda|\xi|^2$, $\forall \xi \in \mathbb{R}^n$ and $\forall x \in \overline{\Omega}$. Let $b_i \in C^{3,\alpha}(\partial\Omega)$, and there exists a constant $\chi > 0$ such that $b_i(x)\gamma_i(x) \geq \chi$, $\forall x \in \partial\Omega$.*

Then there exists a constant N_0, depending only on $\|a_{st}\|_{C^{1,\alpha}}, \|b_i\|_{C^{3,\alpha}}, n, \lambda, \chi$ such that for all $N > N_0$, L^N is a bijection. Furthermore, L^N depends continuously on a_{st}, b_i with respect to the corresponding topologies.

Having the above theorem, by [1, Theorem 7.3], $(L^{u,N})^{-1}$ maps $C^{1,\alpha}(\overline{\Omega}) \times C^{2,\alpha}(\partial\Omega) \times C^{2,\alpha}(\partial\Omega)$ into $C^{5,\alpha}(\overline{\Omega})$, and its norm as a linear map between these spaces is bounded by a constant depends only on $\|a_{st}\|_{C^{2,\alpha}}, \|b_i\|_{C^{3,\alpha}}, \lambda$ and χ. It follows that

$$u \mapsto (L^{u,N})^{-1} R^{u,N}[u]$$

is a compact operator from $\mathcal{O}$ to $C^{4,\alpha}(\overline{\Omega})$.

Moreover, $(\tilde{F}, \tilde{G})[u] = 0$ is the same as $u + (L^{u,N})^{-1} R^{u,N}[u] = 0$, i.e.,

$$\partial\mathcal{O} \cap (Id + (L^{u,N})^{-1} R^{u,N})^{-1}(0) = \partial\mathcal{O} \cap (F,G)^{-1}(0) = \emptyset.$$

Therefore, we can define the degree of (F, G) as the Leray-Schauder degree of the map $u \mapsto u + (L^{u,N})^{-1} R^{u,N}[u]$. More precisely we have the following definition.

Definition 0.3. *Let F, G be operators as above, and $\mathcal{O} \subset C^{4,\alpha}(\overline{\Omega})$ is a bounded open set with $\partial\mathcal{O} \cap (F,G)^{-1}(0) = \emptyset$. We define a degree of (F,G) on $\mathcal{O}$ at 0 by*

$$(0.7) \qquad \deg((F,G), \mathcal{O}, 0) = \deg_{L.S.}(Id + (L^{u,N})^{-1} R^{u,N}, \mathcal{O}, 0),$$

where $N > N_0$, N_0 is the constant in Theorem 2.

We remark that as in [2], this definition of the degree is independent of $N > N_0$ according to the homotopy invariance of the Leray-Schauder degree. And the degree satisfies our desired properties in Theorem 1.

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Rigidity of equality cases in symmetrization inequalities

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<div align="center">(joint work with F. Cagnetti, M. Colombo, and G. De Philippis)</div>

Symmetrization inequalities, and, in particular, necessary conditions for their equality cases, are commonly used to prove symmetry properties of minimizers in geometric variational problems. The archetypical example of this method is Steiner's proof of the isoperimetric inequality, as made rigorous in the context of sets of finite perimeter by De Giorgi [6]. We consider here the problem of understanding *rigidity* of equality cases in a given symmetrization inequality. Sufficient conditions for rigidity of equality cases have been obtained in the case of the Polya-Szego inequality for Dirichlet-type integrals (Brothers and Ziemer [2]), and in the case of the Steiner's perimeter inequality for perimeter of sets (Chlebík, Cianchi and Fusco [5]). We have obtained geometric conditions that actually *characterize* rigidity in two model examples: Ehrhard's symmetrization for Gaussian perimeter [3], and Steiner's symmetrization for Euclidean perimeter [4]. We focus here on the latter problem.

Consider a Borel function $v : \mathbb{R}^{n-1} \to [0, \infty)$, and let $F[v]$ be the sets of points $x = (x', x_n) \in \mathbb{R}^n$ such that $|x_n| < v(x')/2$. Given a set $E \subset \mathbb{R}^n$, one denotes by $E_z = \{t \in \mathbb{R} : (z, t) \in E\}$ the vertical section of E above $z \in \mathbb{R}^{n-1}$, says that E is v-distributed if $v(z) = \mathcal{H}^1(E_z)$ for $\mathcal{H}^{n-1}$-a.e. $z \in \mathbb{R}^{n-1}$, and sets $E^s = F[v]$ for the *Steiner's symmetral of E*. Steiner's inequality gives

$$(0.1) \qquad P(E) \geq P(F[v]), \qquad \text{for } E \subset \mathbb{R}^n \text{ } v\text{-distributed},$$

where $P(E)$ denotes the distributional perimeter of E. (We notice that $P(F[v]) < \infty$ if and only if

$$(0.2) \qquad v \in BV(\mathbb{R}^{n-1}), \qquad \mathcal{H}^{n-1}(\{v > 0\}) < \infty,$$

where $BV(\mathbb{R}^{n-1})$ is the space of functions of bounded variation on $\mathbb{R}^{n-1}$.) Let us denote by $\mathcal{M}(v)$ the set of equality cases in (0.1). The rigidity problem amounts in characterizing those function v as in (0.2) such that

$$(0.3) \qquad \mathcal{M}(v) = \left\{ t\, e_n + F[v] : t \in \mathbb{R} \right\}.$$

The inclusion $\supset$ is, of course, trivial, while the inclusion $\subset$ may fail for various reasons: (i) the projection $\{v > 0\}$ of $F[v]$ could be "disconnected"; (ii) the set $\{v = 0\}$ may "disconnect" the projection; (iii) it could be that the jump set of

v "disconnects" the projection; (iv) the projection could contain an "integrable" Cantorian part of Dv. Chlebík, Cianchi and Fusco in [5] provide a sufficient condition for rigidity by ruling out these four possibilities.

Theorem 0.1 (Chlebík, Cianchi, Fusco [5]). *If (a) Ω is an open connected set, (b) $v \in W^{1,1}(\Omega)$, and (c) the Lebesgue representative of v is positive $\mathcal{H}^{n-2}$-a.e. on Ω, then $P(E; \Omega \times \mathbb{R}) = P(F[v]; \Omega \times \mathbb{R})$ implies the existence of $t \in \mathbb{R}$ such that $E \cap (\Omega \times \mathbb{R}) = (t\, e_n + F[v]) \cap (\Omega \times \mathbb{R})$.*

Notice that assumptions (a) and (c) – together with the choice of working with the localized Steiner's inequality over $\Omega \times \mathbb{R}$ – exclude problems (i) and (ii), while assumption (b) excludes the existence of jump or Cantorian parts of Dv inside Ω, and thus rules out problems (iii) and (iv). Simple examples shows that Theorem 0.1 does not characterize rigidity even in the case of polyhedra. In order to improve on this result one thus needs to understand rigidity in the presence of jumps or Cantorian parts of Dv, or of substantially large regions where v vanishes. Since the various sets involved in this heuristic statements are just Borel sets, we first need to specify in which sense a Borel set K disconnects another Borel set G. Precisely, in [3] we introduce the following definition: if K and G are Borel sets in $\mathbb{R}^m$, then K *essentially disconnects* G if there exists a non-trivial Borel partition $\{G_+, G_-\}$ of G such that

$$G^{(1)} \cap \partial^e G_+ \cap \partial^e G_- \subset_{\mathcal{H}^{m-1}} K \,.$$

(Here, $G^{(t)}$ is the set of point of density $t \in [0,1]$ of G, and $\partial^e G = \mathbb{R}^m \setminus (G^{(0)} \cup G^{(1)})$.) Notice that if $\mathcal{H}^{m-1}(K \Delta K') = \mathcal{H}^m(G \Delta G') = 0$, then K essentially disconnects G if and only if K' essentially disconnects G'. Moreover, we say that G is essentially connected if the empty set does not essentially disconnect G. When G is of finite perimeter, this is equivalent to asking that G is indecomposable in the sense of [7], [1]; see also [8, 4.2.25].

Let us now denote by $v^\wedge$ and $v^\vee$ the lower and upper approximate limits of v (so that $v^\wedge$ and $v^\vee$ are pointwise unambiguously defined on $\mathbb{R}^{n-1}$ in $\mathcal{H}^{n-1}$-equivalence class of v), let $[v] = v^\vee - v^\wedge$ denote the jump of v, and let $S_v = \{[v] > 0\}$. Starting from a sharp regularity result for barycenter functions of sets with segments as sections, see Theorem 0.2, we can use these notions to formulate several characterizations of rigidity.

Theorem 0.2. *If E is a v-distributed set with segments as vertical sections and $b_E(z)$ denotes the barycenter of E_z, then $b_{M,\delta} = \tau_M(1_{\{v > \delta\}}\, b_E) \in BV(\mathbb{R}^{n-1})$ for a.e. $\delta, M > 0$, where $\tau_M(s) = \max\{-s, \min\{M, s\}\}$, $s \in \mathbb{R}$. Moreover, $E \in \mathcal{M}(v)$ if and only if the approximate gradient ∇b_E of b_E vanishes $\mathcal{H}^{n-1}$-a.e. on $\mathbb{R}^{n-1}$, $2\,[b_E] \le [v]$ $\mathcal{H}^{n-2}$-a.e. on $\{v^\wedge > 0\}$, and $D^c b_{M,\delta} = f_{M,\delta}\, D^c v$ for a Borel function $f_{M,\delta} : \mathbb{R}^{n-1} \to [-1/2, 1/2]$.*

Theorem 0.3. *If v satisfies (0.2) and $D^s v \llcorner \{v^\wedge > 0\} = 0$, then rigidity holds if and only if $\{v^\wedge = 0\}$ does not essentially disconnect $\{v > 0\}$. This last condition is in turn equivalent in asking that $F[v]$ is indecomposable.*

Theorem 0.4. *If $F[v]$ is a generalized polyhedron (roughly speaking, v consists of finitely many Sobolev functions over finitely many indecomposable sets of finite perimeter), then rigidity holds if and only if $\{v^\wedge = 0\} \cup \{[v] > \epsilon\}$ does not essentially disconnect $\{v > 0\}$.*

Theorem 0.5. *If $v \in SBV(\mathbb{R}^{n-1})$ (i.e. $D^c v = 0$) and S_v is locally $\mathcal{H}^{n-2}$-finite, then every $E \in \mathcal{M}(v)$ is obtained by countably many vertical translations of $F[v]$ (above disjoint Borel sets in $\mathbb{R}^{n-1}$). In particular, rigidity holds if and only if v has the following mismatched stairway property: If $\{G_h\}_{h \in I}$ is a Borel partition of $\{v > 0\}$ with $\sum_{h \in I} P(G_h \cap B_R \cap \{v > \delta\}) < \infty$ for a.e. $\delta, R > 0$, and if $\{c_h\}_{h \in I} \subset \mathbb{R}$ is a sequence with $c_h \neq c_k$ whenever $h \neq k$, then there exist $h_0, k_0 \in I$ with $h_0 \neq k_0$, and a Borel set Σ with*

$$\Sigma \subset \partial^e G_{h_0} \cap \partial^e G_{k_0} \cap \{v^\wedge > 0\}, \qquad \mathcal{H}^{n-2}(\Sigma) > 0,$$

such that $[v](z) < 2|c_{h_0} - c_{k_0}|$ for every $z \in \Sigma$.

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Uniformization of surfaces with conical singularities

ANDREA MALCHIODI

(joint work with D.Bartolucci, A.Carlotto, F.De Marchis, D.Ruiz)

We study some singular equations, motivated by the problem of the Gaussian curvature prescription, and from some models in physics such as self-dual Chern-Simons theory or Electroweak theory: we prove some existence results exploiting the variational structure of the problem.

Consider a compact surface Σ endowed with a metric g: with the conformal change of metric $\tilde{g} = e^{2w}g$ one has

$$-\Delta_g w + K_g = K_{\tilde{g}}e^{2w},$$

where K_g (resp. $K_{\tilde{g}}$) stands for the Gaussian curvature of g (resp. $\tilde{g}$).

If one wishes to solve the *Uniformization problem*, one is then led to study the Liouville equation

$$-\Delta_g w + K_g = \rho e^{2w},$$

where ρ is a constant determined by the Gauss-Bonnet formula.

We consider the problem of obtaining a given conical structure at a finite set of points $\{p_1, \ldots, p_m\} \subseteq \Sigma$ and to get at the same time constant Gaussian curvature on $\Sigma \setminus \{p_1, \ldots, p_m\}$. Then we reduced to solving the singular PDE

$$(0.1) \qquad\qquad -\Delta_g w + K_g = \rho e^{2w} - 2\pi \sum_{i=1}^{m} \alpha_j \delta_{p_j},$$

where the $\alpha_j > -1$ is related to the conical angle θ_j at p_j by $\theta_j = 2\pi(1 + \alpha_j)$. Normalizing Σ to unit volume, the value of the constant ρ is determined by a modified Gauss-Bonnet formula, see [14]

$$\rho = 2\pi \left[\chi(\Sigma) + \sum_J \alpha_j \right].$$

For applications in physics the constant ρ is determined instead by the physical parameters involved in the problem, see e.g. [13], [15].

Some approach to study the above problem relies on Leray-Schauder degree theory, see [9], [10], while we will exploit the variational structure of (0.1): letting $G_p(x)$ be the Green's function of $-\Delta_g$ on Σ with pole at p

$$-\Delta_g G_p(x) = \delta_p - \frac{1}{|\Sigma|} \quad \text{on } \Sigma, \ \text{ with } \int_\Sigma G_p(x)\, dV_g = 0,$$

by the substitution

$$u \mapsto u + 2\pi \sum_{j=1}^{m} \alpha_j G_{p_j}(x)$$

(0.1) becomes

$$(0.2) \qquad\qquad -\Delta_g u + f(x) = \rho\, \tilde{h}(x) e^{2u} \qquad\qquad \text{on } \Sigma,$$

where $f(x)$ is smooth on Σ and where $\tilde{h}$ is a non-negative function with the asymptotic $\tilde{h}(x) \simeq d_g(x, p_i)^{2\alpha_i}$ near the singularities.

(0.2) is the Euler-Lagrange equation of the functional

$$(0.3) \qquad J_\rho(u) = \int_\Sigma |\nabla_g u|^2 \, dV_g + 2\int_\Sigma f(x)u\, dV_g - \rho \log \int_\Sigma h(x) e^{2u}\, dV_g$$

defined on $H^1(\Sigma, g)$.

To study J_ρ, a singular variant of the classical Moser-Trudinger inequality was proved in [14] (see also [8])

$$(0.4) \quad \log \int_\Sigma \tilde{h}\, e^{2(u - \overline{u})}\, dV_g \leq \frac{1}{4\pi \min\{1, 1 + \min_i \alpha_i\}} \int_\Sigma |\nabla_g u|^2 \, dV_g + C_{\tilde{h}, \Sigma, g}.$$

When ρ is small the latter inequality implies that J_ρ is coercive, so critical points can be found via direct methods of the calculus of variations. We tackle the problem min-max or Morse theory to prove general existence results in non-coercive settings. To apply variational methods some compactness criteria are needed. In [3], [4], extending some previous results by H.Brezis, F.Merle, Y.Li and I.Shafrir to the singular case, it was proved that solutions to (0.2) are uniformly bounded if $\rho \notin \Lambda_{\underline{\alpha}}$, where $\Lambda_{\underline{\alpha}}$ is a discrete set of numbers defined by

$$\Lambda_{\underline{\alpha}} := \left\{ 4k\pi + \sum_{j \in J} 4\pi(1 + \alpha_j) \; : \; k \in \mathbb{N}^*, J \subseteq \{1, \ldots, m\} \right\}.$$

Given $q \in \Sigma$ we define a weighted cardinality in the following way:

$$\chi(q) = \begin{cases} 1 + \alpha_j & \text{if } q = p_j \text{ for some } j = 1, \ldots, m; \\ 1 & \text{otherwise.} \end{cases}$$

χ on finites sets of Σ is obtained extending χ by additivity. We then define a set of *admissible probability measures* by

$$(0.5) \qquad \Sigma_{\rho,\underline{\alpha}} = \left\{ \sum_{q_j \in I} t_j \delta_{q_j} : \sum_{q_j \in I} t_j = 1, \; t_j \geq 0, \; q_j \in \Sigma \quad 4\pi\chi(I) < \rho \right\}.$$

Our main result reads as follows.

Theorem 0.1 ([7]). *Suppose $\Sigma_{\rho,\underline{\alpha}}$ is defined as in (0.5) and that it is endowed with the topology of weak distributions. Then (0.1) has a solution provided $\rho \notin \Lambda_{\underline{\alpha}}$ and $\Sigma_{\rho,\underline{\alpha}}$ is not contractible.*

The above result extends in full generality other previous ones which also used a variational approach, see [1], [2], [5], [6] and [12], and in particular it allows the weights α_i's to have different signs. The main arguments in the proof rely on the construction of suitable test functions parametrized on $\Sigma_{\rho,\underline{\alpha}}$ and on some improvements of the Moser-Trudinger/Troyanov inequality, in the spirit of [11] for the regular case.

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New developments in Nonlinear Potential Theory

GIUSEPPE MINGIONE

(joint work with Tuomo Kuusi)

1. ELLIPTIC ESTIMATES

One of the basic results in nonlinear potential theory is that the typical linear pointwise estimates via fundamental solutions actually find a precise analog in the case of nonlinear equations. This is achieved via Wolff and Riesz potentials, defined for $\beta > 0$ as

$$\mathbf{I}^\mu_\beta(x, R) := \int_0^R \frac{|\mu|(B(x, \varrho))}{\varrho^{n-\beta}} \frac{d\varrho}{\varrho}, \quad \mathbf{W}^\mu_{\beta,p}(x, R) := \int_0^R \left(\frac{|\mu|(B(x, \varrho))}{\varrho^{n-\beta p}} \right)^{1/(p-1)} \frac{d\varrho}{\varrho},$$

respectively. We shall therefore consider quasilinear, possibly degenerate equations of the type

$$(1.1) \qquad -\operatorname{div} a(Du) = \mu \qquad \text{in } \Omega,$$

where $\Omega \subset \mathbb{R}^n$ is an open subset, $n \geq 2$ and the vector field $a \colon \mathbb{R}^n \to \mathbb{R}^n$ is of class C^1 and satisfies the following growth and ellipticity assumptions:

$$(1.2) \qquad |a(z)| + |z||\partial a(z)| \leq L|z|^{p-1}, \qquad \nu|z|^{p-2}|\lambda|^2 \leq \langle \partial a(z)\lambda, \lambda \rangle$$

whenever $z, \lambda \in \mathbb{R}^n$. Here it is $0 < \nu \leq L$ and we concentrate on the case $p \geq 2$; μ denotes a Borel measure with finite total mass. Solutions are initially taken to be of class $W^{1,p}$, while more general ones can be achieved via approximations; in other words we shall concentrate on a priori estimates. Assumptions (1.2) are motivated by the classical p-Laplacean operator given by $\triangle_p u := \operatorname{div}(|Du|^{p-2}Du)$, which is the primary model case we have in mind when $p > 2$. Anyway all the results below are new and nontrivial already in the non degenerate case $p = 2$, since the

main point here is to obtain *linear type estimates for problems that are genuinely nonlinear.* The first nonlinear potential estimate is due to Kilpelainen & Malý [4] who established the following pointwise optimal bound via Wolff potentials:

$$(1.3) \qquad |u(x)| \leq c\mathbf{W}_{1,p}^{\mu}(x,R) + c\left(\fint_{B(x,R)} |u|^{p-1}\, dy\right)^{1/(p-1)}.$$

in the case of non negative measures μ. See also [14] for a different proof and [3] for the case of general measures. The orthodoxy of nonlinear potential theory indeed prescribes that Wolff potentials must be considered when dealing with the p-laplacean operator. The following result breaks up this principle.

Theorem 1.1 (Riesz potential estimate [7, 12]). *Let $u \in W^{1,p}(\Omega)$ be a solution to the equation (1.1), under the assumptions (1.2). There exists a constant $c \equiv c(n,p,\nu,L)$ such that the Riesz potential estimate*

$$(1.4) \qquad |Du(x_0)|^{p-1} \leq c\mathbf{I}_1^{\mu}(x_0,R) + c\left(\fint_{B(x_0,R)} |Du|\, dx\right)^{p-1}$$

holds whenever $B(x_0,R) \subseteq \Omega$ and the right hand side is finite. Moreover, if

$$\lim_{R \to 0} \mathbf{I}_1^{\mu}(x,R) = 0 \quad \text{locally uniformly in } \Omega \text{ w.r.t. } x$$

then Du is continuous in Ω.

The methods linked to Theorem 1.1 also provide information in the vectorial case. An example is given by the following nonlinear version of a classical result of Stein [13] (asserting the continuity of functions in terms of Lorentz space regularity of its derivatives).

Theorem 1.2 (Nonlinear Stein theorem [10]). *Let $u \in W^{1,p}(\Omega, \mathbb{R}^N)$ be a vector valued solution to the p-Laplacean systems $-\triangle_p u = F$, where the components of the vector field F belong to the Lorentz space $L(n,1)$, locally in Ω. Then Du is continuous in Ω.*

Estimates (1.3) and (1.4) are size estimates; it is nevertheless possible to estimate oscillations of solutions via potentials. An example is given in the following:

Theorem 1.3 ([6]). *Let $u \in W^{1,p}(\Omega)$ be a weak solution to the equation (1.1) under assumptions (1.2). Let $B_R \subset \Omega$ be such that $x,y \in B_{R/4}$; then*

$$
\begin{aligned}
|u(x) - u(y)| \quad &\leq \quad \frac{c}{\alpha}\left[\mathbf{I}_{p-\alpha(p-1)}^{\mu}(x,R) + \mathbf{I}_{p-\alpha(p-1)}^{\mu}(y,R)\right]^{1/(p-1)} |x-y|^{\alpha} \\
(1.5) \qquad &\quad + \frac{c}{\alpha}\fint_{B_R} |u|\, d\tilde{x} \cdot \left(\frac{|x-y|}{R}\right)^{\alpha}
\end{aligned}
$$

holds uniformly in $\alpha \in [0,1]$. Indeed, the constant c depend only $n,p,L/\nu$.

The loss of information when $\alpha \to 0$, due to the appearance of c/α, is unavoidable since estimate 1.3 is optimal. This can be fixed by using suitable maximal operators, and we refer to [6, 11] for more estimates.

2. Parabolic estimates

Proving potential estimates in the parabolic case

$$(2.1) \qquad u_t - \operatorname{div} a(Du) = \mu \qquad \text{in } \Omega_T := \Omega \times (-T, 0)$$

requires different ideas for the case $p > 2$, while the case $p = 2$ can be treated along the lines of the elliptic case (see [3]). In particular, when $p > 2$ a careful use of DiBenedetto's intrinsic geometry [1] in the context of potential theory is needed. We therefore consider so called intrinsic cylinders $Q_r^\lambda(x_0, t_0) := B(x_0, r) \times (t_0 - \lambda^{2-p}r^2, t_0)$ for $\lambda > 0$, with the idea that on $Q_r^\lambda(x_0, t_0)$ the norm of the gradient is comparable to λ. Accordingly we define the Riesz type intrinsic caloric potentials:

$$\mathbf{I}_{1,\lambda}^\mu(x_0, t_0; r) := \int_0^r \frac{|\mu|(Q_\varrho^\lambda(x_0, t_0))}{\varrho^{N-1}} \, \frac{d\varrho}{\varrho}, \qquad N := n + 2 \,.$$

Theorem 2.1 ([8, 9])**.** *Let u be a solution to* (2.1) *under the assumptions* (1.2)*. There exist a constant $c > 1$ depending only on n, p, ν, L, such that the following implication holds:*

$$(2.2) \qquad c\mathbf{I}_{1,\lambda}^\mu(x_0, t_0; r) + c \left(\fint_{Q_r^\lambda(x_0, t_0)} |Du|^{p-1} \, dx \, dt \right)^{1/(p-1)} \le \lambda$$
$$\implies |Du(x_0, t_0)| \le \lambda$$

whenever $Q_r^\lambda(x_0, t_0) \subset \Omega_T$ and (x_0, t_0) is Lebesgue point of Du.

The previous result is of intrinsic type, and it is indeed a conditional estimate, but it implies bounds on arbitrary parabolic cylinders. For instance it implies the following local estimate:

$$|Du(x_0, t_0)| \le c\mathbf{I}_1^{|\mu|}(x_0, t_0; r) + c \fint_{Q_r(x_0, t_0)} (|Du| + 1)^{p-1} \, dx \, dt \,,$$

that in the case $\mu \equiv 0$ gives back the classical gradient L^∞-estimate of Di Benedetto & Friedman [2]. Moreover, (2.2) is sharp in the sense that when μ is the Dirac mass charging $(0, 0) \in \mathbb{R}^n \times \mathbb{R}$, it allows to recover the behaviour of the so called Barenblatt solution. For further results we again refer to [9].

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PDEs in metric measure spaces and geometric applications

Andrea Mondino

(joint work with Luigi Ambrosio, Nicola Garofalo, Nicola Gigli, Tapio Rajala, Giuseppe Savaré)

The framework of the talk is given by a metric measure space (m.m.s. for short) $(X, \mathsf{d}, \mathfrak{m})$; i.e. (X, d) is a complete and separable metric space, and $\mathfrak{m}$ is a positive Radon measure on it (for simplicity in this talk we assume also that X is compact and $\mathfrak{m}(X) = 1$, if not specified differently). Metric measure spaces are, a priori, singular objects which naturally arise as limits of Riemannian manifolds in many contexts (as limit spaces in the theory of Cheeger-Colding, as singular objects in geometric flows such as Ricci Flow, as singularities in Physical situations such as black holes in general relativity, etc.). The goal of the talk is to discuss the interplay between analysis and geometry in a m.m.s, more precisely how the analysis of PDEs in a m.m.s. can give geometric informations, and vice-versa how the geometry of a m.m.s can be used to establish estimates on solutions to natural PDEs. We will focus on the notion of Ricci curvature bounded from below.

We denote with $(P(X), W_2)$ the Wasserstein space of probability measure endowed with the quadratic transportation distance. On $(P(X), W_2)$ we have the following entropies:

$$U_{N,\mathfrak{m}}(\rho\mathfrak{m}) \quad := \quad -N \int_X \rho^{1-\frac{1}{N}} \, d\mathfrak{m}, \quad \text{Renyi entropy, } N \in [1, +\infty)$$

$$U_{\infty,\mathfrak{m}}(\rho\mathfrak{m}) \quad := \quad \int_X \rho \log \rho \, d\mathfrak{m}, \quad \text{Shannon entropy.}$$

Analyzing the convexity properties of the above entropies along geodesics in $(P(X), W_2)$, Lott-Villani [13] and Sturm [17]-[18] defined the notion of Ricci curvature bounded below by $K \in \mathbb{R}$ and dimension bounded above by $N \geq 1$ in a

m.m.s.; this is called $\mathsf{CD}(K,N)$-condition. An a priori weaker modification of this notion, given by Bacher-Sturm in [6], called $\mathsf{CD}^*(K,N)$-condition enjoys better localization and tensorization properties.

Since the $\mathsf{CD}(K,N)$ and the $\mathsf{CD}^*(K,N)$ conditions include Finsler structures (see [15] and [19]), in order to isolate the Riemannian-like spaces, Ambosio-Gigli-Savaré [3] (see also [1] for simplifications of the axiomatization and extensions to σ-finite measures) enforced the $\mathsf{CD}(K,\infty)$ with the linearity of the heat flow getting the so-called $\mathsf{RCD}(K,\infty)$ condition (for the definition of heat flow in a metric setting see [2]). An important result of [3] and [1] is that the $\mathsf{RCD}(K,\infty)$ condition is equivalent to ask that the heat flow is the gradient flow of the Shannon entropy $U_{\infty,\mathfrak{m}}$ in the Wasserstein space $(P(X),W_2)$ in a suitable strong sense which implies K-contractivity (the so called EVI_K gradient flow condition). We stress that in this way the *geometric condition* given by asking that $(X,\mathsf{d},\mathfrak{m})$ has Ricci curvature bounded below by K and has a Riemannian-like structure is encoded by an *analytic condition* in terms of gradient flows.

The finite dimensional case has been investigated independently by Erbar-Kuwada-Sturm [7] and Ambrosio-M.-Savaré [5], here we focus on the approach of the second group (the approach of the first group is quite different even if the final results are comparable). In analogy with the infinite dimensional case, one says that $(X,\mathsf{d},\mathfrak{m})$ is a $\mathsf{RCD}^*(K,N)$-space if it satisfies the $\mathsf{CD}^*(K,N)$-condition and if the the heat flow is linear.

A natural question is to ask if there is some relation between the $\mathsf{RCD}^*(K,N)$ condition and the Bochner inequality

$$(0.1) \qquad \frac{1}{2}\Delta|\nabla f|^2 \geq \frac{1}{N}|\Delta f|^2 + K|\nabla f|^2 + \nabla\Delta f \cdot \nabla f$$

properly understood in a weak sense. The answer is that the two notions, under some technical conditions, are equivalent both in the infinite dimensional setting [4] and in the finite dimensional one [5],[7]. The proof of this *geometric result* passes through the theory of gradient flows in Wasserstein spaces and in the finite dimensional case (in the approach of [5]) has the further complication that the gradient flow of the Renyi entropy $U_{N,\mathfrak{m}}$ is the *non linear diffusion flow* defined by the equation

$$\partial_t \rho_t = \Delta((\rho_t)^{1-\frac{1}{N}}).$$

As a consequence of the equivalence between the $\mathsf{RCD}^*(K,N)$ condition and the dimensional Bochner inequality, in collaboration with Garofalo [8] we established Li-Yau and Harnack-type inequalities on the heat flow in $\mathsf{RCD}^*(K,N)$-spaces.

Therefore on one hand the *analysis* in m.m.s. (in particular the theory of gradient flows) was fundamental to get *geometric* informations (Ricci curvature bounds, dimension upper bounds, Riemannian-like behavior, Bochner inequality); on the other hand the *geometry* of the m.m.s. (as above) was fundamental to get estimates on the solutions of natural PDEs (e.g. the aforementioned Li-Yau and

Harack-type inequalities on the heat flow).

We finish the talk by presenting some results concerning static PDEs in a m.m.s. In [9], Gigli studied the Laplacian in m.m.s. and proved that the classical Laplacian comparison theorems hold true also in the non smooth setting; adapting techniques from [9], in collaboration with Gigli we studied the p-Laplace equation in m.m.s. (see [11]). One of our main results is that the minimizers of the associated p-energy satisfy in a suitable weak sense the classical p-Laplace equation, so that they enjoy the natural sheaf property (this was an open problem in non smooth spaces). We apply these techniques to prove that the Buseman functions associated to a line in a m.m.s. are harmonic; this fact played an important role in the proof of the Splitting Theorem in non smooth spaces by Gigli [10], a breakthrough in the theory of $\mathsf{RCD}^*(K, N)$-spaces. Combining the splitting theorem and some ideas of Preiss [16] on iterated tangent cones, in collaboration with Gigli and Rajala [12] we proved that, if $(X, \mathsf{d}, \mathfrak{m})$ is a $\mathsf{RCD}^*(K, N)$-space, then for $\mathfrak{m}$-a.e. $x \in X$ the collection of local blow up contains a euclidean space of dimension at most N. We end by mentioning that, inspired by the duality of differentials and gradients in m.m.s. studied by Gigli in [9], one can define the notion of (a possibly multivalued) angle in a m.m.s. even without assuming curvature conditions; this is the content of [14].

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Characterizations of Bounded Ricci Curvature on Smooth and NonSmooth Spaces

Aaron Naber

There are two primary goals to this talk. In the first part we study smooth metric measure spaces $(M^n, g, e^{-f}dv_g)$ and give several ways of characterizing bounds $-\kappa g \leq \mathrm{Ric} + \nabla^2 f \leq \kappa g$ on the Ricci curvature of the manifold. In particular, we see how bounded Ricci curvature on M controls the analysis of path space $P(M)$ in a manner analogous to how lower Ricci curvature controls the analysis on M. In the second part we develop the analytic tools needed to in order to use these new characterizations to give a definition of bounded Ricci curvature on general metric measure spaces (X, d, m). We show that on such spaces many of the properties of smooth spaces with bounded Ricci curvature continue to hold on metric-measure spaces with bounded Ricci curvature.

In more detail, in the first part of the talk we see that bounded Ricci curvature can be characterized in terms of the metric-measure geometry of path space $P(M)$. The correct notion of geometry on path space is one induced by what we call the parallel gradient, and the measures on path space of interest are the classical Wiener measures. Our first characterization shows that bounds on the Ricci curvature are equivalent to certain parallel gradient estimates on path space. These turn out to be infinite dimensional analogues of the Bakry-Emery gradient estimates. Our second characterization relates bounded Ricci curvature to the stochastic analysis of path space. In particular, we see that bounds on the Ricci curvature are equivalent to the appropriate $C^{\frac{1}{2}}$-time regularity of martingales on $P(M)$. Our final characterization of bounded Ricci curvature relates Ricci curvature to the analysis on path space. Specifically, we study the Ornstein-Uhlenbeck operator L_x, a form on infinite dimensional laplacian on path space, and prove sharp spectral gap and log-sobolev estimates under the assumption of bounded Ricci curvature. Further we show these estimates on L_x are again equivalent to bounds on the Ricci curvature. We have analogous results for d-dimensional bounded Ricci curvature.

In the second part of the talk we study metric measure spaces (X, d, m) and use the structure of the first part of the paper to define the notion of bounded Ricci curvature. A primary technical difficulty is to describe the notion of the parallel gradient in such a setting. Even in the smooth case one requires some deep ideas from stochastic analysis, namely the stochastic parallel translation map, to deal with this. Our replacement for this allows us to sidestep the need for the stochastic parallel translation map, and in particular works on an arbitrary metric space. After this is introduced and studied we spend the rest of the paper proving various structural properties of metric-measure spaces with bounded Ricci curvature. Among others, we will see that spaces with Ricci curvature bounded by κ have lower Ricci curvature bounded from below by $-\kappa$ in the sense of Lott-Villani-Sturm. We will see that spaces with bounded Ricci curvature continue to have well behaved martingales. Further, we will see that not only can one define the Ornstein-Uhlenbeck operator on path space, which still behaves as an infinite dimensional laplacian on path space, but that on spaces with bounded Ricci curvature these operators still enjoy poincare and log-sobolev estimates. In particular, these tools allow us to do analysis on the path space of metric-measure spaces.

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The Yang-Mills Plateau problem in supercritical dimension

Mircea Petrache

(joint work with Tristan Rivière)

Consider a horizontal 5-plane distribution on a principal $SU(2)$-bundle $P \to \partial M$, where M is a 5-dimensional compact manifold with boundary. We consider the question of extending this plane distribution over the interior of M in the "most integrable way". This corresponds to minimizing the L^2-norm of the curvature of the corresponding connection ∇, i.e. the Yang-Mills functional

$$(0.1) \qquad \mathcal{YM}(\nabla) = \int_M |F_\nabla|^2.$$

In dimension ≤ 4 the minimization can be done in the space $\mathcal{A}^{1,2}(M)$ of locally Sobolev connections over associated vector bundles $E \to M$ with structure group $SU(2)$. The space $\mathcal{A}^{1,2}(M)$ has indeed the correct closure property:

Theorem 0.1 ([11],[8],[7],[6]). *If $\nabla_k \in \mathcal{A}^{1,2}(M)$ correspond to L^2-curvatures F_k converging weakly in L^2 to a $\mathfrak{su}(2)$-valued 2-form F then $F = F_\nabla$ for some connection $\nabla \in \mathcal{A}^{1,2}(M)$.*

This result allows to prove, in dimensions $n \leq 4$, the existence in the class $\mathcal{A}^{1,2}(M)$ of a minimizer for the *Yang-Mills Plateau problem*

$$(0.2) \qquad \min \left\{ \int_M |F_A|^2 : \; g(i^*_{\partial M} A) = \bar{A} \right\}.$$

We use the notation F_A for the curvature form which in local trivializations corresponds to connection forms $A = \{A_i\}_{i \in \mathcal{I}}$ and the notation $g(i^*_{\partial M} A) = \bar{A}$ means that there exists a global gauge change g over ∂M such that the local expressions $\bar{A} = \{\bar{A}_i\}_{i \in \mathcal{I}}$ correspond to the local expressions of the restriction $i^*_{\partial M}$ after the gauge change by g, i.e. $A'_i = g^{-1}dg + g^{-1}i^*_{\partial M}A_i g$ for $i \in \mathcal{I}$.

The main phenomenon behind Theorem 0.1 is the fact proved by Uhlenbeck [11] that when the energy $\|F\|^2_{L^2}$ is smaller than a geometric threshold $\epsilon_0 > 0$ we may find coordinates in which the energy controls the connection form:

$$(0.3) \qquad \|A\|_{W^{1,2}} \leq C\|F\|_{L^2}.$$

This control fails in dimensions $n \geq 5$ and the condition that the connection forms A be locally $W^{1,2}$ in some trivialization is *not* preserved under weak L^2-convergence of the curvature forms in dimensions $n \geq 5$. These dimensions are therefore named *supercritical dimensions*.

The correct weakening of the space $\mathcal{A}^{1,2}$ for dimension 5 is the space of weak connections defined by imposing a suitable condition on their slices:

Definition 0.2 (weak curvatures in 5-dimensions). *We say that a L^2-integrable $\mathfrak{su}(2)$-valued 1-form A over $\mathbb{B}^5$ is a* weak connection *if the following conditions hold:*

- *The $\mathfrak{su}(2)$-valued 2-form F satisfying the distributional equation $F \overset{\mathcal{D}}{=} dA + A \wedge A$ is L^2-integrable on $\mathbb{B}^5$.*
- *For all x and almost all $r > 0$ the slice $i^*_S A$ along the 4-sphere $S = \partial B_r(x)$ is gauge-equivalent to an element of $\mathcal{A}^{1,2}(S)$.*

The above space of gauge-equivalence classes of weak connections is denoted by $\mathcal{A}_{SU(2)}(\mathbb{B}^5)$.

We then proved the following closure result, which ensures the existence of minimizers of the Yang-Mills Plateau problem (0.2).

Theorem 0.3 ([6]). *Let F_k be curvatures corresponding to weak connections as above over $\mathbb{B}^5$ with the uniform bound $\|F_k\|_{L^2(\mathbb{B}^5)} \leq C$ and converging weakly in L^2 to a $\mathfrak{su}(2)$-valued 2-form F. Then locally $F = dA + A \wedge A$ for a weak connection A.*

The approach of Ambrosio-Kirchheim to the closure theorem for integral currents [1] and the theory of *scans* by Hardt-Rivière [4], [3] present alogous definitions of weak objects as sets of slices "connected" via a compatibility condition based on an overlying integrable quantity (in our case this control comes from the curvature

2-form F). The closure results for such objects comes from the interplay of three ingredients:

- A *geometric distance* on sliced 1-forms: for A, A' which are L^2-connection forms over $\mathbb{S}^4$ we use the gauge-orbit distance

$$\mathrm{dist}(A, A') := \min\{\|A - g(A')\|_{L^2(\mathbb{S}^4)} : g \in W^{1,2}(\mathbb{S}^4, SU(2))\}.$$

  This corresponds ot the use of the flat distance for the closure theorem of integral currents by Ambrosio-Kirchheim [1].
- The fact that *the above distance interacts well with our energy at the level of slices*, which follows from Theorem 0.1. More precisely we have that sublevels of $A \mapsto \|F_A\|_{L^2(\mathbb{S}^4)}$ are dist-compact. In [1], [4] a similar interaction occurs between the flat distance and the mass of rectifiable currents.
- The *oscillation control on slices* of a fixed weak connection, obtained via the L^2 norm of the overlying weak curvature form 2-form F. More precisely, we have that if we identify $\mathbb{S}^4$ by homothety with each one of the spheres $S := \partial B_t(x), S' := \partial B_{t'}(x')$ then the pullbacks $A(t, x), A(t', x')$ of $i_S^* A, i_{S'}^* A$ satisfy

$$\mathrm{dist}(A(t, x), A(t', x')) \le C\|F\|_{L^2(\mathbb{B}^5)}(|x - x'| + |t - t'|)^{1/2}.$$

  In [1] the corresponding fact is the interpretation of rectifiability as a bound of the metric variation of the slices.

These ingredients allow to conclude the proof of Theorem 0.3 via an abstract compactness theorem of [4], which can be seen as a generalization of the compactness theorem of bounded variation functions. The good control on the distance dist above allows also to define the $\mathcal{A}_{SU(2),\bar{A}}(\mathbb{B}^5)$ as the weak connection classes having trace over $\partial\mathbb{B}^5$ gauge equivalent to the fixed form $\bar{A}$. This class is again closed under weak convercence of the curvature forms.

The second fact attesting the optimality of the space $\mathcal{A}_{SU(2)}(\mathbb{B}^5)$ for the study of the Yang-Mills-plateau problem (0.2) in dimension 5 is the fact that elements of $\mathcal{F}_{\mathbb{Z}}(\mathbb{B}^5)$ can be approximated by classical curvatures over bundles with finitely many defects:

Theorem 0.4 ([6]). *Consider the space $\mathcal{R}^\infty(\mathbb{B}^5)$ consisting of curvature forms on $SU(2)$-bundles over $\mathbb{B}^5 \setminus \Sigma$ for varying finite sets $\Sigma \subset \mathbb{B}^5$. Then any element $F \in \mathcal{F}_{\mathbb{Z}}(\mathbb{B}^5)$ can be approximated by elements $F_k \in \mathcal{R}^\infty(\mathbb{B}^5)$ with respact to the geometric distance*

$$\mathrm{dist}(F, F') := \min\left\{\|F - g^{-1}F'g\|_{L^2(\mathbb{B}^5)} : g : \mathbb{B}^5 \to SU(2) \text{ measurable}\right\}.$$

The proof of this result is an improvement upon the celebrated approximation techniques for nonlinear Sobolev spaces by Bethuel [2] by division into "good" and "bad" regions reminiscent of Calderon-Zygmund decompositions. In our setting the new difficulty is the nonlinear dependence $F \overset{loc.}{=} dA + A \wedge A$ of the curvature form F on the connection form A in local coordinates, and the nonlinear interplay

of different coordinate expressions of the connection form A itself. Together with results of [5], [10], [9], Theorem 0.4 allows to prove the follwoing regularity result.

Theorem 0.5 (optimal partial regularity for Yang-Mills-Plateau minimizers). *Let $\bar{A}$ be a smooth $\mathfrak{su}(2)$-valued connection 1-form over $\partial\mathbb{B}^5$. Then the minimizer of*

$$\inf\left\{\|F_A\|_{L^2(\mathbb{B}^5)} : [A] \in \mathcal{A}_{SU(2),\bar{A}}(\mathbb{B}^5)\right\}$$

belongs to a class $[A] \in \mathcal{A}_{SU(2),\bar{A}}(\mathbb{B}^5)$ which has a representative which is locally smooth connection on a smooth bundle defined outside a set of isolated points.

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Asymptotics of Teichmüller harmonic map flow

MELANIE RUPFLIN

(joint work with Peter M. Topping)

We discuss a new geometric flow, the Teichmüller harmonic map flow, which is designed to evolve maps towards minimal surfaces.

Let M be a closed orientable surface and let (N, g_N) be a Riemannian manifold of arbitrary dimension. In [3] we defined Teichmüller harmonic map flow as the L^2-gradient flow of the Dirichlet energy $E(u,g) := \frac{1}{2}\int_M |du|_g^2\, dv_g$ on the set

$$\mathcal{A} := \{[(u,g)], u : M \to N, \quad g \text{ metric on } M\}$$

of *equivalence classes* identified under the symmetries of E, namely conformal invariance and invariance under pull-back by diffeomorphisms. A canonical representative of the flow is described by

$$(0.1) \qquad \partial_t u = \tau_g(u); \qquad \partial_t g = \frac{1}{4} Re(P_g(\Phi(u,g)))$$

where $\tau_g(u)$ is the tension, $\Phi(u,g) = (|u_{x_g}|^2 - |u_{y_g}|^2 - 2i\langle u_{x_g}, u_{y_g}\rangle)dz_g^2$ the Hopf-differential and P_g the projection onto the space $\mathcal{H}(M,g)$ of holomorphic quadratic differentials on (M,g).

For M a sphere, the flow reduces to classical harmonic map flow, while for a torus the evolution of the domain metric is restricted to a two dimensional manifold and (0.1) can be rewritten to coincide with a flow of Ding-Li-Liu [5]. For surfaces of higher genus the evolution of the domain is much more complicated due to $g \mapsto Re(\mathcal{H}(g))$ being non-integrable as well as due to the non-completeness of Teichmüller space. Nonetheless, as I discussed in the talk, the flow of metrics can be shown to have very strong regularity properties and Teichmüller harmonic map flow is as regular as harmonic map heat flow itself for as long as the domain does not degenerate in moduli space [2]. In certain situations such a degeneration in finite time can be excluded. Conversely, the possibility of degeneration becomes very relevant in the asymptotic analysis.

In the second part of the talk, I thus focused on recent results concerning the asymptotic behaviour of the flow as well as the underlying ideas of the proofs. I explained that global solutions of (0.1) always find critical points of the area functional even if the domain degenerates as $t \to \infty$. More precisely [1, 3], global solutions subconverge to a limiting pair consisting of a surface (Σ, g_∞) and a map $u_\infty : (\Sigma, g_\infty) \to (N, g_N)$ that is both harmonic and weakly conformal and thus either constant or a minimal immersion away from finitely many branch points. In case of degeneration, the limiting domain (Σ, g_∞) is no longer of the same topological type as the original surface M; rather (Σ, g_∞) is a complete, non-compact and possibly disconnected surface of lower genus which results from the collapse of finitely many simple closed geodesics in the evolving domains.

The most delicate aspect of this asymptotic analysis is to show that the obtained limit map is not only harmonic but also conformal. Since the energy decays according to $\frac{d}{dt}E(u(t), g(t)) = -\|\tau_g(u)\|_{L^2}^2 - \frac{1}{32}\|P_g(\Phi(u,g))\|_{L^2}^2$ it is a priori not clear whether the full Hopf-differential, and not just its projection, converges to zero along a sequence $t_i \to \infty$. This issue is further complicated by the fact that the value of $P_g\Phi$ at any given point depends on the properties of Φ on the whole surface, including its degenerating parts. A key aspect of the proof of asymptotics thus consists of a careful and quantitative analysis of the space of holomorphic quadratic differentials, and the corresponding projection operator, for sequences of degenerating hyperbolic surfaces, see [3] and [4]. As I discussed in the talk, the results we obtained can not only be applied to Teichmüller harmonic map flow, but also lead to a uniform Poincaré inequality for quadratic differentials which

seems of independent interest; given any number $\gamma \geq 2$ the inequality

$$\|\Phi - P_g\Phi\|_{L^1(M,g)} \leq C \cdot \|\partial_{\bar{z}}\Phi\|_{L^1(M,g)}$$

holds true for any quadratic differential Φ and any hyperbolic surface (M,g) of the given genus γ. Here C depends only on the topology of the surface, that is on γ, but contrary to the Poincaré inequality for functions is independent of its geometry.

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The supercritical Lane-Emden equation and its gradient flow
Michael Struwe

Consider the Dirichlet problem for the Lane-Emden equation

$$(0.1) \qquad -\Delta u = |u|^{p-2}u \text{ on } \Omega, \ u = 0 \text{ on } \partial\Omega,$$

where Ω is a smoothly bounded domain in $\mathbb{R}^n$, $n \geq 3$. A challenging question attributed to Paul Rabinowitz is whether for domains Ω with nontrivial topology, in particular, for domains sufficiently close to an annulus, there exists a solution $u > 0$ to (0.1) for any $p > 2$ (see [3], p.S19; also see for instance [6], [7], [8], [12], [19], [22], [23], and [25] for some partial results and further background). Formally, solutions to (0.1) correspond to critical points of the functional

$$E_p(u) = \frac{1}{2}\int_\Omega |\nabla u|^2 dx - \frac{1}{p}\int_\Omega |u|^p dx, \ u \in H^1_0 \cap L^p(\Omega).$$

In joint work [2] with Simon Blatt we identify a functional analytic framework for dealing with problem (0.1) in the "supercritical" case when $p > 2^* := 2n/(n-2)$ and to discuss the corresponding heat flow

$$(0.2) \qquad u_t - \Delta u = |u|^{p-2}u \text{ on } \Omega \times [0,\infty[, \ u = 0 \text{ on } \partial\Omega \times [0,\infty[,$$

for given initial data $u_0 \in H^1_0 \cap L^p(\Omega)$ as a suitable gradient flow for E_p.

Much of our work has been inspired by Pacard's paper [21], where he presents a remarkable monotonicity formula and a partial regularity theory for "stationary" solutions to (0.1), the notion of "stationary" solution being modelled on Evans' [9] notion of a stationary weakly harmonic map. In our work [2], moreover, we extend Pacard's [21] partial regularity result to the full range of exponents $p > 2^*$.

Pacard's work motivates the study of (0.1) in the Morrey space $H_0^1 \cap L^{p,\mu}(\Omega)$ defined below, where $\mu = \frac{2p}{p-2}$. Recall that a function $f \in L^p(\Omega)$ on a domain $\Omega \subset \mathbb{R}^n$ belongs to the Morrey space $L^{p,\lambda}(\Omega)$ if

$$(0.3) \qquad \|f\|_{L^{p,\lambda}(\Omega)}^p := \sup_{x_0 \in R^n,\ r>0} r^{\lambda-n} \int_{B_r(x_0)\cap\Omega} |f|^p dx < \infty,$$

where $B_r(x_0)$ denotes the Euclidean ball of radius $r > 0$ centered at x_0, and similarly in the time-dependent setting. By combining Pacard's ideas with the arguments giving monotonicity and partial regularity for the heat flow of harmonic maps developed in [24], we obtain a monotonicity formula similar to Pacard's for the flow (0.2), improving the classical result of Giga-Kohn [13]. From this improved monotonicity formula we then deduce Morrey estimates and partial regularity results.

Combined with Adams' [1] potential theoretic methods our results even give ε-regularity results for weak solutions of class $H_0^1 \cap L^{p,\mu}$ of either (0.1) or (0.2); they can also be applied in the case when the (smooth) solution to (0.2) blows up in finite time. In the latter case the Morrey estimates hold on domains whose size naturally decreases as we approach the blow-up time. This is sufficient to recover partially regular blow-up profiles as "tangent cones" to the flow by rescaling the solution suitably around blow-up points.

A particular consequence of our estimates is that classical solutions of (0.2) on a convex domain always either blow up in finite time or uniformly decay to 0 as $t \to \infty$, which agrees with results of Matano and Merle [18] for global radially symmetric solutions of (0.2).

For so-called "borderline solutions" $u \geq 0$ of (0.2) in the sense of [20] partial regularity results similar to ours previously were obtained by Chou et al. [5], however, exploiting the additional global bounds available in that case.

Further results relevant for (0.2) can be found in [4], [10], [11], [14], [15], [16], [17], [26], and [27].

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The role of Liouville systems in the study of non-abelian Chern-Simons vortices

Gabriella Tarantello

We discuss recent results concerning planar (singular) Liouville systems in connection with the construction of non-abelian Chern-Simons vortex configurations of "non-topological type".

In gauge field theory, the interest towards the existence of non-abelian vortices (or monopole condensates) has grown considerably in recent years, on the basis of their connection with the delicate issue of quark confinement. We shall focus in a non-abelian Chern-Simons model introduced in this context by Gudnason, which involves a rather general guage group (e.g. $G = U(1) \times G'$, with G' a simple group), so to allow for solutions with orientational modes, that are also characterized by a BPS regime, and thus governed by selfdual equations. Our main goal is to establish the existence of static configurations of the self-dual equations that saturate the "minimal" energy allowed by the system. While this program has been successfully carried out in the abelian case, in the non-abelian setting it is not easy to attain. Even for the most popular choice of the gauge group (e.g. $SU(n)$, $U(1) \times SO(2n)$, $U(1) \times USp(2n)$, etc.) the self-dual equations present technical difficulties that are hard to tackle by the available analytical tools. So far, such difficulties have been handled by imposing before hand some ad-hoc structure on the vortex solution sought, consistently with their physical interpretation. In this way one arrives to reduce the whole vortex problem to a Liouville-type system in presence of singular sources, (i.e. Dirac measures supported at the vortex points), for which the unbroken and broken vacua states can be nicely identified.

Thus, as in the abelian case, the model allows for both topological and non-topological planar vortex configurations as well as those of mixed-type. Also periodic configurations are present. While topological vortices are easier to find by means of a variational formulation that allows one to use a minimization procedure to identify them (as recently worked out in collaboration with X. Han, C.S. Lin and Y. Yang, also for the periodic case), much more delicate is the construction of non-topological vortices, whose existence has been justified so far only by some numerical evidence. We illustrate the construction of non-topological solutions for the given elliptic system by means of a "perturbation" approach, which is inspired to the work of Chae-Imanuvilov, in the abelian case.

To pursue this goal, we need to provide very precise information about the solutions of the "limiting" Liouville system, subject to decay (or integrability) conditions at infinity. Such boundary conditions express the asymptotic gauge equivalence of the non-topological vortex solutions to the broken vacua state (at infinity) and ensure finite energy. We observe that the (limiting) Liouville system under exam contains, as particular case, the Toda system, dealing with SU(n)-vortices of non-topological type. We mention that, Toda systems of Liouville type enjoys a stronger conformal invariance property, as recently pointed out by C.S. Lin-C. Wei-D. Ye, who have obtained a complete characterization of the

corresponding solutions, extending (in a non-trivial manner) the result of J. Jost-G. Wang concerning the regular case (i.e. no singular sources), and that of J. Prajapat-G. Tarantello and W. Chen-C. Li relative to single equations. In particular, in the Toda-situation the integral condition (required on each component) is attained at a specific value, implying a "quantization" property for the charges. This is no longer the case away from the Toda-situation, and our effort has been to characterize the sharp set covered by all possible integral values corresponding to radial solutions (about a given single vortex point). This information identifies the amount of energy and charges necessary to develop a non-topological vortex. It also provides a first approximation for the profile of the non-topological one-vortex solution sought. In collaboration with A. Poliakowsky, we have dealt with the so-called "cooperative" case, where all the off diagonal entries of the coupling matrix are assumed positive. In this situation we have identified necessary and sufficient conditions for existence, and turned our problem to a dual coercive problem via a "singular" Log HLS inequality for systems, that also in the radial setting only. Our result extends previous work of E. Lieb, E. Carlen-M. Loss, W. Beckner concerning the single equation, and of M. Chipot-I. Shafrir- G. Wolanski for "regular" systems. Concerning the "collaborative" case (such as the Toda-System), in collaboration with R. Fortini we have dealt with the 2×2 case. We have used a delicate blow-up analysis in order to follow the asymptotic behavior of a solution of the (radial) initial value problem, when the initial data go to plus or minus infinity. We mention that, in the context of systems, the blow-up analysis takes some unusual new turns, even within the radial setting.

Uniqueness of Instantaneously Complete Ricci flows
PETER MILES TOPPING

Hamilton's Ricci flow equation [12, 17]

$$\frac{\partial}{\partial t}g(t) = -2Ric[g(t)]$$

for an evolving smooth Riemannian metric $g(t)$ on a manifold M has a short-time existence and uniqueness theory valid for closed M [12, 6] and also for noncompact M when the initial metric and subsequent flows are taken to be complete and of bounded curvature [16, 1, 14]. The flow will exist until such time as the curvature blows up.

In the case of surfaces, Giesen and the author extended the existence theory to arbitrary initial metrics, building on the work of Hamilton and Chow in the compact case [13, 3] and on the theory of the logarithmic fast diffusion equation (see e.g. [4, 7, 20, 5]):

Theorem 1 ([10])**.** *Let (M, g_0) be any smooth Riemannian surface, possibly incomplete and/or with unbounded curvature. Depending on the conformal type, we*

define $T \in (0, \infty]$ *by*[1]

$$T := \begin{cases} \frac{1}{4\pi\chi(M)} \operatorname{Vol}_{g_0} M & \textit{if } (M, g_0) \cong \mathcal{S}^2, \mathbb{C} \textit{ or } \mathbb{R}P^2, \\ \infty & \textit{otherwise.} \end{cases}$$

Then there exists a smooth Ricci flow $g(t)$ on M, defined for $t \in [0, T)$ such that
  (1) $g(0) = g_0$, and
  (2) $g(t)$ is instantaneously complete (i.e. complete for all $t \in (0, T)$).
In addition, this Ricci flow $g(t)$ is 'maximally stretched' (see [10, 11]), and the Gauss curvature $K_{g(t)}$ satisfies

$$K_{g(t)} \geq -\frac{1}{2t}$$

for $t \in (0, T)$. If $T < \infty$, then we have

$$\operatorname{Vol}_{g(t)} M = 4\pi\chi(M)(T - t) \longrightarrow \quad 0 \quad \textit{as } t \nearrow T,$$

and in particular, T is the maximal existence time.

For examples of how such solutions can look, see [19].

In the talk, we saw how the condition of instantaneous completeness is enough to guarantee uniqueness:

Theorem 2. *Let $g(t)$ and $\bar{g}(t)$ be two instantaneously complete Ricci flows on any surface M, defined for $t \in [0, T]$, with $g(0) = \bar{g}(0)$. Then $g(t) = \bar{g}(t)$ for all $t \in [0, T]$.*

This result extends uniqueness results from [2, 10, 18, 9, 8] and, when restricted to the case of surfaces, [12, 1, 14].

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A new mass and geometric inequalities

GUOFANG WANG

(joint work with Yuxin Ge, Jie Wu, Chao Xia)

In this talk, we report our work on a new mass and related geometric inequalities in [1, 2, 3, 4, 6, 5, 7]. For the related work, please see the reference therein. The project is partly supported by SFB/TR71 "Geometric partial differential equations" of DFG.

Let us consider following Gauss-Bonnet curvature (or Lovelock curvature)

$$L_k := \frac{1}{2^k} \delta^{i_1 i_2 \cdots i_{2k-1} i_{2k}}_{j_1 j_2 \cdots j_{2k-1} j_{2k}} R_{i_1 i_2}{}^{j_1 j_2} \cdots R_{i_{2k-1} i_{2k}}{}^{j_{2k-1} j_{2k}}.$$

Here $R_{ij}{}^{sl}$ is the Riemannian curvature tensor. One can check that L_1 is just the scalar curvature R. When $k = 2$, it is the (second) Gauss-Bonnet curvature

$$L_2 = \|Riem\|^2 - 4\|Ric\|^2 + R^2.$$

When $2k = n$, L_k is in fact the Euler density, which was studied by Chern in his proof of the Gauss-Bonnet-Chern theorem. One can decompose the Gauss-Bonnet curvature in the following way

$$L_k = P_{(k)}^{stjl} R_{stjl},$$

where

$$P_{(k)}^{stlj} := \frac{1}{2^k} \delta_{j_1 j_2 \cdots j_{2k-3} j_{2k-2} j_{2k-1} j_{2k}}^{i_1 i_2 \cdots i_{2k-3} i_{2k-2} st} R_{i_1 i_2}{}^{j_1 j_2} \cdots R_{i_{2k-3} i_{2k-2}}{}^{j_{2k-3} j_{2k-2}} g^{j_{2k-1} l} g^{j_{2k} j}.$$

It is crucial to observe that

- $P_{(k)}$ shares the same symmetry and antisymmetry with the Riemann curvature tensor that

$$P_{(k)}^{stjl} = -P_{(k)}^{tsjl} = -P_{(k)}^{stlj} = P_{(k)}^{jlst}.$$

- $P_{(k)}$ satisfies the first Bianchi identity, i.e., $P^{stjl} + P^{tjsl} + P^{jstl} = 0$.
- $P_{(k)}$ is divergence-free,

$$\nabla_s P_{(k)}^{stjl} = 0.$$

Now we introduce a new mass, which we call the *Gauss-Bonnet-Chern mass*

$$m_k(g) := m_{GBC}(g) = c_k(n) \lim_{r \to \infty} \int_{S_r} P^{ijkl} \partial_l g_{jk} \nu_i dS,$$

where $c_k(n)$ can be decided by the following generalized Schwarzschild metric

$$g_{\text{Sch}}^k = (1 - \frac{2m}{\rho^{\frac{n}{k}-2}})^{-1} d\rho^2 + \rho^2 d\Theta^2,$$

where $d\Theta^2$ is the round metric on $\mathbb{S}^{n-1}$. When $k = 1$ we recover the Schwarzschild solution of the Einstein gravity.

When $k = 1$, m_1 is just the ADM mass. Similar to the work of Bartnik for the ADM mass, we first show that the GBC mass is a geometric invariant.

Theorem 1. ([1]) *Suppose that (M^n, g) $(k < n/2)$ is an asymptotically flat manifold of decay order $\tau > \frac{n-2k}{k+1}$ and L_k is integrable on (M^n, g). Then the Gauss-Bonnet-Chern mass m_{GBC} is well-defined and does not depend on the choice of the coordinates used in the definition.*

Motivated by the work on the Positive Mass Theorem (PMT) and the Penrose Inequality (PI) for the ADM mass, we may propose a

Conjecture 1. *The corresponding PMT and the PI for the Gauss-Bonnet-Chern mass hold for asymptotically flat manifolds.*

We provided the support for this conjecture in the following

Theorem 2. *The Positive Mass Theorem holds for (1) conformally flat manifolds [2] and (2) for asymptotically flat graph [1].*

We define also the Gauss-Bonnet-Chern mass for asymptotically hyperbolic manifold in [5]. A positive mass theorem and a Penrose inequality have been proved in [5] with the help of hyperbolic Alexandrov-Fenchel quermassintegral inequalities in $\mathbb{H}^n$ in [3, 4, 6]

Hyperbolic Alexandrov-Fenchel quermassintegral inequalities. *Let* $1 \leq k \leq n-1$. *Any horospherical convex hypersurface* Σ *in* $\mathbb{H}^n$ *satisfies*

$$(0.1) \qquad \int_{\Sigma} H_k d\mu \geq C_{n-1}^k \omega_{n-1} \left\{ \left(\frac{|\Sigma|}{\omega_{n-1}} \right)^{\frac{2}{k}} + \left(\frac{|\Sigma|}{\omega_{n-1}} \right)^{\frac{2}{k} \frac{(n-k-1)}{n-1}} \right\}^{\frac{k}{2}},$$

where H_k *is the (normalized) higher order mean curvature of* Σ. *Equality holds if and only if* Σ *is a geodesic sphere.*

and

Weighted Hyperbolic Alexandrov-Fenchel quermassintegral inequalities.
Let Σ be a horospherical convex hypersurface in the hyperbolic space $\mathbb{H}^n$. We have

$$(0.2) \qquad \int_{\Sigma} V H_{2k+1} d\mu \geq \omega_{n-1} \left(\left(\frac{|\Sigma|}{\omega_{n-1}} \right)^{\frac{n}{(k+1)(n-1)}} + \left(\frac{|\Sigma|}{\omega_{n-1}} \right)^{\frac{n-2k-2}{(k+1)(n-1)}} \right)^{k+1}.$$

Equality holds if and only if Σ is a centered geodesic sphere in $\mathbb{H}^n$.

Previous related work was done by Brendle-Hung-Wang, de Lima-Girao and Li-Wei-Xiong. In [7] we established a Penrose inequality for graphs over the Kottler space.

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Structure of branch sets of minimal submanifolds and multi-valued harmonic functions

Neshan Wickramasekera

(joint work with Brian Krummel)

The talk reported on the first results of an on-going investigation (joint work with Brian Krummel) on the structure of the set of branch points of certain classes of multiple-valued harmonic functions and minimal submanifolds. The main results presented include countable rectfiability and local-finiteness-of-measure properties for branch sets of two-valued $C^{1,\mu}$ harmonic functions and two-valued $C^{1,\mu}$ solutions to the minimal surface system ([KruWic]; Theorem 0.1 below).

Two-valued functions. Let $\mathcal{A}_2(\mathbb{R}^m)$ denote the space of unordered pairs $\{a_1, a_2\}$ with $a_1, a_2 \in \mathbb{R}^m$ not necessarily distinct. Equip $\mathcal{A}_2(\mathbb{R}^m)$ with the metric

$$\mathcal{G}(a,b) = \min\{\sqrt{|a_1 - b_1|^2 + |a_2 - b_2|^2}, \sqrt{|a_1 - b_2|^2 + |a_2 - b_1|^2}\}$$

for $a = \{a_1, a_2\} \in \mathcal{A}_2(\mathbb{R}^m)$ and $b = \{b_1, b_2\} \in \mathcal{A}_2(\mathbb{R}^m)$. For $a = (a_1, a_2) \in \mathcal{A}_2(\mathbb{R}^m)$, let

$$|a| = \mathcal{G}(a, \{0, 0\}) = \sqrt{|a_1|^2 + |a_2|^2}.$$

A two-valued function on a set $\Omega \subseteq \mathbb{R}^n$ is a map $u : \Omega \to \mathcal{A}_2(\mathbb{R}^m)$.

For each $\mu \in (0, 1]$, define $C^{0,\mu}(\Omega; \mathcal{A}_2(\mathbb{R}^m))$ to be the space of two-valued functions $u : \Omega \to \mathcal{A}_2(\mathbb{R}^m)$ such that

$$[u]_{\mu,\Omega} = \sup_{X,Y \in \Omega, X \neq Y} \frac{\mathcal{G}(u(X), u(Y))}{|X - Y|^\mu} < \infty.$$

Let now $\Omega \subseteq \mathbb{R}^n$ be open. A two-valued function $u : \Omega \to \mathcal{A}_2(\mathbb{R}^m)$ is *differentiable* at $Y \in \Omega$ if there exists a two-valued affine function $\ell_Y : \mathbb{R}^n \to \mathcal{A}_2(\mathbb{R}^m)$, i.e. a two-valued function ℓ_Y of the form $\ell_Y(X) = \{A_1^Y X + b_1^Y, A_2^Y X + b_2^Y\}$ for some $m \times n$ real constant matrices A_1^Y, A_2^Y and constants $b_1^Y, b_2^Y \in \mathbb{R}^m$, such that

$$\lim_{X \to Y} \frac{\mathcal{G}(u(X), \ell_Y(X))}{|X - Y|} = 0.$$

ℓ_Y is unique if it exists. The *derivative* $Du(Y)$ of u at Y is the unordered pair of $m \times n$ matrices $\{A_1^Y, A_2^Y\}$, also denoted $\{Du_1(Y), Du_2(Y)\}$.

Define $C^1(\Omega; \mathcal{A}_2(\mathbb{R}^m))$ to be the space of two-valued functions $u : \Omega \to \mathcal{A}_2(\mathbb{R}^m)$ such that the derivative of u exists and is continuous on Ω as a function taking values in $\mathcal{A}_2(\mathbb{R}^{nm})$. For $\mu \in (0, 1]$, define $C^{1,\mu}(\Omega; \mathcal{A}_2(\mathbb{R}^m))$ as the set of $u \in C^1(\Omega; \mathcal{A}_2(\mathbb{R}^m))$ such that $Du \in C^{0,\mu}(\Omega; \mathcal{A}_2(\mathbb{R}^{nm}))$.

Given $u \in C^1(\Omega; \mathcal{A}_2(\mathbb{R}^m))$, define the branch set $\mathcal{B}_u$ of u to be the set of points $Z \in \Omega$ such that there is no neighborhood of Z in which the values of u are given by two single-valued C^1 functions. It is clear that

$$\mathcal{B}_u \subset \mathcal{K}_u$$

where $\mathcal{K}_u = \{X \in \Omega : u_1(X) = u_2(X), \ Du_1(X) = Du_2(X)\}$.

A two-valued function $u \in C^1(\Omega; \mathcal{A}_2(\mathbb{R}^m))$ is said to be *harmonic* on Ω if for every open ball $B_\rho(Y) \subset \Omega \setminus \mathcal{B}_u$, $u(X) = \{u_1(X), u_2(X)\}$ on $B_\rho(Y)$ for two single-valued harmonic (hence real analytic) functions $u_1, u_2 : B_\rho(Y) \to \mathbb{R}^m$. Clearly if such u_1, u_2 exist, they are unique.

A two-valued function $u \in C^1(\Omega; \mathcal{A}_2(\mathbb{R}^m))$ is said to be a *solution to the minimal surface system* if its graph $G = \operatorname{graph} u = \{(X, Y) \in \Omega \times \mathbb{R}^m : Y \in \{u_1(X), u_2(X)\}\}$ taken with multiplicity function defined by $\theta(X, Y) = 2$ whenever $Y = u_1(X) = u_2(X)$ and $\theta(X, Y) = 1$ whenever $Y \in \{u_1(X), u_2(X)\}$ with $u_1(X) \neq u_2(X)$ is a stationary integral n-varifold on $\Omega \times \mathbb{R}^m$.

Theorem 0.1. *([KruWic]) Let $u = \{u_1, u_2\}$ be a 2-valued $C^{1,\alpha}$ function on a connected, open subset Ω of $\mathbb{R}^n$, with values $u_1, u_2 \in \mathbb{R}^m$. Suppose that $\mathcal{K}_u \neq \Omega$. If either*

(a) u is harmonic or

(b) u is a solutions to the minimal surface system

then for each closed ball $B \subset \Omega$, the set $B \cap \mathcal{K}_u$ is the union of finitely many pairwise disjoint locally compact, locally $(n-2)$-rectifiable subsets (each having locally finite $\mathcal{H}^{n-2}$-measure).

In particular, either $B \cap \mathcal{B}_u = \emptyset$ or $B \cap \mathcal{B}_u$ is a closed set of positive $(n-2)$-dimensional Hausdorff measure (so has Hausdorff dimension precisely equal to $(n-2)$), and $B \cap \mathcal{B}_u$ is the finite union of locally compact, locally $(n-2)$-rectifiable sets.

An application. In view of results of [Wic], case (b) of Theorem 0.1 implies the following:

Let T be an n-dimensional rectifiable current on an open subset $U \subset \mathbb{R}^{n+1}$ with no boundary in U, and suppose that the varifold $|T|$ associated with T is stationary in U, and that the regular part $\operatorname{reg} T$ is stable (in the usual sense of stability of an oriented smooth minimal hypersurface with respect to compactly supported normal deformations). Let $Z_0 \in \operatorname{sing} T$ be such that $|T|$ has a varifold tangent cone equal to a multiplicity 2 hyperplane. Then there exists $\rho_0 > 0$ such that the set of points $Z \in \operatorname{spt} T \cap B_{\rho_0}(Z_0)$ at which $|T|$ has a varifold tangent cone equal to a multiplicity 2 hyperplane is the union of finitely many pairwise disjoint, locally compact, locally $(n-2)$-rectifiable sets.

In particular, if $\mathcal{B}_T$ is the set of density 2 branch point singularities of T, i.e. points $Z \in \operatorname{sing} T$ where $|T|$ has at least one varifold tangent cone equal to a multiplicity 2 hyperplane but $\operatorname{spt} T$ is not, in any neighborhood of Z, the union of two smoothly embedded hypersurfaces, then $\mathcal{B}_T$ has locally positive $(n-2)$-dimensional Hausdorff measure and $\mathcal{B}_T$ is locally the finite union of pairwise disjoint, locally compact, locally $(n-2)$-rectifiable sets.

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