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Group Theory, Measure, and Asymptotic Invariants

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Abstract. The workshop 'Group Theory, Measure, and Asymptotic Invariants' organized by Miklos Abert (Budapest), Damien Gaboriau (Lyon) and Andreas Thom (Leipzig) was held 18 - 24 August 2013. The event was a continuation of the previous Oberwolfach workshop 'Actions and Invariants of Residually Finite Groups: Asymptotic Methods' organized by Miklos Abert (Budapest), Damien Gaboriau (Lyon) and Fritz Grunewald (Dusseldorf) that was held September 5 - September 11, 2010. Fritz Grunewald passed away in March 2010 and Andreas Thom joined the organizing team.

The workshop aimed to study finitely generated groups and group actions using ergodic and measure theoretic methods, incorporating asymptotic invariants, such as ℓ^2-invariants, the rank gradient, cost, torsion growth, entropy-type invariants and invariants coming from random walks and percolation theory.

The participant body came from a wide range of areas: finite and infinite group theory, geometry, ergodic theory, graph theory, topology, probability theory, representation theory, von Neumann algebras and ℓ^2-theory. The participants typically did not speak each other's mathematical dialect fluently. To address this situation, the organizers asked the speakers to put a special emphasis on the first, introductory part of their talks. This aspect worked very well.

As a general rule, the organizers asked speakers to talk about specific subjects, not just any nice piece of their research. In some cases, this meant sacrificing hearing about some new results from excellent mathematicians that were further away from the workshop's main directions.

Mathematics Subject Classification (2010): 20F69, 20E26, 22D40, 37A20, 05C25, 20E05, 22E15, 22E40, 37A50.

Introduction by the Organisers

Invariant random subgroups, limit multiplicities and Benjamini-Schramm convergence of graphs and Riemannian manifolds. A newly emerging topic is invariant random subgroups (IRS's). These are conjugacy invariant probability measures on a space of subgroups of a fixed ambient (Lie or discrete) group. It turns out that weak convergence of IRS's corresponds to Benjamini-Schramm convergence of the quotient spaces. For discrete groups, these are graphs, while for Lie groups, they are locally symmetric spaces.

Tsachik Gelander gave a talk on his joint work with Miklos Abert, Nicolas Bergeron, Ian Biringer, Nikolay Nikolov, Jean Raimbault, and Iddo Samet on Invariant Random Subgroups in higher rank groups. The main result he talked about was to show that for a higher rank simple Lie group G, any sequence of locally symmetric G-spaces of finite volume converges to the symmetric space of G. The proof uses the Stuck-Zimmer theorem. For compact spaces with some natural additional conditions, it also implies the convergence of the normalized Betti numbers. Nicolas Bergeron talked about the same project, but from a representation theoretical point of view. In his talk Limit formulas along BS-converging sequences of X-manifolds he showed how Benjamini-Schramm convergence implies the convergence of the Plancherel measures. The methods employed here only work in the cocompact setting. Tobias Finis talked about his joint work with Erez Lapid and Werner Muller where they prove the corresponding theorems on limit multiplicities for *nonuniform* lattices in SL_n.

Lewis Bowen talked about his work on Cheeger constants and L2-Betti numbers. Here he finds a rather unexpected use of Benjamini-Schramm convergence of Riemannian manifolds (using a generalization of the Lück approximation theorem in this setting, due to Elek) to prove a uniform lower bound on the Cheeger constant of certain natural families of discrete subgroups of Lie groups.

Arie Levit gave a talk on a generalization of the intermediate factor theorem to local fields. His work in particular implies that in the presence of property (T), in the above setting, the Stuck-Zimmer theorem holds.

ℓ^2 Betti numbers, homology growth and spectral measure.

Wolfgang Lück talked about approximating L2-invariants and homological growth. There is an interesting connection between homology growth, L2 Betti numbers and other invariants, like the cost, the rank gradient and various torsions. Lück gave a thorough survey on the known results and also talked about new directions, like understanding the mod p homology growth and its connections to the other invariants.

Balint Virag gave a talk on his joint work with Lukasz Grabowski on how to defy an old conjecture of Lott and Lück on spectral measure. Note that their result still leaves open the more important general conjecture on determinants to hold.

Russell Lyons gave a talk on L2-Betti numbers, cost, and the free uniform spanning forest. After giving a very good introduction to the subject, he also discussed some new results.

Hanfeng Li gave a talk about when the Fuglede-Kadison determinant is equal to 1. Li (partially in joint works with Chung, Kerr and Thom) made new advances on the topic of entropy for principal algebraic actions of amenable and sofic groups.

Henrik Densing Petersen talked about his joint work on L2-invariants of locally compact groups with Kyed and Vaes and also with Valette. Based on a previous work of Gaboriau, they introduce L2 Betti numbers for locally compact groups in a very general setting, using a von Neumann algebraic approach.

Ergodic theory of group actions. David Kerr gave an introductory talk to sofic entropy with a special emphasis on Bernoulli actions.

François Le Maître gave a talk on his exciting result on the topological rank for full groups. He proved that the topological minimal number of generators for the full group of a pmp equivalence relation always equals the floor of its cost plus one.

Robin Tucker-Drob presented a measure theoretic proof of solid ergodicity for Bernoulli shifts. This is a result of Chifan and Ioana who used von Neumann algebraic tools: now there is an elementary proof. This was one of the talks where the speaker presented the full proof for his result.

Orbit and measure equivalence, rigidity. Uri Bader gave a talk on his joint work with Furman on a new perspective on super-rigidity. They give a new, exciting representation theoretic proof of the Margulis supperrigidity theorem, which also leads to natural generalizations. Bader also gave an evening session on the details of the proof. Roman Sauer talked about his joint work with Uri Bader and Alex Furman on L1-measure equivalence of hyperbolic lattices.

Jesse Peterson talked about his work with Thom and another with Creutz on new results on character rigidity. Characters are positive definite, conjugacy invariant functions. They have an intimate connection to von Neumann algebra representations and also to invariant random subgroups (by the work of Vershik). Here rigidity usually means a complete classification of irreducible characters.

Amenable-nonamenable groups. We had three nice talks on amenability-nonamenability.

Kate Juschenko talked about her joint work with Nekrashevych and de la Salle on amenability of groups acting by homeomorphisms on compact spaces. Here they give a very general condition that implies that certain groups are amenable.

Rostislav Grigorchuk talke about his joint work with Benli and Vorobets on random groups of intermediate growth. Here they build an interesting random model that produces natural (uncountable) families of groups of intermediate growth. The speaker analyzed the properties of these random groups.

Mikhail Ershov gave a talk on Tarski numbers. These are the minimal number of pieces needed for a paradoxical decomposition of a nonamenable group. Among other results, the speaker presented how to use L2 Betti numbers to show that Tarski numbers can get arbitrarily large.

There were also some exciting talks that would be hard to group together by subject.

Denis Osin gave a talk on geometric and analytic negative curvature. The story here is that various people studied various properties of group actions on metric spaces and proved numerous theorems in deep papers. Osin showed that these properties are actually all equivalent to what he calls acylindrically hyperbolic groups.

John Wilson talked about ultraproducts of finite simple groups. He proves an array of results, partially jointly with Thom.

Nir Avni gave a talk on his joint work with Aizenbud on the representation growth of arithmetic lattices. This counts the number of rank n irreducible characters of a given discrete group. The authors find some beautiful and unexpected connections between the representation growth of an arithmetic lattice and the singularities of the moduli space of the corresponding local systems on closed surfaces.

Chen Meiri gave a talk on the Group Large Sieve. This is a new sieve method invented by Rivin, Kowalski, Lubotzky and Meiri that can be used to address the asymptotic properties of random elements of discrete groups. Randomness here is achieved by performing a long random walk with respect to some natural generating set.

Workshop: Group Theory, Measure, and Asymptotic Invariants

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Abstracts

Invariant Random Subgroups in higher rank groups

Tsachik Gelander

Based on a joint work with M. Abert, N. Gergeron, I. Biringer, N. Nikolov, J. Raimbault, I. Samet [2, 1]

The aim of this lecture is to explain a new approach in the theory of lattices. The idea is to associate lattices with measures defined on the space of closed subgroups and to study the space of such measures. Remarkably, this naive approach has proven very profitable and was a key to various recent achievements.

Let G be a locally compact second countable group, and recall the compact space of closed subgroups Sub_G with the Chabauty topology. G acts continuously on Sub_G by conjugations. An Invariant Random Subgroup (shortly IRS) of G is a Borel regular G-invariant probability measure on Sub_G.

For any measure preserving action of G on a probability space Ω, it can be shown that almost every stabilizer is a closed subgroup in G, and hence the push forward of the measure from Ω to Sub_G is an IRS of G. It can also be shown (see [2, Theorem 2.4]) that every IRS in G arises in this way. In particular, one can consider (the conjugacy class of) a lattice $\Gamma \leq_L G$ as an example of an IRS — we shall denote by μ_Γ the IRS on G induced by the G action on G/Γ with the normalised measure.

Various people have recently become aware of the importance of IRS's in many branches of group theory, dynamics, geometry and representation theory, and there has been a lot of works studying different aspects of IRS in different context during the last three years. Here I will restrict to the work [2] which makes use of the notion of IRS in order to study the asymptotic of L_2-invariant of lattices in semi-simple Lie groups, and report few results from this work. For simplicity of the formulations of the results below let us restrict again to the case where G is simple.

Some results about lattices can be extended to statement about IRS's. For instance the Borel density theorem can be generalized as follows:

Theorem 1. *(*[2, Theorem 2.5]*) Let G be a simple real algebraic group and let μ be an IRS without atoms*[1]*. Then μ is supported on discrete and Zariski dense subgroups.*

Of significant importance in this approach is the rigidity theorem of Nevo, Stuck and Zimmer (proven in [6] relying on the later work [4]):

Theorem 2. *Let G be a simple Lie group of real rank ≥ 2. Then every non-transitive, ergodic probability measure preserving G-action is essentially free.*

Relying on Theorem 2 and on property (T) it is shown in [2]:

[1] As G is simple the atoms can only be supported on the trivial normal subgroups $\{1\}, G$.

Theorem 3. *(*[2, Section 4]*) Let G be a noncompact simple Lie group of rank ≥ 2. The non-atomic ergodic IRS in G are precisely μ_Γ, $\Gamma \leq_L G$, and the only accomulation point of the set $\{\mu_\Gamma : \Gamma \leq_L G\}$ is the Dirac measure on the trivial group $\{1\}$.*

The following geometric result is a consequence of Theorem 3:

Theorem 4. *(*[2, Corolarry 4.10]*) Let G be as in the previous theorem and let $X = G/K$ be the associated symmetric space. Let Γ_n be a sequence of representatives for the distinct conjugacy classes of lattices in G and let $M_n = \Gamma_n \backslash X$ be the corresponding X-orbifolds. Then for every $R > 0$ we have*

$$\lim_{n\to\infty} \frac{\mathrm{vol}(\{p \in M_n : \mathit{InjRad}_{M_n}(p) \geq R\}}{\mathrm{vol}(M_n)} = 1.$$

Associating a finite volume manifold together with a random point in it with a probability measure on the space of pointed metric spaces, the last result is interpreted as follows: If $\mathrm{rank}(X) \geq 2$, every sequence of X-manifolds, of finite volume tending to infinity, locally converges (in the probabilistic sense of Benjamini and Schramm, see [2] for a precise definition) to the universal cover X. The local convergence to the universal cover implies convergence of certain topological and representation theoretical invariants. When restricting to the subsequence Γ_{n_k} of uniform torsion free lattices (for which the M_{n_k} are compact manifolds) this result is used to study the asymptotic of L_2-invariants of G/Γ_{n_k} and of $M_{n_k} = \Gamma_{n_k}\backslash X$. In particular a uniform version of the de-George–Wallach theorem [3] about multiplicity of unitary representations ([2, Section 7]) and a uniform version of the Lueck approximation theorem ([2, Section 8]) are proved.

A family of lattices is called *uniformly discrete* if the minimal injectivity radius of the corresponding locally symmetric manifolds is uniformly bounded from below. A well known conjecture of Margulis (see [5, Page 322]) suggests that the family of all torsion free arithmetic uniform lattices in a every given semisimple Lie group is uniformly discrete. Two of the main results of [2] are:

Theorem 5. *Let G be as above, and suppose that $\Gamma_n \leq_L G$ are non-conjugate torsion free uniformly discrete lattices. Let π be a unitary representation of G and let $m(\pi, \Gamma)$ be the multiplicity of π in $L_2(G/\Gamma)$. Then*

$$\frac{m(\pi,\Gamma)}{\mathrm{vol}(G/\Gamma)} \to d(\pi)$$

where $d(\pi)$ is the formal degree of π and is nonzero iff π is a discrete series representation.

Theorem 6. *Let G and Γ_n be as above and denote $M_n = \Gamma_n\backslash X$. Then for every $k \leq \dim(X)$ we have*

$$\frac{b_k(M_n)}{\mathrm{vol}(M_n)} \to \beta_k(X)$$

where b_k denotes the k'th betti number and

$$\beta_k(X) = \begin{cases} \frac{\chi(X^d)}{\mathrm{vol}(X^d)} & \delta(G) = 0 \text{ and } k = \frac{1}{2}\dim X \\ 0 & \text{otherwise,} \end{cases}$$

where X^d is the compact dual of X equipped (like X) with the Riemannian metric induced by the Killing form on $\mathrm{Lie}(G)$ *and* $\delta(G) = rank_{\mathbb{C}}(G) - rank_{\mathbb{C}}(K)$.

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Character rigidity and applications

Jesse Peterson

A character on a group Γ is a class function τ of positive type which is normalized so that $\tau(e) = 1$. The set of characters forms a convex space and the extreme points are naturally in bijective correspondence (via the GNS-construction) to unitary representations which generated von Neumann algebra is a finite factor.

In 1964, Thoma [11] initiated the systematic study of characters on infinite discrete groups, classifying all extremal characters for the group of finite permutations of the natural numbers. We'll say a group Γ is character rigid if the only extremal characters correspond to either the left regular representation or else a finite dimensional representation. The first example of character rigid groups were found by Kirillov [7] who showed this property for the groups $PSL_n(k)$ where $n \geq 3$, and k is an arbitrary infinite field. More recently Bekka [2] has shown that, in fact, the group $PSL_n(\mathbb{Z})$ is also character rigid for $n \geq 3$, giving the first such example for an irreducible lattice in a higher rank semi-simple group. This is significant since it was conjectured by Connes (based on the rigidity theorems of Mostow, Margulis, and Zimmer) that all such lattices are character rigid (see the discussion in [6]).

In my talk I discussed two new examples of this phenomenon for lattices, as well as applications of such rigidity properties. This is based on two papers, the first [10] is joint with Andreas Thom, and the second [4] is joint with Darren Creutz. The first new class of examples we consider are similar to Kirillov's and Bekka's results above for the case of PSL_2. For an outline of the proof we refer the reader [9] in the next report.

Theorem 1. *Let R be either an infinite field, or else a ring of algebraic integers with infinitely many units, than $PSL_2(R)$ is character rigid.*

The second class of examples has the advantage of holding for arbitrary lattices in a class of groups. The proof is based on ideas from [3], which in turn go back to ideas from [5], [1], and ultimately to the strategy developed by Margulis for his Normal Subgroup Theorem [8].

Theorem 2. *Let $G = G_1 \times G_2$ where each G_i is a semi-simple group with property (T) and no compact factors. Suppose that G_2 is totally disconnected. Then an arbitrary irreducible lattice $\Gamma < G$ is character rigid.*

The rest of the talk consisted of giving applications of character rigid groups. We state some simple examples below, for the most general statements, and their proofs, consult [10].

Theorem 3. *Suppose $n \geq 2$ and k is a countably infinite discrete field, then any non-trivial ergodic, probability measure preserving action of $PSL_n(k)$ is essentially free.*

Theorem 4. *Suppose $n \geq 2$ and k is an infinite discrete field which is not an algebraic extension of a finite field. Then there is no non-trivial homomorphism from $PSL_n(k)$ into $\mathcal{U}(R)$ where R denotes the hyperfinite II_1 factor.*

Theorem 5. *Suppose $n \geq 3$ and k is an infinite discrete field which is not an algebraic extension of a finite field. Let M be a finite factor with Haagerup's property, e.g., M_j, R, or $L\mathbb{F}_j$, then for all $\varepsilon > 0$ there exists $\delta > 0$, such that if $\pi : PSL_n(k) \to \mathcal{U}(M)$ is such that $\|\pi(g)\pi(h) - \pi(gh)\|_2 < \delta$ for all $g, h \in PSL_n(k)$, then $\|\pi(g) - 1\|_2 < \varepsilon$, for all $g \in PSL_n(k)$.*

Theorem 6. *Suppose $n \geq 2$ and k is an infinite algebraic extension over a finite field, and let $\omega \in \beta\mathbb{N} \setminus \mathbb{N}$ be a non-principle ultrafilter. For each $t \in [0, 1]$ fix a projection $p_n \in \mathcal{P}(R)$ with trace t and consider the homomorphism $\pi_t : PSL_n(k) \to \mathcal{U}(R) \subset \mathcal{U}(R^\omega)$, given by $\pi_t(g) = p_t + \lambda(g)p_t^\perp$. Then every homomorphism $\pi : PSL_n(k)$ is conjugate to some π_t.*

Theorem 7. *Suppose $n \geq 2$ and k is an infinite algebraic extension over a finite field, then for any subgroup $\Sigma < PSL_n(k)$, there exists a sequence $\{g_l\} \subset PSL_n(k)$ such that either $\cup_m \cap_{l>m} g_l \Sigma g_l^{-1} = PSL_n(k)$, or $\cap_m \cup_{l>m} g_l \Sigma g_l^{-1} = \{e\}$.*

Theorem 8. *Suppose $n \geq 2$ and k is an infinite discrete field. Suppose that $x = \sum_g \alpha_g g \in \mathbb{C}(PSL_n(k))$ is a self-adjoint element such that $\alpha_e > 0$, and $\sum_g \alpha_g > 0$. Then x is of the form $x = \sum_j a_i^* a_i + \sum_j (b_j^* b_j - b_j b_j^*)$, for some finite collection of elements $\{a_i, b_j\} \subset \mathbb{C}(PSL_n(k))$.*

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Topological rank for full groups

François Le Maître

Let $\mathcal{R}$ be a pmp equivalence relation on a standard probability space (X, μ). Define its full group $[\mathcal{R}]$ to be the set of automorphisms T of the space X such that $T(x)\,\mathcal{R}\,x$ for all $x \in X$. Then every orbit equivalence yields a conjugation between full groups and vice versa, so that the full group is a complete invariant for pmp equivalence relations. It is thus natural to wonder how its algebraic properties reflect properties of the pmp equivalence relation. For instance, Eigen [Eig81] showed that $\mathcal{R}$ is ergodic iff $[\mathcal{R}]$ is simple.

Furthermore, the full group of a pmp equivalence relation is a Polish group when endowed with the uniform metric $d_u(S,T) = \mu(\{x \in X : S(x) \neq T(x)\}$. So topological properties of $[\mathcal{R}]$ may also give us information on $\mathcal{R}$. In this talk, we are interested in the topological rank $t([\mathcal{R}])$ of the full group of $\mathcal{R}$, that is the minimal number of elements needed to generate a dense full group. Kittrell, Tsankov [KT10] and then Matui [Mat11] had previously obtained bounds for such a number in terms of the cost of the equivalence relations (cf.[Gab00]) , and here we give a definitive answer to this question by showing the following formula :

$$t([\mathcal{R}]) = \lfloor \mathrm{Cost}(\mathcal{R}) \rfloor + 1.$$

It is then natural to wonder about the structure of the set $t([\mathcal{R}])$-tuples which topologically generate $[\mathcal{R}]$. As a motivation, the Schreier-Ulam theorem states that whenever G is a compact metrisable connected group, the set of pairs which generate a dense subgroup is a dense G_δ in G^2. In our setting, such a result cannot hold, for elements in a dense G_δ set can have arbitrarily small support, and hence cannot even generate the equivalence relation $\mathcal{R}$. However, when T has discrete spectrum, the set of U's such that $\overline{\langle T, U\rangle} = [\mathcal{R}_T]$ is a dense G_δ. It would be nice to generalize this result to a wider class of pmp automorphisms T.

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Limit formulas along BS-converging sequences of X-manifolds

Nicolas Bergeron

(joint work with Miklos Abert, Ian Biringer, Tsachik Gelander, Nikolay Nikolov, Jean Raimbault, and Iddo Samet)

Let G be a connected center-free semisimple Lie group without compact factors, $K \leq G$ a maximal compact subgroup and $X = G/K$ the associated Riemannian symmetric space. The subject of the talk was the study of the asymptotics of L^2-invariants of the spaces $\Gamma\backslash X$, where Γ varies over the space of lattices of G.

In Tsachik Gelander's talk BS-convergence was defined. Here we only recall a particularly transparent case of BS-convergence: the case when a sequence of locally symmetric spaces $\Gamma_n\backslash X$ converges to X.

Definition Let (Γ_n) be a sequence of lattices in G. We say that the X-orbifolds $M_n = \Gamma_n\backslash X$ *BS-converge* to X if for every $R > 0$, the probability that the R-ball centered around a random point in M_n is isometric to the R-ball in X tends to 1 when $n \to \infty$. In other words, if for every $R > 0$, we have

$$\lim_{n\to+\infty} \frac{\mathrm{vol}((M_n)_{<R})}{\mathrm{vol}(M_n)} = 0,$$

where $M_{<R} = \{x \in M \ : \ \mathrm{InjRad}_M(x) < R\}$ is the R-thin part of M.

A straightforward, and well studied, particular case is when $\Gamma \leq G$ is a lattice and $\Gamma_n \leq \Gamma$ is a chain of normal subgroups with trivial intersection; in this case, the R-thin part of $\Gamma_n\backslash X$ is empty for large n.

Uniform discreteness. A family of lattices (resp. the associated X-orbifolds) is *uniformly discrete* if there is an identity neighborhood in G that intersects trivially all of their conjugates. For torsion-free lattices Γ_n, this is equivalent to saying that there is a uniform lower bound for the injectivity radius of the manifolds $M_n = \Gamma_n\backslash X$. In particular, any family (M_n) of covers of a fixed compact orbifold is uniformly discrete. Margulis has conjectured that the family of all cocompact torsion-free arithmetic lattices in G is uniformly discrete. This is a weak form of the famous Lehmer conjecture on monic integral polynomials.

BS-convergence and Plancherel measure. The main result discuted in the talk says that BS-convergence to X implies a spectral convergence: namely, the relative Plancherel measure of $\Gamma_n\backslash G$ will converge to the Plancherel measure of G in a strong sense.

For an irreducible unitary representation $\pi \in \widehat{G}$ and a uniform lattice Γ in G let $\mathrm{m}(\pi, \Gamma)$ be the multiplicity of π in the right regular representation $L^2(\Gamma\backslash G)$. Define the relative Plancherel measure of $\Gamma\backslash G$ as the measure

$$\nu_\Gamma = \frac{1}{\mathrm{vol}(\Gamma\backslash G)} \sum_{\pi \in \widehat{G}} \mathrm{m}(\pi, \Gamma)\delta_\pi$$

on $\widehat{G}$. Finally denote by ν^G the Plancherel measure of the right regular representation $L^2(G)$.

Theorem Let (Γ_n) be a uniformly discrete sequence of lattices in G such that the spaces $\Gamma_n\backslash X$ BS-converge to X. Then for every relatively quasi-compact ν^G-regular subset $S \subset \widehat{G}$, we have:

$$\nu_{\Gamma_n}(S) \to \nu^G(S).$$

Note that the Plancherel measure of G depends on a choice of a Haar measure on G as does $\mathrm{vol}(\Gamma\backslash G)$.

Let $d(\pi)$ be the 'multiplicity' — or rather the formal degree — of π in the regular representation $L^2(G)$ with respect to the Plancherel measure of G. Thus, $d(\pi) = 0$ unless π is a discrete series representation. the theorem implies the following:

Corollary Let (Γ_n) be a uniformly discrete sequence of lattices in G such that the spaces $\Gamma_n\backslash X$ BS-converge to X. Then for all $\pi \in \widehat{G}$, we have

$$\frac{m(\pi, \Gamma_n)}{\mathrm{vol}(\Gamma_n\backslash G)} \to d(\pi).$$

In the special situation when (Γ_n) is a chain of normal subgroups with trivial intersection in some fixed cocompact lattice $\Gamma \leq G$, this corollary is the classical theorem of DeGeorge and Wallach. In that very same situation the Theorem is due to Delorme.

The classical theorem of DeGeorge and Wallach implies a corresponding statement on the approximation of L^2-Betti numbers by normalized Betti numbers of finite covers, generalized by Wolfgang Lück to the CW-complex setting. The theorem above implies the following uniform version of it.

Corollary Let (M_n) be a sequence of uniformly discrete compact X-manifolds that BS-converge to X. Then for every $k \leq \dim(X)$ we have

$$\frac{b_k(M_n)}{\mathrm{vol}(M_n)} \to \beta_k^{(2)}(X).$$

In the corollary, $b_k(M_n)$ is the k^{th} Betti number of M_n and

$$\beta_k^{(2)}(X) = \begin{cases} \frac{\chi(X^d)}{\mathrm{vol}(X^d)} & k = \frac{1}{2}\dim X \\ 0 & \text{otherwise,} \end{cases}$$

is the k^{th} *L^2-Betti number of X,* where X^d is the compact dual of X equipped with the Riemannian metric induced by the Killing form on Lie(G).

The material of the talk as well as the abstract are extracted from [1].

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A perspective on super-rigidity

Uri Bader

Margulis super rigidity is a thorem describing the algebraic representations of lattices in higher-rank simple Lie group. In my talk I presented a new proof of this theorem, based on the theory of algebraic representations of ergodic actions. The new method of proof admits also proving various new generalizations of Margulis and Zimmer Super-Rigidity theorems. The talk was based on a joint work with Alex Furman.

When is the Fuglede-Kadison determinant equal to 1?

Hanfeng Li

The Fuglede-Kadison determinant was introduced in [5]. For a nice survey, see [8]. Let Γ be a countable discrete group. Denote by $\mathbb{C}\Gamma$ the complex group ring of Γ, and by tr the canonical trace of $\mathbb{C}\Gamma$ sending $f = \sum_{s\in\Gamma} f_s s$ to f_{e_Γ}, where e_Γ denotes the identity element of Γ. For each $f \in \mathbb{C}\Gamma$, there is a unique Borel probability measure on the interval $[0, \|f\|_1^2]$, called the *spectral measure of f^*f* and denoted by μ_{f^*f}, satisfying

$$\int_0^{\|f\|_1^2} x^n \, d\mu_{f^*f}(x) = \mathrm{tr}((f^*f)^n)$$

for all $n = 0, 1, 2, \ldots$. Here $\|f\|_1 = \sum_{s\in\Gamma} |f_s|$ denotes the ℓ^1-norm of $f = \sum_{s\in\Gamma} f_s s$. The *Fuglede-Kadison determinant* of f, denoted by $\det_{\mathrm{FK}} f$, is defined as

$$\det_{\mathrm{FK}} f = e^{\frac{1}{2}\int_0^{\|f\|_1^2} \log x \, d\mu_{f^*f}(x)}.$$

A crucial property of the Fuglede-Kadison determinant is the multiplicative property:

$$\det_{\mathrm{FK}}(fg) = \det_{\mathrm{FK}} f \cdot \det_{\mathrm{FK}} g$$

for all $f, g \in \mathbb{C}\Gamma$. Lück's modified determinant, denoted by $\det'_{\mathrm{FK}} f$, is defined as

$$\det{}'_{\mathrm{FK}} f = e^{\frac{1}{2}\int_{0+}^{\|f\|_1^2} \log x \, d\mu_{f^*f}(x)}.$$

When the left multiplication of f on $\ell^2(\Gamma)$ is injective, one has $\mu_{f^*f}(\{0\}) = 0$ and hence $\det_{\mathrm{FK}} f = \det'_{\mathrm{FK}} f$.

Denote by $\mathbb{Z}\Gamma$ the integral group ring of Γ. Lück's *Determinant Conjecture* says that $\det'_{\mathrm{FK}} f \geq 1$ for every $f \in \mathbb{Z}\Gamma$. Elek and Szabó proved this conjecture for sofic groups [4]. Recall that Γ is called *sofic* if for any finite subset F of Γ and any $\varepsilon > 0$, there are some positive integer d and some map σ from Γ to the permutation group S_d of $\{1, \ldots, d\}$ satisfying

$$\rho_{\mathrm{Hamming}}(\sigma_s\sigma_t, \sigma_{st}) \leq \varepsilon$$

for all $s, t \in F$ and

$$\rho_{\mathrm{Hamming}}(\sigma_s, \sigma_t) \geq 1 - \varepsilon$$

for all distinct $s, t \in F$. Here ρ_{Hamming} denotes the *Hamming distance* on S_d defined by

$$\rho_{\mathrm{Hamming}}(\varphi, \psi) = \frac{|\{a \in \{1, \ldots, d\} : \varphi(a) \neq \psi(a)\}|}{d}.$$

Amenable groups and residually finite groups are all sofic. For a nice survey about sofic groups, see [9].

From the Elek-Szabó result and the multiplicative property of the Fuglede-Kadison determinant, it follows easily that if Γ is sofic and $f \in \mathbb{Z}\Gamma$ is invertible in $\mathbb{Z}\Gamma$, then $\det_{\mathrm{FK}} f = 1$. Deninger asked the converse question under the additional hypotheses of invertibility in $\ell^1(\Gamma)$ [2, Question 26]:

Question 1. *For a countable discrete group Γ, if $f \in \mathbb{Z}\Gamma$ is invertible in $\ell^1(\Gamma)$ but not invertible in $\mathbb{Z}\Gamma$, then do we have $\det_{\mathrm{FK}} f > 1$?*

Question 1 was answered affirmatively by Deninger and Schmidt for the case Γ is amenable and residually finite [3], by Chung and Li for the case Γ is amenable [1], and by Kerr and Li for the case Γ is residually finite [7]. Now we answer it for all sofic groups:

Theorem 2. *For a countable sofic group Γ, if $f \in \mathbb{Z}\Gamma$ is invertible in $\ell^1(\Gamma)$ but not invertible in $\mathbb{Z}\Gamma$, then $\det_{\mathrm{FK}} f > 1$.*

The proof of Theorem 2 uses sofic entropy. For a countable sofic group Γ, we fix a sequence of maps $\Sigma = \{\sigma_i : \Gamma \to S_{d_i}\}_{i \in \mathbb{N}}$, called a *sofic approximation sequence of* Γ, satisfying the following conditions:

(1) for any $s, t \in \Gamma$, one has $\lim_{i\to\infty} \rho_{\mathrm{Hamming}}(\sigma_i(s)\sigma_i(t), \sigma_i(st)) = 0$;
(2) for any distinct $s, t \in \Gamma$, one has $\lim_{i\to\infty} \rho_{\mathrm{Hamming}}(\sigma_i(s), \sigma_i(t)) = 1$;
(3) $\lim_{i\to\infty} d_i = \infty$.

The existence of such a sofic approximation sequence is equivalent to the soficity of Γ. Let α be a continuous action of Γ on a compact metrizable space X. Let ρ

be a compatible metric on X. For any $d \in \mathbb{N}$, we define a metric ρ_2 on $X^{\{1,\dots,d\}}$ by

$$\rho_2(\varphi, \psi) = (\frac{1}{d}\sum_{j=1}^{d}(\rho(\varphi(j), \psi(j)))^2)^{1/2}.$$

For any map $\sigma : \Gamma \to S_d$, any $\delta > 0$, and any finite $F \subseteq \Gamma$, we denote by $\mathrm{Map}(\rho, F, \delta, \sigma)$ the set of all $\varphi \in X^{\{1,\dots,d\}}$ satisfying $\rho_2(\varphi \circ \sigma_s, \alpha_s \circ \varphi) < \delta$ for all $s \in F$. For any $\varepsilon > 0$, denote by $N_\varepsilon(\mathrm{Map}(\rho, F, \delta, \sigma), \rho_2)$ the largest cardinality of subset Y of $\mathrm{Map}(\rho, F, \delta, \sigma)$ satisfying $\rho_2(\varphi, \psi) \geq \varepsilon$ for all distinct $\varphi, \psi \in Y$. Then the *topological entropy of* α *with respect to* Σ, denoted by $h_\Sigma(X, \Gamma)$, is defined by

$$h_\Sigma(X, \Gamma) = \sup_{\varepsilon>0} \inf_{F} \inf_{\delta>0} \limsup_{i\to\infty} \frac{\log N_\varepsilon(\mathrm{Map}(\rho, F, \delta, \sigma_i), \rho_2)}{d_i},$$

where F in $\inf_F$ ranges over all finite subsets of Γ [6]. It does not depend on the choice of the metric ρ.

For any countable group Γ and any $f \in \mathbb{Z}\Gamma$, denote by X_f the Pontryagin dual of the countable discrete abelian group $\mathbb{Z}\Gamma/\mathbb{Z}\Gamma f$. It is a compact metrizable abelian group and can be described explicitly as the closed subgroup of $(\mathbb{R}/\mathbb{Z})^\Gamma$ consisting of all elements x satisfying $xf^* = 0$. The left $\mathbb{Z}\Gamma$-module structure of $\mathbb{Z}\Gamma/\mathbb{Z}\Gamma f$ induces an action of Γ on X_f by continuous automorphisms, called a *principal algebraic action* of Γ. Under the above identification of X_f with a subgroup of $(\mathbb{R}/\mathbb{Z})^\Gamma$, this action is simply the restriction of the left translation to X_f.

Using combinatorial independence, Kerr and Li proved the following [7, Proposition 4.16 and Theorem 6.7]:

Lemma 3. *Let* Γ *be a countable sofic group and* Σ *a sofic approximation sequence of* Γ. *Let* $f \in \mathbb{Z}\Gamma$ *be invertible in* $\ell^1(\Gamma)$ *but not invertible in* $\mathbb{Z}\Gamma$. *Then* $h_\Sigma(X_f, \Gamma) > 0$.

Recently we proved

Lemma 4. *Let* Γ *be a countable sofic group and* Σ *a sofic approximation sequence of* Γ. *Let* $f \in \mathbb{Z}\Gamma$ *such that the left multiplication of* f *on* $\ell^2(\Gamma)$ *is injective. Then* $h_\Sigma(X_{f^*f}, \Gamma) \leq 4 \log \det_{\mathrm{FK}} f$.

Now Theorem 2 follows from Lemmas 3 and 4 easily.

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ℓ^2-Betti numbers, cost, and the free uniform spanning forest

Russell Lyons

Let G be a Cayley graph of a group Γ with respect to a finite symmetric generating set S. If B_n denotes the ball of radius n about the identity, then $B_m \cdot B_n = B_{m+n}$, whence $|B_m| \cdot |B_n| \ge |B_{m+n}|$, and so by Fekete's lemma, the *exponential growth rate* $\mathrm{gr}(G) := \lim_{n\to\infty} |B_n|^{1/n}$ exists. This rate depends on S, but whether it is > 1 does not. In 1981, Gromov [4] asked whether $\mathrm{gr}(G) > 1$ (*exponential growth*) implies that $\inf_S \mathrm{gr}(\Gamma, S) > 1$ (*uniform exponential growth*). Over the years, this implication was shown to hold for groups in the following classes: free; word hyperbolic; solvable; linear; elementary amenable; and others. However, finally in 2004, Wilson [10] gave a counter-example that had other interesting properties: it contained the free group on two letters, $\mathbf{F}_2$, and $\inf_{|S|=2} \mathrm{gr}(\Gamma, S) = 1$.

For $K \subset \Gamma$, define its *external vertex boundary* $\partial_S K := \{x \notin K\,;\ xS \cap K \ne \emptyset\}$. Let the *expansion constant* of G be $\Phi(\Gamma, S) := \inf\{\frac{|\partial_S K|}{|K|}\,;\ \emptyset \ne K \subset \Gamma,\ K \text{ finite}\}$. This depends on S, but whether it is positive does not and is equivalent to Γ being non-amenable. Since $|B_{n+1}|/|B_n| \ge 1 + \Phi(G)$, we have $\mathrm{gr}(G) \ge 1 + \Phi(G)$. Thus, non-amenable groups have exponential growth. However, there exist groups of exponential growth that are amenable.

When is a group *uniformly non-amenable*, i.e., for which classes of groups does $\Phi(\Gamma, S) > 0$ imply that $\inf_S \Phi(\Gamma, S) > 0$? Work of [8, 9, 1, 2] showed that this implication holds for the following classes: free; word hyperbolic; linear; and others, but that certain Baumslag-Solitar groups are counter-examples. Wilson's group is also a counter-example since it is non-amenable (as it contains $\mathbf{F}_2$) but does not have uniformly exponential growth.

Lyons, Pichot and Vassout [7] showed that $\Phi(\Gamma, S) \ge 2\beta_1(\Gamma)$, where the latter is the first ℓ^2-Betti number of Γ and does not depend on S. This is sharp for free groups. We gave the proof of this inequality in our talk. This depends on the fact [5, 6] that the expected degree of every vertex in the *free uniform spanning forest*, FUSF, is equal to $2\beta_1(\Gamma) + 2$.

In place of $\beta_1(\Gamma)$, one can use 1 less than the *cost* of Γ, which is the infimum of $\mathbf{E}[\deg_{\mathcal{G}} o]/2$ over all random graphs $\mathcal{G}$ with vertex set Γ that are connected and have a Γ-invariant law. Here, o is any element of Γ, say, the identity. However, this is not known to be an improvement. That is, Gaboriau [3] has shown that for all Γ, we have $\beta_1 + 1$ is at most the cost of Γ and [3] asked whether they are equal. He noted that this would follow if for every $\epsilon > 0$, there exists an invariant connected random graph $\mathcal{G} = \mathfrak{F} \cup \omega$, where $\mathfrak{F} \sim \mathsf{FUSF}$ and $\mathbf{E}[\deg_\omega o] < \epsilon$.

The *cost* of the random graph $\mathcal{G}$ is the infimum of $\mathbf{E}[\deg_{\mathcal{H}} o]/2$ over all random connected graphs $\mathcal{H}$ on Γ that are Γ-equivariant factors of $\mathcal{G}$. The *fixed-price problem* of [3] asks whether the cost of $\mathcal{G}$ is equal to the cost of Γ for all invariant connected $\mathcal{G}$. This would follow if the previous problem has a positive answer and, in addition, FUSF is a Γ-equivariant factor of IID. This latter question is interesting for several other reasons as well. A positive answer could help to answer currently open questions about FUSF and provide a new technique for showing existence of factor maps. On the other hand, if not, a negative answer could exhibit a new technique for proving that processes are not factors.

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A measure theoretic proof of solid ergodicity for Bernoulli shifts

Robin Tucker-Drob

Chifan and Ioana have shown that the orbit equivalence relation $\mathcal{S}_G$ associated to the Bernoulli shift action of a countable group G on $([0,1]^G, m^G)$ has the following fundamental property, known as solid ergodicity: for any subequivalence relation $\mathcal{R}$ of $\mathcal{S}_G$ there is a countable partition $X_0, X_1, X_2, \dots$ of $[0,1]^G$ into $\mathcal{R}$-invariant Borel sets such that $\mathcal{R}|X_0$ is hyperfinite and $\mathcal{R}|X_i$ is strongly ergodic for all $i > 0$. Their proof uses the machinery of von Neumann algebras and it was unclear whether one could give a direct proof which stays within the category of measure preserving equivalence relations. In this talk I will indicate such a direct proof, which proceeds from the perspective of measure preserving actions of equivalence relations. I will discuss the conceptual benefits and wide applicability of this perspective.

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Cheeger constants and L^2-Betti numbers

Lewis Bowen

The *Cheeger constant* of an infinite volume manifold X is the the infimum of $\frac{area(\partial M)}{vol(M)}$ over all compact submanifolds $M \subset X$. It is denoted by $h(X)$. Now, suppose that X is a Riemannian manifold and $\mathcal{C}$ is a collection of subgroups of the isometry group of X. We also require that for every $\Gamma \in \mathcal{C}$ that X/Γ is a manifold and the quotient map $X \to X/\Gamma$ is a covering space. In this case, we define the uniform Cheeger constant $h(X|\mathcal{C}) = \inf\{h(X/\Gamma) : \Gamma \in \mathcal{C}\}$. This leads to the main problem of this talk: determine whether $h(X|\mathcal{C}) = 0$ or $h(X|\mathcal{C}) > 0$ for some interesting cases. For example, an interesting case occurs when X is real hyperbolic n-space $\mathbb{H}^n$ and Γ is the family of all free subgroups of the isometry group which act nicely (properly discontinuously and freely).

Here are some motivations for this problem. The Cheeger constant is bounded above and below by explicit quadratic functions of $\lambda_0(X)$, the zero-th eigenvalue of the Laplacian. In the special case in which $X = \mathbb{H}^n/\Gamma$ is a real hyperbolic manifold and Γ is geometrically finite, it is also related to the Hausdorff dimension of the limit set of Γ via an explicit formula. So bounding the Cheeger constant leads to bounds on λ_0 and the Hausdorff dimension of the limit set (and similarly, the critical exponent).

It is well-known that one can start with a convex cocompact surface group in $Isom(\mathbb{H}^3)$ and continuously deform it so that the Hausdorff dimension of the limit set tends to 2, the maximum dimension. A natural question is: can this be done one dimension higher? Does there exist a hyperbolic 3-manifold group inside $Isom(\mathbb{H}^4)$ whose Hausdorff dimension of the limit set is close to 3? It is also natural to replace the source group here with a free group or a surface group, as this appears to be related to the well-known problem: does there exist a real hyperbolic 4-manifold which fibers over a surface (with fiber a surface)?

For one last motivation, let us note that Phillips-Sarnak and Doyle established that $h(\mathbb{H}^n|Schottky) > 0$ for $n \geq 3$. That is to say: there is a uniform lower bound (depending on dimension) for the Cheeger constant of $\mathbb{H}^n/\Gamma$ where Γ is a special kind of free group called a Schottky group.

The main result of this talk is the following. Let $\mathcal{G}_d$ be the set of all groups Γ such that

- Γ is residually finite;

- for any $\Gamma' < \Gamma$ and any $\epsilon > 0$ there exists a finite-index subgroup $N \lhd \Gamma'$ with $b_d(N)[\Gamma' : N]^{-1} < \epsilon$.

Then if X is any contractible complete Riemannian manifold and there exists a cocompact lattice $\Lambda < Isom(X)$ such that Λ is residually finite and $b_d^{(2)}(\Lambda) > 0$ then $h(X|\mathcal{G}_d) > 0$. This implies, for example, that $h(\mathbb{H}^4|Free) > 0$. More generally, there exists a uniform lower bound on $h(\mathbb{H}^4/\Gamma)$ where Γ is any hyperbolic 3-manifold group.

The key idea is a generalization of Gabor Elek's result on the continuity of normalized Betti-numbers with respect to Benjamini-Schramm convergence.

Approximating L^2-invariants and homological growth

Wolfgang Lück

Let G be a group together with an inverse system $\{G_i \mid i \in I\}$ of normal subgroups of G directed by inclusion over the directed set I such that $[G : G_i]$ is finite.

We discuss the following result taken from [1, Theorem 1.1].

Theorem Let $F \xrightarrow{j} X \xrightarrow{f} B$ be a fibration of connected CW-complexes. Consider a homomorphism $\phi\colon \pi_1(X) \to G$. Let $p\colon \overline{X} \to X$ be the associated G-covering. Let $G_1(F) \subseteq \pi_1(F)$ be Gottlieb's subgroup of the fundamental group of F. Suppose that the image of $G_1(F)$ under the composite $\phi \circ \pi_1(j)\colon \pi_1(F) \to G$ is infinite.

If d is a natural number such that the $(d+1)$-skeleton of X is finite, then:

(1) We get for all $n \le d$
$$\lim_{i \in I} \frac{d\big(H_n(G_i\backslash\overline{X})\big)}{[G : G_i]} = 0;$$

(2) We get for all $n \le d$
$$\lim_{i \in I} \frac{\ln\Big(\big|\mathrm{tors}\big(H_n(G_i\backslash\overline{X})\big)\big|\Big)}{[G : G_i]} = 0;$$

(3) We get for all $n \le d$
$$b_n^{(2)}\big(\overline{X};\mathcal{N}(G)\big) = \lim_{i \to \infty} \frac{b_n(G_i\backslash X;K)}{[G : G_i]} = 0;$$

(4) Suppose that X is a connected finite CW-complex. Then
$$\lim_{i \in I} \frac{\rho^{(2)}\big(G_i\backslash\overline{X};\mathcal{N}(\{1\})\big)}{[G : G_i]} = \lim_{i \in I} \frac{\rho^{\mathbb{Z}}\big(G_i\backslash\overline{X}\big)}{[G : G_i]} = 0;$$

(5) Suppose that both F and B are connected finite CW-complexes and that $\rho^{(2)}(\overline{F};\mathcal{N}(H)) = 0$, where H is the image of the composite $\pi_1(F) \xrightarrow{\pi_1(j)} \pi_1(E) \xrightarrow{\phi} G$ and $\overline{F}$ is the covering associated to the induced epimorphism $\pi_1(F) \to H$. Then the L^2-torsion $\rho^{(2)}\big(\overline{X};\mathcal{N}(G)\big)$ is defined and satisfies
$$\rho^{(2)}\big(\overline{X};\mathcal{N}(G)\big) = 0.$$

Here $d(\Gamma)$ is the minimal number of generators of a group Γ, $b_n^{(2)}\big(\overline{X};\mathcal{N}(G)\big)$ and $\rho^{(2)}\big(\overline{X};\mathcal{N}(G)\big)$ denote the nth L^2-Betti number and the L^2-torsion of the G-CW-complex $\overline{X}$ and $\rho^{\mathbb{Z}}(G_i\backslash X_i)$ is $\sum_{n\geq 0}(-1)^n\cdot\ln\big(|\operatorname{tors}(H_n(X/G_i))|\big)$.

Let $L \xrightarrow{j} X \xrightarrow{f} B$ be a fibration with a connected Lie group L as fiber such that $\pi_1(L)$ is infinite and $\pi_2(B)=0$. Suppose that $G=\pi_1(X)$, $\phi=\mathrm{id}$ so that $\overline{X}$ is the universal covering of X. Then all assumptions in the theorem above are satisfied and all the assertions hold.

We also get the following result, see [1, Corollary 1.13].

Theorem Let M be an aspherical closed manifold with fundamental group $G=\pi_1(M)$. Suppose that M carries a non-trivial S^1-action or suppose that G contains a non-trivial elementary amenable normal subgroup. Then we get for all $n\geq 0$

$$\begin{aligned}
\lim_{i\to\infty}\frac{b_n(G_i\backslash\widetilde{M};K)}{[G:G_i]} &= 0;\\
\lim_{i\in I}\frac{d\big(H_n(G_i\backslash\widetilde{M})\big)}{[G:G_i]} &= 0;\\
\lim_{i\in I}\frac{\ln\big(\big|\operatorname{tors}\big(H_n(G_i\backslash\widetilde{M})\big)\big|\big)}{[G:G_i]} &= 0;\\
\lim_{i\in I}\frac{\rho^{(2)}\big(G_i\backslash\widetilde{M};\mathcal{N}(\{1\})\big)}{[G:G_i]} &= 0;\\
\lim_{i\in I}\frac{\rho^{\mathbb{Z}}\big(G_i\backslash\widetilde{M}\big)}{[G:G_i]} &= 0;\\
b_n^{(2)}(\widetilde{M};\mathcal{N}(G)) &= 0;\\
\rho^{(2)}(\widetilde{M};\mathcal{N}(G)) &= 0.
\end{aligned}$$

We discuss the following two conjectures and why the results above give evidence for them.

Conjecture *Approximation Conjecture for L^2-torsion*
Let X be a finite connected CW-complex and let $\overline{X}\to X$ be a G-covering.

(1) If the G-CW-structure on $\overline{X}$ and for each $i\in I$ the CW-structure on $G_i\backslash\overline{X}$ come from a given CW-structure on X, then

$$\rho^{(2)}(\overline{X};\mathcal{N}(G))=\lim_{i\to\infty}\frac{\rho^{(2)}(G_i\backslash\overline{X};\mathcal{N}(\{1\}))}{[G:G_i]};$$

(2) If X is a closed Riemannian manifold and we equip $G_i\backslash\overline{X}$ and $\overline{X}$ with the induced Riemannian metrics, one can replace the torsion in the equality appearing above by the analytic versions;

(3) If $b_n^{(2)}(\overline{X};\mathcal{N}(G))$ vanishes for all $n \geq 0$, then
$$\rho^{(2)}(\overline{X};\mathcal{N}(G)) = \lim_{i\to\infty} \frac{\rho^{\mathbb{Z}}(G_i\backslash\overline{X})}{[G:G_i]}.$$

Conjecture *Homological growth and L^2-torsion for aspherical closed manifolds* Let M be an aspherical closed manifold of dimension d and fundamental group $G = \pi_1(M)$. Then

(1) For any natural number n with $2n \neq d$ we have
$$b_n^{(2)}(\widetilde{M};\mathcal{N}(G)) = \lim_{i\to\infty} \frac{b_n(G_i\backslash\widetilde{M};K)}{[G:G_i]} = 0.$$
If $d = 2n$ is even, we get
$$b_n^{(2)}(\widetilde{M};\mathcal{N}(G)) = \lim_{i\to\infty} \frac{b_n(G_i\backslash\widetilde{M};K)}{[G:G_i]} = (-1)^n \cdot \chi(M) \geq 0;$$

(2) For any natural number n with $2n+1 \neq d$ we have
$$\lim_{i\in I} \frac{\ln\big(\big|\mathrm{tors}\big(H_n(G_i\backslash\widetilde{M})\big)\big|\big)}{[G:G_i]} = 0.$$
If $d = 2n+1$, we have
$$\lim_{i\in I} \frac{\ln\big(\big|\mathrm{tors}\big(H_n(G_i\backslash\widetilde{M})\big)\big|\big)}{[G:G_i]} = (-1)^n \cdot \rho^{(2)}\big(\widetilde{M};\mathcal{N}(G)\big) \geq 0.$$

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Amenability of groups acting by homeomorphisms on compact spaces

Kate Juschenko

(joint work with V. Nekrashevych and Mikael de la Salle)

The subject of amenability essentially begins in 1900's with Lebesgue. He asked whether the properties of his integral are really fundamental and follow from more familiar integral axioms. This led to the study of positive, finitely additive and translation invariant measure on different spaces. In particular the study of isometry-invariant measure led to the Banach-Tarski decomposition theorem in 1924. The class of amenable groups was introduced and studied by von Neumann in 1929 and he explained why the paradox appeared only in dimensions greater or equal to three. In 1940's and 1950's a major contribution was made by M. Day in his paper on amenable semigroups.

In 1940's the amenability theory shifted into the field of functional analysis, mainly due to the fact that integration agains a positive, finitely additive measure on a space X produces a continuous linear functional μ on $l_\infty(X)$ such that $m(1) =$

$1 = \|\mu\|$. Currently amenability theory appears in many fields of mathematics, most notably in operator algebras, functional analysis, ergodic theory, probability theory, harmonic analysis. Many conjectures are verified to be true on amenable groups. In many cases when a statement is true for a particular amenable group, for example for $\mathbb{Z}$, it turns out to be true for all amenable groups. In spite of the large list of equivalent definitions of amenability, it is frequently hard and challenging to decide whether a particular group is amenable.

Our recent research develops a technique that can be used to prove amenability of several classes of groups. Since simple groups are building blocks in the group theory it is a natural question to try to find examples of simple amenable groups. The classical example of finitely supported alternating group $A(\infty)$ is simple and amenable, however $A(\infty)$ is not finitely generated. Surprisingly the question of existence of finitely generated simple and amenable group remained open for decades. Recently, in collaboration with N. Monod, [6], we solved this longstanding problem by showing that the full topological group of a Cantor minimal system is amenable. The amenability of this group was previously conjectured by R. Grigorchuk and K. Medynets, [4]. The algebraic properties of the full topological group were studied by H. Matui, [], who proved that the commutator subgroup of $[[T]]$ is simple and finitely generated for any Cantor minimal subshift $[[T]]$. Two systems (T_1, X_1) and (T_2, X_2) are *flip-conjugate* if T_1 is conjugate to T_2 or T_2^{-1}. By a result of Giordano-Putnam-Skau, [3], the full topological group is a complete invariant of flip-conjugacy for (T, X). Thus combining our result with results of Giordano-Putnam-Skau and Matui we obtain $2^{\aleph_0}$ pairwise non-isomorphic infinite amenable simple finitely generated groups.

Continuing our work on amenability, together with V. Nekrashevych and M. de la Salle, [5], we developed a machinery which produced even more new examples of amenable groups with interesting properties and answered a sequence of open problems. Our proofs are more of probabilistic nature: the main ingredient is to find and an action of a group on a discrete set such that all connected components of the Schreier graph of this action are recurrent (as simple random walk).

The main theorem covers amenability of several important classes of groups that act on rooted trees: bounded automorphisms, groups generated by finite automata of linear and quadratic growth. This covers and extends the main results of L. Bartholdi, V. Kaimanovich and V. Nekrashevych, [2], as well as of Amir, Angel and Virag, [1],. Their proof of amenability is very technical. In a sense, our methods provide more direct and unified proof. Using our general approach we also prove amenability of the groups which naturally appear in dynamic: one is a holonomy group of the stable foliation of the Julia set of a Hénnon map, the other is the iterated monodromy group of a mating of two quadratic polynomials. Even though the technique is very general (it covered all known non-elementary amenable groups!) we are convinced that it can be modified to cover many other important examples, which we plan to chase in our future research.

An important and difficult question is to verify that the groups that satisfy the conditions of our main theorem imply *Liouville property*, which is our work in progress. We also plan to develop further the existence of invariant means for the cases when the *Scheier graph* of the action is not recurrent. The question of the existence of means in the transient case is important for understanding amenability of interval exchange transformation group and Thompson group F.

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Geometric and analytic negative curvature

Denis Osin

The action of a group G on a metric space S is called *acylindrical* if for every $\epsilon > 0$ there exist $R, N > 0$ such that for every two points x, y with $\mathrm{d}(x,y) \geq R$, there are at most N elements $g \in G$ satisfying $\mathrm{d}(x, gx) \leq \varepsilon$ and $\mathrm{d}(y, gy) \leq \varepsilon$. Informally, one can think of this condition as a kind of properness of the action on $S \times S$ minus a "thick diagonal".

In the recent years, many interesting results were obtained for groups that admit a non-elementary action on a hyperbolic space which is acylindrical or satisfies certain similar assumptions such as weak acylindricity introduced by Hamenstädt, weak proper discontinuity introduced by Bestvina and Fujiwara, existence of weakly contracting elements in the sense of Sisto, or existence of non-degenerate hyperbolically embedded subgroups introduced by Dahmani, Guirardel, and Osin. I will explain that these classes are essentially the same and coincide with the class of acylindrically hyperbolic groups which can be defined as follows: A group is acylindrically hyperbolic if it admits a non-elementary acylindrical action on a hyperbolic space. I will also discuss the relation between acylindrical hyperbolicity and various classes of groups having "analytic negative curvature", namely the Monod-Shalom class $\mathcal{C}_{reg}$ and the class $\mathcal{D}_{reg}$ introduced by Thom.

Lamplighter groups, random Schrödinger operators, and the Lott-Lück conjecture

Bálint Virág
(joint work with Lukasz Grabowski)

A random Schrödinger operator is a perturbation of the adjacency matrix of a lattice such as

$$A + \alpha V,$$

where V is an i.i.d. diagonal matrix. In other versions edges are perturbed. It has been known is the RSO literature, that the expected spectral measure $\mathbb{E}\mu$ satisfies

- $\mathbb{E}\mu(x, x+\varepsilon) \leq \frac{c}{|\log \varepsilon|}$ for most V when $A = \mathbb{Z}^d$
- $\mathbb{E}\mu$ has bounded density when V does
- $\mathbb{E}\mu(x, x+\varepsilon) \leq c\varepsilon^\gamma$ for most V when $A = \mathbb{Z}$
- $\mathbb{E}\mu(x, x+\varepsilon) \geq c\varepsilon^\gamma$ for some x, some γ, certain Bernoulli V and $A = \mathbb{Z}$
- $\mathbb{E}\mu(0, 0+\varepsilon) \geq \frac{c}{|\log \varepsilon|^3}$ for some bond models.

The lamplighter product of a graph G and a rooted graph H has state space $\{(\eta, x) \in H^G \times G \mid \eta(g) = \text{root}(H) \text{ for all but finitely many } g\}$. There are two edge sets $A = \{(\eta, x) \sim (\eta, x') \mid x \sim x \text{ in } G\}$ and $S = \{(\eta, x) \sim (\eta', x) \mid \eta(x) \sim \eta'(x) \text{ in } H, \text{ and otherwise } \eta = \eta'\}$.

Lamplighter products are spectrally equivalent to RSO:

$$\sigma_{H^G \times G, p(A,S)} = \mathbb{E}\sigma_{p(A,\tilde{S})}$$

where p is a noncommutative polynomial and $\tilde{S}$ is the multiplication operator by an i.i.d. random saple of σ_H.

The spectral equivalence resolves some open problems:

- the Lott-Lück conjecture does not hold: $\mu_G(0, \varepsilon) \geq \frac{1}{|log\varepsilon|^3}$ for some group.
- existence of point spectrum is sensitive to generators.
- $\exists$ groups with singular continuous spectra.
- relaxation-time asymptotics for finite lamplighter groups.

L^2-invariants of locally compact groups

Henrik Densing Petersen

In this talk we explain how to define L^2-Betti numbers for any locally compact, unimodular, second countable (henceforth abbreviated lcus) group G. The definition is motivated in part by a well-known theorem of D. Gaboriau: Given any two lattices Γ, Λ in a locally compact group G,

$$\beta^n_{(2)}(\Gamma) = \frac{\operatorname{covol}\Gamma}{\operatorname{covol}\Lambda} \cdot \beta^n(2)(\Lambda)$$

for all $n \in \mathbb{N}$. It then seems natural to look for an appropriate definition of $\beta^n_{(2)}(G)$ satisfying

$$\beta^n_{(2)}(\Gamma) = \operatorname{covol}\Gamma \cdot \beta^n_{(2)}(G) \tag{1}$$

for any lattice Γ. In [2] this was introduced as follows.

Recall that the continuous cohomology $H^n(G, \mathcal{E})$ of a lcus group G with coefficients in a locally convex G-module $\mathcal{E}$ is computed as the n'th homology of the complex of inhomogeneous continuous cochains

$$0 \longrightarrow \mathcal{E} \xrightarrow{\partial^0} C(G, \mathcal{E}) \xrightarrow{\partial^1} C(G^2, \mathcal{E}) \xrightarrow{\partial^2} \cdots$$

where the coboundary maps are given by

$$\begin{aligned}(\partial^n \xi)(g_1, \ldots, g_{n+1}) \quad &= \quad g_1.\xi(g_2, \ldots, g_{n+1}) + \sum_{i=1}^{n} (-1)^i \xi(g_1, \ldots, g_i g_{i+1}, \ldots, g_{n+1}) + \\ &\quad + (-1)^{n+1} \xi(g_1, \ldots, g_n).\end{aligned}$$

Then the L^2-Betti numbers of G are by definition (extending directly the definition for discrete groups)

$$\beta^n_{(2)}(G) = \dim_{LG} H^n(G, L^2 G).$$

Here LG is the group von Neumann algebra of G and the dimension $\dim_{LG}$ is the extended von Neumann dimension, applied to the right-LG-module $H^n(G, L^2G)$.

Then equation (1) was shown to hold for G totally disconnected and/or admitting a cocompact lattice in [2], and the result was extended to general G in [1] in joint work with David Kyed and Stefaan Vaes. The general proof of (1) given in [1] relies on the existence of cocompact lattices "in a measurable sense" in any lcus group G, and the methods developed there then allows direct generalizations of several results on L^2-Betti numbers of countable groups and equivalence relations.

For a short summary of the results of [2] see either the introduction of the thesis itself, or [3]. In joint work with Alain Valette [4] we give a complete description of the L^2-Betti numbers of type I groups in terms of cohomology with coefficients in irreducible representations.

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The intermediate factor theorem over local fields

Arie Levit

In the talk we discussed a generalization of the intermediate factor theorem to the local field case.

Recall that given a locally compact group G, a *G-space* is a standard Borel space (X, μ) with a Borel G-action such that μ is quasi-invariant under the action. Moreover, the action is *essentially transitive* if there exists a conull orbit. An ergodic action is *properly ergodic* if it is not essentially transitive. The action is *irreducible* if every non-centeral normal subgroup acts ergodically. It is *faithful* if for every $g \in G$, $gx \neq x$ holds for x of positive measure. Finally, the action is *essentially free* if μ-a.e. $x \in X$ has trivial stabilizer in G.

Given a fixed local field k, the following is a generalization of the classical intermediate factor theorem [5] for k-groups:

Theorem 5. *Let $\mathbb{G}$ be a semisimple connected algebraic k-group without k-anisotropic factors and of k-rank ≥ 2. Let X and Y be $\mathbb{G}_k$-spaces such that X is irreducible, ergodic and has an invariant probability measure. Let $\mathbb{P} \leq \mathbb{G}$ be a minimal k-parabolic subgroup of $\mathbb{G}$. Given a sequence of $\mathbb{G}_k$-factor maps*

$$(\mathbb{G}_k/\mathbb{P}_k) \times X \to Y \to X$$

whose composition is the projection to X, there exists a parabolic k-subgroup $\mathbb{Q}$ containing $\mathbb{P}$ such that

$$Y \cong (\mathbb{G}_k/\mathbb{Q}_k) \times X$$

as $\mathbb{G}_k$-spaces, and moreover the maps $(\mathbb{G}_k/\mathbb{P}_k) \times X \to Y$ and $Y \to X$ are the natural ones.

Furthermore, in the sitation that $\mathbb{G}_k$ has property (T), the following is a generalization of the Stuck-Zimmer theorem to local fields (see [4]):

Theorem 6. *Let $\mathbb{G}$ be as in the preivous theorem, and assume that $\mathbb{G}_k$ has property (T). Then every faithful, properly ergodic, irreducible and finite measure-preserving $\mathbb{G}_k$-action is essentially free.*

Note that the factor theorem was proved by Margulis in the generality of local fields [2] while both the intermediate factor and the Stuck-Zimmer theorems were proved only for real Lie groups.

The original proof of the intermediate factor theorem in [5] contains a gap, that was later noticed in [3] where an alternative proof of that theorem for real Lie groups was also given. Our proof in [1] completes the gap in a way that is closer in spirit to the proof of the factor theorem and extends to local fields.

We remark that the theorem for real Lie groups follows from the general case.

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Ultraproducts of finite simple groups: abstract and metric

John Wilson

Let G be an abstract ultraproduct $\prod S_i/\mathcal{U}$ of a family $\{S_i \mid i \in I\}$ of finite simple groups and suppose that G is infinite. From the classification of the finite simple groups and a result of F. Point, if the groups S_i have bounded rank, then G is a Chevalley group (possibly twisted) over an ultraproduct of finite fields. Such groups turn out to be precisely the infinite simple groups that satisfy all first-order sentences that hold in all finite groups, by a result of the author and Ryten.

Suppose instead that the groups S_i have unbounded rank. In this case, it can be assumed that the family $\{S_i \mid i \in I\}$ is contained in one of the following: (1) the family of alternating groups; (2) the family of groups $\mathrm{PSL}_n(q)$; (3) the family of finite simple classical groups not of type $\mathrm{PSL}_n(q)$. It was shown by Nikolov that G has a unique infinite simple image S, and that S is a metric ultraproduct of the groups S_i with respect to the metric on S_i given by conjugacy length, defined by $\ell(x) = \log|x^{S_i}|/\log|S_i|$ for $x \in S_i$. Current work of A. Thom and the speaker was described, on recognition from the structure of centralizers in S which of cases (1), (2) or (3) holds, and, in cases (2), (3), whether some prime predominates as the defining characteristic for the groups S_i.

On random groups of intermediate growth

Rostislav Grigorchuk
(joint work with Mustafa G. Benli, Yaroslav Vorobets)

Let G be a finitely generated group and S a finite set of generators of G. For $g \in G$, let $|g|_S$ be the minimal number n such that $g = s_1 \dots s_n$ where $s_i \in S^{\pm}$. The growth function of G (with respect to the generating set S) is the function $\gamma_G^S(n) = \#\{g \in G \mid |g|_S \le n\}$. For any two functions f, g let us write $f \preceq g$ if there exists $C > 0$ such that $f(n) \le Cg(Cn)$ for all $n \in \mathbb{N}$, and write $f \sim g$ if both $f \preceq g$ and $g \preceq f$. It can be observed that the growth functions of a group corresponding to different generating sets are $\sim$ equivalent.

In 1968 it became apparent that all known examples of groups have either *exponential* growth or *polynomial* growth. That is $\gamma_G \sim e^n$ or $\gamma_G \preceq n^d$ for some d. John Milnor asked whether groups of intermediate growth exists. The first examples of groups with intermediate growth were constructed by the author in [Gri84] and in [Gri85].

Let p be a a prime and let $\Omega_p = \{0, 1, \ldots, p\}^{\mathbb{N}}$ denote the set of infinite sequences over the alphabet $\{0, 1, \ldots, p\}$ endowed with its natural topology and the shift map $\tau : \Omega_p \to \Omega_p$ for which $\tau(\omega)_n = \omega_{n+1}$. For each $\omega \in \Omega_p$, the speaker constructed in [Gri84, Gri85] a finitely generated group G_ω with a set $S_\omega = \{a, b_\omega, c_\omega\}$ of three generators, acting on the unit interval $[0, 1]$ by Lebesgue measure preserving transformations. One of the specific features of this construction is that if two sequences $\omega, \eta \in \Omega_p$, which are not eventually constant, have the same prefix of length n, then the corresponding groups G_ω, G_η have isomorphic Cayley graphs in the neighborhood of identity of radius 2^{n-1}. Therefore, removing the groups G_ω when ω is eventually constant and and replacing them by the corresponding limit groups in the space of 3-marked groups $\mathcal{M}_3$, one obtains a compact subset $\mathcal{G}_p = \{(G_\omega, S_\omega) \mid \omega \in \Omega_p\} \subset \mathcal{M}_3$ which is homeomorphic to Ω_p via the map $(G_\omega, S_\omega) \mapsto \omega$. Let $\Omega_{p,\infty}$ denote the set of all sequences in which all symbols $\{0, 1, \ldots, p\}$ appear infinitely often. In [Gri84] (for $p = 2$) and [Gri85] (for $p > 2$), it was shown that the groups G_ω for $\omega \in \Omega_{p,\infty}$ are examples of periodic p-groups of intermediate growth hence making these examples interesting and related to the Milnor problem mentioned in the previous paragraph and also to the Burnside problem about periodic groups.

The growth rates of the groups in these families show a wide range of different behavior. In [Gri84] it was shown that the set of growth rates in the family $\mathcal{G}_2$ contains an uncountable chain and also an uncountable anti-chain. It was also shown in [Gri84] that for any subexponential function f, there exists $G_\omega \in \mathcal{G}_2$ having growth not slower than f. On the other hand, in [Gri84, Gri85] it was observed that if the symbols $\{0, 1, \ldots, p\}$ are "uniformly" distributed in a sequence $\omega \in \Omega_p$, then the growth of the corresponding group G_ω is no more than of a function of the form e^{n^α} for some $\alpha < 1$.

The space $\mathcal{G}_p$ with the transformation $T(G_\omega, S_\omega) = (G_{\tau\omega}, S_{\tau\omega})$ induced by the shift $\tau : \Omega_p \to \Omega_p$ and T invariant probability measure on it constitutes a model for a random group from the family $\mathcal{G}_p$. A natural question is what group theoretic properties are typical (in the sense of measure and in the sense of Baire category) in $\mathcal{G}_p$. Some properties (such as amenability or periodicity) are known to be typical and this talk is concerned with the growth rate of a typical group.

Given $\omega \in \Omega_p$, we denote the growth function of the group G_ω (with respect to the generating set S_ω) by γ_ω. The first result is:

Theorem 1. Suppose μ is a Borel probability measure on Ω_p that is invariant and ergodic relative to the shift transformation $\tau : \Omega_p \to \Omega_p$.

a) If the measure μ is supported on $\Omega_{p,\infty}$, then there exists $\alpha = \alpha(\mu, p) < 1$ such that $\gamma_\omega(n) \preceq e^{n^\alpha}$ for $\mu-$almost all $\omega \in \Omega_p$.
b) In the case μ is the uniform Bernoulli measure on Ω_2, one can take $\alpha = 0.999$

In fact, there is nothing special about the space $\mathcal{M}_3$ and the following holds:

Theorem 1'. For any $k \geq 2$ and prime p, $\mathcal{M}_k$ contains a compact subset $\mathcal{K}_k = \{(M_\omega, L_\omega) \mid \omega \in \Omega_p\}$ homeomorphic to Ω_p (via the map $\omega \to (M_\omega, L_\omega)$) such that if μ is a measure supported on $\Omega_{p,\infty}$, invariant and ergodic relative to the shift, then there exists $\alpha = \alpha(\mu, p) < 1$ such that $\gamma_{M_\omega}(n) \preceq e^{n^\alpha}$ for μ-almost all $\omega \in \Omega_p$.

Given two functions $f_1, f_2 : \mathbb{N} \to \mathbb{N}$ such that $f_1(n) \preceq f_2(n) \preceq e^n$, we say that a group G has *oscillating growth of type* (f_1, f_2) if $f_1 \not\preceq \gamma_G$ and $\gamma_G \not\preceq f_2$. The existence of groups with oscillating growth follows from the results of [Gri84]. The results of [KP13] and of [BE12, Bri11] provide more information in this direction. Our next result deals with the growth of a typical group from the family $\mathcal{G}_p$, from a topological point of view.

Let $\theta_2 = \log(2)/\log(2/x_0)$ where x_0 is the real root of the polynomial $x^3 + x^2 + x - 2$. We have $\theta_2 < 0.767429$. For a prime $p \geq 3$, let $\theta_p = \frac{\log(p^{p+1})}{\log(p^{p+1}) - \log(3/4)} < 1$.

Theorem 3.

a) For any and $\theta > \theta_p$ and any function f satisfying $e^{n^\theta} \prec f(n) \prec e^n$, there exists a dense G_δ subset $\mathcal{Z}_p \subset \mathcal{G}_p$ such that any group in $\mathcal{Z}_p$ has oscillating growth of type $\left(e^{n^\theta}, f\right)$.

b) For every θ and β with $\theta_p < \theta < \beta < 1$, there exists a dense G_δ subset of $\mathcal{G}_p$ which consists of groups with oscillating growth of type $\left(e^{n^\theta}, e^{n^\beta}\right)$.

c) Given any $\epsilon > 0$ and function f satisfying $\exp\left(\frac{n}{\log^{1-\epsilon} n}\right) \prec f(n) \prec e^n$, there is a dense G_δ subset $\mathcal{E} \subset \{(G_\omega, S_\omega) \mid \omega \in \{0,1\}^{\mathbb{N}}\} \subset \mathcal{G}_2$ such that any group in $\mathcal{E}$ has oscillating growth of type $\left(\exp\left(\frac{n}{\log^{1-\epsilon} n}\right), f\right)$.

Again, one can generalize these to arbitrary $k \geq 2$ and in particular, the following holds.

Theorem 3'. for each $k \geq 2, \theta > \theta_p$ and function f satisfying $e^{n^\theta} \prec f(n) \prec e^n$, $\mathcal{M}_k$ contains a compact subset $\mathcal{C}_k$ homeomorphic to a Cantor set such that there exists a dense G_δ subset $\mathcal{C}'_k \subset \mathcal{C}_k$ which consists of groups with oscillating growth of type $\left(e^{n^\theta}, f\right)$.

The reason why oscillating groups are typical in the categorical sense is the existence of a countable dense subset in $\mathcal{G}_p$ consisting of groups of exponential growth and also a dense subset of groups with the growth equivalent to the growth of the groups $G_{(01\ldots p)^\infty}$ whose growths are bounded above by $e^{n^{\theta_p}}$ by results of [Bar98] and [MP01]. To prove part c) of Theorem 3, we use instead a result of Erschler [Ers04] stating that the growth of the group G_ω for $\omega = (01)^\infty \in \Omega_2$ is slower than $\exp\left(\frac{n}{\log^{1-\epsilon} n}\right)$ for all $\epsilon > 0$.

We see that the growth of a typical group in the families $\mathcal{G}_p$ from a measure-theoretic sense is radically different compared to the growth of a typical group in the sense of category.

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On Tarski numbers

Mikhail Ershov

Let G be a discrete group. It is well known that G is non-amenable if and only if G admits a paradoxical decomposition, that is, there exist positive integer n and m, disjoints subsets $A_1, \dots, A_n, B_1, \dots, B_m$ of G and elements $g_1, \dots, g_n, h_1, \dots, h_m$ of G such that

$$G = (\sqcup_{i=1}^n A_i) \sqcup (\sqcup_{j=1}^m B_j) = \sqcup_{i=1}^n g_i A_i = \sqcup_{j=1}^m B_j.$$

If G is non-amenable, the minimal number of pieces in its paradoxical decomposition, that is, the minimal value of $n+m$, is called the Tarski number of G and denoted by $\mathcal{T}(G)$.

The following properties of Tarski numbers are well known (and easy to check):

(a) $\mathcal{T}(G) \le \mathcal{T}(H)$ if H is either a subgroup or a quotient of G;
(b) $\mathcal{T}(G) \ge 4$ for any G, and equality holds if and only if G contains a non-abelian free subgroup.

In addition, in [1] it was shown that $\mathcal{T}(G) \ge 6$ for any torsion group G.

In my talk, I discussed the following two new results on Tarski numbers:

Theorem 1. *The set of possible values of Tarski numbers is unbounded.*

Theorem 2. *There exist groups whose Tarski number is equal to* 6.

To the best of my knowledge, Theorem 2 provides the first examples of groups whose Tarski number has been computed precisely and is not equal to 4.

Theorem 1 has been established by Ozawa and Sapir in a recent mathoverflow post [6]. Ozawa first observed that if G is a non-amenable group such that every m-generated subgroup is amenable, then $\mathcal{T}(G) > m+2$, and Sapir noticed that such groups do exist for an arbitrary m. The latter is a consequence of the following two results on Golod-Shafarevich groups:

(i) Every Golod-Shafarevich group is non-amenable [2].
(ii) For every m there exists an $(m+1)$-generated Golod-Shafarevich group all of whose m-generated subgroups are finite.

I explained how to prove Theorem 1 essentially from scratch using only the fact that there exists a Golod-Shafarevich group with property (T) (the latter is one of several ingredients of the proof of non-amenability of Golod-Shafarevich groups). The same argument shows that there exists a non-amenable group G such that the values of Tarski numbers of finite index subgroups of G are unbounded.

Theorem 2 is a direct consequence of the following two results:

Theorem 3. *Let G be a 3-generated group with $b_1^{(2)}(G) \geq \frac{3}{2}$ (where $b_1^{(2)}(G)$ is the first L^2-Betti number of G). Then $\mathcal{T}(G) \leq 6$. If in addition G is torsion, then $\mathcal{T}(G) = 6$.*

Theorem 4. *For any integer $d \geq 2$ and any $\varepsilon > 0$ there exists a d-generated group G with $b_1^{(2)}(G) > d - 1 - \varepsilon$.*

Theorem 4 was established by Osin [5]; in fact, groups with such property can be explicitly constructed. In my talk, I briefly outlined the proof of Theorem 3. The main ingredient in the proof is the result of Lyons [4] which asserts that for any finitely generated group G and any finite generating set S of G, the expected degree of the free uniform spanning forest on the Cayley graph $\mathrm{Cay}(G, S)$ is equal to $2b_1^{(2)}(G) + 2$.

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Limit multiplicities for $\mathbf{SL}(n)$

Tobias Finis

(joint work with Erez Lapid, Werner Müller)

The limit multiplicity problem, which goes back to DeGeorge and Wallach, concerns the asymptotic behavior of the spectra of lattices Γ (discrete subgroups of finite covolume $\mathrm{vol}(\Gamma\backslash G)$) in a fixed semisimple Lie group G in the situation where $\mathrm{vol}(\Gamma\backslash G) \to \infty$. In a great number of cases, the normalized discrete spectra μ_Γ converge then to the Plancherel measure μ_{pl} of the group G, which is defined purely in terms of the decomposition of the space $L^2(G)$, i.e., without any reference to discrete subgroups.

For uniform lattices Γ (lattices for which the quotient $\Gamma\backslash G$ is compact), general results on this problem have been known for some time. The case of normal towers, i.e., of descending sequences of finite index normal subgroups of a given uniform lattice with trivial intersection, was completely resolved by Delorme [4]. Recently, limit multiplicity has been shown for much more general sequences of uniform lattices [1, 2].

In the case of non-compact quotients $\Gamma\backslash G$, where the spectrum also contains a continuous part, much less is known. In a recent joint preprint of the author with E. Lapid and W. Müller [9], this case has been analyzed in a rather general setup. An extension of these results will appear in a forthcoming paper of E. Lapid and the author. (See [9, §1] for previous results in the literature.) The new approach is based on a careful study of the spectral side of Arthur's trace formula in the recent form given in [5, 7]. The results are unconditional only for the groups $\mathrm{GL}(n)$ and $\mathrm{SL}(n)$, but in the general case a substantial reduction of the problem is obtained.

1. The limit multiplicity property

Let G be a connected linear semisimple Lie group with a fixed choice of Haar measure. Since the group G is of type I, we can write unitary representations of G on separable Hilbert spaces as direct integrals (with multiplicities) over the unitary dual $\Pi(G)$, the set of isomorphism classes of irreducible unitary representations of G with the Fell topology. The regular representation of $G \times G$ on $L^2(G)$ decomposes as the direct integral of the tensor products $\pi \otimes \pi^*$ against the *Plancherel measure* μ_{pl} on $\Pi(G)$. The support of the Plancherel measure is called the *tempered dual* $\Pi(G)_{\mathrm{temp}} \subset \Pi(G)$. The Plancherel measure and the tempered dual are well understood, mainly by the work of Harish-Chandra.

By definition, a Jordan measurable subset of $\Pi(G)_{\mathrm{temp}}$ is a bounded set A such that $\mu_{\mathrm{pl}}(\bar{A} - A^\circ) = 0$. We say that a collection $\mathfrak{M}$ of Borel measures μ on $\Pi(G)$ has the *limit multiplicity property* (property (LM)) if the following two conditions are satisfied:

(1) For any Jordan measurable Borel set $A \subset \Pi(G)_{\mathrm{temp}}$ we have[1]

$$\mu(A) \to \mu_{\mathrm{pl}}(A), \quad \mu \in \mathfrak{M}.$$

(2) For any bounded Borel set $A \subset \Pi(G) \setminus \Pi(G)_{\mathrm{temp}}$ we have

$$\mu(A) \to 0, \quad \mu \in \mathfrak{M}.$$

We will apply this setup to the regular representations R_Γ of G on $L^2(\Gamma\backslash G)$ for lattices Γ in G. Consider the discrete part $L^2_{\mathrm{disc}}(\Gamma\backslash G)$ of $L^2(\Gamma\backslash G)$, namely the sum of all irreducible subrepresentations, and denote by $R_{\Gamma,\mathrm{disc}}$ the corresponding restriction of R_Γ. For any $\pi \in \Pi(G)$ let $m_\Gamma(\pi)$ be the multiplicity of π in $L^2(\Gamma\backslash G)$. These multiplicities are known to be finite, at least if either G has no compact

[1] Here convergence means that for any $\varepsilon > 0$ the set of $\mu \in \mathfrak{M}$ with $|\mu(A) - \mu_{\mathrm{pl}}(A)| \geq \varepsilon$ is finite.

factors or if Γ is arithmetic. We define the discrete spectral measure on $\Pi(G)$ with respect to Γ by

$$\mu_\Gamma = \frac{1}{\mathrm{vol}(\Gamma\backslash G)} \sum_{\pi\in\Pi(G)} m_\Gamma(\pi)\delta_\pi,$$

where δ_π is the Dirac measure at π.

2. Density principle and trace formula

We want to study property (LM) for the measures μ_Γ, where Γ ranges over a collection of lattices in G. A basic approach to this problem is to use integration against test functions on G and the trace formula. Let K be a maximal compact subgroup of G. For a test function $f \in C^\infty_{c,\mathrm{fin}}(G)$, the space of smooth, compactly supported bi-K-finite functions on G, we define its "Fourier transform" on the unitary dual by taking traces: $\hat{f}(\pi) = \mathrm{tr}\,\pi(f)$, $\pi \in \Pi(G)$. This defines $\mu(\hat{f})$ for Borel measures μ on $\Pi(G)$ (of course $\mu(\hat{f})$ might in general be divergent). In particular we have $\mu_{\mathrm{pl}}(\hat{f}) = f(1)$ by Plancherel inversion and

$$\mu_\Gamma(\hat{f}) = \frac{1}{\mathrm{vol}(\Gamma\backslash G)} \mathrm{tr}\, R_{\Gamma,\mathrm{disc}}(f),$$

which is known to be convergent for arithmetic lattices Γ. Sauvageot's density principle [10], a refinement of the work of Delorme, amounts to the following:

Theorem 1 (Sauvageot)**.** *Let $\mathfrak{M}$ be a collection of Borel measures on G and assume that for all $f \in C^\infty_{c,\mathrm{fin}}(G)$ we have*

$$\mu(\hat{f}) \to \mu_{\mathrm{pl}}(\hat{f}) = f(1), \quad \mu \in \mathfrak{M}.$$

Then $\mathfrak{M}$ satisfies (LM).

For the purpose of illustration let now Γ be a cocompact lattice in G. For a finite index subgroup Δ of Γ and $\gamma \in \Gamma$ set

$$c_\Delta(\gamma) = |\{\delta \in \Delta\backslash\Gamma : \delta\gamma\delta^{-1} \in \Delta\}|.$$

Combining the density principle with the Selberg trace formula, which expresses $\mathrm{tr}\, R_\Delta(f)$ in terms of orbital integrals of f associated to the conjugacy classes of Δ, we can reduce the limit multiplicity problem for collections $\mathfrak{D}$ of finite index subgroups Δ of Γ to the following purely group-theoretical question: do we have

$$\frac{c_\Delta(\gamma)}{[\Gamma : \Delta]} \to 0, \quad \Delta \in \mathfrak{D},$$

for any $\gamma \in \Gamma$, $\gamma \neq 1$? Note that for central elements γ (in particular for $\gamma = 1$), we have obviously $c_\Delta(\gamma) = [\Gamma : \Delta]$.

Can we expect that for irreducible arithmetic lattices the limit multiplicity property holds for any collection of subgroups not containing non-trivial central elements? For congruence subgroups of cocompact lattices (or for arbitrary finite index subgroups in the higher rank case) this follows from [1, 2]. In the real rank one case there are counterexamples if we allow arbitrary finite index subgroups.

An independent proof of the congruence subgroup case will be contained in work in preparation of E. Lapid and the author (see [6]).

For the non-cocompact lattices $\mathrm{SL}(n, \mathfrak{o}_F) \subset \mathrm{SL}(n, F \otimes \mathbb{R})$, where F is a number field, we can show the following:

Theorem 2. *Let F be a number field. The collection of measures μ_Γ, where Γ runs over all congruence subgroups of $\mathrm{SL}(n, \mathfrak{o}_F)$ not containing non-trivial central elements, has the limit multiplicity property.*

Note that for $n \geq 3$ and F not totally complex, every finite index subgroup of $\mathrm{SL}(n, \mathfrak{o}_F)$ is in fact a congruence subgroup.

It seems very likely that this result generalizes to the lattices $\mathrm{SL}(m, \mathfrak{o}_D)$, where D is a division algebra with center F that splits at the infinite places (we are planning to include this case in a revised version of [9]). Unlike the results of [1, 2] for cocompact lattices, our current proof does not cover more general sequences of lattices, where infinitely many distinct commensurability classes are allowed, although it might be possible to include this case by making the dependence of all parameters on D and F explicit.

As mentioned already above, the proof of Theorem 2 is based on Arthur's trace formula [3], an elaborate extension of the Selberg trace formula to the non-cocompact case. One needs to control both its geometric and its spectral side. It is the spectral side which poses genuinely new problems. The contribution from the continuous spectrum to R_Γ involves generalized logarithmic derivatives of intertwining operators. In [5, 7] it was shown that those can be rewritten in terms of usual logarithmic derivatives $A^{-1}(s)A'(s)$ of operator-valued functions $A(s)$ of one variable. Each such operator can be decomposed as a product of a scalar normalizing factor, which at least for $\mathrm{GL}(n)$ and $\mathrm{SL}(n)$ can be expressed in terms of automorphic L-functions, and of a tensor product (over all places of the ground field F) of locally defined normalized intertwining operators, whose matrix coefficients are essentially rational functions. The necessary control of the scalar factors can be deduced from the theory of automorphic L-functions. Regarding the local operators, the fact that only first derivatives occur implies that we only need to bound the degrees of their matrix coefficients in terms of the level of the congruence subgroup Δ, which was achieved (for $\mathrm{GL}(n)$ and $\mathrm{SL}(n)$) in [8]. The remaining group-theoretic ingredient is essentially a bound of the form $O([\Gamma : \Delta]^{-\varepsilon})$ for the average value of $c_\Delta(\gamma)$ as γ runs over $U_P \cap \Gamma$, where P is a parabolic subgroup of G for which $U_P \cap \Gamma$ is a lattice in U_P. Such a bound is provided by our group-theoretic analysis (cf. [6]), as well as by the alternative method of [1, 2].

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Representation growth and LocSys

Nir Avni
(joint work with A. Aizenbud)

For a group Γ, let $r_n(\Gamma)$ be the number of rank-n irreducible complex characters of Γ. The representation growth of Γ is the asymptotic behavior of the sequence $r_n(\Gamma)$. We find connections between the representation growth of an arithmetic lattice of the form $G(\mathbb{Z})$ and the singularities of the moduli space of G-local systems on closed surfaces. Our main theorem is the following:

Theorem Let G be a semisimple algebraic group defined over $\mathbb{Z}$ and let k be a natural number. Let $\Phi : G^{2k} \to G$ be the map $\Phi(x_1, y_1, \ldots, x_k, y_k) = [x_1, y_1] \cdots [x_k, y_k]$. For every p, denote the Haar measure on the group $G(\mathbb{Z}_p)$ by λ_p. The following conditions are equivalent:

(1) For all p, we have $r_n(G(\mathbb{Z}_p)) = o(n^{2k-2})$.
(2) For all p, the push-forward of the Haar measure $\lambda_p^{\otimes 2k}$ under Φ has a continuous density with respect to λ_p.
(3) The map Φ is flat and all of its fibers have rational singularities.
(4) The fiber $\Phi^{-1}(1)$ has rational singularities.
(5) For all p, the moduli space of $G(\mathbb{Z}_p)$-local systems on a closed surface of genus k has finite Atiyah–Bott volume.

We also show that these conditions hold for $G = SL_d$ if $k \geq 12$. As an application, we show that $r_n(SL_d(\mathbb{Z})) = o(n^{22})$ for every $d \geq 3$.

L^1-measure equivalence of hyperbolic lattices

Roman Sauer
(joint work with Uri Bader, Alex Furman)

Two finitely generated groups Γ and Λ are *measure equivalent* if there is a Lebesgue space (Ω, ν) – called a (Γ, Λ)-coupling – endowed with ν-preserving, commuting, essentially free actions of Γ and Λ that admit ν-finite fundamental domains.

The basic example are two lattices in the same locally compact group G with G serving as a coupling.

The choice of measurable fundamental domains $X \subset \Omega$ and $Y \subset \Omega$ leads to *cocycles* $\alpha : \Gamma \times Y \to \Lambda$ and $\beta : \Lambda \times X \to \Gamma$ that are defined by the equation

$$\gamma y = \alpha(\gamma, y)Y$$

and similarly for β. With $l : \Lambda \to \mathbb{R}^{\geq 0}$ being the length function on Λ coming from a word-metric on Λ, we call Ω a *integrable* (Γ, Λ)-coupling if there exist fundamental domains such that the associated cocycles are integrable in the sense that

$$\int_Y l \circ \alpha(\gamma, y) dm(y) < \infty \text{ for every } \gamma \in \Gamma,$$

and similarly for β. Two groups Γ, Λ are called *L^1-measure equivalent*, if they possess an integrable (Γ, Λ)-coupling.

Similarly to the classification of groups up to quasi-isometry or measure equivalence, we would like to classify groups up to L^1-measure equivalence. A natural class to consider are lattices in semi-simple Lie groups. Alex Furman showed that lattices in simple Lie groups of higher rank are rigid with respect to measure equivalence [2]. More precisely, if Λ is measure equivalent to a lattice in a simple Lie group G of higher rank, then Λ surjects onto a lattice in G with finite kernel. This result also yields the L^1-classification of such lattices. Concerning Lie groups of rank 1 we prove the following [1]:

Theorem: *Let G be the isometry group of real hyperbolic n-space with $n \geq 2$ and let $\Gamma < G$ be a lattice. In the case $n = 2$ we additionally assume that Γ is cocompact. If Λ is a finitely generated group that is L^1-measure equivalent to Γ, then there are a finite group F and a lattice $\bar{\Lambda} < G$ and a short exact sequence*

$$1 \to F \to \Lambda \to \bar{\Lambda} \to 1.$$

If $n = 2$, then $\bar{\Lambda}$ is a cocompact lattice.

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The Group Large Sieve

Chen Meiri

The sieve method, which is a classic one in Number Theory, can be applied to study random walks on linear groups (and on general groups via their representations). Let Γ be a linear group generated by some finite symmetric set S. Let w_k be the k^{th}-step of a random walk on the Cayley graph $\mathrm{Cay}(\Gamma, S)$. The main interest is to estimate the probability that w_k belongs to some subset $T \subseteq \Gamma$. The starting point of the random walk is the identity so for every $k \geq 0$,

$$\text{(1)} \qquad \mathrm{Prob}(w_k \in T) = \frac{|\{(s_1, \ldots, s_k) \in S^k \mid s_1 s_2 \cdots s_k \in T\}|}{|S|^k}.$$

There are two kinds of sieving techniques for groups. The first is called the *Affine Sieve* and was developed by Jean Bourgain, Alex Gamburd and Peter Sarnak [BGS]. This method deals with arithmetic questions. A typical result of the affine sieve is:

Theorem 1 (Bourgain-Gamburd-Sarnak)**.** *Let $\Gamma \leq \mathrm{SL}(n, \mathbb{Z})$ be a finitely generated Zariski-dense subgroup with a finite generating set S. There are constants $r, t > 0$ for which*

$$\mathrm{Prob}\left(\mathrm{tr}(w_k) \textit{ has at most } r \textit{ prime factors}\right) \sim \frac{1}{k^t}.$$

The second technique is called the *Group Large Sieve* and was developed by Igor Rivin [Ri], Emmanuel Kowalski [Ko], Alex Lubotzky and Chen Meiri. This method deals with algebraic questions. A typical result of the group large sieve is:

Theorem 2 (Rivin, Kowalski)**.** *Let $\Gamma \leq \mathrm{SL}(n, \mathbb{Z})$ be a finitely generated Zariski-dense subgroup with a finite generating set S. There are constants $\alpha, c > 0$ for which*

$$\mathrm{Prob}\left(\textit{the characteristic polynomial of } w_k \textit{ is irreducible}\right) \leq ce^{-\alpha k}.$$

The constants α and c depend on the generating set. To avoid the need to specify the generating set we call a subset T *exponentially small* if for every finite symmetric generating set there exist constants $\alpha, c > 0$ for which

$$\text{(2)} \qquad \mathrm{Prob}(w_k \in T) \leq ce^{-\alpha k}.$$

The basic idea behind the sieve method is to study the properties of random walks via their images in finite quotients. The key point is that the distributions of the random walks in these finite quotients converge very fast to the uniform ones.

Theorem 3 (Varju)**.** *Let Γ be a finitely generated Zariski-dense subgroup of $\mathrm{SL}(n, \mathbb{Z})$ with a finite symmetric generating set S. There exist constants $d \in \mathbb{N}$*

and $\alpha, \beta > 0$ *such that for every* $k, q \in \mathbb{N}$ *for which* $\gcd(q, d) = 1$ *and* $q \leq e^{\beta k}$ *and every* $T \subseteq \mathrm{SL}(n, q)$*,*

$$\mathrm{Prob}(\pi_q(w_k) \in T) = \frac{|T|}{|\,\mathrm{SL}(n, q)|} + o(e^{-\alpha k}). \tag{3}$$

where $\pi_q : \mathrm{SL}(n, \mathbb{Z}) \to \mathrm{SL}(n, q)$ *is the modulo-q homomorphism.*

Theorem 3 can be used in an obvious way. For example, in order to bound the probability that the characteristic polynomial of w_k is reducible, it is enough to bound the probability that for all primes p the characteristic polynomial of $\pi_p(w_k)$ is reducible.

The proof of Theorem 3 (and its generalizations to other linear groups) is based on the strong approximation theorem of Boris Weisfeiler [We] and Madhav Nori [No] and on the recent developments in the theory of expansion in groups and spectral gaps. The development in the theory of expansion in groups is due to Harald Helfgott [He], Emmanuel Breuillard, Ben Green and Terence Tao [BGT] and by László Pyber and Endre Szabó [PS]. New methods for proving spectral gap results by using the theory of expansion in groups were developed by Jean Bourgain, Alex Gamburd and Peter Sarnak [BGS], Peter Varju [Va] and Alireza Salehi-Golsefidy and Varju [SGV].

Finally, we would like to state some results which were proven by using the group large sieve method. The Galois group of an element $g \in \mathrm{SL}(n, \mathbb{Z})$ is defined to be the Galois group of the splitting field of its characteristic polynomial.

Theorem 4 (Rivin, Kowalski)**.** *The subset of* $\mathrm{SL}(n, \mathbb{Z})$ *consisting of the elements whose Galois group is not isomorphic to the symmetric group Sym*(n) *is exponentially small.*

For a generalizations of this theorem to other arithmetic lattices see [JKZ] and [LR].

The quantitative nature of the sieve method might be helpful for answering non-quantitative questions. For example, a theorem of Ehud Hrushovski, Peter Kropholler, Alex Lubotzky and Aner Shalev [HKLS] states that if Γ is a non-(virtually-solvable) linear group and $s \geq 2$ then the set $\{g^m \mid g \in \Gamma \ \&\ 2 \leq m \leq s\}$ does not contain a coset of a finite index subgroup. The proof of this theorem uses the profinite topology and cannot be extended to the set of all powers. Using the group large sieve, Alex Lubotzky and Chen Meiri generalized this result.

Theorem 5 (Lubotzky-Meiri)**.** *Let* $\Gamma \leq \mathrm{GL}(n, \mathbb{C})$ *be a non-(virtually-solvable) finitely generated group. The set* $\{g^m \mid g \in \Gamma \ \&\ 2 \leq m\}$ *is exponentially small. In particular, it does not contain a coset of a finite index subgroup.*

The Group Large Sieve method is also fruitful for groups which are not necessarily linear. For example, the action of the Mapping Class Group MCG on the first homology of the surface Σ_g induces a representation $\rho : \mathrm{MCG} \to \mathrm{Sp}(2g, \mathbb{Z})$. Igor Rivin [Ri] and Emmanuel Kowalski [Ko] used this representation to show:

Theorem 6 (Rivin, Kowlaski)**.** *The set of non-(Pseudo-Anosov) elements in the Mapping Class Group is exponentially small.*

The kernel of ρ is called the Torelii subgroup and it plays an important role in the study of the mapping class group. Kowalski [Ko] asked if the same result is true for the Torelli subgroup. Using the action of the Torelli group of the first homology of covers of Σ_g, Justin Malestein and Juan Souto [MS] and Lubotzky-Meiri [LM2] proved:

Theorem 7 (Malestein-Souto, Lubotzky-Meiri)**.** *The set of non-(Pseudo-Anosov) elements in the Torelli group is exponentially small.*

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Bernoulli actions and sofic entropy

David Kerr

For a probability-measure-preserving action $G \curvearrowright (X,\mu)$ of a countable amenable group, the Kolmogorov-Sinai entropy is defined by taking a Følner sequence $\{F_n\}$ for G and setting

$$h_\mu(T) = \sup_{\mathcal{P}} \lim_{n\to\infty} \frac{1}{|F_n|} H\bigg(\bigvee_{s\in F_n} s^{-1}\mathcal{P} \bigg).$$

where $H(\cdot)$ is the Shannon entropy and $\mathcal{P}$ ranges over the finite measurable partitions of X. The Følner property enables one to prove the Kolmogorov-Sinai theorem, which asserts that the above supremum is achieved on every finite generating partition. As a consequence, the entropy of a Bernoulli action $G \curvearrowright (Y,\nu)^G$ is equal to the Shannon entropy of the base (Y,ν). By work of Ornstein in the case $G = \mathbb{Z}$ [8, 9] and by Ornstein and Weiss in the general amenable case [10], Bernoulli actions are classified by their entropy and every factor of a Bernoulli action is Bernoulli.

By externalizing the averaging in the Kolmogorov-Sinai definition to an abstract finite set on which the group approximately acts, Bowen introduced a more general notion of measure entropy that applies to actions of countable sofic groups [2]. This is defined as follows, in the generator-free formulation of [6]. Let $G \curvearrowright (X,\mu)$ be a measure-preserving action of a countable sofic group. Soficity means that there exists a sequence Σ of maps $\sigma_i : G \to \mathrm{Sym}(d_i)$ into finite permutation groups which are asymptotically multiplicative and free in the sense that

$$\lim_{i\to\infty} \frac{1}{d_i} \big|\{k \in \{1,\dots,d_i\} : \sigma_{i,st}(k) = \sigma_{i,s}\sigma_{i,t}(k)\}\big| = 1$$

for all $s,t \in G$, and

$$\lim_{i\to\infty} \frac{1}{d_i} \big|\{k \in \{1,\dots,d_i\} : \sigma_{i,s}(k) \neq \sigma_{i,t}(k)\}\big| = 1$$

for all distinct $s,t \in G$. Fixing such a Σ, we define $\mathrm{Hom}_\mu(\mathcal{P},F,\delta,\sigma_i)$ to be the set of all homomorphisms from the algebra generated by $\mathcal{P}$ to the algebra of subsets of $\{1,\dots,d_i\}$ which, to within δ in a summable sense, are approximately F-equivariant and approximately pull back the uniform probability measure on $\{1,\dots,d_i\}$ to μ. For a partition $\mathcal{Q} \leq \mathcal{P}$, write $|\mathrm{Hom}_\mu(\mathcal{P},F,\delta,\sigma_i)|_{\mathcal{Q}}$ for the cardinality of the set of restrictions of elements of $\mathrm{Hom}_\mu(\mathcal{P},F,\delta,\sigma_i)$ to $\mathcal{Q}$. We then define the measure entropy

$$h_{\Sigma,\mu}(X,G) = \sup_{\mathcal{Q}} \inf_{\mathcal{P}\geq\mathcal{Q}} \inf_{F,\delta} \limsup_{i\to\infty} \frac{1}{d_i} \log |\mathrm{Hom}_\mu(\mathcal{P},F,\delta,\sigma_i)|_{\mathcal{Q}}$$

where $\mathcal{Q}$ ranges over the finite measurable partitions of X and F over the finite subsets of G. As in the amenable case,

- the entropy of a Bernoulli action of a countable sofic group is equal to the Shannon entropy of its base [2, 7, 6],

- Bernoulli actions of nontorsion countable sofic groups are classified by their entropy [2].

Unlike in the amenable case,

- if G is nonamenable then there are Bernoulli actions of G which factor onto every Bernoulli action of G [1],
- if G contains F_2 then any two nontrivial Bernoulli actions of G factor onto one another [5],
- many nonamenable groups, including property (T) groups, have Bernoulli actions with non-Bernoulli factors [11, 12, 13, 14].

An action of a sofic group has *completely positive entropy* if every nontrivial factor has positive entropy with respect to every sofic approximation sequence Σ. A Bernoulli action $G \curvearrowright (Y, \nu)^G$ of an amenable G has completely positive entropy because all factors are Bernoulli. Although it is possible for Bernoulli actions of sofic groups to admit non-Bernoulli factors by the last point above, we nevertheless show the following.

Theorem 1. *A Bernoulli action $G \curvearrowright (Y, \nu)^G$ of a sofic group has completely positive entropy.*

In [4] Bowen introduced an entropy-type invariant for free groups called the *f-invariant*. He showed in [3] that the f-invariant coincides with a version of sofic entropy which is locally computed by averaging over all sofic approximations on a finite set instead of using a given sofic approximation. Using this fact and the above theorem, we derive the following consequence.

Corollary 2. *If a nontrivial factor of a Bernoulli action of a free group possesses a finite generating partition, then it has strictly positive f-invariant.*

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