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Matrix Factorizations in Algebra, Geometry, and Physics

Organised by
Ragnar-Olaf Buchweitz, Toronto
Kentaro Hori, Kashiwa
Henning Krause, Bielefeld
Christoph Schweigert, Hamburg

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Abstract. Let W be a polynomial or power series in several variables, or, more generally, a nonzero element in some regular commutative ring. A matrix factorization of W consists of a pair of square matrices X and Y of the same size, with entries in the given ring, such that the matrix product XY is W multiplied by the identity matrix. For example, if X is a matrix whose determinant is W and Y is its adjoint matrix, then (X, Y) is a matrix factorization of W.

Such matrix factorizations are nowadays ubiquitous in several different fields in physics and mathematics, including String Theory, Commutative Algebra, Algebraic Geometry, both in its classical and its noncommutative version, Singularity Theory, Representation Theory, Topology, there in particular in Knot Theory.

The workshop has brought together leading researchers and young colleagues from the various input fields; it was the first workshop on this topic in Oberwolfach. For some leading researchers from neighboring fields, this was their first visit to Oberwolfach.

Mathematics Subject Classification (2010): 18Dxx, 18E30, 81T40, 14F05, 13Dxx, 16Gxx.

Introduction by the Organisers

The workshop *Matrix Factorizations in Algebra, Geometry, and Physics*, organised by Ragnar-Olaf Buchweitz (Toronto), Kentaro Hori (Kashiwa), Henning Krause (Bielefeld) and Christoph Schweigert (Hamburg) has brought together 50 particpants from the fields of algebra, geometry and mathematical physics.

The concept of matrix factorization as here understood was introduced in 1980 by D. Eisenbud. He showed that matrix factorizations of W are essentially the same as maximal Cohen-Macaulay modules over the associated hypersurface ring and that consequently any projective resolution of a module over a local hypersurface ring becomes eventually 2–periodic, the periodic part given by a matrix factorization.

During the 1980's, algebraic as well as geometric aspects of matrix factorizations were studied by M. Auslander, R.-O. Buchweitz, D. Eisenbud, G. M. Greuel, J. Herzog, H. Knörrer, I. Reiten, F.-O. Schreyer and many others, with important applications to Singularity Theory, Algebraic Geometry and Representation Theory.

In particular, it was shown that (homotopy classes of) matrix factorizations of W form a triangulated category that is equivalent to the stabilized derived category of the usually singular hypersurface $W = 0$, as well as to the homotopy category of complete resolutions or to the stable category of maximal Cohen–Macaulay modules, with the last three equivalences holding true more generally for Gorenstein rings (Buchweitz). Also, it was shown that the categories of matrix factorizations of W and of $W + xy$ are equivalent (Knörrer periodicity).

In the early 2000's, the relevance of matrix factorizations to String Theory was recognized — supersymmetric boundary conditions (D-branes) in the topological Landau-Ginzburg model with superpotential W are described as matrix factorizations of W (Kontsevich, Orlov). Independently, Orlov also generalized the affine results for Gorenstein rings to a large class of schemes.

There were 18 contributed talks to the workshop, of a length of 60 minutes each. 7 of these talks were selected by the organizers before the workshop. On Monday evening, 36 participants gave short presentations of five minutes each from which the remaining 11 full talks were selected. The session comprising the short presentations was generally perceived as stimulating and instructive by the participants of the workshop. Ten talks were given by young participants; for some of them, the talk was their first contribution to a workshop in Oberwolfach.

In the remaining part of the introduction, we briefly describe the mathematical interrelations of the full talks.

A series of contributions was strongly rooted in *singularity theory*: H. Lenzing discussed categories of matrix factorizations for Brieskorn singularities. K. Ueda's talk was devoted to the relation between non-commutative matrix factorizations and dimer models, while A. Takahashi reported on algebraic and geometric aspects of mirror symmetry for Landau-Ginzburg orbifolds for invertible polynomials in three variables. M. Kalck discussed relative singularity categories that measure the "difference" between a non-commutative resolutation and the smooth part of a Gorenstein singularity. In this context, also O. Iyama's short contribution on higher-dimensional Geigle-Lenzing spaces should be mentioned. Walker demonstrated in his talk how ideas in singularity theory, in particular results by Buchweitz and van Straten on the Milnor fibre, can be usefully applied to arithmetic questions.

Triangulated categories have featured prominently in the contributions of T. Dyckerhoff and M. Kapranov. Using intrinsic combinatorial structure in triangulated categories, they have used standard constructions from topological field theory to associate a dg-enrichted triangulated category to triangulated surfaces with marked points which is a combinatorial version of a Fukaya category. Kapranov in particular explained the role of a particular 2-Segal cyclic object. F. Haiden introduced a notion of a dynamical entropy for endofunctors on triangulated categories. A. Polishchuck has used matrix factorizations to provide an algebraic construction, analogous to an analytic construction of Fan-Jarvis-Ruan, of cohomology classes on the moduli space of stable pointed curves.

In this talk, the *bicategorical structure* (with dualities) on matrix factorizations was important. Based on general ideas in two-dimensional field theories, this structure was introduced in N. Carqueville's contribution. In fact, structures inspired by low-dimensional quantum field theories played an important role also in other talks with various directions of mathematical impact, in particular in the contributions of Dyckerhoff-Kapranov and of Pantev. Murfet showed how to use the structure of a cut system to implement linear logic. Hanno Becker discussed the relation of two categorified knot invariants: Khovanov-Rozansky homology that is defined via matrix factorizations and Mazorchuk-Stroppel-Sussan homology based on the Bernstein-Gelfand-Gelfand category $\mathcal{O}$. O. Schnürer presented a construction of motivic measure from matrix factorizations.

The interrelation of questions arising in string theory and in mathematics was reflected in the contribution of E. Sharpe and U. Isik who discussed aspects of Kuznetsov's homological projective duality from the point of view of string theory and projective geometry. GIT-quotients and their variations were discussed using gauge-theoretic methods by J. Knapp; I. Shipman has constructed autoequivalences for them.

Numerous discussions among the participants, in particular among participants belonging to different mathematical communities, have contributed to the workshop in an essential way. Thus the workshop provided an ideal atmosphere for fruitful interaction and exchange of ideas. It is a pleasure to thank the administration and the staff of the Oberwolfach Institute for their efficient support and hospitality.

Workshop: Matrix Factorizations in Algebra, Geometry, and Physics

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Abstracts

Khovanov-Rozansky homology via stable Hochschild homology of Soergel bimodules

Hanno Becker

Since the discovery of Khovanov homology [4] the study of categorified link invariants has become a very active field of research that has been approached by an amazing variety of different perspectives and techniques. Given an n-variable polynomial link invariant $\mathcal{P}$ like the Reshetikhin-Turaev invariant $\mathcal{P}_{\mathfrak{g},V}$ associated with a complex simple Lie algebra $\mathfrak{g}$ and a finite-dimensional simple representation V of its quantum group $\mathcal{U}_q(\mathfrak{g})$, see [10], a *categorification* of $\mathcal{P}$ is an $(n+1)$-variable polynomial link invariant from which $\mathcal{P}$ is obtained by taking Euler characteristic, i.e. by specializing the new variable to -1. In this sense, Khovanov homology is a categorification of the Jones polynomial, and by now categorifications have been found for all $\mathcal{P}_{\mathfrak{g},V}$ as well as for the HOMFLYPT polynomial defined in [3] (subsuming all $\mathcal{P}_{\mathfrak{sl}(k),\mathrm{nat}}$). Still, there are quite a few open questions, some concerning the *existence* of categorifications (e.g. is there a categorification of the Kauffman polynomial?), some concerning the *enhancement* of categorified link invariants by additional structure like differentials or algebra actions, and finally those concerning the *uniqueness* of categorifications of a given link invariant. This last question of uniqueness is already very interesting in the case of $\mathcal{P}_{\mathfrak{sl}(k),\mathrm{nat}}$ for which many categorifications are known, coming from representation theory, algebraic geometry, algebraic topology and commutative algebra. The author's talk was concerned with the comparison of two particular such: on the one hand, Mazorchuk-Stroppel-Sussan's categorifications [13, 8, 14] based on Bernstein-Gelfand-Gelfand category $\mathcal{O}$, and on the other hand, Khovanov-Rozansky's categorification KR_k [5] constructed using matrix factorizations.

The construction of Khovanov-Rozansky homology KR_k goes as follows: Given an oriented link L, one chooses firstly a triple-point free projection of L onto the plane. Secondly, one cuts the projection into pieces each of which looks like an unknotted single strand ↑ or one of the two crossings ⤱ or ⤲, and assigns a variable to any point where a cut was made. Thirdly, to each of the pieces just obtained one associates a certain fixed complex of $\mathbb{Z}$-graded matrix factorizations, the ground ring being the polynomial ring over $\mathbb{Q}$ over the variables attached to the open ends of the piece. Finally, one takes the tensor product of all these complexes to obtain a complex of matrix factorizations of potential 0. Taking total cohomology in each of its matrix factorization components, one gets a complex of graded vector spaces; $\mathrm{KR}_k(L) \in \mathbb{Z}[a^{\pm 1}, q^{\pm 1}]$ is then defined as the graded Poincaré series of the cohomology of this complex.

Given any ring S and a central element $w \in Z(S)$, the homotopy category $\underline{\mathrm{MF}}(S,w)$ of matrix factorizations of type (S,w) arises as the derived category $\mathbf{D}^{\mathrm{ctr}}\,\mathrm{LF}(S,w)$ of a suitable model structure on the abelian category $\mathrm{LF}(S,w)$ of all linear factorizations of type (S, w), i.e. diagrams $M^0 \xrightarrow{\delta} M^1 \xrightarrow{\delta} M^0$ of not necessarily free S-modules M^0, M^1 satisfying $\delta^2 = w \cdot \mathrm{id}$. This category $\mathbf{D}^{\mathrm{ctr}}\,\mathrm{LF}(S, w)$ is called the *contraderived category* of linear factorizations; it was introduced by Positselski in [9] and further studied in [1]. The important point is that even though $\delta^2 \neq 0$ there exists a reasonable notion of weak equivalence of linear factorizations which upon localization turns $\mathrm{LF}(S, w)$ into $\mathrm{HMF}(S, w)$. The canonical functor $S/(w)$-Mod $\to \mathrm{LF}(S, w) \to \mathbf{D}^{\mathrm{ctr}}(S, w) \cong \mathrm{HMF}(S, w)$ is called the *stabilization functor* and denoted $(-)^{\{w\}}$; if S is regular local and $w \in \mathfrak{m}_S \setminus \{0\}$, it coincides with the classical stabilization functor from commutative algebra. Further, there is a derived tensor product on the $\mathbf{D}^{\mathrm{ctr}}\,\mathrm{LF}(S, -)$ giving rise to the following definition of "stable" Hochschild homology:

Definition. Let K be a field and A be a commutative K-algebra with enveloping algebra $A^{\mathrm{en}} = A \otimes_K A$. Further, let $w \in A$ and M be an A-bimodule such that $w.m = m.w$ for all $m \in M$. Then the stabilization $M^{\{w^{\mathrm{en}}\}}$ of M with respect to $w^{\mathrm{en}} := w \otimes 1 - 1 \otimes w \in A^{\mathrm{en}}$ is defined, and we call

$$\mathrm{sHH}^*_w(M) \;:=\; \mathrm{H}^*[\Delta^{\{-w^{\mathrm{en}}\}} \overset{\mathbb{L}}{\otimes}_{A^{\mathrm{en}}} M^{\{w^{\mathrm{en}}\}}]$$

the *w-stable Hochschild homology* of M.

Khovanov-Rozansky homology now admits the following description:

Theorem. Let β be a braid with labels x_i resp. y_i on the upper resp. lower ends of its strands. Then the complex of matrix factorizations $\mathrm{KR}_k(\beta)$ is termwise contraderived equivalent to the stabilization, with respect to $\sum x_i^{k+1} - y_i^{k+1}$, of the Rouquier complex of Soergel bimodules $\mathrm{RC}(\beta)$ associated to β, [11, 12].

In particular, one recovers the following Theorem of Webster [15]:

Corollary. Given an oriented link L presented as the closure of an n-strand braid word β with writhe $w(\beta)$, its Khovanov-Rozansky homology $\mathrm{KR}_k(L)$ equals

$$(a^{-1}q^{k+1})^{w(\beta)} \sum_{i,j\in\mathbb{Z}} \dim_{\mathbb{Q}} \mathrm{H}^i\,[\mathrm{sHH}^*_k\,\mathrm{RC}(\beta)_j]\, a^i q^j \in \mathbb{Z}[a^{\pm 1}, q^{\pm 1}], \tag{1}$$

where sHH_k denotes stable Hochschild homology with respect to $\sum x_i^{k+1} - y_i^{k+1}$.

This theorem is analogous to the description [7] of triply graded Khovanov-Rozansky homology [6] (categorifying the HOMFLYPT-polynomial) as ordinary Hochschild homology of Rouquier complexes of Soergel bimodules.

It turns out that the expression (1) defines an invariant of oriented links for any base field K with $\mathrm{char}\, K \nmid k+1$, and that its invariance under the Markov moves can be checked rather quickly working in the contraderived category. More generally, we have the following:

Theorem. Let R be a Noetherian $\mathbb{Z}[\frac{1}{k+1}]$-algebra. Then, for an n-strand braid word β with writhe $w(\beta)$, the complex

$$\Sigma^{-w(\beta)}\rho_n\,[\mathrm{sHH}^*_k\;\;\mathrm{RC}_R(\beta)]\,\langle (k+1)\, w(\beta)\rangle$$

has finitely generated cohomology over R. Moreover, its isomorphism class in $\mathbf{D}(R[x_1, x_2, ...]\text{-Mod})$ is invariant under the Markov moves, hence an invariant of oriented links. Here, ρ_n denotes the functor induced by the homomorphism $R[x_1, x_2, ...] \to R[x_1, x_2, ..., x_n]$, given by $x_i \mapsto x_i$ for $i \leq n$ and $x_i \mapsto x_n$ for $i \geq n$, and RC_R denotes the Rouquier complex defined over R.

If $k+1 = 0$ in R, it turns out that (1) still defines an invariant of oriented links after a suitable renormalization, and that it agrees with Khovanov-Rozansky's triply graded categorification of the HOMFLYPT-polynomial after some specialization; this is because the canonical spectral sequence from ordinary to k-stable Hochschild homology degenerates on the E_1-page in this case.

The above results can be considered a first step in a comparison of Khovanov-Rozansky homology with the categorifications defined by Mazorchuk, Stroppel and Sussan. The latter is based on shuffling functors restricted to certain parabolic versions of Bernstein-Gelfand-Gelfand category $\mathcal{O}$, and it is known that these shuffling functors can be described in terms of Rouquier complexes of Soergel bimodules before restriction to parabolic category $\mathcal{O}$. It therefore remains to be studied whether and in what sense restriction of shuffling functors to parabolic subcategories of $\mathcal{O}$ is equivalent to stabilization of the corresponding Rouquier complexes.

The results described in this abstract will appear in [2].

References

[1] H. Becker, *Models for singularity categories*, Preprint, arXiv:1205.4473.

[2] H. Becker, *Khovanov-Rozansky homology via stable Hochschild homology of Soergel bimodules*, in preparation.

[3] P. Freyd, D. Yetter, J. Hoste, W. B. R. Lickorish, K. Millett, A. Ocneanu, *A new polynomial invariant of knots and links*, Bulletin of the American Mathematical Society **12** (1985), 239–246.

[4] M. Khovanov, *A categorification of the Jones polynomial*, Duke Mathematical Journal **101** (2000), 359–426.

[5] M. Khovanov, L. Rozansky, *Matrix factorizations and link homology*, Fundamenta Mathematicae **199** (2008), 1–91.

[6] M. Khovanov, L. Rozansky, *Matrix factorizations and link homology. II*, Geometry & Topology **12** (2008), 1387–1425.

[7] M. Khovanov, *Triply-graded link homology and Hochschild homology of Soergel bimodules*, International Journal of Mathematics **18** (2007), 869–885.

[8] W. Mazorchuk, C. Stroppel, *A combinatorial approach to functorial quantum $\mathfrak{sl}_k$ knot invariants*, American Journal of Mathematics **131** (2009), 1679–1713.

[9] L. Positselski, *Two kinds of derived categories, Koszul duality, and comodule-contramodule correspondence*, Memoirs of the American Mathematical Society **212** (2011).

[10] N. Reshetikhin, V. Turaev, *Ribbon graphs and their invariants derived from quantum groups*, Communications in Mathematical Physics **127** (1990), 1–26.

[11] R. Rouquier, *Categorification of $\mathfrak{sl}_2$ and braid groups*, Trends in representation theory of algebras and related topics, Contemp. Math. **406** (2006), 137–167.

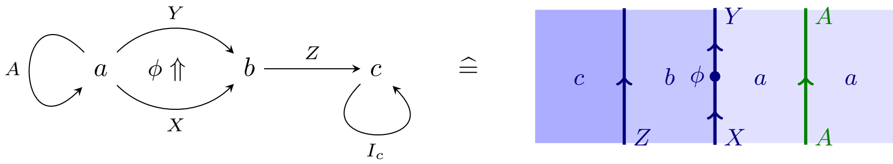
[12] W. Soergel, *Kazhdan-Lusztig-Polynome und unzerlegbare Bimoduln über Polynomringen*, Journal of the Institute of Mathematics of Jussieu **6** (2007), 501–525.

[13] C. Stroppel, *Categorification of the Temperley-Lieb category, tangles, and cobordisms via projective functors*, Duke Mathematical Journal **126** (2005), 547–596.
[14] J. Sussan, *Category $\mathcal{O}$ and $\mathfrak{sl}(k)$ link invariants*, PhD Thesis at Yale University (2007).
[15] B. Webster, *Khovanov-Rozansky homology via a canopolis formalism*, Algebraic & Geometric Topology **7** (2007), 673–699.

Equivariant completion

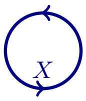
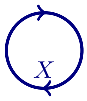
Nils Carqueville
(joint work with Ingo Runkel)

Recall the basic setting of a bicategory $\mathcal{B}$, which one may think of as a 'monoidal category with labels.' We denote objects of $\mathcal{B}$ by $a, b, \ldots$, generic 1-morphisms by $X, Y, \ldots$, units by I_a, and 2-morphisms by $\phi, \psi, \ldots$. Composition of 1-morphisms is written as $\otimes$, and we have two (Poincaré dual) graphical depictions at our disposal:



For our purposes the latter is more convenient, where every such string diagram is read from bottom to top and from right to left.

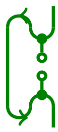
We assume that in our bicategory $\mathcal{B}$ for each $X \in \mathcal{B}(a,b)$ there is $X^\dagger \in \mathcal{B}(b,a)$ together with 2-morphisms  $: X \otimes X^\dagger \to I_b$ and  $: I_a \to X^\dagger \otimes X$ exhibiting $X^\dagger$ as the *right adjoint* of X. Furthermore we ask that $\mathcal{B}$ be *pivotal* in the sense that there are natural monoidal 2-isomorphisms between the units and the double-adjoint $(-)^{\dagger\dagger}$. It follows that every 1-morphism also has a left adjoint, and we can define the *left* and *right quantum dimensions* as

$$\dim_{\mathrm{l}}(X) = $$  $$\in \operatorname{End}(I_a)\,, \quad \dim_{\mathrm{r}}(X) = $$  $$\in \operatorname{End}(I_b)\,.$$

Our main example is the bicategory of Landau-Ginzburg models $\mathcal{LG}_k$, with potentials $W \in k[x]$ as objects and 1- and 2-morphisms given by matrix factorisation categories $\mathcal{LG}(W,V) = \operatorname{hmf}(V - W)$. We mostly adopt the notation of [4] where it was shown that $\mathcal{LG}_k$ is 'graded' pivotal and that for a matrix factorisation (X, d_X) of $V(z_1, \ldots, z_m) - W(x_1, \ldots, x_n)$ we have

$$\dim_{\mathrm{r}}(X) = (-1)^{\binom{m+1}{2}} \operatorname{Res}\left[\frac{\operatorname{str}(\partial_{x_1} d_X \ldots \partial_{x_n} d_X \, \partial_{z_1} d_X \ldots \partial_{z_m} d_X) \, \underline{\mathrm{d}x}}{\partial_{x_1} W, \ldots, \partial_{x_n} W} \right].$$

Returning to the general setting, recall that $A \in \mathcal{B}(a,a)$ is an *algebra* if it comes with maps  $: A \otimes A \to A$ and  $: I_a \to A$ which are associative and unital. Similarly A is a *coalgebra* if there are appropriate maps  and . A 1-morphism which is both an algebra and a coalgebra is *Frobenius* if  =  = , *separable* if  = , and *symmetric* if  = . A is symmetric iff its *Nakayama automorphism*

$\gamma_A =$  $: A \longrightarrow A \qquad$ equals 1_A .

Note that it is no accident that the defining relations for a separable Frobenius algebra are reminiscent of the moves  $\leftrightarrow$  and  $\leftrightarrow$ which allow to pass between any two given triangulations of a surface, see [5, Sect. 3.3].

A *right A-module* is a 1-morphism $X \in \mathcal{B}(a,b)$ together with a map $X \otimes A \to X$ that is compatible with the multiplication and unit of A. There are also no surprises as to how left modules, bimodules, and (bi)module maps are defined.

Our first use of algebras is the following characterisation of equivariant matrix factorisations (for which there is a version for each element of $H^2(G, U(1))$, cf. [2]):

Proposition 1 ([5, 1])**.** Let $W \in \mathcal{LG}_k$ and $G \subset \{g \in \operatorname{Aut}(k[x]) \,|\, g(W) = W\}$ be a finite group acting diagonally on the variables x_i. Then (i) $A_G := \bigoplus_{g \in G} {}_g I$ naturally has the structure of a separable Frobenius algebra, (ii) $\gamma_{A_G} = \sum_{g \in G} \det(g)^{-1} \cdot 1_{{}_g I}$, and (iii) $\operatorname{hmf}(W)^G \cong \operatorname{mod}(A_G)$.

Inspired by orbifolds of two-dimensional quantum field theories with defects and building on the foundational work summarised in [7] the following construction generalises the equivariantisation procedure:

Definition 2 ([5])**.** The *equivariant completion* of a pivotal, idempotent complete bicategory $\mathcal{B}$ is the bicategory $\mathcal{B}_{\text{eq}}$ whose objects are pairs (a, A) with $a \in \mathcal{B}$ and $A \in \mathcal{B}(a,a)$ separable Frobenius; 1-morphisms $(a, A) \to (b, B)$ are B-A-bimodules $X \in \mathcal{B}(a,b)$ and 2-morphisms are bimodule maps; horizontal composition is the tensor product over the intermediate algebra, and for the units we have $I_{(a,A)} = A$.

The justification of the name 'completion' lies in the fact that $\mathcal{B}$ embeds into $\mathcal{B}_{\text{eq}}$ via $a \mapsto (a, I_a)$ and that there is an equivalence $(\mathcal{B}_{\text{eq}})_{\text{eq}} \cong \mathcal{B}_{\text{eq}}$. Furthermore, every $X \in \mathcal{B}_{\text{eq}}((a,A),(b,B))$ has left and right adjoints $(X^\dagger)_{\gamma_B^{-1}}$ and ${}_{\gamma_A}(X^\dagger)$, which differ from $X^\dagger$ only by their bimodule structure which is twisted by precomposing with the (inverse of the) Nakayama automorphism. As a corollary one finds that the category of A-modules has a Serre functor given by ${}_{\gamma_A}(-)$.

Every symmetry group G as in Proposition 1 leads to an A_G and thus to an object in $(\mathcal{LG}_k)_{\text{eq}}$. We wish to construct separable Frobenius algebras that are not of the form A_G. For this the following variant of the monadicity theorem is central.

Theorem 3 ([5])**.** Let $\mathcal{B}$ be pivotal and $X \in \mathcal{B}(a,b)$ have invertible right quantum dimension. Then $A := X^\dagger \otimes X \in \mathcal{B}(a,a)$ is a separable symmetric Frobenius algebra, and $X \otimes_A X^\dagger \cong I_b$ in $\mathcal{B}$.

In this situation it follows immediately that $(a, A) \cong (b, I_b)$ in $\mathcal{B}_{\mathrm{eq}}$, which we call a *(generalised) orbifold equivalence*. Furthermore, if $\mathcal{B}$ has a trivial object 0 (such as the monoidal unit $W = 0$ of $\mathcal{B} = \mathcal{LG}_k$), then we have

$$\mathcal{B}(0, b) \equiv \mathcal{B}_{\mathrm{eq}}\big((0, I_0), (b, I_b)\big) \cong \mathcal{B}_{\mathrm{eq}}\big((0, I_0), (a, A)\big) = \mathrm{mod}(A)\,.$$

Using our explicit residue expression for the quantum dimension in $\mathcal{LG}_k$ it is easy to check the invertibility condition of Theorem 3 in practice. In particular, with sufficient stamina one can construct three matrix factorisations X in $\mathrm{hmf}(W^{(\mathrm{E}_6)} - u^{12})$, $\mathrm{hmf}(W^{(\mathrm{E}_7)} - u^{18})$ and $\mathrm{hmf}(W^{(\mathrm{E}_8)} - u^{30})$ where

$$W^{(\mathrm{E}_6)} = x^3 + y^4 + z^2\,, \quad W^{(\mathrm{E}_7)} = x^3 + xy^3 + z^2\,, \quad W^{(\mathrm{E}_8)} = x^3 + y^5 + z^2$$

and most importantly $\dim_{\mathrm{r}}(X) \in \mathbb{C}^*$. Hence the simple singularities E_6, E_7, E_8 are orbifolds of A_{11}, A_{17}, A_{29}, respectively, complementing the more classical result of [9, 8, 5] that there is a $\mathbb{Z}_2$-orbifold between D_{d+1} and A_{2d-1}. Computing $X^\dagger \otimes X$ explicitly with the method of [3] and invoking Theorem 3 one arrives at:

Theorem 4 ([6])**.** For $d \in \mathbb{Z}_{\geqslant 2}$ let $\eta_d = \mathrm{e}^{2\pi\mathrm{i}/d}$, $D = \{0, 1, \ldots, d-1\}$, $S \subset D$, and define $P_S^{(d)}$ to be the matrix factorisation of $u^d - v^d$ with twisted differential

$$\begin{pmatrix} 0 & \prod_{j\in S}(u - \eta_d^j v) \\ \prod_{j\in D\setminus S}(u - \eta_d^j v) & 0 \end{pmatrix}.$$

Then we have

$$\begin{aligned}
\mathrm{hmf}\big(W^{(\mathrm{E}_6)}\big) &\cong \mathrm{mod}\left(I_{u^{12}} \oplus P^{(12)}_{\{-3,-2,\ldots,3\}}\right),\\
\mathrm{hmf}\big(W^{(\mathrm{E}_7)}\big) &\cong \mathrm{mod}\left(I_{u^{18}} \oplus P^{(18)}_{\{-4,-3,\ldots,4\}} \oplus P^{(18)}_{\{-8,-7,\ldots,8\}}\right),\\
\mathrm{hmf}\big(W^{(\mathrm{E}_8)}\big) &\cong \mathrm{mod}\left(I_{u^{30}} \oplus P^{(30)}_{\{-5,-4,\ldots,5\}} \oplus P^{(30)}_{\{-9,-8,\ldots,9\}} \oplus P^{(30)}_{\{-14,-13,\ldots,14\}}\right)
\end{aligned}$$

where the separable Frobenius algebras on the right are not of the form A_G.

References

[1] I. Brunner, N. Carqueville, and D. Plencner, *Orbifolds and topological defects*, arXiv:1307.3141.

[2] I. Brunner, N. Carqueville, and D. Plencner, *Discrete torsion defects*, in preparation.

[3] N. Carqueville and D. Murfet, *Computing Khovanov-Rozansky homology and defect fusion*, to appear in Algebraic & Geometric Topology, arXiv:1108.1081.

[4] N. Carqueville and D. Murfet, *Adjunctions and defects in Landau-Ginzburg models*, arXiv:1208.1481.

[5] N. Carqueville and I. Runkel, *Orbifold completion of defect bicategories*, arXiv:1210.6363.

[6] N. Carqueville, A. Ros Camacho, and I. Runkel, in preparation.

[7] J. Fröhlich, J. Fuchs, I. Runkel, and C. Schweigert, *Defect lines, dualities, and generalised orbifolds*, Proceedings of the XVI International Congress on Mathematical Physics, Prague, August 3–8, 2009, arXiv:0909.5013.

[8] H. Kajiura, K. Saito, and A. Takahashi, *Matrix Factorizations and Representations of Quivers II: type ADE case*, Adv. Math. **211** (2007), 327–362, math.AG/0511155.

[9] I. Reiten and C. Riedtmann, *Skew group algebras in the representation theory of Artin algebras*, J. Algebra **92** (1985), 224–282.

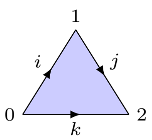
Triangulated surfaces in triangulated categories

Tobias Dyckerhoff
(joint work with Mikhail Kapranov)

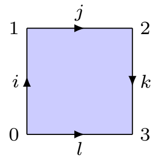
0.1. **State sums in algebras.** Let $\mathbf{k}$ be a field, and let A be an associative finite dimensional $\mathbf{k}$-algebra with chosen basis $E = \{e_1, e_2, \ldots, e_r\}$. The multiplication law of A is numerically encoded in the structure constants $\lambda_{ij}^k \in \mathbf{k}$ defined via $e_i e_j = \sum_k \lambda_{ij}^k e_k$. Associativity is then expressed by the equations

$$\sum_t \lambda_{ij}^t \lambda_{tk}^l = \lambda_{ijk}^l = \sum_t \lambda_{it}^l \lambda_{jk}^t \tag{1}$$

where the generalized structure constants $\{\lambda_{ijk}^l\}$ are given by $e_i e_j e_k = \sum_l \lambda_{ijk}^l e_l$. We can think of the numbers $\{\lambda_{ij}^k\}$ and $\{\lambda_{ijk}^l\}$ as numerical invariants attached to triangles and squares, respectively, where the set of vertices is ordered and the edges are labeled by E as illustrated in



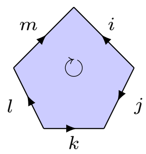
$\mapsto \quad \lambda_{ij}^k,$



$\mapsto \quad \lambda_{ijk}^l.$

Equation (1) is then geometrically reflected by the fact that $\{\lambda_{ijk}^l\}$ can be computed in terms of $\{\lambda_{ij}^k\}$ via two different formulas corresponding to the two possible triangulations of the square. Similarly, this observation extends to yield numerical invariants of planar convex polygons with ordered vertices and E-labeled edges which can be computed in terms of $\{\lambda_{ij}^k\}$ via any chosen triangulation.

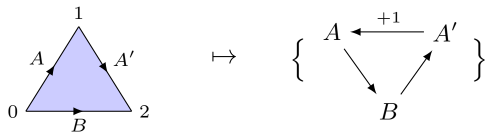
Assume now that A carries a Frobenius structure, i.e., a non-degenerate trace map $\mathrm{tr} : A \to \mathbf{k}$ such that, for every $a, b \in A$, we have $\mathrm{tr}(ab) = \mathrm{tr}(ba)$. Then we can introduce a dual basis $E^* = \{e_1^*, e_2^*, \ldots, e_r^*\}$ of A such that $\mathrm{tr}(e_i e_j^*)$ equals 1 if $i = j$ and 0 otherwise. This allows us to enlarge the range of definition of the above system of invariants to include planar polygons with oriented E-labeled edges such as



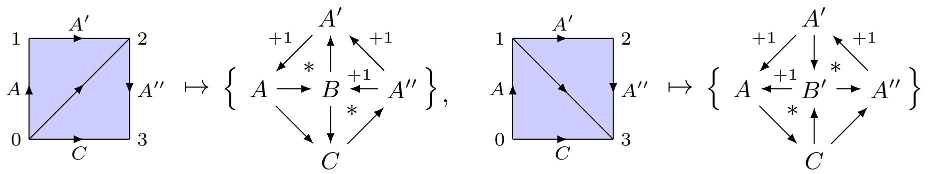
$\mapsto \quad \mathrm{tr}(e_m e_i^* e_j e_k^* e_l)$

where, due to the cyclic invariance of the trace expression, no linear ordering of the vertices is needed as long as we remember the orientation of the polygon. Again, these invariants can be computed by choosing any triangulation involving the vertices of the polygon. This suggests that, given an oriented closed surface S with a finite set of marked points M, we can define a generalized structure constant $\lambda_{(S,M)}$ which is a numerical invariant of the marked surface and can be computed by choosing any triangulation of S involving the vertices M. These invariants exist and originate in physics where they are called *partition functions* which we have computed via *state sums*. The collection of all partition functions associated to a given Frobenius algebra form a so-called topological field theory [5, 6].

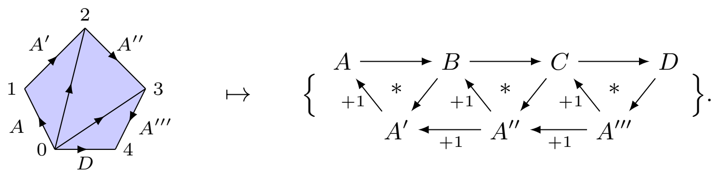
0.2. **State sums in triangulated categories.** The central observation of [2] is that certain symmetries in 2-periodic triangulated categories can be exploited to define invariants of oriented surfaces via a state sum formalism similar to §1. We start with an informal discussion. Let $\mathcal{T}$ be a triangulated category with set of objects $E = \{A, B, \ldots, A', B', \ldots\}$. We associate to an E-labeled triangle the collection



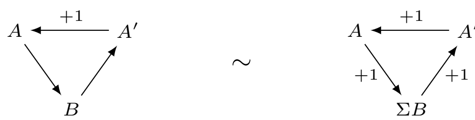
of all distinguished triangles in $\mathcal{T}$ involving the objects determined by the edge labels. To a triangulated square with E-labeled edges, we attach the following collections of diagrams



where the $*$-marked triangles are distinguished, the unmarked triangles commute, and the objects B and B' are allowed to vary. The two types of diagrams correspond to the upper and lower cap of an octahedron. On a heuristic level, the role of the associativity in Equation (1) will now be played by the *octahedral axiom* which allows us to pass from one triangulation of the square to the other. More generally, to an E-labeled polygon with a chosen triangulation we associate the collection of certain *Postnikov systems* [4] such as



The analog of the Frobenius structure in §1 turns out to be a 2-periodic structure on $\mathcal{T}$, i.e., an isomorphism of functors $\Sigma^2 \simeq \mathrm{id}$. This structure allows us to rewrite any distinguished triangle as



where the right-hand form exhibits a cyclic symmetry similar to the symmetry of the trace expression $\mathrm{tr}(e_i e_j e_k^*)$ from §1.

These heuristics suggest the existence of invariants of marked oriented surfaces associated with any 2-periodic triangulated category $\mathcal{T}$. Further, state sum formulas should lead to a description of these invariants in terms of *surface Postnikov systems*: collections of distinguished triangles in $\mathcal{T}$ parametrized by a chosen triangulation of the surface. The following results of [2] give a precise account of the above informal discussion and show the existence of the expected invariants.

Theorem 1. Let $\mathcal{T}$ be a triangulated category equipped with a differential $\mathbb{Z}/2\mathbb{Z}$-graded enhancement $\mathcal{A}$. Denote by $S(\mathcal{A})$ the simplicial space given by Waldhausen's S-construction. Then

(1) $S(\mathcal{A})$ is a 2-Segal space in the sense of [3],
(2) $S(\mathcal{A})$ admits a canonical cyclic structure in the sense of Connes [1].

In other words, for each n-gon, we can define a *classifying space* of Postnikov systems in $\mathcal{T}$. Letting n vary, we obtain a simplicial space well-known in algebraic topology: Waldhausen's S-construction. The 2-Segal property then reflects the fact that these classifying spaces do not depend on a chosen triangulation while the cyclic structure formalizes the symmetries heuristically observed above. In the proof, a key role is played by a certain cocyclic 2-Segal object in the Morita model category of differential $\mathbb{Z}/2\mathbb{Z}$-graded categories given as

$$\mathcal{E} : \Lambda \longrightarrow \mathrm{dgcat}^{(2)}, \quad \langle n \rangle \mapsto \mathrm{MF}(\mathbf{k}[z], z^{n+1})$$

where the right-hand side denotes the dg category of $\mathbb{Z}/(n+1)$-graded matrix factorizations of the polynomial z^{n+1} in one variable.

The following result of [2] shows that the expected surface invariants can indeed be defined and computed in terms of a homotopy limit formula which should be regarded as the analog of a state sum.

Theorem 2. Let $\mathbf{C}$ be a combinatorial model category and let X be a cyclic 2-Segal object in $\mathbf{C}$. Let (S, M) be a closed oriented surface with a non-empty finite set of marked points M where, in the case when S is a sphere, $|M| \geq 3$. Then there exists an object $X_{(S,M)}$ in $\mathrm{Ho}(\mathbf{C})$ which, for every triangulation $\Delta(S, M)$ of (S, M), comes equipped with canonical isomorphism

$$X_{(S,M)} \xrightarrow{\cong} \operatorname*{holim}_{\Lambda^n \to \Delta(S,M)} X_n.$$

Further, the mapping class group of (S, M) acts on $X_{(S,M)}$ via automorphisms in $\mathrm{Ho}(\mathbf{C})$.

As an application of our theory, we can use the cocyclic 2-Segal object $\mathcal{E}$ to associate to any marked oriented surface (S, M) a differential $\mathbb{Z}/2\mathbb{Z}$-graded category which can be interpreted as a purely topological variant of the Fukaya category of the surface. The state sum formula given by the homotopy limit in Theorem 2 can then be regarded as implementing a 2-dimensional instance of Kontsevich's program on localizing the Fukaya category along a singular Lagrangian spine (given in our context as the dual graph of the chosen triangulation).

References

[1] A. Connes, *Noncommutative Geometry*, Academic Press, San Diego, New York, London (1994).
[2] T. Dyckerhoff, M. Kapranov, *Triangulated surfaces in triangulated categories*, Arxiv-Preprint 1306.2545 (2013).
[3] T. Dyckerhoff, M. Kapranov, *Higher Segal spaces*, Arxiv-Preprint 1212.3563 (2012).
[4] S. I. Gelfand, Y. I. Manin. *Methods of Homological Algebra*, Springer-Verlag, Berlin (2003).
[5] C. I. Lazaroiu, *On the structure of open closed topological field theory in two dimensions*, Nuclear Physics B **603**, Issue 3 (2001), 497–530.
[6] G. W. Moore, *Some comments on branes, G-flux and K-theory*, Int.J.Mod.Phys. **A16** (2001), 936–944.

Entropy of endofunctors

Fabian Haiden

(joint work with George Dimitrov, Ludmil Katzarkov, Maxim Kontsevich)

A fundamental notion in the theory of dynamical systems is **entropy**. The precise definition of this quantity depends on the types of spaces one is considering: measure spaces, metric spaces, algebraic sets, ... In [5] we consider the context of triangulated categories, possibly enhanced by a dg-/A_∞-structure, thought of as **formal non-commutative spaces**. Thus the entropy, which is in fact a function $h_t(F) : \mathbb{R} \to [-\infty, +\infty)$, is assigned to an exact endofunctor F. The definition is naturally invariant under conjugation by autoequivalences.

The above set-up is quite general, and it is beneficial to give special attention to the more well-behaved case when $\mathcal{T}$ is the homotopy category of a **smooth and proper** dg-/A_∞-category (and the exact functor is induced by a dg-/A_∞-functor). This includes categories of the form $D^b(X)$ where X is a smooth projective variety, and $MF(X, f)$ where X is smooth and quasi-projective and f is proper with compact critical locus. We show that

$$h_t(F) = \lim_{N\to\infty} \frac{1}{N} \log \sum_{n\in\mathbb{Z}} \dim \operatorname{Ext}^n(G, F^N G)\mathrm{e}^{-nt} \tag{1}$$

where G is any generator of $\mathcal{T}$. Another feature of the smooth and proper case is a lower bound

$$\log \rho(HH_*(F)) \leq h_0(F) \tag{2}$$

which holds under a generic technical condition on $HH_*(F)$. Here we assume that $\mathcal{T}$ is defined over $\mathbb{C}$ and $\rho(HH_*(F))$ denotes the spectral radius of the induced map on Hochschild homology. The proof makes use of the Lefschetz fixed point theorem for Hochschild homology [7].

The quantity $h_0(F)$ is shown to be related to **topological entropy** [1] in the following contexts. First, if X is a smooth projective variety over $\mathbb{C}$, $f : X \to X$ a regular map, and f^* the pullback functor on $D^b(X)$, then

$$h_t(f^*) = \text{const.} = \log \rho(H^*(f; \mathbb{Q})) \tag{3}$$

under the generic assumption that (2) can be applied. The RHS is known to coincide with the topological entropy of f. Second, if X is a closed oriented surface and $f : X \to X$ a pseudo-Anosov map with stretch factor $\lambda > 1$ (see [6]), then f induces an autoequivalence f^* of the $\mathbb{Z}/2$-graded Fukaya category of X. Using (1) we show

$$h_0(f^*) = \log \lambda \tag{4}$$

and $\log \lambda$ is the topological entropy of f by results of [6].

Recall that a **Serre functor** S (see [3]) on a triangulated category $\mathcal{T}$ is, if it exists, unique up to natural isomorphism, and thus its entropy an invariant of $\mathcal{T}$. If X is a smooth projective variety and $\mathcal{T} = D^b(X)$, then $h_t(S) = \dim(X)t$. In general, $h_t(S)$ does not have to be of the form nt for some integer n though. For example, if $\mathcal{T} = D^b(kQ)$ where Q is a quiver of Dynkin type, then $h_t(S) = (h-2)t/h$ where h is the Coxeter number of Q. In this example, the category $\mathcal{T}$ is also a category of graded matrix factorizations over an ADE singularity. A natural question is to find a geometric interpretation of $h_t(S)$ for more general categories of graded matrix factorizations.

Another question concerns the **algebraicity** of $\exp(h_0(F))$, which holds in fact in all examples we consider. Is this a general phenomenon, and if so, under what conditions does it hold? Furthermore, one can study the set

$$\{h_0(F) | F \in \mathrm{Aut}(T)\} \tag{5}$$

of entropies of all autoequivalences, c.f. [2] in the context of birational geometry. In which cases is it discrete, and if it is, what is its smallest positive element?

As a final remark, let us discuss a possible connection with Bridgeland's theory of stability conditions on triangulated categories [4]. Fixing such a stability condition σ on $\mathcal{T}$, every object $E \in \mathcal{T}$ has a **mass** $m(E) = \sum |Z(G_i)|$, where G_i are the semistable components of E. For an endofunctor F consider

$$h_\sigma = \sup_{0 \neq E \in \mathcal{T}} \left\{ \limsup_{n \to \infty} \frac{1}{n} \log m(F^n(E)) \right\} \tag{6}$$

which measures the exponential increase in mass under the action of F. One verifies that h_σ is invariant under deformations of σ and that

$$h_\sigma(F) \leq h_0(F) \tag{7}$$

holds. In many cases, the above inequality is actually an equality, but identifying natural conditions under which this should be true is still an open problem.

References

[1] R. L. Adler, A. G. Konheim, and M. H. McAndrew, Topological Entropy, *Trans. AMS* 114 (1965), no. 2, 309–319.

[2] J. Blanc and S. Cantat, *Dynamical degrees of birational transformations of projective surfaces*, arXiv:1307.0361

[3] A. I. Bondal, M. M. Kapranov, *Representable functors, Serre functors, and mutations*, Izv. Akad. Nauk SSSR Ser. Mat. 53 (1989), no. 6, 1183–1205.

[4] T. Bridgeland, Stability conditions on triangulated categories, *Annals of Math.* 166 (2007), 317–345.

[5] G. Dimitrov, F. Haiden, L. Katzarkov, M. Kontsevich, *Dynamical systems and categories*, arXiv:1307:8418

[6] A. Fathi, F. Laudenbach, and V. Poénaru, *Travaux de Thurston sur les surfaces*, Asterisque 66–67 (1979)

[7] V. A. Lunts, Lefschetz fixed point theorems for Fourier-Mukai functors and DG algebras, *J. Algebra* 256 (2012), 230–256.

Homological Projective Duality for GIT quotients

M. Umut Isik

(joint work with Matthew Ballard, Dragos Deliu, David Favero, Ludmil Katzarkov)

Homological Projective Duality (HPD), introduced by Kuznetsov [Kuz07], is a homological phenomenon involving semi-orthogonal decompositions of the derived category $\mathrm{D^b}(X)$ of a smooth scheme X together with a morphism $X \to \mathbb{P}(V)$ and the derived category of a homological projective dual $Y \to \mathbb{P}(V^*)$. The basic assumption is that $\mathrm{D^b}(X)$ has a special kind of semi-orthogonal decomposition, called a *Lefschetz decomposition*, of the form:

$$\mathrm{D^b}(X) = \langle \mathcal{A}_0, \mathcal{A}_1(1), \dots, \mathcal{A}_i(i) \rangle,$$

where $\mathcal{A}_0 \supset \mathcal{A}_1 \supset \dots \supset \mathcal{A}_i$ is a filtered sequence of subcategories and $\mathcal{A}_k(k)$ denotes the tensor product of $\mathcal{A}_k$ by the pullback of $\mathcal{O}_{\mathbb{P}(V)}(k)$ to X. The main virtue of a Lefschetz decomposition is that it behaves well with respect to taking hyperplane sections of X. That is, for every hyperplane $H \subset \mathbb{P}(V)$, there is a semi-orthogonal decomposition

$$\mathrm{D^b}(X \times_{\mathbb{P}(V)} H) = \langle \mathcal{C}_H, \mathcal{A}_1(1), \dots, \mathcal{A}_i(i) \rangle.$$

The same holds for the family $\mathcal{X} \subset X \times_{\mathbb{P}(V)} \mathbb{P}(V^*)$ of all hyperplane sections of X. So, there is a semi-orthogonal decomposition

$$\mathrm{D^b}(\mathcal{X}) = \langle \mathcal{C}, \mathcal{A}_1(1) \boxtimes \mathrm{D^b}(\mathbb{P}(V^*)), \dots, \mathcal{A}_i(i) \boxtimes \mathrm{D^b}(\mathbb{P}(V^*)) \rangle. \tag{1}$$

If $Y \to \mathbb{P}(V^*)$ is such that the subcategory $\mathcal{C}$ is the image of $\mathrm{D^b}(Y)$ under a $\mathbb{P}(V^*)$-linear Fourier-Mukai functor $\mathrm{D^b}(Y) \to \mathrm{D^b}(\mathcal{X})$, then Y is called a *homological projective dual* (HPD) to X. Note that this relationship also depends on the morphisms and the Lefschetz decomposition.

HPD for 2-Veronese embeddings have been studied in [Kuz05].

Once such a relationship is established between two possibly non-commutative varieties then not only do we have that Y is smooth and has a *dual Lefschetz decomposition* which makes X HPD to Y but also we have semi-orthogonal decomposition relationships between the derived categories of any generic linear sections of X and the corresponding dual linear sections of Y. We call this Kuznetsov's *Fundamental Theorem of HPD.* This enables one to prove the existence of many interesting semi-orthogonal decompositions in algebraic geometry, including some previously mysterious ones arising in physics.

The first observation of the work [BDFIK13] is that, when $X = \mathbb{P}(W)$ considered with the d-Veronese embedding $X \to \mathbb{P}(S^dW)$ with $d \leq \dim W$ and the Lefschetz decomposition is the one obtained natually from the Beilinson decomposition of $\mathrm{D}^{\mathrm{b}}(\mathbb{P}(W))$, the decomposition (1) would be the relative version of a well-known theorem of Orlov [Orl09], which would give a semi-orthogonal decomposition

$$\mathrm{D}^{\mathrm{b}}(\mathcal{X}) = \langle \mathrm{MF}^{\mathbb{C}^\times}(W \times \mathbb{P}(S^dW^*), w), \mathcal{A}_1(1) \boxtimes \mathrm{D}^{\mathrm{b}}(\mathbb{P}(S^dW^*)), \ldots \\ \ldots, \mathcal{A}_i(i) \boxtimes \mathrm{D}^{\mathrm{b}}(\mathbb{P}(S^dW^*))\rangle$$

We prove this by using the Calabi-Yau-Landau-Ginzburg (CY-LG) correspondence from [Isi12, Shi12] and the recent developments in considering the behaviour of derived categories under variations of GIT quotients [BFK12, H-L12]. We also prove the decompositions of the Fundamental Theorem of HPD in this case. So, in this sense, the $\mathbb{C}^\times$-equivariant Landau-Ginzburg pair $(W \times \mathbb{P}(S^dW^*), w)$, where the $\mathbb{C}^\times$ action is by dilation along the fibers and w is the universal degree-d polynomial, is HPD to the Veronese embedding.

The previous observation is not a coincidence. It stems directly from the fact that the Beilinson decomposition itself is obtained from considering a variation of GIT quotients and the variation of GIT quotients of the space obtained from the CY-LG correspondence used when proving the relative Orlov theorem is induced by the variation used to obtain the Beilinson decomposition in the first place. This leads to the following: starting with a variety X given as a quotient $X = [Q^{\mathrm{ss}}(\mathcal{M})/G]$, where Q is a smooth variety with an action of G and $\mathcal{M}$ is a G-equivariant invertible sheaf, and an elementary wall-crossing, i.e. a simple kind of variation of the GIT quotient $X//^{\mathcal{M}}G$, we can consider X with the morphism to projective space induced by the bundle $\mathcal{M}$. Then, under mild assumptions on the elementary wall crossing, we prove that X has a Lefschetz decomposition. We then construct a Landau-Ginzburg pair (Y, w) which is a homological projective dual to X with respect to this Lefschetz decomposition.

Going back to the d-Veronese case, we consider a 'local generator' of the matrix factorization category of the pair $(W \times \mathbb{P}(S^dW^*), w)$. Calculating its graded sheaf-endomorphism algebra over $\mathbb{P}(S^dW^*)$ and using homological perturbation techniques, we obtain a sheaf $\mathcal{A}$ of $\mathcal{A}_\infty$-algebras over $\mathbb{P}(S^dW^*)$. When $d > 2$, we have

$$\mathcal{A} = \mathrm{Sym}(u\mathcal{O}_{\mathbb{P}(S^dW^*)}(1), u^{-1}\mathcal{O}_{\mathbb{P}(S^dW^*)}(-1)) \otimes \Lambda^\bullet W^*,$$

where

$$\mu^d(1 \otimes v_{i_1}, \ldots, 1 \otimes v_{i_d}) = \frac{u}{d!}\frac{\partial^d w}{\partial x_{i_1} \ldots \partial x_{i_d}}$$

and $\mu^i = 0$ for $2 < i < d$. All the higher products are determined by this dth product. This gives a different description of the HPD. When $d = 2$, we recover the HPD obtained in [Kuz05].

References

[BDFIK13] M. Ballard, D. Deliu, D. Favero, M.U. Isik, L. Katzarkov *Homological Projective Duality via Variation of Geometric Invariant Theory Quotients*, arXiv.org preprint, arXiv:1306.3957

[BFK12] M. Ballard, D. Favero, L. Katzarkov. *Variation of Geometric Invariant Theory quotients and derived categories.* arXiv.org preprint: arXiv:1203.6643.

[H-L12] D. Halpern-Leistner. *The derived category of a GIT quotient.* arXiv.org preprint: arXiv:1203.0276.

[Isi12] M. U. Isik. *Equivalence of the derived category of a variety with a singularity category.* Int Math Res Notices (2012), doi:10.1093/imrn/rns125

[Kuz05] A. Kuznetsov. *Derived categories of quadric fibrations and intersections of quadrics.* Adv. Math. 218 (2008), no. 5, 1340-1369.

[Kuz06] A. Kuznetsov. *Homological projective duality for Grassmannians of lines.* arXiv.org preprint: arXiv:math/0610957.

[Kuz07] A. Kuznetsov. *Homological projective duality.* Publ. Math. Inst. Hautes Études Sci. No. 105 (2007), 157-220.

[Orl09] D. Orlov. *Derived categories of coherent sheaves and triangulated categories of singularities.* Algebra, arithmetic, and geometry: in honor of Yu. I. Manin. Vol. II, 503-531, Progr. Math., 270, Birkhäuser Boston, Inc., Boston, MA, 2009.

[Shi12] I. Shipman. *A geometric approach to Orlov's theorem.* Compositio Math. 148 (2012), no. 5, 1365-1389.

Relative singularity categories

Martin Kalck

(joint work with Osamu Iyama, Michael Wemyss and Dong Yang)

1. Relative singularity categories (with D. Yang)

Setup. Let k be an algebraically closed field and let $(R, \mathfrak{m})$ be a commutative complete local Gorenstein k-algebra with an isolated singularity in $\mathfrak{m}$ and residue field $R/\mathfrak{m} \cong k$. We assume that R admits a noncommutative resolution (NCR) $A = \mathsf{End}_R(R \oplus M)$, i.e. A has finite global dimension and M is a maximal Cohen–Macaulay R-module (MCM). We remark that Van den Bergh's *noncommutative crepant resolutions* (NCCR) are examples of NCRs in many cases.

We consider the following well-known inclusions of triangulated categories

$$\mathcal{D}^b(R) \hookleftarrow \mathsf{Perf}(R) \xrightarrow{-\otimes_R (R\oplus M)} \mathcal{D}^b(A) \tag{1}$$

The triangulated quotient category $\mathcal{D}_{sg}(R) := \mathcal{D}^b(R)/\mathsf{Perf}(R)$ associated with the left inclusion is the singularity category of Buchweitz and Orlov. It may be viewed as a measure for the complexity of the singularities of $\mathsf{Spec}(R)$. For hypersurface singularities $R = S/(f)$ the singularity category of R is equivalent to the *homotopy category of matrix factorizations* of f by work of Buchweitz and Eisenbud.

Motivated by this construction, we study the *relative singularity category*

$$\Delta_R(A) := \mathcal{D}^b(A)/\mathsf{Perf}(R)$$

associated with the right inclusion in (1) in joint work with Igor Burban [2]. It may be seen as a measure for the size of the categorical resolution $\mathcal{D}^b(A)$.

The following theorem provides relations between the notions of singularity categories and relative singularity categories, see [5, Theorem 5.19.].

Theorem. *Let R and R' be complete Gorenstein k-algebras with only finitely many isomorphism classes of indecomposable* MCM*s. Let $A = \mathsf{Aus}(\mathsf{MCM}(R))$ and $A' = \mathsf{Aus}(\mathsf{MCM}(R'))$ be the corresponding Auslander algebras - they are known to be NCRs by work of Auslander. Then the following statements are equivalent.*

- (i) *There is an equivalence $\mathcal{D}_{sg}(R) \cong \mathcal{D}_{sg}(R')$ of triangulated categories.*
- (ii) *There is an equivalence $\Delta_R(A) \cong \Delta_{R'}(A')$ of triangulated categories.*

The implication $(ii) \Rightarrow (i)$ holds more generally for NCRs A and A' of arbitrary isolated Gorenstein singularities R and R', respectively (see also [6]*).*

Remark. (a) *Knörrer's periodicity yields a wealth of non-trivial examples for (i).* (b) *The implication $(i) \Rightarrow (ii)$ follows from an equivalence $\Delta_R(\mathsf{Aus}(\mathsf{MCM}(R))) \cong \mathsf{per}(B(R))$, where the dg algebra $B(R)$ is determined by the Auslander–Reiten quiver of $\mathcal{D}_{sg}(R)$. $B(R)$ was also determined for NCCRs of certain Gorenstein quotient singularities R by Thanhoffer de Völcsey & Van den Bergh* [6]*, in order to express singularity categories as generalized cluster categories (see also* [1]*).*

2. Rational surface singularities (with O. Iyama, M. Wemyss & D. Yang)

This part of my talk was inspired by some of the techniques developed in the context of relative singularity categories.

Setup. Let R be a complete local rational surface singularity over $\mathbb{C}$ (i.e. $H^1(X, \mathcal{O}_X) = 0$ for a resolution of singularities $X \to \mathsf{Spec}(R)$) and let E be the exceptional fibre of the minimal resolution $Y \xrightarrow{\pi} \mathsf{Spec}(R)$. It is well-known that $E = \bigcup E_i$ is a tree of rational $(-n)$-curves with $n \leq 2$.

Definition. A *special Cohen–Macaulay* (SCM) R-module M is a reflexive module satisfying $\mathsf{Ext}^1_R(M, R) = 0$. (Note that R Gorenstein $\Rightarrow \mathsf{SCM}(R) = \mathsf{MCM}(R)$).

The following results indicate that SCMs over arbitrary rational surface singularities play a role analogous to MCMs over rational double points:

- (a) There is a natural bijection between the irreducible exceptional curves E_i and the indecomposable non-free SCMs, restricting to the classical McKay correspondence in the case of rational double points (see Wunram [7],...).
- (b) The natural exact structure on $\mathsf{SCM}(R) \subseteq \mathsf{MCM}(R)$ is *Frobenius* with indecomposable projective-injective objects R and all M_i corresponding to exceptional $(-n)$-curves E_i with $n < 2$, see Iyama & Wemyss [4].
- (c) In particular, the stable category $\underline{\underline{\mathsf{SCM}}}(R) := \mathsf{SCM}(R)/\mathsf{proj}\,\mathsf{SCM}(R)$ is triangulated by work of Happel. Iyama & Wemyss [4] observed that the

Auslander–Reiten quiver of $\underline{\underline{\mathsf{SCM}}}(R)$ is a finite union of doubles of ADE-Dynkin quivers and that in many cases there are additive equivalences

$$\underline{\underline{\mathsf{SCM}}}(R) \cong \underline{\mathsf{MCM}}(R'), \tag{2}$$

where R' is a rational Gorenstein surface singularity.

The following result may be seen as an explanation for the observations in (c). In particular, we show that there is always a *triangle* equivalence (2).

Theorem. *Let R be a rational surface singularity with minimal resolution Y. Let X be obtained from Y by contracting the exceptional (-2)-curves. It is well-known that $\mathsf{Sing}(X)$ consists of isolated singularities, which are rational double points.*

Then there are equivalences of triangulated categories

$$\underline{\underline{\mathsf{SCM}}}(R) \cong \mathcal{D}_{sg}(X) \cong \bigoplus_{x \in \mathsf{Sing}(X)} \underline{\mathsf{MCM}}\big(\widehat{\mathcal{O}}_x\big).$$

In particular, $\underline{\underline{\mathsf{SCM}}}(R)$ is 1-CY and there is a natural isomorphism $[2] \cong \mathsf{id}$.

Remark. (a) *The second equivalence follows from work of Orlov, see also* [2]*. The first equivalence combines an algebraic result (a Morita-type Theorem for Frobenius categories admitting a noncommutative resolution, which was inspired by our techniques for relative singularity categories) with a geometric statement (X admits a tilting bundle).*

(b) *Our result shows that the Auslander–Reiten quiver of $\underline{\underline{\mathsf{SCM}}}(R)$ is the double quiver of the dual intersection graph of the exceptional fibre of the contraction $Y \to X$. This may be viewed as a generalization of Auslander's algebraic McKay correspondence to all rational surface singularities.*

References

[1] C. Amiot, O. Iyama, I. Reiten, *Stable categories of Cohen–Macaulay modules and cluster categories*, arXiv:1104.3658.

[2] I. Burban, M. Kalck, *Relative singularity category of a non-commutative resolution of singularities*, Adv. in Math. **231** (2012), no. 1, 414–435.

[3] O. Iyama, M. Kalck, M. Wemyss, D.Yang, *Frobenius categories, Gorenstein algebras and rational surface singularities*, arXiv:1209.4215.

[4] O. Iyama, M. Wemyss, *A New Triangulated Category for Rational Surface Singularities*, Illinois J. Math. **55** (2011), no. 1, 325–341.

[5] M. Kalck, D. Yang, *Relative Singularity Categories I: Auslander Resolutions*, arXiv:1205.1008.

[6] L. de Thanhoffer de Völcsey, M. Van den Bergh, *Explicit models for some stable categories of maximal Cohen-Macaulay modules*, arXiv:1006.2021.

[7] J. Wunram, *Reflexive modules on quotient surface singularities*, Math. Ann. **279** (1988), no. 4, 583–598.

Cyclic 2-Segal spaces and matrix factorizations

Mikhail Kapranov
(joint work with Tobias Dycherhoff)

The construction of Connes' cyclic category Λ given by Drinfeld uses the concept of a $\mathbb{Z}_+$-category, a setting in which one can speak about matrix factorizations. The 2-Segal property, expressing independence of data on a triangulation of a polygon, can also be traced to the properties of cyclic orders. In the talk, I explain how using this approach leads to a construction of a particular 2-Segal cyclic object ϵ in the category of dg-categories. This leads to a construction of a dg-category $\epsilon^{S,M}$ for every marked surface (S, M), which is a combinatorial version of the Fukaya category of $S - M$.

Exotic Calabi-Yaus from non-abelian gauge theories

Johanna Knapp
(joint work with Kentaro Hori)

We construct new compact Calabi-Yau (CY) threefolds with one Kähler parameter by making use of a supersymmetric non-abelian gauge theory - the linear sigma model (LSM). Using physics methods we analyze different regions of the Kähler moduli space. This leads to equivalences between Pfaffian or determinantal CY varieties and hybrid models which are Landau-Ginzburg fibrations over Fano manifolds. We conjecture that their corresponding D-brane categories are equivalent. A surprising result is that two CY threefolds which do not have the same Hodge numbers have the same Kähler moduli space. This is a summary of [1].

1. The non-abelian LSM

We consider an LSM [2] with gauge group $G = U(1) \times H$, where we will focus on $H = SU(2)$ or $H = O(2)$. The matter content consists of M chiral fields p^i ($i = 1, \ldots, M$) with $U(1)$-charge q_{p^i} which do not transform under H, and N chiral fields x_j^a ($j = 1, \ldots, N$, $a = 1, 2$) with $U(1)$-charges q_{x_i} which transform in the fundamental representation of H. Furthermore there are twisted chirals $\{\sigma_{U(1)}, \sigma_H\}$. Interactions are encoded in the classical potential

$$\begin{aligned} V &= \frac{1}{2e^2}\mathrm{Tr}[\sigma_H, \sigma_H^\dagger]^2 + \frac{e^2}{2}|D|^2 + \frac{1}{2}\sum_{j=1}^{N} x_j^\dagger\{\sigma_H^\dagger, \sigma_H\}x_j + |F|^2 \\ &\quad + \sum_{i=1}^{M} q_{p^i}^2|\sigma_{U(1)}|^2|p^i|^2 + \sum_{i=j}^{N} q_{x_j}^2|\sigma_{U(1)}|^2||x_j||^2, \end{aligned} \tag{1}$$

where e is the gauge coupling, $()^\dagger$ denotes hermitean conjugation, $||()||$ implies summation over the group indices, and D and F depend on p, x. The classical vacuum of the theory is determined by $V = 0$. We distinguish two kinds of solutions: on the Higgs branch $\sigma = 0, D = 0, F = 0$, whereas on the Coulomb branch $x = 0$, $p = 0$ and σ_H has to take values in the maximal torus of H. The D-term equation $D = 0$ associated to the $U(1)$-factor is

$$\sum_{i=1}^{M} q_{p^i}|p^i|^2 + \sum_{j=1}^{N} q_{x_j}||x_j||^2 = r_{U(1)}, \tag{2}$$

where $r \in \mathbb{R}$ is the Fayet-Illiopoulos (FI) parameter of the supersymmetric gauge theory, which can be combined with a further 2π-periodic parameter – the theta angle – to $t = r + i\theta$. This will be identified with the Kähler modulus of the CY. In the cases of interest we can always choose $q_{p^i} < 0$ and $q_{x_i} \geq 0$. Depending on the sign of the FI-parameter all the x- or p-fields will not be allowed to vanish simultaneously in order for (2) to be satisfied. There D-terms associated to H are

$$SU(2): \;\; xx^\dagger - \frac{1}{2}||x||^2\mathbf{1}_2 = 0 \qquad O(2): \;\; xx^\dagger - (xx^\dagger)^T = 0. \tag{3}$$

For the vacuum space to be compact, there must be a non-zero superpotential in the LSM. In our case, this has the following structure

$$SU(2): \;\; W = \sum_{ij} A^{ij}(p)[x_i x_j] \qquad O(2): \;\; W = \sum_{ij} S^{ij}(p)(x_i x_j). \tag{4}$$

Here $[x_i x_j]$ and $(x_i x_j)$ are bilinear invariants of $SU(2)$ and $O(2)$, respectively. $A(p)$ and $S(p)$ are (skew-)symmetric $N \times N$ matrices whose entries are homogeneous polynomials in the p-fields. Their degrees are determined by the condition that W is invariant under G. The F-term equations $F = 0$ are

$$\frac{\partial W}{\partial p^i} = 0 \qquad \frac{\partial W}{\partial x_i^a} = 0. \tag{5}$$

In order for the classical vacuum to be CY, the following condition has to be satisfied

$$\sum_{i=1}^{M} q_{p^i} + 2\sum_{j=1}^{N} q_{x_j} = 0. \tag{6}$$

For a three-dimensional CY one requires

$$M - 1 - \frac{2(2 \pm 1)}{2} = 3 \qquad +/-\ldots SU(2)/O(2). \tag{7}$$

Given this data, the vacuum space, i.e. the solutions of the D-term and F-term equations, will be a CY threefold. Unless there is a duality, different field content and gauge groups will lead to different CYs. By tuning the FI parameter r we can probe the Kähler moduli space of the CY. Different regions, which lead to different solutions of (2), (3), and (5), are called phases of the LSM. Going from one phase to another typically changes the topology of the CY. The corresponding D-brane categories are conjectured to be equivalent.

An estimate for the Hodge numbers of the CY is given by counting degrees of freedom of the theory:

(8) $h^{1,1} \leftrightarrow$ number of FI parameters

(9) $h^{2,1} \leftrightarrow$ number of monomials in W modulo reparametrizations

The singular loci in the Kähler moduli space can be determined from the Coulomb branch. The classical potential (1) is zero, which would leave us with non-compact directions for σ. These are however lifted by quantum corrections except at certain points which are determined by the critical locus of the effective potential

$$\widetilde{W}_{eff} = -\sum_{\chi} \chi(\sigma)(\log(\chi(\sigma)) - 1) + \ldots, \tag{10}$$

where χ denotes the characters of the representation the fields transform in with respect to the maximal torus and ... denote further terms linear in σ and the FI parameter [1]. A further important datum is the sphere partition function of the LSM which has recently been related to the quantum corrected Kähler potential of the CY moduli space [3]: $Z_{S^2} \sim e^{-K(t,\bar{t})}$. In geometric phases of the LSM, this provides a way to extract the periods and Gromov-Witten invariants without having to rely on mirror symmetry. Furthermore, the sphere partition function can be used to determine the leading behavior of the Kähler metric on the moduli space in a given phase. A further helpful tool in the analysis is a duality discovered in [4] which shows that the same vacuum structure, i.e. CY, can be obtained by pairs of LSMs with different field content and different gauge groups.

2. New examples and correspondences

In [1] we have constructed five new one-parameter LSMs. We use the shorthand notation (A^k_q) or $(\mathrm{S}^{k,\bullet}_q)$, where A and S indicate whether the symmetry behavior of the p-dependent matrix in (4), k is the rank of H, q denotes the $U(1)$-charges of the matter fields and $\bullet$ can be $0, \pm$, where 0 is for $SO(2)$ and $\pm$ denotes the two possible sign choices for $O(2) \simeq SO(2) \rtimes \mathbb{Z}_2$. The models we found are

$(\mathrm{A}^2_{(-1)^4,(-2)^3,1^5})$, $(\mathrm{A}^2_{(-1)^6,(-2),1^4,0})$, $(\mathrm{A}^2_{(-2)^7,3,1^4})$, $(\mathrm{A}^2_{(-2)^5,(-4)^2,3^2,1^3})$, $(\mathrm{S}^{2,+}_{(-1)^2,(-2)^3,1^4})$

For all these models we have performed the program outlined above. The $r \ll 0$ phases of the first four models can be identified with the Pfaffian CYs discussed in [5]. The $r \gg 0$ phases are of the first three models are hybrids which are $\mathbb{Z}_2$ Landau-Ginzburg orbifolds over a Fano base, where the limiting point is a cusp singularity at infinite distance and the periods are those of a geometric CY in the large radius limit. The $r \gg 0$ phase of the fourth model is a "pseudo-hybrid" model, where the limiting point is at finite distance in the moduli space. The last model has not been discussed in the literature before, as far as we can tell. The Hodge numbers are $h^{1,1} = 1$ and $h^{2,1} = 23$. The $r \ll 0$ phase is a symmetric determinantal variety, the $r \gg 0$ phase is again a $\mathbb{Z}_2$ Landau-Ginzburg orbifold over a Fano base. We conjecture that the categories associated to the respective phases are equivalent. Furthermore we have encountered an unexpected surprise: the Kähler moduli spaces of the second and the last model are the same, even though $h^{2,1}$ is different. Using the duality of [4] we could show that the sphere partition functions of the two models are the same. In particular, the partition function in the $r \gg 0$ phase of one model is the same as for the $r \ll 0$ phase of the other and vice versa. We could further show that also the singular points in the moduli space match.

References

[1] K. Hori, J. Knapp, *Linear Sigma Models With Strongly Coupled Phases – One Parameter Models*, arXiv:1308.6265[hep-th]

[2] E. Witten, *Phases of N=2 theories in two-dimensions*, Nucl. Phys. **B403** (1993) 159–222

[3] H. Jockers, V. Kumar, J. M. Lapan, D. R. Morrison, and M. Romo, *Two-Sphere Partition Functions and Gromov-Witten Invariants*, arXiv:1208.6244 [hep-th]

[4] K. Hori, *Duality In Two-Dimensional (2,2) Supersymmetric Non-Abelian Gauge Theories*, arXiv:1104.2853 [hep-th]

[5] A. Kanazawa, *On Pfaffian Calabi-Yau Varieties and Mirror Symmetry*, Commun. Number Theory Phys. **6** (2012) 661–696

Categories of matrix factorizations for Brieskorn (and triangle) singularities

Helmut Lenzing
(joint work with Dirk Kussin, Hagen Meltzer)

Let k be an algebraically closed field. We consider the (universally graded) Brieskorn singularity $f = x_1^{a_1} + x_2^{a_2} + \cdots + x_t^{a_t}$, $t \geq 3$, for a t-tupel of integers $a_1, \ldots, a_t$ which are greater or equal 2. The integer $d := t-2$ will be referred to as *dimension*. The algebras $T = k[x_1, \ldots, x_t]$ and $S = k[x_1, \ldots, x_t]/(f)$ are equipped with the grading by the rank-one abelian group $\mathbb{L}$ with generators $\vec{x}_1, \ldots, \vec{x}_t$, satisfying $a_1\vec{x}_1 = \cdots = a_t\vec{x}_t =: \vec{c}$, where the generators x_i of T and S are given degree $\vec{x}_i$, such that f gets degree $\vec{c}$. We call $\vec{c}$ the *canonical* and $\vec{\omega} = \vec{c} - \sum_{i=1}^{t} \vec{x}_i$ the *dualizing element* of $\mathbb{L}$. By sheafification, or Serre construction, we obtain a category $\operatorname{coh}\mathbb{X} = \operatorname{mod}^{\mathbb{L}}(S)/\operatorname{mod}_0^{\mathbb{L}}(S)$ of coherent sheaves on some (non-commutative) space $\mathbb{X}$. Here $\operatorname{mod}^{\mathbb{L}}(S)$ (resp. $\operatorname{mod}_0^{\mathbb{L}}(S)$) refers to the category of finitely generated $\mathbb{L}$-graded S-modules (resp. those of finite length). By a result of Buchweitz [2] the (graded) singularity category od S is equivalent to the *stable category of graded matrix factorizations* $\mathrm{MF}^{\mathbb{L}}(T, f)$. Moreover, by a graded variant of a theorem of Orlov [8], the bounded derived category $D^b(\operatorname{coh}\mathbb{X})$ and the category $\mathrm{MF}^{\mathbb{L}}(T, f)$ are related by *Orlov correspondence* through semi-orthogonal decompositions which depend on the value of the *Gorenstein parameter* $\gamma = (\prod_{i=1}^{t} a_i)(\sum_{i=1}^{t} \frac{1}{a_i} - 1)$ of S. We are going to clarify the following aspects.

- (A1) Understand the geometry of $\mathbb{X}$.
- (A2) To what extent is the geometry of $\mathbb{X}$ visible in the category $\mathrm{MF}^{\mathbb{L}}(T, f)$?
- (A3) Develop an explicit understanding of the relationship between $D^b(\operatorname{coh}\mathbb{X})$ and $\mathrm{MF}^{\mathbb{L}}(T, f)$.

Concerning (A1) it turns out that $\mathbb{X}$ is a *weighted projective space* $\mathbb{P}^d\langle a_1, \dots, a_t\rangle$, obtained from the underlying projective d-space $\mathbb{P}^d$ by *weight insertion* in $t = d+2$ hyperplanes H_i ($i = 1, \dots, t$) in general position. In more detail, we have the following properties, that can be derived in analogy to [1]. Independently, these results were obtained by [HIMO] (=Herschend-Iyama-Minamoto-Oppermann), compare [5], [7]. For $t = 3$ the theory reduces to the study of *triangle singularities* where $\mathbb{X}$ is a weighted projective line with three weights, see [6].

Theorem 1. With the above assumptions, the following holds.

(1) By means of the map $\vec{x} \mapsto \mathcal{O}_{\mathbb{X}}(\vec{x})$, the grading group $\mathbb{L}$ is isomorphic to the Picard group of $\mathbb{X}$.
(2) The space $\mathbb{X}$ is *smooth of dimension* d, that is, the category $\operatorname{coh}\mathbb{X}$ has global dimension d. Moreover, the category has *Serre duality* in the form $D\operatorname{Ext}^j(X,Y) = \operatorname{Ext}^{d-j}(Y, X(\vec{\omega}))$ for any integer j.
(3) The category $\operatorname{coh}\mathbb{X}$ has a *tilting object* $T = \bigoplus_{0\le\vec{x}\le d\vec{c}}\mathcal{O}_{\mathbf{X}}(\vec{x})$ consisting of line bundles. (The endomorphism ring of T is called d-canonical by [HIMO]).

Concerning (A2), we define a *restriction functor* ρ from $\operatorname{coh}\mathbb{X}$ to $\operatorname{coh}\mathbb{P}^d$ which sends vector bundles to vector bundles. We say that a vector bundle E on $\mathbb{X}$ is $\mathbb{P}^d$*-split* if ρE is a direct sum of line bundles on $\mathbb{P}^d$. By $\operatorname{vect}_{sp}\mathbb{X}$ we denote the full subcategory of $\mathbb{P}^d$-split vector bundles on $\mathbb{X}$. (For $t = 3$ this is the category of all vector bundles on $\mathbb{X}$). We obtain the following results:

Theorem 2. Keeping the above notations, the following assertions hold.

(1) By sheafification the category $\mathrm{CM}^{\mathbb{L}} S$ of $\mathbb{L}$-graded Cohen-Macaulay modules over S is equivalent to the category $\operatorname{vect}_{sp}\mathbb{X}$.
(2) The exact Frobenius structure on $\mathrm{CM}^{\mathbb{L}} S$, induced from the ambient module category, translates under this equivalence to a Frobenius structure on $\operatorname{vect}_{sp}\mathbb{X}$.
(3) The *stable category of* $\mathbb{P}^d$ *− split vector bundles* on $\mathbb{X}$ is equivalent to the stable category $\underline{\mathrm{CM}}^{\mathbb{L}} S$, hence to the stable category of graded matrix factorizations $\mathrm{MF}^{\mathbb{L}}(T, F)$.
(4) The category $\mathrm{MF}^{\mathbb{L}}(T, F)$ is equivalent to the derived category $D^b(\operatorname{mod} A)$, where A is the tensor product over k of the path algebras of equioriented Dynkin quivers $\vec{\mathbb{A}}_{a_i-1}$, $i = 1, \dots, t$, of type $\mathbb{A}$.

For $t = 3$ property (4) is due to [6]. For $t \ge 3$ and k the field of complex numbers it is due to [3]. Including the complete intersection case, the general assertion is due to [HIMO, unpublished]. As an immediate consequence, the categories $\mathrm{MF}^{\mathbb{L}}(T, F)$ are fractional Calabi-Yau. Moreover, using [4], the weights > 2, however not the dimension D, can be recovered from the category $\mathrm{MF}^{\mathbb{L}}(T, F)$.

References

[1] Dagmar Baer. Tilting sheaves in representation theory of algebras. *Manuscripta Math.*, 60(3):323–347, 1988.

[2] Ragnar-Olaf Buchweitz. Maximal Cohen-Macaulay Modules and Tate-Cohomology over Gorenstein Rings. preprint, 1986, `http://hdl.handle.net/1807/16682`, 1986.
[3] Masahiro Futaki and Kazushi Ueda. Homological mirror symmetry for Brieskorn-Pham singularities. *Selecta Math. (N.S.)*, 17(2):435–452, 2011.
[4] Lutz Hille and Jürgen Müller. On tensor products of path algebras of type A. `arXiv:1206.1152`, 2012.
[5] Osamu Iyama. "Geigle-Lenzing spaces and canonical algebras in dimension d". Talk at the second ARTA Conference, Torun, September 2013.
[6] Dirk Kussin, Helmut Lenzing, and Hagen Meltzer. Triangle singularities, ADE-chains, and weighted projective lines. *Adv. Math.*, 237:194–251, 2013.
[7] Steffen Oppermann. d-dimensional geigle-lenzing spaces and canonical algebras. Talk at the Auslander Conference, Woodshole, April 2013.
[8] Dmitri Orlov. Derived categories of coherent sheaves and triangulated categories of singularities. Volume 270 of *Progr. Math.*, pages 503–531. Birkhäuser, Boston, MA, 2009.

Computation and geometry

Daniel Murfet

The work which is the subject of this abstract has as its aim the construction of interpretations of Girard's linear logic [4] in bicategories of geometric origin, in order to provide a bridge between computation and geometry. The main example of such a bridge is the connection between cut-elimination in linear logic and what we call *cut systems* on bicategories. We give a sketch of the general theory of cut systems and the motivating example involving matrix factorisations.

In a bicategory $\mathcal{B}$ there are objects, 1-morphisms and 2-morphisms, where for every pair of objects a, b the 1-morphisms from a to b and the 2-morphisms between them form a category $\mathcal{B}(a, b)$. Bicategories arise naturally in geometry: the objects $a, b, c, \ldots$ are spaces, the 1-morphisms between a and b are integral kernels of some kind on $a \times b$, and the 2-morphisms are transformations between kernels. The composition rule for 1-morphisms is given by the convolution of kernels.

Our main example is the bicategory $\mathcal{LG}_k$ of Landau-Ginzburg models [2], which is roughly speaking a bicategory of isolated hypersurface singularities and integral kernels. Objects are pairs $(k[x], W)$ consisting of a polynomial ring and a potential $W(x) \in k[x]$, 1-morphisms from $W(x) \longrightarrow V(z)$ are finite rank matrix factorisations of $V - W$ over $k[x, z]$, and 2-morphisms are homotopy equivalences classes of homomorphisms of matrix factorisations. Composition of a pair

$$X : W(x) \longrightarrow V(z), \qquad Y : V(z) \longrightarrow U(y) \tag{1}$$

is given by the tensor product

$$Y \circ X = Y \otimes_{k[z]} X\,.$$

This composition rule is "denotational" in the sense that $Y \circ X$ is defined to be the unique (up to isomorphism) finite rank matrix factorisation homotopy equivalent to $Y \otimes_{k[z]} X$ (the differential on which is given by an infinite matrix over $k[x, y]$) but this unique finite model is not prescribed as part of the data of the bicategory. The cut system on $\mathcal{LG}_k$ presented below gives a coherent algorithm for constructing these finite models, refining earlier work with Toby Dyckerhoff [3].

The bicategory $\mathcal{LG}_k$ has various applications, for example in the setting of topological field theory with defects, but mathematically its existence is amply justified by the work on generalised orbifolding between ADE-type singularities presented at this workshop by Nils Carqueville and Ingo Runkel.

The rest of this report is structured as follows: we first define cut systems, then construct the natural cut system related to $\mathcal{LG}_k$, and finally sketch how this relates to cut-elimination. Throughout k is a $\mathbb{Q}$-algebra.

0.1. **Cut systems.** Let $\mathcal{CL}$ denote the category of Clifford algebras over k which are Morita trivial, or in other words, the associative algebras $C(n)$ for $n \geq 0$ generated by symbols $\theta_1, \ldots, \theta_n, \theta_1^*, \ldots, \theta_n^*$ subject to the anti-commutation relations

$$\{\theta_i, \theta_j\} = 0, \{\theta_i^*, \theta_j^*\} = 0, \{\theta_i, \theta_j^*\} = \delta_{ij}\,.$$

A morphism $C(n) \longrightarrow C(m)$ is an isomorphism class of $C(m)$-$C(n)$-bimodules and composition is given by the tensor product of bimodules. This category is a very simple one: the structure is determined by the morphisms

$$\Lambda_n = \bigwedge(k\theta_1 \oplus \cdots \oplus k\theta_n) : k = C(0) \longrightarrow C(n)$$

where $C(n)$ acts on the exterior algebra Λ_n by wedge products and contraction. The bimodule Λ_n is an isomorphism with inverse $\Lambda_n^* = \operatorname{Hom}_k(\Lambda_n, k)$.

A category *fibered* over $\mathcal{CL}^{\mathrm{op}}$ is a special kind of functor $F : \mathcal{T} \longrightarrow \mathcal{CL}^{\mathrm{op}}$. For the moment it is enough to recall that if F is such a fibered category, we get from Λ_n and its dual a pair of mutually inverse maps

$$F_{\Lambda_n} : \mathcal{T}_0 \longrightarrow \mathcal{T}_n, \qquad F_{\Lambda_n^*} : \mathcal{T}_n \longrightarrow \mathcal{T}_0$$

where $\mathcal{T}_n$ denotes the set of objects $X \in \mathcal{T}$ with $F(X) = C(n)$.

Let $\mathcal{B}$ be a bicategory, in which the composition of Y with X is denoted $Y|X$.

Definition 1. A *cut system* on $\mathcal{B}$ consists of the following data:

(a) a fibered category $\pi : \mathcal{B}(a,b) \longrightarrow \mathcal{CL}^{\mathrm{op}}$ for each pair of objects $a, b \in \mathcal{B}$.
(b) an object $\pi(a)$ of $\mathcal{CL}$ for each object $a \in \mathcal{B}$
(c) for each pair of composable 1-morphisms $X : a \longrightarrow b, Y : b \longrightarrow c$ a natural isomorphism of Clifford algebras

$$\pi(Y|X) \cong \pi(Y) \otimes_k \pi(b) \otimes_k \pi(X)\,.$$

subject to various conditions which we omit.

If $\pi(X) = k$ we say X is *cut free* or *normal*. The family of fibrations π associates to X with $\pi(X) = C(n)$ a normal 1-morphism $F_{\Lambda_n^*}(X)$, called the *normalisation*.

The bicategory $\mathcal{LG}_k^{cut}$ with its cut system π is defined as follows. The objects are the same as $\mathcal{LG}_k$ while 1-morphisms are pairs (X, ρ) consisting of a 1-morphism X in $\mathcal{LG}_k$ together with the action ρ of a Clifford algebra on X. We define $\pi(X, \rho)$ to be this Clifford algebra and to the object $(k[x_1, \ldots, x_n], W)$ we associate $C(n)$. The maps F_{Λ_n} and $F_{\Lambda_n^*}$ of the fibration are defined respectively by splitting idempotents and by tensoring with a graded vector space.

The composition is the interesting part: in the simplest case where X, Y as in (1) are viewed as cut free morphisms by equipping them with the trivial action of $k = C(0)$, the composition in $\mathcal{LG}_k^{cut}$ is given by the pair

$$Y|X := (Y \otimes_{k[z]} k[z]/(\partial_{z_1} V, \dots, \partial_{z_m} V) \otimes_{k[z]} X, \rho)$$

where the action ρ of $C(m)$ is defined by explicit formulas written in terms of the matrices giving the differentials on X and Y. The main result is:

Theorem 2. The normalisation of $Y|X$ is a finite rank matrix factorisation naturally homotopy equivalent to the 1-morphism $Y \circ X$ composed in $\mathcal{LG}$.

That is, the normalisation of $Y|X$ is the desired finite model for composition of 1-morphisms in $\mathcal{LG}$. It is important to note that the normalisation itself is not prescribed, only the larger object $Y|X$ together with a Clifford action. Nonetheless the cut system gives important structural information about $\mathcal{LG}$, and it is explicit enough to implement in the computer algebra package Singular [1].

0.2. **Interpretation of linear logic.** Linear logic is an extension of classical logic [4, 5] and the standard mathematical frameworks for computation like the lambda calculus. A central part of linear logic is an equivalence relation on proofs called *cut-elimination*, introduced in the sequent calculus by Gentzen. This corresponds to the process of normalisation or β-reduction of terms in the lambda calculus, and given the central importance of these concepts in computation it has, since the seminal work of Girard, been an important problem to find nontrivial mathematical models of cut-elimination.

There is an interpretation in $\mathcal{LG}_k^{cut}$ of the multiplicative fragment of linear logic annotated with cuts: the bare sequents $\vdash \Gamma$ are mapped to objects $\langle \Gamma \rangle$, possible cut annotations Δ are mapped to Clifford algebras $\langle \Delta \rangle$, and a proof of the annotated sequent $\vdash [\Delta]\Gamma$ is mapped to a 1-morphism X with $\pi(X) = \langle \Delta \rangle$. The cut rule in linear logic is interpreted as composition of 1-morphisms, and so the axiom (c) of a cut system reflects the introduction of annotations by instances of the cut rule. The novelty of this interpretation of linear logic is that cut-elimination is modelled by the process of normalising 1-morphisms, in the sense described above.

The embedding of computation into $\mathcal{LG}$ has various applications: for example, once the interpretation is extended to include the exponential connectives of linear logic, every computable function on the integers will determine a functor between categories of matrix factorisations.

References

[1] N. Carqueville and D. Murfet, *Computing Khovanov-Rozansky homology and defect fusion*, to appear in Algebraic & Geometric Topology, [arXiv:1108.1081].

[2] N. Carqueville and D. Murfet, *Adjunctions and defects in Landau-Ginzburg models*, [arXiv:1208.1481].

[3] T. Dyckerhoff and D. Murfet, *Pushing forward matrix factorizations*, Duke Math. J. Volume 162, Number 7 (2013), 1249–1311, [arXiv:1102.2957].
[4] Jean-Yves Girard, *Linear logic*, Theoretical Computer Science, 50: 1–102, 1987.
[5] Paul-André Melliès, *Categorical semantics of linear logic*.

The complex moduli of a Landau-Ginzburg model

Tony Pantev
(joint work with L.Katzarkov and M.Kontsevich)

The talk aims to understand the local structure of the moduli space of complex Landau-Ginzburg models. Such a Landau-Ginzburg model is determined by a pair (Y, w), where Y is a complex quasi-projective variety, and $\mathsf{w} : Y \to \mathbb{A}^1$ is a holomorphic function on Y. When Y has a trivial canonincal class $K_Y \cong \mathcal{O}_Y$, the category of matrix factorization $\mathbf{MF}(Y, \mathsf{w})$ of the potential w can be viewed [KKP08] as the category of coherent sheaves on a smooth compact non-commutative Calabi-Yau variety.

The main result of the talk is that the versal deformation space of such a non-commutative Calabi Yau, that is the versal deformation space of the category $\mathbf{MF}(Y, \mathsf{w})$, is unobstructed. This result extends the classical Tian-Todorov theorem [Tia87, Tod89] to the non-commutative context. This result is natural from the point of view of mirror symmetry. Indeed, a Landau-Ginzburg pair (Y, w) as above will typically arise as the mirror of a symplective manifold (X, ω_X) underlying a projective Fano variety. The homological mirror symmetry conjecture [Kon95] predicts that the Fukaya category $\mathbf{Fuk}(X, \omega_X)$ of (X, ω_X) will be equivalent to the category $\mathbf{MF}(Y, \mathsf{w})$. In particular the deformation theories of the Fukaya category and of the category of matrix factorizations will be identified, and versal deformation space of the Fukaya category is manifestly smooth since it is an open cone in the space of harmonic 2-forms on X.

By the work of Orlov [Orl04, Orl05, Orl12] the category $\mathbf{MF}(Y, \mathsf{w})$ is the coproduct of the categories of singularities of the singular fibers of w. This interpretation indicates that flat deformations of the geometric data (Y, w) will not necessarily give rise to flat deformations of $\mathbf{MF}(Y, \mathsf{w})$. Indeed, when we deform (Y, w) geometrically, the singularities of fibers of w can coalesce and more importantly can run away to infinity. This suggests that we should only consider geometric deformations of (Y, w) that are anchored at infinity. Indeed, if $((\mathsf{Z}, \mathsf{f}), D_{\mathsf{Z}})$ is a compactification of (Y, w) with a normal crossings boundary, then the deformations of the pair (Z, f) that fix the boundary divisor D_{Z} will give deformations of (Y, w) for which the associated categories $\mathbf{MF}(Y, \mathsf{w})$ vary flatly. This allows us to study the moduli of $\mathbf{MF}(Y, \mathsf{w})$ by studying the deformations of the compactification (Z, f).

The main geometric result of the talk is the following

Theorem A *Let* Z *be a smooth projective variety,* $\mathsf{f} : \mathsf{Z} \to \mathbb{P}^1$ *a flat morphism, and* $D_{\mathsf{Z}} \subset \mathsf{Z}$ *a reduced anti-canonical divisor with strict normal crossings. Assume moreover that crit*(f) *does not intersect the horizontal part of* D_{Z}*, that the vertical* *part of D_{Z} coincides with the scheme theoretic fiber $\mathsf{f}^{-1}(\infty)$ of f over $\infty \in \mathbb{P}^1$. Let $\mathcal{M}$ be the versal space parametrizing deformations of (Z, f) keeping D_{Z} fixed. Then $\mathcal{M}$ is smooth.*

To prove Theorem A we identify the L_∞-algebra that controls the relevant deformation theory and show that this L_∞-algebra is homotopy abelian. We argue that, as in the case of compact Calabi-Yau manifolds, the latter statement can be reduced to a Hodge theoretic property: the double degeneration property for the Hodge-to-De Rham spectral sequence associated with the complex of f-adapted logarithmic forms. By definition a meromorphic a-form α on Z with poles at most on D_{Z} is called an f-*adapted logarithmic form* if both α and $\alpha \wedge d\mathsf{f}$ have logarithmic poles along D_{Z}. If $\Omega^a_{\mathsf{Z}}(\log D_{\mathsf{Z}}, \mathsf{f})$ denotes the sheaf of f-adapted logarithmic forms, then the double degeneration property is given by

Theorem B *Let $a \geq 0$. Under the assumptions of Theorem A, the dimesion*

$$\dim_{\mathbb{C}} \mathbb{H}^a \left(\mathsf{Z}, \left[\Omega^\bullet_{\mathsf{Z}}(\log D_{\mathsf{Z}}, \mathsf{f}), c_1 \cdot d_{DR} + c_2 \cdot d\mathsf{f} \wedge (\bullet)\right]\right)$$

is independent of the choice of $(c_1, c_2) \in \mathbb{C}^2$.

Our proof of this statement relies on the method of Deligne-Illusie [DI87] and on a topological argument for limits of logarithmic complexes. A different proof of a stronger version of the double degeneration theorem was recently given by Esnault, Sabbah, Yu, and Saito [ESY13].

The double degeneration property together with an analysis of the homological mirror correspondence predict the following

Conjecture *Let (Y, w) be an n-dimensional Landau-Ginzburg mirror of a symplectic Fano variety (X, ω_X) of complex dimension n. Suppose (Z, f) is a compactification of (Y, w). Then*

$$h^{p,n-q}(X) = \dim_{\mathbb{C}} \mathbb{H}^p \left(\mathsf{Z}, \Omega^a_{\mathsf{Z}}(\log D_{\mathsf{Z}}, \mathsf{f})\right),$$

for all p, q.

References

[DI87] P. Deligne and L. Illusie. Relèvements modulo p^2 et décomposition du complexe de de Rham. *Invent. Math.*, 89(2):247–270, 1987.

[ESY13] H. Esnault, C. Sabbah, and J.-D. Yu. 'E_1-degeneration of the irregular Hodge filtration (with an appendix by Morihiko Saito). *ArXiv e-prints*, February 2013, 1302.4537. 55 pp.

[KKP08] L. Katzarkov, M. Kontsevich, and T. Pantev. Hodge theoretic aspects of mirror symmetry. In *From Hodge theory to integrability and TQFT tt*-geometry*, volume 78 of *Proc. Sympos. Pure Math.*, pages 87–174. Amer. Math. Soc., Providence, RI, 2008.

[Kon95] M. Kontsevich. Homological algebra of mirror symmetry. In *Proceedings of the International Congress of Mathematicians, Vol. 1, 2 (Zürich, 1994)*, pages 120–139, Basel, 1995. Birkhäuser.

[Orl04] D. Orlov. Triangulated categories of singularities and D-branes in Landau-Ginzburg models. *Tr. Mat. Inst. Steklova*, 246(Algebr. Geom. Metody, Svyazi i Prilozh.):240–262, 2004.

[Orl05] D. Orlov. Triangulated categories of singularities and equivalences between Landau-Ginzburg, 2005. math.AG/0503630.

[Orl12] D. Orlov. Matrix factorizations for nonaffine LG-models. *Math. Ann.*, 353(1):95–108, 2012.

[Tia87] G. Tian. Smoothness of the universal deformation space of compact Calabi-Yau manifolds and its Petersson-Weil metric. In *Mathematical aspects of string theory (San Diego, Calif., 1986)*, volume 1 of *Adv. Ser. Math. Phys.*, pages 629–646. World Sci. Publishing, Singapore, 1987.

[Tod89] A. Todorov. The Weil-Petersson geometry of the moduli space of SU($n \geq 3$) (Calabi-Yau) manifolds. I. *Comm. Math. Phys.*, 126(2):325–346, 1989.

Matrix factorizations and cohomological field theories

Alexander Polishchuk

(joint work with Arkady Vaintrob)

In this talk I discussed the approach to the Fan-Jarvis-Ruan-Witten (*FJRW*) theory via categories of matrix factorizations developed in [3]. Recall that the FJRW theory is an analog of the Gromov-Witten theory where the role of the target space is played by a quasihomogeneous polynomial with isolated singularity. It is an example of *Cohomological Field Theory*, which consists of a *state space* H together with a collection of operations

$$\Lambda_{g,n} : H^{\otimes n} \to H^*(\overline{\mathcal{M}}_{g,n}, \mathbb{C}), \tag{1}$$

where $\overline{\mathcal{M}}_{g,n}$ is the moduli space of stable pointed curves. These operations should satisfy some gluing axioms that make use of a nondegenerate pairing on H. In the case of the Gromov-Witten theory with target X the state space is $H^*(X, \mathbb{C})$. For the FJRW theory the state space is related to the Hochschild homology space of the category of matrix factorizations.

More precisely, the FJRW theory depends on a pair (W, G), where $W(x_1, \dots, x_N)$ is a quasihomogeneous polynomial with isolated singularity and G is a finite group of diagonal symmetries of W (so G is a subgroup of $(\mathbb{C}^*)^N$). For each $\gamma \in G$ we consider the subspace of γ-invariants $\mathbb{A}^\gamma \subset \mathbb{A}^N$ and set $W_\gamma = W|_{\mathbb{A}^\gamma}$. The polynomial W_γ still has an isolated singularity and our state space is given by

$$H = \bigoplus_{\gamma \in G} H_\gamma, \quad \text{where}$$

$$H_\gamma = HH_*(\mathrm{MF}(W_\gamma)).$$

Here $\mathrm{MF}(W_\gamma)$ denotes the dg-category of matrix factorizations of W_γ, and HH_* denotes Hochschild homology.

F?r each collection of elements of G, $\gamma_1, \dots, \gamma_n$, we consider a certain finite covering $S_g(\gamma_1, \dots, \gamma_n) \to \overline{\mathcal{M}}_{g,n}$ and construct a canonical object $\mathbf{P}$ of the derived category of matrix factorizations on $S_g(\gamma_1, \dots, \gamma_n) \times \mathbb{A}^{\gamma_1} \times \dots \mathbb{A}^{\gamma_n}$ of the potential $-W_{\gamma_1} \oplus \dots \oplus W_{\gamma_n}$. Then we use $\mathbf{P}$ to construct a functor

$$\mathrm{MF}(W_{\gamma_1}) \otimes \dots \otimes \mathrm{MF}(W_{\gamma_1}) \to D(S_g(\gamma_1, \dots, \gamma_n)).$$

The map (1) is defined by passing to the Hochschild homology map induced by this functor (the original definition of [1] used a different analytic approach).

Up until recently the explicit computations of the FJRW classes were only done in the so called *concave case*, i.e., when certain line bundles on the universal curve over $S_g(\gamma_1, \dots, \gamma_n)$ have no global sections when restricted to each particular curve. In the talk I discussed the recent work of Guéré [2] where the FJRW classes were calculated in many nonconcave cases with W being *invertible* (i.e., such that the number of monomials in W is equal to the number of variables).

References

[1] H. Fan, T. Jarvis, Y. Ruan, *The Witten equation, mirror symmetry and quantum singularity theory*, Annals of Math. **178** (2013), 1–106.

[2] J. Guéré, *A Landau-Ginzburg mirror theorem without concavity*, arXiv:1307.5070.

[3] A. Polishchuk, A. Vaintrob, *Matrix factorizations and cohomological field theories*, arXiv.

Matrix factorizations, semiorthogonal decompositions, and motivic measures

Olaf M. Schnürer

(joint work with Valery A. Lunts)

Let k be an algebraically closed field of characteristic zero. The Grothendieck group $K_0(\mathrm{Var}_k)$ of varieties over k is the free abelian group on isomorphism classes $[X]$ of varieties X over k modulo the subgroup generated by the "scissor relations" $[X] - [X \setminus Y] - [Y]$ whenever Y is a closed subvariety of a variety X over k. It becomes a commutative unital ring by defining $[X] \cdot [Y] = [X \times Y]$. In order to understand this Grothendieck ring of k-varieties better one may construct **motivic measures**, i. e. morphisms of rings from $K_0(\mathrm{Var}_k)$ to some other ring.

Consider the map that sends a smooth projective k-variety X to its bounded derived category $D^b(\mathrm{Coh}(X))$ of coherent sheaves. A beautiful result due to A. Bondal, M. Larsen and V. Lunts says that this map can be turned (uniquely) into a motivic measure $K_0(\mathrm{Var}_k) \to K_0(\mathrm{sat})$ if one replaces $D^b(\mathrm{Coh}(X))$ by its "injective" enhancement (see [2]). Here $K_0(\mathrm{sat})$ denotes the Grothendieck group of saturated (= proper, smooth, and triangulated) differential $\mathbb{Z}$-graded (k-)categories with relations coming from semiorthogonal decompositions. Its ring structure is induced by the tensor product of differential $\mathbb{Z}$-graded categories (and by passing to the triangulated envelope).

Our aim is to establish a similar motivic measure using categories of matrix factorizations. Let X be a smooth quasi-projective variety over k together with a morphism $W\colon X \to \mathbb{A}^1 = \mathbb{A}^1_k$. We define the **category of singularities of** W as

$$\mathbf{MF}(W) = \prod_{a \in k} \mathbf{MF}(X, W - a).$$

Here $\mathbf{MF}(X, W - a)$ is the category of (global) matrix factorizations of $W - a$ on X. We have $\mathbf{MF}(W) = 0$ if and only if W is smooth. We denote by $\mathbf{MF}(W)^{\mathrm{dg}}$

a suitable enhancement (in the differential $\mathbb{Z}_2$-graded setting) of $\mathbf{MF}(W)$ (for example defined using injective quasi-coherent sheaves), and by $\mathbf{MF}(W)^{\mathrm{dg},\natural}$ the triangulated envelope of $\mathbf{MF}(W)^{\mathrm{dg}}$.

Consider the Grothendieck group $K_0(\mathrm{Var}_{\mathbb{A}^1})$ of varieties over $\mathbb{A}^1$ defined similarly as the group $K_0(\mathrm{Var}_k)$ above. It is turned into a commutative unital ring by defining

$$[X \xrightarrow{W} \mathbb{A}^1] \cdot [Y \xrightarrow{V} \mathbb{A}^1] := [X \times Y \xrightarrow{W*V} \mathbb{A}^1]$$

where $(W*V)(x,y) = W(x)+V(y)$. On the other hand we consider the Grothendieck ring $K_0(\mathrm{sat}_2)$ of saturated differential $\mathbb{Z}_2$-graded categories defined similarly as $K_0(\mathrm{sat})$ above. Now we can state our main theorem.

Theorem 1 (see [4])**.** There is a unique morphism

$$K_0(\mathrm{Var}_{\mathbb{A}^1}) \to K_0(\mathrm{sat}_2)$$

of rings (= a Landau-Ginzburg motivic measure) that maps $[X \xrightarrow{W} \mathbb{A}^1]$ to the class of $\mathbf{MF}(W)^{\mathrm{dg},\natural}$ whenever X is a smooth variety and $W\colon X \to \mathbb{A}^1$ is a proper morphism.

We prove first that $\mathbf{MF}(W)^{\mathrm{dg},\natural}$ is indeed saturated if X is a smooth variety and $W\colon X \to \mathbb{A}^1$ is a proper morpism. Additivity is based on an alternative description of $K_0(\mathrm{Var}_{\mathbb{A}^1})$ in terms of "blow-up relations" (see [1]) and on semiorthogonal decompositions for categories of matrix factorizations on blowing-ups and projective space bundles (see [3] and below). Multiplicativity needs a Thom-Sebastiani result for such categories of singularities and some compactification argument.

Let us explain the semiorthogonal decompositions obtained from blowing-ups in more detail. Let $\pi\colon \widetilde{X} \to X$ be the blowing-up of a smooth quasi-projective variety X along a smooth connected closed subvariety Y of codimension r. Let $j\colon E \hookrightarrow \widetilde{X}$ be the inclusion of the exceptional divisor, and let $p\colon E \to Y$ be the obvious morphism. The usual construction of the blowing-up endows $\widetilde{X}$ with a line bundle $\mathcal{O}_{\widetilde{X}}(1)$. We denote its restriction to E by $\mathcal{O}_E(1)$. Let $W\colon X \to \mathbb{A}^1$ be a morphism. Denote its pullback functions to Y and $\widetilde{X}$ by the same symbol. The following theorem is the main result of the article [3] and the analog of a well-known result for bounded derived categories of coherent sheaves.

Theorem 2 (see [3])**.** The category $\mathbf{MF}(\widetilde{X}, W)$ has the following semiorthogonal decomposition into admissible subcategories,

$$\begin{aligned}\mathbf{MF}(\widetilde{X}, W) = \big\langle & j_*(\mathcal{O}_E(-r+1) \otimes p^*(\mathbf{MF}(Y,W))), \dots, \\ & j_*(\mathcal{O}_E(-1) \otimes p^*(\mathbf{MF}(Y,W))), \pi^*(\mathbf{MF}(X,W))\big\rangle.\end{aligned}$$

In our talk we also discussed the relation between the motivic measure from [2] and the Landau-Ginzburg motivic measure from Theorem 1. For more details and our future plans we refer the reader to the articles [3, 4].

References

[1] Franziska Bittner, *The universal Euler characteristic for varieties of characteristic zero*, Compos. Math. **140** (2004), no. 4, 1011–1032.
[2] Alexey I. Bondal, Michael Larsen, and Valery A. Lunts, *Grothendieck ring of pretriangulated categories*, Int. Math. Res. Not. (2004), no. 29, 1461–1495.
[3] Valery A. Lunts and Olaf M. Schnürer, *Matrix-factorizations and semi-orthogonal decompositions for blowing-ups*, Preprint (2012), arXiv:1212.2670.
[4] Valery A. Lunts and Olaf M. Schnürer, *Matrix factorizations and motivic measures*, in preparation.

Matrix factorizations and homological projective duality in physics

Eric Sharpe

(joint work with Tony Pantev, others)

'Gauged linear sigma models' (GLSM's) are one of the central tools used by physicists to describe strings propagating on spaces. They were originally developed about twenty years ago by E. Witten [1], but have recently undergone a revolution. For example, prior to around 2007, it was believed that gauged linear sigma models

- could only describe geometries presented as global complete intersections,
- in which those geometries were realized as the critical locus of a 'superpotential,'
- and any two geometries related by a GLSM were necessarily birational.

Over the last few years, counterexamples to all of these claims have been found (see *e.g.* [2, 3, 4] for some early work), and the more subtle ideas replacing them revolve around aspects of Kuznetsov's homological projective duality [5, 6, 7].

In this talk we will give a basic introduction to some of these phenomena and their consequences, largely following [4]. Instead of working with GLSM's, we will instead translate to 'Landau-Ginzburg (LG) models,' which are defined by a complex Kähler manifold X together with a holomorphic function $W : X \to \mathbb{C}$ known as the superpotential. Another set of theories, known as 'nonlinear sigma models' (NLSM's), are defined just by specifying just a complex Kähler manifold, without a superpotential. String propagation on a space is described by a nonlinear sigma model. Given a Landau-Ginzburg model, we can sometimes (though not always) construct a nonlinear sigma model by an operation called 'renormalization group flow,' which generates an effective theory describing just the low-energy fluctuations of the Landau-Ginzburg model.

As a warm-up, let us describe a Landau-Ginzburg model associated to a quintic Calabi-Yau hypersurface in $\mathbb{P}^4$. The Landau-Ginzburg model is defined on

$$\mathrm{Tot}\left(\mathcal{O}(-5) \xrightarrow{\pi} \mathbb{P}^4\right),$$

with superpotential $W = p\pi^* s$, $s \in \Gamma(\mathcal{O}(5))$, p a fiber coordinate. This theory contains a potential V of the form

$$V \;=\; |dW|^2 \;=\; |s|^2 \;+\; |pds|^2,$$

and for a smooth hypersurface, the zero-energy locus ($V = 0$) is $\{p = 0\} \cap \{s = 0\}$. We say that this theory "renormalization-group flows" to a nonlinear sigma model on $\{s = 0\} \subset \mathbb{P}^4$, that describes its low energy behavior. This ("perturbative") analysis has been the standard technique for two decades.

If instead we perform a birational transformation on the space underlying the Landau-Ginzburg model, we get a Landau-Ginzburg model on

$$\mathrm{Tot}\ \left(\mathcal{O}(-1)^5 \longrightarrow B\mathbb{Z}_5 \right) \ = \ [\mathbb{C}^5/\mathbb{Z}_5],$$

with the same superpotential. The 'Landau-Ginzburg/Calabi-Yau' relationship often cited in the literature relates this Landau-Ginzburg model, on $[\mathbb{C}^5/\mathbb{Z}_5]$, to the previously-described nonlinear sigma model on $\{s = 0\} \subset \mathbb{P}^4$.

Schematically, we can outline the relations between these theories in the following diagram:

$$\begin{array}{ccc}
\text{LG model on Tot}\left(\mathcal{O}(-5) \xrightarrow{\pi} \mathbb{P}^4 \right) & \dashrightarrow & \text{LG model on } [\mathbb{C}^5/\mathbb{Z}_5] \\
\Big\downarrow{\scriptstyle \mathrm{RG}} & & \Big\| \\
\text{NLSM on } \{s = 0\} \subset \mathbb{P}^4 & \dashrightarrow & \text{LG model on } [\mathbb{C}^5/\mathbb{Z}_5].
\end{array}$$

Now, let us consider a different case, another warm-up. Consider a Landau-Ginzburg model on

$$\mathrm{Tot}\ \left(\mathcal{O}(-2)^2 \xrightarrow{\pi} \mathbb{P}^3 \right),$$

with superpotential $W = p_1 \pi^* Q_1 + p_2 \pi^* Q_2$, where the p_a are fiber coordinates and $Q_a \in \Gamma(\mathcal{O}(2))$. The same perturbative analysis as above yields that this theory at low energies is described by a nonlinear sigma model on a complete intersection of two quadrics in $\mathbb{P}^3$, which is to say, an elliptic curve.

Now, consider the Landau-Ginzburg model on the birational space

$$\mathrm{Tot}\ \left(\mathcal{O}(-1)^4 \longrightarrow \mathbb{P}^1_{[2,2]} \right),$$

with the same superpotential, which is now usefully rewritten in the form

$$W \ = \ \sum_{i,j} \phi_i \phi_j A^{ij}(p_a),$$

where ϕ_i are fiber coordinates and p_a homogeneous coordinates on $\mathbb{P}^1_{[2,2]}$.

The superpotential above appears to define a 'mass' term for the ϕ_i, which to a physicist means naively that the renormalization group would remove them, leading to a nonlinear sigma model on $\mathbb{P}^1$, which cannot be correct, because (for more subtle physics reasons) the result needs to be a Calabi-Yau.

Instead, we utilize the fact that $\mathbb{P}^1_{[2,2]}$ has a $\mathbb{Z}_2$ gerbe structure, so everywhere that the ϕ_i's are massive, *i.e.* $\{\det A \neq 0\}$, physics sees a double cover, applying the 'decomposition conjecture' described in [8] for strings on gerbes. Putting this together, we have a branched double cover of $\mathbb{P}^1$, branched over the degree four locus $\{\det A = 0\}$, which is another elliptic curve.

If we start with a complete intersection of three quadrics in $\mathbb{P}^5$, then proceeding in the same fashion, we are led to a pair of K3 surfaces, one the complete intersection of quadrics, the other a branched double cover of $\mathbb{P}^2$.

Proceeding to a complete intersection of four quadrics in $\mathbb{P}^7$, we find something more interesting. Naively the same analysis would, in the birational model, lead to a branched double cover of $\mathbb{P}^3$, branched over a degree 8 locus given as $\{\det A = 0\}$ for a symmetric 8×8 matrix A. However, there is a subtlety involving mismatched singularities. Mathematically, the branched double cover above has singularities at solutions of

$$\det A \,=\, 0, \quad d \det A \,=\, 0,$$

whereas the physical theory can be shown to have singularities at points where there exists a vector v which is simultaneously a null eigenvector of both A and dA. A singularity in the physics implies a singularity in the mathematics, but not conversely.

The fix in the physical interpretation involves understanding matrix factorizations in the Landau-Ginzburg model. Working locally on $\mathbb{P}^3$, the superpotential over any point is quadratic, hence matrix factorizations form a module over a sheaf of Clifford algebras [9], in fact a sheaf of even parts of Clifford algebras, defined by the symmetric matrix A. Briefly, this is the defining property of the noncommutative resolution of the branched double cover described in [6], so we interpret this physical theory as describing a string on a noncommutative resolution.

The examples outlined here are the prototypes for a number of examples, both Calabi-Yau and non-Calabi-Yau, appearing in GLSM's. All such examples, relating various complete intersections of quadrics to (noncommutative resolutions of) branched double covers, are examples of homological projective duality.

References

[1] E. Witten, "Phases of N=2 theories in two dimensions," Nucl. Phys. **B403** (1993) 159-222, `hep-th/9301042`.

[2] K. Hori, D. Tong, "Aspects of non-abelian gauge dynamics in two-dimensional N=(2,2) theories," JHEP 0705 (2007) 079, `hep-th/0609032`.

[3] R. Donagi, E. Sharpe, "GLSMs for partial flag manifolds," J. Geom. Phys. **58** (2008) 1662-1692, `arXiv: 0704.1761`.

[4] A. Caldararu, J. Distler, S. Hellerman, T. Pantev, and E. Sharpe, "Non-birational twisted derived equivalences in abelian GLSMs," Comm. Math. Phys. **294** (2010) 605-645, `arXiv: 0709.3855`.

[5] A. Kuznetsov, "Homological projective duality," `math.AG/0507292`.

[6] A. Kuznetsov, "Derived categories of quadric fibrations and intersections of quadrics," `math.AG/0510670`.

[7] A. Kuznetsov, "Homological projective duality for Grassmannians of lines," `math.AG/0610957`.

[8] S. Hellerman, A. Henriques, T. Pantev, E. Sharpe, M. Ando, "Cluster decomposition, T-duality, and gerby CFT's," Adv. Theor. Math. Phys. **11** (2007) 751-818, `hep-th/0606034`.

[9] A. Kapustin, Y. Li, "D-branes in Landau-Ginzburg models and algebraic geometry," `hep-th/0210296`.

Autoequivalences arising from variation of GIT quotient

Ian Shipman
(joint work with Daniel Halpern-Leistner)

Homological mirror symmetry predicts, in certain cases, that the bounded derived category of coherent sheaves on an algebraic variety should admit *twist autoequivalences* corresponding to a spherical object [8]. The autoequivalences predicted by mirror symmetry have been widely studied, and the notion of a spherical object has been generalized to the notion of a spherical functor [1]. We apply recently developed techniques for studying the derived category of a geometric invariant theory (GIT) quotient [2, 3, 4, 5, 7] to the construction of autoequivalences, and our investigation leads to general connections between the theory of spherical functors and the theory of semiorthogonal decompositions and mutations.

We consider an algebraic stack which arises as a GIT quotient of a smooth quasiprojective variety X by a reductive group G. By varying the G-ample line bundle used to define the semistable locus, one gets a birational transformation $X^{ss}_-/G \dashrightarrow X^{ss}_+/G$ called a variation of GIT quotient (VGIT). We study a simple type of VGIT, which we call a *balanced wall crossing.*

Under a hypothesis on ω_X, a balanced wall crossing gives rise to an equivalance $\psi_w : \mathbf{D}^b(X^{ss}_-/G) \to \mathbf{D}^b(X^{ss}_+/G)$ which depends on a choice of $w \in \mathbb{Z}$, and the composition $\Phi_w := \psi^{-1}_{w+1}\psi_w$ defines an autoequivalence of $\mathbf{D}^b(X^{ss}_-/G)$. Autoequivalences of this kind have been studied recently under the name window-shifts [3, 7]. We generalize the observations of those papers in showing that Φ_w is always a spherical twist.

Recall that if B is an object in a dg-category, then we can define the twist functor

$$T_B : F \mapsto \mathrm{Cone}(\mathrm{Hom}^\bullet(B, F) \otimes_{\mathbb{C}} B \to F)$$

If B is a spherical object, then T_B is by definition the spherical twist autoequivalence defined by B. More generally, if $S : \mathcal{A} \to \mathcal{B}$ is a spherical functor, then one can define a twist autoequivalence $T_S := \mathrm{Cone}(S \circ S^R \to \mathrm{id}_{\mathcal{B}})$ of $\mathcal{B}$, where S^R denotes the right adjoint. We refer to a twist autoequivalence corresponding to a spherical functor simply as a "spherical twist." A spherical object corresponds to the case where $\mathcal{A} = \mathbf{D}^b(k - \mathrm{vect})$.

It was noticed immediately [8] that if B were instead an exceptional object, then T_B is the formula for the left mutation equivalence ${}^\perp B \to B^\perp$ coming from a pair of semiorthogonal decompositions $\langle B^\perp, B\rangle = \langle B, {}^\perp B\rangle$. In fact, we will show that there is more than a formal relationship between spherical functors and mutations. If $\mathcal{C}$ is a pre-triangulated dg category, then the braid group on n-strands acts by left and right mutation on the set of length n semiorthogonal decompositions $\mathcal{C} = \langle \mathcal{A}_1, \ldots, \mathcal{A}_n\rangle$ with each $\mathcal{A}_i$ admissible. Mutating by a braid gives equivalences $\mathcal{A}_i \to \mathcal{A}'_{\sigma(i)}$, where σ is the permutation that the braid induces on end points. In particular if one of the semiorthogonal factors is the same subcategory before and after the mutation, one gets an autoequivalence $\mathcal{A}_i \to \mathcal{A}_i$.

Theorem 1 (spherical twist=mutation=window shifts)**.** If $\mathcal{C}$ is a pre-triangulated dg category admitting a semiorthogonal decomposition $\mathcal{C} = \langle \mathcal{A}, \mathcal{G} \rangle$ which is fixed by the 4-twist braid (acting by mutations):

$$\langle \mathcal{A}, \mathcal{G} \rangle = \langle \mathcal{G}, \mathcal{A}' \rangle = \langle \mathcal{A}', \mathcal{G}' \rangle = \langle \mathcal{G}', \mathcal{A} \rangle$$

then the autoequivalence of $\mathcal{G}$ induced by mutation is the twist T_S corresponding to a spherical functor $S : \mathcal{A} \to \mathcal{G}$. Conversely, if $S : \mathcal{A} \to \mathcal{B}$ is a spherical functor, then there is a larger category $\mathcal{C}$ admitting a semiorthogonal decomposition fixed by this braid which recovers S and T_S.

In the context of a balanced GIT wall crossing, the category $\mathcal{C}$ arises naturally as a subcategory of the equivariant category $\mathbf{D}^b(X/G)$, defined in terms of "grade restriction rules". The resulting autoequivalence agrees with the window shift Φ_w and corresponds to a spherical functor $f_w : \mathbf{D}^b(Z/L)_w \to \mathbf{D}^b(X^{ss}_-/G)$, where Z/L is the "critical locus" of the VGIT, which is unstable in both quotients.

Next, we revisit the prediction of derived autoequivalences from mirror symmetry. Spherical twist autoequivalences of $\mathbf{D}^b(V)$ for a Calabi-Yau V correspond to loops in the moduli space of complex structures on the mirror Calabi-Yau $V^\vee$, and flops correspond, under the mirror map, to certain paths in that complex moduli space. We review these predictions, first studied in [6] for toric varieties, and formulate corresponding predictions for flops coming from VGIT in which an explicit mirror may not be known.

By studying toric flops between toric Calabi-Yau varieties of Picard rank 2, we find that mirror symmetry predicts more autoequivalences than constructed in Theorem 1. The expected number of autoequivalences agrees with the length of a full exceptional collection on the critical locus Z/L of the VGIT. Motivated by this observation, we introduce a notion of "fractional grade restriction windows" given the data of a semiorthogonal decomposition on the critical locus. This leads to

Theorem 2 (Factoring spherical twists)**.** Given a full exceptional collection

$$\mathbf{D}^b(Z/L)_w = \langle E_0, \ldots, E_N \rangle,$$

the objects $S_i := f_w(E_i) \in \mathbf{D}^b(X^{ss}_-/G)$ are spherical, and

$$\Phi_w = T_{S_0} \circ \cdots T_{S_n}.$$

This is a general phenomenon as well. Let $S = \mathcal{E} \to \mathcal{G}$ be a spherical functor of dg-categories and let $\mathcal{E} = \langle \mathcal{A}, \mathcal{B} \rangle$ be a semiorthogonal decomposition such that there is also a semiorthogonal decomposition $\mathcal{E} = \langle F_S(\mathcal{B}), \mathcal{A} \rangle$, where F_S is the cotwist autoequivalence of $\mathcal{E}$ induced by S. Then the restrictions $S_\mathcal{A} : \mathcal{A} \to \mathcal{G}$ and $S_\mathcal{B} : \mathcal{B} \to \mathcal{G}$ are spherical as well, and $T_S \simeq T_{S_\mathcal{A}} \circ T_{S_\mathcal{B}}$.

References

[1] R. Anno, T. Logvinenko, *Spherical DG-functors*, `arXiv:1309.5035 [math.AG]` (2013).

[2] M. Ballard, D. Favero, L. Katzarkov, *Variation of geometric invariant theory quotients and derived categories*, `arXiv:1203.6643 [math.AG]`, (2012).

[3] W. Donovan, E. Segal, *Window shifts, flop equivalences and Grassmannian twists*, arXiv:1206.0219 [math.AG], (2012).
[4] D. Halpern-Leistner, *The derived category of a GIT quotient*, arXiv:1203.0276 [math.AG], (2012).
[5] M. Herbst, K. Hori, D. Page, *B-type D-branes in toric Calabi-Yau varieties*, Homological Mirror Symmetry, Lecture Notes in Phys. **757** (2009), 27–44.
[6] R. P. Horja, *Derived category automorphisms from mirror symmetry*, Duke Math. J. **127** (2005), no. 1, 1–34.
[7] E. Segal, *Equivalences between GIT quotients of Landau-Ginzburg B-models*, Comm. Math. Phys. **304** (2011), no. 2, 411-432.
[8] R. Thomas, P. Seidel, *Braid group actions on derived categories of coherent sheaves.*, Duke Math. J. **108** (2001), no. 1, 37–108.

Algebra and Geometry of LG Orbifolds for Invertible Polynomials in Three Variables

Atsushi Takahashi

We report our recent study on the mirror symmetry of Landau–Ginzburg orbifolds for invertible polynomials in three variables. In this abstract, after recalling some notations and terminologies, we shall list some of our results.

Let $f(x_1,\dots,x_n)$ be a weighted homogeneous complex polynomial. This means that there are positive integers $w_1,\dots,w_n$ and d such that $f(\lambda^{w_1}x_1,\dots,\lambda^{w_n}x_n) = \lambda^d f(x_1,\dots,x_n)$ for $\lambda\in\mathbb{C}^*$. We call $(w_1,\dots,w_n;d)$ a system of *weights*.

Definition 1. A weighted homogeneous polynomial $f = f(x_1,\dots x_n)$ which defines an isolated singularity at the origin in $\mathbb{C}^n$ is called *invertible* if the number of variables coincides with the number of monomials in the polynomial f, namely,

$$f(x_1,\dots,x_n)=\sum_{i=1}^n a_i\prod_{j=1}^n x_j^{E_{ij}},\quad a_i\in\mathbb{C}^*,\quad E_{ij}\in\mathbb{Z}_{\geq 0},\ i,j=1,\dots,n,$$

and if the matrix $E := (E_{ij})$ is invertible over $\mathbb{Q}$.

It is useful to consider the *canonical system of weights*, which is the unique system of weights such that $d = \det(E)$. Set $c_f := \gcd(w_1,\dots,w_n,d)$ and $\epsilon_f := \left(\left(\sum_{i=1}^n w_i\right)-d\right)/c_f$.

Definition 2. The *maximal grading* L_f of the invertible polynomial f is the abelian group generated by the symbols $\vec{x_i}$ for the variables x_i for $i=1,\dots,n$ and the symbol $\vec{f}$ for the polynomial f defined by the quotient

$$L_f := \left(\bigoplus_{i=1}^n \mathbb{Z}\vec{x_i}\oplus\mathbb{Z}\vec{f}\right)\Bigg/\left(\vec{f}-\sum_{j=1}^n E_{ij}\vec{x_j};1\leq i\leq n\right).$$

The *maximal abelian symmetry group* G_f of f is a finite abelian group defined by

$$G_f := \left\{(\lambda_1,\dots,\lambda_n)\in(\mathbb{C}^*)^n\ \middle|\ \prod_{j=1}^n\lambda_j^{E_{1j}}=\cdots=\prod_{j=1}^n\lambda_j^{E_{nj}}=1\right\}.$$

Note that the polynomial f is invariant under the natural action of G_f on the variables. Namely, we have

$$f(\lambda_1 x_1, \dots, \lambda_n x_n) = f(x_1, \dots, x_n), \quad (\lambda_1, \dots, \lambda_n) \in G_f.$$

and the short exact sequence

$$\{1\} \longrightarrow G_f \longrightarrow \mathrm{Spec}(\mathbb{C}L_f)(\mathbb{C}) \longrightarrow \mathbb{C}^* \longrightarrow \{1\}.$$

It is important that G_f always contains the exponential grading operator

$$g_0 := (\mathbf{e}[q_1], \dots, \mathbf{e}[q_n]), \quad \mathbf{e}[-] := e^{2\pi\sqrt{-1}\cdot -}, \quad q_i := \frac{w_i}{d},\ i = 1, \dots, n.$$

Denote by G_0 the subgroup of G_f generated by g_0.

Definition 3. The *Berglund–Hübsch transpose* f^T is an invertible polynomial defined by the transpose E^T of the matrix E, namely,

$$f^T(x_1, \dots, x_n) = \sum_{i=1}^n a_i \prod_{j=1}^n x_j^{E_{ji}}.$$

For a subgroup $G \subset G_f$, the *Berglund–Henningson dual group* G^T is defined by

$$G^T := \mathrm{Hom}(G_f/G, \mathbb{C}^*).$$

From now on, set $n = 3$ and assume that G contains G_0, which is equivalent to that G^T is a subgroup of $\mathrm{SL}_3(\mathbb{C}) \cap G_{f^T}$.

Aim: Understand the *Mirror Symmetry* of pairs (f, G).

- Compare *algebraic* objects associated to (f, G) with *geometric* objects associated to (f^T, G^T).
- Describe these objects *combinatorially* in terms of E and G.

First we consider the algebraic aspects. Under the assumption $G \supset G_0$, we have an abelian group L which is a quotient of L_f. Set $S := \mathbb{C}[x_1, x_2, x_3]$ and $R_f := S/(f)$. Note that S and R_f are naturally L-graded.

Consider the stack $\mathcal{C}_{(f,G)} := [(\mathrm{Spec}(R_f)\backslash\{0\})/\mathrm{Spec}(\mathbb{C}L)]$ whose underlying curve is smooth since f has an isolated singularity only at the origin. The genus of the curve is denoted by $g_{(f,G)}$. The orders $A_{(f,G)} := (a_1, \dots, a_r)$ of the isotropy groups of the orbifold points of $\mathcal{C}_{(f,G)}$ is called the *Dolgachev numbers*.

We have the following results on $\mathcal{C}_{(f,G)}$ and the Dolgachev numbers $A_{(f,G)}$:

Theorem 4 ([1]). There exists a triple $A' = (a'_1, a'_2, a'_3)$ of positive integers and an isomorphism of stacks $\mathcal{C}_{(f,G_f)} \cong \mathbb{P}^1_{A'}$ where $\mathbb{P}^1_{A'}$ denotes the Geigle–Lenzing weighted projective line of type A'. In particular, we have $A_{(f,G_f)} = A'$. □

Theorem 5 ([2]). Let $H_i \subset G_f$ be the minimal subgroup containing G and the isotropy group of the point p_i, $i = 1, 2, 3$ on $\mathcal{C}_{(f,G_f)}$. Then we have the following formula for the Dolgachev numbers:

$$A_{(f,G)} = \left(\frac{a'_i}{|H_i/G|} * |G_f/H_i|, i = 1, 2, 3 \right).$$

□

We also consider L-graded matrix factorizations. Denote by $\mathrm{HMF}^L_S(f)$ the category of L-graded matrix factorizations of f. We have the following results:

Theorem 6 ([6]). $\mathrm{HMF}^{L_f}_S(f)$ admits a full strongly exceptional collection. □

Theorem 7 ([7, 10, 4]). If $\varepsilon_f > 0$, then $\mathrm{HMF}^{L}_S(f)$ admits a full strongly exceptional collection. □

Theorem 8 ([5, 9]). Suppose that $\varepsilon_f = 0$ and $g_{(f,G)} = 0$. We have an isomorphism of stacks $\mathcal{C}_{(f,G)} \cong \mathbb{P}^1_{A,\Lambda}$ for some A and Λ where $\mathbb{P}^1_{A,\Lambda}$ denotes the Geigle–Lenzing weighted projective line of type (A, Λ). □

By the L-graded version of Orlov's theorem, Theorem 8 implies the following corollary, which is also proven by [8] without using Orlov's theorem.

Corollary 9. Suppose that $\varepsilon_f = 0$ and $g_{(f,G)} = 0$. Then $\mathrm{HMF}^{L}_S(f)$ admits a full strongly exceptional collection. □

Next we consider the geometric aspects. Consider f^T as a holomorphic map. Since G^T is a subgroup of $SL_3(\mathbb{C})$ under which f^T is invariant, we obtain the holomorphic map $\widetilde{f}^T : \widetilde{\mathbb{C}^3/G} \longrightarrow \mathbb{C}$, where $\widetilde{\mathbb{C}^3/G}$ is a crepant resolution of $\mathbb{C}^3/G$ (e.g. G-Hilb$(\mathbb{C}^3)$). What we really want to study is the relative homology group $H_i(\widetilde{\mathbb{C}^3/G}, (\widetilde{f}^T)^{-1}(1); \mathbb{Q})$, however, it is so difficult in general. Instead, we take the holomorphic map $\widehat{f} := \widetilde{f}^T - cx_1x_2x_3 : \widetilde{\mathbb{C}^3/G} \longrightarrow \mathbb{C}$, $c \gg 0$, and study the relative homology group $H_i := H_i(\widetilde{\mathbb{C}^3/G^T}, \widehat{f}^{-1}(1); \mathbb{Q})$. We have the following results. We omit important details due to lack of space. See references for precise statements.

Theorem 10 ([1]). By a "suitable" change of coordinates z_1, z_2, z_3, we have

$$\widehat{f} = z_1^{\gamma'_1} + z_2^{\gamma'_2} + z_2^{\gamma'_3} - c'z_1z_2z_3,\ c' \gg 0,$$

for positive integers $\gamma'_1, \gamma'_2, \gamma'_3$ given explicitly in terms of E^T. □

Theorem 11 ([3]). By the McKay correspondence, we have

$$\dim_{\mathbb{Q}} H_2 = \dim_{\mathbb{Q}} H_4 = j_{G^T},$$

where $j_{G^T} := \#\{g \in G^T \mid \mathrm{age}(g) = 1,\ \mathrm{Fix}(g) = \{0\}\}$. □

Since the map $H_3 \longrightarrow H_2(\widehat{f}^{-1}(1); \mathbb{Q})$ is injective, H_3 has an intersection form.

Theorem 12 ([3]). There is a subset $\mathcal{B}$ of H_3 consisting of vanishing classes which represents a $\mathbb{Q}$-basis of $H_3/\langle\delta_0\rangle$ whose intersection numbers are given by the "star Coxeter-Dynkin diagram" where δ_0 is a cycle in the radical. □

The lengths of arms of the star Coxeter-Dynkin diagram for (f^T, G^T) is called the *Gabrielov numbers* for (f^T, G^T) and is denoted by $\Gamma_{(f^T,G^T)}$. It is given by

$$\Gamma_{(f^T,G^T)} := \left(\frac{\gamma'_i}{|G^T/K_i|} * |K_i|, i = 1, 2, 3 \right),$$

where K_i denotes the maximal subgroup of G^T fixing the i-th coordinate z_i. The duality between H_i and K_i yields a generalization of Arnold's strange duality:

Theorem 13 ([2]). We have $g_{(f,G)} = j_{G^T}$ and $A_{(f,G)} = \Gamma_{(f^T,G^T)}$. □

References

[1] W. Ebeling, A. Takahashi, *Strange duality of weighted homogeneous polynomials*, Compositio Math. **147** (2011), 1413 – 1433.

[2] W. Ebeling, A. Takahashi, *Mirror Symmetry between Orbifold curves and Cusp Singularities with Group action*, Internat. Math. Res. Notices **2013(10)** (2013), 2240–2270.

[3] W. Ebeling, A. Takahashi, *A geometric definition of Gabrielov numbers*, arXiv:1305.6268.
[4] M. Futaki and K. Ueda *Homological mirror symmetry for Brieskorn-Pham singularities*, Sel. Math. New Ser. **17** (2011), 435–452.
[5] W. Geigle and H. Lenzing, *A class of weighted projective curves arising in representation theory of finite-dimensional algebras*, Singularities, representation of algebras, and vector bundles (Lambrecht, 1985), 9–34, Lecture Notes in Math. **1273**, Springer, Berlin, 1987.
[6] Y. Hirano, A. Takahashi, in preparation.
[7] H. Kajiura, K. Saito, A. Takahashi, *Matrix Factorizations and Representations of Quivers II*, Advances in Mathematics **211** (2007), 327–362.
[8] H. Oda, K. Saito, A. Takahashi, in preparation.
[9] A. Takahashi, unpublished note.
[10] K. Ueda, *Homological Mirror Symmetry and Simple Elliptic Singularities*, arXiv:0604361.

Dimer models and matrix factorizations

Kazushi Ueda
(joint work with Akira Ishii)

A dimer model is a bicolored graph on a real 2-torus which encodes the information of a quiver with relations. One can show that for any smooth quasi-projective toric Calabi-Yau 3-fold Y, there is a dimer model G such that

- the moduli space $\mathcal{M}_\theta$ of θ-stable representations of the quiver Γ with relations associated with G of dimension vector $(1, \dots, 1)$ is isomorphic to Y for a suitable choice of a stability parameter θ, and
- the direct sum $\mathcal{E} = \bigoplus_v \mathcal{E}_v$ of the tautological bundles is a tilting object whose endomorphism algebra is isomorphic to the path algebra $\mathbb{C}\Gamma$ of the quiver Γ with relations.

This gives a combinatorial description of the derived category of toric Calabi-Yau 3-fold;

$$D^b \operatorname{coh} Y \cong D^b \operatorname{mod} \mathbb{C}\Gamma.$$

See e.g. [1] and references therein for the proof of these facts.

Let Y_0 be the union of all the toric divisors of Y, and $W \in H^0(\mathcal{O}_\mathcal{M})$ be the defining function of Y_0, which can also be considered as a central element of $\operatorname{End} \mathcal{E} \cong \mathbb{C}\Gamma$. The restriction $\mathcal{E}|_{Y_0}$ is a tilting object in $D^b \operatorname{coh} Y_0$, and one has an isomorphism $\operatorname{End}(\mathcal{E}|_{Y_0}) \cong \mathbb{C}\Gamma/(W)$ of algebras [2]. This gives an equivalence

$$D^b \operatorname{coh} Y_0 \cong D^b \operatorname{mod} \mathbb{C}\Gamma/(W),$$

of derived categories, which in turn induces an equivalence

$$D^b_{\mathrm{sing}}(\operatorname{coh} Y_0) \cong D^b_{\mathrm{sing}}(\operatorname{mod} \mathbb{C}\Gamma/(W)). \tag{1}$$

of singularity categories. The left hand side of (1) is equivalent to the the triangulated category of non-affine matrix factorizations of W on Y, whereas the right hand side is equivalent to the triangulated category of non-commutative matrix factorizations of W over $\mathbb{C}\Gamma$.

References

[1] Akira Ishii and Kazushi Ueda, *Dimer models and the special McKay correspondence*, arXiv:0905.0059.

[2] Akira Ishii and Kazushi Ueda, *Dimer models and exceptional collections*, arXiv:0911.4529.

On the vanishing of Hochster's theta invariant

Mark E. Walker

Let R be an isolated hyper-surface singularity. That is, $R = (R, \mathfrak{m})$ is a local ring that can be written as $R = Q/f$, where Q is a regular local ring and f is a non-zero-divisor, and $R_\mathfrak{p}$ is regular for all $\mathfrak{p} \neq \mathfrak{m}$. These assumptions ensure that given finitely generated R-modules M and N, we have

$$\mathrm{Tor}_i^R(M,N) \cong \mathrm{Tor}_{i+2}^R(M,N), \text{ for } i \gg 0$$

and $\mathrm{Tor}_i^R(M,N)$ has finite length, for $i \gg 0$. We define *Hochster's theta invariant* of M and N to be

$$\theta^R(M,N) = \mathrm{length}_R \mathrm{Tor}_{2i}^R(M,N) - \mathrm{length}_R \mathrm{Tor}_{2i+1}^R(M,N), \text{ for } i \gg 0.$$

The theta invariant is closely related to the Euler characteristic $\chi(-,-)$ for the category of matrix factorizations for (Q,f). Namely, if $\mathbb{E} = (E_1 \xrightarrow{\alpha} E_0 \xrightarrow{\beta} E_1)$ is a matrix factorization for (Q,f), then $\mathrm{coker}(\mathbb{E}) := \mathrm{coker}(E_1 \xrightarrow{\alpha} E_0)$ is a maximum Cohen-Macaulay (MCM) R-module, and we have

$$\chi(\mathbb{E}^*, \mathbb{E}') = \theta^R(\mathrm{coker}(\mathbb{E}), \mathrm{coker}(\mathbb{E}')).$$

(Here $\mathbb{E}^*$ denotes the dual matrix factorization.)

In recent work, Buchweitz and van Straten [1] relate Hochster's theta invariant for isolated hyper-surface singularities of the form $\mathbb{C}\{x_0, \dots, x_n\}/f$, where $\mathbb{C}\{x_0, \dots, x_n\}$ denotes the ring of convergent power series, to the linking form on the link of the singularity. In particular, they prove θ^R is identically 0 if n is even. Part of their proof relies on the fact that the Milnor fiber of the singularity has the homotopy type of a bouquet of n-spheres and hence, if n is even, its odd degree cohomology vanishes.

In this talk, I present a purely algebraic version of some of the results of Buchweitz and van Straten. In particular, I prove θ^R is identically 0 for a large class of hypersurfaces of even dimension, confirming a conjecture of Hailong Dao [2]. My proof relies on the following algebraic analogues of some of the standard notions used in the analytic study of singularities in characteristic 0:

Assumptions 1. For the rest of this document, we adopt the following notations and assumptions:

- V is a Henselian dvr with algebraically closed residue field k, field of fractions F and uniformizing parameter t — this is the analogue of a small disk D in the complex plane.

- $\mathrm{Spec}(\tilde{Q}) \to \mathrm{Spec}(V)$ is a flat morphism of finite type of relative dimension n that is smooth away from a specified closed point $\mathfrak{m}$ of $\mathrm{Spec}(\tilde{Q})$. We also assume this map is a local complete intersection near $\mathfrak{m}$.
- $Q = \tilde{Q}^{hen}_{\mathfrak{m}}$ is the Henselization of $\tilde{Q}$ at $\mathfrak{m}$. The map $\mathrm{Spec}(Q) \to \mathrm{Spec}(V)$ is the algebraic analogue of a good representation $f : X \to D$ of an isolated singularity. We let $f \in Q$ be the image of the uniformizing parameter $t \in V$ under this map.
- The generic fiber $\mathrm{Spec}(Q[\frac{1}{f}]) = \mathrm{Spec}(Q \otimes_V F) \to \mathrm{Spec}(F)$ is the algebraic analogue of the Milnor fibration.
- The geometric generic fiber of this map, namely $\mathrm{Spec}(Q \otimes_V \overline{F})$, is the algebraic analogue of the Milnor fiber.
- Let $R = Q/f$, so that $\mathrm{Spec}(R)$ is the closed fiber. Its punctured spectrum $\mathrm{Spec}(R) \setminus \mathfrak{m}$ is the algebriac analogue of the link of the singularity.

We associated to an MCM R-module M a pair of classes in K-theory. The first is easy to describe: the coherent sheaf on $\mathrm{Spec}(R) \setminus \mathfrak{m}$ determined by such an M is locally free and hence determines a class $[M]_{K_0} \in K_0(\mathrm{Spec}(R) \setminus \mathfrak{m})$. Since Q and R are local, an MCM R-module is the cokernel of a matrix factorization of the form $Q^r \xrightarrow{A} Q^r \xrightarrow{B} Q^r$. The matrix A becomes invertible in $Q[\frac{1}{f}]$ and hence determines a class in $K_1(Q[\frac{1}{f}])$. Let $[M]_{K_1} = [A] \in K_1(Q[\frac{1}{f}])$.

Let ℓ be a prime distinct from $\mathrm{char}(k)$ and assume $\ell \geq 5$ (to avoid complications in the multiplication rules for K-theory with coefficients). The classes $[M]_{K_0}$ and $[M]_{K_1}$ also determine classes in K-theory with $\mathbb{Z}/\ell$ coefficients, and we use the same notation for them:

$$[M]_{K_0} \in K_0(\mathrm{Spec}(R) \setminus \mathfrak{m}, \mathbb{Z}/\ell) \text{ and } [M]_{K_1} \in K_1(\mathrm{Spec}(Q[\tfrac{1}{f}]), \mathbb{Z}/\ell).$$

Since the map $\mathrm{Spec}(Q) \setminus \mathfrak{m} \to \mathrm{Spec}(V)$ is smooth with generic fiber $\mathrm{Spec}(Q[\frac{1}{f}])$ and closed fiber $\mathrm{Spec}(R) \setminus \mathfrak{m}$, we have a *specialization map* in K-theory with finite coefficients

$$\sigma : K_1(\mathrm{Spec}(Q[\tfrac{1}{f}]), \mathbb{Z}/\ell) \to K_1(\mathrm{Spec}(R) \setminus \mathfrak{m}, \mathbb{Z}/\ell).$$

Explicitly, σ is given by

$$\sigma(\gamma) = \partial(\gamma \cup [f]),$$

where $[f] \in K_1(Q[\frac{1}{f}])$ is the class determined by the unit f, $\cup$ is the product rule for the ring $K_*(Q[\frac{1}{f}], \mathbb{Z}/\ell)$, and $\partial : K_2(\mathrm{Spec}(Q[\frac{1}{f}]), \mathbb{Z}/\ell) \to K_1(\mathrm{Spec}(R) \setminus \mathfrak{m}, \mathbb{Z}/\ell)$ is a boundary map in the evident K-theory localization long exact sequence.

Our vanishing result is based on the following two theorems:

Theorem 2. Under Assumptions 1, the specialization map factors through $K_1(\mathrm{Spec}(Q \otimes_V \overline{F}), \mathbb{Z}/\ell)$, the K-theory of the algebraic analogue of the Milnor fiber.

Define

$$\chi : K_1(\mathrm{Spec}(R) \setminus \mathfrak{m}, \mathbb{Z}/\ell) \to \mathbb{Z}/\ell$$

to be the composition of the boundary map $K_1(\mathrm{Spec}(R) \setminus \mathfrak{m}, \mathbb{Z}/\ell) \to K_0(R/\mathfrak{m}, \mathbb{Z}/\ell)$ in the evident long exact localization sequence and the canonical isomorphism $K_0(R/\mathfrak{m}, \mathbb{Z}/\ell) \cong \mathbb{Z}/\ell$.

Theorem 3. Under Assumptions 1, we have

$$\theta^R(M,N) = \chi\left((\sigma\left([M]_{K_1}\right) \cup [N]_{K_0}\right) \pmod{\ell}$$

where $\cup$ is the multiplication rule for the ring $K_*(\mathrm{Spec}(R) \setminus \mathfrak{m}, \mathbb{Z}/\ell)$.

All of the precious constructions and results, in particular the previous two theorems, remain valid if we replace algebraic K-theory with finite coefficients, $K_*(-,\mathbb{Z}/\ell)$, with étale K-theory with $\mathbb{Z}/\ell$ coefficients, $K_*^{\text{ét}}(-,\mathbb{Z}/\ell)$.

To deduce the vanishing of θ^R when $\dim(R)$ is even from these results, we use also the following two theorems. The first, due to Illusie [3], is the algebraic analogue of Milnor's theorem, that the Milnor fiber has the homotopy type of a bouquet of n-dimensional spheres:

Theorem 4 (Illusie)**.** Under Assumptions 1, $H^p_{\text{ét}}(\mathrm{Spec}(Q \otimes_V \overline{F}), \mathbb{Z}/\ell) = 0$ unless $p = 0$ or $p = n$.

The second theorem was proved originally by Thomason [5] under more restrictive assumptions, and it was extended by Rosenschon-Østvær [4] to the case we need:

Theorem 5 (Thomason/Rosenschon-Ostvaer)**.** There is a strongly convergent spectral sequence

$$E_2^{p,q} \Longrightarrow K^{\text{ét}}_{q-p}(\mathrm{Spec}(Q \otimes_V \overline{F}), \mathbb{Z}/\ell)$$

where

$$E_2^{p,q} = \begin{cases} H^p_{\text{ét}}(X, \mu_\ell^{\otimes i}) & \text{if } q = 2i \text{ and} \\ 0 & \text{if } q \text{ is odd.} \end{cases}$$

Combining these two theorems gives immediately that $K_1^{\text{ét}}(\mathrm{Spec}(Q \otimes_V \overline{F}), \mathbb{Z}/\ell) = 0$ if $n = \dim(R)$ is even. Theorems 2 and 3 then yield:

Corollary 6. With R as in Assumptions 1, $\theta^R(M,N) = 0$ for all pairs of finitely generated R-modules, provided $n = \dim(R)$ is even.

References

[1] Ragnar-Olaf Buchweitz and Duco Van Straten, *An index theorem for modules on a hypersurface singularity*, Mosc. Math. J. **12** (2012), no. 2, 237–259, 459.

[2] Hailong Dao, *Decent intersection and Tor-rigidity for modules over local hypersurfaces*, Trans. Amer. Math. Soc. **365** (2013), no. 6, 2803–2821.

[3] Luc Illusie, *Perversité et variation*, Manuscripta Math. **112** (2003), no. 3, 271–295.

[4] Andreas Rosenschon and P. A. Østvær, *Descent for K-theories*, J. Pure Appl. Algebra **206** (2006), no. 1-2, 141–152.

[5] R. W. Thomason, *Algebraic K-theory and étale cohomology*, Ann. Sci. École Norm. Sup. (4) **18** (1985), no. 3, 437–552.

Reporter: Louis-Philippe Thibault

Participants

Prof. Dr. Matthew R. Ballard
Department of Mathematics
University of South Carolina
1523 Greene St.
Columbia, SC 29208
UNITED STATES

Hanno Becker
Mathematisches Institut
Universität Bonn
Endenicher Allee 60
53115 Bonn
GERMANY

Dr. Nathan Broomhead
Institut für Algebraische Geometrie
Leibniz Universität Hannover
Welfengarten 1
30167 Hannover
GERMANY

Prof. Dr. Ragnar-Olaf Buchweitz
Computer & Mathematical Sciences
Dept.
University of Toronto Scarborough
1265 Military Trail
Toronto Ont. M1C 1A4
CANADA

Prof. Dr. Igor Burban
Mathematisches Institut
Universität zu Köln
Weyertal 86 - 90
50931 Köln
GERMANY

Dr. Jesse Burke
Department of Mathematics
University of California at
Los Angeles
405 Hilgard Avenue
Los Angeles, CA 90095-1555
UNITED STATES

Dr. Nils Carqueville
Simons Center for Geometry & Physics
Stony Brook University
Stony Brook NY 11794-3840
UNITED STATES

Dr. Hailong Dao
Department of Mathematics
University of Kansas
625 Snow Hall
1460 Jayhawk Blvd.
Lawrence, KS 66045-7567
UNITED STATES

Dr. Tobias Dyckerhoff
Department of Mathematics
Yale University
Box 208 283
New Haven, CT 06520
UNITED STATES

Dr. Eleonore Faber
Computer & Mathematical Sciences
Dept.
University of Toronto Scarborough
1265 Military Trail
Toronto Ont. M1C 1A4
CANADA

Fabian Haiden
Fakultät für Mathematik
Universität Wien
Nordbergstr. 15
1090 Wien
AUSTRIA

Prof. Dr. Lutz Hille
Mathematisches Institut
Universität Münster
48149 Münster
GERMANY

Dr. Mehmet Umut Isik
Institut für Mathematik
Universität Wien
Strudlhofgasse 4
1090 Wien
AUSTRIA

Prof. Dr. Osamu Iyama
Graduate School of Mathematics
Nagoya University
Chikusa-ku, Furo-cho
Nagoya 464-8602
JAPAN

Dr. Hans Jockers
Bethe Center for Theoretical Physics
Physikalisches Institut
Nussallee 12
53115 Bonn
GERMANY

Martin Kalck
Fakultät für Mathematik
Universität Bielefeld
Postfach 100131
33501 Bielefeld
GERMANY

Prof. Dr. Mikhail Kapranov
Department of Mathematics
Yale University
Box 208 283
New Haven, CT 06520
UNITED STATES

Prof. Dr. Ludmil Katzarkov
Fakultät für Mathematik
Universität Wien
Oskar-Morgenstern-Platz 1
1090 Wien
AUSTRIA

Dr. Dmitry Kerner
Department of Mathematics
Ben-Gurion University of the Negev
P.O.Box 653
84105 Beer-Sheva
ISRAEL

Dr. Johanna Knapp
Institut für Theoretische Physik
Technische Universität Wien
Wiedner Hauptstr. 8-10/136
1040 Wien
AUSTRIA

Prof. Dr. Henning Krause
Fakultät für Mathematik
Universität Bielefeld
Postfach 100131
33501 Bielefeld
GERMANY

Prof. Dr. Andrey Lazarev
Department of Mathematics
University of Leicester
University Road
Leicester LE1 7RH
UNITED KINGDOM

Prof. Dr. Calin I. Lazaroiu
Institute for Basic Science
Center for Geometry and Physics
Phang University of Science
Pohang 790 784
KOREA, REPUBLIC OF

Prof. Dr. Manfred Lehn
Institut für Mathematik
Johannes-Gutenberg Universität Mainz
55128 Mainz
GERMANY

Prof. Dr. Helmut Lenzing
Institut für Mathematik
Universität Paderborn
Warburger Str. 100
33098 Paderborn
GERMANY

Prof. Dr. Wolfgang Lerche
Theory Division
CERN
Route de Meyrin 385
1211 Geneve 23
SWITZERLAND

Prof. Dr. Graham J. Leuschke
Mathematics Department
Syracuse University
Syracuse NY 13244-1200
UNITED STATES

Prof. Dr. Daniel Murfet
UCLA
Department of Mathematics
Los Angeles, CA 90095-1555
UNITED STATES

Dr. Tony Pantev
Department of Mathematics
David Rittenhouse Laboratory
University of Pennsylvania
209 South 33rd Street
Philadelphia PA 19104-6395
UNITED STATES

Dr. Markus Perling
Fakultät für Mathematik
Universität Bielefeld
Postfach 100131
33501 Bielefeld
GERMANY

Prof. Dr. Alexander Polishchuk
Department of Mathematics
University of Oregon
Eugene, OR 97403-1222
UNITED STATES

Prof. Dr. Andreas Recknagel
Department of Mathematics
King's College London
Strand
London WC2R 2LS
UNITED KINGDOM

Dr. Daniel Roggenkamp
Institut für theoretische Physik
Universität Heidelberg
Philosophenweg 19
69120 Heidelberg
GERMANY

Ana Ros Camacho
Department Mathematik
Universität Hamburg
Bundesstr. 55
20146 Hamburg
GERMANY

Prof. Dr. Ingo Runkel
Fachbereich Mathematik
Universität Hamburg
Bundesstr. 55
20146 Hamburg
GERMANY

Dr. Emanuel Scheidegger
Mathematisches Institut
Universität Freiburg
Eckerstr. 1
79104 Freiburg
GERMANY

Olaf Schnürer
Mathematisches Institut
Universität Bonn
Endenicher Allee 60
53115 Bonn
GERMANY

Prof. Dr. Christoph Schweigert
Fachbereich Mathematik
Universität Hamburg
Bundesstr. 55
20146 Hamburg
GERMANY

Prof. Dr. Eric Sharpe
Virginia Polytechnic Institute and
State University
Department of Physics
910 Drillfield Drive
Blacksburg, VA 24061
UNITED STATES

Dr. Ian Shipman
Department of Mathematics
University of Michigan
530 Church Street
Ann Arbor, MI 48109-1043
UNITED STATES

Dr. Greg Stevenson
Fakultät für Mathematik
Universität Bielefeld
Universitätsstr. 25
33615 Bielefeld
GERMANY

Prof. Dr. Catharina Stroppel
Mathematisches Institut
Universität Bonn
Endenicher Allee 60
53115 Bonn
GERMANY

Prof. Dr. Atsushi Takahashi
Department of Mathematics
Graduate School of Science
Osaka University
Machikaneyama 1-1, Toyonaka
Osaka 560-0043
JAPAN

Prof. Dr. Ryo Takahashi
Nagoya University
Graduate School of Mathematics
Nagoya 464-8612
JAPAN

Louis-Philippe Thibault
Computer & Mathematical Sciences
Dept.
University of Toronto Scarborough
1265 Military Trail
Toronto Ont. M1C 1A4
CANADA

Prof. Dr. Kazushi Ueda
Department of Mathematics
Graduate School of Science
Osaka University
Machikaneyama 1-1, Toyonaka
Osaka 560-0043
JAPAN

Prof. Dr. Duco van Straten
Institut für Mathematik
Johannes-Gutenberg Universität Mainz
Staudingerweg 9
55128 Mainz
GERMANY

Prof. Dr. Mark E. Walker
Department of Mathematics
University of Nebraska, Lincoln
Lincoln NE 68588
UNITED STATES

Prof. Dr. Michael Wemyss
School of Mathematics
University of Edinburgh
King's Buildings
Mayfield Road
Edinburgh EH9 3JZ
UNITED KINGDOM

Prof. Dr. Yuji Yoshino
Department of Mathematics
Faculty of Science
Okayama University
3-1-1 Tsushima-naka
Okayama 700-8530
JAPAN