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Noncommutative Geometry

Organised by
Alain Connes, Paris
Joachim Cuntz, Münster
Marc A. Rieffel, Berkeley
Guoliang Yu, College Station

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Abstract. Noncommutative Geometry applies ideas from geometry to mathematical structures determined by noncommuting variables. This meeting emphasized the connections of Noncommutative Geometry to number theory and ergodic theory.

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Introduction by the Organisers

Noncommutative Geometry applies ideas from geometry to mathematical structures determined by noncommuting variables. Within mathematics, it is a highly interdisciplinary subject drawing ideas and methods from many areas of mathematics and physics. Natural questions involving noncommuting variables arise in abundance in many parts of mathematics and theoretical quantum physics. On the basis of ideas and methods from algebraic and differential topology and Riemannian geometry, as well as from the theory of operator algebras and from homological algebra, an extensive machinery has been developed which permits the formulation and investigation of the geometric properties of noncommutative structures. This includes K-theory, cyclic homology and the theory of spectral triples. Areas of intense research in recent years are related to topics such as index theory, quantum groups and Hopf algebras, the Novikov and Baum-Connes conjectures as well as to the study of specific questions in other fields such as number theory, modular forms, topological dynamical systems, renormalization theory, theoretical high-energy physics and string theory. Many results elucidate important properties of specific classes of examples that arise in many applications. But the properties of many important classes of examples still remain mysterious, and are currently under intense investigation.

This meeting covered selected aspects of the topics described above. We put a special emphasis this time on methods and developments in Noncommutative Geometry which are related to number theory and ergodic theory. This reflects the fact that there are some very interesting recent developments in that direction, including the new role of cyclic homology in several aspects of number theory including regulators and L-functions.

We decided to dedicate the meeting to the memory of Jean-Louis Loday who passed away in an accident on June 6, 2012. This is in honour of his important contributions to the theory of cyclic homology which is a central element of Noncommutative Geometry.

Workshop: Noncommutative Geometry

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Abstracts

Algebraic and Hermitian K-theory of $\mathcal{K}$-rings

Max Karoubi
(joint work with Mariusz Wodzicki)

Let A be a complex C^*-algebra and $\mathcal{K} = \mathcal{K}(H)$ be the ideal of compact operators on a separable complex Hilbert space H. One of us conjectured around 1977 that the natural comparison map between the algebraic and topological K-groups

$$\epsilon_n \colon K_n(\mathcal{K}\bar{\otimes}A) \longrightarrow K_n^{\mathrm{top}}(\mathcal{K}\bar{\otimes}A) \simeq K_n^{\mathrm{top}}(A)$$

is an isomorphism for a suitably completed tensor product $\bar{\otimes}$. The conjecture was announced in [4] where accidentally only $\mathcal{K}\widehat{\otimes}_\pi A$ was mentioned, and it was proved there for $n \leq 0$. In [5] the conjecture was established for $n \leq 2$ and $\mathcal{K}\bar{\otimes}A$ having the meaning of the C^*-algebra completion of the algebraic tensor product (in view of *nuclearity* of $\mathcal{K}$, there is only one such completion).

The C^*-algebraic form of the conjecture was established for all $n \in \mathbf{Z}$ in 1990 [6], [7]. A year later the conjecture was proved also for all $n \in \mathbf{Z}$ and $\mathcal{K}\widehat{\otimes}_\pi A$ where A was only assumed to be a Banach $\mathbf{C}$-algebra with one-sided bounded approximate identity [8].

Here we prove analogous theorems for K-theory of real Banach and C^*-algebras, and then deduce similar results in Hermitian K-theory.

We set the stage by introducing the concept of a $\mathcal{K}$-*ring*, which is a slight generalization and a modification to what was called a "stable" algebra in [5]. We also introduce a novel notion of a *stable retract*. Then we proceed to demonstrate that the comparison map between algebraic and topological K-groups in degrees $n \leq 0$ is an isomorphism for Banach algebras that are stable retracts of $\mathcal{K}\otimes_{\max}A$ for some C^*-algebra A, or of $\mathcal{K}\widehat{\otimes}_\pi A$, for some Banach algebra A.

We recall how to endow $K_*(\mathcal{K})$ with a canonical structure of a $\mathbf{Z}$-graded, associative, graded commutative, and unital ring. If a $\mathcal{K}$-ring is H-unital as a $\mathbf{Q}$-algebra, we equip its $\mathbf{Z}$-graded algebraic K-groups with a structure of a $\mathbf{Z}$-graded unitary $K_*(\mathcal{K})$-module. We distinguish two cases: $\mathcal{K}_{\mathbf{R}}$-rings and $\mathcal{K}_{\mathbf{C}}$-rings, where $\mathcal{K}_{\mathbf{R}}$ stands for the ring of compact operators on a real Hilbert space, and $\mathcal{K}_{\mathbf{C}}$ on the complex Hilbert space.

These structures are used to prove several results, two of them: 2-periodicity of algebraic K-groups for $\mathcal{K}_{\mathbf{C}}$-rings and a comparison theorem for Banach $\mathcal{K}_{\mathbf{C}}$-rings. Both results were known before [8], [7].

Relying on a rather delicate argument employing K-theory with coefficients mod 16 and certain classical results of stable homotopy theory, we detect the existence of an element $v_8 \in K_8(\mathcal{K}_{\mathbf{R}})$ which maps onto a generator of $K_8^{\mathrm{top}}(\mathcal{K}_{\mathbf{R}}) \simeq \mathbf{Z}$. We showed that $K_{-8}(\mathcal{K}_{\mathbf{R}}) \simeq \mathbf{Z}$. The element v_8 is the multiplicative inverse of a generator of that group. This allows us to establish 8-periodicity of algebraic K-groups for arbitrary $\mathcal{K}_{\mathbf{R}}$-rings, and to prove that the comparison map is an isomorphism for Banach $\mathcal{K}_{\mathbf{R}}$-rings ("the real case of Karoubi's Conjecture"). Both results are new.

The remaining sections are devoted to Hermitian K-theory. The goal is to deduce comparison theorems in Hermitain K-theory from the corresponding results in algebraic K-theory (for complex C^*-algebras this was partially done in [2]). The primary tool in this task is a pair of *Comparison Induction Theorems* which are among the consequences of the "Fundamental Theorem of Hermitian K-theory" [3]. We also provide a rather thorough discussion of Hermitian K-groups for nonunital rings. We compare two approaches to defining relative Hermitian K-groups and, in the end, we deduce excision properties in Hermitian K-theory from the corresponding properties in algebraic K-theory, cf. [1], [2]. This is achieved with a different kind of Induction Theorems.

We would like to point out that the results hold also for the corresponding algebraic and Hermitian K-groups with finite coefficients. The hypothesis of H-unitality over $\mathbf{Q}$ can be dropped since $\mathbf{Q}$-algebras satisfy excision in algebraic and Hermitian K-theory with finite coefficients.

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K-theory of crossed products by certain group actions on totally disconnected spaces and semigroup algebras

Siegfried Echterhoff

(joint work with Joachim Cuntz, Xin Li)

This report is very similar to [5]. Let $e \in P \subseteq G$ be a sub-semigroup of the countable group G. The left-reduced C*-semigroup algebra $C^*_\lambda(P)$ is defined as the sub-C*-algebra of the bounded operators $\mathcal{B}(\ell^2(P))$ generated by the set of isometries $\{V_p : p \in P\}$ with

$$V_p : \ell^2(P) \to \ell^2(P); V_p\delta_q = \delta_{pq},$$

where $\{\delta_q : q \in P\}$ denotes the standard orthonormal basis of $\ell^2(P)$. An important example which motivated much of our work is given by the class of $ax+b$-semigroups $R \rtimes R^\times \subseteq K \rtimes K^\times$ in which R is the ring of integers in a number field K and $R^\times = R \setminus \{0\}$. In this case the semigroup C*-algebra has been studied extensively by Cuntz, Deninger, and Laca in [1] since it encodes interesting number theoretic features of the field K. We refer to the report of the lecture by Xin Li for further details on this example.

Let $P \subseteq G$ be given. For $X \subseteq P$ let $E_X : \ell^2(P) \to \ell^2(X) \subseteq \ell^2(P)$ denote the orthogonal projection. Then

$$V_p E_X V_p^* = E_{pX} \quad \text{and} \quad V_p^* E_X V_p = E_{p^{-1}X}$$

where $p^{-1}X := \{q \in P : pq \in X\}$ denotes the inverse image of X under multiplication with p. Let $\mathcal{J}_P$ denote the smallest set of subsets $X \subseteq P$ which contains $\emptyset$ and P, and which is closed under finite intersections and the operations $X \mapsto pX, p^{-1}X$ as considered above. We call $\mathcal{J}_P$ the set of *constructible right ideals in* P. Since we assume that the unit e of G lies in P, we get $E_P = V_e \in C^*_\lambda(P)$, from which it then follows that $\{E_X : X \in \mathcal{J}_P\} \subseteq C^*_\lambda(P)$. In fact, $C^*_\lambda(P)$ equals the reduced semi-group crossed product $D_P \rtimes_\lambda P$ where $D_P \subseteq C^*_\lambda(P)$ denotes the C*-algebra generated by $\{E_X : X \in \mathcal{J}_P\}$.

Following ideas of Laca and Li (see [7, 8]) we dilate the semigroup action of P on D_P to an action of G on some algebra $D_{P\subseteq G}$. For this let $\mathcal{J}_{P\subseteq G}$ denote the G-saturation of $\mathcal{J}_P$ as subsets of G and let $D_{P\subseteq G}$ denote the commutative C*-subalgebra of $\mathcal{B}(\ell^\infty(G))$ generated by the orthogonal projections $\{E_Y : Y \in \mathcal{J}_{P\subseteq G}\}$. By an argument due to Fell the reduced crossed product $D_{P\subseteq G} \rtimes_\lambda G$ is faithfully represented in $\mathcal{B}(\ell^2(G))$ by the canonical representation in such a way that

$$C^*_\lambda(P) \subseteq E_P(D_{P\subseteq G} \rtimes_\lambda G)E_P, \tag{1}$$

where the orthogonal projection $E_P : \ell^2(G) \to \ell^2(P)$ is always a full projection in $D_{P\subseteq G} \rtimes_\lambda G$. We say that $P \subseteq G$ satisfies the *weak Toeplitz condition* if we have equality in (1) (this condition is implied by the stronger Toeplitz condition of [8, Definition 4.1]).

We want to use this picture for computing the K-theory of $C^*_\lambda(P)$. For this we observe that $D_{P\subseteq G} \cong C_0(\Omega_{P\subseteq G})$ for some totally disconnected space $\Omega_{P\subseteq G}$. So we look to general actions of a countable group G on a totally disconnected space Ω. We denote by $C_c^\infty(\Omega)$ the set of locally constant functions on Ω with compact supports. We show in [4] that there exists a linear basis $\mathcal{P} = \{p_i : i \in I\}$ of $C_c^\infty(\Omega)$ consisting of projections in $C_0(\Omega)$ which is closed under multiplication (up to 0), and it is not to hard to show that this implies that

$$K_0(C_0(\Omega)) = \bigoplus_{i\in I} \mathbb{Z}[p_i] \quad \text{and} \quad K_1(C_0(\Omega)) = \{0\},$$

i.e., the K_0-classes of the projections $[p_i]$ form a base of the K-theory of $C_0(\Omega)$. Assume now that we can find such a basis $\mathcal{P}$ which is G-invariant. Then there is an action of G on the discrete space I such that $g \cdot p_i = p_{gi}$ for all $i \in I$. Moreover, there is a unique $*$-homomorphism

$$\mu : C_0(I) \to C_0(\Omega) \otimes \mathcal{K}(\ell^2(I))$$

which sends a Dirac-function $\delta_i \in C_0(I)$ to the projection $p_i \otimes d_i$ if $d_i : \ell^2(I) \to \mathbb{C}\delta_i \subseteq \ell^2(I)$ denotes the orthogonal projection to the subspace spanned by δ_i. The action of G on I induces a unitary representation $U : G \to \mathcal{U}(\ell^2(I))$ and the homomorphism μ becomes G-equivariant with respect to the given action on $C_0(I)$ and the action $\tau \otimes \operatorname{Ad} U$ on $C_0(\Omega) \otimes \mathcal{K}(\ell^2(I))$, where τ denotes the action on $C_0(\Omega)$. Thus μ determines a class $[\mu] \in KK^G\big(C_0(I), C_0(\Omega) \otimes \mathcal{K}(l^2(I))\big) \cong KK^G\big(C_0(I), C_0(\Omega)\big)$. Our central result is the following (see [4, §3]):

Theorem 1. *Let G, Ω, and $\{p_i : i \in I\}$ be as above and let $\alpha : G \to \operatorname{Aut}(A)$ be an action of G on a C^*-algebra A. Assume in addition that G satisfies the Baum-Connes conjecture for $A \otimes C_0(I)$ and $A \otimes C_0(\Omega)$. Then the descent*

$$[\mathrm{id}_A \otimes \mu] \rtimes_\lambda G \in KK\big((A \otimes C_0(I)) \rtimes_\lambda G, (A \otimes C_0(\Omega)) \rtimes_\lambda G\big)$$

induces an isomorphism of K-theory groups

$$K_*\big((A \otimes C_0(I)) \rtimes_\lambda G\big) \cong K_*\big((A \otimes C_0(\Omega) \rtimes_\lambda G\big).$$

Since I is discrete, it follows from Green's imprimitivity theorem that there is a Morita equivalence

$$(A \otimes C_0(I)) \rtimes_\lambda G \sim_M \bigoplus_{[i] \in G \backslash I} A \rtimes_\lambda G_i,$$

where $G_i = \{g \in G : gi = i\}$ is the stabilizer of $i \in I$. Thus, if the conditions of the above theorem are satisfied, we obtain an isomorphism

$$K_*\big((A \otimes C_0(\Omega)) \rtimes_\lambda G\big) \cong \bigoplus_{[i] \in G \backslash I} K_*(A \rtimes_\lambda G_i).$$

The proof of the theorem uses a principle observed in [2] which, using the Baum-Connes assumption, allows to reduce the above theorem to the case of actions of finite groups and finite dimensional algebras, in which the result can be shown by some more or less elementary combinatorics. We refer to [4, §3] for more details. We should point out that by a seminal theorem of Higson and Kasparov ([6]) the Baum-Connes assumption is always satisfied if G is a-T-menable (or amenable). Moreover, under some extra condition which we don't explain here, we can even obtain KK-equivalence of the crossed products.

It turned out that the condition on the existence of a G-invariant and multiplicatively closed (up to 0) basis $\{p_i : i \in I\}$ of $C_c^\infty(\Omega)$ is quite restrictive for general actions on totally disconnected spaces. For example, it is never satisfied if an amenable group G acts minimally on the Cantor set Ω (see [4, Proposition 3.18]). But somehow surprisingly, the condition is very often satisfied for the dilated action of G on $\Omega_{P \subseteq G}$ if we start with a sub-semigroup $P \subseteq G$ as above. It is then implied by the following independence condition for $P \subseteq G$:

Definition. *We say that* $P \subseteq G$ *satisfies the* independence condition *if the set* $\mathcal{J}_{P\subseteq G}$ *of constructible right* P*-ideals in* G *is independent in the following sense: If* $X, X_1, \ldots, X_l \in \mathcal{J}_{P\subseteq G}$ *such that* $X = \cup_{i=1}^{l} X_i$*, then* $X = X_{i_0}$ *for some* $i_0 \in \{1, \ldots, l\}$.

Namely, if $P \subseteq G$ satisfies this independence condition, then $\{E_X : X \in I_{P\subseteq G}\}$ with $I_{P\subseteq G} := \mathcal{J}_{P\subseteq G} \setminus \{\emptyset\}$ is a basis for $C_c^\infty(\Omega_{P\subseteq G})$ as desired. We then get

Theorem 2. *Suppose that* $P \subseteq G$ *satisfies the (weak) Toeplitz condition and the independence condition. Suppose further that* G *acts on the* C^**-algebra* A *such that* G *satisfies the Baum-Connes conjecture for* $A \otimes C_0(I_{P\subseteq G})$ *and* $A \otimes C_0(\Omega_{P\subseteq G})$*. Then*

$$K_*(A \rtimes_\lambda P) \cong \bigoplus_{[X]\in G\backslash I_{P\subseteq G}} K_*(A \rtimes_\lambda G_X)$$

where $G_X = \{g \in G : gX = X\}$*. In particular, in case* $A = \mathbb{C}$ *we get*

$$K_*(C_\lambda^*(P)) \cong \bigoplus_{[X]\in G\backslash I_{P\subseteq G}} K_*(C_\lambda^*(G_X)).$$

It is shown in [4] that many interesting classes of sub-semigroups $P \subseteq G$ satisfy these conditions. Among them are the quasi-lattice ordered semigroups $P \subseteq G$ which are characterized by the conditions $P \cap P^{-1} = \{e\}$ and for all $g \in G$ there exists a $p \in P$ with $P \cap gP = pP$. In this case G acts freely and transitively on $I_{P\subseteq G}$ and hence our results imply that $K_*(A \rtimes_\lambda P) \cong K_*(A)$ if G satisfies the Baum-Connes for $A \otimes C_0(\Omega_{P\subseteq G})$. For a number of other interesting applications we refer to [3, 4, 10] and [11]. For our motivating example $R \rtimes R^\times \subseteq K \rtimes K^\times$ we get

Theorem 3. *Let* R *be a Dedekind domain and let* $K = \mathcal{Q}(R)$ *denote its quotient field. Let* $R \rtimes R^\times \subseteq K \rtimes K^\times$ *denote the corresponding* $ax+b$*-semigroup and let* Cl_K *denote the ideal class group of* K*. Then*

$$K_*(C_\lambda^*(R \rtimes R^\times)) \cong \bigoplus_{\gamma\in Cl_K} K_*(C_\lambda^*(I_\gamma \rtimes R^*)),$$

where $I_\gamma \subseteq R$ *denotes a representative for* γ *and* R^* *denotes the group of units in* R*. In fact, the isomorphism is induced by a* KK*-equivalence between* $C_\lambda^*(R \rtimes R^\times)$ *and* $\bigoplus_{\gamma\in Cl_K} C_\lambda^*(I_\gamma \rtimes R^*)$*.*

These K-theory results have been used by Xin Li in [9] to classify the semigroup algebra $C_\lambda^*(R \rtimes R^\times)$ of the ring of integers R in the number field K in terms of the field K, the showing that the algebra encodes basically all number theoretic information on the field K. He was also able to extend the results on $ax+b$-semigroups to a much bigger class of rings R (see [10]).

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C*-algebras of semigroups coming from rings

Xin Li

Given a subsemigroup P of a group G, we can construct its left or right reduced semigroup C*-algebra $C^*_\lambda(P)$ or $C^*_\rho(P)$. The construction is analogous to the group case: First, we form the left or right regular representation of P on $\ell^2(P)$. It will be given by isometries. Then, we take the C*-algebra generated by the isometries from the left or right regular representation.

Let us know look at a special class of examples: Let R be an integral domain, i.e. a commutative ring with unit, which has no zero-divisors. We will only consider countable rings. The semigroup of interest is the $ax+b$-semigroup $P = R \rtimes R^\times$. We want to study the semigroup C*-algebras $C^*_\lambda(R \rtimes R^\times)$ and $C^*_\rho(R \rtimes R^\times)$. Let Q be the quotient field of R, and let $G = Q \rtimes Q^\times$ be the $ax+b$-group over Q. P sits as a subsemigroup in G in a canonical way. Moreover, $P \subseteq G$ satisfies the Toeplitz condition from [4], both for the left and right versions. Apart from the Toeplitz condition, we also need the independence condition:

Set $\mathcal{I} := \{(x_1R) \cap \ldots \cap (x_nR) \colon x_i \in Q^\times\}$. These are divisorial ideals of R.

Definition. *R satisfies independence if for every I, I_1, ..., I_n in $\mathcal{I}$, $I = \bigcup_{i=1}^n I_i$ implies that $I = I_i$ for some $1 \leq i \leq n$.*

For example, rings of algebraic integers in number fields satisfy this condition. But the ring $R = \mathbb{Z}[i\sqrt{3}]$ does not satisfy independence (it is not integrally closed).

Now let R be the ring of algebraic integers in a number field. Here is a list of known results about $C^*_\lambda(R \rtimes R^\times)$:

(i) $C^*_\lambda(R \rtimes R^\times) \cong \mathcal{O}_\infty \otimes C^*_\lambda(R \rtimes R^\times)$ has been proven in [2],
(ii) the primitive ideal space for $C^*_\lambda(R \rtimes R^\times)$ has been computed in [3],
(iii) K-theory for $C^*_\lambda(R \rtimes R^\times)$ and $C^*_\rho(R \rtimes R^\times)$ has been determined in [2],

(iv) a classification result for $C^*_\lambda(R \rtimes R^\times)$ has been established in [5],
(v) KMS-states for a canonical dynamical system $(C^*_\lambda(R \rtimes R^\times), \sigma)$ have been computed in [1].

It turns out that (i), (ii) and (iii) have natural generalizations to more general rings, for instance Krull rings. A noetherian domain is a Krull ring if and only if it is integrally closed. It was proven in [6] that every Krull ring also satisfies the independence condition. Also, every integral domain which contains an infinite field satisfies independence (see [6]). Both for Krull rings and also for integral domains containing infinite fields, natural generalizations of the results (i), (ii) and (iii) were obtained in [6].

Here is the precise statement concerning (iv):

Theorem. *Let K and L be number fields which are Galois extensions of $\mathbb{Q}$, and let R and S be rings of algebraic integers in K and L. Assume that $\#\mu_K = \#\mu_L$, i.e., K has the same number of roots of unity as L. Then $C^*_\lambda(R \rtimes R^\times)$ is isomorphic to $C^*_\lambda(S \rtimes S^\times)$ if and only if K is isomorphic to L.*

The proof of this result uses topological K-theory in an essential way. The details of the proof can be found in [5].

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Asymptotically Proper Actions

Rufus Willett

(joint work with Erik Guentner, Guoliang Yu)

Let Γ be a discrete group acting on a compact space X. We are interested in situations in which this action can be 'decomposed' into simpler 'pieces'. In particular, we would like a notion of 'decomposability' for $\Gamma \curvearrowright X$ that unifies the following ideas (and their applications to K-theory) as much as possible.

- Properness of an action [3].
- Amenability of an action [1, 8, 7].
- Gromov's finite asymptotic dimension [5, 10].
- Long-thin covers of an action [4, 2].

An action of a group often cannot be decomposed into simpler actions of groups in a meaningful way; here the extra flexibility offered by *groupoids* is very useful. Let $X \rtimes \Gamma$ be the transformation groupoid associated to an action $\Gamma \curvearrowright X$, so $X \rtimes \Gamma$ is homeomorphic to $X \times \Gamma$, and equipped with a partially defined multiplication. The following is our main definition.

Definition. *The action* $\Gamma \curvearrowright X$ *is* asymptotically proper *if there exists* $d \geq 0$ *(to be thought of as a dimension for the action) such that for any finite subset* F *of* Γ *there exists a finite subset* E *of* Γ *and an open cover* $\{U_0, ..., U_d\}$ *of* $X \times F \subseteq X \rtimes \Gamma$ *such that:*

- *for each* $x \in X$, $\{x\} \times F$ *is contained in some* U_i*;*
- *the subgroupoid of* $X \rtimes \Gamma$ *generated by each* U_i *is contained in* $X \times E$*.*

We have shown that if Γ has finite asymptotic dimension then it admits an asymptotically proper action on a compact space. We have also shown that actions admitting long thin covers, and a restricted class of amenable actions, are asymptotically proper. Thus there is a large class of examples. There are also interesting connections to the nuclear dimension of Winter and Zacharias [9].

Our main goal is applications to K-theory (both algebraic and operator algebraic). We have shown the following result.

Theorem. *If* $\Gamma \curvearrowright X$ *is asymptotically proper, then there is a concrete procedure to compute the* K*-theory groups of the reduced crossed products* $C(X) \rtimes_r \Gamma$*.*

The proof is based on controlled K-theory and associated Mayer-Vietoris sequences, similarly to [10].

This result is strong enough to imply the Baum-Connes conjecture for Γ with coefficients in X. Moreover, using a new descent technique (based roughly on that of [7]), if Γ admits an asymptotically proper action on a compact space, then the Baum-Connes assembly map for Γ is split injective. In all the examples we understand the results on Baum-Connes are known (due to Tu [8] and Higson [7]), but our proof is simpler than existing ones.

We are also interested in the Farrell-Jones conjecture, where we expect to get genuinely new results. We expect we can prove similar results on split injectivity of the Farrell-Jones assembly map in algebraic K-theory. Moreover, we have generalized the metric notion of finite decomposition complexity from [6] to a notion of 'finite dynamical complexity', and expect similar (although more complicated) controlled K-theory arguments should imply quite new results on the Farrell-Jones conjecture. These results on Farrell-Jones are currently only outlined: we are in the process of writing them down and checking the details.

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Tangent Groupoids and Pseudodifferential Calculi

Erik van Erp
(joint work with Bob Yuncken)

Osculating Groups

Let M be a smooth manifold equipped with a distribution $H \subseteq TM$, i.e., a sub-vectorbundle of the tangent bundle. H could be a foliation, or a contact structure, but other structures are of interest too and we allow H to be arbitrary.

Fix a point $m \in M$, and choose coordinates $(x_1, \ldots, x_n)$ near m such that the coordinate tangent vectors $\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_p}$ span the fiber H_m at the point m. Note that in general the coordinate vector fields may not span the fibers of H except at m.

Definition 1. *A* parabolic arrow at m is an equivalence class of curves $c(t)$ with $c(0) = m$ and $c'(0) \in H_m$, where two curves c_1 and c_2 are equivalent if $c_1'(0) = c_2'(0)$ and $c_1''(0) - c_2''(0) \in H_m$.

Denote the equivalence class of $c(t)$ by $[c]$, and let $G_m = \{[c]\}$ be the set of parabolic arrows at m. Remarkably, G_m is a *group*. To define composition of two parabolic arrows $[c_1], [c_2]$ we extend them to local flows Φ_t, Ψ_t.

Definition 2. *If $[c_1], [c_2] \in G_m$ are two parabolic arrows at $m \in M$, and if $c_1(t) = \Phi_t(m), c_2(t) = \Psi_t(m)$ for two local flows Φ_t, Ψ_t, then the product of $[c_1], [c_2]$ in G_m is the parabolic arrow $[c_1] \cdot [c_2] = [c_3]$, where $c_3(t) = (\Phi_t \circ \Psi_t)(m)$.*

The flows Φ_t, Ψ_t must be chosen such that the time derivatives $\frac{\partial}{\partial t}|_{t=0}\Phi_t(x) \in H_x$ are tangent vectors in H for all x near m.

Composition of parabolic arrows is well defined. I refer to the groups G_m as *osculating groups*, borrowing a term from Folland and Stein. As is clear from the definition, the group structure of G_m depends on the germ of the distribution H near m. For example, all osculating groups are abelian if and only if H is a foliation, and all osculating groups are isomorphic to the Heisenberg group if and only if H is a contact structure. In general, the osculating groups at different points need not be isomorphic.

We denote by $T_H M := \sqcup_{m \in M} G_m$ the disjoint union of all osculating groups. The bundle $T_H M$ has a natural smooth structure, making it into a Lie groupoid.

Tangent Groupoid
The tangent groupoid was introduced by Alain Connes. Algebraically it is the disjoint union

$$\mathbf{T}M = TM \sqcup M \times M \times (0, \infty)$$

where TM is the tangent bundle (interpreted as a bundle of abelian groups), and $M \times M$ is the pair groupoid. The family of pair groupoids $M \times M \times (0, \infty)$ is parametrized by $t > 0$, and is glued to the $t = 0$ fiber TM in such a way that a path (a_t, b_t, t) converges to a tangent vector $v \in T_m M$ if both paths a_t and b_t in M converge to the same point m, and moreover,

$$\lim_{t \to 0} \frac{a_t - b_t}{t} = v \in T_m M$$

Analogously, we define a *parabolic tangent groupoid* associated to the bundle of osculating groups as follows. Algebraically,

$$\mathbf{T}_H M = T_H M \sqcup M \times M \times (0, \infty)$$

This time we glue a path (a_t, b_t, t) to a parabolic arrow $v \in G_m$ if both paths a_t, b_t converge to m, and moreover the tangent vectors $a'(0)$ and $b'(0)$ are in H_m. Then we have parabolic arrows $[a], [b]$, and we require that

$$[a] \cdot [b]^{-1} = v \in G_m$$

There is a smooth structure on $\mathbf{T}_H M$ that makes it into a Lie groupoid. Exactly as for Connes' tangent groupoid, the smooth structure is defined by means of an exponential map

$$\exp^H : T_H M \to M \times M$$

The crucial property of an exponential map for the osculating groups is that it must be adapted to H in the following sense: If $c(t) = \exp^H(\delta_t v)$ for some parabolic arrow $v \in G_m$, then we must have $[c] = v$. Here δ_t denotes the natural dilation structure of the osculating groups induced by rescaling of curves $\delta_t[a] = [b]$ if $b(s) = a(ts)$.

Symbols
Let P be a differential operator on M of degree d. Associated to P we construct a differential operator $\mathbf{P}$ on the tangent groupoid $\mathbf{T}M$ as follows. If $t > 0$, then $\mathbf{P}$ is the differential operator $t^d P$ acting on $M \times \{m\} \times \{t\}$ for every $m \in M$. Note that this defines a *right-invariant* operator on $M \times M \times (0, \infty)$.

The coefficients of the differential operator $\mathbf{P}$ extend smoothly (and, of course, in a unique manner) to the $t = 0$ fiber TM. At $t = 0$ we get the *co-symbol* of P, which is a smooth family $\hat{\sigma}(P) = \{P_m, m \in M\}$ of constant coefficient operators on the vector spaces $T_m M$, each homogeneous of degree d. The cosymbol $\hat{\sigma}(P)$ represents the highest order part of P, and the usual principal symbol $\sigma(P)$ is the

Fourier transform of the cosymbol, which is a polynomial on the cotangent bundle T^*M. In this way, the symbolic calculus for differential operators is built into the tangent groupoid formalism. Note that it is a priori clear that the cosymbol is invariantly defined, and that we automatically have $\hat{\sigma}(PQ) = \hat{\sigma}(P)\hat{\sigma}(Q)$.

Exactly the same construction works on the parabolic tangent groupoid $\mathbf{T}_H M$. Starting with $\mathbf{P}$ defined as the right-invariant operator $t^d P$ on $M \times M \times (0, \infty)$, as above, extend $\mathbf{P}$ to the fiber at $t = 0$, which is now the bundle of osculating groups $T_H M$. At $t = 0$ one obtains a smooth family $\hat{\sigma}_H(P) = \{P_m, m \in M\}$ of right-invariant differential operators P_m on the osculating groups G_m, homogeneous of degree d for the dilation structure δ_t of G_m. This family $\hat{\sigma}_H(P)$ is precisely the cosymbol of P in the *Heisenberg calculus* for H.

Pseudodifferential Operators

The groupoid $\mathbf{T}_H M$ has an action of the multiplicative group $\mathbf{R}_+^\times$,

$$\alpha\colon \mathbf{R}_+^\times \to \mathrm{Aut}(\mathbf{T}_H M)$$

defined by

$$\alpha_s(x, y, t) := (x, y, s^{-1}t) \qquad t > 0$$

$$\alpha_s(x, v, 0) := (x, \delta_s v, 0) \qquad v \in G_m$$

This induces an action of $\mathbf{R}_+^\times$ by automorphisms on the convolution algebra $C_c^\infty(\mathbf{T}_H M)$.

The right invariant operator $\mathbf{P}$ on the groupoid $\mathcal{G} = \mathbf{T}_H M$ defined above can be thought of as a smooth family of distributions on the source fibers $\mathcal{G}^{m,t} = s^{-1}(m, t)$ for $(m, t) \in \mathcal{G}^{(0)}$. We denote the collection of smooth families of distributions on the source fibers of $\mathcal{G}$ by

$$A = C^\infty(\mathcal{G}^{(0)}, \mathcal{E}'_{reg}(\mathcal{G}^{(m,t)}))$$

where $\mathcal{E}'_{reg}$ denotes distributions with compact support that are regular in the sense that they are C^∞ functions away from the units of $\mathcal{G}$. Note that $\mathbf{P} \in A$, and that the operator $\mathbf{P}$ is homogeneous for the action α,

$$\alpha_s \mathbf{P} = s^{-(d+Q)} \mathbf{P}$$

Here $Q = p + 2(n - p)$ is the homogeneous dimension of G_m, with $p = \dim H_m$.

Given $k \in A$, we say that $k_1 \in C^\infty(M, \mathcal{E}'_{reg}(M))$ (the restriction of k to $t = 1$) is the Schwartz kernel of an *H-Pseudodifferential operator* on M of order d if k is α-quasihomogeneous in the sense that

$$\alpha_s k - s^{-(d+Q)} k \in C^\infty(\mathcal{G})$$

We prove that the collection of such k_1 defines a filtered algebra with a symbolic calculus (the principal H-symbol of k_1 is k_0), with asymptotic expansions of full symbols, etc. If $H = TM$ we recover the classical pseudodifferential calculus, and if H has codimension 1 we recover the Heisenberg calculus.

Higher rho invariants and the moduli space of positive scalar curvature metrics

Zhizhang Xie
(joint work with Guoliang Yu)

Let M be a closed smooth manifold. Suppose M carries a metric of positive scalar curvature. It is well known that the space of all Riemannian metrics on M is contractible, hence topologically trivial. To the contrary, the space of all positive scalar curvature metrics on M, denoted by $\mathcal{R}^+(M)$, often has very nontrivial topology. For example, the homotopy groups of $\mathcal{R}^+(M)$ often contain many nontrivial elements (cf. [1]). In particular, $\mathcal{R}^+(M)$ is often not connected and in fact has infinitely many connected components. For example, by using the Cheeger-Gromov L^2-rho invariant [2] and the delocalized eta invariant of Lott [4], Piazza and Schick showed that $\mathcal{R}^+(M)$ has finitely many connected components, if M is a closed spin manifold with $\dim M = 4k+3 \geq 5$ and $\pi_1(M)$ contains torsion [5, Theorem 1.3].

Following Stolz [8, 7], Weinberger and Yu introduced an abelian group $P(M)$ to measure the size of the space of positive scalar curvature metrics [9]. In addition, they used the finite part of K-theory of the maximal group C^*-algebra $C^*_{\max}(\pi_1(M))$ to give a lower bound of the rank of $P(M)$. A special case of their theorem states that the rank of $P(M)$ is ≥ 1, if M is a closed spin manifold with $\dim M = 2k+1 \geq 5$ and $\pi_1(M)$ contains torsion [9, Theorem 4.1]. In particular, this implies the above theorem of Piazza and Schick.

In this talk, I will report my recent joint work with Guoliang Yu [10] on results concerning the moduli space of positive scalar curvature metrics by using the finite part of K-theory of $C^*_{\max}(\pi_1(M))$ and the higher rho invariant (cf. [3, 6, 11]). These results are inspired by the results of Piazza and Schick [5] and Weinberger and Yu [9]. Recall that the group of diffeomorphisms on M, denoted by $\mathrm{Diff}(M)$, acts on $\mathcal{R}^+(M)$ by pulling back the metrics. The moduli space of positive scalar curvature metrics is defined to be the quotient space $\mathcal{R}^+(M)/\mathrm{Diff}(M)$. Similarly, $\mathrm{Diff}(M)$ acts on the group $P(M)$ and we denote the coinvariant of the action by $\widetilde{P}(M)$. That is, $\widetilde{P}(M) = P(M)/P_0(M)$ where $P_0(M)$ is the subgroup of $P(M)$ generated by elements of the form $[x] - \psi^*[x]$ for all $[x] \in P(M)$ and all $\psi \in \mathrm{Diff}(M)$. We call $\widetilde{P}(M)$ the moduli group of positive scalar curvature metrics on M. It measures the size of the moduli space of positive scalar curvature metrics on M. The following conjecture gives a lower bound for the rank of the abelian group $\widetilde{P}(M)$.

Conjecture. *Let M be a closed spin manifold with $\pi_1(M) = \Gamma$ and $\dim M = 2k+1 \geq 5$, which carries a positive scalar curvature metric. Then the rank of the abelian group $\widetilde{P}(M)$ is $\geq N_{\mathrm{fin}}(\Gamma)$, where $N_{\mathrm{fin}}(\Gamma)$ is the cardinality of the following collection of positive integers:*

$$\big\{d \in \mathbb{N}_+ \mid \exists \gamma \in \Gamma \text{ such that } \mathrm{order}(\gamma) = d \text{ and } \gamma \neq e\big\}.$$

We prove that the conjecture holds for all strongly finitely embeddable groups. The notion of strongly finite embeddability into Hilbert space for a group is a stronger version of the notion of finite embeddability into Hilbert space, which was introduced by Weinberger and Yu in [9].

We recall the notion strongly finite embeddability in the following. Before that, we recall the notion of coarse embeddability due to Gromov.

Definition 1 (Gromov)**.** *A countable discrete group* Γ *is said to be coarsely embeddable into Hilbert space* H *if there exists a map* $f : \Gamma \to H$ *such that*

(1) *for any finite subset* $F \subseteq \Gamma$*, there exists* $R > 0$ *such that if* $\gamma^{-1}\beta \in F$*, then* $\|f(\gamma) - f(\beta)\| \leq R$*;*
(2) *for any* $S > 0$*, there exists a finite subset* $F \subseteq \Gamma$ *such that if* $\gamma^{-1}\beta \in \Gamma - F$*, then* $\|f(\gamma) - f(\beta)\| \geq S$*.*

Weinberger and Yu introduced the following notion of finite embeddability for groups, which is more flexible than the notion of coarse embeddability.

Definition 2 ([9])**.** *A countable discrete group* Γ *is said to be finitely embeddable into Hilbert space* H *if for any finite subset* $F \subseteq \Gamma$*, there exists a group* Γ' *that is coarsely embeddable into* H *and there is a map* $\phi : F \to \Gamma'$ *such that*

(1) *if* γ, β *and* $\gamma\beta$ *are all in* F*, then* $\phi(\gamma\beta) = \phi(\gamma)\phi(\beta)$*;*
(2) *if* γ *is a finite order element in* F*, then* $\mathrm{order}(\phi(\gamma)) = \mathrm{order}(\gamma)$*.*

Now we define the notion of strongly finite embeddability. For a countable discrete group Γ, observe that any set of n automorphisms of Γ, say $\psi_1, \cdots, \psi_n \in \mathrm{Aut}(\Gamma)$, induces a natural action of F_n the free group of n generators on Γ. Equivalently, if we denote the set of generators of F_n by $\{s_1, \cdots, s_n\}$, then we have a homomorphism $F_n \to \mathrm{Aut}(\Gamma)$ by $s_i \mapsto \psi_i$. This homomorphism induces an action of F_n on Γ. We denote by $\Gamma \rtimes_{\{\psi_1,\cdots,\psi_n\}} F_n$ the semi-direct product of Γ and F_n with respect to the above action.

Definition 3. *A countable discrete group* Γ *is said to be strongly finitely embeddable into Hilbert space* H*, if* $\Gamma \rtimes_{\{\psi_1,\cdots,\psi_n\}} F_n$ *is finitely embeddable into Hilbert space for all* $n \in \mathbb{N}$ *and any* $\psi_1, \cdots, \psi_n \in \mathrm{Aut}(\Gamma)$*.*

We point out that the class of groups that are strongly finitely embeddable into Hilbert space includes all residually finite groups, amenable groups, hyperbolic groups, and virtually torsion free groups (e.g. $Out(F_n)$). It is an open question if every countable discrete group is strongly finitely embeddable into Hilbert space. It is also open if finite embeddability into Hilbert space implies strongly finite embeddability into Hilbert space.

Now we state the main results from [10].

Theorem 1. *Let* M *be a closed spin manifold which carries a positive scalar curvature metric with* $\dim M = 2k + 1 \geq 5$*. If the fundamental group* $\Gamma = \pi_1(M)$ *of* M *is strongly finitely embeddable into Hilbert space, then the rank of the abelian group* $\widetilde{P}(M)$ *is* $\geq N_{\mathrm{fin}}(\Gamma)$*.*

For general groups, we prove the following weaker version of the above conjecture. This result is motivated by a theorem of Piazza and Schick [5]. They used a different method to show that the moduli space $\mathcal{R}^+(M)/\mathrm{Diff}(M)$ has infinitely many connected components when $\dim M = 2k+1 \geq 5$ and the fundamental group $\pi_1(M)$ is not torsion free [5].

Theorem 2. *Let M be a closed spin manifold which carries a positive scalar curvature metric with $\dim M = 2k+1 \geq 5$. If $\Gamma = \pi_1(M)$ is not torsion free, then the rank of the abelian group $\widetilde{P}(M)$ is ≥ 1.*

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On Furstenberg's measure rigidity conjecture

Manfred Einsiedler

In this talk we describe Furstenberg's measure rigidity conjecture and partial results towards it and its generalizations to higher dimension and homogeneous spaces.

It is well known that for instance the dynamical system defined by $T_p(x) = px$ modulo 1 for $x \in \mathbb{T} = \mathbb{R}/\mathbb{Z}$ has many closed invariant sets and many invariant measures. For instance the standard middle third Cantor set (defined by disallowing the digit 1 in the ternary digit expansion) is a T_3-invariant and closed, and one can construct many more such examples by disallowing other blocks of digits.

Furstenberg [4] proved in 1967 the following amazing rigidity for the joint action of two such transformations.

Theorem 1. *Let $A \subset \mathbb{T}$ be a closed subset such that $T_p(A) \subset A$ and $T_q(A) \subset A$ for two multiplicative independent integers p, q. Then either $A \subset \mathbb{Q}$ is finite or $A = \mathbb{T}$.*

Both in the original proof and also in the shorter proof by Boshernitzan [1] the fact that $\mathbb{N}\log p + \mathbb{N}\log q$ is (although discrete) rather dense far right on the real line plays an important role. To see the importance of this assume for simplicity that 0 is an accumulation point of A. Then we can find for every $\epsilon > 0$ some large $R > 0$ such that $\mathbb{N}\log p + \mathbb{N}\log q$ is ϵ-dense in $[R, \infty)$, find some $x \in A \setminus \{0\}$ so close to zero such that $p^{m_1}q^{n_1}x$ has distance $< \epsilon$ to 0 for the first (m_1, n_1) with $p^{m_1}q^{n_1} > e^R$, and consider the points $p^m q^n x$ with $p^m q^n \geq p^{m_1}q^{n_1}$. It is then not hard to deduce from the ϵ-density of $(\mathbb{N}\log p + \mathbb{N}\log q) \cap [R, \infty)$ in $[R, \infty)$ that these points are at least 2ϵ-dense in $\mathbb{T}$. The main part of the proof consists of reducing the problem to this special case.

Furstenberg also asked what the invariant measures are for this dynamical system. Namely is it true that apart from probability measures with finite support, the Lebesgue measure, and all convex combinations of these, there are no other invariant measures. The first partial result was obtained by Lyons [9]. In 1990 Rudolph [10] and Johnson [5] proved the following result.

Theorem 2. *Let $p, q \geq 2$ be two multiplicatively independent integer. Let μ be a probability measure on $\mathbb{T}$ such that μ is invariant under T_p (i.e. $\mu \circ T_p^{-1} = \mu$) and invariant under T_q and is ergodic with respect to the joint dynamical system (i.e. if $A \subset \mathbb{T}$ is measurable and $T_p^{-1}A = A$ and $T_q^{-1}A = A$ then $\mu(A) \in \{0, 1\}$). If $h_\mu(T_p) > 0$ (or $h_\mu(T_q) > 0$) then μ is the Lebesgue measure.*

Let us briefly recall the definition of the measure-theoretic entropy. Let $\mathcal{P}$ be a finite partition of a probability space (X, μ). The entropy of the partition is defined by

$$H_\mu(\mathcal{P}) = -\sum_{P \in \mathcal{P}} \mu(P) \log \mu(P).$$

The ergodic-theoretic entropy for T_p is now defined by

$$h_\mu(T_p) = \lim_{n \to \infty} \frac{1}{n} H_\mu \left(\left\{ [0, \frac{1}{p^n}), \ldots, [\frac{p^n - 1}{p^n}, 1) \right\} \right),$$

where we note that the partition that is used here is naturally obtained from the partition $\{[0, \frac{1}{p}), \ldots, [\frac{p-1}{p}, 1)\}$ and the dynamics of the transformation T_p. Roughly speaking the entropy equals (in this case) the dimension of the measure μ times the logarithm of the eigenvalue p (or q).

In the higher dimensional case partial results were obtained by A. Katok and Spatzier [6, 7], but the full analogue of Rudolph's theorem was obtained in joint work with E. Lindenstrauss [3].

To state this we let $\alpha : \mathbb{Z}^d \to \mathrm{GL}_n(\mathbb{Z})$ be a homomorphism. We say that the induced action of α on $\mathbb{T}^n$ is irreducible if there is no proper rational subspace in $\mathbb{R}^d$ that is invariant under α, and is totally irreducible if the same holds after restriction to any finite index subgroup of $\mathbb{Z}^d$. We also say that α is virtually-cyclic if $\alpha(\mathbb{Z}^d)$ has a cyclic subgroup with finite index.

Theorem 3. *Let α be a totally irreducible, not virtually-cyclic $\mathbb{Z}^d$-action by automorphisms on $\mathbb{T}^n$. Let μ be an α-ergodic measure. Then either $\mu = \lambda$ is the Haar measure of X, or the entropy $h_\mu(\alpha(\mathbf{m})) = 0$ vanishes for all $\mathbf{m} \in \mathbb{Z}^d$.*

Another generalization of Rudolph's theorem was obtained in joint work with A. Katok and E. Lindenstrauss [2]. For this we let $A < \mathrm{SL}(k, \mathbb{R})$ be the full diagonal subgroup.

Theorem 4. *Let μ be an A-invariant and ergodic measure on $X = \mathrm{SL}(k, \mathbb{R})/\mathrm{SL}(k, \mathbb{Z})$ for $k \geq 3$. Assume that there is some one parameter subgroup of A which acts on X with positive entropy. Then μ is algebraic (i.e. the Haar measure on a closed finite-volume orbit of an intermediate subgroup).*

Especially the latter theorem and its twins for similar spaces are of particular interest due to their connections to number theoretic problems (see e.g. [8, 2]). For instance Littlewood's conjecture in the theory of higher dimensional Diophantine approximation would follow from the conjectured classification of invariant measures (without the positive entropy assumption) and Theorem 4 gives a partial result towards this conjecture.

Over the last years many generalizations of Rudolph's theorem have been proven, which in turn lead to different new proofs of Rudolph's theorem. However, all of the arguments stop working (actually quite early in the various proofs) if the unknown measure has zero entropy. For that reason it would even be interesting to obtain a new partial classification replacing the entropy assumption by a different assumption as for this completely new ideas would have to be used.

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Totally non-free actions and factor-representations of the countable groups

Anatoly Vershik

We explain the new notion of totally non-free action of the measure preserving groups and give the list of such actions and conjugation-invariant measures on the lattice of subgroup of the infinite symmetric group.

CONTENT

1. Two types of the representations of semi-direct product C^*-algebras generated by measure preserving actions of the groups.
2. The definition of the totally non-free actions (TNF) of the group. Lattice $L(G)$ of the subgroups of the group G; and conjugation action of the group on $L(G)$, connection with TNF actions.
3. Problem about existence of the continuous measures which are invariant under conjugation (TNF-measures on $L(G)$ and IRS. Known results.
4. Factor-representations and characters generated by TNF-actions of the groups
5. List of the TNF and $RTNF$ measures for infinite symmetric group.
6. References.

1. Suppose $(X, \mathfrak{A}, \mu)$ is a standard space with continuous finite or sigma-finite measure and G countable group which acts on X preserving the measure μ. Let A is commutative separable C^*-algebra of the complex measurable functions on X (f.e continuous if X is compact space) s.t. if $f \in A$ then $(U_g f)(x) \equiv f(gx) \in A$ and algebra A distinguish the points of x. Let

$$C^*(A; G) \equiv C^*(G) \leftthreetimes A$$

is semidirect product of group algebra $C^*(G)$ and algebra A.

There are well-known two $*$-representation of algebra $C^*(A; G)$ — Koopman in $L^2(X, \mu)$ and von Neumann (or groupoid, or orbit, or trajectory)-representation in $L^2_\nu(\mathcal{X})$, where $\mathcal{X}$ is a graph of orbit equivalence relation on $X \times X$ equipped with measure ν which has measure μ as marginal measure and invariant under the natural action of $G \times G$.

If action of G is ergodic then Koopman representation is irreducible representation of algebra $C^*(A; G)$; and von Neumann representation is irreducible representation of $C^*(A; G) \bigotimes C^*(A; G)$. In the last case restriction of the representation on both multipliers are factor-representation of type II_1 is μ is probability invariant measure etc.

2. Define a map $\Psi : X \to L(G)$ where $L(G)$ is a lattice of all subgroups of the group G equipped with weak topology, and

$$\Psi(x) \equiv Stab_x = \{g \in G : gx = x\}.$$

If $\Psi(x) \equiv \{e\}$ the action called free action.

The action of the group G called *totally non free (TNF)* if for $x, y \in X; x \neq y$ we have: $Stab_x \neq Stab_y$. In this case Ψ is isomorphisms of (X, μ) and $L(G), \Psi_*(\mu) \equiv M_\mu$ and we can consider the triple $(L(G), M_\mu, Conj(G))$ where $Conj(G)$ action of G with conjugation on $L(G)$ — as isomorphic model of the initial action of G on $(X, \mu$. We an say that measure M_μ on$L(G)$ is complete metric invariant of the TNF action.

The criteria of TNF-action is the following: sigma-field of measurable sets, generated by the all fixed-points sets: $Fix_g = \{x : gx = x\}, g \in G$ coincide with whole sigma-field $\mathfrak{A}$.

3. There are two natural questions that have arisen:

I. For what group G there exist conjugation-invariant continuous measures? and

II. For such a group G to describe all conjugation-invariant measures.

It is less known about the first question. In [1] the case of $SL(2, R)$ was considered with name IRS-invariant random subgroup; for the group of rank more than 1 there are no such measures as was proved by R.Zimmer). In the paper [2] nontrivial examples for the free groups was studied. As to the second question we can say only that the list of such measures for infinite symmetric group is obtained in my paper [3, 4] (see below).

Important to emphasize the difference between an arbitrary (even ergodic) measure on $L(G)$ which is invariant under conjugation and measures on $L(G)$ for which conjugation is TNF-action:

Lemma. *The action of the group G by conjugation on $L(G)$ which is equipped with continuous conjugation-invariant measure M is TNF-action iff $M(\{H \in L(G) : N^2(H) = N(H)\}$, where $N(H)$ is normalizer of the subgroup H, and $N^2(H)$ is the second normalizer of H.*

For each measure preserving action of the group G there exists (transfinite in general) sequences of the quotient actions (consecutive factorization over homomorphism Ψ above) whose limit is TNF-action (may be identical action), This follows from the fact that for each subgroup $H \subset G$ there exist a minimal self-normalizer subgroup N^ω which contains H.

4. One of the main reason to consider the TNF-action consist in the following theorem (see [4]):

Theorem 1. *For ergodic TNF-action of the group G denote $\pi_L(C^*(A; G)$ (correspondingly $\pi_R(C^*(A; G))$ the restriction of the groupoid factor-representation of the algebra $C^*(A; G) \bigotimes C^*(A; G)$ in the Hilbert space $L^2_\nu(\mathcal{X})$ on the left (right) multiplier. Then we have reduction:*

$$W^*(\pi_L(C^*(A;G))) = W^*\pi_L(C^*(G)), W^*(\pi_R(C^*(A;G))) = W^*\pi_R(C^*(G)),$$

(Recall that $C^*(G) \subset C^*(A;G)$ as subalgebra) In other words the factor generated by the representation of the semidirect product can be generated by the restriction of the representation of the group itself.

This theorem gives the method of the construction of factor-representations of some groups using its TNF-actions. A remarkable formula for the trace (characters) is the following:

Corollary. *Each ergodic TNF-action of the group G with invariant probability measure defines an indecomposable character of the group G which generates a factor-representation of type II_1:*

$$\chi_\mu(g) = \mu(Fix_g).$$

5. Now we will formulate the main theorem about list of all ergodic measures on $L(G)$ which is invariant under conjugation for infinite symmetric group and also TNF-actions of the this group.

Recall that infinite symmetric group is the countable group $\mathfrak{S}_{\mathbb{N}}$ of the finite permutations of the countable set, say $\mathbb{N}$ or inductive limit of finite symmetric groups with natural embedding. The structure of the lattice of subgroups $L(\mathfrak{S}_{\mathbb{N}})$ is very complicate, but it happened that the continuous conjugation-invariant measures on that lattice is possible to describe using the description of the family of subgroups which can be supports of those measures. We define a map from the space $Part_s(\mathbb{N})$ — of all partitions of the natural ($\mathbb{N}$) to $L(\mathfrak{S}_{\mathbb{N}})$, and then introduce the set of measures on $Part_s(\mathbb{N})$ images of which give the need measure on $L(\mathfrak{S}_{\mathbb{N}})$ (or random invariant subgroup.

Construction. Consider a partition ξ on $\mathbb{N}$ with finite or countable number of infinite blocks (elements)

$\xi = \{N_i; i \in I$ The number of blocks is finite or infinite: $|I| \in \mathbb{N} \cup \infty\}$

Divide the list of blocks I onto several parts: $I = I_0 \cup I^+ \cup I^- \cup I^!$, where

$$I^!(\equiv I \setminus (I_0 \cup I^+ \cup I^-)) = \bigcup_{s \in S} I_s; \quad I_s \subset I^! \quad I_s \cap I_t = \varnothing \quad s \neq t$$

The set of partitions ξ of $\mathbb{N}$ with decomposition of the list of blocks we denoted as $Part_0$

Now we define to each ξ a subgroup H_ξ of $\mathfrak{S}_{\mathbb{N}}$ as follow -this is direct product of the subgroups:

$$H_\xi = \prod_{i \in I^+} \mathfrak{S}(N_i) \times \prod_{i \in I^-} \mathfrak{S}^-(N_i) \times \prod_{s \in S} \mathfrak{S}^\pm(\bigcup_{i \in I_s} N_i),$$

where $\mathfrak{S}^-(N_i)$ is alternation group of finite permutations of N_i; and

$\mathfrak{S}^\pm(\bigcup_{i \in I_s} N_i)$ is the subgroup of index 2 of the product $\prod_{i \in I_s} \mathfrak{S}(N_i)$ of those permutations of $N_i, i \in I_s$, which has the same signature for all i. Remark that on the I_0 our subgroup acts identically.

Now suppose we choose a random partition of $\mathbb{N}$ of Bernoulli type. This means that for probability vector $\alpha = \{\alpha_i\}; \sum_i \alpha = 1, \alpha > 0$ we define the sequence of independent variables $x = \{x_k\}$ with values in $0, 1, \dots$ and probabilities $Prob\{x_k = i\} = \alpha_i, r = 1 \dots.$ Then the partition ξ_x has blocks $N_i = \{k \in \mathbb{N} : x_k = i.$ Finally we can divide (in arbitrary manner) the set of the blocks on the positive, negative blocks, and others The formula above gives the random subgroup H_{ξ_x} and corresponding measure on $L(\mathfrak{S}_{\mathbb{N}})$ which we denote as $M_{\alpha,\xi}$. The invariance this, measure under the conjugation follows immediately from Bernoulli property.

Theorem 2. *Each ergodic Borel continuous probability measure on $L(\mathfrak{S}_{\mathbb{N}})$ which is invariant under the conjugation is one of the measures $M_{\alpha,\xi}$.*

The proof uses classical theorems (like Jordan theorem about primitive subgroup) and other facts.

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Remarks on real and p-adic Fuglede–Kadison determinants

Christopher Deninger

In the talk we reported on the calculation of Fuglede–Kadison determinants of "algebraic" elements in certain crossed product von Neumann algebras [1]. We also made some advertisment for a p-construction which may be viewed as a p-adic analogue of the Fuglede–Kadison determinant [2]. In different terms such a construction is also contained in greater generality in the work of Karoubi as explained by the latter during the conference. We thank Karoubi very much for sharing his insights.

We begin by recalling the Oseledets multiplicative ergodic theorem. Let γ be a measure preserving ergodic automorphism of a standard probability space $(\Omega, \mathcal{A}, \mu)$. For a measurable map $A : \Omega \to \mathrm{GL}_N(\mathbb{C})$ let $(A_n(\omega))_{n\in\mathbb{Z}}$ be the cocycle attached to A:

$$\begin{aligned} A_n(\omega) &= A(\omega)\cdots A(\gamma^{n-1}(\omega)) \quad \text{for } n > 0 \\ A_n(\omega) &= A(\gamma^{-1}(\omega))^{-1}\cdots A(\gamma^{n}(\omega))^{-1} \quad \text{for } n < 0 \end{aligned}$$

and $A_0(\omega) = 1$. Setting $\log^+ x = \max(\log x, 0)$ we have:

Theorem 1 (Oseledets). *Assume that $\log^+ \|A^{\pm 1}(\omega)\|_\sigma$ is integrable over Ω. Then there are:*

(a) a measurable γ-invariant subset Ω' of Ω with $P(\Omega \setminus \Omega') = 0$

(b) *an integer* $1 \leq M \leq N$ *and real numbers* $\chi_1 < \chi_2 < \ldots < \chi_M$*, the Ljapunov exponents of* A

(c) *positive integers* $r_1, \ldots, r_M$ *with* $r_1 + \ldots + r_M = N$*, the multiplicities of the* χ_j

(d) *measurable maps* $V_j : \Omega' \to \mathrm{Gr}_{r_j}(\mathbb{C}^N)$ *into the Grassmannian space of* r_j*-dimensional subspaces of* $\mathbb{C}^N$*, such that the following assertions hold for all* $\omega \in \Omega'$

(i) $\mathbb{C}^N = \bigoplus_{j=1}^M V_j(\omega)$

(ii) $V_j(\omega)A(\omega) = V_j(\gamma(\omega))$ *and hence* $V_j(\omega)A_n(\omega) = V_j(\gamma^n(\omega))$ *for all* $n \in \mathbb{Z}, 1 \leq j \leq M$.

(iii) *For* $v \in V_j(\omega), v \neq 0$ *we have*

$$\lim_{n\to\pm\infty} \frac{1}{n} \log \frac{\|vA_n(\omega)\|}{\|v\|} = \chi_j \quad \textit{uniformly in } v .$$

(iv) $\sum_{j=1}^M r_j \chi_j = \int_\Omega \log|\det A(\omega)|\, dP(\omega)$.

For the proof see [5] V (2.1) Theorem and remark (5) and App. A. If A is constant, the Ljapunov exponents are the logarithms of the absolute values of the eigenvalues of A. For $N = 1$ and A given by a scalar function $a : \Omega \to \mathbb{C}^*$ the only Ljapunov exponent is:

$$\chi = \int_\Omega \log|a(\omega)|\, dP(\omega) .$$

This follows from *(iv)* or from Birkhoff's ergodic theorem applied to $\log|a(\omega)|$.

Consider the von Neumann algebra $\mathcal{R} = L^\infty(\Omega) \rtimes_\gamma \mathbb{Z}$ with its canonical normalized trace τ and the von Neumann algebra $M_N(\mathcal{R}) = M_N(L^\infty(\Omega)) \ltimes_\gamma \mathbb{Z}$ with trace $\tau_N = \tau \circ \mathrm{tr}$. Note that $\tau_N(1) = N$. The following result calculates the Fuglede–Kadison determinant and more generally the Brown measure of the operator $AU \in M_N(\mathcal{R})$ where A as above is viewed as an element of $M_N(L^\infty(\Omega)) \subset M_N(\mathcal{R})$ and U is the standard unitary operator corresponding to the γ-operation.

Theorem 2. *Let* γ *be a measure preserving ergodic automorphism of a probability space* $(\Omega, \mathcal{A}, P)$*. Assume that* $(\Omega, \mathcal{A})$ *is a standard Borel space. Let* $A : \Omega \to \mathrm{GL}_N(\mathbb{C})$ *be a measurable map for which* $\|A(\omega)\|_\sigma$ *is essentially bounded and* $\log^+ \|A^{-1}(\omega)\|_\sigma$ *integrable over* Ω*. Viewing* A *as an element of* $M_N(L^\infty(\Omega)) \subset M_N(\mathcal{R})$ *we have the following formula:*

$$\log \det_{M_N(\mathcal{R})}(1 - AU) = \sum_{j=1}^M r_j \chi_j^+ . \tag{1}$$

Set $M_j = \exp \chi_j$ and let μ be the Haar measure on S^1. The Brown measure of AU is given by

$$\mu_{AU} = \sum_{i=1}^{M} r_j (M_{j^*} \mu) \; .$$

Its support is the union of the circles $\{|\lambda| = M_j\}$ for $1 \le j \le M$. If $M = 1$, so that there is only one Ljapunov exponent, we have

$$\log \det{}_{M_N(\mathcal{R})}(1 - AU) = \left(\int_\Omega \log |\det A(\omega)| \, dP(\omega) \right)^+ \; . \tag{2}$$

On the other hand, if all Ljapunov exponents are non-negative we have:

$$\log \det{}_{M_N(\mathcal{R})}(1 - AU) = \int_\Omega \log |\det A(\omega)| \, dP(\omega) \; . \tag{3}$$

Sketch of proof: Using the technique of Margulis' proof of the Oseledets theorem [5], Appendix, one reduces to triangular A and from there to scalar valued functions $a = A$ i.e. to $N = 1$. With the help of suitable automorphisms one may assume that $a = |a|$ is non-negative. Using a result of Dykema and Schultz [4] on the Brown measure of $|a|U$, one concludes the result.

Consider an element Φ of $\mathcal{R}$ of the form

$$\Phi = a_N U^N + \ldots + a_1 U + 1 \quad \text{with } a_i \in L^\infty(\Omega)$$

and let A_Φ be its companion matrix in $M_N(L^\infty(\Omega))$. Then we have the following result:

Theorem 3. *Let γ be a measure preserving ergodic automorphism of a probability space $(\Omega, \mathcal{A}, P)$ where $(\Omega, \mathcal{A})$ is a standard Borel space. Assume that $\log |a_N|$ is integrable. Then A_Φ satisfies the assumptions of theorems 1 and 2 and we have the formula*

$$\log \det{}_{\mathcal{R}} \Phi = \sum_j r_j \chi_j^+ \; . \tag{4}$$

Here the χ_j's and r_j's are the Ljapunov exponents of A_Φ and their multiplicities.

Writing the von Neumann algebra of the discrete Heisenberg group as a direct integral over rotation algebras, one obtains formulas for the Fuglede–Kadison determinants of certain elements in the integral group algebra. As entropies the same expressions had been found previously by Lind and Schmidt. This was the motivation for the above results because by [3] those entropies are given by logarithms of Fuglede–Kadison determinants.

As for the second topic of the lecture we will be short. For a countable discrete group Γ let $c_0(\Gamma)$ be the following p-adic Banach algebra with $\| \sum a_\gamma \gamma \|_p = \max_\gamma |a_\gamma|_p$

$$c_0(\Gamma) = \left\{ \sum_{\gamma \in \Gamma} a_\gamma \gamma \mid a_\gamma \in \mathbb{Q}_p \; , \; |a_\gamma|_p \to 0 \text{ for } \gamma \to \infty \right\} \; .$$

It is a p-adic substitute for the L^1-group algebra of Γ. In [2] we defined a homomorphism

$$\log_p \det_\Gamma : c_0(\Gamma)^\times \longrightarrow \mathbb{Q}_p$$

which for $\Gamma \cong \mathbb{Z}$ reduces to a p-adic Snirelman integral. Since we used deep facts about algebraic K-theory we had to assume that Γ was residually finite, elementary amenable and torsion free. The homomorphism $\log_p \det_\Gamma$ is an analogue of the homomorphism

$$\log \det_{\mathcal{N}\Gamma} : L^1(\Gamma)^\times \longrightarrow \mathbb{R} .$$

Set $c_0(\Gamma, \mathbb{Z}_p) = \big\{ \sum_{\gamma\in\Gamma} a_\gamma \gamma \mid a_\gamma \in \mathbb{Z}_p \, , \, |a_\gamma|_p \to 0 \, \text{for} \, \gamma \to \infty \big\}$. On the one-units $U^1_\Gamma = 1 + pc_0(\Gamma, \mathbb{Z}_p)$ of the $\mathbb{Z}_p$-Banach algebra $c_0(\Gamma, \mathbb{Z}_p)$, the map $\log_p \det_\Gamma$ is given by the convergent series

$$\log_p \det_\Gamma(f) = \sum_{\nu=1}^{\infty} \frac{(-1)^{\nu-1}}{\nu} \tau(1-f)^\nu . \tag{5}$$

Here $\tau : c_0(\Gamma) \to \mathbb{Q}_p$ is the trace $\tau(\sum_\gamma a_\gamma \gamma) = a_0$. Using the p-adic Baker–Campbell–Hausdorff formula one sees that (5) defines a homomorphism

$$\log_p \det_\Gamma : U^1_\Gamma \longrightarrow \mathbb{Q}_p$$

for every countable discrete group Γ.

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Interpolating Carlitz zeta values

Federico Pellarin

(joint work with Bruno Anglès, Floric Tavares-Ribeiro)

In this talk we have discussed some variants of the sums

$$\sum_{n\geq 1} n^{-k} \in \mathbb{R}, \quad k > 1 \tag{1}$$

in the framework of function fields of positive characteristic. The starting point of our investigations is contained in some papers by Leonhard Carlitz (1907-1999), most of them written in the years 1935-1941 [5, 6, 7, 8]. Carlitz introduced some convergent series offering a naive resemblance with the values (1). We begin by introducing our settings. We then partially review Carlitz's theory also in the light of some recent advances by Taelman and finally, we discuss our contributions: we shall introduce a generalization of these Carlitz zeta values which can be seen at once as elements in Tate algebras and entire functions having some similarity with the Riemann zeta function. Some results about these new zeta values will be described.

1. Settings

Let $\mathbb{F}_q$ be the finite field with q elements and characteristic p, let θ be an indeterminate over $\mathbb{F}_q$. We shall use the ring $A = \mathbb{F}_q[\theta]$, its fraction field $K = \mathbb{F}_q(\theta)$ and its completion at the infinite prime $K_\infty = \mathbb{F}((\theta^{-1}))$. Over K_∞, we have the absolute value $|\cdot|$ associated to the θ^{-1}-adic valuation v_∞ which is uniquely determined by setting $|\theta| = q$.

We shall work in the field $\mathbb{C}_\infty$, completion of an algebraic closure of K_∞ for the unique extension of v_∞; this field is algebraically closed and complete. It is for us a valid substitute of the field of complex numbers $\mathbb{C}$. Notice however that $\mathbb{C}_\infty$ is not locally compact; also, $\mathbb{C}_\infty$ is an infinite dimensional vector space over K_∞.

2. Some contribution by Carlitz

Just as $\mathbb{Z}$ in $\mathbb{R}$, the ring A is co-compact in K_∞. This feature prompted Carlitz to introduce the following series:

$$\zeta_C(n) = \sum_{a \in A^+} a^{-k} = \prod_P \left(1 - \frac{1}{P}\right)^{-1}, \quad k > 0,$$

where the sum is over the set A^+ of monic polynomials of A and the product is over the *primes* of A, that is, the irreducible polynomials of A^+. The series above are easily proved to define elements in K_∞.

2.1. Link with the Exponential.

The link, first noticed by Carlitz, is the following. Consider the sequence of polynomials $D_0 = 1$ and $D_n = (\theta^{q^n} - \theta)D_{n-1}^q$ for $n > 0$. We set, for $x \in \mathbb{C}_\infty$:

$$\exp_C(X) = \sum_{n \geq 0} D_n^{-1} X^{q^n}.$$

It is easily seen that $\exp_C$ defines an entire, $\mathbb{F}_q$-linear, surjective map $\mathbb{C}_\infty \to \mathbb{C}_\infty$. Let us also set (choice of the $(q-1)$-th root of $-\theta$ in $\mathbb{C}_\infty$):

$$\widetilde{\pi} = \theta(-\theta)^{\frac{1}{q-1}} \prod_{i>0} (1 - \theta^{1-q^i})^{-1} \in K_\infty.$$

It is easy to show that $\widetilde{\pi}$ is transcendental over K. Carlitz proved that $\mathrm{Ker}(\exp_C) = \widetilde{\pi}A$ (an $\mathbb{F}_q$-vector space of infinite dimension). Furthermore, there exists a unique structure $C(\mathbb{C}_\infty)$ of A-module over $\mathbb{C}_\infty$, called the *Carlitz module* for which $\exp_C$ becomes an homomorphism of A-modules. The Carlitz module is defined by the action of θ: we have that $\phi_\theta \in \mathrm{End}_{\mathbb{F}_q\text{-lin.}}(\mathbb{G}_a(\mathbb{C}_\infty))$ is defined by $\phi_\theta = \theta + \tau$, where $\tau(x) = x^q$ for all $x \in \mathbb{C}_\infty$.

Theorem 1 (Carlitz)**.** *Let $k>0$ be an integer divisible by $q-1$. Then, $\zeta_C(k)\in K^\times\widetilde{\pi}^k$.*

Carlitz recognized, in the rational factors of proportionality of the above result, familiar structures leading to what are called now *Bernoulli-Carlitz polynomials.* These objects are known to satisfy e.g. a variant of Clausen-von Staudt Theorem.

2.2. **Link with the logarithm.** If $X\in\mathbb{C}_\infty$ is such that $|X|<|\widetilde{\pi}|$, the formal local reciprocal $\log_C$ of $\exp_C$ at zero converges; it is called *Carlitz's logarithm.* Carlitz proved:

Theorem 2 (Carlitz)**.** *We have $\exp_C(\zeta_C(1))=1$. Furthermore, $\zeta_C(1)=\log_C(1)$.*

Again, one can prove that $\log_C(1)$ is transcendental over K. Very recently, Lenny Taelman [3] provided an interpretation of Carlitz formula as an instance of the *Class number formula.* He introduced, in the broader framework where, instead of the Carlitz module $C_\theta=\theta+\tau$ we have a *Drinfeld A-module* $\phi_\theta=\theta+\alpha_1\tau+\cdots+\alpha_r\tau^r$ with $\alpha_1,\ldots,\alpha_r\in\mathbb{C}_\infty$ integral over A, a *unit A-module* and a *class A-module.* He associated to ϕ an *L-series value* $L(\phi)$ which he decomposed in product of a *regulator* in the unit module times a generator of the Fitting ideal of the class module. In the case $\phi=C$, we have $L(C)=\zeta_C(1)$, the class module is trivial, and the regulator precisely is $\log_C(1)$, whence the fact that Carlitz formula in Theorem 2 is an instance of Class number formula.

3. Results

We have extended the above investigations to a wider class of zeta values. We give here some examples of these new values. Let us consider s indeterminates $t_1,\ldots,t_s$ and denote by $\mathbb{T}_s$ the Tate algebra of formal series of $\mathbb{C}_\infty[[t_1,\ldots,t_s]]$ which converge in the unit polydisk of radius one with boundary, containing $(0,\ldots,0)$. The s-dimensional Tate algebra, for the Gauss absolute value $\|\cdot\|$, is a $\mathbb{C}_\infty$-Banach algebra and an ultrametric ring. For $i=1,\ldots,s$, let us denote by χ_{t_i} the ring homomorphism $A\to\mathbb{T}_s$ defined, for $a=\sum_i a_i\theta^i\in\mathbb{F}_q[\theta]$, by $\chi_{t_i}(a)=\sum_i a_it^i$. Our new "zeta values" are then (see [2, 1]), for all $k>0$ and $s\geq 0$:

$$L(k,s)=\sum_{a\in A^+}\chi_{t_1}(a)\cdots\chi_{t_s}(a)a^{-k}\in\mathbb{T}_s.$$

the set of these values includes the Carlitz zeta values: $L(k,0)=\zeta_C(k)$. There exists a unique $\mathbb{F}_q[t_1,\ldots,t_s]$-linear extension of Carlitz module to the Tate algebra, denoted by $C(\mathbb{T}_s)$. The exponential function $\exp_C$ extends $\mathbb{F}_q[t_1,\ldots,t_s]$-linearly to a continuous endomorphism of $\mathbb{T}_s$. The torsion of this extended C is particularly interesting. For instance, we mention, in the case of $s=1$ (case in which we write $\mathbb{T}$ for $\mathbb{T}_1$ and $t=t_1$) that the $(t-\theta)$-torsion point

$$\omega(t)=\exp_C\left(\frac{\widetilde{\pi}}{\theta-t}\right)\in\mathbb{T}$$

precisely is *Anderson-Thakur function*, a function whose arithmetic properties are also related to the arithmetic of the special values of the so-called Thakur's *geometric gamma function.* Let $s \geq 0$ be an integer as above. We proved the following result, where we analyzed the case $k = 1$:

Theorem 3. *(1) If $s \equiv 1 \pmod{q-1}$, $s \geq 2$, then*

$$L(1,s)\omega(t_1)\cdots\omega(t_s) \in \widetilde{\pi}A[t_1,\ldots,t_s] \setminus \{0\}.$$

(2) If $s \not\equiv 1 \pmod{q-1}$, then

$$\exp_C(L(1,s)\omega(t_1)\cdots\omega(t_s)) \in \omega(t_1)\cdots\omega(t_s)A[t_1,\ldots,t_s] \setminus \{0\}.$$

The first part of our Theorem is a special case of results in [1] while the second part will appear in a forthcoming paper of the three contributors of this collaboration. The remaining zeta-value to consider is $L(1,1)$ and a variant of (1) in the above Theorem holds true, the formula (see [2]):

$$L(1,1) = \frac{\widetilde{\pi}}{(\theta - t)\omega(t)}. \tag{2}$$

In the above results, we have viewed the series $L(1,s)$ as "numbers" in Tate algebras (sums of series which are convergent for the Gauss absolute value). Of course, we can also look at them as functions of their variables. Then, the identities we obtained at the point (1) of the above Theorem can be considered as some kind of functional equations. The function ω^{-1} is entire with simple zeros at $t = \theta, \theta^q, \theta^{q^2}, \ldots$. In particular, (2) yields entire continuation of the function $L(1,1)$ in the variable t, information about its trivial zeroes (corresponding to the poles of ω) and explicit formulas for special values according to Theorem 1. Goss [4] already introduced a "zeta" function allowing to continuously interpolate the Carlitz zeta values. Our construction leads however to a different class of functions.

As for the point (2) of our Theorem, it generalizes Theorem 2 and nicely agrees with part of the predictions of Stark conjectures.

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First-order condition and a semi-group of inner fluctuations

Walter D. van Suijlekom
(joint work with Ali Chamseddine, Alain Connes)

We generalize inner fluctuations to real spectral triples that fail on the first-order condition. In this case, the usual prescription [1] does not apply, since the operator $D + A \pm JAJ^{-1}$ with gauge potential $A = \sum_j a_j[D, b_j]$ $(a_j, b_j \in \mathcal{A})$ does not behave well with respect to the action of the gauge group $\mathcal{U}(\mathcal{A})$. In fact, one would require that conjugation of the fluctuated Dirac operator by the unitary operator $U := uJuJ^{-1}$ for $u \in \mathcal{U}(\mathcal{A})$ can be implemented by a usual type of gauge transformation $A \mapsto A^u = u[D, u^*] + uAu^*$ so that

$$D + A \pm JAJ^{-1} \mapsto U(D + A \pm JAJ^{-1})U^* \equiv D + A^u \pm JA^uJ^{-1}$$

However, the simple argument only works if $[JuJ^{-1}, A] = 0$ for gauge potentials A of the above form and $u \in \mathcal{U}(\mathcal{A})$, that is, if the first-order condition is satisfied.

For real spectral triples that possibly fail on the first-order condition one starts with a self-adjoint, *universal* one-form

$$A = \sum_j a_j \delta(b_j); \qquad (a_j, b_j \in \mathcal{A}). \tag{1}$$

The inner fluctuations of a real spectral triple $(\mathcal{A}, \mathcal{H}, D; J)$ are then given by

$$D' = D + A_{(1)} + \tilde{A}_{(1)} + A_{(2)} \tag{2}$$

where

$$A_{(1)} := \sum_j a_j[D, b_j],$$
$$\tilde{A}_{(1)} := \sum_j \hat{a}_j[D, \hat{b}_j]; \qquad \hat{a}_i = Ja_iJ^{-1}, \qquad \hat{b}_i = Jb_iJ^{-1},$$
$$A_{(2)} := \sum_j \hat{a}_j[A_{(1)}, \hat{b}_j] = \sum_{j,k} \hat{a}_j a_k[[D, b_k], \hat{b}_j].$$

Clearly $A_{(2)}$ which depends quadratically on the fields in $A_{(1)}$ vanishes when the first order condition is satisfied, thus reducing to the usual formulation of inner fluctuations. As such, we will interpret the terms $A_{(2)}$ as non-linear corrections to the *first-order*, linear inner fluctuations $A_{(1)}$ of $(\mathcal{A}, \mathcal{H}, D; J)$.

The need for such quadratic terms can also be seen from the structure of *pure gauge* fluctuations $D \mapsto UDU^*$ with $U = uJuJ^{-1}$ and $u \in \mathcal{U}(\mathcal{A})$. Indeed, in the absence of the first order condition we find that

$$UDU^* = u[D, u^*] + \hat{u}[D, \hat{u}^*] + \hat{u}[u[D, u^*], \hat{u}^*].$$

In the above prescription this corresponds to taking as a universal one-form $A = u\delta(u^*)$.

On a fluctuated Dirac operator D' such a gauge transformation acts in a similar way as $D' \mapsto UD'U^*$. By construction, it is implemented by the gauge transformation

$$A \mapsto uAu^* + u\delta(u^*)$$

in the universal differential calculus. In particular, this implies that

$$A_{(1)} \mapsto uA_{(1)}u^* + u[D, u^*]$$

so the first-order inner fluctuations transform as usual. For the term $A_{(2)}$ we compute that a gauge transformation acts as

$$A_{(2)} \mapsto JuJ^{-1}A_{(2)}Ju^*J^{-1} + JuJ^{-1}[u[D, u^*], Ju^*J^{-1}]$$

where the $A_{(2)}$ on the right-hand-side is expressed using the gauge transformed $A_{(1)}$. This non-linear gauge transformation for $A_{(2)}$ confirms our interpretation of $A_{(2)}$ as the non-linear contribution to the inner fluctuations.

It turns out [2] that inner fluctuations come from the action on operators in Hilbert space of a semi-group $\mathrm{Pert}(\mathcal{A})$ of *inner perturbations* which only depends on the involutive algebra $\mathcal{A}$ and extends the unitary group of $\mathcal{A}$. More precisely, the semi-group $\mathrm{Pert}(\mathcal{A})$ consists of *normalized self-adjoint* elements in $\mathcal{A} \otimes \mathcal{A}^{\mathrm{op}}$:

$$\mathrm{Pert}(\mathcal{A}) := \left\{ \sum_j a_j \otimes b_j^{\mathrm{op}} \in \mathcal{A} \otimes \mathcal{A}^{\mathrm{op}} : \sum_j a_j b_j = 1, \quad \sum_j a_j \otimes b_j^{\mathrm{op}} = \sum_j b_j^* \otimes a_j^{*\mathrm{op}} \right\}$$

with $\mathcal{A}^{\mathrm{op}}$ the involutive algebra $\mathcal{A}$ but with the opposite product $(ab)^{\mathrm{op}} = b^{\mathrm{op}}a^{\mathrm{op}}$. The semi-group product is inherited from the multiplication in the algebra $\mathcal{A} \otimes \mathcal{A}^{\mathrm{op}}$, that is:

$$\left(\sum_i a_i \otimes b_i^{\mathrm{op}} \right) \left(\sum_j a'_j \otimes (b'_j)^{\mathrm{op}} \right) = \sum_{i,j} a_i a'_j \otimes (b'_j b_i)^{\mathrm{op}},$$

which indeed respects the above normalization and self-adjointness condition. Note that the unitary group of $\mathcal{A}$ is mapped to $\mathrm{Pert}(\mathcal{A})$ by sending a unitary u to $u \otimes u^{*\mathrm{op}}$.

Given a spectral triple $(\mathcal{A}, \mathcal{H}, D)$ an inner fluctuation of D by an element $\sum_j a_j \otimes b_j^{\mathrm{op}}$ in $\mathrm{Pert}(\mathcal{A})$ is now simply given by

$$D \mapsto \sum_j a_j D b_j.$$

This covers both cases of ordinary spectral triples and real spectral triples (*i.e.* those which are equipped with the operator J). In the latter case one simply uses the natural homomorphism of semi-groups $\mu : \mathrm{Pert}(\mathcal{A}) \to \mathrm{Pert}(\mathcal{A} \otimes \hat{\mathcal{A}})$ given by $\mu(A) = A \otimes \hat{A}$. Explicitly, this implies for real spectral triples the following transformation rule:

$$D \mapsto \sum_{i.j} a_i \hat{a}_j D b_i \hat{b}_j$$

which can indeed be shown [2, Proposition 5] to coincide with the above (2).

The structure of a semi-group implies in particular that inner fluctuations of inner fluctuations are still inner fluctuations —a fact which is not at all direct when looking at Equation (2)— and that the corresponding algebraic rules are unchanged by passing from ordinary spectral triples to real spectral triples.

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Equations for Fourier

Jean-François Burnol

1. Dirac-Klein-Gordon equation

We study finite energy solutions to the two dimentional Klein-Gordon equation. It is convenient to use the Dirac form

$$\partial_t\phi + \partial_x\phi = -\psi \ , \tag{1}$$

$$\partial_t\psi - \partial_x\psi = +\phi \tag{2}$$

It can be shown that that part of the energy (a conserved quantity) which, on the time slice $t = T$, comes from *outside* of the light cone, goes to zero when $T \to \infty$. Related to this is an equivalent unitary representation of the L^2 space of Cauchy data on the $t = 0$ line with a suitable L^2 space either on the future or the past light cone.

Especially: solutions inside the Rindler wedge, $0 < |t| < x$ are determined either by the (boundary) values $\phi(-x, x)$ on the past light cone, or the values $\psi(x, x)$ on the future light cone, or the Cauchy data $\begin{bmatrix} \psi(x,0) \\ \phi(x,0) \end{bmatrix}$ at time $t = 0$ $(x > 0)$, and they are related by the unitary equivalences

$$\int_0^\infty |\phi(0,x)|^2 + |\psi(0,x)|^2\,dx = 2\int_0^\infty |\psi(x,x)|^2\,dx = 2\int_0^\infty |\phi(-x,x)|^2\,dx \ . \tag{3}$$

The unitary operator $\mathcal{H}$ transforming the past data $\phi(-x, x)$ into the future data $\psi(x, x)$ is the one with kernel $J_0(2\sqrt{xy})$. It can also be computed using the Mellin transform $\widehat{f}(s) = \int_0^\infty f(x)x^{-s}\,dx$: $\widehat{\mathcal{H}f}(s) = \frac{\Gamma(1-s)}{\Gamma(s)}\widehat{f}(1-s)$.

Starting with arbitrary Cauchy data (ψ, ϕ) at $t = 0$ vanishing on $(0, 2a)$ we thus, by causality, generate all $\mathcal{H}$-transform pairs $f(x) = \phi(-x, x)$, $g(x) = \psi(x, x)$ $(x > 0)$ with $f|_{(0,a)} \equiv 0$ and $g|_{(0,a)} \equiv 0$.

Ref.: *The results in this section are all contained in* [2].

2. Spaces $\boldsymbol{K_{a,\nu}}$, integral equations and linear systems

We study more generally the operator $\mathcal{H}_\nu$ with kernel $J_\nu(2\sqrt{xy})$ ($\nu > -1$). One has:

$$\widehat{\mathcal{H}_\nu f}(s) = \frac{\Gamma(1-s+\frac{\nu}{2})}{\Gamma(s+\frac{\nu}{2})}\widehat{f}(1-s) \qquad (\Re(s)=\tfrac{1}{2}) \tag{4}$$

This equation holds only in the L^2-sense when f is a general square integrable function. Let however $K_{a,\nu}$ for each $a>0$ be the subspace of the f's with $f|_{(0,a)} \equiv 0$ and $\mathcal{H}_\nu(f)|_{(0,a)} \equiv 0$. For such an f its Mellin transform $\widehat{f}$ extends to an *entire function* and it has "trivial zeros" at the poles of $\Gamma(s+\frac{\nu}{2})$.

The equations in $L^2(0,a;dx)$:

$$f(x) \pm \int_0^a J_\nu(2\sqrt{xy}) f(y)\,dy = J_\nu(2\sqrt{ax}) \tag{5}$$

have unique solutions f_a^+ and f_a^- which are analytic functions (with a branch $x^{\nu/2}$). They thus make sense for $x>a$ and we then define (for a given ν):

$$\mathcal{A}_a(s) = \Gamma(s+\frac{\nu}{2})\frac{\sqrt{a}}{2}\left(a^{-s} + \int_a^\infty f_a^+(x)x^{-s}\,dx\right) \tag{6}$$

$$-i\mathcal{B}_a(s) = \Gamma(s+\frac{\nu}{2})\frac{\sqrt{a}}{2}\left(a^{-s} - \int_a^\infty f_a^-(x)x^{-s}\,dx\right) \tag{7}$$

These functions, initially defined in a right half-plane, are entire and have all their zeros (which are simple) *on the critical line.* Let's now consider:

$$\mu_\nu(a) = a\frac{d}{da}\log\frac{\det 1+\mathcal{H}_\nu|_{(0,a)}}{\det 1-\mathcal{H}_\nu|_{(0,a)}} \tag{8}$$

The following system holds:

$$\left(a\frac{d}{da}+\mu_\nu(a)\right)\mathcal{A}_a(\frac{1}{2}+i\gamma) = -\gamma\mathcal{B}_a(\frac{1}{2}+i\gamma) \tag{9}$$

$$\left(a\frac{d}{da}-\mu_\nu(a)\right)\mathcal{B}_a(\frac{1}{2}+i\gamma) = +\gamma\mathcal{A}_a(\frac{1}{2}+i\gamma) \tag{10}$$

This allows to interpret the phase functions on the critical line

$$\pm\frac{\Gamma(1-s+\frac{\nu}{2})}{\Gamma(s+\frac{\nu}{2})} \tag{11}$$

as the associated scatterings defined by the corresponding Schrödinger equations (or Dirac system). The (imaginary part of the) zeros (all on the critical line) arise as spectrum of the associated second order operator under boundary condition at a (or rather $\log a$).

Ref. *the special case of the cosine and sine transforms was done in* [1], *and the generally applicable technique is completely detailed for $\nu = 0$ in* [3] *(some further portions of this paper are specific to $\nu = 0$.)*

3. Non linear equations

The following non-linear differential system holds:

$$(a\frac{d}{da} - \nu)\frac{\mu_\nu}{2a} = -(1 - \frac{\mu_\nu^2}{4a^2})\mu_{\nu+1} \tag{12}$$

$$(a\frac{d}{da} + \nu + 1)\frac{\mu_{\nu+1}}{2a} = +(1 - \frac{\mu_{\nu+1}^2}{4a^2})\mu_\nu \tag{13}$$

The quotient $Q = \frac{\mu_{\nu+1}(a)}{\mu_\nu(a)}$ satisfies a PIII equation:

$$\frac{d}{da}a\frac{d}{da}\log Q = 4a(Q^2 - Q^{-2}) - (4\nu + 2)(Q - Q^{-1}) + 2(Q + Q^{-1})\ , \tag{14}$$

and $1 - \frac{\mu_\nu^2}{4a^2}$ is a PV transcendent.

Replacing $\Gamma(s)^{-1}$ by a rational function $s(s+1)\dots(s+n-1)$, one ends up with functions $\mu_{\nu,n}(a)$ (which are rational functions of a and a^ν), for each $n \geq 1$, and

$$\boxed{a^2 \mapsto a\frac{\mu_{\nu+1,n}(a)}{\mu_{\nu,n}(a)} \text{ is a } \mathsf{PVI} \text{ function.}}$$

Ref. *The μ functions associated with the cosine and sine transforms were defined in* [1], *expressed as here in terms of relevant Fredholm determinants, and related to scattering. The non-linear system was obtained in January 2008 in letters to Philippe Biane, who confirmed that they implied a Painlevé equation and also suggested to study the 'toy' Gamma functions. The functions $\mu_{\nu,n}$ were then defined and studied by the author in further letters to Philippe Biane, during February-April 2008, concluding in the PVI theorem. Another approach to the definition of the functions $\mu_{\mu,n}$, with a proof of the PVI property, was then published in* [4].

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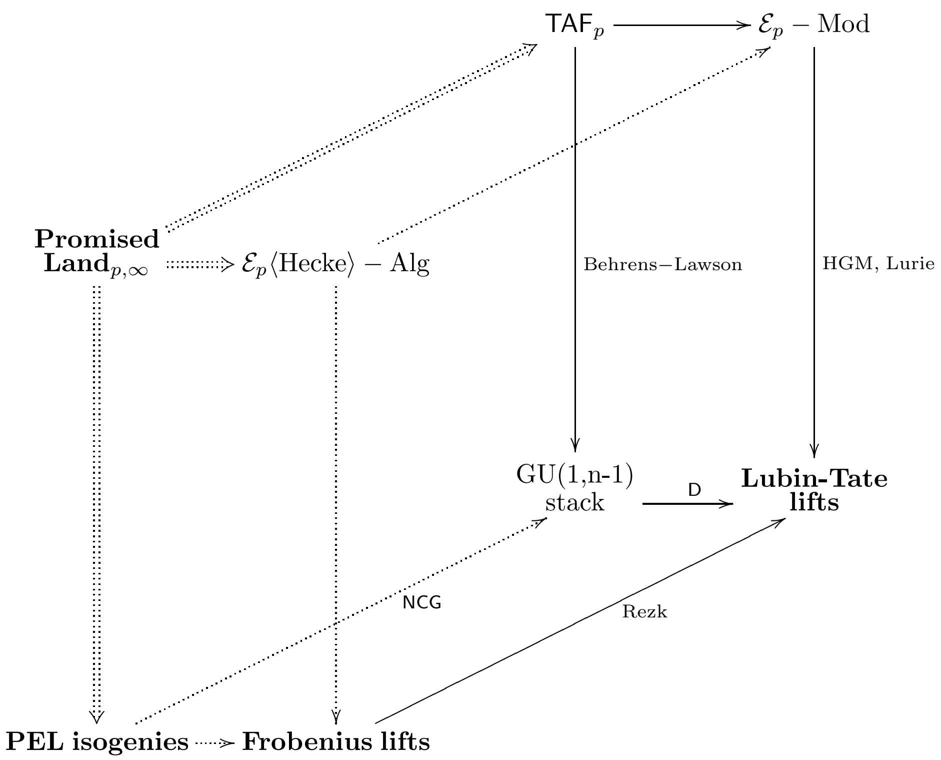
Distributions on Lubin-Tate groups

Jack Morava

A (sketch for kind of) BLUEPRINT over Spec $(\mathbb{F}_1 \times_{\$_1} \$_p)$



The far wall of this diagram displays some recent work in stable homotopy theory as a fiber product of categories. The diagram itself conjectures a second pullback, using recent work on Hecke operations in **un**stable homotopy theory, defining an enriched site defined by a hypothetical Shimura **monoid**, hopefully possessing interesting KMS states.

Construing the diagram

1 Blueprint is meant here **not** as a technical term, and Spec $(\mathbb{F}_1 \times_{\$_1} \$_p)$ is similarly a metaphor (following Toën & Vaquié) for a kind of *trait*, with one finite and one infinite place: where we can hope to speak of both C^*-algebras and p-local homotopy theory.

2 A **pile** is a generalization of a stack, but with morphisms which are **almost** invertible (e.g. isogenies, or Fredholm maps . . .). The base of the diagrams consists of piles and stacks, while the top displays categories of sheaves (of modules or algebras) over them.

3 Chronologically, the lower back right-hand corner is the natural place to start:

Quillen showed that a **geometric cycle map** for **complex divisors** (defined for some nice cohomology theory or ring-spectrum) corresponds to a **one-dimensional formal group law**.

At a finite prime, a minimal such thing is a **lift** of a formal group over a finite field, of height $1 \leq n \leq \infty$. **Lubin** and **Tate** constructed a beautiful moduli stack for these lifts, and since then topologists (Hopkins-Miller, Goerss, Lurie . . .) have constructed a sheaf of nice commutative ring-spectra over its etalé site.

4 The vertical arrows in the diagram indicate categories of modules (over these structured ring-spectra). **Drinfel'd** constructed the morphism along the lower back wall: the unitary Shimura stack at the lower left far corner classifies [BL §6.3] polarized B-linear Abelian varieties A, where B is a division algebra of dimension n^2, with a positive involution of the second kind, over an imaginary quadratic extension $F/\mathbb{Q}$ in which $p = u \cdot u^c$ splits, and such that B splits over u and u^c. At such p, D splits a one-dimensional height n formal group $A(u)$ off the formal completion of A.

Behrens and **Lawson** define **topological automorphic forms** (generalizing the **topological modular forms**, ie **elliptic cohomology**, of Hopkins *et al*), at the upper left corner of the back wall – roughly, by pulling back the structure sheaf on the Lubin-Tate stack along Drinfel'd's map.

5 **Rezk**, working in a different direction, studied lifts of formal group laws together with lifts of their Frobenius endomorphisms (an idea going back to Serre and Tate). He proved a far-reaching generalization of Wilkerson's characterization of a Λ-ring, linking homotopy theory and the model for $\mathbb{F}_1$-geometry advocated by **Borger**. This fits naturally into a parametrization of cohomological power operations (pioneered by Ando and Joyal-Bisson) by isogenies and Hecke operations (following Baker and Ganter, and recently rediscovered by Kohlhaase).

6 The moduli stack underlying the Behrens-Lawson theory of topological automorphic forms has polarized (PEL) Abelian varieties as objects, and B-linear prime-to-p isogenies as morphisms. **Presumably** some pullback (along Rezk's forgetful map) of Drinfel'd's morphism D defines a Shimura **monoid** (analogous to constructions of Connes-Consani-Marcolli, Ha-Paugam, and possibly Yalkinoglu) for an imaginary quadratic extension $F/\mathbb{Q}$.

7 Note, however, that at the moment our pretensions are **purely** (quasi)-**local**! Our modest **hope** is to define an arithmetic BC system, not for this global object, but for a C^*-algebra associated to its p-component: or, more precisely, to identify its KMS states with suitable Katz-Kubert distributions on (the Tate module of the Cartier dual to) $A(u)$.

In fact it seems interesting enough to forget about this whole grandiose program altogether, and to concentrate on the purely arithmetic question of KMS states for Lubin-Tate groups of local number fields; in particular, to interpret the work of Connes-Consani on p-adic polylogarithms in terms of the Lubin-Tate theory of $\mathbb{Q}_p$.

Here are some **references** and a timeline for some of the work cited in the diagram (arranged in three threads: the local Langlands program, algebraic topology, and noncommutative geometry):

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• Local Langlands program

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Distribution of prime ideals and the space of adele classes

Sergey Neshveyev

(joint work with Jeffrey C. Lagarias)

For a global field K, the space of adeles $\mathbb{A}_K$ appeared explicitly in Tate's thesis [9]. The action of K^* on $\mathbb{A}_K$ played a key role in his proof of the functional equation for Hecke L-functions. More recently, this action, together with the associated space $\mathbb{A}_K/K^*$ of adele classes, was used by Connes [2] to construct a Hilbert space that provides a spectral interpretation of the critical zeros of Hecke L-functions. One of the major difficulties in constructing such a Hilbert space, as well as in trying to prove the Riemann hypotheses following Connes' approach, is that the quotient space $\mathbb{A}_K/K^*$ is quite tricky. The precise statement for $K = \mathbb{Q}$ is that the action of $\mathbb{Q}^*$ on $\mathbb{A}_\mathbb{Q}$ is ergodic with respect to the Haar measure on $\mathbb{A}_\mathbb{Q}$, so from the point of view of classical measure theory the space $\mathbb{A}_\mathbb{Q}/\mathbb{Q}^*$ consists of just one point. This is a known byproduct of the analysis of the Bost-Connes system [1, 8]. We prove the following extension of this result [5]. Let K be a global field, and denote by $\mu_\mathbb{A}$ the Haar measure on $\mathbb{A}_K$.

Theorem 1. *The action of K^* on $(\mathbb{A}_K, \mu_\mathbb{A})$ by multiplication is ergodic.*

As it turns out, for number fields the ergodicity problem is related to the question of distribution of angles of prime ideals.

Example 1. *Consider the field $K = \mathbb{Q}(i)$. The corresponding ring $\mathcal{O}$ of integers is $\mathbb{Z}[i]$. The structure of prime ideals in $\mathbb{Z}[i]$ is well-known: in addition to the prime ideal $(1 + i)$, every odd prime number p either remains prime in $\mathbb{Z}[i]$ or gives rise to two prime ideals $(a+ib)$ and $(a-ib)$ such that $p = a^2+b^2$, depending on whether $p \equiv 3 \pmod 4$ or $p \equiv 1 \pmod 4$. The corresponding norms of ideals are $N(p) = p^2$ and $N(a \pm ib) = a^2 + b^2 = p$. Therefore if we order prime ideals by the norm, then the ideals of the first type become sparse, while the distribution of norms of ideals of the second type follows from Dirichlet's theorem on arithmetic progressions: the number of prime ideals of norm $\le x$ grows like $x/\log x$. But in addition to distribution of norms of ideals it makes sense to ask how the angles are distributed. More precisely, every ideal $\mathfrak{p}$ of the second type has the form $(a + ib)$*

for uniquely defined $a > 0$ and $b > 0$, and so defines an angle $\theta(\mathfrak{p}) \in (0, \pi/2)$. If we order the ideals by the norm, it turns out that the angles become equidistributed in $(0, \pi/2)$.

Results of this type go back to Hecke [3, 4]. The most general result is due to Mitsui [7]. In a more modern terminology it can be formulated as follows.

Set $n = [K : \mathbb{Q}] = r_1 + 2r_2$, so $\mathbb{A}_K$ has $r_1 + r_2$ Archimedean places, with r_2 of them complex. Write $x = (x_v)_v$ for elements of $\mathbb{A}_K$, where v runs over the valuations of K. The idelic norm map $N\colon \mathbb{A}_K^* \to \mathbb{R}_+^*$ sends $(x_v)_v \mapsto \prod_v ||x||_v$. For every nonzero integral ideal $\mathfrak{m} = \prod_{v\in S(\mathfrak{m})} \mathfrak{p}_v^{m_v}$, the quotient norm map

$$N_\mathfrak{m} : \mathbb{A}_K^*/K^* W_\mathfrak{m} \to \mathbb{R}_+^*,$$

where

$$W_\mathfrak{m} = \prod_{v\notin S(\mathfrak{m})} \mathcal{O}_v^* \times \prod_{v\in S(\mathfrak{m})} (1 + \hat{\mathfrak{p}}_v^{m_v}),$$

is well-defined, and we put

$$\Gamma_\mathfrak{m} = \ker N_\mathfrak{m} \subset \mathbb{A}_K^*/K^* W_\mathfrak{m}.$$

The norm map $N\colon \mathbb{A}_K^* \to \mathbb{R}_+^*$ has a (noncanonical) splitting homomorphism $s\colon \mathbb{R}_+^* \to \mathbb{A}_K^*$. We fix such a continuous homomorphism, for example, letting $s(t)_v = 1$ for finite places v and $s(t) = (t^{1/n}, t^{1/n}, \dots, t^{1/n})$ at the $r_1 + r_2$ Archimedean places $v|\infty$ (note that by definition $||t^{1/n}||_v = t^{2/n}$ at complex places). Using this homomorphism we can identify $\mathbb{A}_K^*/K^* W_\mathfrak{m}$ with $\mathbb{R}_+^* \times \Gamma_\mathfrak{m}$. Then, identifying the group $I_\mathfrak{m}$ of fractional ideals that are relatively prime to $\mathfrak{m}$ with a subgroup of $\mathbb{A}_{K,f}/\hat{\mathcal{O}}^*$, the embedding

$$I_\mathfrak{m} \hookrightarrow \mathbb{R}_+^* \times \Gamma_\mathfrak{m} = \mathbb{A}_K^*/K^* W_\mathfrak{m}$$

takes the form $\mathfrak{a} \mapsto (N(\mathfrak{a})^{-1}, \rho_\mathfrak{m}(\mathfrak{a}))$ for some homomorphism $\rho_\mathfrak{m}\colon I_\mathfrak{m} \to \Gamma_\mathfrak{m}$. Here $N(\mathfrak{a})$ denotes the norm of the ideal $\mathfrak{a}$, so if $x \in \mathbb{A}_{K,f}^*$ is an idele representing $\mathfrak{a}$, then $N(\mathfrak{a}) = N(x)^{-1}$. The group $\Gamma_\mathfrak{m}$ is a compact abelian Lie group, whose connected components have real dimension $n - 1$. The points $\rho_\mathfrak{m}(\mathfrak{a}) \in \Gamma_\mathfrak{m}$ can be thought of as measuring generalized angles of ideals $\mathfrak{a} \in I_\mathfrak{m}$.

The result of Mitsui can be formulated as follows, see also a closely related result in Lang [6, Theorem XV.5.6].

Theorem 2. *The image under $\rho_\mathfrak{m}$ of the set of prime ideals in $\mathcal{O}$, ordered by the norm, is equidistributed in $\Gamma_\mathfrak{m}$. In other words, for any continuous function f on $\Gamma_\mathfrak{m}$ we have*

$$\frac{1}{|\{\mathfrak{p} \notin S(\mathfrak{m}) : N(\mathfrak{p}) \le x\}|} \sum_{\mathfrak{p}\notin S(\mathfrak{m}) : N(\mathfrak{p}) \le x} f(\rho_\mathfrak{m}(\mathfrak{p})) \to \int_{\Gamma_\mathfrak{m}} f\, d\lambda_\mathfrak{m} \quad \textit{as} \quad x \to \infty,$$

where $\lambda_\mathfrak{m}$ is the normalized Haar measure on the compact group $\Gamma_\mathfrak{m}$.

For number fields, Theorem 1 is deduced from this result using dynamical systems methods, by computing the asymptotic range of a certain $(\mathbb{R}_+^* \times \Gamma_\mathfrak{m})$-valued cocycle for every ideal $\mathfrak{m}$. For function fields the proof is similar, but also involves a bit of class field theory.

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Pseudo-differential operators and diagonals

Marius Junge
(joint work with Tao Mei, Javier Parcet)

The aim of this talk is to report on first progress on singular integral operators on noncommutative spaces. In the classical theory singular integral operators and pseudodifferential operators are used to study solutions of partial differential equations on $\mathbb{R}^n$, open subsets of $\mathbb{R}^n$, and Riemanian manifolds. The theory of singular integrals and classical harmonic analysis are important in identifying the right spaces for solutions and perturbations, in particular elliptic regularity results.

Review of classical results:

Let K be a kernel defined on $\mathbb{R}^n \times \mathbb{R}^n \setminus \{(x,x) : x \in \mathbb{R}^n\}$, i.e. K is well-defined of the diagonal. A classical result in harmonic analysis tells us that if

i) the integral operator $T_K(f)(x) = \int K(x,y)f(y)dy$ $(x \notin \mathrm{supp}(f))$, extends to a bounded operator on $L_2(\mathbb{R}^n)$, and
ii) $\max_{j=1,..,n} \max\{|\frac{d}{dx_j}K(x,y)|, |\frac{d}{dy_j}K(x,y)|\} \le C|x-y|^{-(n+1)}$,

then T_K extends to a bounded operator on $L_p(\mathbb{R}^n)$.

In fact there are several competing conditions for singular integral operator, such as Hörmander's condition, and the assumption for the Mihlin-multiplier theorem, or Sobolev type conditions, and all of them are satisfied in the situation above, see e.g. [St] for more details.

The most prominent examples are Fourier multipliers, where $K(x,y) = K(x-y)$. In this particular case one can use Plancharel's formula to deduce the assumptions above from smoothness conditions of the symbol $m = \hat{K}$, i.e.

$$\sup_{\xi} |\xi|^{\alpha} |(\partial_{\alpha} m)(\xi)| \ \le \ c_{\alpha} \tag{1}$$

holds for the partial derivatives $\partial_{\alpha} = \frac{d^{|\alpha|}}{d\xi_1^{\alpha_1} \cdots d\xi_n^{\alpha_n}}$ of order $|\alpha| = \alpha_1 + \cdots \alpha_n \ \le \ m$. Here m is the smallest integer bigger than $n/2$. Using only very rough tools it is easy to see that $m = n+1$ derivatives are enough to deduce from (1) the conditions required for the Fourier transform

$$K(x) \ = \ \int e^{-i(x,\xi)} m(\xi) \frac{d\xi}{\sqrt{2\pi}^{n/2}} \ .$$

In general it is agreed that for a manifold the number of derivatives required for singular integrals is related to the dimension of the underlying manifold. The dimension improvement from $n+1$ to $(n+1)/2$ (roughly), is based on a smooth decomposition of the identity

$$\xi \ = \ \sum_{j \in \mathbb{Z}} \psi_j(\xi), \quad \xi \neq 0,$$

where ψ_j is supported in a compact interval $[a^j, a^{j+1}]$ for some $a > 1$. In many cases ψ_j is constructed from dilates $\phi_j(\xi) = \phi(2^j \xi)$ of a given fixed bump function. Although technical at first, this decomposition often enters in analyzing different norms for Hardy spaces and Besov spaces (see [Ta] on how these are used to prove *elliptic regularity* results). A key tool in this analysis is the so-called Littlewood-Paley theory. Indeed, let $K_j = \hat{\psi}_j$ then

$$\|f\|_p \sim \|(\sum |T_{k_j}(f)|^2)^{1/2}\|_p \ . \tag{2}$$

Littlewood-Paley theory is also useful in studying the Sobolev spaces $\|f\|_{H_p^s} = \|(1+|\Delta|)^{s/2} f\|_p$, Δ the Laplace operator. Based on this description of Sobolev spaces, Bourdaud showed in 1988 that pseudo-differential operators in the class $S^0_{1,1} \cap (S^0_{1,1})^*$ form an operator algebra and act on $L_p(\mathbb{R}^n)$ for all $1 < p < \infty$. A Schwartz function p belongs to the class $S^m_{\rho,\delta}$ if

$$|(\partial^x_{\beta} \partial^{\xi}_{\alpha} p)(x,\xi)| \ \le \ c_{\alpha,\beta} (1+|\xi|^2)^{\frac{m-\rho|\alpha|+\delta\beta}{2}}$$

holds for some constant $C_{\alpha,\beta}$. The corresponding operator is given by the operator

$$T_p(f)(x) \ = \ \int p(x,\xi) e^{i(\xi,x)} \hat{f}(\xi) d\xi \ .$$

The operators associated to symbols from pseudo-differential calculus and singular integrals are essentially the same. In fact, up to a clever change of variables we have $T_p = T_{K_p}$ for

$$K_p(x,y) \ = \ (2\pi)^{-n} \int p(x,\xi) e^{i(x-y,\xi)} d\xi \ .$$

Although innocent at first, the ξ-variable for p carries the information off the diagonal. Indeed, using integration by parts we find

$$(x-y)^\alpha K(x,y) \ = \ \int e^{i(x-y,\xi)} \partial_\alpha^\xi p(x,\xi) d\xi \ .$$

This implies that for $p \in S^0_{1,1}$, the so-called forbidden class, the first smoothness condition ii) is satisfied, i.e. $|\frac{d}{dx_j} K_p(x,y)| \ \le \ C|x-y|^{-(n+1)}$. The other condition follows from p^* being in $S^0_{1,1}$.

Similar results on noncommutative spaces:

The starting point for harmonic analysis on noncommutative spaces for us is a semigroup of completely positive, selfadjoint, trace preserving maps $T_t = e^{-tA}$ on a finite von Neumann algebra N. Following the work of Sauvageot, in particular [CS], we may define the gradient form

$$2\Gamma(x,y) \ = \ A(x^*)y + x^*A(y) - A(x^*y) \ .$$

It is well-known that such a gradient form comes with a gradient bimodule $X_\Gamma = \mathrm{dom}(A^{1/2}) \otimes_\Gamma N$ with inner product $\langle x_1 \otimes x_2, y_1 \otimes y_2 \rangle \ = \ x_2^* \Gamma(x_1,y_1) y_2$. This gradient form also shows up in the rigidity/deformation theory of Popa and his coauthors (see [Pe] for more definitions). Then $\delta(a) \ = \ a \otimes 1$ turns out to be a derivation. In a recent preprint with Ricard and Shlyakhtenko, we show that δ is indeed a real derivation which can be implemented with a limit $\delta(x) = ([\xi_\alpha, x])^\bullet$ of commutators, such that $N \subset M_\alpha$ is an inclusion of finite von Neumann algebras. On $\mathcal{H} = L_2(N) \oplus L_2(X_\Gamma)$, one then finds the canconically associated real spectral triple (H, D, π, J), where J is canonical and

$$D \ = \ \begin{pmatrix} 0 & \delta^* \\ \delta & 0 \end{pmatrix} \ .$$

In order to find a spectral triple one has to assume that

$$\|[D,x]\| \ = \ \max\{\|\Gamma(x,x)^{1/2}\|, \|\Gamma(x^*,x^*)^{1/2}\}$$

is finite for a (weakly dense) *-algebra $\mathcal{A}$ of N. This is satisfied for all semigroups coming from group cocycles.

Armed with this technology, we analyze the space $M = B(L_2(\mathbb{R}^n))$ for $n = 2d$. The heat semigroup is defined on the usual Weyl operator

$$W(\zeta)(f)(x) \ = \ e^{i(\zeta_1,x)} f(x-\zeta_2)$$

and the heat semigroup $T_t(W(\zeta)) = e^{-t|\zeta|^2} W(\zeta)$. Alternatively, we can obtain these unitaries by looking at $L_\infty(\mathbb{R}^d) \rtimes \mathbb{R}^d$, or as a fiber of the Heisenberg group. In both cases, we find the correct trace and $L_p(M) = S_p$ spaces given by the classical Schatten classes. A suitable diagonal in this particular case is given by a commutative subalgebra $\sigma(L_\infty(\mathbb{R}^n)) \subset M^{op}$ given by $\sigma(e_\zeta) \ = \ W(\zeta) \otimes W(\zeta)^*$, where $e_\zeta(x,y) = e^{i(\zeta_1,x)+(\zeta_2,y)}$ is the usual character on $\mathbb{R}^{2d} = \mathbb{R}^n$. With the help of this * homomorphism we can identify pseudo-differential operators and symbols on M in the form $a(\zeta) \in M$ such that

$$T_a(W(\zeta)) \ = \ a(\zeta)W(\zeta) \ .$$

By proving a noncommutative version of the Mihlin-multiplier theorem, we derive the following result.

Theorem. *Let a and a^* be in $S^0_{1,1}$ and T_a be bounded on S_2. Then T_a is bounded on S_p for all $1 < p < \infty$.*

We feel that the boundedness of T_a is redundant, but this required extending Bourdaud's [Bo] results to this noncommutative context.

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Cyclic homology, Serre's local factors and the λ-operations

Alain Connes

(joint work with Caterina Consani)

In my talk I explained two recent results in collaboration with C. Consani. In the first, we show that for a smooth, projective variety X defined over a number field K, cyclic homology with coefficients in the ring $\mathbb{A}_\infty = \prod_{\nu|\infty} K_\nu$, provides the right theory to obtain, using the λ-operations, Serre's archimedean local factors of the complex L-function of X as regularized determinants. Our main result is the following, where HC^{ar} is a variant of the cyclic theory HC for schemes.

Let X be a smooth, projective variety of dimension d over an algebraic number field K and let $\nu|\infty$ be an archimedean place of K. Then, the action of the operator Θ on the archimedean cyclic homology of X_ν satisfies the following formula

$$(1) \qquad \prod_{0 \le w \le 2d} L_\nu(H^w(X), s)^{(-1)^{w+1}} = \frac{det_\infty(\frac{1}{2\pi}(s-\Theta))|_{HC^{\mathrm{ar}}_{\mathrm{even}}(X_\nu)}}{det_\infty(\frac{1}{2\pi}(s-\Theta))|_{HC^{\mathrm{ar}}_{\mathrm{odd}}(X_\nu)}}, \qquad s \in \mathbb{R}.$$

The left-hand side of (1) is the product of Serre's archimedean local factors of the complex L-function of X. On the right-hand side, det_∞ denotes the regularized determinant and one sets $HC^{\mathrm{ar}}_{\mathrm{even}}(X_\nu) = \bigoplus_{n=2k\ge 0} HC^{\mathrm{ar}}_n(X_\nu)$, $HC^{\mathrm{ar}}_{\mathrm{odd}}(X_\nu) = \bigoplus_{n=2k+1\ge 1} HC^{\mathrm{ar}}_n(X_\nu)$.

The endomorphism Θ has two constituents: the natural grading in cyclic homology and the action of the multiplicative semigroup $\mathbb{N}^\times$ on cyclic homology of *commutative* algebras given by the λ-operations $\Lambda(k)$, $k \in \mathbb{N}^\times$. More precisely, the action u^Θ of the multiplicative group $\mathbb{R}_+^\times$ generated by Θ on cyclic homology, is uniquely determined by its restriction to the dense subgroup $\mathbb{Q}_+^\times \subset \mathbb{R}_+^\times$ where it is given by the formula

$$k^\Theta|_{HC_n(X_\nu)} = \Lambda(k)\, k^{-n}\,,\ \forall n \geq 0,\ \ k \in \mathbb{N}^\times \subset \mathbb{R}_+^\times. \tag{2}$$

By taking into account the fact that cyclic homology of a finite product of algebras (or disjoint union of schemes) is the direct sum of their cyclic homologies, (1) determines the required formula for the product of all archimedean local factors in terms of cyclic homology with coefficients in the ring $\mathbb{A}_\infty = \prod_{\nu|\infty} K_\nu$.

In the second contribution, we show that the cyclic and epicyclic categories which play a key role in the encoding of cyclic homology and the lambda operations, are obtained from projective geometry in characteristic one over the infinite semifield of "max-plus integers" $\mathbb{Z}_{\max}$. Finite dimensional vector spaces are replaced by modules defined by restriction of scalars from the one-dimensional free module, using the Frobenius endomorphisms of $\mathbb{Z}_{\max}$. The associated projective spaces are *finite* and provide a mathematically consistent interpretation of J. Tits' original idea of a geometry over the absolute point. The self-duality of the cyclic category and the cyclic descent number of permutations both acquire a geometric meaning.

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Cyclic homology and *K*-theory via *cdh* cohomology

Chuck Weibel

This was a survey talk. The basic principle presented is that we can use cyclic homology to calculate K-groups of commutative algebras.

The fact that the cyclic homology of $\mathbb{C}$-algebras (taken over $\mathbb{Q}$) extends to the cyclic homology of schemes over $\mathbb{C}$ was presented in Alain Connes' talk, and dates to [9]. Briefly, there is a cochain complex of sheaves $\underline{\mathrm{HC}} = \oplus \underline{\mathrm{HC}}^{(i)}$ and $HC_n(X)$ is the Zariski hypercohomology $H^{-n}(X, \underline{\mathrm{HC}})$. A similar definition works for Hochschild homology: $HH_n(X) = H^{-n}(X, \underline{\mathrm{HH}})$. If X is smooth, we have

$$HH_n^{(i)} = H^{i-n}(X, \Omega^i_{X/\mathbb{Q}}) \quad \text{and} \quad HC_n^{(i)} = H^{2i-n}(X, \mathcal{O}_X \to \cdots \to \Omega^i_{X/\mathbb{Q}}).$$

If Z is closed in X, there are relative groups $HC_n(X,Z)$ and $HH_n(X,Z)$, defined using the usual constructions in homological algebra.

Theorem. *There is a covariant cyclic theory* $\mathbb{H}C_n(A)$ *for commutative* $\mathbb{C}$*-algebras, contravariant in schemes, and a corresponding Hochschild theory* $\mathbb{H}H_n(A)$*, satisfying the following properties:*

(1) There are natural maps $HH_n \to \mathbb{H}H_n$ *and* $HC_n \to \mathbb{H}C_n$*. If* X *is smooth, the maps* $HH_n(X) \to \mathbb{H}H_n(X)$ *and* $HC_n(X) \to \mathbb{H}C_n(X)$ *are isomorphisms.*

(2) The λ*-operations act on* $\mathbb{H}H_n$ *and* $\mathbb{H}C_n$ *compatibly with the maps in (1), and determine decompositions* $\mathbb{H}H_n(X) \cong \oplus \mathbb{H}H_n^{(i)}(X)$ *and* $\mathbb{H}C_n(X) \cong \oplus \mathbb{H}C_n^{(i)}(X)$*. The usual SBI yoga goes through in this setting.*

(3) (Excision) If $p : \tilde{X} \to X$ *is a map such that* $\tilde{X} - \tilde{Z} \to X - Z$ *is an isomorphism, where* $\tilde{Z} = Z \times_X \tilde{X}$*, then* $\mathbb{H}H_*(X, Z) \cong \mathbb{H}H_*(\tilde{X}, \tilde{Z})$ *and* $\mathbb{H}C_*(X, Z) \cong \mathbb{H}C_*(\tilde{X}, \tilde{Z})$*.*

Excision (3) is equivalent to the existence of Mayer-Vietoris sequences. Taking $Z = X_{red}$ and $\tilde{X} = \emptyset$, we see that $\mathbb{H}C_*(X) \cong \mathbb{H}C_*(X_{red})$. Thus we may restrict to reduced schemes.

There is a practical algorithm for computing these groups by induction on the dimension of X; $\dim(X) = 0$ is clear. If $\tilde{X}$ is a resolution of singularities and Z the singular locus, we have the Mayer-Vietoris sequence with $\dim Z, \dim \tilde{Z} < \dim(X)$.

$$\to \mathbb{H}C_{n+1}(\tilde{Z}) \to \mathbb{H}C_n(X) \to \mathbb{H}C_n(Z) \oplus \mathbb{H}C_n(\tilde{X}) \to \mathbb{H}C_n(\tilde{Z}) \to$$

In fact, there is a Grothendieck topology on schemes of finite type over $\mathbb{C}$, called the *cdh* topology, and $\mathbb{H}C_n(X)$ is defined to be $H^{-n}_{cdh}(X, \underline{\mathrm{HC}})$. This definition was introduced in [1]. The topology, and its name, is due to Voevodsky: 'cd' for Nisnevich's completely decomposed topology and 'h' for Voevodsky's h-topology.

The map in (1) is the hypercohomology of a morphism of complexes. If $F_*(X)$ denotes the cohomology of the shifted mapping cone, the vector spaces $F_n(X)$ fit into the long exact sequence

$$\mathbb{H}C_{n+1}(X) \to F_n(X) \to HC_n(X) \to \mathbb{H}C_n(X) \to .$$

By (1), F_* measures the "singular" part of cyclic homology; see [2, 3, 4].

Let $KH_n(X)$ denote the homotopy invariant K-theory of X, where "homotopy invariant" means that $KH_*(A) = KH_*(A[t])$. This theory was introduced in [8]. If A is regular then $KH_*(A) = KH_*(A[t])$. Every commutative C^*-algebra has this property too, and conjecturally so does every C^*-agebra.

Theorem. *There is a long exact sequence*

$$KH_{n+1}(X) \to F_{n-1}(X) \to K_n(X) \to KH_n(X) \to F_{n-2}(X)$$

This sequence is frequently split, in which case $K_n(X) \cong KH_n(X) \oplus F_n(X)$*.*

This result allows us to calculate K-groups which were previously inaccessible. This theory was worked out in [2, 5, 6, 7].

Example: Let A be the coordinate ring of a conic cone in 3-space, say represented by the equation $xy = z^2$. It was know in the 1960s that $K_0(A) = \mathbb{Z}$, and discovered in the 1980's that $K_1(A)$ is nontrivial. In fact, $K_1(A) \cong K_1(\mathbb{C}) \oplus \mathbb{C}$, where $a \in \mathbb{C}$ corresponds to the class of the matrix $\begin{pmatrix} 1+az & -ax \\ ay & 1-az \end{pmatrix}$. There is an injection from $\Omega^n_{\mathbb{C}}$ into $K_{n+1}(A)$ sending $adb_1 \wedge ... \wedge db_n$ to the product of this matrix with the Milnor symbols $\{b_1, ..., b_n\}$. For $n \geq 1$,

$$K_{n+1}(A) \cong K_{n+1}(\mathbb{C}) \oplus \Omega^n_{\mathbb{C}} \oplus \Omega^{n-2}_{\mathbb{C}} \oplus \cdots .$$

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Karoubi's relative Chern character and regulators

Georg Tamme

Regulators are certain natural transformations from the algebraic K-theory of a variety defined over a number field or a local field to some suitable cohomology theory. Beilinson's conjectures relate the so-called Beilinson regulator, which has values in real Deligne-Beilinson cohomology, to special values of L-functions up to some nonzero rational multiple [1]. To recover the p-adic valuation of the rational factor one also has to consider p-adic regulators.

Recent work of Connes and Consani [2] suggests that cyclic homology should play an important role in the study of L-functions. The purpose of the talk was to present some relations between cyclic homology and regulators, in the classical and in the p-adic setting.

In a series of papers [4, 5, 3] Karoubi and Connes-Karoubi constructed the so-called *relative Chern character* for archimedean or non-archimedean Banach algebras A. It is defined on the *relative K-theory* of A, i.e. the homotopy fibre of the natural comparison map between algebraic and topological K-theory of A, and has values in cyclic homology:

$$\mathrm{ch}^{\mathrm{rel}}_i \colon K^{\mathrm{rel}}_i(A) \to HC_{i-1}(A).$$

Consider now a smooth variety X over $\mathbb{C}$. We define the relative K-theory as the homotopy fibre of the comparison map between the algebraic K-theory of X and the complex connective K-theory of the associated complex manifold $X(\mathbb{C})$.

An analogous construction as in the Banach algebra case gives relative Chern character maps

$$\mathrm{ch}^{\mathrm{rel}}_{n,i}\colon K^{\mathrm{rel}}_i(X) \to H^{2n-i-1}_{dR}(X/\mathbb{C})/F^n.$$

The target is the quotient of the (algebraic) de Rham cohomology of X by the n-th step of (Deligne's) Hodge filtration, also called relative de Rham cohomology. There is a natural map

$$H^{2n-i-1}_{dR}(X/\mathbb{C})/F^n \to H^{2n-i-1}\left(X, \Omega^0_X \xrightarrow{d} \dots \xrightarrow{d} \Omega^{n-1}_X\right) \tag{1}$$

to the hypercohomology of the truncated de Rham complex which is an isomorphism if X is proper. The cohomology on the right hand side of (1) is isomorphic to the weight $n-1$ part $HC^{(n-1)}_{i-1}(X)$ in the λ-decomposition of the cyclic homology of X [9].

There is also a natural map from the relative de Rham cohomology to Deligne-Beilinson cohomology $H^{2n-i}_{\mathcal{D}}(X, \mathbb{R}(n))$ which sits in a long exact sequence

$$\dots \to H^{2n-i-1}(X(\mathbb{C}), \mathbb{R}(n)) \to H^{2n-i-1}_{dR}(X/\mathbb{C})/F^n \to H^{2n-i}_{\mathcal{D}}(X, \mathbb{R}(n)) \to \dots$$

We have the following comparison result:

Theorem 1 ([7]). *The diagram*

$$\begin{array}{ccc} K^{\mathrm{rel}}_i(X) & \longrightarrow & K_i(X) \\ \downarrow{\scriptstyle \mathrm{ch}^{\mathrm{rel}}_{n,i}} & & \downarrow{\scriptstyle \textit{Beilinson's regulator}} \\ H^{2n-i-1}_{dR}(X/\mathbb{C})/F^n & \longrightarrow & H^{2n-i}_{\mathcal{D}}(X, \mathbb{R}(n)) \end{array}$$

commutes.

Let us now describe the p-adic analog of this result. We consider a finite extension K of $\mathbb{Q}_p$ with ring of integers $\mathcal{O}_K$ and uniformizer π. Let X be a smooth, separated scheme of finite type over $\mathcal{O}_K$. Using the π-adic completion of X we have a Karoubi-Villamayor type definition of the topological K-theory of X and hence also the relative K-theory. To simplify the exposition we now assume that X is proper. As before, we construct relative Chern character maps

$$\mathrm{ch}^{\mathrm{rel}}_{n,i}\colon K^{\mathrm{rel}}_i(X) \to H^{2n-i-1}_{dR}(X_K/K)/F^n$$

whose target may be identified with $HC^{(n-1)}_{i-1}(X)$. (For non-proper schemes the construction still works, but the target is more complicated to describe.)

The p-adic analog of Beilinson's regulator with values in Deligne-Beilinson cohomology is the regulator with values in rigid syntomic cohomology $H^{2n-i}_{\mathrm{syn}}(X, n)$. There is also a natural map from relative de Rham to rigid syntomic cohomology and we have:

Theorem 2 ([8]). *The diagram*

$$\begin{array}{ccc} K_i^{\mathrm{rel}}(X) & \longrightarrow & K_i(X) \\ {\scriptstyle \mathrm{ch}_{n,i}^{\mathrm{rel}}}\downarrow & & \downarrow{\scriptstyle \textit{syntomic regulator}} \\ H_{dR}^{2n-i-1}(X_K/K)/F^n & \longrightarrow & H_{\mathrm{syn}}^{2n-i}(X,n) \end{array}$$

commutes.

Finally, we explained the comparison of the relative Chern character and the p-adic étale regulator via the Bloch-Kato exponential map which follows from Theorem 2 using work of Nizioł [6].

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Index, determinant and torsion of commuting n-tuples of operators

Ryszard Nest

(joint work with Jens Kaad)

1. Introduction

Basic data: n-tuple $A = (A_1, \ldots, A_n)$ of commuting operators on a Hilbert space $\mathcal{H}$.

We will denote by $K(A)$ - the Koszul complex of A: $(\mathcal{H} \otimes \Lambda^* \mathbb{C}^n, d_A)$ where the differential is given by $d_A = \sum_i A_i \otimes \iota_{e_i}$, $(e_1, \ldots, e_n)$ the standard basis of $\mathbb{C}^n$. Due to Taylor (see [Tay70a]), there exists a multivariable holomorphic functional calculus

$$\phi_A : \mathcal{O}_A \to B(\mathcal{H})$$

where $\mathcal{O}_A$ is the ring of germs of functions holomorphic in a neighbourhood of $\sigma(A)$.

2. Local index theorem

Let us start with the known global results. Given an n-tuple $g \in \mathcal{O}\big(\mathrm{Sp}(A)\big)_A^m$

Theorem 1 (Eschmeier, Putinar, Levy). *([Lev89], [EsPu96])*

(1) $g(A)$ is Fredholm iff the set $g^{-1}(\{0\}) \cap \sigma_{ess}(A) = \emptyset$.
(2) Suppose that $g(A)$ is Fredholm. Then $g^{-1}(\{0\}) \cap \sigma_F(A)$ is finite and

$$\mathrm{Index}(g(A)) = \sum_{\lambda \in Z(g)} \deg_\lambda(g) \mathrm{Index}(A - \lambda)$$

where $\deg_\lambda(g)$ is the intersection multiplicity of $\{g_1 = \ldots = g_n = 0\}$ at λ.

The holomorphic functional calculus gives the Hilbert space $\mathcal{H}$ the structure of a $\mathcal{O}\big(\mathrm{Sp}(A)\big)_A$-module. For each $\lambda \in \sigma(A)$ set

$$\mathfrak{p}_\lambda = \{f \in \mathcal{O}\big(\mathrm{Sp}(A)\big) \mid f(\lambda) = 0\}$$

The Koszul complex of A localizes, hence

Definition. *The local index at λ is*

$$\mathrm{Index}_\lambda(g(A)) := -\sum_i (-1)^i \mathrm{Dim}_{\mathbb{C}}(H_i(g, \mathcal{H}_{\mathfrak{p}_\lambda})).$$

Here $H_*(g, H_{\mathfrak{p}_\lambda})$ stand for the homology groups of Koszul complex $K(g(A))$ localised at $\mathfrak{p}_\lambda$. Note that the homology groups are finite dimensional as a consequence of the Fredholmness of $g(A)$.

Theorem 2. *Suppose that $g(A)$ is Fredholm and that $g(\lambda) = 0$. Then the homology groups $H_*(g, \mathcal{H}_\lambda)$ are finite dimensional and*

$$\mathrm{Ind}_\lambda(g(A)) = \deg_\lambda(g)\mathrm{Ind}(A - \lambda),$$

where $\deg_\lambda(g)$ is the intersection multiplicity of $\{g_1 = \ldots = g_n = 0\}$ at λ.

3. Determinants

Given a Fredholm operator $T : H_+ \to H_-$, the corresponding determinant line is the one dimensional complex vector space

$$Det(T) = \Lambda^{top}(Ker(T)) \otimes \Lambda^{top}(Coker(T))^*$$

Let us recall some basic facts.

(1) $T \to Det(T)$ is a continuous line bundle over the spaces of Fredholm operators (in fact analytic in the case of $H_+ = H_-$)

(2) Suppose that

$$S:\ 0 \longrightarrow V_0 \longrightarrow V_1 \longrightarrow V_2 \longrightarrow 0$$

is an exact complex of finite dimensional vector spaces. Then there exists a natural isomorphism

$$Det(S) : Det(V_1) \to Det(V_0) \otimes Det(V_2).$$

(3) The determinant functor

$$Det : \mathcal{V}ect_0 \Longrightarrow \mathcal{P}ic$$

is given by

$$\mathcal{C} \Longrightarrow \ldots \otimes Det(H_i(\mathcal{C})) \otimes Det(H_{i+1}(\mathcal{C}) \otimes \ldots$$

Fact: *The same construction works in the category $\mathcal{F}red$ of* finite *Fredholm complexes, i.e. of the form*

$$\mathcal{C} = \ \ldots \longrightarrow \mathcal{H}_k \xrightarrow{d_k} \mathcal{H}_{k-1} \xrightarrow{d_{k-1}} \mathcal{H}_{k-2} \xrightarrow{d_{k-2}} \ldots$$

where, for each k, $Rg(d_k)$ is closed of finite codimension in $Ker(d_{k-1})$.

4. Torsion

$\mathcal{F}red$ is a triangulated category, with

- $\mathcal{C} \to \mathcal{C}[1]$ (shift the grading to the left)
- and exact triangles: given a morphism $f : \mathcal{A} \to \mathcal{B}$ in $\mathcal{F}red$ there is a mapping cone $\mathcal{C}_f$, the total complex of the double complex

$$\begin{array}{ccccccccc} \ldots & \longrightarrow & \mathcal{A}_k & \longrightarrow & \mathcal{A}_{k-1} & \longrightarrow & \mathcal{A} & \longrightarrow & \ldots \\ & & \downarrow{\scriptstyle -f} & & \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle -f} & & \\ \ldots & \longrightarrow & \mathcal{B}_k & \longrightarrow & \mathcal{B}_{k-1} & \longrightarrow & \mathcal{B} & \longrightarrow & \ldots \end{array}$$

which fits into an exact triangle

$$\Delta_f : \mathcal{C}_f[-1] \longrightarrow \mathcal{A} \xrightarrow{f} \mathcal{B} \longrightarrow \mathcal{C}_f$$

- Exact triangle as above induces a natural isomorphism

$$Det(\Delta_f) : Det(\mathcal{C}_f) \to Det(\mathcal{B}) \otimes Det(\mathcal{A})[1].$$

Some examples for later.

Example 2. *(1) Suppose that $\mathcal{A} = \mathcal{B}$. Then $Det(\Delta)$ is the trivialization of $Det(\mathcal{C}_f)$ given by*

$$\begin{array}{ccc} Det(\mathcal{C}_f) & \longrightarrow & Det\mathcal{A} \otimes Det\mathcal{A}[1] \\ & \searrow^{Det(\Delta)} & \downarrow \simeq \\ & & Det(\mathcal{A}) \otimes Det(\mathcal{A})^{\dagger} \\ & & \downarrow \simeq \\ & & \mathbb{C} \end{array}$$

(2) Suppose moreover f is a quasi-isomorphism, i. e. $H_*(\mathcal{C}_f) = 0$. Then the composition

$$\mathbb{C} = Det(\mathcal{C}_f) \xrightarrow{Det(\Delta)} Det(\mathcal{A}) \otimes Det(\mathcal{A})[1] = \mathbb{C}$$

given (up to a sign) by the multiplication with

$$det(f) = \prod_n det(f_{2n}) det(f_{2n+1})^{-1}$$

5. Local formula for torsion

Back to our Koszul complexes. Let $h = (h_1, \dots, h_{n-1}) \in \mathcal{O}_A^{n-1}$ and $f, g \in \mathcal{O}_A$ Then

Lemma. *The Koszul complex $K(h(A), f(A), g(A))$ coincides with both*

(1) the mapping cone $\mathcal{C}_f$ of

$$f(A) : K(h(A), g(A)) \to K(h(A), g(A))$$

and

(2) the mapping cone $\mathcal{C}_g$ of

$$g(A) : K(h(A), f(A)) \to K(h(A), f(A)).$$

In particular, by above example, we get two isomorphisms

$$Det(\Delta_f) : Det(K(h(A), f(A), g(A))) \to \mathbb{C}$$

and

$$Det(\Delta_g) : Det(K(h(A), f(A), g(A))) \to \mathbb{C}.$$

Theorem 3. *([Kaa12]) The torsion of the n+1 tuple $(h_1, \dots, h_{n-1}, f, g)$ is*

$$Tor(f, g) = Det(\Delta_f) Det(\Delta_g)^{-1} \ (\in \mathbb{C}^*)$$

The following generalizes the results from [CaPi99].

Theorem 4. *Suppose that $h_1, h_2, \dots, f, g \in \mathcal{O}_A$ are such that*

$$V(h, f) \cap \sigma_{ess}(A) = \emptyset \text{ and } V(h, g) \cap \sigma_{ess}(A) = \emptyset,$$

where $V(f,\ldots)$ stands for the common set of zero's of $(f,\ldots)$. Suppose moreover that either the homology groups of $K(f(A))$ are Hausdorff or that they are concentrated in a single degree (for example in the case of a commuting family of Toeplitz operators). Then

$$Tor(f,g) = \epsilon \frac{\prod_{\lambda \in V(h,g)} f(\lambda)^{deg_\lambda V(h,g) Index(A-\lambda)}}{\prod_{\mu \in V(h,f)} g(\mu)^{deg_\mu (h,f) Index(A-\mu)}}$$

where the sign ϵ is given by

$$(-1)^{Ind(K(f(A),h(A)))Ind(K(g(A),h(A)))}.$$

Remark. *In the case of $\mathcal{H} = L^2_{hol}(\overline{\mathbb{D}})^n$ and A given by the coordinate functions $(z_1,\ldots,z_n)$ this can be interpreted as the* Tate tame symbol *of the n+1-tuple of holomorphic functions $(h_1,h_2,\ldots,f,g)$.*

Sketch of proof. Suppose first that the n+1 tuple $(h_1,h_2,\ldots,f,g)$ has no common zeroes within $\sigma(A)$. Then $K(h(A),f(A),g(A))$ is contractible, hence

$$Det(\Delta_f) = \text{determinant of } f(A)\text{:} \quad \circlearrowright \qquad H_*(h(A),g(A);\mathcal{H})$$

Since $H_*(h(A),g(A);\mathcal{H}) = \oplus_{\lambda \in V(h,g)} H_*(h(A),g(A);\mathcal{H})(\lambda)$ with $f(A)$ acting on the components $H_*(h(A),g(A);\mathcal{H})(\lambda)$ as upper triangular matrices with constant diagonals $f(\lambda)$, the claim follows easily. □

The general case (up to the sign) follows from continuity of the determinant bundle. The sign requires more careful study, since it in general depends on the parity of the individual homology groups.

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An equivariant index theorem for Lie groupoids and the van Est map

Xiang Tang
(joint work with Markus Pflaum, Hessel Posthuma)

Inspired by the Atiyah-Singer family index theorem, it is very natural to study differential operators which are not elliptic on the whole manifold, but only elliptic when they are restricted to a smooth family of submanifolds. As such an operator D is only elliptic along submanifolds of a closed manifold, D does not define a Fredholm operator. However, C^*-algebra K-theory provides a powerful tool to formulate and study the "index" of such an operator.

A successful and influential example is the longitudinal index theory for foliations developed by Connes [3] and Connes-Skandalis [10]. Let $\mathcal{F}$ be a foliation on a manifold M, and D be a leafwise elliptic operator. The "index" of D can be made precise as a K-theory element of the groupoid C^*-algebra of the holonomy groupoid associated to $\mathcal{F}$. Connes and his coauthors [3], [10] gave a beautiful description of the topological information of this C^*-algebraic analytic index, which is a milestone result in noncommutative geometry.

When $\mathcal{F}$ is not regular, i.e. the dimensions of leaves of $\mathcal{F}$ are not constant, a natural generalization of Connes' foliation index theory is the index theory for G-elliptic operators on a general Lie groupoid G, where orbits of G are leaves of $\mathcal{F}$. There are many natural interesting examples of elliptic G-differential operators [19], e.g. the leafwise signature operator. The index of an elliptic G-differential operator, D, defines a K-theory element, $\mathrm{Ind}(D)$ of the (reduced) convolution C^*-algebra $C_r^*(\mathsf{G})$ of the groupoid G, which is called the analytic index of D. In [7, Chap. II, Sec. 10.ε], Connes raised the index problem for general Lie groupoids as a generalization of his foliation index theorem and many other famous results. After Connes' work, many interesting results about foliation longitudinal index theory have appeared in literature, e.g., [1], [2], [13]-[15], [16]. However, the index problem for a general Lie groupoid beyond a foliation and an étale groupoid is still wide open.

Recently, we have made some progress [17] on understanding Connes' question. The index $\mathrm{Ind}(D)$ can be constructed to live in the K-theory of the subalgebra $C_c^\infty(\mathsf{G})$ of $C_r^*(\mathsf{G})$. Our main idea is to study the topological information of $\mathrm{Ind}(D)$ by pairing it with a differentiable groupoid cohomology class, $H^\bullet_{\mathrm{diff}}(\mathsf{G}; L_\mathfrak{g})$ ([11]), of G with coefficient in $L_\mathfrak{g}$, where $L_\mathfrak{g} = \bigwedge^{\mathrm{top}} T^*M \otimes \bigwedge^{\mathrm{top}} \mathfrak{g}$ is the "transverse densities" line bundle [12] of the associated Lie algebroid $\mathfrak{g}$.

Theorem 1 ([17])**.** *Let $\mathsf{G} \rightrightarrows M$ be a Lie groupoid over a compact manifold M with the Lie algebroid $\mathfrak{g} \to M$. Suppose that D is a G-elliptic differential operator. Then, for $\alpha \in H^{2k}_{\mathrm{diff}}(\mathsf{G}; L_\mathfrak{g})$, the following equality holds true;*

$$\langle \alpha, \mathrm{Ind}(D)\rangle = \frac{1}{(2\pi\sqrt{-1})^k}\int_{\mathfrak{g}^*} \pi^*\Phi_{\mathfrak{g}^*}(\alpha) \wedge \mathrm{Td}^{\pi^!\mathfrak{g}}(\pi^!\mathfrak{g}\otimes\mathbb{C}) \wedge \mathrm{ch}^{\pi^!\mathfrak{g}}(\sigma_{\mathrm{pr}}(D)).$$

On the right hand side, $\pi^!\mathfrak{g}$ is the pull-back Lie algebroid of $\mathfrak{g}$ along the projection $\pi : \mathfrak{g}^* \to M$, and $\Phi_\mathfrak{g} : H^\bullet_{\mathrm{diff}}(\mathsf{G}; L_\mathfrak{g}) \to H^\bullet(\mathfrak{g}; L_\mathfrak{g})$ is the van Est map for the Lie groupoid G, and $\pi^* : H^\bullet(\mathfrak{g}; L_\mathfrak{g}) \to H^\bullet(\pi^!\mathfrak{g}; \pi^* L_\mathfrak{g})$ is the obvious pull-back map, an isomorphism on the level of cohomology. $\sigma_{\mathrm{pr}}(D)$ is the principal symbol of D. The individual terms are certain Lie algebroid classes, analogous to the usual Todd genus and the Chern character, and they combine to a compactly supported top degree form over $\mathfrak{g}^*$, which can be integrated.

In the special case of the fundamental groupoid, $\Pi_1(M) \rightrightarrows M$, the above theorem recovers the Connes-Moscovici Higher Index Theorem [9] for an elliptic operator on M. In the case of foliations, our above index theorem provides an interesting generalization of the Connes foliation index theorem [3], [10] for measured foliations.

Recently [18] we have generalized the above Theorem to study the index of an invariant elliptic operator for a proper cocompact G-action, which includes both the Connes foliation index theorem and the Connes-Moscovici L^2-index theorem [8], [20] for homogeneous spaces as special cases. In the case of a proper cocompact Lie group G action on a manifold M, the theorem is stated as follows.

Theorem 2 ([18])**.** *Let G be a Lie group acting properly and cocompactly on a manifold M. Suppose that D is an elliptic G-invariant differential operator on M, and $[\varphi] \in H^{2k}_{\mathrm{diff}}(G; L_\mathfrak{g})$. The index pairing between* $\mathrm{Ind}(D)$ *and* $[\varphi]$ *is computed to be*

$$\langle [\varphi], \mathrm{Ind}(D)\rangle = \frac{1}{(2\pi\sqrt{-1})^k}\int_{T^*M} c\Phi([\varphi]) \wedge \widehat{A}(T_\mathbb{C}M) \wedge \mathrm{ch}(\sigma_{\mathrm{pr}}(D)),$$

where $c \in C_c^\infty(M)$ is a cut-off function, and $\Phi : H^\bullet_{\mathrm{diff}}(G; L_\mathfrak{g}) \to H^\bullet_G(M)$, the de Rham cohomology of G-invariant differential forms on M, is the van Est map for the G-action.

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Scalar curvature, Connes' trace theorem, and Einstein-Hilbert action for noncommutative 4-tori

Masoud Khalkhali
(joint work with Farzad Fathizadeh)

In my Oberwolfach talk I gave a report on our joint (with Farzad Fathizadeh) new paper *Scalar Curvature for Noncommutative Four-Tori* [9]. In this paper we study the curved geometry of noncommutative 4-tori $\mathbb{T}^4_\theta$. We use a Weyl conformal factor to perturb the standard volume form and obtain the Laplacian that encodes the local geometric information. Connes' pseudodifferential calculus is then used to explicitly compute the terms in the small time heat kernel expansion of the perturbed Laplacian which correspond to the volume and scalar curvature of $\mathbb{T}^4_\theta$. We establish the analogue of Weyl's law, define a noncommutative residue, prove the analogue of Connes' trace theorem, and find explicit formulas for the local functions that describe the scalar curvature of $\mathbb{T}^4_\theta$. We also study the analogue of the Einstein-Hilbert action for these spaces and show that metrics with constant scalar curvature are critical for this action.

Spectral geometry has played an important role in the development of metric aspects of noncommutative geometry [2, 3]. After the seminal paper [5], in which the analogue of the Gauss-Bonnet theorem is proved for noncommutative two tori $\mathbb{T}^2_\theta$ with complex parameter $\tau = i$, there has been much progress in understanding the local differential geometry of these noncommutative spaces [6, 4, 7, 8, 13]. In these works, the flat geometry of $\mathbb{T}^2_\theta$ is conformally perturbed by means of a Weyl factor given by a positive invertible element in $C^\infty(\mathbb{T}^2_\theta)$ (see also [1] for a preliminary version). Connes' pseudodifferential calculus for C^*-dynamical systems is employed crucially to apply heat kernel techniques to geometric operators on $\mathbb{T}_\theta^2$ to derive small time heat kernel expansions that encode local geometric information such as scalar curvature. A purely noncommutative feature is the appearance of the modular automorphism of the Tomita-Takesaki theory for the KMS state implementing the conformal perturbation of the metric in the computations and in the final formula for the curvature [4, 7]. Among other results, in [9] we show that modular automorphism appears also in the final formula for the scalar curvature of the noncommutative 4-tori.

We consider the noncommutative 4-torus $\mathbb{T}_\theta^4$ with the simplest structure of a noncommutative abelian variety. We perturb the standard volume form on this space conformally and analyse the corresponding perturbed Laplacian. Using Connes' pseudodifferential calculus for $\mathbb{T}_\theta^4$ we derive the small time heat kernel expansion for the perturbed Laplacian. This enables us to prove the analogue of Weyl's law for $\mathbb{T}_\theta^4$ by studying the asymptotic distribution of the eigenvalues of the perturbed Laplacian on this space. We define a noncommutative residue on the algebra of classical pseudodifferential operators on $\mathbb{T}_\theta^4$, and show that it gives the unique continuous trace on this algebra. We also prove the analogue of Connes' trace theorem for $\mathbb{T}_\theta^4$ by showing that this noncommutative residue and the Dixmier trace coincide on pseudodifferential operators of order -4. We have performed the computation of the scalar curvature for $\mathbb{T}_\theta^4$, and found explicit formulas for the local functions that describe the curvature in terms of the modular automorphism of the conformally perturbed volume form and derivatives of the logarithm of the Weyl factor. Then, by integrating this curvature, we define and find an explicit formula for the analogue of the Einstein-Hilbert action for $\mathbb{T}_\theta^4$. Finally, we show that the extremum of this action occurs at metrics with constant scalar curvature. We record here some of the main results of our paper [9] that were presented in my talk in Oberwolfach.

Theorem 1. *(Noncommutative Weyl's law) The eigenvalue counting function N of the Laplacian $\triangle_\varphi$ on $\mathbb{T}_\theta^4$ satisfies*

$$N(\lambda) \sim \frac{\pi^2 \varphi_0(e^{-2h})}{2} \lambda^2 \qquad (\lambda \to \infty). \tag{1}$$

Theorem 2. *(Noncommutative Connes' trace theorem) Let ρ be a classical pseudodifferential symbol of order -4 on $\mathbb{T}_\theta^4$. Then P_ρ is a measurable operator in $\mathcal{L}^{1,\infty}(\mathcal{H}_0)$, and under the assumption that all nonzero entries of θ are irrational, we have*

$$⨍ P_\rho = \frac{1}{4}\,\mathrm{res}(P_\rho).$$

Following [3, 4, 7] we define the scalar curvature of $\mathbb{T}_\theta^4$ equipped with the perturbed Laplacian $\triangle_\varphi$ as follows.

Definition. *The scalar curvature of the noncommutative 4-torus equipped with the perturbed volume form is the unique element $R \in C^\infty(\mathbb{T}_\theta^4)$ such that*

$$\mathrm{res}_{s=1}\mathrm{Trace}(a\triangle_\varphi^{-s}) = \varphi_0(aR),$$

for any $a \in C^\infty(\mathbb{T}^4_\theta)$.

Theorem 3. *The scalar curvature* R *of* $\mathbb{T}^4_\theta$*, up to a factor of* π^2*, is equal to*

$$e^{-h}K(\nabla)\Big(\sum_{i=1}^{4}\delta_i^2(h)\Big)+e^{-h}H(\nabla_{(1)},\nabla_{(2)})\Big(\sum_{i=1}^{4}\delta_i(h)^2\Big), \tag{2}$$

where

$$K(s)=\frac{1-e^{-s}}{2s},$$
$$H(s,t)=-\frac{e^{-s-t}\left(\left(-e^s-3\right)s\left(e^t-1\right)+\left(e^s-1\right)\left(3e^t+1\right)t\right)}{4st(s+t)}.$$

A natural analogue of the Einstein-Hilbert action for $\mathbb{T}^4_\theta$ is $\varphi_0(R)$, where R is the scalar curvature given by (2). In the following theorem we find an explicit formula for this action.

Theorem 4. *A local expression for the Einstein-Hilbert action for* $\mathbb{T}^4_\theta$*, up to a factor of* π^2*, is given by*

$$\varphi_0(R)=\frac{1}{2}\varphi_0\Big(\sum_{i=1}^{4}e^{-h}\delta_i^2(h)\Big)+\varphi_0\Big(\sum_{i=1}^{4}G(\nabla)(e^{-h}\delta_i(h))\delta_i(h)\Big), \tag{3}$$

where $G(s)=\frac{-4s-3e^{-s}+e^s+2}{4s^2}$.

The Einstein-Hilbert action $\varphi_0(R)$ attains its maximum if and only if the Weyl factor e^{-h} is a constant. This is done by combining the two terms in the explicit formula (3) for $\varphi_0(R)$, and observing that it can be expressed by a non-negative function. We note that the function G in (3), is neither bounded below nor bounded above.

Theorem 5. *The maximum of the Einstein-Hilbert action is equal to* 0*, and it is attained if and only if the Weyl factor is a constant. That is, for any Weyl factor* $e^{-h}, h=h^*\in C^\infty(\mathbb{T}^4_\theta)$*, we have*

$$\varphi_0(R)\le 0,$$

and the equality happens if and only if h *is a constant.*

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Dynamics and Algebraic Properties of Algebraic Actions

Nhan-Phu Chung

Actions of countable discrete groups Γ on compact (metrizable) groups X by (continuous) automorphisms are a rich class of dynamical systems, and have drawn much attention since the beginning of ergodic theory. The fact that $\mathbb{Z}\mathbb{Z}^d$ is a commutative factorial Noetherian ring plays a vital role for such study, as it makes the machinery of commutative algebra available.
Recently, via operator algebra method, especially group von Neumann algebras, much progress has been made towards understanding the algebraic actions of general countable groups. In this talk I will describe recent results about dichotomies between dynamics property of algebraic actions and algebraic property of the corresponding modules over group rings.

Let Γ be a countable group. An action of Γ on a compact abelian group by automorphisms is called an *algebraic action.* For a locally compact abelian group X, we denote by $\widehat{X}$ its Pontryagin dual. Then we have the following correspondences:

{countable left $\mathbb{Z}\Gamma$-modules $\widehat{X}$} $\leftrightarrow$ { Γ-actions on discrete abelian groups $\widehat{X}$},

and

{ Γ-actions on discrete abelian groups $\widehat{X}$} $\leftrightarrow$ { Γ-actions on compact abelian groups X}.

Example 3. *(1) For each $k \in \mathbb{N}$, we may identify the Pontryagin dual $\widehat{(\mathbb{Z}\Gamma)^k}$ of $(\mathbb{Z}\Gamma)^k$ with $((\mathbb{R}/\mathbb{Z})^k)^\Gamma = ((\mathbb{R}/\mathbb{Z})^\Gamma)^k$ naturally. Under this identification, the canonical action of Γ on $\widehat{(\mathbb{Z}\Gamma)^k}$ is just the left shift action on $((\mathbb{R}/\mathbb{Z})^k)^\Gamma$. If J is a left $\mathbb{Z}\Gamma$-submodule of $(\mathbb{Z}\Gamma)^k$, then $\widehat{(\mathbb{Z}\Gamma)^k/J}$ is identified with*

$$\{(x_1,\ldots,x_k) \in ((\mathbb{R}/\mathbb{Z})^\Gamma)^k : x_1 g_1^* + \cdots + x_k g_k^* = 0, \text{ for all } (g_1,\ldots,g_k) \in J\}.$$

(2) *Principal algebraic action: Let $f \in \mathbb{Z}\Gamma$. Put $X_f := \widehat{\mathbb{Z}\Gamma/\mathbb{Z}\Gamma f} = \{g \in (\mathbb{R}/\mathbb{Z})^\Gamma : f \cdot g = 0\}$. Let α_f be the right shift action, i.e $\alpha_{f,\gamma}(g) = g \cdot \gamma^{-1}, \forall \gamma \in \Gamma, g \in \widehat{\mathbb{Z}\Gamma/\mathbb{Z}\Gamma f}$. Then $\alpha_f : \Gamma \curvearrowright X_f = \widehat{\mathbb{Z}\Gamma/\mathbb{Z}\Gamma f}$ is called a principal algebraic action.*

An action of Γ on a compact space X is called *expansive* if there is a constant $c > 0$ such that $\sup_{s\in\Gamma} \rho(sx, sy) > c$ for all distinct x, y in X, where ρ is a compatible metric on X. The definition does not depend on the choice of ρ.

Previously, characterizations of expansiveness for algebraic actions have been obtained in various special cases, such as the case $\Gamma = \mathbb{Z}^d$ for $d \in \mathbb{N}$ [11], the case Γ is abelian [10], the case $\widehat{X} = \mathbb{Z}\Gamma/J$ for a finitely generated left ideal J of $\mathbb{Z}\Gamma$ [6], the case X is connected and finite-dimensional [1], and the case $\widehat{X} = \mathbb{Z}\Gamma/\mathbb{Z}\Gamma f$ for some $f \in \mathbb{Z}\Gamma$ [5].

Lemma ([2])**.** *Let $k \in \mathbb{N}$, and $A \in M_k(\mathbb{Z}\Gamma)$ be invertible in $M_k(\ell^1(\Gamma))$. Denote by J the left $\mathbb{Z}\Gamma$-submodule of $(\mathbb{Z}\Gamma)^k$ generated by the rows of A. Then the canonical action α of Γ on $X_A := \widehat{(\mathbb{Z}\Gamma)^k/(\mathbb{Z}\Gamma)^k A}$ is expansive.*

The canonical actions α of Γ on $X_A = \widehat{(\mathbb{Z}\Gamma)^k/(\mathbb{Z}\Gamma)^k A}$ as above are the largest expansive algebraic actions in the sense that every expansive algebraic action is the restriction of one of these actions to a closed invariant subgroup.

Theorem 1 ([2])**.** *Let Γ act on a compact abelian group X by automorphisms. Then the following are equivalent:*

(1) *the action is expansive;*
(2) *there exist some $k \in \mathbb{N}$, some left $\mathbb{Z}\Gamma$-submodule J of $(\mathbb{Z}\Gamma)^k$, and some $A \in M_k(\mathbb{Z}\Gamma)$ being invertible in $M_k(\ell^1(\Gamma))$ such that the left $\mathbb{Z}\Gamma$-module $\widehat{X}$ is isomorphic to $(\mathbb{Z}\Gamma)^k/J$ and the rows of A are contained in J;*
(3) *the left $\mathbb{Z}\Gamma$-module $\widehat{X}$ is finitely generated, and $\ell^1(\Gamma) \otimes_{\mathbb{Z}\Gamma} \widehat{X} = \{0\}$.*

Theorem 2 ([2])**.** *Suppose that Γ is amenable. Let α be an action of Γ on a compact abelian group X by automorphisms and $f \in \mathbb{Z}\Gamma$. Then f is a non-zero divisor of $\mathbb{Z}\Gamma$ if and only if $h(\alpha_f)$ is finite, where $h(\alpha_f)$ is the topological entropy of α_f .*

The left group von Neumann algebra $\mathfrak{L}\Gamma$, is defined as the closure of $\mathbb{C}\Gamma$ under the strong operator topology. Explicitly, $\mathfrak{L}$ consists of $T \in B(\ell^2(\Gamma))$ commuting with the right regular representation of Γ on $\ell^2(\Gamma)$, i.e., $(T(h\gamma))_{\gamma'} = (Th)_{\gamma'\gamma}$ for all $h \in \ell^2(\Gamma)$ and $\gamma, \gamma' \in \Gamma$, where $(h\gamma)_{\gamma''} = h_{\gamma''\gamma}$ for all $\gamma, \gamma'' \in \Gamma$. The algebra $\mathfrak{L}\Gamma$ has a canonical tracial state $\mathrm{tr}_{\mathfrak{L}\Gamma}$ defined as $\mathrm{tr}_{\mathfrak{L}\Gamma}(T) = \langle Te_\Gamma, e_\Gamma\rangle$.
The *Fuglede-Kadison determinant* for an invertible $u \in \mathfrak{L}\Gamma$ is defined as

$$\det{}_{\mathrm{FK}}(u) = \exp(\mathrm{tr}_{\mathfrak{L}\Gamma} \log |u|) = \exp(\frac{1}{2}\mathrm{tr}_{\mathfrak{L}\Gamma} \log(u^* u)),$$

where $|u| = (u * u)^{1/2}$ is the absolute part of u.
When $f \in \mathfrak{L}\Gamma$ and is not invertible, we define $\det_{\mathrm{FK}}(f) = \lim_{\varepsilon\to 0} \det_{\mathrm{FK}}(|\mathrm{f}| + \varepsilon)$.

Example 4. *(1) For any $n \in \mathbb{N}$ and any invertible $u \in B(\ell^2_n)$, one has* $\det_{FK}(u) = |\det u|^{1/n}$.

(2) In the case $\Gamma = \mathbb{Z}^d$ for some $d \in \mathbb{N}$: Let $f \in \mathbb{Z}\Gamma$. Using Fourier transform we can calculate trace of f by integral on d-tori: $\mathrm{tr}(f) = \int_{\mathbb{T}^d} f(s)ds$. Then for any f is invertible in A, $\det_{\mathrm{FK}}(f) = \mathbb{M}(f)$, where $\mathbb{M}(f) = \exp(\int_{\mathbb{T}^d} \log|f(s)|\, ds)$ is the Mahler measure of f.

Theorem 3 ([8])**.** *Let Γ be an amenable group and $f \in \mathbb{Z}\Gamma$. If f is a non-zero divisor of $\mathbb{Z}\Gamma$ then $h(\alpha_f) = \log \det_{\mathrm{FK}}(f)$.*

The relation between entropy of principal algebraic action and Fuglede-Kadison determinant was established before for special cases such as: when $\Gamma = \mathbb{Z}^d$ [9]; when f is invertible in $\ell^1(\Gamma)$ and Γ is virtually nilpotent [4]; and for general amenable groups and f is invertible in $\mathfrak{L}\Gamma$ [7].

Let $\mathcal{M}$ be a countable left $\mathbb{Z}\Gamma$-module. We define $\rho(\mathcal{M}) := h(\Gamma \curvearrowright \widehat{\mathcal{M}}) \in [0, \infty]$, where $h(\Gamma \curvearrowright \widehat{\mathcal{M}})$ is the topological entropy. The quantity $\rho(\mathcal{M})$ is called *torsion* of $\mathcal{M}$. Put $\mu_p := \mathbb{Z}[\frac{1}{p}]/\mathbb{Z}$. Note that there is an exact sequence

$$0 \to \mathbb{Z} \to \mathbb{Z}[\tfrac{1}{p}] \to \mu_p \to 0,$$

which induces an exact sequence

$$0 \to \mathrm{Tor}(\mu_p, \mathcal{M}) \to \mathcal{M} \to \mathbb{Z}[\tfrac{1}{p}] \otimes_{\mathbb{Z}} \mathcal{M} \to \mu_p \otimes_{\mathbb{Z}} \mathcal{M} \to 0,$$

for any abelian group $\mathcal{M}$. Here, we have $\mathrm{Tor}(\mu_p, \mathcal{M}) = \{x \in \mathcal{M} \mid \exists k \in \mathbb{N}\, p^k x = 0\}$. If $\mathcal{M}$ is a $\mathbb{Z}\Gamma$-module satisfying $\rho(\mathcal{M}) < \infty$, we set

$$\rho_p(\mathcal{M}) := \rho(\mathrm{Tor}(\mu_p, \mathcal{M})) - \rho(\mu_p \otimes_{\mathbb{Z}} \mathcal{M}).$$

In anology to the finite places, we set $\rho_\infty(\mathcal{M}) = \rho(\mathbb{Q} \otimes_{\mathbb{Z}} \mathcal{M})$ for any $\mathbb{Z}\Gamma$-module $\mathcal{M}$.

Theorem 4 ([3])**.** *Let $\mathcal{M}$ be a $\mathbb{Z}\Gamma$-module with finite torsion. Then, we have*

$$\rho(\mathcal{M}) = \rho_\infty(\mathcal{M}) + \sum_p \rho_p(\mathcal{M}).$$

Moreover, for any exact sequence $0 \to \mathcal{M}' \to \mathcal{M} \to \mathcal{M}'' \to 0$ of $\mathbb{Z}\Gamma$-modules with finite torsion, we have $\rho_p(\mathcal{M}) = \rho_p(\mathcal{M}') + \rho_p(\mathcal{M}'')$ for any prime p, and $\rho_\infty(\mathcal{M}) = \rho_\infty(\mathcal{M}') + \rho_\infty(\mathcal{M}'')$.

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Arithmetic and Automorphic *L*-functions

James W. Cogdell

I. Arithmetic L-functions. Why do we care about L-functions? One reason is that they are wonderful local to global interpolation devices.

For the Riemann zeta function we input the primes through the Euler product $\zeta(s) = \prod_p (1-p^{-s})^{-1}$, which only converges for $Re(s) > 1$. But the interesting information obtainable from $\zeta(s)$ is in the region $0 \leq Re(s) \leq 1$. The non-vanishing of $\zeta(s)$ on the line $s = 1+it$ gives the Prime Number Theorem and the Riemann hypothesis is that the (non-trivial) zeroes of $\zeta(s)$ lie on $s = \frac{1}{2}+it$ [8, 5].

If k is a number field and $\zeta_k(s) = \prod_{\mathfrak{p}}(1-N(\mathfrak{p})^{-s})^{-1}$ is its Dedekind zeta function, converging for $Re(s) > 1$, Dirichlet's analytic class number formula tells us the residue of the pole of $\zeta_k(s)$ at $s = 1$ contains the most important global invariants of the field: the class number h_k, the regulator R_k, the number of roots of unity in k, and the absolute value of its discriminant d_k [5].

If $E/\mathbb{Q}$ is an elliptic curve, then its L-function $L(s, E)$ is also defined by an Euler product converging for $Re(s) > 1$ (if properly normalized) where the p^{th} Euler factor depends on the number of points of (the reduction of) E over the finite field $\mathbb{F}_p$. By the Mordell-Weil theorem we know that the group of rational points $E(\mathbb{Q})$ is a finitely generated abelian group, and the rank r of the free part of $E(\mathbb{Q})$ is a most fundamental global invariant of the curve. The simplest statement of the Birch–Swinnerron-Dyer Conjecture [11] says that the order of vanishing of $L(s, E)$ at $s = \frac{1}{2}$ is precisely r.

What are the properties of an L-function that give it such interpolation properties? Experience tells us that they are the following. Let M be an arithmetic or geometric object. (M of course stands for motive, but only in a vague sense.) Then (i) $L(s, M) = \prod_p L_p(s, M)$ should be given by an Euler product, convergent for $Re(s) >> 0$, where the p^{th} Euler factor is defined by local information about M "mod p", (ii) there should be an associated archimedean factor, or Γ–factor $\Gamma_\infty(s, M)$, essentially made up of products of shifts of the classical $\Gamma(s)$, such that (iii) the completed L-function $\Lambda(s, M) = \Gamma_\infty(s, M)L(s, M)$ is *nice*, in the sense that (a) $\Lambda(s, M)$ has a meromorphic continuation to all of $\mathbb{C}$, entire if M

is irreducible (and not “trivial”) and (b) $\Lambda(s,M)$ satisfies a functional equation $\Lambda(s,M)=\varepsilon(s,M)\Lambda(1-s,M^\vee)$. It is the functional equation that mixes the various local pieces together and accounts for the interpolation properties – a “hidden symmetry” among the disparate local pieces.

Currently, the local factors $L_v(s,M)$ and $\Gamma_v(s,M)$ are all independently locally defined; there is no global definition of $\Lambda(s,M)$. Will some version of the Connes-Consani cyclic homology theory with adelic coefficients provide such a global definition? If so, it will need come with a method of continuation, a definition of local $\varepsilon_v(s,M)$ factors, and a global duality to give the functional equation.

II. Arithmetic L-functions $\leftrightarrow$ Automorphic L-functions. The only way we currently know to obtain the analytic continuation and functional equation necessary for $\Lambda(s,M)$ is to somehow associate to M an *analytic object*, a classical modular form f or an automorphic representation π, such that $\Lambda(s,M)=\Lambda(s,f)$ or $\Lambda(s,M)=\Lambda(s,\pi)=L(s,\pi)$, because we have analytic tools from Riemann, Hecke, Rankin, Selberg, Jacquet, Piatetski-Shapiro, Shalika, Shahidi ... to show $\Lambda(s,f)$ or $L(s,\pi)$ are nice. Riemann wrote the completed zeta function as the Mellin transform of Jacobi's theta series [8]; after Wiles proof of Fermat, we know the completed L-function of an elliptic curve over $\mathbb{Q}$ is the Mellin transform of a weight 2 holomorphic modular form [4]. This problem of arithmetic $\leftrightarrow$ automorphic correspondence is a central part of the Langlands Program [1]. For example, the local and global Langlands correspondences/conjectures make a very precise connection between n–dimensional Galois representations and automorphic forms on GL_n such that their L-functions correspond.

III. Automorphic L-functions. Let us return to our arithemtic interpolation problem. There is a natural space, or family of spaces, that should offer this interpolation. These are spaces made from adeles. For example, for the field $\mathbb{Q}$ we have $\mathbb{A}=\mathbb{R}\prod_p'\mathbb{Q}_p$ a locally compact topological ring, with $\mathbb{Q}$ embedding diagonally as a co-compact discrete subring. The restricted product decomposition of $\mathbb{A}$ sees all the local places (like an Euler product) and the diagonal action of $\mathbb{Q}$ then provides global mixing (like a functional equation). This is the space that Tate used in his thesis [10]. More generally we can look at $GL_n(\mathbb{A})=GL_n(\mathbb{R})\prod_p' GL_n(\mathbb{Q}_p)$, a locally compact topological group, with $GL_n(\mathbb{Q})$ embedding as a discrete subgroup which has finite co-volume (modulo the center). One can of course do the same over any global field k and for any reductive algebraic group G.

One way to try to understand these spaces is to understand an appropriate class of functions on them. For our purposes, this is the space of (smooth) automorphic forms $\mathcal{A}^\infty(G(k)\backslash G(\mathbb{A}))$ [1, 2]. These are functions on $G(\mathbb{A})$ (so they see the product decomposition) that are invariant by $G(k)$ (so they understand interpolation) and satisfy suitable regularity and growth properties. The group $G(\mathbb{A})$ acts on this space by right translation and one can decompose it into irreducible $G(\mathbb{A})$ representations. One has a (rough) decomposition

$$\mathcal{A}^\infty=\mathcal{A}^\infty_{cusp}\oplus\mathcal{A}^\infty_{res}\oplus\mathcal{A}^\infty_{cont}$$

where $\mathcal{A}^\infty_{cusp} = \oplus m(\pi)V_\pi$ is the space of cusp forms and the pieces (π, V_π) are the cuspidal automorphic representation, $\mathcal{A}^\infty_{res}$ are the residual forms, and $\mathcal{A}^\infty_{cont}$ is continuously spanned by the Eisenstein series. Note that a cuspidal representation (or any irreducible admissible representation) of $G(\mathbb{A})$ decomposes as $\pi \simeq \otimes'\pi_v$ where π_v is an irreducible admissible representation of $G(k_v)$, so we already have an "Euler product" expansion at this level.

Langlands understood two basic constructions very early on. (i) For cuspidal $\pi \simeq \otimes'\pi_v$, for almost all places v of k Langlands associated to π_v an Euler factor $L(s, \pi_v)$ and then to π a partial Euler product $L^S(s, \pi) = \prod_{v \notin S} L(s, \pi_v)$ such that $L^S(s, \pi)$ converged in a right half place $Re(s) >> 0$ [6]. (Here S is essentially the finite set of places where anything relevant is ramified.) (ii) Langlands understood the theory of Eisenstein series [7]. These are automorphic forms on $G(\mathbb{A})$ that are pulled in from the "boundary" of the space $G(k)\backslash G(\mathbb{A})$. This boundary is parameterized by parabolic subgroups $P \subset G$ whose Levi factors $M \subset P$ are again reductive algebraic groups and to cuspidal representations σ of $M(\mathbb{A})$ one associates an Eisenstein series $E(g; s, f)$ for cusp forms $f \in V_\sigma$ and $s \in \mathbb{C}$ a complex parameter. Langlands showed that these Eisenstein series converge for $Re(s) >> 0$, have a meromorphic continuation to $\mathbb{C}$, and satisfy a functional equation, roughly of the form $E(g; s, f) = E(g; 1-s, M(s)f)$ with $M(s)$ an intertwining operator. (In fact, Langlands L-functions in (i) came out of considering the constant terms of Eisenstein series.)

Where are the L-functions and how do they obtain an analytic continuation and functional equation? There are two schools. (1) The *Langlands-Shahidi* method is based on the observation that the Fourier coefficients of $E(g; s, f)$ as above contain $L(s, \sigma)^{-1}$. Then one deduces the meromorphic continuation and functional equation of $L(s, \sigma)$ from those of the Eisenstein series [9]. (This is a vast simplification of the work of Shahidi!). This method is well understood by now and works well for L-functions $L(s, \sigma)$ of cuspidal representations σ of groups M that occur as Levi subgroups of parabolic subgroups P inside a larger reductive group G. (2) The method of *integral representations* is a vast generalization of the work of Hecke, Rankin, and Selberg, pioneered by Jacquet, Piatetski-Shapiro, and Shalika [2, 3]. For a general group G, other than GL_n, these integral representations take the following form. Begin with a cusp form $f \in V_\pi$ on $G(\mathbb{A})$ and an Eisenstein series $E(g; s)$ on an auxiliary group $H(\mathbb{A})$, and say $H \supset G$ for example. Then, in rough outline, the integral representation would relate

$$L(s, \pi) \sim \int_{G(k)\backslash G(\mathbb{A})} f(g) E(g; s)\, dg$$

and again we deduce continuation and functional equation from that of $E(g; s)$. Given $L(s, \pi)$, finding an integral representation for it is quite mysterious. It is more of an art form than a "method" or a science at this moment.

So even to obtain the analytic continuation and functional equation for the general cuspidal automorphic representation of $G(\mathbb{A})$, we need (i) a better understanding of the space of cusp forms $\mathcal{A}^{\infty}_{cusp}(G(k)\backslash G(\mathbb{A}))$ and (ii) a better understanding of the Eisenstein series $E(g;s)$ coming from the boundary of $G(k)\backslash G(\mathbb{A})$. Can NCG help us better understand these as analytic spaces? In particular, can the non-commutative boundary of these spaces give us more Eisenstein series to work with either to expand the Langlands-Shahidi method or give us new integral representations?

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Brown measure, Haagerup–Schultz subspaces and upper triangularization in finite von Neumann algebras

Ken Dykema
(joint work with Fedor Sukochev, Dmitriy Zanin)

Aronszajn and Smith [1] showed that every compact operator on Banach space has a nontrivial invariant subspace. (Note that this has been generalized and an easier proof given by Lomonosov [8]. See also [9] for a survey of Lomonosov's technique and some extensions.) Ringrose [10] used this result to prove that for every compact operator T on Hilbert space, there is a maximal increasing family $(p_t)_{0\le t\le 1}$ of T–invariant projections such that $T=N+Q$, where $N=\sum_{0<t\le 1}(p_t-p_{t-})T(p_t-p_{t-})$, with $p_{t-}=\vee_{s<t}p_s$, is a normal operator and Q is quasinilpotent.

Our goal (in joint work with Fedor Sukochev and Dmitriy Zanin) is to prove some sort of analogue of this result for elements of finite von Neumann algebras. We do so by using the Haagerup–Schultz invariant projections [7]. These are intimately bound up with Brown measure [2], which is in turn derived from the Fuglede–Kadison determinant [6].

We work in a von Neumann algebra $\mathcal{M}$ having a fixed normal, faithful tracial state τ. The Fuglede–Kadison determinant of $T\in\mathcal{M}$ is

$$\Delta(T)=\exp(\tau(\log|T|)),$$

which should be interpretted as 0 if T has a nonzero kernel or if the (unbounded) operator $\log|T|$ does not have finite trace. Letting $f(\lambda)=\Delta(T-\lambda I)$, where I is the identity operator, the Brown measure of T is constructed as $\mu_T=\frac{1}{2\pi}\nabla^2 f$, where the Laplacian is taken in the sense of distributions. It is the unique compactly supported probability measure satisfying

$$\log\Delta(T-\lambda)=\int\log|z-\lambda|\,d\mu_T(z)$$

for all $\lambda\in\mathbf{C}$. In fact, the support of μ_T is contained in the spectrum, $\sigma(T)$, of T and Brown's analogue of Lidskii's theorem holds: $\tau(T)=\int z\,d\mu_T(z)$. In two examples the Brown measure is a familiar obejct: (1) if $T\in M_n(\mathbf{C})$, then

$$\mu_T=\frac{1}{n}\sum_{j=1}^{n}\delta_{\lambda_j},$$

where $\lambda_1,\ldots,\lambda_n$ are the eigenvalues of T, listed according to algebraic multiplicity; (2) if $N\in\mathcal{M}$ is a normal operator, then

$$\mu_N=\tau\circ(\text{spectral measure of }N).$$

Here are three versions of the invariant subspace problem for (infinite dimensional, separable) Hilbert space $\mathcal{H}$ and all of them are open:

(I) The invariant subspace problem: is there, for arbitrary $T\in B(\mathcal{H})$, a projection $p\in B(\mathcal{H})$ such that $p\notin\{0,1\}$ and $Tp=pTp$?

(II) The invariant subspace problem relative to a von Neumann algebra: is there, for arbitrary $T \in B(\mathcal{H})\backslash \mathbf{C}I$, a projection p belonging to the von Neumann algebra generated by T such that $p \notin \{0,1\}$ and $Tp = pTp$?

(III) The hyperinvariant subspace problem: is there, for arbitrary $T \in B(\mathcal{H})\backslash \mathbf{C}I$, a projection $p \in B(\mathcal{H})$ such that $p \notin \{0,1\}$ and $Sp = pSp$ for all $S \in B(\mathcal{H})$ such that $ST = TS$?

A projection p satisfying the condition given in (I) is called T-invariant. A projection p satisfying the condition given in (III) is called T-hyperinvariant. It is not difficult to show that every T-hyperinvariant projection belongs to the von Neumann algebra generated by T.

In [7], Haagerup and Schultz proved that for $T \in \mathcal{M}$ and for a Borel set $B \subseteq \mathbf{C}$, if $0 < \mu_T(B) < 1$, then there is a unique projection $p_B = p_B(T) \in \mathcal{M}$ such that

(i) p_B is T-hyperinvariant
(ii) $\tau(p_B) = \mu_T(B)$
(iii) treating Tp_B as an element of $p_B\mathcal{M}p_B$ equipped with the renormalized trace $\tau(p_B)^{-1}\tau\restriction_{p_B\mathcal{M}p_B}$, its Brown measure is $\mu_{Tp_B} = \frac{1}{\mu_T(B)}\mu_T\restriction_B$
(iv) treating $(1-p_B)T$ as an element of $(1-p_B)\mathcal{M}(1-p_B)$ equipped with the renormalized trace, its Brown measure is $\mu_{(1-p_B)T} = \frac{1}{1-\mu_T(B)}\mu_T\restriction_{B^c}$,

where the restriction $\nu\restriction_E$ of a measure ν to a subset E is, of course, the measure $A \mapsto \nu(A \cap E)$. We call these projections p_B the *Haagerup–Schultz* projections of T.

A consequence of this result of Haagerup and Schultz is that all elements of a finite von Neumann algebra, except possibly those whose Brown measures are concentrated at single points, have nontrivial hyperinvariant subspaces. If the Brown measure of $T \in \mathcal{M}$ is δ_λ for some $\lambda \in \mathbf{C}$, then $T - \lambda$ has Brown measure δ_0. In [7], Haagerup and Schultz also proved that for $T \in \mathcal{M}$, $\mu_T = \delta_0$ if and only if

$$\text{s.o.t.-}\lim_{n\to\infty} ((T^*)^n T^n)^{1/2n} = 0, \tag{1}$$

where the limit is in strong operator topology. Compare this to the characterization for $T \in B(\mathcal{H})$ that T is quasinilpotent (i.e. has spectrum $\{0\}$) if and only if $\lim_{n\to\infty} \|((T^*)^n T^n)^{1/2n}\| = 0$. For this reason, we call an operator $T \in B(\mathcal{H})$ *s.o.t.-quasinilpotent* if (1) holds. (This terminology was introduced in [3]).

Here is are an anlogue of Ringrose's theorem in finite von Neumann algebras:

Theorem 1 ([4])**.** *Let $T \in \mathcal{M}$. There there is an increasing family $(q_t)_{0\le 1\le 1}$ of Haagerup–Schultz invariant projections of T with $q_0 = 0$ and $q_1 = 1$ such that, letting $\mathcal{D} = W^*(\{q_t \mid 0 \le t \le 1\})$ and letting $N = \mathrm{Exp}_{\mathcal{D}}(T)$, where $\mathrm{Exp}_{\mathcal{D}}$ is the τ–preserving conditional expectation from $\mathcal{M}$ onto $\mathcal{D}$, we have*

(a) N is normal
(b) $\mu_N = \mu_T$
(c) $T = N + Q$ where Q is s.o.t.–quasinilpotent.

This result is not a complete analogue of Ringrose's result, because our nest $(q_t)_{0\leq t\leq 1}$ need not be maximal. In particular, if μ_T has atoms, then there are gaps in the values of the traces $\tau(q_t)$.

The main idea of the proof is to choose a Peano curve $\rho : [0,1] \to \{|z| \leq \|T\|\}$ and to let $q_t = p_{\rho([0,t])}(T)$. Note that (a) is clear; to show (b) uses straightforward approximation arguments; our proof of (c) relies on the following Lemma.

Lemma. *If $T \in \mathcal{M}$ and if $(p_t)_{0\leq t\leq 1}$ is any increasing, right-continuous family of T-invariant projections in $\mathcal{M}$ with $p_0 = 0$ and $p_1 = 1$, then letting $\mathcal{D} = W^*(\{p_t \mid 0 \leq t \leq 1\})$ and letting $\mathrm{Exp}_{\mathcal{D}'}$ denote the conditional expectation onto the relative commutant $\mathcal{D}' \cap \mathcal{M}$ of $\mathcal{D}$ in $\mathcal{M}$, we have*

$$\Delta(T) = \Delta(\mathrm{Exp}_{\mathcal{D}'}(T)),$$

and, consequently, $\mu_T = \mu_{\mathrm{Exp}_{\mathcal{D}'}(T)}$.

Finally, we have the following further results for the decomposition.

Theorem 2 ([5]). *Let $T \in \mathcal{M}$ and let $T = N + Q$ be as in Theorem 1.*

(i) If h is a complex-valued function that is holomorphic on a neighborhood of the spectrum of T, then $h(T) = h(N)+Q_h$,, where Q_h is s.o.t.-quasinilpotent.

(ii) If $0 \notin \mathrm{supp}\mu_T$ (so that N is invertible), then $T = N(I+N^{-1}Q)$, and $N^{-1}Q$ is s.o.t.-quasinilpotent.

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Regularity in C*-algebras and topological dynamics

Wilhelm Winter

Dynamical systems and operator algebras are intimately related. On the one hand, dynamical systems are one of the main sources of natural examples of operator algebras; on the other hand, the latter can be used to systematically isolate and analyze key properties of the former. While this has been carried out with tremendous success by using von Neumann algebras in ergodic theory, the connections between C*-algebras and topological dynamics are somewhat less well developed. This is partially due to the fact that topological dynamical systems are much less likely to be determined by their associated crossed product C*-algebras. In particular, there are substantial additional complications related to high-dimensional topological phenomena (which do not seem to arise in the ergodic setup in the same manner). Consequently, the case of actions on the Cantor set is much better understood from the C*-algebraic point of view than higher dimensional situations. Indeed, Cantor minimal systems (with $\mathbb{Z}$-actions) are determined up to strong orbit equivalence by their ordered K-groups. This was first observed by Giordano, Putnam and Skau, who showed that up to strong orbit equivalence Cantor minimal systems are determined by their crossed product C*-algebras (see [3]); classification results then entailed that the latter are determined by their Elliott invariants. Only later, Glasner and Weiss gave a proof which did not rely on C*-algebras.

For more general group actions, the first problem is that even for $\mathbb{Z}^2$ it is not at all obvious how to define the large subalgebras (as in [5]) used by Giordano–Putnam–Skau to "break orbits". For more general spaces, the notion of strong orbit equivalence tends to degenerate. This phenomenon seems to be related to the fact that for base spaces which are not totally disconnected, in general there are no Rokhlin towers. The notion of Rokhlin dimension introduced by Hirshberg, Zacharias and myself is at least partially motivated by these problems, see [4].

Even in situations when the crossed product C*-algebra does retain sufficient information about the underlying dynamical system, one would still be left with the problem of classifying the former. Although the classification program for nuclear C*-algebras is by no means complete, there has been huge progress in recent years. In particular, we now know that classifiability (by K-theoretic invariants) is closely related to the presence of certain regularity properties, namely finite topological dimension, $\mathcal{Z}$-stability, and strict comparison. These may be interpreted as topological, analytic and algebraic regularity properties of C*-algebras. They turn out to be equivalent in stunning generality; in fact, A. Toms and myself have conjectured them to be equivalent for all simple nuclear C*-algebras (see [2, 7]); at least currently, they seem to be indispensable for any sort of classification result in this direction. Perhaps even more surprising, this 'perfect' regularity seems to be the key to handle amenable C*-algebraic classification in a way which is much more than just formally analogous to Connes' celebrated classification of injective factors, as pointed out by N. Brown.

It should be noted at this point that on the C*-side the theory is best established in the simple case, which corresponds to the case of factors on the von Neumann side and to (free) minimal actions on the dynamics side. This restriction reflects the current state of the art more than any sort of principal obstruction; for example, the regularity conjecture can also be formulated in the non-simple situation, only assuming that there are no elementary subquotients; for dynamical systems this would correspond to absence of periodic points.

While the above regularity properties have been (and still are) exploited successfully to push forward classification results for nuclear C*-algebras, in this talk I explain how they can be defined for dynamical systems themselves, and that even at this level they remain closely related.

More precisely, I define notions of topological dimension, $\mathcal{Z}$-stability and comparison for classical dynamical systems (i.e., $\mathbb{Z}$-actions on compact metrizable spaces); under suitable additional assumptions (like minimality or unique ergodicity) I derive several implications between these. Moreover, I explain how these definitions still make sense for $\mathbb{Z}^d$-actions; by a recent result by Szabó, finite dynamic dimension always occurs if the action is free and the space is finite dimensional, see [6]. For actions of hyperbolic groups, dynamic dimension is closely related to the existence of long thin covers as considered by Bartels–Lück–Reich in their work on the Farrell–Jones conjecture for hyperbolic groups, see [1].

The questions to consider next are whether the dynamic regularity properties play a role for the structure theory of dynamical systems similar to that of nuclear C*-algebras, whether they can be used with similar success, and whether they provide a new passway between crossed products and the underlying dynamics.

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Tannaka-Kreĭn duality for quantum homogeneous spaces

Makoto Yamashita
(joint work with Kenny De Commer)

An ergodic action of a compact quantum group $\mathbb{G}$ on an operator algebra A can be interpreted as a quantum homogeneous space for $\mathbb{G}$. In this project we aim at classifying the ergodic actions of compact quantum groups from a categorical viewpoint, taking into account of the natural Morita equivalence relations.

In the general part [1], we show that such ergodic actions can be classified by the category $\mathcal{D}_A$ of finite $\mathbb{G}$-equivariant Hilbert modules over A. This category has a natural structure of module over $\mathrm{Rep}(\mathbb{G})$, the tensor category of finite-dimensional representations of $\mathbb{G}$. Following the Tannaka–Kreĭn reconstruction theorem of S.L. Woronowicz, one reconstructs A from the fiber functor over $\mathrm{Rep}(\mathbb{G})$, the $\mathrm{Rep}(\mathbb{G})$-module structure of $\mathcal{D}_A$, and the distinguished irreducible object $A \in \mathcal{D}_A$. Different choices of irreducible objects correspond to the equivariantly and strongly Morita equivalent algebras. Moreover, the $\mathbb{G}$-equivariant $*$-homomorphisms of ergodic $\mathbb{G}$-algebras are encoded as $\mathrm{Rep}(\mathbb{G})$-module functors sending the prescribed irreducible object of one category to that of another. This gives a global approach to the duality theory for ergodic actions as developed by C. Pinzari and J. Roberts.

For the case of quantum SU(2) groups, the above general theory specializes to an interesting combinatorial theory. In [2], we obtain a classification of their quantum homogeneous spaces in terms of weighted oriented graphs.

Specifically, we consider an oriented graph $\Gamma = (V, E, s, t)$, together with the weight function $w\colon E \to \mathbb{R}^+$ on the edge set, subject to the following conditions. First, for any given vertex v, the sum $\sum_e w(e)$ over the edges with $s(e) = v$ have to be equal to $\left|q + q^{-1}\right|$. Next, there has to be an involution $e \mapsto \bar{e}$ on the edge set satisfying $r(\bar{e}) = s(e)$ and $w(\bar{e}) = w(e)^{-1}$. Additionally, when $q > 0$, we require that this involution is fixed-point free $\bar{e} \neq e$.

Such a graph without weights describes the structure of the equivariant K-group as an $R(\mathrm{SU}_q(2))$-module, and the weight function represents a more precise data encoding how the self duality of the fundamental representation behaves. The equivariant maps between these quantum homogeneous spaces can be characterized by certain quadratic equations associated with the natural transformation of module structures. We show that, for $|q|$ close to 1, all quantum homogeneous spaces are realized by coideals up to strong Morita equivalence. This generalizes a famous theorem by A. Wassermann for the case of SU(2), but our proof is easier even in that case.

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Participants

Dr. Abhishek Banerjee
Mathématiques
College de France
(Annexe)
3, rue d'Ulm
75005 Paris Cedex 05
FRANCE

Selcuk Barlak
Mathematisches Institut
Universität Münster
Einsteinstr. 62
48149 Münster
GERMANY

Dr. Jyotishman Bhowmick
Department of Mathematics
University of Oslo
P. O. Box 1053 - Blindern
0316 Oslo
NORWAY

Prof. Dr. Jean-Francois Burnol
U. F. R. Mathématiques
Université de Lille 1
Bat. M2
59655 Villeneuve d'Ascq Cedex
FRANCE

Dr. Alcides Buss
Mathematisches Institut
Universität Münster
Einsteinstr. 62
48149 Münster
GERMANY

Prof. Dr. Alan Carey
Mathematical Sciences Institute
Australian National University
Canberra ACT 0200
AUSTRALIA

Sayan Chakraborty
Mathematisches Institut
Universität Münster
Einsteinstr. 62
48149 Münster
GERMANY

Dr. Nhan-Phu Chung
Max-Planck-Institut für Mathematik
in den Naturwissenschaften
Inselstr. 22 - 26
04103 Leipzig
GERMANY

Prof. Dr. James W. Cogdell
Department of Mathematics
The Ohio State University
100 Mathematics Building
231 West 18th Avenue
Columbus, OH 43210-1174
UNITED STATES

Prof. Dr. Alain Connes
I.H.E.S.
Le Bois Marie
35, route de Chartres
91440 Bures-sur-Yvette
FRANCE

Prof. Dr. Caterina Consani
Department of Mathematics
Johns Hopkins University
Baltimore, MD 21218-2689
UNITED STATES

Prof. Dr. Gunther Cornelissen
Mathematisch Instituut
Universiteit Utrecht
P.O.Box 80.010
3508 TA Utrecht
NETHERLANDS

Prof. Dr. Joachim Cuntz
Mathematisches Institut
Universität Münster
Einsteinstr. 62
48149 Münster
GERMANY

Prof. Dr. Christopher Deninger
Mathematisches Institut
Universität Münster
Einsteinstr. 62
48149 Münster
GERMANY

Prof. Dr. Ken J. Dykema
Department of Mathematics
Texas A & M University
College Station, TX 77843-3368
UNITED STATES

Prof. Dr. Siegfried Echterhoff
Mathematisches Institut
Universität Münster
Einsteinstr. 62
48149 Münster
GERMANY

Prof. Dr. Manfred Einsiedler
Departement Mathematik
ETH-Zentrum
Rämistr. 101
8092 Zürich
SWITZERLAND

Prof. Dr. Alexander Gorokhovsky
Department of Mathematics
University of Colorado
Boulder, CO 80309-0395
UNITED STATES

Dr. Huichi Huang
Mathematisches Institut
Universität Münster
Einsteinstr. 62
48149 Münster
GERMANY

Prof. Dr. Marius Junge
Department of Mathematics
University of Illinois at
Urbana-Champaign
273 Altgeld Hall, MC-382
1409 West Green Street
Urbana, IL 61801-2975
UNITED STATES

Prof. Dr. Max Karoubi
U.F.R. de Mathématiques
Case 7012
Université de Paris 7
175 rue du Chevaleret
75251 Paris Cedex 13
FRANCE

Prof. Dr. Masoud Khalkhali
Department of Mathematics
The University of Western Ontario
London Ont. N6A 5B7
CANADA

Prof. Dr. Giovanni Landi
Dip. di Matematica e Informatica
Universita di Trieste
Via Alfonso Valerio 12/1
34127 Trieste
ITALY

Prof. Dr. Eric Leichtnam
Directeur de Recherche au CNRS
Institut Mathématiques de Jussieu
Etage 7 E
175 Rue de Chevaleret
75013 Paris
FRANCE

Dr. Xin Li
Mathematisches Institut
Universität Münster
Einsteinstr. 62
48149 Münster
GERMANY

Dr. Franz Luef
Fakultät für Mathematik
Universität Wien
Nordbergstr. 15
1090 Wien
AUSTRIA

Prof. Dr. Ralf Meyer
Mathematisches Institut
Georg-August-Universität Göttingen
Bunsenstr. 3-5
37073 Göttingen
GERMANY

Prof. Dr. Jack Morava
Department of Mathematics
Johns Hopkins University
Baltimore, MD 21218-2689
UNITED STATES

Prof. Dr. Sergey Neshveyev
Department of Mathematics
University of Oslo
P. O. Box 1053, Blindern
0316 Oslo
NORWAY

Prof. Dr. Ryszard Nest
Institut for Matematiske Fag
Kobenhavns Universitet
Universitetsparken 5
2100 Kobenhavn
DENMARK

Dr. Bogdan Nica
Mathematisches Institut
Georg-August-Universität Göttingen
Bunsenstr. 3-5
37073 Göttingen
GERMANY

Dr. Federico Pellarin
Université de St. Etienne
Mathématiques
42023 St. Etienne
FRANCE

Prof. Dr. Mira Peterka
Department of Mathematics
University of Kansas
Snow Hall
Lawrence, KS 66045-7567
UNITED STATES

Prof. Dr. Markus Pflaum
Department of Mathematics
University of Colorado
Boulder, CO 80309-0395
UNITED STATES

Prof. Dr. Michael Puschnigg
Institut de Mathématiques de Luminy
UMR 6206
Université de la Mediterranee
163 Avenue de Luminy
13288 Marseille Cedex 9
FRANCE

Prof. Dr. Bahram Rangipour
Dept. of Mathematics & Statistics
The University of New Brunswick
Fredericton NB E3B 5A3
CANADA

Karen Strung
Mathematisches Institut
Universität Münster
Einsteinstr. 62
48149 Münster
GERMANY

Gabor Szabo
Mathematisches Institut
Universität Münster
Einsteinstr. 62
48149 Münster
GERMANY

Dr. Georg Tamme
Fakultät für Mathematik
Universität Regensburg
93040 Regensburg
GERMANY

Prof. Dr. Xiang Tang
Department of Mathematics
Washington University in St. Louis
Campus Box 1146
Cupples I, Room 100
1 Brookings Drive
St. Louis, MO 63130
UNITED STATES

Thomas Timmermann
Fakultät für Mathematik
Institut für Informatik
Universität Münster
Einsteinstr. 62
48149 Münster
GERMANY

Dr. Erik van Erp
Department of Mathematics and
Computer Science
Dartmouth College
27 N. Main Street
Hanover, NH 03755-3551
UNITED STATES

Dr. Walter D. van Suijlekom
IMAPP
Radboud Universiteit Nijmegen
Huygens Bldg.
Heyendaalseweg 135
6525 AJ Nijmegen
NETHERLANDS

Prof. Dr. Anatoli M. Vershik
Steklov Mathematical Institute
PDMI
Fontanka 27
St. Petersburg 191 023
RUSSIAN FEDERATION

Prof. Dr. Charles A. Weibel
Department of Mathematics
Rutgers University
Busch Campus, Hill Center
New Brunswick, NJ 08854-8019
UNITED STATES

Prof. Dr. Rufus E. Willett
Department of Mathematics
University of Hawaii
2565 The Mall
Honolulu, HI 96822
UNITED STATES

Prof. Dr. Wilhelm Winter
Mathematisches Institut
Universität Münster
Einsteinstr. 62
48149 Münster
GERMANY

Dr. Jianchao Wu
Mathematisches Institut
Universität Münster
Einsteinstr. 62
48149 Münster
GERMANY

Prof. Dr. Zhizhang Xie
Department of Mathematics
Texas A & M University
College Station, TX 77843-3368
UNITED STATES

Prof. Dr. Quanhua Xu
Faculté des Sciences et Techniques
Laboratoire Mathématiques de Besancon
Université de Franche-Comte
16, route de Gray
25030 Besancon Cedex
FRANCE

Dr. Bora Yalkinoglu
Institut de Recherche
Mathématique Avancee
Université de Strasbourg et CNRS
7, rue Rene Descartes
67084 Strasbourg Cedex
FRANCE

Dr. Makoto Yamashita
School of Mathematics
Ochanomizu University
2-1-1, Otsuka
Bunkyo-ku
Tokyo 112-8610
JAPAN

Dr. Yi-Jun Yao
School of Mathematical Sciences
Fudan University
220 Hadan Road
Shanghai 200 433
CHINA

Prof. Dr. Guoliang Yu
Department of Mathematics
Texas A & M University
College Station, TX 77843-3368
UNITED STATES