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Cluster Algebras and Related Topics

Organised by
Bernhard Keller, Paris
Bernard Leclerc, Caen
Jan Schröer, Bonn

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Abstract. Cluster algebras are a class of commutative algebras intoduced by Fomin and Zelevinsky in 2000. Their original purpose was to obtain a combinatorial approach to Lusztig's dual canonical bases of quantum groups and to total positivity. Since then numerous connections between other areas of mathematics have been discovered. The aim of this workshop was to further strengthen these connections and to develop interactions.

Mathematics Subject Classification (2010): 13F60.

Introduction by the Organisers

The workshop *Cluster Algebras and Related Topics*, organised by Bernhard Keller (Paris 7), Bernard Leclerc (Caen) and Jan Schröer (Bonn) was attended by 52 participants coming from various different areas of mathematics. There were 23 one hour lectures given at the meeting allowing ample time for questions and discussions.

Cluster algebras were introduced by Fomin and Zelevinsky in 2000. They are by definition a class of commutative algebras with an inductively constructed set of algebra generators called *cluster variables*. These are grouped together in finite overlapping subsets (called *clusters*) of a given size. Starting with an initial cluster the other clusters and cluster variables are obtained via a combinatorially defined process called *mutation*.

The positivity conjecture (saying that all cluster variables are positive Laurent polynomials in any given cluster) was solved only very recently by Schiffler and Lee. Schiffler presented their proof in the opening talk of the conference.

The search for bases of cluster algebras with favourable properties (such as positivity) is still wide open. However substantial progress has been made for some classes of cluster algebras. Thurston presented in his talk a construction of a positive basis for skein algebras, which are intimately related to cluster algebras arising from triangulations of marked surfaces. For arbitrary skew-symmetric cluster algebras Plamondon presented his joint work with Cerulli, Keller and Labardini showing that all cluster monomials are linearly independent. The proof uses a categorification of cluster algebras via generalized cluster categories.

Geiß explained in his talk that for most mutation-finite quivers there exists up to equivalence only one non-degenerate potential. This implies that for such quivers there is essentially just one generalized cluster category. Labardini (joint work with Zelevinsky) presented a new approach via representations of species in an attempt to generalize Derksen, Weyman and Zelevinsky's additive categorification via Jacobian algebras from skew-symmetric to skew-symmetrizable cluster algebras. Iyama (joint work with Reiten and Adachi) presented a report on τ-tilting, a new framework for categorifying cluster algebras.

An important question is which algebras carry natural cluster algebra structures. Yakimov (joint work with Goodearl) presented a new ring theoretical approach for the construction of quantum cluster algebra structures on numerous quantum coordinate algebras arising in Lie theory. A different link between cluster algebras and Lie theory was the content of Gekhtman's talk on Poisson-Lie groups and cluster algebras.

The homogeneous coordinate rings of Grassmannians carry a natural cluster algebra structure by work of Scott. King presented a new categorification of these Grassmann cluster algebras, in terms of Cohen-Macauley modules over a twisted group ring. The interaction between Grassmann cluster algebras, the classical combinatorics of Grassmannians and dimer models have been the leading theme of the talks of Baur, Muller, Musiker and L. Williams.

The link between complex integrable systems and cluster algebras was the topic of Fock's talk and of Soibelman's report on his ground breaking work with Kontsevich. The strong connections between mathematical physics and cluster algebras were also discussed in the talks by Neitzke and H. Williams and Di Francesco.

Other interesting topics related with cluster algebras have been presented: cohomology of cluster varieties (Chapoton), complex volume of knots (Inoue), generalized friezes and cluster categories (Jorgensen, joint work with Holm), noncommutative cluster algebras (Retakh, joint work with Berenstein).

Last not least the week was filled with many informal discussions. The workshop provided a perfect atmosphere for exchanging ideas and strengthen interactions. It us our pleasure to thank the administration and the staff of the Oberwolfach Institute for their support and hospitality.

Workshop: Cluster Algebras and Related Topics

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Abstracts

Dimer models with boundary and Grassmannian cluster categories

Karin Baur
(joint work with Alastair King, Robert J. Marsh)

A dimer model can be defined as a quiver with an embedding of it into a compact oriented surface, giving rise to a tiling of the surface. Such dimer models can also be considered in the case of a surface with boundary. To Q we associate a natural potential W, obtained by using the oriented faces of the embedded quiver. Modifying the usual approach, we only take cyclic derivatives of W w.r.t. internal arrows: every internal arrow $\alpha \in Q_1$ belongs to two unique oriented cycles αp_α^+, αp_α^- (of opposite orientations), where $p_\alpha^\pm$ are paths from the head of α to the tail of α. Let ∂W be the set of relations $p_\alpha^+ = p_\alpha^-$ for all internal arrows α. We then define the *dimer algebra* A_Q of Q to be the quotient of path algebra $\mathbb{C}(Q)$ by the relations ∂W. In the boundary case, the idea of only considering relation with respect to internal arrows has arisen independently in work of Franco ([3]) and of Demonet-Luo ([2]).

In this work, we are interested in dimer models arising from alternating strand diagrams on a disk. They are collections of curves in a disk satisfying certain axioms. Alternating strand diagrams have been introduced by Postnikov for arbitrary permutations of n ([9]). Such diagrams also appear in this volume ([11], [7], [6]). In particular, we are interested in diagrams arising from the permutation $i \mapsto i + k$, for $1 \leq i \leq n$, with $k < n$ fixed, as used in Scott's work [10] on the cluster structure of the homogeneous coordinate ring of $\mathrm{Gr}(k, n)$. We call them (k, n)-diagrams.

We recall the algebra B appearing in [5] in this volume: B is the quotient of the preprojective algebra $\Pi(\tilde{A}_{n-1})$ of type $\tilde{A}_{n-1}$ on $2n$ generators $x_1, \ldots, x_n$, with $x_i : i \to i+1$, $y_1, \ldots, y_{n-1}$, with $y_i : i+1 \to i$ by the n (additional) relations $x^k = y^{n-k}$ (subscripts omitted for brevity).

By [4], there is a bijection between k-subsets of $\{1, \ldots, n\}$ and rigid indecomposable rank 1 modules in the category $\mathrm{CM}(B)$ of maximal Cohen-Macaulay modules for B. Furthermore, it is shown in [4, Prop 5.6] that two k-subsets are non-crossing if and only if the corresponding rank 1-modules have vanishing extension spaces.

We recall that any (k, n)-diagram D corresponds to a maximal collection of non-crossing k-subsets of $\{1, \ldots, n\}$ ([10], [8]). So the maximal collection of non-crossing k-subsets associated to a (k, n)-diagram gives rise to a cluster tilting object T_D of $\mathrm{CM}(B)$.

With this, we are ready to formulate the main result of this talk:

Theorem (B-K-M [1], §9)**.** Let D be an arbitrary (k, n)-diagram of reduced type. Let $A_{Q(D)}$ be the associated dimer algebra and T_D the associated B-module. Then there is an isomorphism $A_{Q(D)} \cong \mathrm{End}_B(T_D)$.

From this, we can deduce that the algebra B^{op} is isomorphic to the idempotent subalgebra $eA_{Q(D)}e$ by the idempotent e of all boundary vertices of $Q(D)$. In particular, the idempotent subalgebra is independent of the choice of D.

Thus, we obtain a description of B via an arbitrary (k, n)-diagram.

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Cohomology of fibers of cluster varieties

Frédéric Chapoton

Cluster algebras are commutative algebras, and it is natural to try to understand them from the point of view of algebraic geometry. In the case of acyclic cluster algebras, a finite presentation by generators and relations has been given in [1]. This allows to consider the spectrum of these cluster algebras as sub-schemes of affine spaces.

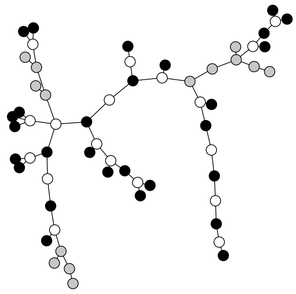
One very classical and useful invariant of algebraic varieties is the cohomology ring of their complex points. Our main question is therefore: what can be said about the cohomology of spectra of acyclic cluster algebras ? There are several additional motivations for asking this question. The first one is that some interesting integrals are related to these varieties, and understanding algebraic differential forms is a necessary first step to say more about them. The second reason is that there are already known differential forms, showing that at least part of the cohomology has a nice description.

As a first step towards the general case of acyclic cluster algebras, one restricts to the case of cluster algebras admitting a seed whose quiver is a tree. This is of course a strong limitation, as many cluster algebras do not have such a seed. But it remains a rather large family, where cohomology seems already complicated to describe. As explained later, our study involves nice combinatorial aspects of trees, that do not extend easily to more general quivers.

It is also well-known that cohomology of complex points of varieties defined over $\mathbb{Q}$ is strongly related (by famous results of Deligne) to counting the number of points over finite fields, but the latter is usually much simpler, as more tools are available.

To define the varieties that we want to study, it is necessary to first consider in depth the combinatorics of independent sets and matching of trees. The following result is due independently to J. Zito [2] and to M. Bauer and S. Coulomb [4].

Theorem 1. *For any tree T, there is a unique partition of the vertices of T into three subsets, called green, orange and red vertices, such that*
- Every red vertex $\bullet$ has only green neighbors.
- Every green vertex $\circ$ has at least two red neighbors.
- There is a perfect matching on the induced forest on orange vertices $\bullet$.



In this theorem, a matching is a set of disjoint edges, called dominos, which is perfect if every edge belongs to a domino. This coloring admits an alternative description using matchings of maximum cardinality (*maximum matchings*), and yet another one using independent sets of maximal cardinality. In particular, one can show that vertices not covered by a maximum matching must be red.

Let us now choose a tree T and a maximum matching of T. From this data, one can make an ice quiver as follows. The mutable vertices will be the vertices of T and frozen vertices will be new vertices, one for every vertex of T not covered by the matching. Every frozen vertex is attached by an edge to the corresponding vertex of T. This extended graph is still a tree, and is endowed with the alternating orientation.

One can then consider two kinds of varieties. The first one is the spectrum of the ring defined by the known presentation of the cluster algebra (where coefficient variables attached to frozen vertices are assumed to be invertible). For example, for the tree with one vertex, the generators are x, x' and the invertible coefficient α, with unique relation $x\,x' = 1 + \alpha$. This define an open set in the affine space of dimension 2.

The second sort of varieties one can look at are the fibers of the morphism to an algebraic torus defined by the coefficient variables. In the same example, this means fixing a complex value for α. This is smooth if the choice is generic.

One can generalize this as follows. The red-orange-green coloring of vertices of a tree T allows to define the red-green components of T as the components of the induced forest on vertices that belong to at least one red-green edge. One can then choose for every red-green component C either to let the coefficients variables attached to C stay variables, or to give them generic values.

Theorem 2. *The variety defined in this way does not depend on the chosen matching, up to isomorphism. All these varieties are smooth.*

The genericity assumption on the fixed values of coefficient variables can be made explicit, and is necessary to ensure smoothness. Consider for example the tree with 3 vertices. Its coloring is red–green–red. A maximum matching covers only two vertices, so there is a coefficient variable α attached to the remaining vertex. If the value of α is fixed to 1, then the variety is singular.

Theorem 3. *For any variety X in this family, the number of points of X over the finite fields $\mathbb{F}_q$ is a polynomial in q.*

One can show that there is an algebraic torus acting freely on these varieties. The dimension N of this torus is given by the sum, over red-green components where coefficients have been given a fixed generic value, of the number of red points minus the number of green points.

Theorem 4. *The number of points of X over the finite fields $\mathbb{F}_q$ is $(q-1)^N$ times a reciprocal polynomial in q.*

Let us now turn to cohomology. One is interested in the explicit description of algebraic differential forms, that should be a basis of the algebraic de Rham cohomology. Because our varieties are smooth, the cohomology with compact support (which is more closely connected to the counting of points over finite fields) can also be used to get some information. All the cohomology groups involved carry a mixed Hodge structure. One can show by induction that this mixed Hodge structure is of Tate type, *i.e.* an iterated extension of pure Hodge structures $\mathbb{Q}(i)$ for some integers i.

The main tools that can be used to compute cohomology are the Mayer-Vietoris long exact sequence for a covering by two open sets, and the similar spectral sequence attached to a covering by more open sets.

The cohomology ring contains 1-forms $\frac{d\alpha}{\alpha}$ for every coefficient variable (those not with a fixed value). It also contains the Weil-Petersson 2-form introduced in [3] and defined by the following sum over all edges:

$$\mathrm{WP} = \sum_{i \to j} \frac{dx_i dx_j}{x_i x_j}, \tag{1}$$

where one has to include frozen vertices, but exclude those that bear a fixed coefficient. In general these differential forms do not generate the full cohomology ring.

Let us now describe some results and conjectures, first for linear trees (Dynkin diagrams of type $\mathbb{A}$):

- For n even, the cohomology ring is generated by WP.
- For n odd and one free coefficient variable α, it is generated by WP and $\frac{d\alpha}{\alpha}$.
- For n odd and a generic fixed value for α, one conjectures that the cohomology ring is generated by WP and $n-1$ classes in top degree.

For trees of type $\mathbb{D}_n$ with n odd and one generic coefficient, the cohomology ring is generated by WP, an element of degree 3 and an element of top degree.

There is currently work in progress about understanding the case of trees given by adding one edge between two linear trees. A precise description of the cohomology of varieties attached to a general tree remains elusive so far.

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Fusion Products and Cluster Algebra

Philippe Di Francesco
(joint work with Rinat Kedem)

The characters of special finite-dimensional modules of simple Lie algebras $\mathfrak{g}$, or corresponding quantum affine algebras or Yangians, satisfy recursion relations, known as the Q-systems [1]. In the simply-laced case, the Q-system reads:

$$Q_{i,k+1}Q_{i,k-1} = Q_{i,k}^2 - \prod_{j\tilde{i}} Q_{j,k} \qquad (j = 1, 2, ..., r; k \in \mathbb{Z}_+) \tag{1}$$

where the index i labels the nodes of the corresponding Dynkin diagram, whereas the last product extends over the neighbors j of the node i, and r is the rank of the Lie algebra. For initial conditions $Q_{i,0} = 1$, $Q_{i,k}$ as determined by the above system is the character of the Kirillov-Reshetikhin module $KR_{i,k}$. The latter enjoy special properties such as cyclicity of any tensor product thereof.

The equations (1), up to a trivial rescaling and upon relaxing the initial condition, were shown to form a subset of the mutations of a suitably defined cluster algebra [2, 3]. For the simply-laced case, the latter has rank $2r$, and the initial exchange matrix $B = \begin{pmatrix} 0 & -C \\ C & 0 \end{pmatrix}$, where C is the corresponding Cartan matrix, while the initial cluster is $\{Q_{i,0}, Q_{i,1}\}_{i=1}^r$.

The associated quantum cluster algebra induces a natural deformation of these relations, called the quantum Q-system. For instance, for sl_2, the latter reads:

$$tQ_{k+1}Q_{k-1} = Q_k^2 - 1 \qquad (k \in \mathbb{Z}) \tag{2}$$

$$Q_kQ_{k+1} = tQ_{k+1}Q_k \qquad (k \in \mathbb{Z}) \tag{3}$$

for some fixed quantum parameter $t \in \mathbb{C}^*$. The non-commuting variables Q_k can no longer be interpreted as characters of the corresponding Kirillov-Reshetikhin modules. Moreover, from the Laurent property of quantum cluster algebra, they are expressible as Laurent polynomials of the initial cluster (Q_0, Q_1).

The solutions of the quantum Q-system turn out to play a crucial role in understanding the graded tensor product multiplicities of Kirillov-Reshetikhin modules in their decomposition onto fundamental ones, known as Feigin and Loktev's fusion products, defined as follows. We start from the action of the current algebra $\mathfrak{g}[z]$ on the tensor product of Kirillov-Reshetikhin modules $V_1, ..., V_N$ localized at generic complex points $z_1, ..., z_N$. The action of the current $x \otimes z^m$ is simply via usual coproduct for the Lie algebra part x, and z_i^m on the localized module $V_i(z_i)$ for the current part z^m. The homogeneous degree in z induces a grading of the tensor product (which is a cyclic module by the Kirillov-Reshetikhin property), and the fusion product, denoted by $V_1 \star V_2 \star \cdots \star V_N$ is simply the associated graded space $\mathcal{F}_m = F[m]/F[m-1]$ where $F[m]$ denotes the total degree m part of the tensor product. This in turn allows to define graded tensor product multiplicities in the decomposition of the fusion product onto irreducible modules V_λ as the Hilbert series:

$$N_{\{V_i\};V_\lambda}(q) = \sum_{m\geq 0} q^m \dim\left(\mathrm{Hom}\left(\mathcal{F}_m, V_\lambda\right)\right)$$

The fusion product was shown to be independent of the localization parameters z_i, henceforth we may drop them, and rewrite

$$V_1 \star V_2 \star \cdots \star V_N = \star_{i,k} KR_{i,k}^{\star n_{i,k}}$$

Our main result is a compact constant term formula for the graded tensor product multiplicities [4], taking the following form in the case of sl_2, where the Kirillov-Reshetikhin modules $KR_{1,k}$ are simply the irreducible modules $V_{k\omega_1}$:

Theorem.

$$N_{\{n_i\};\ell}(q) = q^{f(\{n_i\},\ell)}\, \phi\left(Q_1 Q_0^{-1} \prod_{i=1}^{k} Q_i^{n_i}\, \xi^{\ell+1}\right)$$

where the notation $N_{\{n_k\};\ell}$ stands for $N_{\{\prod_k V_{k\omega_1}^{n_k}\};V_{\ell\omega_1}}$ and t^f is a normalization factor depending on the n's and ℓ only. The Q_i denote the solution of the quantum Q-system (2) with $t = q^{\frac{1}{2}}$, expressed as a Laurent polynomial of the initial variables Q_0, Q_1 while $\xi = \lim_{m\to\infty} Q_m Q_{m+1}^{-1} \in \mathbb{C}[Q_0, Q_0^{-1}] \otimes \mathbb{C}((Q_1^{-1}))$. Here ϕ denotes the left evaluation at $Q_0 = 1$ of the constant term in Q_1, namely for any formal power series $f = \sum_{m\geq -N} c_m(Q_0, Q_0^{-1}) Q_1^{-m} \in \mathbb{C}[Q_0, Q_0^{-1}] \otimes \mathbb{C}((Q_1^{-1}))$, we have $\phi(f) = c_0(1, 1)$.

A straightforward generalization exists for simply laced Lie algebras [4], and we believe the non-simply-laced case should pose no other problems than technical ones.

The formula above gives a very simple way of computing the graded fusion numbers $N_{\{V_i\};V_\lambda}(t)$. By defining suitable generating functions thereof, we have managed to derive difference equations that completely characterize them up to Cauchy type initial conditions. The latter generating series resemble standard Whittacker functions and are subject to a discrete version of the Toda equations these satisfy.

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Classification of non-degenerate potentials for mutation-finite quivers

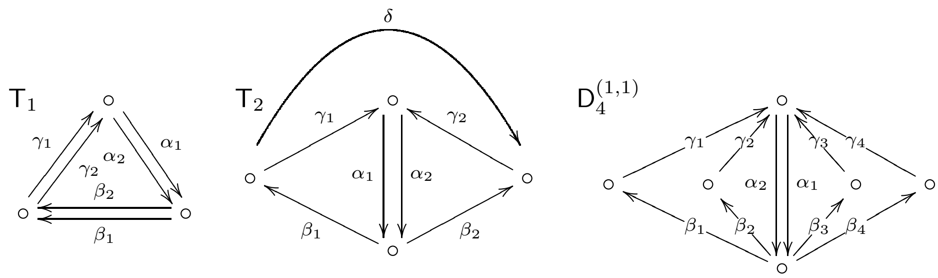
Christof Geiss

(joint work with Daniel Labardini, Jan Schröer)

1. Mutation finite quivers

We say that a 2-acyclic quiver is mutation finite if its mutation class contains only finitely many isomorphism classes of quivers. For example all quivers with at most two vertices are trivially mutation finite.

Fomin, Shapiro and Thurston presented in [4] an important class of mutation finite quivers coming from triangulations of marked oriented surfaces with possibly non-empty boundary. Roughly speaking, the vertices of such a quiver correspond to the arcs of the triangulation, and there is an arrow $i \to j$ if the arcs i and j are consecutive sides of a triangle in clockwise sense. Since, somewhat simplifying, flipping arcs corresponds to mutation of quivers, this class is closed under mutation. It contains all quivers of the finite types A_n and D_n ($n \geq 4$) as well as all quivers of the affine types $\mathsf{A}_n^{(1)}$ and $\mathsf{D}_n^{(1)}$ ($n \geq 4$). Further interesting examples are the following quivers



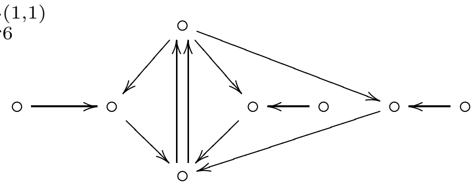
which come from triangulations of a once punctured torus, a torus with one boundary component and one marked point, and a sphere with 4 punctures, respectively.

Note that the mutation class of T_1 and T_2 consists of only one isomorphism class of quivers, and the mutation class of $\mathsf{D}_4^{(1,1)}$ consists of four isomorphism classes of quivers. We have the following remarkable classification [3]:

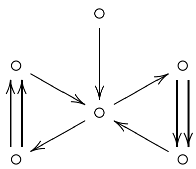
Theorem (Felikson, Shapiro, Tumarkin)**.** Let Q be a connected, mutation finite quiver with at least 3 vertices which does not come from a triangulation of a surface, then Q belongs to the mutation class of precisely one quiver of the following list:

- a quiver of finite type E_6, E_7 or E_8 (any orientation),
- a quiver of affine type $\mathsf{E}_6^{(1)}$, $\mathsf{E}_7^{(1)}$ or $\mathsf{E}_8^{(1)}$ (any orientation),
- one of the following 5 non-acyclic quivers:

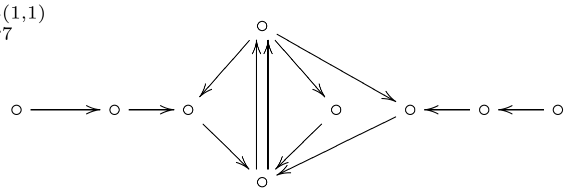
$\mathsf{E}_6^{(1,1)}$



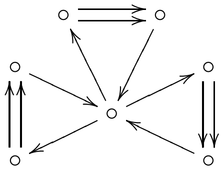
X_6



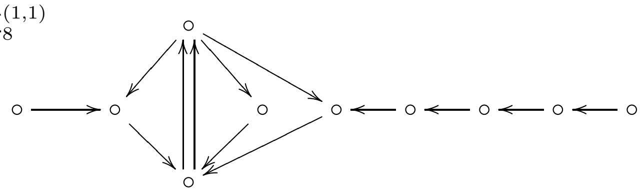
$\mathsf{E}_7^{(1,1)}$



X_7



$\mathsf{E}_8^{(1,1)}$



2. Quivers with potential

In this section we briefly review several fundamental constructions from [1]. Let K be an algebraically closed field and Q a quiver. We denote by $K\langle\langle Q\rangle\rangle$ the complete path algebra of Q and write $\mathfrak{m}(Q)$ for the corresponding Jacobson radical. $K\langle\langle Q\rangle\rangle$ has the paths of Q as a topological basis, so that possibly infinite linear combinations of paths of increasing length are allowed. Following [1, Rem. 6.8], a *potential* for Q is by definition an element of $\mathcal{C}(Q) := \mathfrak{m}(Q)/\{K\langle\langle Q\rangle\rangle, K\langle\langle Q\rangle\rangle\}$, where $\{K\langle\langle Q\rangle\rangle, K\langle\langle Q\rangle\rangle\} \subset \mathfrak{m}(Q)$ is the closure of the set of commutators $\{a\cdot a' - a'\cdot a \mid a, a' \in K\langle\langle Q\rangle\rangle\}$. Thus the cyclic paths of Q, up to rotation, form a topological basis of $\mathcal{C}(Q)$. Next, denote by $\mathrm{Aut}_{Q_0}(K\langle\langle Q\rangle\rangle)$ the group of automorphisms of $K\langle\langle Q\rangle\rangle$ which fix pointwise trivial paths. $\mathrm{Aut}_{Q_0}(K\langle\langle Q\rangle\rangle)$ acts naturally on $\mathcal{C}(Q)$. Two potentials $W, W' \in \mathcal{C}(Q)$ are *right equivalent*, if there exists $\phi \in \mathrm{Aut}_{Q_0}(K\langle\langle Q\rangle\rangle)$ with

$W' = \phi(W)$. We say that W and W' are *weakly right equivalent* if $W' = t\phi(W)$ for some $t \in K^*$ and $\phi \in \mathrm{Aut}_{Q_0}(K\langle\langle Q\rangle\rangle)$.

For example, the potentials $W_t^{(1)} := \gamma_1\beta_1\alpha_1 + \gamma_2\beta_2\alpha_2 - t\gamma_2\beta_1\alpha_2\gamma_1\beta_2\alpha_1$ for the quiver T_1 are not right equivalent for $t \neq t'$. However, $W_t^{(1)}$ is weakly right equivalent to $W_1^{(1)}$ if and only if $t \in K^*$.

In [1] also mutations of loop-free quivers with potential (Q, W) are introduced. Let $k \in Q_0$ and suppose that k is not incident to any 2-cycle in Q, and that W involves no 2-cycles. In this case $(Q', W') = \mu_k(Q, W)$ with the same properties is defined. However, Q' depends on W, and W' is well defined up to right equivalence. In particular, for non-generic choices of W, the quiver Q' *may* have 2-cycles. If this does not occur, Q' is given by the usual quiver mutation of Q in direction k. A potential W for a 2-acyclic quiver is called *non-degenerate* if after arbitrary sequences of the above mutation procedure, for quivers with potential, the resulting quiver is still 2-acyclic. Being non-degenerate for a potential comes down to avoiding the zeros of infinitely many polynomial equations on the vector space $\mathcal{C}(Q)$. Thus, a standard argument shows that over uncountable fields for every 2-acyclic quiver there exists a non-degenerate potential.

In general, it is quite difficult to identify a given potential as non-degenerate, however D. Labardini-Fragoso managed to produce for each quiver coming from a triangulated surface a non-degenerate potential, essentially by showing that these potentials are compatible with arbitrary QP-mutations [6].

Non-degenerate potentials are relevant for cluster algebras since in this situation one obtains for example the Laurent expansions of cluster variables in terms of quiver Grassmannians over the corresponding Jacobian algebra $\mathcal{P}(Q, W)$. This observation allowed Derksen, Weyman and Zelevinsky to prove many difficult conjectures about cluster algebras with skew symmetric exchange matrix, see [2].

3. Main Result

In the following theorem we collect our findings about the classification of non-degenerate potentials for mutation finite quivers [5].

Theorem. Let Q be a connected quiver.

(a) For Q there exists up to right equivalence a unique non-degenerate potential W in the following cases:
  - Q comes from a triangulation of a surface with non-empty boundary except if Q is isomorphic to the quiver T_2.
  - Q is mutation equivalent to one of the following exceptional quivers: E_q^* with $* \in \{\emptyset, (1), (1,1)\}$ and $q \in \{6, 7, 8\}$ or X_6.

(b) For Q there exists up to weak right equivalence a unique non-degenerate potential in the following cases:
  - Q comes from a triangulation of a closed surface of positive genus and at least 3 punctures.
  - Q comes from a triangulation of a sphere with at least 5 punctures.

(c) For the quiver T_2 the potentials $W = (\gamma_1\beta_1 + \gamma_2\beta_2)\alpha_1 + \gamma_2\delta\beta_1\alpha_2$ and $S = \gamma_1\beta_1\alpha_1 + \gamma_2\beta_2\alpha_2$ form a complete system of representatives for the (weak) right equivalence classes of non-degenerate potentials.
(d) For the quiver $\mathsf{D}_4^{(1,1)}$ a complete system of representatives of the (weak) right equivalence classes of non-degenerate potentials is given by $W_t := (\gamma_1\beta_1 + \gamma_2\beta_2 + \gamma_3\beta_3)\alpha_1 + (t\gamma_2\beta_2 + \gamma_3\beta_3 + \gamma_4\beta_4)\alpha_2$ with $t \in K \setminus \{0, 1\}$.
(e) If Q comes from a closed surface of positive genus with exactly one puncture, then Q admits at least two classes of non-degenerate potentials up to weak right equivalence.

Comments.

- Part (a) and (b) are [5, Thm. 1.4], part (c) is [5, Sec. 9.7.1], part (d) follows from the discussion in [5, Sec. 9.9], part (e) is [5, Prp. 9.19].
- By the theorem of Felikson-Shapiro-Tumarkin, the above theorem covers all mutation finite quivers except those coming from triangulations of a closed surface of positive genus with exactly 2 punctures, and the two quivers in the mutation class of X_7. We conjecture that in the first case there exists a unique non-degenerate potential up to weak right equivalence.
- By [1] for mutation equivalent quivers Q and Q' the (weak) right equivalence classes of non-degenerate potentials for Q and Q' respectively are in bijection. Thus a first step towards the classification problem is to identify in each mutation class a quiver with convenient properties. For example, if Q is mutation acyclic, then there exists for Q a unique non-degenerate potential up to right equivalence.
- For triangulations of a closed surface with at least 3 punctures it is important to observe that in this case there exists a triangulation with no loops and such that each vertex has valency at least 4. The corresponding quivers have a convenient constellation of cycles which simplifies the classification.
- The potentials $W_1^{(1)}$ and $W_0^{(1)}$ for the quiver T_1 from the previous section are typical examples of two non-degenerate potentials which are not weakly right equivalent in the situation of part (e) of our Theorem.

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Cluster structures on Poisson-Lie groups

Michael Gekhtman
(joint work with Michael Shapiro, Alek Vainshtein)

This talk can be viewed as a progress report on the conjecture formulated in [7], that state each class in the Belavin-Drinfeld classification of Poisson-Lie structures on a complex simple group $\mathcal{G}$ corresponds to a cluster structure in $\mathcal{O}(\mathcal{G})$. Our approach, developed and documented in [5, 6], is on the notion of a Poisson bracket compatible with a cluster structure. If an algebraic Poisson variety $(\mathcal{M}, \{\ ,\ \})$ possesses a coordinate chart that consists of regular functions whose logarithms have pairwise constant Poisson brackets (*log-canonical coordinates*), then one can try to use this chart to define a cluster structure $\mathcal{C}_\mathcal{M}$ compatible with $\{\ ,\ \}$. Algebraic structures corresponding to $\mathcal{C}_\mathcal{M}$ (the cluster algebra and the upper cluster algebra) are closely related to the ring $\mathcal{O}(\mathcal{M})$ of regular functions on $\mathcal{M}$.

Let $\mathcal{G}$ be a Lie group equipped with a Poisson bracket $\{\ ,\ \}$. $\mathcal{G}$ is called a *Poisson–Lie group* if the multiplication map

$$\mathcal{G} \times \mathcal{G} \ni (x, y) \mapsto xy \in \mathcal{G}$$

is Poisson. We are interested in the case when $\mathcal{G}$ be a simple complex Lie group and a Poisson-Lie structure is associated with a classical R-matrix, $r \in \mathfrak{g} \otimes \mathfrak{g}$, a solution of the *classical Yang-Baxter equation* which satisfy an additional condition that $r + r^{21}$ is an element of $\mathfrak{g} \otimes \mathfrak{g}$ that defines an invariant nondegenerate inner product on $\mathcal{G}$. (Here $\mathfrak{g} = Lie(\mathcal{G})$ and r^{21} is obtained from r by switching factors in tensor products.) Classical R-matrices were classified, up to an automorphism, by Belavin and Drinfeld in [1]. Let $\mathfrak{h}$ be a Cartan subalgebra of $\mathcal{G}$ and Δ be the set of positive simple roots in the root system associated with $\mathcal{G}$. A *Belavin-Drinfeld triple* $T = (\Gamma_1, \Gamma_2, \Gamma)$ consists of two subsets Γ_1, Γ_2 of Δ and an isometry $\Gamma : \Gamma_1 \to \Gamma_2$ nilpotent in the following sense: for every $\alpha \in \Gamma_1$ there exists $m \in \mathbb{N}$ such that $\Gamma^j(\alpha) \in \Gamma_1$ for $j = 0, \ldots, m-1$, but $\Gamma^m(\alpha) \notin \Gamma_1$. To each T there corresponds a set $\mathcal{R}_T$ of classical R-matrices that we call the *Belavin-Drinfeld class* corresponding to T. Two R-matrices in $\mathcal{R}_T$ differ by an element from $\mathfrak{h} \otimes \mathfrak{h}$ satisfying a linear relation specified by T. We denote by $\{\ ,\ \}_r$ the Poisson-Lie bracket associated with $r \in \mathcal{R}_T$.

In [7] we conjectured that there exists a classification of regular cluster structures on $\mathcal{G}$ that is completely parallel to the Belavin-Drinfeld classification.

Conjecture 1. *For any Belavin-Drinfeld triple $T = (\Gamma_1, \Gamma_2, \gamma)$ there exists a cluster structure $(\mathcal{C}_T, \varphi_T)$ on $\mathcal{G}$ compatible with $\{\ ,\ \}_r$ for any $r \in \mathcal{R}_T$. Furthermore, the corresponding upper cluster algebra $\overline{\mathcal{A}}_\mathbb{C}(\mathcal{C}_T)$ is naturally isomorphic to $\mathcal{O}(\mathcal{G})$.*

We called a cluster structure compatible with non-standard Poisson-Lie brackets *exotic.*

This conjecture has been verified

- for any $\mathcal{G}$ in the case of the standard Poisson-Lie structure (trivial Belavin-Drinfeld data) [7];

- for all Belavin-Drinfeld classes in $SL(n), n < 6$ [7, 3];
- for the Cremmer-Gervais Poisson-Lie structure on $SL(n)$: $\Gamma_1 = \{\alpha_2, \dots, \alpha_{n-1}\}$, $\Gamma_2 = \{\alpha_1, \dots, \alpha_{n-2}\}$ and $\gamma(\alpha_i) = \alpha_{i-1}$ for $i = 2, \dots, n-1$ [8, 9].

To construct an initial cluster in the Cremmer-Gervais case, we needed an insight that comes from analyzing the corresponding Poisson-Lie structure in *the Drinfeld double* of $\mathcal{G}$. It is associated with the *Manin triple* $(D(\mathfrak{g}), d_{\mathfrak{g}}, \mathfrak{g}_r)$, where $D(\mathfrak{g}) = \mathfrak{g} \oplus \mathfrak{g} = d_{\mathfrak{g}} \dot{+} \mathfrak{g}_r$ is equipped with the invariant nondegenerate inner product $\langle\langle(\xi, \eta), (\xi', \eta')\rangle\rangle = \langle \xi, \xi' \rangle - \langle \eta, \eta' \rangle$, an isotropic subalgebra $d_{\mathfrak{g}}$ is the image of $\mathfrak{g}$ in $D(\mathfrak{g})$ under the diagonal embedding and $\mathfrak{g}_r$ is an isotropic subalgebra of $D(\mathfrak{g})$ given by $\mathfrak{g}_r = \{(R_+(\xi), R_-(\xi)) \ : \xi \in \mathfrak{g}\}$, where $R_\pm \in \operatorname{End} \mathfrak{g}$ are given by $\langle R_+\eta, \zeta \rangle = -\langle R_-\zeta, \eta \rangle = \langle r, \eta \otimes \zeta \rangle$. $(\mathcal{G}, \{\ ,\ \}_r)$ is a Poisson-Lie subgroup of $D(\mathcal{G})$ under the diagonal embedding. Another Poisson-Lie subgroup is $\mathcal{G}_r = \operatorname{Exp}(\mathfrak{g}_r)$.

Working with the Drinfeld double, one is able to recognize stable variables for the exotic cluster structure as two-sided semi-invariants of the $\mathcal{G}_r$-action and to obtain an initial cluster as a restriction to the diagonal subgroup of a certain family of regular functions log-canonical on the double. This also hints at a possibility of endowing the double itself with a cluster structure. However, our current work in progress shows that while it is possible to construct a regular log-canonical coordinate system in the double in all the cases in which Conjecture 1 is established, in order to stay within the ring of regular functions, one is forced to replace one of the the exchange relations with a *generalized exchange relation* in the sense of [2]:

$$xx' = u_- p \left(\frac{u_+}{u_-} \right) ,$$

where x is a cluster variable being transformed, $u_\pm$ are monomials in variables from the same cluster and p is a polynomial of degree d (the case $d = 1$ corresponds to the usual cluster exchange relation featured in the definition of the cluster algebra [4]). We can now formulate

Conjecture 2. *For any Belavin-Drinfeld triple $T = (\Gamma_1, \Gamma_2, \gamma)$ there exists a generalized cluster structure on $\mathcal{D}(G)$ compatible with the corresponding Poisson-Lie structure.*

Theorem. *Conjecture 2 is valid for the trivial and Cremmer-Gervais Belavin-Drinfeld data.*

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Cluster algebra and complex volume of knots

Rei Inoue
(joint work with Kazuhiro Hikami)

The cluster algebra is widely studied from the viewpoint of geometry, such as Teichmüller theory [1] and triangulated surface [3, 4]. The key observation is that a cluster mutation corresponds to a flip (a change of triangulation). In [9], the cluster algebraic techniques are used to describe hyperbolic structures of punctured surface bundles on S^1. There a flip in hyperbolic three-space is interpreted as attachment of an ideal tetrahedron, and its modulus is identified with a cluster y-variable. Inspired by this work, in [6, 7] we study the complex volume by using cluster variables with coefficients. The complex volume of a hyperbolic three-manifold M is a complexification of hyperbolic volume given by

$$\mathrm{Vol}(M) + \mathrm{i}\,\mathrm{CS}(M),$$

where $\mathrm{Vol}(M)$ is the hyperbolic volume and $\mathrm{CS}(M)$ is the Chern–Simons invariant of M.

In this talk we introduce the idea to compute the complex volume of the two-bridge knot complement based on [6]. First, we explain some basics of three-dimensional hyperbolic geometry: an ideal tetrahedron $\triangle$, its modulus $z \in \mathbb{C}$, and hyperbolic volume. We also introduce the tools to compute complex volume: an oriented ideal tetrahedron and its flattening $(z; p, q) \in \mathbb{C} \times \mathbb{Z}^2$ [10]. A flattening is an element of the extended pre-Bloch group which is, roughly speaking, a quotient of the free $\mathbb{Z}$-module on $\mathbb{C} \times \mathbb{Z}^2$ by the (generalized) five-term relation. When a cusped hyperbolic three-manifold M is decomposed into ideal tetrahedra $\{\triangle_\nu\}_\nu$, the complex volume of M is

$$\mathrm{i}\,(\mathrm{Vol}(M) + \mathrm{i}\,\mathrm{CS}(M)) = \sum_\nu \mathrm{sgn}(\nu) L(z_\nu; p_\nu, q_\nu).$$

Here $(z_\nu; p_\nu, q_\nu)$ is the flattening of $\triangle_\nu$, $\mathrm{sgn}(\nu) = \pm 1$ is determined by the orientation of $\triangle_\nu$, and L is the extended Rogers dilogarithm function given by

$$L(z; p, q) = \mathrm{Li}(z) + \frac{1}{2} \log z \log(1-z) + \frac{\pi\,\mathrm{i}}{2}\big(q \log z + p \log(1-z)\big) - \frac{\pi^2}{6},$$

where $\mathrm{Li}(z)$ is the dilogarithm function.

Next we introduce the cluster algebraic interpretation for hyperbolic structure of a punctured surface bundle. To compute complex volume, we use the cluster variables with coefficients where coefficients belong to a tropical semifield. Then our main observation is that the cluster variable with coefficients is closely related to Zickert's formulation of flattening [13], and that the complex volume is obtained from the cluster variable.

We use the ideal tetrahedral decomposition of two-bridge knot complement given by a 4-punctured sphere bundle on an interval (with a specific gluing at the two boundaries) [11]. We translate it into a sequence of mutations of seed which reduces to algebraic equations for cluster variables. By taking an appropriate solution of the equations, we get the complex volume. We remark that a punctured surface bundle on S^1 is translated into a sequence of mutations of seed with a periodic boundary condition (cf. [9]).

In closing, we mention some related topics and future problems. It would be interesting to study hyperbolic three-manifolds by applying quantum cluster algebra [2] or cluster algebra related to higher Teichmüller theory [1]. Actually in the case of once-punctured torus bundle on S^1, it is studied in [12] that a classical limit of adjoint action of mutations, from which complex volume and A-polynomial are obtained. A generalization to higher rank should be related to [5]. On the other hand, in [7] we define the R-operators in terms of cluster mutation, which satisfy the braid relation. We also study its quantization [8] using quantum cluster algebra. This R-operator may be helpful in studying a relationship between hyperbolic geometry and quantum invariants.

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Mutation and g-vectors in τ-tilting theory

Osamu Iyama

(joint work with Takahide Adachi, Idun Reiten)

Let A_Q be a cluster algebra associated with a quiver Q. For a potential W, the cluster category $\mathcal{C}_{Q,W}$ is defined by the Ginzburg dg algebra $\Gamma_{Q,W}$. We denote by c-tilt $\mathcal{C}_{Q,W}$ the set of isomorphism classes of basic cluster-tilting objects, and c-tilt$_0\,\mathcal{C}_{Q,W}$ the subset consisting of *reachable* cluster-tilting objects (i.e. cluster-tilting objects which are obtained from $\Gamma_{Q,W}$ by iterated mutation). When (Q,W) is non-degenerate, we have a bijection

$$\text{c-tilt}_0\,\mathcal{C}_{Q,W} \to \{\text{clusters in } A_Q\} \tag{1}$$

which commutes with mutation [CKLP] after works by a number of authors.

τ-tilting theory [AIR] provides another framework of categorifying cluster algebras, which is simpler since it deals with the Jacobian algebra $P_{Q,W}$ instead of the Ginzburg dg algebra $\Gamma_{Q,W}$. It is interesting from a representation theoretic viewpoint since it works for arbitrary finite dimensional algebras.

1. τ-tilting theory Let k be an algebraically closed field and Λ a basic finite dimensional k-algebra. We denote by $\mathsf{mod}\,\Lambda$ the category of finitely generated left Λ-modules, and $\mathsf{proj}\,\Lambda$ (respectively, $\mathsf{inj}\,\Lambda$) the full subcategory consisting of projective (respectively, injective) modules. We have an equivalence $\nu := (D\Lambda)\otimes_\Lambda - : \mathsf{proj}\,\Lambda \to \mathsf{inj}\,\Lambda$ called *Nakayama* functor. For $M \in \mathsf{mod}\,\Lambda$, we denote by $P_1^M \xrightarrow{f} P_0^M \to M \to 0$ a minimal projective presentation. The *AR-translation* of M is defined by an exact sequence $0 \to \tau M \to \nu(P_1^M) \xrightarrow{\nu(f)} \nu(P_0^M)$.

Clearly $\tau M = 0$ if M is projective. Moreover τ gives a bijection between the set of isomorphism classes of indecomposable non-projective Λ-modules and the set of isomorphism classes of indecomposable non-injective Λ-modules.

The notion of τ-rigid modules appeared in an old work by Auslander-Smalø:

Definition 1 Let $M \in \mathsf{mod}\,\Lambda$. We call M *τ-rigid* if $\mathrm{Hom}_\Lambda(M,\tau M) = 0$. We call M *τ-tilting* if it is τ-rigid and $|M| = |\Lambda|$ holds, where $|M|$ is the number of non-isomorphic indecomposable direct summands of M. We call M *support τ-tilting* if there exists an idempotent e of Λ such that M is a τ-tilting $(\Lambda/\langle e\rangle)$-module.

We denote by sτ-tilt Λ the set of isomorphism classes of basic support τ-tilting Λ-modules. We give a few examples.

Example 2 (a) Λ and 0 always belong to sτ-tilt Λ.
(b) If Λ is local, then sτ-tilt $\Lambda = \{\Lambda, 0\}$.
(c) [M] Let Π be a preprojective algebra of a Dynkin quiver Q, W the corresponding Weyl group, and $I_i := \langle 1 - e_i\rangle$ a two-sided ideal of Π for the vertex i of Q. Then

there exists a bijection $W \to \mathrm{s}\tau\text{-tilt}\,\Pi$ sending $w \in W$ to $I_w := I_{i_1} \cdots I_{i_\ell}$, where $w = s_{i_1} \cdots s_{i_\ell}$ is an arbitrary reduced expression of w.
(d) For the path algebra kQ of the quiver Q of type A_2, $\mathrm{s}\tau\text{-tilt}(kQ)$ consists of 5 elements kQ, $D(kQ)$, S_1, S_2 and 0. This is a special case of the next (e).
(e) Let $\mathcal{C}$ be a 2-Calabi-Yau triangulated category with a cluster-tilting object T. Then we have a bijection

$$\text{c-tilt}\,\mathcal{C} \to \mathrm{s}\tau\text{-tilt}\ \mathrm{End}_{\mathcal{C}}(T),\ U \mapsto \mathrm{Hom}_\Lambda(T,U). \tag{2}$$

When $\mathcal{C} = \mathcal{C}_{Q,W}$ and $T = \Gamma_{Q,W}$ for a non-degenerate Jacobi-finite potential (Q,W), by combining (1) and (2) we have a bijection

$$\mathrm{s}\tau\text{-tilt}_0 P_{Q,W} \to \{\text{clusters in } A_Q\}$$

for the Jacobian algebra $P_{Q,W}$, where $\mathrm{s}\tau\text{-tilt}_0 P_{Q,W}$ is the set of *reachable* support τ-tilting $P_{Q,W}$-modules.

To introduce mutation of support τ-tilting modules, we need to deal with pairs (M,P) with $M \in \mathsf{mod}\,\Lambda$ and $P \in \mathsf{proj}\,\Lambda$. We call (M,P) *τ-rigid* if M is τ-rigid and $\mathrm{Hom}_\Lambda(P,M) = 0$. We call (M,P) *support τ-tilting* if (M,P) is τ-rigid and $|M| + |P| = |\Lambda|$.

Theorem 3 Let (M,P) be a basic τ-rigid-pair for Λ.
(a) (M,P) is a direct summand of some $(N,Q) \in \mathrm{s}\tau\text{-tilt}\,\Lambda$ (i.e. M and P are direct summands of N and Q respectively).
(b) If $|M| + |P| = |\Lambda| - 1$, then (M,P) is a direct summand of precisely two elements $(N_i,Q_i) \in \mathrm{s}\tau\text{-tilt}\,\Lambda$ $(i = 1,2)$.
(c) [J] There exist finite dimensional k-algebra Γ with $|M| + |P| = |\Lambda| - |\Gamma|$ and a bijection $\{(N,Q) \in \mathrm{s}\tau\text{-tilt}\,\Lambda \mid (N,Q) \text{ has } (M,P) \text{ as a direct summand}\} \to \mathrm{s}\tau\text{-tilt}\,\Gamma$.

Note that (b) is a special case of (c) since $|\Gamma| = 1$ implies $\mathrm{s}\tau\text{-tilt}\,\Gamma = \{\Gamma, 0\}$.

We call (N_1,Q_1) and (N_2,Q_2) in (b) above *mutation* of each other. The *exchange graph* of Λ has the set $\mathrm{s}\tau\text{-tilt}\,\Lambda$ of vertices and edges correspond to mutation.

It is important to know the number of connected components of the exchange graph. The partial order gives an effective tool.

2. Partial order The partial order on tilting modules due to Riedtmann-Schofield and Happel-Unger can be extended to support τ-tilting modules.

A *torsion class* is a full subcategory $\mathcal{T}$ in $\mathsf{mod}\,\Lambda$ which is closed under factor modules and extensions. We call a torsion class *functorially finite* if there exists $M \in \mathsf{mod}\,\Lambda$ such that $\mathcal{T} = \mathsf{Fac}\,M$, where $\mathsf{Fac}\,M$ is the full subcategory of $\mathsf{mod}\,\Lambda$ consisting of all factor modules of finite direct sums of copies of M.

Theorem 4 (a) There exists a bijection from $\mathrm{s}\tau\text{-tilt}\,\Lambda$ to the set of all functorially finite torsion classes in $\mathsf{mod}\,\Lambda$. Hence $\mathrm{s}\tau\text{-tilt}\,\Lambda$ has a natural partial order, i.e. we define $M \geq N$ if and only if $\mathsf{Fac}\,M \supset \mathsf{Fac}\,N$. Clearly Λ is a unique maximal element and 0 is a unique minimal element.
(b) [DIJ] Let $M \in \mathrm{s}\tau\text{-tilt}\,\Lambda$, and let $\mathcal{T}$ be a torsion class in $\mathsf{mod}\,\Lambda$. If $\mathsf{Fac}\,M \supsetneq \mathcal{T}$ (respectively, $\mathsf{Fac}\,M \subsetneq \mathcal{T}$), then there exists a mutation N of M such that $\mathsf{Fac}\,M \supsetneq \mathsf{Fac}\,N \supset \mathcal{T}$ (respectively, $\mathsf{Fac}\,M \subsetneq \mathsf{Fac}\,N \subset \mathcal{T}$).

(c) The exchange graph of sτ-tiltΛ coincides with the Hasse graph of sτ-tiltΛ.

Note that (c) is an easy consequence of (b).

We call a finite dimensional k-algebra Λ *τ-rigid-finite* if there exists only finitely many indecomposable τ-rigid Λ-modules, or equivalently, sτ-tiltΛ is a finite set [DIJ]. For example, any representation-finite algebra and any local algebra are τ-rigid-finite. Any preprojective algebra Π of Dynkin type is also τ-rigid-finite (Example 2(c)), and the partially ordered set sτ-tiltΠ is isomorphic to W with respect to the opposite of weak order.

It is an interesting question to classify τ-rigid-finite algebras.

Another easy consequence of (b) above is the following result.

Corollary 5 [DIJ] A finite dimensional k-algebra Λ is τ-rigid-finite if and only if any torsion class in $\mathsf{mod}\Lambda$ is functorially finite. In this case, the exchange graph of sτ-tiltΛ is connected.

This is an analog of a classical result: A finite dimensional k-algebra Λ is representation-finite if and only if any subcategory in $\mathsf{mod}\Lambda$ is functorially finite.

3. g-vectors A combinatorial invariant of τ-rigid pairs is given by g-vectors. Let Λ be a basic finite dimensional k-algebra such that $1 = e_1 + \cdots + e_n$ for primitive orthogonal idempotents $e_1, \ldots, e_n$. The Grothendieck group $K_0(\mathsf{proj}\Lambda)$ of an additive category $\mathsf{proj}\Lambda$ is a free abelian group with a basis $[e_1\Lambda], \ldots, [e_n\Lambda]$.

Theorem 6 [AIR, DIJ] (a) If (M, P) is a τ-rigid pair, then P_0^M and $P_1^M \oplus P$ have no non-zero common direct summands. We define the *g-vector* (or *index*) of (M, P) as

$$g^{(M,P)} := [P_0^M] - [P_1^M \oplus P] \in K_0(\mathsf{proj}\Lambda).$$

(b) τ-rigid pairs are determined by their g-vectors.
(c) Let $(M, P) \in$ sτ-tiltΛ. Then g-vectors of indecomposable direct summands of (M, P) give a basis of $K_0(\mathsf{proj}\Lambda)$. Let $C(M, P)$ be the cone in $K_0(\mathsf{proj}\Lambda) \otimes_{\mathbf{Z}} \mathbf{R}$ spanned by these basis elements.
(d) Different cones intersect only at their boundaries.
(e) If Λ is τ-rigid-finite, then $\bigcup_{(M,P)\in \mathrm{s}\tau\text{-tilt}\Lambda} C(M, P) = K_0(\mathsf{proj}\Lambda) \otimes_{\mathbf{Z}} \mathbf{R}$.

For the preprojective algebra Π of Dynkin type, the cones $C(M, P)$ are precisely Weyl chambers.

Note that τ-rigid pairs for Λ have a structure of a simplicial complex, and a geometric realization is given by g-vectors. If Λ is τ-rigid-finite, then it is homeomorphic to an $(n-1)$-sphere as an easy consequence of (d) and (e) above.

We conjecture that g-vectors determine the partial order on sτ-tiltΛ.

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Generalised friezes and a modified Caldero-Chapoton map depending on a rigid object

Peter Jørgensen
(joint work with Thorsten Holm)

This report is based on [4]. Let Q be a finite quiver without loops and 2-cycles. The (original) Caldero-Chapoton (CC) map is a map from the set of objects of the cluster category $\mathsf{C}(Q)$ to a ring of Laurent polynomials over $\mathbb{Z}$. If Q has no cycles then the CC map restricts to a bijection from the set of rigid indecomposable objects to the set of cluster variables of the cluster algebra $\mathcal{A}(Q)$, see [3, thm. 4].

We show how to modify the CC map by replacing the cluster tilting object which appears in the original definition with a basic rigid object. The modified CC map turns out to be what we call a *generalised frieze*. We show that when applied to $\mathsf{C}(A_n)$, the cluster category of Dynkin type A_n, the modified CC map recovers the generalised friezes defined by combinatorial means in [1].

Abstract setup of the CC map. Let k be an algebraically closed field, C a k-linear Hom-finite Krull-Schmidt triangulated category with Auslander-Reiten (AR) triangles. Note that the cluster category $\mathsf{C}(Q)$ is an example, but there are many others.

Let R be an object of C and write $E = \operatorname{End} R$. There is a functor $G : \mathsf{C} \to \operatorname{mod} E$ given by $G(c) = \operatorname{Hom}(R, \Sigma c)$ where Σ is the suspension functor of C.

Let A be a commutative ring and let $\alpha : \operatorname{obj} \mathsf{C} \to A$ and $\beta : \mathrm{K}_0(\operatorname{mod} E) \to A$ be maps which are "exponential" in the sense that $\alpha(0) = 1$, $\alpha(c \oplus d) = \alpha(c)\alpha(d)$ and $\beta(0) = 1$, $\beta(e + f) = \beta(e)\beta(f)$.

The CC map defined by these data is the map $\rho : \operatorname{obj} \mathsf{C} \to A$ given by

$$\rho(c) = \alpha(c) \sum_e \chi\big(\operatorname{Gr}_e(Gc)\big)\beta(e). \tag{1}$$

The sum is over $e \in \mathrm{K}_0(\operatorname{mod} E)$. By χ is denoted the Euler characteristic defined by étale cohomology with proper support, and $\operatorname{Gr}_e(Gc)$ is the Grassmannian of submodules $M \subseteq Gc$ with $[M] = e$.

The original CC map is the following special case. Assume that C is 2-Calabi-Yau. Let R be a basic cluster tilting object of C, set $A = \mathbb{Z}[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$ where n is the number of indecomposable direct summands of R, and define α and β on the basis of certain distinguished triangles in C which involve R, see [6, lem. 4.1].

The modified CC map is the following special case. Let R be a basic object of C which is rigid, that is, $\operatorname{Hom}(R, \Sigma R) = 0$. This is much more general than being

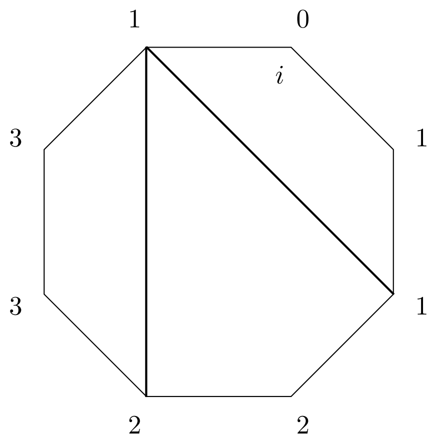

FIGURE 1. An 8-gon with a fixed vertex i and a polygon dissection. For each vertex j, the corresponding number m_{ij} is shown next to j.

cluster tilting. Set $A = \mathbb{Z}$, and let α and β be identically equal to 1. Then

$$\rho(c) = \sum_e \chi\big(\operatorname{Gr}_e(Gc)\big).$$

Theorem A. *The modified CC map ρ is a generalised frieze, that is,*

(1) $\rho(b_1 \oplus b_2) = \rho(b_1)\rho(b_2)$,

(2) *if $\Delta = \tau c \to b \to c$ is an AR triangle, then $\rho(\tau c)\rho(c) - \rho(b) \in \{0, 1\}$.*

More precisely, $\rho(\tau c)\rho(c) - \rho(b)$ is 0 if $G(\Delta)$ is a split short exact sequence and 1 otherwise.

Dynkin type A_n. Consider the cluster category $\mathsf{C}(A_n)$ of Dynkin type A_n. By [2] there is a bijection between the indecomposable objects of $\mathsf{C}(A_n)$ and the diagonals of an $(n+3)$-gon P. Let R be a basic rigid object of $\mathsf{C}(A_n)$. Under the bijection, the indecomposable summands of R correspond to a set D of pairwise non-crossing diagonals, that is, to a polygon dissection D of P.

In [1, def. 3.1] we described the following inductive procedure. Consider a vertex i of P and set $m_{ii} = 0$. Let $j \neq i$ be another vertex of P. If i and j belong to the same piece π of P, as defined by the dissection D, then set $m_{ij} = 1$. If i and j belong to different pieces, then we can assume inductively that j belongs to a piece, π, which has two vertices, k and ℓ, such that m_{ik} and $m_{i\ell}$ have already been defined, and we set $m_{ij} = m_{ik} + m_{i\ell}$. Figure 1 shows an example.

There is a map φ from the set of indecomposable objects of $\mathsf{C}(A_n)$ to $\mathbf{Z}$, given by

$$\varphi(c) = m_{ij}$$

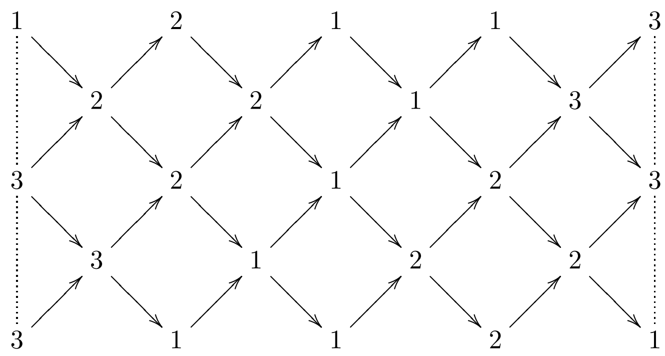

Figure 2. The generalised frieze in Dynkin type A_5 arising from the polygon dissection in Figure 1. The slice starting at the south-west corner contains the numbers $3, 3, 2, 2, 1$ obtained from Figure 1. Since the figure shows the AR quiver of $\mathsf{C}(A_5)$, the dotted lines should be identified with opposite orientations.

where the indecomposable object c corresponds to the diagonal between i and j. This makes sense because $m_{ij} = m_{ji}$ by [1, thm. 3.3]. We extend φ to all objects of $\mathsf{C}(A_n)$ by requiring $\varphi(b_1 \oplus b_2) = \varphi(b_1)\varphi(b_2)$. Figure 2 shows the AR quiver of $\mathsf{C}(A_5)$ with the values of φ arising from the dissection in Figure 1.

Theorem B. *Let ρ and φ be the maps* $\operatorname{obj}\mathsf{C}(A_n) \to \mathbf{Z}$ *defined above. Then* $\rho = \varphi$.

In particular, φ is a generalised frieze by Theorem A. This was shown by combinatorial means in [1, thm. 5.1].

Further developments. Going back to the abstract setup, the modified CC map in Equation (1) depends on choices for A, α, and β. In [5] we show that if R is a rigid object in a suitable 2-Calabi-Yau category C, then A can be chosen as a ring of Laurent polynomials over $\mathbf{Z}$ and α and β can be chosen to be non-constant maps. This produces generalised friezes with values in Laurent polynomials.

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Grassmannian Cluster Categories

Alastair King
(joint work with Bernt Jensen and Xiuping Su)

We find an additive categorification of the cluster algebra structure of Fomin-Zelevinsky [1] and Scott [4] on the homogeneous coordinate ring $\mathbb{C}\big[\widehat{G}_k^n\big]$ of the Grassmannian of k-subspaces of $\mathbb{C}^n$, extending the categorification by Geiss-Leclerc-Schröer [2] for the affine coordinate ring $\mathbb{C}[N]$ of the open cell.

Theorem. *(G-L-S [2])* Let Π be the preprojective algebra of type A_{n-1}. Then $\mathbb{C}[N]$ is categorified by the category $\operatorname{Sub} Q_k$ of Π-modules with socle at k.

Since $\mathbb{C}[N] = \mathbb{C}\big[\widehat{G}_k^n\big]/(\phi_0 - 1)$, we see that ϕ_0 is the one Plücker coordinate missed by $\operatorname{Sub} Q_k$: it corresponds to the zero module, not a rigid indecomposable.

Theorem. *(J-K-S [3])* Let B be the twisted group ring $\mathbb{C}[x,y]/(x^k - y^{n-k}) * C_n$, where C_n is a diagonal cyclic subgroup of $SL_2(\mathbb{C})$. Then $\mathbb{C}\big[\widehat{G}_k^n\big]$ is categorified by the category $\mathrm{CM}(B)$ of B-modules that are free over the centre $Z(B) = \mathbb{C}[xy]$.

The proof uses the fact that B has an idempotent e_0 such that B/Be_0B is a quotient of Π supporting $\operatorname{Sub} Q_k$. Then the functor $M \mapsto M/Be_0M$ identifies $\operatorname{Sub} Q_k$ as the quotient of $\mathrm{CM}(B)$ by the projective $P_0 = Be_0$. Note: P_0 is precisely the new rigid indecomposable corresponding to the missed coordinate ϕ_0.

For example, when $k = 2$, the Auslander-Reiten quiver of $\mathrm{CM}(B)$ has a structure that mirrors that of a Coxeter-Conway frieze pattern, e.g. when $n = 5$:

$$\begin{tikzcd}
15 \arrow[dr] & & 12 \arrow[dr] & & 23 \arrow[dr] & & 34 \arrow[dr] & & 45 \\
& 25 \arrow[ur] \arrow[dr] \arrow[rr, dash, dashed] & & 13 \arrow[ur] \arrow[dr] \arrow[rr, dash, dashed] & & 24 \arrow[ur] \arrow[dr] \arrow[rr, dash, dashed] & & 35 \arrow[ur] & \\
& & 35 \arrow[ur] \arrow[dr] \arrow[rr, dash, dashed] & & 14 \arrow[ur] \arrow[dr] \arrow[rr, dash, dashed] & & 25 \arrow[ur] & & \\
& & & 45 \arrow[ur] & & 15 \arrow[ur] & & &
\end{tikzcd}$$

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Strongly primitive species with potentials: aims and limitations

Daniel Labardini-Fragoso
(joint work with Andrei Zelevinsky)

This talk is based on [4]. We define mutations of strongly primitive species with potentials, and show that if B is any $n \times n$ integer matrix admitting a skew-symmetrizer with pairwise coprime diagonal entries, then for any given sequence of mutations which one wants to perform there exists a species realization A of B which admits a potential S such that the mutations of (A, S) along the given sequence are compatible with matrix mutation.

An $n \times n$ integer matrix B is said to be *skew-symmetrizable* if there exist positive integers $d_1, \dots, d_n$ such that DB is skew-symmetric, where $D = \mathrm{diag}(d_1, \dots, d_n)$. Such D is said to be a *skew-symmetrizer* of B.

A *weighted quiver* is a pair $(Q, \mathbf{d})$, where $Q = (Q_0, Q_1, t, h)$ is a loop-free quiver and $\mathbf{d} = (d_i)_{i \in Q_0}$ is a tuple of positive integers attached to the vertices of Q. We call $\mathbf{d}$ the *weight tuple* of $(Q, \mathbf{d})$.

If B is a skew-symmetrizable matrix with skew-symmetrizer D, we define a weighted quiver $(Q_B, \mathbf{d})$ as follows: The vertex set of Q_B is $\{1, \dots, n\}$, and Q_B has exactly

$$\frac{\gcd(d_i, d_j) b_{ij}}{d_j}$$

arrows from j to i whenever $b_{ij} \geq 0$. The tuple $\mathbf{d}$ is defined to be the tuple of diagonal entries of D.

If D (and hence $\mathbf{d}$) is kept fixed, the assignment $B \mapsto (Q_B, \mathbf{d})$ is easily seen to be a bijection between the set of $n \times n$ integer matrices that are skew-symmetrized by D and the set of 2-acyclic weighted quivers with weight tuple $\mathbf{d}$. This means that there is a weighted quiver counterpart of the notion of matrix mutation:

Definition 1. Let $(Q, \mathbf{d})$ be a weighted quiver. For $k \in Q_0$, define the *mutation of* $(Q, \mathbf{d})$ *with respect to* k to be the weighted quiver $\mu_k(Q, \mathbf{d})$ with vertex set Q_0 and weight tuple $\mathbf{d}$, obtained as the result of performing the following 3-step procedure:

(Step 1) For each pair of arrows $a : j \to k$ and $b : k \to i$ of Q, introduce $\frac{\gcd(d_i, d_j) d_k}{\gcd(d_i, d_k) \gcd(d_k, d_j)}$ "composite" arrows from j to i;
(Step 2) replace each $c \in Q_1$ incident to k with an arrow c^* going in the opposite direction;
(Step 3) choose a maximal collection of disjoint 2-cycles and remove them.

Let d be the least common multiple of the tuple $\mathbf{d} = (d_i)_{i \in Q_0}$, F be a finite field, and E be the unique degree-d field extension of F. For $i \in Q_0$, set F_i to be the unique degree-d_i field subextension of E/F, and for every pair of vertices $i, j \in Q_0$, set

$$A_{ij} = \bigoplus_{a : j \to i} F_i \otimes_{F_i \cap F_j} F_j.$$

Define

$$R = \bigoplus_{i \in Q_0} F_i \quad \text{and} \quad A = \bigoplus_{i,j \in Q_0} A_{ij}.$$

Then R is a semisimple ring and A is an R-R-bimodule. We say that A is the *species of* $(Q, \mathbf{d})$ over E/F.

Remark 2. The data $((F_i)_{i \in Q_0}, (A_{ij})_{i,j \in Q_0}, (A^*_{ij})_{i,j \in Q_0})$ constitutes a species (or *modulation*) of the valued quiver of B in the sense of Dlab-Ringel [3], hence our use of the term "species".

The complete tensor algebra of A over R is called the *complete path algebra* of A and denoted $R\langle\langle A\rangle\rangle$. Thus we have

$$R\langle\langle A\rangle\rangle = \prod_{\ell=0}^{\infty} A^{\ell}$$

as an R-R-bimodule, where A^{ℓ} denotes the ℓ-fold tensor product $A \otimes_R \ldots \otimes_R A$. If an element $S \in \prod_{\ell=1}^{\infty} A^{\ell}$ satisfies $S = \sum_{i \in Q_0} e_i S e_i$, where e_i is the idempotent sitting in the i^{th} component of R, we say that S is a *potential* on A and that (A, S) is a *species with potential*.

From now on we assume that $(Q, \mathbf{d})$ is *strongly primitive*, that is, we suppose that $\gcd(d_i, d_j) = 1$ for all $i \neq j$. We shall say that A, the species of $(Q, \mathbf{d})$ over E/F, is strongly primitive as well, and that (A, S) is a *strongly primitive species with potential* whenever S is a potential on A. We also assume that the characteristic of F is congruent to 1 modulo d. This implies that there exists an element $v \in E$ such that the set $\mathcal{B} = \{1, v, v^2, \ldots, v^{d-1}\}$ is an *eigenbasis* of E/F, that is, an F-basis of E consisting of eigenvectors of all elements of the Galois group $\mathrm{Gal}(E/F)$. This eigenbasis is quite useful to obtain an explicit description of the elements of $R\langle\langle A\rangle\rangle$; indeed, setting $\mathcal{B}_i = \mathcal{B} \cap F_i$ for $i \in Q_0$ we have:

Lemma 3. For every $\ell \geq 0$, the set $\{\omega_0 a_1 \omega_1 a_2 \ldots \omega_{\ell-1} a_\ell \omega_\ell \mid t(a_m) = h(a_{m+1})$ for $m = 1, \ldots, \ell - 1$, $\omega_0 \in \mathcal{B}_{h(a_1)}$ and $\omega_m \in \mathcal{B}_{t(a_m)}$ for $m = 1, \ldots, \ell\}$, whose elements we call *paths of length* ℓ, is a basis of A^{ℓ} as an F-vector space. Consequently, every element of $R\langle\langle A\rangle\rangle$ has a unique expression as a possibly infinite F-linear combination of paths. In particular, every potential has a unique expression as a possibly infinite F-linear combination of cyclic paths of positive length.

Given a 2-acyclic strongly primitive weighted quiver $(Q, \mathbf{d})$ and a vertex $k \in Q_0$, for each pair of arrows $a : j \to k$ and $b : k \to i$ of Q, the composite arrows introduced in the first step of the weighted-quiver mutation with respect to k are denoted with the symbols $[b\omega a]$, where ω runs in the set $\mathcal{B}_k$.

We can state now the mutation rule for strongly primitive species with potentials.

Definition 4. Let $(Q, \mathbf{d})$ be a 2-acyclic strongly primitive weighted quiver, and let A be its species over E/F. For $k \in Q_0$, let $\widetilde{\mu}_k(A)$ denote the species over E/F of the weighted quiver obtained from $(Q, \mathbf{d})$ by applying only the first two steps of weighted-quiver mutation. If S is a potential on A, we define a potential $\widetilde{\mu}_k(S)$ on $\widetilde{\mu}_k(A)$ according to the formula

$$\widetilde{\mu}_k(S) = [S] + \sum_{\xrightarrow{a} k \xrightarrow{b}} \sum_{\omega \in \mathcal{B}_k} \omega^{-1} b^* [b\omega a] a^*.$$

The species with potential which is the *reduced part* of $(\widetilde{\mu}_k(A), \widetilde{\mu}_k(S))$ will be called the *mutation* of (A, S) with respect to k and denoted $\mu_k(A, S)$.

We refer the reader to [4] for the definition of reduced parts as well as for a proof of their existence. Note that the underlying species of $\mu_k(A, S)$ is again strongly primitive, although its underlying weighted quiver may have oriented 2-cycles.

Definition 5. Given a finite sequence $(k_1, \ldots, k_m)$ of vertices of Q, we say that (A, S) is $(k_m, \ldots, k_1)$-*non-degenerate* if the quivers underlying the species with potentials (A, S), $\mu_{k_1}(A, S)$, $\mu_{k_2}\mu_{k_1}(A, S)$, …, $\mu_{k_m} \ldots \mu_{k_2}\mu_{k_1}(A, S)$, are 2-acyclic (hence well-defined).

The following is the main result of the talk.

Theorem 6. [4] *If $(Q, \mathbf{d})$ is a 2-acyclic strongly primitive weighted quiver, then for every finite sequence $(k_1, \ldots, k_m)$ of vertices of Q there exists a finite-degree field extension K/F which is linearly disjoint from E/F and has the property that the species $A_{KE/K}$ of $(Q, \mathbf{d})$ over KE/K admits a potential S such that $(A_{KE/K}, S)$ is $(k_1, \ldots, k_m)$-non-degenerate.*

Besides Theorem 6, mutations of strongly primitive species with potentials share many other properties with the mutations of quivers with potentials of Derksen-Weyman-Zelevinsky. For example, they are well-defined up to right-equivalence and involutive up to right-equivalence, and they preserve Jacobi-finiteness. Following the spirit of [2], the notion of mutation is further lifted in [4] to the representation-theoretic level.

The class of skew-symmetrizable matrices whose associated weighted quivers are strongly primitive includes several examples of matrices that do not admit *global unfoldings* whatsoever (a global unfolding is an unfolding which is compatible with all possible sequences of mutations). So, the species framework in [4] provides a representation-theoretic approach to (the skew-symmetrizable matrices of) several cluster algebras where approaches via unfoldings do not work.

Now, how about matrices that admit global unfoldings?, and how about species with potentials for weighted quivers that are not strongly primitive? The answer to the first question is provided by the work [1] of L. Demonet, who developed an approach to mutations via *group species with potentials*. The framework of group species differs from the one we have presented above in that a group species attaches group algebras to the vertices of Q rather than fields; the bimodules attached to the arrows are hence also different in nature from the bimodules A_{ij} above. Group species with potentials provide a representation-theoretic approach that can be successfully applied to those matrices that admit global unfoldings through group actions. Unfortunately, for matrices that do not admit global unfoldings one needs a framework different from the group species framework.

Regarding the second question: if $\gcd(d_i, d_j)$ is not assumed to be equal to 1 for all $i \neq j$, one can easily construct an example of a weighted quiver $(Q, \mathbf{d})$ that has a 2-cycle ab which is cyclically equivalent to 0 in $R\langle\langle A\rangle\rangle$, and this implies that none of the arrows a, b belongs to the Jacobian ideal of any potential on the species A of $(Q, \mathbf{d})$ over E/F. This means that none of a, b can be deleted from Q. Ultimately, this yields an example of a 2-acyclic species (not strongly primitive) such that, no matter which potential on it we take, when we try to perform the three steps of weighted-quiver mutation at the level of species with potentials, the species framework presented above fails to delete 2-cycles from the underlying weighted quiver.

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Locally acyclic cluster algebras and Grassmannians

Greg Muller

(joint work with David Speyer)

Cluster algebras exhibit a curious dichotomy. Many of the first examples of cluster algebras - the simple or motivational examples - are algebraically well-behaved: finitely generated and with mild or no singularities. However, one does not need to wander too far into the general theory to encounter cluster algebras and upper cluster algebras which are non-Noetherian and badly singular [2, 5, 7]. The class of *locally acyclic cluster algebras* attempts to capture the well-behaved cluster algebras.

Locally acyclic cluster algebras are 'locally elementary', in a sense we now make precise. A **cluster localization** $\mathcal{A}'$ of a cluster algebra $\mathcal{A}$ is a localization which coincides with freezing a subset of a cluster in $\mathcal{A}$. The seeds and cluster variables of $\mathcal{A}'$ are contained in the seeds and cluster variables of $\mathcal{A}$; hence, one should typically expect that $\mathcal{A}'$ is a simpler cluster algebra than $\mathcal{A}$.

Geometrically, the spectrum $Spec(\mathcal{A}')$ of a cluster localization describes an open affine patch in the spectrum $Spec(\mathcal{A})$. Given a cover of $Spec(\mathcal{A})$ by such patches, we can verify many algebraic properties by checking them locally on the cover. Ideally, this cover consists of cluster localizations for which these properties are already known; so that these results lift to $\mathcal{A}$ without any cleverness on our part.

A cluster algebra $\mathcal{A}$ is **locally acyclic** if there is a set of cluster localizations $\{\mathcal{A}_i\}$ of $\mathcal{A}$ such that each $\mathcal{A}_i$ is an acyclic cluster algebra, and $\cup Spec(\mathcal{A}_i) = Spec(\mathcal{A})$ [5]. Remarkably, this class of cluster algebras is the same if we replace *acyclic* with several other notions of 'elementary' cluster algebra; for example, *locally tree-type*, *locally finite-type* and *locally isolated* are all equivalent to being locally acyclic.[1]

As intended, many properties of acyclic cluster algebras lift to locally acyclics.

Proposition. Let $\mathcal{A}$ be locally acyclic.

(1) [5] $\mathcal{A}$ is finitely generated and locally a complete intersection.
(2) [5, 4] The cluster algebra $\mathcal{A}$ equals its own upper cluster algebra.[2]
(3) [5] $\mathcal{A}$ is normal.
(4) [5] If $\mathcal{A}$ is full-rank, then $Spec(\mathcal{C}\otimes\mathcal{A})$ is smooth.
(5) [1] In general, $Spec(\mathcal{C}\otimes\mathcal{A})$ has at worst 'canonical singularities'.[3]

Locally acyclic cluster algebras contain many important examples of cluster algebras. Acyclic cluster algebras already contain many of the first examples of cluster algebras, such as the finite-type cluster algebras. In [5], it was shown that the cluster algebra of a triangulable marked surface with at least two marked points on the boundary was locally acyclic.

The main subject of my talk is new work (joint with David Speyer) which establishes the local acyclicity of a broad class of important cluster algebras: those from *Postnikov diagrams in the disc*. Given a permutation $\pi : \{1, 2, ..., n\} \to \{1, 2, ..., n\}$, a **Postnikov diagram** for π is a collection of generic oriented strands in the disc with marked points indexed by $\{1, 2, ..., n\}$, with the following properties.

- The unique strand which begins at i ends at $\pi(i)$.
- No strand intersects itself except possibly at its endpoints.
- There are no pairs of intersections between the same two strands, such that the strands are both oriented from one intersection to the other.
- Travelling along any strand, the orientations of the crossings alternate.

In his seminal unpublished work, Postnikov associates to each such diagram a seed, and proves that the resulting cluster algebra only depends on the underlying permutation π. This formalism includes many well-studied families of cluster algebras, including coordinate rings of Grassmannians [6], double Bruhat cells in SL_n [2], and higher Teichmüller space of A-type [3].

The idea of the proof is based on a combinatorial criterion for local acyclicity, which can be checked on the level of quivers. Define the class of **SB quivers** to be the smallest class of quivers with the following properties.

(1) The mutation of an SB quiver is SB.
(2) The empty quiver is SB.
(3) The union of an SB quiver with an isolated vertex is SB.

[1] A cluster algebra is *isolated* if, for any seed, there are no arrows between mutable vertices.

[2] The proof which appears in [5] has a gap. To use the analogous result from [2], one needs to refine an acyclic cover until it is also totally coprime, though, this is always possible. The note [4] gives a self-contained proof that $\mathcal{A} = \mathcal{U}$ for locally acyclic cluster algebras.

[3] A singularity in a normal variety is *canonical* if, in any resolution of singularities, every exceptional divisor has non-negative discrepancy.

(4) Let s be a source in Q, and t the target of some arrow out of s. Let Q_s and Q_t denote the induced subquivers of Q on the complement of s and t, respectively. If Q_s and Q_t are SB, then Q is SB.

Any cluster algebra with a seed whose mutable quiver is SB is locally acyclic.

In practice, the main obstacle for applying this criterion to Postnikov diagrams is to find an equivalent seed with a source. Different Postnikov diagrams with the same permutation π describe a connected component of the exchange graph, and in some cases, there is a seed with a source among these seeds. However, this won't work for all permutations.

A way around this obstacle is to observe that distinct permutations π_1 and π_2 can have Postnikov diagrams with the same underlying mutable quiver. If π_1 has a different Postnikov diagram with a source, then the cluster algebra associated to π_2 has a seed with a source (though it may not be described by a Postnikov diagram). An extension of this argument produces a mutation-equivalent seed with a source for every Postnikov diagram.

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Connections between cluster algebras and string theory

Gregg Musiker

(joint work with In-Jee Jeong, Sicong Zhang)

In this talk, we gave a glimpse of the connections between cluster algebras and string theory. In particular, string theorists study certain bipartite graphs on a torus, referred to as a *brane tiling*, to describe world volumes of D_3 branes and certain $(3 + 1)$-dimensional superconformal field theories. The geometry of the extra dimensions around D-branes determines the structure of the quantum field theories living on them. This geometry also gives rise to a Calabi-Yau 3-fold with singularities which can be described by a polygon, known as a *toric diagram.*

Following a construction in Goncharov-Kenyon [GK] and appearing in the physics literature, such as [FHKVW] and [HK], one can take a toric diagram, compute normal vectors, and draw them as directed lines on a torus. By then deforming oriented regions to black or white vertices and labeling the remaining regions, one can obtain a brane tiling, i.e. a tesselation of the torus. This construction also appeared in earlier work of Kenyon, Okounkov, and Sheffield [KOS06], as cited in [HK].

Given such a brane tiling, we take the dual to get a quiver hence a cluster algebra, in the sense of Fomin and Zelevinsky [FZ02a]. With an eye towards the corresponding geometry, physicists such as R. Eager, S. Franco, A, Hanany, K.D. Kennaway, R.-K. Seong, D. Vegh, B. Wecht, and others [FHKVW, DHP10, HS12] study certain special families of quivers, for instance those coming from reflexive polygons.

In joint work with University of Minnesota REU students I. Jeong and S. Zhang, we investigated several such examples, including a six-vertex quiver associated to the $dP3$ lattice and periodic quivers coming from Gale-Robinson sequences [BPW09, Gal91], defined by the recurrence

$$x_n x_{n-m} = x_{n-r} x_{n-m+r} + x_{n-s} x_{n-m+s}.$$

We obtain combinatorial formulas for cluster varibles with principal coefficients as subgraphs of these brane tilings. Forthcoming work [JMZ], also presented at FPSAC 2013 [JMZ13], focuses on these cases and other special period-1 and period-2 cases. Associated REU reports from 2011 and 2012 are also available [J, Z].

In particular, for the $dP3$ lattice, also known as the honeycomb lattice, mutation periodically by $1, 2, 3, 4, 5, 6, 1, 2, \dots$ leads to a sequence of cluster variables whose Laurent expansions have combinatorial interpretations in terms of perfect matchings of certain graphs. One assigns weights and heights to each such matching, and using a certain labeling of *Aztec Dragons*, a family of graphs appearing in work of Cottrell-Young [CY], M. Ciucu [C03], and J. Propp [P99]. This definition of weights appears previously in D. Speyer's discussion of the Octahedron Recurrence [S07] and the heights appear in a few places such as [P]. Using graphical condensation [K04], one obtains the desired formulas. These cluster variables were also studied by S. Franco and R. Eager [E11, EF], where they are referred to as Colored BPS Pyramid Partition functions, and a related construction of a *shadow* of a brane tiling is described. A joint project with S. Franco is currently underway which investigates the exact connections between subgraphs and shadows of the brane tilings.

In 2013, University of Minnesota REU students M. Leoni, S. Neel, and P. Turner extended results for the $dP3$ lattice even further by inventing a two-parameter family of graphs known as *Aztec Castles* which yield combinatorial interpretations for a much larger class of cluster variables [LNT]. In particular, by mutating at antipodal vertices of the $dP3$ quiver, one obtains the same quiver back up to a cyclic rotation. This allows us to consider sequences of composite mutations τ_1, τ_2, τ_3 which each represent mutation at a different antipodal pair. They satisfy relations of the affine A_2 Weyl group, e.g. $(\tau_1\tau_2)^3 = \tau_3^2 = 1$. However, after quotienting out by these relations, there is a two-parameter family of τ-mutation sequences whose cluster variables correspond to Aztec Castles.

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Exact WKB analysis and cluster algebras

Tomoki Nakanishi
(joint work with Kohei Iwaki)

We develop the mutation theory in the exact WKB analysis using the framework of cluster algebras. Under a continuous deformation of the potential of the Schrödinger equation on a compact Riemann surface, the Stokes graph may undergo discontinuous deformations, which we call mutations of Stokes graphs. Along the mutations of Stokes graphs, the Voros symbols, which are monodromy data of the equation, also mutate due to the Stokes phenomenon. We show that the Voros symbols mutate as variables of a cluster algebra with surface realization. As an application, we obtain the identities of the Stokes automorphisms associated with periods of cluster algebras.

This talk is based on our forthcoming paper [1].

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Abelianization and cluster-like coordinate systems

Andrew Neitzke

Abelianization is a way of studying moduli spaces $\mathcal{M}(C, GL(K))$ of flat $GL(K)$-connections over a surface C by relating them to moduli spaces $\mathcal{M}(\Sigma, GL(1))$ of flat $GL(1)$-connections over K-fold branched coverings $\Sigma \to C$. Roughly the idea is that a generic flat $GL(K)$-connection can be obtained by *pushing forward* a flat $GL(1)$-connection from Σ and then "correcting" the result. The correction process amounts to cutting-and-gluing along a network of curves on C ("spectral network"), using flat unipotent endomorphisms as the gluing maps. In this way one eliminates unwanted monodromy around branch points of $\Sigma \to C$, while not introducing monodromy anywhere else. For each spectral network $\mathcal{W}$ we thus obtain a map

$$\Psi_{\mathcal{W}} : \mathcal{M}(\Sigma, GL(1)) \to \mathcal{M}(C, GL(K)).$$

This map is expected to preserve the Poisson structure; moreover there are many examples of spectral networks $\mathcal{W}$ for which $\Psi_{\mathcal{W}}$ is 1-1 onto an open dense subset $U_{\mathcal{W}} \subset \mathcal{M}(C, GL(K))$. Because $\mathcal{M}(\Sigma, GL(1))$ is a complex torus, we thus get a local coordinate system on $U_{\mathcal{W}}$, which we call "spectral coordinate system." The coordinate functions X_γ in the spectral coordinate system are interpreted as holonomies of the $GL(1)$-connection around cycles $\gamma \in H_1(\Sigma, \mathbb{Z})$.

The spectral coordinate systems have many delightful properties. In particular, they include the coordinate systems studied by Fock-Goncharov in [1], which belong to the atlas of cluster coordinate systems on $\mathcal{M}(C, GL(K))$. One also obtains other, new coordinate systems — some belonging to the cluster atlas, others not (for example, in the case $K = 2$ one can get Fenchel-Nielsen coordinates as spectral coordinates, but these are not cluster coordinates.) The various spectral coordinate systems are related to one another by various transformations, which include cluster mutations as well as some "generalized mutations" of the form

$$X_\gamma \longmapsto X_\gamma \prod_{n=1}^{\infty} (1 - X_{n\gamma'})^{n\langle\gamma,\gamma'\rangle\Omega(n\gamma')}$$

The coefficients $\Omega(\cdot)$ appearing here are generalized Donaldson-Thomas invariants in the sense of [2, 3].

These general expectations were worked out in [4, 5], but relatively few examples have been explored. One simple class of examples which should be interesting is the following: by considering connections over $C = \mathbb{CP}^1$ with a single irregular singularity of a particular kind, we can arrange that the moduli space $\mathcal{M}(C, GL(K))$ is a certain torus quotient of a Grassmannian $Gr(k, n)/(\mathbb{C}^\times)^{n-1}$. By considering connections equipped with a certain extra decoration we can "undo" the quotient, obtaining the affine cone over $Gr(k, n)$. We conjecture that the spectral coordinates in this case will recover the cluster structure on homogeneous coordinate rings of Grassmannians [6]. In particular, there should be a spectral network corresponding to each cluster. This should give some insight into (some special cases of) the proposal of [7] for how the clusters are described.

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On linear independence of cluster monomials

Pierre-Guy Plamondon
(joint work with Giovanni Cerulli Irelli, Bernhard Keller and Daniel Labardini-Fragoso)

One of the main problems in the theory of cluster algebras, developed by S. Fomin and A. Zelevinsky [7], is that of finding a "good" basis for these algebras. Among the good properties that these bases are expected to possess is the following: they should contain the *cluster monomials*, that is, the products of cluster variables belonging to a common cluster. Thus the following conjecture:

Conjecture 1 (Conjecture 4.16 of [8]). *Let $\mathcal{A}$ be a cluster algebra defined over a coefficient semifield $\mathbb{P}$. Then the cluster monomials of $\mathcal{A}$ are linearly independent over $\mathbb{ZP}$.*

The aim of this report is to sketch the proof of the following result, using methods from the additive categorification of cluster algebras:

Theorem 2 ([4]). *If the defining matrix of $\mathcal{A}$ is skew-symmetric, then its cluster monomials are linearly independent over $\mathbb{ZP}$.*

1. The proper Laurent property

The proof of Theorem 2 is done by showing that an apparently stronger condition, described in the following definition, is satisfied.

Definition 3 ([5], see also [12]). Let $\mathcal{A}$ be any cluster algebra.

(1) Let $\mathbf{u} = (u_1, \ldots, u_n)$ be a cluster in $\mathcal{A}$. A *proper Laurent monomial* in $\mathbf{u}$ is an element of the form $u_1^{a_1} \cdots u_n^{a_n}$, where at least one of the a_i's is negative.

(2) The cluster algebra $\mathcal{A}$ has the *proper Laurent monomial property* if, for any two clusters $\mathbf{u}$ and $\mathbf{u}'$, any cluster monomial in $\mathbf{u}'$ in which at least one element of $\mathbf{u}' \setminus \mathbf{u}$ appears with positive power is a proper Laurent monomial in $\mathbf{u}$.

The importance of this definition for our purposes comes from the following result.

Theorem 4 (Theorem 6.4 of [5]). *If $\mathcal{A}$ has the proper Laurent monomial property, then its cluster monomials are linearly independent over $\mathbb{ZP}$.*

Our aim is thus to prove that skew-symmetric cluster algebras have the proper Laurent monomial property. Note that if the cluster algebra defined by a matrix B over principal coefficients has the property, then so does any cluster algebra defined by the same B over arbitrary coefficients, thanks to the "separation of additions" of [9, Theorem 3.7].

2. Additive categorification

Our proof relies on the categorification of cluster algebras using (generalized) cluster categories. In this section, we will briefly describe the relevant results of this theory.

Let us work over the field $\mathbb{C}$. To any skew-symmetric cluster algebra B, one can associate a quiver Q without oriented cycles of length 1 or 2. Let W be a non-degenerate potential on Q [6]. Then C. Amiot [1] has constructed a triangulated category $\mathcal{C}_{Q,W}$, the *(generalized) cluster category*, with suspension functor Σ. This category comes equipped with a basic rigid object $\Gamma = \Gamma_1 \oplus \ldots \oplus \Gamma_n$ whose endomorphism algebra is isomorphic to the Jacobian algebra $J(Q, W)$. Inside $\mathcal{C}_{Q,W}$ lies a full subcategory $\mathcal{D}$, whose objects are those M such that

(1) $Hom_{\mathcal{C}}(\Gamma, \Sigma M)$ is finite-dimensional;
(2) there is a triangle $T_1^M \to T_0^M \to M \to \Sigma T_1^M$, with T_1^M and T_0^M in $add(\Gamma)$;
(3) there is a triangle $T_M^0 \to T_M^1 \to \Sigma M \to \Sigma T_M^0$, with T_M^0 and T_M^1 in $add(\Gamma)$.

The category $\mathcal{D}$ contains Γ and is Krull–Schmidt, although it is not triangulated. The *index* of an object M of Γ is $ind_\Gamma M = [T_0^M] - [T_1^M] \in K_0(add(\Gamma))$; since $K_0(add(\Gamma))$ is isomorphic to $\mathbb{Z}^n$, we view the index of M as a vector of n integers.

Define the *cluster character* CC as being the map taking an object M of $\mathcal{D}$ and sending it to the Laurent polynomial

$$CC(M) = \sum_{\mathbf{e} \in \mathbb{N}^n} \chi\Big(Gr_{\mathbf{e}}\big(Hom_{\mathcal{C}}(\Gamma, \Sigma M)\big)\Big)\mathbf{x}^{ind_\Gamma M + B\cdot\mathbf{e}},$$

where $Hom_{\mathcal{C}}(\Gamma, \Sigma M)$ is viewed as a right module over $J(Q, W)$, $Gr_{\mathbf{e}}(-)$ is the quiver Grassmannian (a projective variety whose points parametrize submodules of dimension vector $\mathbf{e}$), and χ is the Euler characteristic.

The main result for categorification of cluster algebras is the following:

Theorem 5 ([2][3][10][11][4])**.** *The map CC induces a bijection*

$$\{\textit{indecomposable reachable rigid objects of } \mathcal{D}\}/isom \to \{\textit{cluster variables of } \mathcal{A}\}.$$

As a result, any cluster monomial of $\mathcal{A}$ is the image by CC of a rigid object in $\mathcal{D}$.

Note that this theorem allows for the categorification of any cluster algebra of geometric type, by viewing them as coming from a quiver with "frozen vertices". In particular, the result applies to cluster algebras with principal coefficient.

3. Sketch of the proof of the main theorem

We now want to prove that a cluster algebra $\mathcal{A}$ (say with trivial coefficients for simplicity) has the proper Laurent monomial property. Let z be a cluster monomial in $\mathcal{A}$ involving non initial cluster variables. Then, by Theorem 5, there exists a rigid object R in $\mathcal{D}$ but not in $add(\Gamma)$ such that $CC(R) = z$. We need to prove the following:

Theorem 6 ([4])**.** *With the above assumptions, $CC(R)$ is a linear combination of proper Laurent monomials in the initial cluster.*

The idea of the proof is as follows. In view of the formula for $CC(R)$, we need to prove that whenever $Gr_{\mathbf{e}}\big(Hom_{\mathcal{C}}(\Gamma, \Sigma R)\big)$ is non-empty (and thus has possibly non-zero Euler characteristic), the vector $(ind_\Gamma R + B \cdot \mathbf{e})$ has a negative entry.

If $\mathbf{e}$ is zero, then the vector becomes $ind_\Gamma R$, which has a negative entry since we assumed that R is not in $add(\Gamma)$.

If $\mathbf{e}$ is non-zero, we prove the following stronger statement: the integer $\mathbf{e} \cdot (ind_\Gamma R + B \cdot \mathbf{e})$ is negative. Since B is skew-symmetric, this integer simplifies to $\mathbf{e} \cdot (ind_\Gamma R)$.

By our assumptions, $Hom_{\mathcal{C}}(\Gamma, \Sigma R)$ has a non-zero submodule of dimension vector $\mathbf{e}$; let L be an object of $\mathcal{D}$ such that $Hom_{\mathcal{C}}(\Gamma, \Sigma L)$ is that submodule. Applying the functor $Hom_{\mathcal{C}}(?, \Sigma L)$ to the triangle $T_1^R \to T_0^R \to R \to \Sigma T_1^R$, we get an exact sequence

$$0 \to (R, \Sigma L)/(\Sigma\Gamma) \to (T_0^R, \Sigma L) \to (T_1^R, \Sigma L) \to (\Gamma)(\Sigma^{-1}R, \Sigma L) \to 0,$$

where we write (X, Y) instead of $Hom_{\mathcal{C}}(X, Y)$. This yields the equality

$$\dim(T_0^R, \Sigma L) - \dim(T_1^R, \Sigma L) = \dim(R, \Sigma L)/(\Sigma\Gamma) - \dim(\Gamma)(\Sigma^{-1}R, \Sigma L).$$

A direct computation shows that the left-hand side is equal to $\mathbf{e} \cdot (ind_\Gamma R)$. To finish the proof, one uses the fact that R is rigid to show that $(R, \Sigma L)/(\Sigma\Gamma)$ is zero, and the fact that $Hom_{\mathcal{C}}(\Gamma, \Sigma L)$ is a submodule of $Hom_{\mathcal{C}}(\Gamma, \Sigma R)$ to prove that $(\Gamma)(\Sigma^{-1}R, \Sigma L)$ is non-zero. This shows that the right-hand side is negative, which is sufficient to finish the proof.

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Noncommutative Laurent phenomenon, triangulations and surfaces

Vladimir Retakh
(joint work with Arkady Berenstein)

In this talk we introduce noncommutative triangulations of oriented surfaces, relate them by noncommutative Ptolemy relations, and prove the Noncommutative Laurent Phenomenon. This gives a foundation of noncommutative cluster theory of these surfaces. As a surprising byproduct, we obtain a new topological invariant of closed oriented surfaces with punctures. Another application is the proof of Laurentness and positivity of a noncommutative Kontsevich recursion.

This is a joint work with Arkady Berenstein from University of Oregon. The goal of this talk is to introduce *noncommutative triangulations* of oriented surfaces (with marked boundary points and punctures). This will provide a foundation of noncommutative cluster theory of these geometric objects (this is the main theme of the paper in preparation [2]).

Since each surface can be obtained by gluing edges of a polygon (actually, in many ways), the most important object of study are noncommutative triangulations of a given polygon, or simply *noncommutative polygons*.

In the commutative case, cluster structure (of type A_{n-3}) on triangulations of an n-gon is based on the *Ptolemy relations*:

$$x_{ik}x_{j\ell} = x_{ij}x_{k\ell} + x_{i\ell}x_{jk} \tag{1}$$

for all quadrilaterals (i, j, k, ℓ) inscribed in a circle, $1 \leq i, j, k, \ell \leq n$, so that the chords (ik) and $(j\ell)$ are diagonals of the quadrilateral, and $x_{ij} = x_{ji}$, $i \neq j$ is the Euclidean length of the chord (ij). The Ptolemy relations (1) can also be interpreted as Plücker identities for $2 \times n$ matrices.

In our noncommutative version we do not assume that $x_{ij} = x_{ji}$ and we think of x_{ij} as a measurement of a directed chord from i to j. We suggest the following noncommutative generalization of the Ptolemy identity based on the theory of noncommutative quasi-Plücker coordinates developed in [5]:

$$x_{ik}x_{jk}^{-1}x_{j\ell} = x_{i\ell} + x_{ij}x_{kj}^{-1}x_{k\ell}. \tag{2}$$

for every quadrilateral (i, j, k, ℓ), in which (i, k) and (j, ℓ) are the diagonals.

Note that since elements x_{ij} correspond to directed arrows, the products of the form $x_{ij}x_{k\ell}^{-1}$, $x_{\ell k}^{-1}x_{ji}$ make sense only when $\ell = j$.

The noncommutative Ptolemy relations are not enough for developing a reasonable theory of noncommutative cluster algebras, in particular, for establishing the noncommutative Laurent Phenomenon. However, the Phenomenon holds if we additionally impose the *triangular relations* (also suggested by properties of quasi-Plücker coordinates):

$$x_{ij}x_{kj}^{-1}x_{ki} = x_{ik}x_{jk}^{-1}x_{ik} \tag{3}$$

for all distinct i, j, k (of course, (3) is redundant in the commutative case).

The triangular relations (3) are of fundamental importance because they allow to introduce *non-commutative angles* $T_i^{j,k} := x_{ji}^{-1}x_{jk}x_{ik}^{-1}$ in each triangle (i, j, k) so that $T_i^{j,k} = T_i^{k,j}$ due to (3). That is, the noncommutative angle at a vertex of a triangle does not depend on the order of the remaining two vertices. The "commutative" angles were introduced by Penner in [8, Section 3] (where they were called "h-lengths"), where each $x_{ij} = x_{ji}$ was viewed as the λ-length of the side (ij) of an ideal triangle (i, j, k) (see also [4, Section 12] and [6, Section 1.2], where the term "angle" was used, apparently, for the first time) and thus noncommutative angles together with the "noncommutative λ-lengths" x_{ij} can be thought of as a totally noncommutative metric on the Lobachevsky plane. The term "angle" is justified by the following observation. The noncommutative Ptolemy relations (2) together with the triangular relations (3) are equivalent to:

$$T_j^{ik} = T_j^{i\ell} + T_j^{k\ell}$$

for every quadrilateral (i, j, k, ℓ), in which (i, k) and (j, ℓ) are the diagonals. In other words, the (both commutative and noncommutative) angles are additive, which justifies the name. Using additivity of noncommutative angles, we establish the first instance of the noncommutative Laurent Phenomenon for the n-gon with vertices $1, \ldots, n$:

$$x_{ij} = \sum_{k=i}^{j-1} x_{i,1}T_1^{k,k+1}x_{1,j}$$

for all $2 \le i < j \le n-1$, e.g., each x_{ij} is a noncommutative Laurent polynomial in $x_{1,k}$, $x_{k,1}$, $k = 2, \ldots, n-1$ and all $x_{i,i\pm1}$. In fact, the latter elements correspond to a triangulation of the n-gon where each triangle has a vertex at 1. We generalize this to any triangulation of the n-gon and, as expected, the commutative version of this result (with all $x_{ij} = x_{ji}$) specializes to the Schiffler formula ([9, Theorem 1.2]).

These arguments extend verbatim if we replace a polygon with a surface Σ with marked points in the boundary and possibly with some punctures. That is, for each such Σ one defines a $\mathbb{Z}$-algebra $\mathcal{A}_\Sigma$ generated by $x_\gamma^{\pm1}$, where γ runs over homotopy classes of curves on Σ between marked points subject to the triangular and noncommutative Ptolemy relations. The Laurent Phenomenon asserts that for a given triangulation Δ of Σ each x_γ belongs to the subalgebra generated by all $x_{\gamma'}^{\pm1}$, $\gamma' \in \Delta$.

A surprising byproduct of our approach is that the corresponding group $\mathbb{T}_\Delta$ (generated by all t_γ, $\gamma \in \Delta$ subject to the triangular relations) does not depend on the triangulation of Σ, therefore, is a topological invariant of Σ, which we denote by $\mathbb{T}(\Sigma)$. It turns out that $\mathbb{T}_\Delta$ is either free or a one-relator group. It looks like the fundamental group of Σ, however it is different from $\pi_1(\Sigma)$. For instance, if Σ_n is the sphere S^2 with n punctures, then $\mathbb{T}(\Sigma_3)$ is a free group in 5 generators and $\mathbb{T}(\Sigma_n)$ is a 1-relator torsion-free group in $4n-7$ generators if $n \geq 4$. In fact, $\mathbb{T}(\Sigma)$ is related to the fundamental group of a ramified two-fold cover of Σ. In any case, the association $\Sigma \mapsto \mathbb{T}(\Sigma)$ defines a functor from the (topological) category of surfaces with marked points to the category of finitely generated groups.

For each marked point i on Σ and each triangulation Δ we also introduce a *total (noncommutative) angle* $T_i^\Delta \in \mathcal{A}_\Sigma$ to be the sum of noncommutative angles of all adjacent triangles. Similarly to the commutative case, we establish that the total angles do not depend on the choice of a triangulation Δ. Thus the collection of the total angles $\{T_i\}$ can be thought of as a noncommutative version of a (hyperbolic) Riemann structure on Σ. Using them we define algebra $\mathcal{U}_\Sigma$ to be the subalgebra of $\mathcal{A}_\Sigma$ generated by all noncommutative edges x_γ, the inverses of the boundary edges and all noncommutative angles T_i and verify that $\mathcal{U}_\Sigma$ is an totally noncommutative analogue of the *upper cluster algebra* corresponding to Σ (see e.g., [1]).

As an application of our noncommutative Laurent phenomenon, taking Σ to be an annulus with no punctures, one marked point on the inner boundary and k marked points on the outer boundary, we prove Laurentness of the following noncommutative recursion from [7] thus answering a question of M. Kontsevich. Let noncommutative variables U_n, $n \in \mathbb{Z}$ satisfy the system

$$\begin{cases} U_{n-k}U_n = 1 + U_{n-1}U_{n-k+1} & \text{if } n \text{ is even} \\ U_nU_{n-k} = 1 + U_{n-k+1}U_{n-1} & \text{if } n \text{ is odd} \end{cases} \tag{4}$$

where $k \geq 3$ is a fixed odd natural number. We prove the conjecture of Kontsevich that each U_n is a noncommutative Laurent polynomial in $U_0, \ldots, U_k$ with positive coefficients.

Another proof of the Laurentness for this system was given in [3].

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Positivity for cluster algebras

Ralf Schiffler
(joint work with Kyungyong Lee)

Cluster algebras were introduced by Fomin and Zelevinsky in 2002 to provide a better understanding of Lusztig's canonical basis in Lie Theory. By definition, a cluster algebra is a subalgebra of a field of rational functions in several variables, given by constructing a set of generators, the so-called cluster variables. These cluster variables are constructed recursively using a procedure called mutation. Although this construction is elementary, it is very difficult to compute cluster variables in general because of the recursive nature of the construction.

In 2002, Fomin and Zelevinsky have proved that the cluster variables are Laurent polynomials with integer coefficients and conjectured that these coefficients are non-negative. This is known as the positivity conjecture.

In a joint work with Kyungyong Lee [3], we give a proof of the positivity conjecture using only elementary methods. Our prove is building on our earlier results, namely a combinatorial formula for rank 2 obtained in [1], and a proof of the positivity conjecture in the case of rank 3 in [2].

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Complex integrable systems and tropical geometry

Yan Soibelman

Complex integrable system is usually understood as a holomorphic generically surjective map $\pi : (X, \omega^{2,0}) \to B$ of a complex analytic symplectic manifold of dimension $2n$ to a complex analytic manifold of dimension n such that generic fibers are holomorphic Lagrangian submanifolds. Non-degenerate fibers are parametrized by an open subset $B^0 \subset B$. There is a local system $\underline{\Gamma} \to B^0$ of lattices with fibers given by $H_1(\pi^{-1}(b), \mathbb{Z})$. Integrating $\omega^{2,0}$ over first homology we obtain a closed 1-form on the base. In case if it can be globally represented as $d_b Z, b \in B^0$ we say that Z is the central charge of our integrable system.

My talk was devoted to various conjectures about integrable systems with central charge, mainly following [1] (joint paper with M. Kontsevich).

Here is the summary of the talk.

1) Under certain conditions (mainly on the discriminant locus $B^{sing} = B - B^0$) one can associate with any $b \in B^0$ and $\gamma \in \underline{\Gamma}_b$ an integer $\Omega_b(\gamma)$. Collection of these integers (tropical Donaldson-Thomas invariants) satisfy wall-crossing formulas from our paper [2]. Tropical DT-invariants are defined by counting certain tropical trees on B with external vertices in B^{sing}.

2) In the case of Hitchin integrable systems numbers $\Omega_b(\gamma)$ are Donaldson-Thomas invariants of the Fukaya category associated with the spectral curve.

3) Consider the flow on B^0 given by

$$\frac{d}{dt}Re(Z_b) = -Re(Z_b).$$

Under some reality assumptions on Z the flow extends continuously to B and contracts it to a unique point $b_{min} \in B$. Assume $b_{min} \in B^0$. Then every smooth irreducible component D_i of B^{sing} gives rise to an integer vector $\gamma_i \in T_{b_{min}}B^0$. Thus we obtain a collection of integer vectors in a vector space endowed with skew-symmetric integer form. These data give rise to a quiver. Hence we recover the set-up of cluster theory. The whole framework is a special case of the notion of *wall-crossing structure* introduced in [1].

4) In the case of Hitchin integrable systems we can consider the mirror dual to the real symplectic manifold $(X, Re(\omega^{2,0}))$. It is a cluster variety, which is in fact an affine scheme of finite type. Its algebra of functions is isomorphic to the cluster algebra associated with the quiver described in 3).

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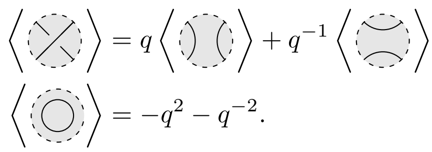
A positive basis for surface skein algebras

Dylan P. Thurston

For a compact oriented surface Σ (possibly with boundary), the *Kauffman bracket skein algebra*, denoted $\mathrm{Sk}_q(\Sigma)$, is the $\mathbb{Z}[q^{\pm 1}]$-module spanned by framed links in $\Sigma \times [0,1]$ modulo the local relations

(1)
(2)



Vertical stacking of links makes $\mathrm{Sk}_q(\Sigma)$ into an algebra: to form $\langle D_1 \rangle \cdot \langle D_2 \rangle$, superimpose D_1 onto D_2, making D_1 cross over D_2.

This skein algebra was first defined by Przytycki [2] and Turaev [3] as an extension of the Jones polynomial of knots in S^3 to knots in a surface cross an interval. When specialized to $q = \pm 1$, the algebra becomes commutative and we no longer need to record crossing information. For $q = -1$, we essentially get the algebra of functions on the $SL_2(\mathbb{R})$ character variety of Σ [4, 5, 6]. $\mathrm{Sk}_1(\Sigma)$ can be thought of as the algebra of functions on the twisted $SL_2(\mathbb{R})$ character variety.

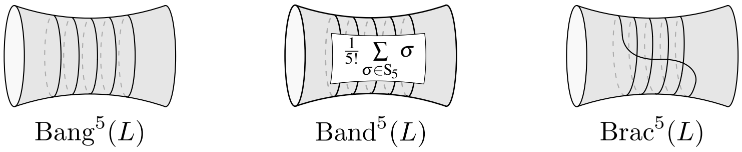


FIGURE 1. Examples of bangle, band, and bracelet operations applied to the core curve of an annulus. The bangle has parallel copies, the band averages over all ways of joining, and the bracelet wraps multiple times.

Definition 1. A *twisted $SL_2(\mathbb{R})$ representation* of a surface Σ is a representation of $\pi_1(UT\Sigma)$, the fundamental group of the unit tangent bundle of Σ, into $SL_2(\mathbb{R})$, with the property that rotation by 2π acts by $-1 \in SL_2(\mathbb{R})$.

For instance, a hyperbolic structure on Σ gives a canonical twisted $SL_2(\mathbb{R})$ representation.

Our main result is that $\mathrm{Sk}_1(\Sigma)$ (now called $\mathrm{Sk}(\Sigma)$) has a *positive basis*.

Definition 2. For an algebra A over $\mathbb{Z}$ (free as a $\mathbb{Z}$-module), a basis $\{x_i\}$ is *positive* if

$$x_i \cdot x_j = \sum_k m_{ij}^k x_k$$

where $m_{ij}^k \geq 0$.

The *bracelets basis* of the skein algebra is positive. This basis is *not* made of crossingless curves. In Fig. 1, instead of *bangles* we use *bracelets*.

Theorem 3. *The bracelets basis is a natural positive basis for* $\mathrm{Sk}(\Sigma)$.

There is an extension of the skein algebra to surfaces with marked points, which may be on the boundary or in the interior, and arcs ending at the marked points, with tagging by a local orientation near the marked points in the interior (the *punctures*). Such surfaces have an associated cluster algebra [7, 8], whose cluster variables correspond to certain of the tagged arcs [9].

In this context, the (tagged) skein algebra is intermediate between the cluster algebra and the upper cluster algebra associated to the surface:

$$\mathcal{A}(\Sigma) \subset \mathrm{Sk}(\Sigma) \subset \overline{\mathcal{A}}(\Sigma).$$

Both inclusions can be strict, for instance for the once-punctured torus. However, we make the following conjecture.

Conjecture 4. *For Σ any surface with a triangulation that is not a closed surface with exactly one puncture,* $\mathrm{Sk}(\Sigma) = \overline{\mathcal{A}}(\Sigma)$.

The bracelets basis was first introduced by Musiker, Schiffler, and Williams in the context of cluster algebras [10].

In order to prove Theorem 3, we prove a stronger theorem.

Theorem 5. *For any diagram D on Σ, the expansion of D in terms of the bracelets basis is sign-coherent. If C has no null-homotopic components or nugatory crossings, then the expansion in terms of the bracelets basis is positive.*

Here, *sign-coherent* means that either all terms are positive or all are negative. A *nugatory crossing* is a crossing that cuts off a null-homotopic loop.

The proof proceeds by taking a diagram with crossings and picking a crossing to resolve, being careful to avoid introducing a negative sign.

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Q-Systems, Double Bruhat Cells, and $\mathcal{N} = 2$ Yang-Mills Theory

Harold Williams

In this talk we discuss connections between cluster algebras and $\mathcal{N} = 2$ quantum field theory in a basic but rich class of examples. The relevant exchange matrices are of the form $\begin{bmatrix} 0 & C \\ -C & 0 \end{bmatrix}$ for a finite-type Cartan matrix C. These matrices appeared independently in mathematics and physics during the past decade. Mathematically they arose in the study of Q-systems as cluster algebra relations by Di Francesco and Kedem [4]. Physically, they were discovered in computations of BPS spectra of $\mathcal{N} = 2$ Yang-Mills theory [1].

A remarkable feature of these exchange matrices is that they possess a special integrable mutation sequence, which underlies the iteration of the Q-system recurrence or the spectrum generator of the corresponding Yang-Mills theory. We explain recent work clarifying this integrability by identifying these exchange matrices with those of quotients of certain Coxeter double Bruhat cells [5]. The restrictions of conjugation invariant functions form an integrable Hamiltonian system (of Toda type) on this quotient, and provide conserved quantities for the corresponding sequence of cluster transformations. This generalizes previous work of Gekhtman, Shapiro, Vainshtein in the setting of planar networks [2].

On the other hand, the spectral networks of Gaiotto, Neitzke, Moore extract cluster coordinates on moduli spaces of flat connections from Seiberg-Witten systems of certain $\mathcal{N} = 2$ field theories [3]. In these examples the Seiberg-Witten systems are (irregular) Hitchin systems, and for $\mathcal{N} = 2$ Yang-Mills this is a Hitchin system on $\mathbb{CP}^1$ with irregular singularities at 0 and ∞. We show the example of gauge group $SU(3)$, and show that the corresponding spectral network parametrizes the holonomy around the unit circle by specifying that it factors exactly like an element of a Coxeter double Bruhat cell. Thus we see that the two a priori unrelated appearances of the exchange matrices we consider are in fact due to a common underlying structure, which we can access from the point of view of spectral networks.

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Cluster duality and mirror symmetry for Grassmannians

Lauren Williams

(joint work with Konstanze Rietsch)

We consider the Grassmannian $X = Gr_{n-k}(\mathbb{C}^n)$ and a mirror dual Landau-Ginzburg model $(\check{X}, W_q : \check{X} \to \mathbb{C})$, where $\check{X}$ is the complement of a particular anti-canonical divisor in a Langlands dual Grassmannian, and W_q is the *superpotential.* Let $N = k(n-k)$. For each *reduced plabic graph* G of type (k,n) (in the sense of Postnikov), we associate a *plabic chart* $\Phi_G : ((\mathbb{C}^*)^N) \to X$ and a cluster chart $\check{\Phi}_G : ((\mathbb{C}^*)^N) \to \check{X}$. On the A-model side, we use the plabic chart Φ_G and a corresponding valuation to define a set of points in $\mathbb{Z}^N$ which are the lattice points of a convex polytope, the *Newton-Okounkov body* NO_G. On the B-model side, we use the cluster chart and the superpotential W_Q to define a polytope $Q_G \subset \mathbb{Z}^N$ as the intersection of some halfspaces. Our main result is that the two polytopes coincide, i.e. $NO_G = Q_G$.

This work builds on work of [3], [1], and our proof uses some results of [2].

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Cluster algebra structures on quantum double Bruhat cells

Milen Yakimov

(joint work with Kenneth R. Goodearl)

We give a proof of the Berenstein–Zelevinsky conjecture [2] on the existence of upper cluster algebra structures on quantized coordinate rings of double Bruhat cells and show that the corresponding quantum cluster algebras coincide with the upper quantum cluster algebras. The starting point is the following definition:

Definition 1. An iterated skew polynomial extension over an arbitrary field $\mathbb{K}$

$$R = \mathbb{K}[x_1][x_2; \sigma_2, \delta_2] \cdots [x_N; \sigma_N, \delta_N]$$

is called a *quantum nilpotent algebra* if it is equipped with a rational action of a $\mathbb{K}$-torus $\mathcal{H}$ by $\mathbb{K}$-algebra automorphisms satisfying the following conditions:

(a) The elements $x_1, \ldots, x_N$ are $\mathcal{H}$-eigenvectors.

(b) For every $2 \leq k \leq N$, δ_k is a locally nilpotent σ_k-derivation of

$$R_{k-1} := \mathbb{K}[x_1] \cdots [x_{k-1}; \sigma_{k-1}, \delta_{k-1}]$$

(c) For every $1 \leq k \leq N$, there exists $h_k \in \mathcal{H}$ such that $\sigma_k = (h_k \cdot)$ and the h_k-eigenvalue of x_k, to be denoted by λ_k, is not a root of unity.

Such an algebra will be called a *symmetric quantum nilpotent algebra* if the above conditions are satisfied for the reverse order of adjoining its generators

$$R = \mathbb{K}[x_N][x_{N-1}; \sigma^*_{N-1}, \delta^*_{N-1}] \cdots [x_1; \sigma^*_1, \delta^*_1]$$

which is essentially equivalent to an abstract form of the Levendorskii–Soibelman straightening law from quantum group theory.

A noncommutative analog of the notion of unique factorization domain was introduced by Chatters in [3]. A nonzero, non-invertible element p of a domain R is called prime if $pR = Rp$ and the factor R/Rp is a domain. A noetherian domain R is called a unique factorization domain (UFD) if every nonzero prime ideal of R contains a prime element. By [9], all quantum nilpotent algebras are (equivariant $\mathcal{H}$)-UFDs.

Theorem 2. *Every symmetric quantum nilpotent algebra R satisfying two minor technical conditions (see* [5, Theorem 8.2] *for details) possesses a canonical structure of quantum cluster algebra for which no frozen cluster variables are inverted and its initial cluster consists of the set of all homogeneous prime elements of the subalgebras $R_1, \dots, R_N$ (taken up to scalar multiples).*

Furthermore, this quantum cluster algebra aways coincides with the corresponding upper quantum cluster algebra. After an appropriate rescaling, each of the generators x_k of the algebra R is a cluster variable.

Many additional facts about the above cluster algebras are collected in [5, Theorem 8.2]. The theorem also has an analog for cluster algebra structures using a notion of Poisson UFDs, see [6, Theorem 8].

The previous approaches to the problem of the construction of (quantum) cluster algebra structures on (quantized) coordinate rings dealt with explicit families of algebras. The power of the above theorem is that it concerns an axiomatic class of algebras which contains many important families as special cases, for example the family of quantum Schubert cell algebras $\mathcal{U}^+[w]$. There is one such algebra for each Kac–Moody algebra $\mathfrak{g}$ and Weyl group element w. Theorem 2 constructs cluster algebra structures on them which extends the result of Geiß, Leclerc and Schröer [4] from the case of symmetric Kac–Moody algebra $\mathfrak{g}$.

Theorem 3. (Berenstein–Zelevinsky conjecture, [2]) *For any complex simple Lie group G and a pair of Weyl group elements (w, v), the quantum double Bruhat cell algebra $R_q[G^{w,v}]$ possesses a canonical structure of quantum cluster algebra for which all frozen cluster variables are inverted and the initial cluster consists of the Berenstein–Fomin–Zelevinsky set of quantum minors* [1, 2].

Furthermore, this quantum cluster algebra coincides with the corresponding upper quantum cluster algebra.

We briefly sketch the main idea in the proof of Theorem 3. Using results of Joseph [7], we first prove that $R_q[G^{w,v}]$ is a localization of

$$(S_w^+ \bowtie S_v^-) \# \mathbb{K}[\Delta_1^{\pm 1}, \dots, \Delta_r^{\pm 1}]$$

where S_w^+ and S_v^- are the Joseph algebras associated to w, v, see [7], and $\Delta_1, \dots, \Delta_r$ are certain quantum minors. The bicrossed and smash products are defined using the Drinfeld R-matrix commutation relations of $R_q[G]$. We then use the isomorphisms

$$S_w^+ \cong \mathcal{U}^-[w]^{\mathrm{op}} \quad \text{and} \quad S_v^- \cong \mathcal{U}^+[v]$$

from [10]. The key point is to prove that $\mathcal{U}^-[w]^{\mathrm{op}} \bowtie \mathcal{U}^+[v]$ is a symmetric quantum nilpotent algebra and to establish that the set of homogeneous prime elements in Theorem 2 is given by the Berenstein–Fomin–Zelevinsky set of quantum minors [1] (after a quantum twist isomorphism).

The notion of *maximal green sequence* of cluster mutations was introduced by Keller [8] in relations to applications to refined Donaldson–Thomas invariants and quantum dilogarithm identities. A further development of the above construction proves [11] the existence of a maximal green sequence for the cluster algebra structure on each double Bruhat cell $G^{w,v}$. The main idea in the proof is to construct such a sequence that starts from a new initial seed and contains the Berenstein–Fomin–Zelevinsky seed at an intermediate step.

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Reporters: Jan Geuenich and Charlotte Ricke

Participants

Prof. Dr. Karin Baur
Institut f. Mathematik & wissenschaftl. Rechnen
Karl Franzens Universität Graz
Heinrichstrasse 36
8010 Graz
AUSTRIA

Prof. Dr. Arkady Berenstein
Department of Mathematics
University of Oregon
Eugene, OR 97403-1222
UNITED STATES

Prof. Dr. Thomas Brüstle
Département de Mathématiques
Faculté des Sciences
Université de Sherbrooke
2500 boul de l'Universite
Sherbrooke Quebec J1K 2R1
CANADA

Dr. Aslak Bakke Buan
Department of Mathematical Sciences
NTNU
7491 Trondheim
NORWAY

Dr. Giovanni Cerulli-Irelli
Mathematisches Institut
Universität Bonn
Endenicher Allee 60
53115 Bonn
GERMANY

Prof. Dr. Frédéric Chapoton
Institut Camille Jordan
Université Lyon 1
Bat. Braconnier
21, Avenue Claude Bernard
69622 Villeurbanne Cedex
FRANCE

Dr. Ben Davison
École Polytechnique Fédérale de Lausanne
Station 8
1015 Lausanne
SWITZERLAND

Prof. Dr. Philippe Di Francesco
Department of Mathematics
University of Illinois at Urbana-Champaign
1409 West Green Street
Urbana IL 61801
UNITED STATES

Dr. Anna Felikson
Dept. of Mathematical Sciences
Durham University
Science Laboratories
Durham DH1 3LE
UNITED KINGDOM

Prof. Dr. Vladimir V. Fock
Institut de Mathématiques
Université de Strasbourg
7, rue Rene Descartes
67084 Strasbourg Cedex
FRANCE

Prof. Dr. Sergey Fomin
Department of Mathematics
University of Michigan
530 Church Street
Ann Arbor, MI 48109-1043
UNITED STATES

Prof. Dr. Christof Geiss
Instituto de Matematicas
U.N.A.M.
Circuito Exterior
Ciudad Universitaria
04510 México, D.F.
MEXICO

Prof. Dr. Michael Gekhtman
Department of Mathematics
University of Notre Dame
Notre Dame, IN 46556-5683
UNITED STATES

Jan Geuenich
Mathematisches Institut
Universität Bonn
Endenicher Allee 60
53115 Bonn
GERMANY

Dr. Thorsten Holm
Institut für Algebra, Zahlentheorie
und Diskrete Mathematik
Leibniz Universität Hannover
Welfengarten 1
30167 Hannover
GERMANY

Prof. Dr. Rei Inoue
Department of Mathematics and
Informatics, Faculty of Science
Chiba University
1-33 Yayoi-cho, Inage
Chiba-Shi 263-8522
JAPAN

Prof. Dr. Osamu Iyama
Graduate School of Mathematics
Nagoya University
Chikusa-ku, Furo-cho
Nagoya 464-8602
JAPAN

Prof. Dr. Peter Jorgensen
School of Mathematics and Statistics
The University of Newcastle
Newcastle upon Tyne NE1 7RU
UNITED KINGDOM

Prof. Dr. Alastair D. King
Department of Mathematical Sciences
University of Bath
Claverton Down
Bath BA2 7AY
UNITED KINGDOM

Dr. Daniel Labardini-Fragoso
Instituto de Matemáticas
Universidad Nacional Autónoma de
México
04510 México D.F.
MEXICO

Dr. Sefi Ladkani
Mathematisches Institut
Universität Bonn
Endenicher Allee 60
53115 Bonn
GERMANY

Lisa Lamberti
University of Oxford
Andrew Wiles Building
Radcliffe Observatory Quarter
Woodstock Road
Oxford OX2 6GG
UNITED KINGDOM

Dr. Philipp Lampe
Fakultät für Mathematik
Universität Bielefeld
Universitätsstr. 25
33615 Bielefeld
GERMANY

Prof. Dr. Bernard Leclerc
Dept. de Mathématiques
Université de Caen
14032 Caen Cedex
FRANCE

Dr. Kyungyong Lee
Department of Mathematics
Wayne State University
656 West Kirby Avenue
Detroit, MI 48202
UNITED STATES

Prof. Robert J. Marsh
School of Mathematics
University of Leeds
Leeds LS2 9JT
UNITED KINGDOM

Dr. Sophie Morier-Genoud
Institut de Mathématiques de Jussieu
Case 247
Université de Paris VI
4, Place Jussieu
75252 Paris Cedex 05
FRANCE

Dr. Gregory Muller
Department of Mathematics
University of Michigan
Ann Arbor MI 48108
UNITED STATES

Prof. Dr. Gregg Musiker
School of Mathematics
University of Minnesota
127 Vincent Hall
206 Church Street S. E.
Minneapolis MN 55455-0436
UNITED STATES

Prof. Dr. Tomoki Nakanishi
Graduate School of Mathematics
Nagoya University
Chikusa-ku
Nagoya 464-8602
JAPAN

Prof. Dr. Andrew Neitzke
Department of Mathematics
The University of Texas at Austin
1 University Station C1200
Austin, TX 78712-1082
UNITED STATES

Dr. Pierre-Guy Plamondon
Laboratoire LMNO
CNRS UMR 6139
Université de Caen
BP 5186
14032 Caen Cedex
FRANCE

Romain Ponchon
Institut de Recherche
Mathématique Avancée
Université de Strasbourg et CNRS
7, rue Rene Descartes
67084 Strasbourg Cedex
FRANCE

Dr. Fan Qin
Institut de Mathématiques
Université de Strasbourg
7, rue René Descartes
67084 Strasbourg Cedex
FRANCE

Dr. Yu Qiu
Dept. of Mathematical Sciences
Norwegian University of Science
and Technology
A. Getz vei 1
7491 Trondheim
NORWAY

Dr. Nathan Reading
Department of Mathematics
North Carolina State University
SAS Hall
Raleigh NC 27695-8205
UNITED STATES

Prof. Dr. Idun Reiten
Department of Mathematical Sciences
NTNU
7491 Trondheim
NORWAY

Prof. Dr. Vladimir Retakh
Department of Mathematics
Rutgers University
Hill Center, Busch Campus
110 Frelinghuysen Road
Piscataway, NJ 08854-8019
UNITED STATES

Charlotte Ricke
Mathematisches Institut
Universität Bonn
Endenicher Allee 60
53115 Bonn
GERMANY

Dr. Sarah Scherotzke
Mathematisches Institut
Universität Bonn
Endenicher Allee 60
53115 Bonn
GERMANY

Prof. Dr. Ralf Schiffler
Department of Mathematics
University of Connecticut
196 Auditorium Road
Storrs, CT 06269-3009
UNITED STATES

Prof. Dr. Jan Schröer
Mathematisches Institut
Universität Bonn
Endenicher Allee 60
53115 Bonn
GERMANY

Prof. Dr. Michael Shapiro
Department of Mathematics
Michigan State University
Wells Hall
East Lansing, MI 48824-1027
UNITED STATES

Prof. Dr. Yan Soibelman
Department of Mathematics
Kansas State University
Manhattan, KS 66506-2602
UNITED STATES

Prof. Dr. Hugh Thomas
Department of Mathematics
University of New Brunswick
P.O. Box 4400
Fredericton, N.B. E3B 5A3
CANADA

Prof. Dr. Dylan Thurston
Department of Mathematics
Indiana University at Bloomington
531 E. Third St.
Bloomington, IN 47405
UNITED STATES

Prof. Dr. Gordana Todorov
Department of Mathematics
Northeastern University
567 Lake Hall
Boston MA 02115-5000
UNITED STATES

Prof. Dr. Alek Vainshtein
Department of Mathematics &
Department of Computer Sciences
University of Haifa
Mount Carmel
Haifa 31905
ISRAEL

Prof. Dr. Michel van den Bergh
Department of Mathematics
Limburgs Universitair Centrum
Universitaire Campus
3590 Diepenbeek
BELGIUM

Prof. Dr. Lauren K. Williams
Department of Mathematics
University of California, Berkeley
Evans Hall
Berkeley, CA 94720-3840
UNITED STATES

Harold Williams
Department of Mathematics
University of California
Berkeley CA 94720-3840
UNITED STATES

Prof. Dr. Milen Yakimov
Department of Mathematics
Louisiana State University
Baton Rouge LA 70803-4918
UNITED STATES