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Combinatorics

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Abstract. Combinatorics is a fundamental mathematical discipline which focuses on the study of discrete objects and their properties. The current workshop brought together researchers from diverse fields such as Extremal and Probabilistic Combinatorics, Discrete Geometry, Graph theory, Combinatorial Optimization and Algebraic Combinatorics for a fruitful interaction. New results, methods and developments and future challenges were discussed. This is a report on the meeting containing abstracts of the presentations and a summary of the problem session.

Mathematics Subject Classification (2010): 05xx.

Introduction by the Organisers

The workshop Combinatorics organized by Jeff Kahn (Piscataway), Angelika Steger (Zürich), and Benny Sudakov (Zürich) was held January 5th - January 11th, 2014. Despite the early point in the year the meeting was extremely well attended with roughly 50 participants from the US, Canada, Australia, UK, Israel, and various European countries. The program consisted of 13 plenary lectures, accompanied by 17 shorter contributions and the special session of the presentations by Oberwolfach Leibniz graduate students. There was also a vivid problem session led by Andrew Thomason. The plenary lectures were intended to provide overviews of the state of the art in various combinatorial areas and/or in-depth treatments of major new results. The short talks ranged over a wide variety of topics including graph theory, coding theory, discrete geometry, extremal combinatorics, Ramsey

theory, theoretical computer science, and probabilistic combinatorics. Special attention was paid throughout to providing a platform for younger researchers to present themselves and their results.

This report contains extended abstracts of the talks and the statements of the problems that were posed during the problem session. This was a particularly successful edition of the meeting Combinatorics, in large part because of the exceptional strength and range of the results discussed. Here we mention just two of these, each of which involved spectacular progress on some well-known, longstanding conjecture. These two snapshots also provide a nice, if small, sample of the large variety of topics and methodologies that were presented during a fascinating week in Oberwolfach. We add in passing that we deliberating did not offer a long talk for one of the most spectacular – and far reaching – results in combinatorics, proved just two years ago: the so called hypergraph container theorem. While the results (and the authors!) certainly would have deserved this, we felt we would bore a large part of the audience (who likely heard reports on hypergraph containers already several times) by adding a formal talk. Thus, this result was just present at lunch and dinner tables and, indirectly, in various small talks.

Now back to the two talks that we would like to explain in a little detail. A Steiner system with parameters (n, q, r) is a set S of q-subsets of an n-set X, such that every r-subset of X belongs to exactly one element of S. The question of whether there is a Steiner system with given parameters is one of the oldest problems in combinatorics, dating back to work of Plücker (1835), Kirkman (1846) and Steiner (1853). More generally, we say that a set S of q-subsets of an n-set X is a *design* with parameters (n, q, r, λ) if every r-subset of X belongs to exactly λ elements of S.

There are some obvious necessary 'divisibility conditions' for the existence of such S, namely that $\binom{q-i}{r-i}$ divides $\lambda\binom{n-i}{r-i}$ for every $0 \leq i \leq r-1$ (fix any i-subset I of X and consider the sets in S that contain I). It is a very old Existence conjecture that the divisibility conditions are also sufficient, apart from a finite number of exceptional n given fixed q, r and λ. The case $r = 2$ has received particular attention because of its connections to statistics, under the name of 'balanced incomplete block designs'. The Existence Conjecture even for $r = 2$ was a long-standing open problem, eventually resolved by Wilson in 70's in a series of papers that revolutionized Design Theory. In his talk at the workshop Peter Keevash announced a proof of the Existence Conjecture in general, via a new method, which can be called Randomized Algebraic Constructions.

In 1912 George Birkhoff introduced a function $\chi_q(G)$, defined for all positive integers q and finite graphs G, which counts the number of proper colorings of G with q colors. It turns out that $\chi_q(G)$ is a polynomial of q with integer coefficients. This polynomial is called the chromatic polynomial of G. It appears that coefficients of the chromatic polynomial have many intriguing properties. One such property was conjectured more than 40 years ago by Read. A sequence of real numbers $a_0, a_1, \ldots, a_n$ is called *log*-concave if for all $0 < i < n$,

$$a_{i-1}a_{i+1} \leq a_i^2.$$

In 1968 Read conjectured that if $\chi_q(G) = a_n q^n - a_{n-1} q^{n-1} + \cdots + (-1)^n a_0$ is the chromatic polynomial of the graph G, then the sequence $a_0, a_1, \ldots, a_n$ is *log*-concave. At the workshop, June Huh discussed how tools from Algebraic Geometry can be used to attack such problems and presented a proof of Read's conjecture.

On behalf of all participants, the organizers would like to thank the staff and the director of the Mathematisches Forschungsinstitut Oberwolfach for providing such a stimulating and inspiring atmosphere.

Workshop: Combinatorics

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Abstracts

Sum-free subgroups

Noga Alon
(joint work with Jean Bourgain)

The study of sum and product problems in finite fields, starting with [2], motivates the investigation of additive structures in multiplicative subgroups of such fields. It is easy to see that any multiplicative subgroup of size at least $q^{3/4}$ in the finite field F_q must contain an additive relation $x + y = z$, here we show that there are infinitely many examples of sum-free multiplicative subgroups of size $\Omega(p^{1/3})$ in prime fields F_p. More complicated additive relations are studied as well. One representative result is the fact that the elements of any multiplicative subgroup H of size at least $q^{3/4+o(1)}$ of F_q can be arranged in a cyclic permutation so that the sum of any pair of consecutive elements in the permutation belongs to H. Here is a precise formulation of these results. Additional variants of the first theorem appear in the full paper.

Theorem 1. There exists an absolute positive constant c so that for any prime power q and for any multiplicative subgroup A of the finite field F_q of size

$$|A| = d \geq c\frac{q^{3/4}(\log q)^{1/2}(\log\log\log q)^{1/2}}{\log\log q}$$

there is a numbering $a_0, a_1, \ldots, a_{d-1}$ of the elements of A so that $a_i + a_{i+1} \in A$ for all i, where the indices are reduced modulo d.

Theorem 2. For any positive integer d which is not divisible by 6 there is a set $E(d)$ of at most $\frac{d^2}{\log d}$ primes so that for any prime $p \equiv 1(\text{mod } d)$ which does not belong to $E(d)$, there are no x, y, z in the subgroup H of cardinality d of Z_p^* that satisfy $x + y = z$ (equivalently, there are no $x', y' \in H$ with $x' + y' = 1$.) Every member of $E(d)$ is smaller than 3^d and in addition the following holds.

(i) For most values of d that are not divisible by 6 the smallest prime $p \equiv 1(\text{mod } d)$, $p \notin E(d)$ satisfies $p \leq O(d^3)$. The same assertion holds for most primes d. Here "most d" means that the proportion of values of d in the range $[D, 2D]$ for which the above assertion fails is $o(D)$.

(ii) For every integer d which is not divisible by 6 the smallest prime $p \equiv 1(\text{ mod } d)$, $p \notin E(d)$ satisfies $p \leq O(d^5)$.

(iii) There is a fixed $\epsilon > 0$ so that for every integer d which is not divisible by 6 and does not have a prime factor that exceeds d^ϵ, the smallest prime $p \equiv 1(\text{mod } d)$, $p \notin E(d)$ satisfies $p \leq O(d^3)$.

The proofs combine combinatorial techniques based on the spectral properties of Cayley sum-graphs, whose eigenvalues can be expressed as character sums (see, e.g., [1]), with tools from algebraic and analytic number theory including some

properties of integers in cyclotomic fields and results and techniques applied in the study of the Linnik problem.

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Tverberg plus constraints

Günter M. Ziegler
(joint work with Pavle V. M. Blagojević and Florian Frick)

Tverberg's celebrated theorem from 1966 [7], its topological version by Bárány, Shlosman & Szűcs [1] and Özaydin [4] 1981/1987, and the 2009 optimal colored version by Blagojević, Matschke & Ziegler [3], may together be summarized as follows:

Theorem (Tverberg [7]; topological Tverberg [1] [4]; optimal colored Tverberg [3])**.** *Let $d \geq 1$ and $r \geq 2$, and let $N \geq N_0 := (r-1)(d+1)$.*

(1) *For any affine map $f : \Delta_N \to \mathbb{R}^d$ the N-dimensional simplex Δ_N contains r points $x_1, \ldots, x_r$ that lie in r vertex-disjoint faces $\sigma_1, \ldots, \sigma_r$ of Δ whose images coincide: $f(x_1) = \cdots = f(x_r)$.*
(2) *If $r \geq 2$ is a prime power, then this holds more generally for arbitrary continuous maps $f : \Delta_N \to \mathbb{R}^d$.*
(3) *If $r \geq 2$ is a prime, and $f : \Delta_N \to \mathbb{R}^d$ a continuous map, then the faces σ_i may in addition be required to have distinct vertex colors (that is, to be "rainbow faces") for any coloring of the vertex set of Δ_N whose color classes have size at most $r-1$.*

In our lecture we presented a new, simple and elementary, proof technique (see [2]) that establishes many of the known extensions of these theorems for maps of a simplex Δ_N *of higher dimension* N directly from these *optimal* results for $N = N_0$. Indeed, we even obtain sharpened and improved results from this.

Our new technique relies on a concept of "Tverberg unavoidable subcomplexes":

Definition 1 (Tverberg unavoidable subcomplexes)**.** Let $r \geq 2$, $d \geq 1$, and N be integers, and let $f : \Delta_N \to \mathbb{R}^d$ be a continuous map with at least one Tverberg r-partition. Then a subcomplex $\Sigma \subseteq \Delta_N$ is *Tverberg unavoidable* if for every Tverberg partition $\{\sigma_1, \ldots, \sigma_r\}$ for f there is at least one face σ_j that lies in Σ.

Examples of Tverberg unavoidable complexes are obtained using the pigeonhole principle. For example, for any set S of at most $2r-1$ vertices in Δ_N the subcomplex of faces with at most one vertex in S is Tverberg unavoidable. And if $r(k+2) > N+1$, then the k-skeleton $\Delta_N^{(k)}$ of Δ_N is Tverberg unavoidable.

If N is large enough, then we can require points of Tverberg coincidence to equalize additional constraint functions, via maps to an extended target space:

Lemma 1 (Key lemma #1)**.** Let r be a prime power, $d \geq 1$, and $c \geq 0$. Let $N \geq N_c := (r-1)(d+1+c)$ and let $f : \Delta_N \to \mathbb{R}^d$ and $g : \Delta_N \to \mathbb{R}^c$ be continuous. Then there are r points $x_i \in \sigma_i$, where $\sigma_1, \ldots, \sigma_r$ are pairwise disjoint faces of Δ_N with $g(x_1) = \cdots = g(x_r)$ and $f(x_1) = \cdots = f(x_r)$.

Using the symmetry of the problem, we now obtain a Tverberg partition that does not only have one, but *all* of its faces in the unavoidable subcomplex:

Lemma 2 (Key lemma #2)**.** Let r be a prime power, $d \geq 1$, and $N \geq N_1 = (r-1)(d+2)$. Assume that $f : \Delta_N \to \mathbb{R}^d$ is continuous and that the subcomplex $\Sigma \subseteq \Delta_N$ is Tverberg unavoidable for f. Then there are r pairwise disjoint faces $\sigma_1, \ldots, \sigma_r$ of Δ_N, all of them contained in Σ, such that $f(\sigma_1) \cap \cdots \cap f(\sigma_r) \neq \emptyset$.

For example, our Ansatz produces directly from the topological Tverberg theorem a colored version that is stronger than Živaljević & Vrećica's 1992 "colored Tverberg's theorem"? [9] that is valid for all prime powers r:

Theorem 1 (Weak colored Tverberg)**.** Let r be a prime power, $d \geq 1$, $N \geq N_{d+1} = (r-1)(2d+2)$ and let $f : \Delta_N \to \mathbb{R}^d$ be continuous. If the vertices of Δ_N are colored by $d+1$ colors, where each color class has cardinality at most $2r-1$, then there are r pairwise disjoint rainbow faces $\sigma_1, \ldots, \sigma_r$ of Δ_N such that $f(\sigma_1) \cap \cdots \cap f(\sigma_r) \neq \emptyset$.

Similarly we obtain directly from the topological Tverberg theorem a strengthened version of the "generalized van Kampen–Flores theorem"? of Sarkaria [5] and Volovikov [8] from 1991/1996:

Theorem 2 (Generalized van Kampen–Flores)**.** Let r be a prime power, $d \geq 1$, $N \geq N_1 = (r-1)(d+2)$, and $k \geq \lceil \frac{r-1}{r} d \rceil$. Then for every continuous $f : \Delta_N \to \mathbb{R}^d$ there are r pairwise disjoint faces $\sigma_1, \ldots, \sigma_r$ of Δ_N, with $\dim \sigma_i \leq k$ for $1 \leq i \leq r$, such that $f(\sigma_1) \cap \cdots \cap f(\sigma_r) \neq \emptyset$.

As another example, our machinery reproves Soberón's 2013 "Tverberg theorem with equal barycentric coordinates"? and at the same time produces a new topological version of this result.

Details appear in our paper [2].

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Two approaches to Sidorenko's conjecture

Choongbum Lee
(joint work with Jeong Han Kim, Joonkyung Lee)

In this talk, we study a beautiful conjecture of Sidorenko [12] on a correlation inequality related to bipartite graphs. The conjecture states that, for every bipartite graph H on $\{1, 2, \cdots, k\}$,

$$\int \prod_{(i,j)\in E(H)} h(x_i, y_j) d\mu^{|V(H)|} \geq \left(\int h(x,y)\, d\mu^2 \right)^{|E(H)|} \tag{1}$$

holds, where μ is the Lebesgue measure on $[0,1]$ and h is a bounded, non-negative, symmetric, measurable function on $[0,1]^2$.

Sidorenko [12, 13] noted that the functional on the left-hand side of the correlation inequality (1) often appears in various fields of science: Feynman integrals in quantum field theory [16], Mayer integrals in classical statistical mechanics, and multicenter integrals in quantum chemistry [3].

An equivalent discrete form expresses the conjecture in terms of graph homomorphisms. For two graphs H and G, a *homomorphism* from H to G is a mapping $g : V(H) \to V(G)$ such that $\{g(v), g(w)\}$ is an edge in G whenever $\{v, w\}$ is an edge in H. Let $Hom(H, G)$ denote the set of all homomorphisms from H to G, and let $t_H(G)$ be the probability that a uniform random mapping from H to G is a homomorphism, i.e.,

$$t_H(G) = \frac{|Hom(H,G)|}{|V(G)|^{|V(H)|}}.$$

The discrete form of Sidorenko's conjecture states that for every bipartite graph H,

$$t_H(G) \geq t_{K_2}(G)^{|E(H)|} \quad \text{for all graphs } G. \tag{2}$$

If G is the Erdős-Rényi random graph $G(n,p)$, the mean of $t_H(G(n,p))$ is $p^{|E(H)|}$ plus an error term bounded from above by $|V(H)|^2/n$. Thus (2) roughly asserts that $t_H(G)$ is minimized when G is a random graph.

Erdős and Simonovits [4, 15] made a similar conjecture for bipartite graphs in 1984. They conjectured that if H is a bipartite graph, then there exists a positive constant c_H depending only on H such that

$$t_H(G) \geq c_H t_{K_2}(G)^{|E(H)|} \tag{3}$$

for all graphs G (their conjecture was originally stated in terms of injective homomorphisms but is equivalent to this form). It turns out that this conjecture is equivalent to Sidorenko's conjecture, as Sidorenko himself showed in [12] using a tensor power trick.

Sidorenko's conjecture is known to be true only for a few bipartite graphs H. We say that a bipartite graph H has *Sidorenko's property* if (2) holds for all graphs G. That paths have Sidorenko's property [1, 9] was proved around 1960, earlier than Sidorenko suggested the conjecture. Sidorenko himself [12] showed that trees, even cycles, and complete bipartite graphs have Sidorenko's property. He also proved that, for a bipartite graph H with bipartition $A \cup B$, H has Sidorenko's property if $|A| \leq 4$. Recently, Hatami [7] proved that hypercubes have Sidorenko's property by developing a concept of norming graphs. He proved that every norming graph has Sidorenko's property, and that all hypercubes are norming graphs. Conlon, Fox, and Sudakov [2] proved that if H is a bipartite graph with a bipartition $A \cup B$ and there is a vertex in A adjacent to all vertices in B, then H has Sidorenko's property. Sidorenko [12] and Li and Szegedy [17] introduced some recursive processes that construct a new graph from a collection of graphs so that the new one has Sidorenko's property whenever all the graphs in the collection have the property. Li and Szegedy [17] introduced some recursive processes that construct a new graph from a collection of graphs so that the new one has Sidorenko's property whenever all the graphs in the collection have the property. On the other hand, the simplest graph not known to have Sidorenko's property is $K_{5,5} \setminus C_{10}$, a 3-regular graph on 10 vertices.

In this talk, we further study Sidorenko's conjecture by taking two different approaches. The first approach uses normalizations by certain conditional expectations and Jensen's inequality for logarithmic functions. This approach is partly motivated by Li and Szegedy [17]. For a bipartite graph H with bipartition $A \cup B$, H is tree-arrangeable if the family of neighborhoods of vertices in A has a certain tree-like structure. To be more precise, a bipartite graph H is tree-arrangeable if there exists a tree T on A such that for each $v \in B$, the set $N_H(v)$ induces a subtree of T. We show that all tree-arrangeable bipartite graphs have Sidorenko's property. For instance, if there is a vertex in A adjacent to all vertices in B, then H is tree-arrangeable with a star as the corresponding tree. Hence our result generalizes the result of Conlon, Fox, and Sudakov [2].

Second, we develop a recursive procedure that preserves Sidorenko's property. For two graphs H_1 and H_2, let the *Cartesian product* $H_1 \square H_2$ (also known as the

box product) be the graph over the vertex set $V(H_1)\times V(H_2)$ such that two vertices (u_1, u_2) and (v_1, v_2) are adjacent if and only if (i) u_1 and v_1 are adjacent in H_1 and $u_2 = v_2$, or (ii) u_2 and v_2 are adjacent in H_2 and $u_1 = v_1$. We prove that if T is a tree and H is a bipartite graph with Sidorenko's property, then $T\Box H$ also has Sidorenko's property. Since paths of all lengths are known to have Sidorenko's property, by repeatedly applying our theorem with paths $P_1, P_2, \cdots, P_d$ of various lengths, we obtain that the d-dimensional grid $P_1\Box P_2\Box\cdots\Box P_d$ has Sidorenko's property. This approach especially yields a simple proof of the statement that the hypercube $K_2\Box K_2\Box\cdots\Box K_2$ satisfies Sidorenko's property, which was first proven by Hatami [7].

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Probabilistic existence of regular combinatorial structures

Ron Peled

We show the existence of regular combinatorial objects which previously were not known to exist. Specifically, for a wide range of the underlying parameters, we show the existence of non-trivial orthogonal arrays, t-designs, and t-wise permutations. In all cases, the sizes of the objects are optimal up to polynomial overhead. The proof of existence is probabilistic. We show that a randomly chosen structure has the required properties with positive yet tiny probability. Our method allows also to give rather precise estimates on the number of objects of a given size and this is applied to count the number of orthogonal arrays, t-designs and regular hypergraphs. The main technical ingredient is a special local central limit theorem for suitable lattice random walks with finitely many steps.

Blocks, profiles and tangles

Reinhard Diestel
(joint work with Johannes Carmesin, Fabian Hundertmark, Maya Stein)

Considering systems of separations in a graph that separate every pair of a given set of vertex sets that are themselves not separated by these separations, we determine conditions under which such a separation system contains a nested subsystem that still separates those sets and is invariant under the automorphisms of the graph.

As an application, we show that the k-blocks – the maximal vertex sets that cannot be separated by at most k vertices – of a graph G live in distinct parts of a suitable tree-decomposition of G of adhesion at most k, whose decomposition tree is invariant under the automorphisms of G. This extends recent work of Dunwoody and Krön and, like theirs, generalizes a similar theorem of Tutte for $k = 2$.

Under mild additional assumptions, which are necessary, our decompositions can be combined into one overall tree-decomposition that distinguishes, for all k simultaneously, all the k-blocks of a finite graph. These results are described in [4].

These results were extended by Hundertmark [6] to existence proofs of canonical tree-decompositions distinguishing dense objects – so-called *profiles* – in more general combinatorial structures. These profiles generalize both blocks and tangles in graphs and matroids. Hundertmark's theorem thus extends and strengthens the fundamental (non-canonical) tree-decomposition theorem of Robertson and Seymour [7] that distinguishes all the maximal tangles in a graph, as well as its analogue for matroids recently obtained by Geelen, Gerards and Whittle [5].

In [1, 2] we explore these decompositions further. In particular, we investigate what we can say about their parts: whereas in Tutte's theorem, the case $k = 3$ of our result, the 3-blocks of a graph make up the entire part of the tree-decomposition containing them, this is no longer the case for $k \geq 4$. Moreover, the canonicity of our decompositions forces them to have parts that contain no

k-block or tangle at all.[1] In order to describe these parts, or indeed the portion of any part outside the k-block it may contain, we need to understand what other properties force a graph to contain a k-block, or either a k-block or a tangle of order k. More generally, we would like to understand better how

$$\beta(G) := \max\{\, k \mid G \text{ contains a } k\text{-block} \,\}$$

interacts with other graph invariants. Some tentative beginnings of such an investigation are made in [3].

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Cycle packing

David Conlon

(joint work with Jacob Fox and Benny Sudakov)

Packing and covering problems have a rich history in graph theory and many of the oldest and most intensively studied topics in this area (see [6]) relate to packings and coverings with paths and cycles. For example, in 1968, Lovász [5] proved the following fundamental result about decompositions or edge-partitions of a graph into paths and cycles.

Theorem 1 (Lovász)**.** Every graph on n vertices can be decomposed into at most $\frac{n}{2}$ paths and cycles.

Theorem 1 easily implies that there is a decomposition of any graph on n vertices into $n-1$ paths. This was subsequently improved [1, 7] to $\lfloor \frac{2n}{3} \rfloor$ paths, a result which is sharp for a disjoint union of triangles. However, Lovász' original motivation for studying such decompositions, a problem of Gallai which asks whether every connected graph on n vertices can be decomposed into $\lfloor \frac{n+1}{2} \rfloor$ paths, remains open. The following old and well-known conjecture of Erdős and Gallai [2, 4] concerning the analogous question for cycles also remains open.

[1] For example, a star has many path-decompositions into K^2s. But any canonical tree-decomposition that distinguishes all its K^2s, i.e. one that is invariant under the automorphisms of the star, has to have the centre of the star as a central singleton part.

Conjecture 1 (Erdős-Gallai)**.** Every graph on n vertices can be decomposed into $O(n)$ cycles and edges.

Let $f(n)$ be the minimum number such that every graph on n vertices can be decomposed into at most $f(n)$ cycles and edges. The Erdős-Gallai conjecture states that $f(n) = O(n)$. An example of Gallai (see [4]) shows that $f(n) \geq (\frac{4}{3} - o(1))n$ and Erdős [2] later remarks that there is an example showing that $f(n) \geq (\frac{3}{2} - o(1))n$. As noted in [4], it is easy to see that $f(n) = O(n \log n)$. Indeed, Erdős and Gallai [3] showed that every graph with n vertices and $m > \ell(n-1)/2$ edges contains a cycle of length at least ℓ. By greedily removing cycles of longest length, we see, after removing $O(n)$ cycles, that the graph that remains will be acyclic or have at most half the edges. The bound $f(n) = O(n \log n)$ follows from a simple iteration. Here we make the first progress on the Erdős-Gallai conjecture, showing that $f(n) = O(n \log \log n)$. This is a corollary of the following stronger result.

Theorem 2. Every graph on n vertices with average degree d can be decomposed into $O(n \log \log d)$ cycles and edges.

We will also prove the Erdős-Gallai conjecture in certain special cases. In particular, we may exploit the fact that random graphs are good expanders to prove that they satisfy the Erdős-Gallai conjecture. The random graph $G(n,p)$ on vertex set $[n] = \{1, \ldots, n\}$ is constructed by taking each potential edge independently with probability p. We say that $G(n,p)$ possesses a property $\mathcal{P}$ asymptotically almost surely, or a.a.s. for short, if the probability that $G(n,p)$ possesses $\mathcal{P}$ tends to 1 as n grows to infinity.

Theorem 3. There exists a constant $c > 0$ such that for any probability $p := p(n)$ the random graph $G(n,p)$ a.a.s. can be decomposed into at most cn cycles and edges.

Building on the ideas used to prove the Erdős-Gallai conjecture in random graphs, we also prove the Erdős-Gallai conjecture for graphs of linear minimum degree.

Theorem 4. Every graph G on n vertices with minimum degree cn can be decomposed into at most $O(c^{-12}n)$ cycles and edges.

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Counting connected hypergraphs via the probabilistic method

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(joint work with Béla Bollobás)

For $n \geq r \geq 2$ and $0 < p < 1$, let $H^r_{n,p}$ denote the random r-uniform hypergraph with vertex set $[n] = \{1, 2, \ldots, n\}$ in which each of the $\binom{n}{r}$ possible hyperedges is present independently with probability p. This is the natural hypergraph generalization of the (binomial) Erdős–Rényi random graph model $G_{n,p}$, which is simply the special case $r = 2$. When studying the phase transition in $G_{n,p}$ it is convenient to write $p = \lambda/n$; in this more general setting, with $r \geq 2$ fixed the analogous normalization is to write

$$p = p(n) = \lambda(r-2)!n^{-r+1}$$

where $\lambda = \lambda(n)$. Schmidt-Pruzan and Shamir [12] showed that the critical value of λ above which a giant component emerges is then $\lambda = 1$. For $\lambda > 1$ let

$$\rho_{r,\lambda} = 1 - (1 - \rho_\lambda)^{1/(r-1)},$$

where ρ_λ is the survival probability of a Poisson Po(λ) Galton–Watson branching process, i.e., ρ_λ is the positive solution to $1 - \rho_\lambda = e^{-\lambda\rho_\lambda}$. Writing $L_1(H)$ for the number of vertices in the largest component of a hypergraph H (chosen according to any rule if there is a tie), Behrisch, Coja-Oghlan and Kang [2] showed that for $\lambda > 1$ constant, $L_1(H^r_{n,p})$ is concentrated around $\rho_{r,\lambda}n$. In fact, they proved much more: both central and local limit theorems for L_1. Then, using a remarkable trick relating $H^r_{n,p}$ to the 'size' model where the number of edges is fixed, and considering the number of edges *outside* the giant component, they managed in [3] to use these univariate results to deduce a bivariate local limit theorem for $L_1(H^r_{n,p})$ and $M_1(H^r_{n,p})$, the number of edges in the giant component. This local limit result is essentially equivalent to an asymptotic formula for the number of connected hypergraphs with a given number of vertices, and a given 'excess' or 'nullity' (roughly speaking, number of extra 'overlaps' compared to a tree) which is of the same order as the number of vertices.

Earlier, Karoński and Łuczak [6, 7] had proved enumerative results for connected hypergraphs with much smaller excess, and then used these to study the phase transition in $H^r_{n,p}$ *just* above the window, when $\lambda = 1 + \varepsilon$ where $\varepsilon = \varepsilon(n)$ satisfies $\varepsilon^3 n \to \infty$ but more slowly than $\log n/\log\log n$. Here we close the gap between these results, proving bivariate central and local limit theorems for $L_1(H^r_{n,p})$ and $M_1(H^r_{n,p})$ throughout the supercritical regime, i.e., when $\lambda = 1 + \varepsilon$ with $\varepsilon = o(1)$ and $\varepsilon^3 n \to \infty$. This leads to an asymptotic formula for the number of connected hypergraphs with a given number n of vertices and given excess ℓ, whenever $\ell = o(n)$ and $\ell \to \infty$.

The central limit theorem is proved by extending the univariate results proved in [4, 5], themselves based on exploration, random walk and martingale arguments due to Martin-Löf [9], Karp [8], Aldous [1] and Nachmias and Peres [10]. We deduce the local limit theorem by probabilistic smoothing techniques. Some of these smoothing techniques are reminiscent of those used by Behrisch, Coja-Oghlan and Kang, but applied in a very different way. The analogous result for the special case of graphs was essentially proved by Pittel and Wormald [11], using very different methods.

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A Short Proof of Gowers' Lower Bound for the Regularity Lemma

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(joint work with Guy Moshkovitz)

Szemerédi's regularity lemma asserts that every graph can be partitioned into a bounded number of vertex sets $Z_1, \ldots, Z_k$, so that the graphs between almost all pairs (Z_i, Z_j) behave "randomly". More precisely, given two vertex sets in a graph G let $d_G(A,B) = e(A,B)/|A||B|$ where $e(A,B)$ is the number of edges in G with one vertex in A and the other in B. We say that the pair (A,B) is ϵ-regular if $|d_G(A,B) - d_G(A',B')| \leq \epsilon$ for all $A' \subseteq A$ and $B' \subseteq B$ satisfying $|A'| \geq \epsilon|A|$ and $|B'| \geq \epsilon|B|$. A partition $\mathcal{Z} = \{Z_1, \ldots, Z_k\}$ of the vertex set of a graph is called an *equipartition* if all the sizes of the sets Z_i differ by at most 1.

The *order* of an equipartition $\mathcal{Z}$, denoted $|\mathcal{Z}|$, is the number of sets in it (k above). An equipartition $\mathcal{Z}$ is ϵ-regular if all but ϵk^2 of the pairs (Z_i, Z_j) are ϵ-regular[1]. Szemerédi's regularity lemma then states the following.

Theorem 3 (Szemerédi [5])**.** For every $\epsilon > 0$ there is $M = M(\epsilon)$ so that every graph has an ϵ-regular equipartition of order at most M.

Despite its apparent simple statement (and proof) the regularity lemma has become one of the most widely used tools in extremal graph theory, as well as in many other fields (see [4] for a survey). Unfortunately, the proof in [5] only showed that $M(\epsilon) \leq \mathrm{twr}(O(1/\epsilon^5))$ where $\mathrm{twr}(x)$ is a tower of exponents of height x. Hence, the numerous applications of the lemma were all of asymptotic nature and supplied very weak effective bounds.

For a long time it was not known whether the tower-type bound for $M(\epsilon)$ was unavoidable until Gowers proved [3] that (surprisingly) this is indeed the case. Gowers' paper contained two proofs. The "first" proof used a simple construction with a short proof of correctness, but it only showed that $M(\epsilon) \geq \mathrm{twr}(c\log(1/\epsilon))$. The "second" proof established the much stronger bound $M(\epsilon) \geq \mathrm{twr}(1/\epsilon^c)$ thus showing that $M(\epsilon)$ indeed grows as a tower of exponents of height polynomial in $1/\epsilon$ ([3] obtains $c = 1/16$). However, the second proof of the stronger bound used a far more complicated construction with a significantly more involved proof of correctness, and was dubbed a tour-de-force in the laudatio to Gowers' Fields medal [1]. Conlon and Fox [2] gave another proof of the fact that $M(\epsilon) \geq \mathrm{twr}(1/\epsilon^c)$ (with $c = 1$), but their proof was equally involved.

While the proof of Gowers' first construction used an inductive approach, in the concluding remarks to his paper [3] he explained that "the proof for the second construction is so much more complicated than the proof for the first" since one cannot use a similar inductive approach in the second construction. Our main contribution here is a new proof that $M(\epsilon) \geq \mathrm{twr}(1/\epsilon^c)$ (we obtain $c = 1/6$). At a high level, our proof is almost identical to the first proof in [3] using a very similar inductive approach. However, the proofs differ is several subtle aspects which make it possible to execute the inductive argument $1/\epsilon^c$ times and not only $\log(1/\epsilon)$ times as in the first proof of [3]. We finally note that the second construction in [3] as well as the one in [2] prove lower bounds for weaker versions of the regularity lemma. It would be interesting to see if one could use the ideas in our new proof to give simple proofs of comparable lower bounds for weaker versions of the regularity lemma.

Let us say that an equipartition $\mathcal{Z} = \{Z_1, \ldots, Z_k\}$ is ϵ-*nice* if for every $Z \in \mathcal{Z}$ all but ϵk of the sets $Z' \in \mathcal{Z}$ are such that (Z, Z') is ϵ-regular. Let $M'(\epsilon)$ be so that every graph has an ϵ-nice equipartition of order at most $M'(\epsilon)$. It is a well-known (and easy) observation that $M'(\epsilon) \leq M(\epsilon^3)$. Hence, to prove that $M(\epsilon) \geq \mathrm{twr}(1/\epsilon^{1/6})$ it would suffice to prove the following.

[1]Note that we do not require the sets Z_i, Z_j in the pairs (Z_i, Z_j) to be distinct. Therefore, we do not need a lower bound on the order of the partition in Theorem 3.

Theorem 4. There is a constant $c > 0$ so that $M'(\epsilon) \geq \mathrm{twr}(c/\epsilon^{1/2})$ for every $0 < \epsilon < c$.

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The tropical Laplacian and unimodality conjectures

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(joint work with Eric Katz)

Let G be a geometric graph with vertices $\mathbf{v}_1, \ldots, \mathbf{v}_k$ in $\mathbb{R}^n$. A positive edge weight w for G is said to be *balanced* if there is a number d_i satisfying

$$d_i \mathbf{v}_i = \sum_{i \sim j} w_{ij} \mathbf{v}_j$$

for each vertex $\mathbf{v}_i$, where the sum is over all neighbors of the vertex $\mathbf{v}_i$ in G. A geometric graph may not admit any balanced weight, but many interesting ones do. For example, the graph of a convex polytope admits a balanced weight.

Definition 2. Let w be a balanced edge weight for a geometric graph G with vertices $\mathbf{v}_1, \ldots, \mathbf{v}_k$. The *tropical Laplacian* of w is the $k \times k$ matrix L with entries

$$L_{ij} = \begin{cases} d_i, & \text{if } i \text{ is } j, \\ -w_{ij}, & \text{if } ij \text{ is an edge}, \\ 0, & \text{if } ij \text{ is not an edge}, \end{cases}$$

where

$$d_i \mathbf{v}_i = \sum_{i \sim j} w_{ij} \mathbf{v}_j.$$

In our language, the paper [Lov01] shows that, among other things, the graph of a three dimensional convex polytope admits a balanced weight whose tropical Laplacian has exactly one negative eigenvalue. The same statement remains valid for convex polytopes of any dimension.

At the workshop, I stated that the analogous result for matroids implies several unimodality conjectures. The analogy is most claerly visible in tropical geometric framework, but it is not difficult to formulate the needed statement in elementary terms.

Let $\mathbf{e_1}, \ldots, \mathbf{e}_n$ be the standard basis of $\mathbb{R}^n$. To each subset F of $\{0, 1, \ldots, n\}$, we associate the vector

$$\mathbf{e}_F := \sum_{i \in F} \mathbf{e}_i,$$

where

$$\mathbf{e}_0 := -\sum_{i=1}^{n} \mathbf{e}_i.$$

Let M be a matroid of rank $r+1$ on the ground set $\{0, 1, \ldots, n\}$.

Definition 3. The graph of M is the following geometric graph in $\mathbb{R}^n$:

$$\begin{array}{rcl} \big\{\text{vertices}\big\} & := & \big\{\mathbf{e}_F \mid F \text{ is a flat of } M \text{ of rank } r \text{ or } r-1\big\}, \\ \big\{\ \text{edges}\ \big\} & := & \big\{\overline{\mathbf{e}_{F_1}\mathbf{e}_{F_2}} \mid F_1 \subsetneq F_2\big\}. \end{array}$$

We define the edge weight w of the graph by

$$w(\overline{\mathbf{e}_{F_1}\mathbf{e}_{F_2}}) = \big|\mu(F_1)\big|, \quad F_1 \subsetneq F_2,$$

where μ is the Möbius function of the lattice of flats of M.

Using the matroid axioms, one can check that the edge weight is balanced for the graph of M. This defines the tropical Laplacian of a matroid.

Conjecture 2. The tropical Laplacian of a matroid has exactly one negative eigenvalue.

Using methods of [HK12], one can show that the conjecture is true for all matroids which are representable over some field. We have verified the conjecture for all rank 3 matroids, and for all matroids on a ground set of size at most 9. The above conjecture implies, for example, Conjectures 15.2.1 and 15.2.7 of [Oxl11].

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The Green-Tao theorem and a relative Szemerédi theorem

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(joint work with David Conlon and Jacob Fox)

The celebrated Green-Tao theorem [4] states that the primes contain arbitrarily long arithmetic progressions. The main technical result in their work is a relative Szemerédi theorem. Szemerédi's theorem [5] states that any subset of the integers with positive upper density contains arbitrarily long arithmetic progressions. A relative Szemerédi theorem is a result where the ground set is no longer $\mathbb{Z}$ but some sparse pseudorandom subset (or more generally some measure).

Green and Tao proved a relative Szemerédi theorem where the ground set satisfies certain pseudorandomness conditions known as the linear forms condition and the correlation condition. They then constructed a majorizing measure to the primes, using ideas from the work of Goldston and Yıldırım [3], so that this majorizing measure satisfies the desired pseudorandomness conditions.

In our work [1, 8], we significantly simplify the proof of the Green-Tao theorem. We show that a relative Szemerédi theorem holds under much weaker pseudorandomness hypotheses. Namely, we only need a weaker version of Green-Tao's linear forms condition, and we completely get rid of the need for a correlation condition.

Here is a weighted formulation of Szemerédi's theorem (it is equivalent to the usual statement about subsets of $\mathbb{Z}$ with positive density).

Theorem 1 (Szemerédi's theorem, weighted version)**.** Let $k \geq 3$ and $0 < \delta \leq 1$ be fixed. Let $f\colon \mathbb{Z}_N \to [0,1]$ be a function satisfying $\mathbb{E}[f] \geq \delta$. Then

$$\mathbb{E}_{x,d\in\mathbb{Z}_N}[f(x)f(x+d)f(x+2d)\cdots f(x+(k-1)d)] \geq c(k,\delta) - o_{k,\delta}(1) \tag{1}$$

for some constant $c(k,\delta) > 0$ which does not depend on f or N.

For a relative Szemerédi theorem, we have a pseudorandom majorizing measure $\nu\colon \mathbb{Z}_N \to [0,\infty)$, and $f\colon \mathbb{Z}_N \to [0,\infty)$ is assumed to satisfy $0 \leq f \leq \nu$ pointwise. Here the function ν is normalized so that $\mathbb{E}[\nu] = 1 + o(1)$. For instance, one can think of ν as $\frac{N}{|S|}1_S$ for some pseudorandom subset $S \subseteq \mathbb{Z}_N$, and f as $1_A\nu$ with some $A \subseteq S$.

Definition 4 (Linear forms condition)**.** A nonnegative function $\nu = \nu^{(N)} : \mathbb{Z}_N \to \mathbb{R}_{\geq 0}$ is said to obey the *k-linear forms condition* if one has

$$\mathbb{E}_{x_1^{(0)},x_1^{(1)},\ldots,x_k^{(0)},x_k^{(1)}\in\mathbb{Z}_N}\Big[\prod_{j=1}^{k}\prod_{\omega\in\{0,1\}^{[k]\setminus\{j\}}}\nu\Big(\sum_{i=1}^{k}(i-j)x_i^{(\omega_i)}\Big)^{n_{j,\omega}}\Big] = 1 + o(1) \tag{2}$$

for any choice of exponents $n_{j,\omega} \in \{0,1\}$.

Example 1. For $k = 3$, condition (2) says that

$$\begin{aligned}\mathbb{E}_{x,x',y,y',z,z'\in\mathbb{Z}_N}[&\nu(y+2z)\nu(y'+2z)\nu(y+2z')\nu(y'+2z')\cdot\\ &\cdot\nu(-x+z)\nu(-x'+z)\nu(-x+z')\nu(-x'+z')\cdot\\ &\cdot\nu(-2x-y)\nu(-2x'-y)\nu(-2x-y')\nu(-2x'-y')] = 1+o(1)\end{aligned} \tag{3}$$

and similar conditions hold if one or more of the twelve ν factors in the expectation are erased.

Observe that (3) is the density of $K_{2,2,2}$ in some tripartite graph constructed from ν.

Here is our main result. (This version is from [8]. In [1] we prove the same result without the additional conclusion that $c(k,\delta)$ can be made the same as in Theorem 1.)

Theorem 2 (Relative Szemerédi theorem)**.** Let $k \geq 3$ and $0 < \delta \leq 1$ be fixed. Let $\nu\colon \mathbb{Z}_N \to [0,\infty)$ satisfy the k-linear forms condition. Assume that N is sufficiently large and relatively prime to $(k-1)!$. Let $f\colon \mathbb{Z}_N \to [0,\infty)$ satisfy $0 \leq f(x) \leq \nu(x)$ for all $x \in \mathbb{Z}_N$ and $\mathbb{E}[f] \geq \delta$. Then

$$\mathbb{E}[f(x)f(x+d)f(x+2d)\cdots f(x+(k-1)d)|x,d \in \mathbb{Z}_N] \geq c(k,\delta) - o_{k,\delta}(1), \tag{4}$$

where $c(k,\delta)$ is the same constant which appears in Theorem 1. The rate at which the $o_{k,\delta}(1)$ term goes to zero depends not only on k and δ but also the rate of convergence in the k-linear forms condition for ν.

Here is an overview of the proof. Starting with $0 \leq f \leq \nu$, by a *dense model theorem*, one can approximate f by some $\tilde{f}\colon \mathbb{Z}_N \to [0,1]$. Note that f is bounded by 1, so it is a dense object. The dense model $\tilde{f}$ is a good approximation of f in terms of a certain cut norm, which we won't define here. A counting lemma (which is the main new technical advance in our work) implies that the k-AP count in f must be similar to the k-AP count in $\tilde{f}$. By Theorem 1, the k-AP count in $\tilde{f}$ can be bounded from below by $c(k,\delta) - o(1)$, and thus the same must be true for the k-AP count in f due to the counting lemma.

This is the same strategy taken in the original Green-Tao proof. However, instead of approximating f by $\tilde{f}$ in terms of a cut norm, they rely on a notion based on the Gowers uniformity norm. By using the cut norm, we obtain a cheaper dense model theorem, at the cost of a trickier counting lemma. This simplifies the proof significantly.

The strategy described above was the approach taken in [8]. A different approach was taken in our earlier paper [1] where we transfer the hypergraph removal lemma to obtain a sparse relative hypergraph removal lemma, from which we deduce the relative Szemerédi theorem (this is similar in spirit to Tao's proof [6, 7] that the Gaussian primes contain arbitrary constellations).

We are currently preparing an exposition [2] of the complete proof of the Green-Tao theorem (assuming Szemerédi's theorem), incorporating all the simplifications that have taken place since the original proof.

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Noncrossing k-tuples

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(joint work with Francisco Santos, Christian Stump)

1. Associahedra

The classical associahedron arose in the work of Tamari and Stasheff in the 1950's motivated by questions in homotopy theory. The history of the object is rich and in particular its various realizations as polytopes are still intensively studied. We refer the reader to the article by Cellabos and Ziegler [3] for details. In its original definition the associahedron is a simple polytope, but throughout this report, when we speak of the associahedron, we mean the simplicial polytope that is the polytope dual of the original definition. As a simplicial complex the associahedron $\Delta_{n,2}$ is given in the following way. Consider the set of diagonals $\Omega_{n,2} = \{(i,j) \mid 1 \leq i < j \leq n\} \setminus \{(i,i+1) \mid 1 \leq i \leq n-1\} \cup \{(1,n)\}$ of the n-gon. Then a subset A of $\Omega_{n,2}$ is a simplex in $\Delta_{n,2}$ if no two diagonal from A cross in the interior of the n-gon. This shows that the maximal simplices of $\Delta_{n,2}$ correspond to triangulations of the n-gon. Using this point of view the first realizations of the associahedron as simplicial polytopes were constructed (see [3]). Then the work of Fomin and Zelevinsky [6] led to the definition and construction of associahedra for general finite Coxeter groups. Another generalization of associahedra arose in the work of Dress, Klucznik, Koolen and Moulton [5]. It should be noted that the essential combinatorics for the definition had been independently developed by Capolyleas and Pach [1] ten years before [5]. Consider the set $\Omega_{n,k} = \{(i,j) \mid 1 \leq i < j \leq n\} \setminus \{(i,i+\ell) \mid 1 \leq \ell \leq k-1\}$ with $i+\ell$ read modulo n and $(i,i+\ell)$ rearranged in ascending order. On ground set $\Omega_{n,k}$ on defines $\Delta_{n,k}$ as the simplicial complex of all subsets $A \subseteq \Omega_{n,k}$ such that no k diagonals in A cross pairwise. $\Delta_{n,k}$ is known to be a triangulation of a sphere, but its polytopality is wide open. There now several proofs of sphericity, perhaps the most conceptional one arises from the fact that $\Delta_{n,k}$ is a subword complexes in the sense of Miller and Knutson (see [2] for more details and references).

2. Noncrossing k-tuples

We define a new generalization of the associahedron on ground set $\Omega_n^k = \{(i_1, \ldots, i_k) \mid 1 \leq i_1 < \cdots < i_k \leq n\} \setminus \{(i, i+1, \ldots, i+k-1) \mid 1 \leq i \leq n\}$. Again entries of $(i, i+1, \ldots, i+k-1)$ should be read modulo n and rearranged in ascending order. Now we call two k-tuples $(i_1, \ldots, i_k)$, $(j_1, \ldots, j_k) \in \Omega_n^k$ noncrossing if for all $1 \leq a < c \leq n$ such that $i_b = j_b$ for $a < b < c$ the diagonals (i_a, i_c) and (j_a, j_c) do not cross in the above sense. Then we define Δ_n^k as the simplicial complex on ground set Ω_n^k as the set of all $A \subseteq \Omega_n^k$ which do not contain a pair of crossing k-tuples.

Clearly, for $k = 2$ we have $\Omega_n^2 = \Omega_{n,2}$ and $\Delta_n^2 = \Delta_{n,2}$.

Theorem: (Santos, Stump, Welker) *The simplicial complex Δ_n^k is the boundary complex of a $k(n-k)-n+1$ dimensional simplicial polytope with*

$$Cat_{n-k,k} := \frac{0!\ 1!\ \cdots (k-1)!}{(n-1)!\ (n-2)!\ \cdots (n-k)!}\big(k(n-k)\big)! \quad .$$

facets.

The motivation for studying Δ_n^k is the following relation of Δ_n^k to algebra.

Theorem: (Santos, Stump, Welker) *Let $P_{k,n}$ be the ideal of Plücker relations defining the coordinate ring of the Grassmannian $G_{k,n}$ of k-planes in complex n-space in its Plücker embedding. Then there is a term order $\prec$ such that the Stanley-Reisner ideal of Δ_n^k is an initial ideal of $P_{k,n}$ with respect to $\prec$.*

We refer the reader to [7] for a related results for $\Delta_{n,k}$.

Surprisingly, the complex Δ_n^k also connects to a simplicial complex studied in the context of quasicommuting quantum Plücker coordinates. This complex was introduced by Leclerc and Zelevinsky [8]. They call two k-tuples $(i_1, \ldots, i_k)$, $(j_1, \ldots, j_k) \in \Omega_n^k$ weakly separable if for the sets $I = \{i_1, \ldots, i_k\}$ and $J = \{j_1, \ldots, j_k\}$ we have either $I - J$ is the disjoint union of I' and I'' such that $I' < J - I < I''$ or $J - I$ is the disjoint union of J' and J'' such that $J' < I - J < J''$. For two sets A and B we mean by $A < B$ that all elements of A are smaller than all elements of B.

Let Γ_n^k be the simplical complex of all subsets $A \subseteq \Omega_n^k$ such that any two elements in A are weakly separable. By work of Danilov, Karzanov and Koshevoy [4] (see also parallel work of Oh, Postnikov and Speyer [9]) it follows that Γ_n^k is pure and of dimension $k(n-k)-n+1$. Now the following result is an immediate consequence of the above theorems.

Corollary: *Γ_n^k is a pure full dimensional subcomplex of Δ_n^k.*

Note: After this report was written, the author was informed that there is a substantial overlap with the work [10].

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Finitely forcible graphons

Roman Glebov
(joint work with Tereza Klimošová, Daniel Kral, and Jan Volec)

Recently, a theory of limits of combinatorial structures emerged and attracted substantial attention. The most studied case is that of limits of dense graphs initiated by Borgs, Chayes, Lovász, Sós, Szegedy and Vesztergombi [1, 2, 9], which we also address in this talk. A sequence of graphs is convergent if the density of every graph as a subgraph in the graphs contained in the sequence converges. A convergent sequence of graphs can be associated with an analytic object (graphon) which is a symmetric measurable function from the unit square $[0,1]^2$ to $[0,1]$. The development of the graph limit theory is also reflected in a recent monograph by Lovász [6].

In this talk, we are concerned with finitely forcible graphons, i.e., those that are uniquely determined (up to a natural equivalence) by finitely many subgraph densities. Such graphons are related to uniqueness of extremal configurations in extremal graph theory as well as to other problems. For example, the classical result of Chung, Graham and Wilson [3] asserting that a large graph is pseudorandom if and only if the homomorphic densities of K_2 and C_4 are the same as in the Erdős-Rényi random graph $G_{n,1/2}$ can be cast in the language of graphons as follows: The graphon identically equal to $1/2$ is uniquely determined by homomorphic densities of K_2 and C_4, i.e., it is finitely forcible. Another example that can be cast in the language of finite forcibility is the asymptotic version of the theorem of Turán [10]: There exists a unique graphon with edge density $\frac{r-1}{r}$ and zero density of K_{r+1}, i.e., it is finitely forcible.

The result of Chung, Graham, and Wilson [3] was generalized by Lovász and Sós [7] who proved that any graphon that is a stepfunction is finitely forcible. This result was further extended and a systematic study of finitely graphons was initiated by Lovász and Szegedy [8] who found first examples of finitely forcible graphons that are not stepfunctions. In particular, they observed that every example of a finitely forcible graphon that they found had a somewhat finite structure. Formalizing this intuition, they associated typical vertices of a graphon W with the topological space $T(W) \subseteq L_1[0,1]$ and observed that their examples of finitely forcible graphons W have compact and at most 1-dimensional $T(W)$. This led them to make the following conjectures [8, Conjectures 9 and 10].

Conjecture 3 (Lovász and Szegedy)**.** If W is a finitely forcible graphon, then $T(W)$ is a compact space.

They noted that they could not even prove that $T(W)$ had to be locally compact. We show a construction of a finitely forcible graphon W such that $T(W)$ fails to be locally compact, in particular, $T(W)$ is not compact. Details can be found in our paper [5].

Theorem 1. There exists a finitely forcible graphon W_R such that the topological space $T(W_R)$ is not locally compact.

They also made the following conjecture on the dimension of $T(W)$ of a finitely forcible graphon W.

Conjecture 4 (Lovász and Szegedy)**.** If W is a finitely forcible graphon, then $T(W)$ is finite dimensional.

Lovász and Szegedy noted in their paper that they did not want to specify the notion of dimension they had in mind. Since every non-compact subset of $L_1[0,1]$ has infinite Minkowski dimension, Theorem 1 also provides a partial answer to Conjecture 4. We also show a graphon $W_{\boxplus}$, which we call a hypercubical graphon, such that $W_{\boxplus}$ is finitely forcible and both $T(W_{\boxplus})$ and $\overline{T}(W_{\boxplus})$ contain subspaces homeomorphic to $[0,1]^\infty$.

Theorem 2. The hypercubical graphon $W_{\boxplus}$ is finitely forcible and the topological spaces $T(W_{\boxplus})$ and $\overline{T}(W_{\boxplus})$ contain subspaces homeomorphic to $[0,1]^\infty$.

This result can be found on our paper [4].

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Coarse thresholds for monotone properties with bounded minterms

Ehud Friedgut
(joint work with Jeff Kahn, Clara Shikhelman)

In this talk I addressed the topic of understanding the threshold behavior of monotone properties with minterms of bounded size, say k. We prove that for any such non-trivial property the ratio $p_\epsilon/p_{1-\epsilon}$ is of order at most $\epsilon^{O(1/k)}$), where p_x is the unique p for which the p-measure of the family is x.

The above result can be derived in three manners.

- As result of a structure theorem of the following flavor: Either the family contains a large subfamily for which the Payley-Zygmund (second-moment) bound gives the correct order of magnitude for the measure, or it may be well-approximated by a family with smaller minterms, for which, by induction, the bound holds.
- By a theorem relating the *Fractional Expectation* of the family with its measure. This theorem is proven using LP duality and a weighted second-moment method.
- As kindly pointed out to me by Oliver Riordan, an application of the Russo-Margulis method implies that the function $\mu_p(A)/p^k$ is monotone decreasing, which implies the result with optimal constants.

As is often true, I find all three auxiliary theorems above (lemmas?) to be more interesting than the fundamental theorem that they imply.

Packing bounded degree trees

Julia Böttcher
(joint work with Jan Hladký, Diana Piguet, Anusch Taraz)

A family of graphs $\mathcal{H} = (H_1, \dots, H_k)$ is said to *pack* into a graph G if there exist pairwise edge-disjoint copies of $H_1, \dots, H_k$ in G. An old and surprising conjecture of Gyárfás (see [2]) concerns packings of trees into the complete graph.

Conjecture 5 (Tree Packing Conjecture)**.** Any family $(T_1, T_2, \dots, T_n)$ of trees with $v(T_j) = j$ for all $j \in [n]$ packs into K_n.

A related conjecture of Ringel [1] dating back to 1963 deals with packing many copies of the same tree.

Conjecture 6. Any $2n + 1$ identical copies of any tree of order $n + 1$ pack into K_{2n+1}.

Note that both conjectures are best possible in the sense that they deal with *perfect packings*, i.e., the total number of edges packed equals the number of edges in the host graph. Moreover, the fact that two spanning stars do not pack into the complete graph shows that further requirements than this necessary condition are needed.

A packing of many of the small trees from Conjecture 5 was obtained by Bollobás [3], who showed that any family of trees $T_1, \dots, T_s$ with $v(T_i) = i$ and $s < n/\sqrt{2}$ can be packed into K_n. He also observed that the validity of the famous conjecture of Erdős and Sós would imply that one can improve the bound to $s < \frac{1}{2}\sqrt{3}n$. The solution of the Erdős-Sós Conjecture for large trees was announced by Ajtai, Komlós, Simonovits, and Szemerédi in the early 1990s. In a similar direction, Yuster [4] proved that any sequence of trees $T_1, \dots, T_s$, $s < \sqrt{5/8}n$ can be packed into $K_{n-1,n/2}$. This improves upon a result of Caro and Roditty [5].

Packing the large trees of Conjecture 5 is a much more challenging task. Balogh and Palmer [6] proved that any family of trees T_n, T_{n-1}, …, $T_{n-\frac{1}{10}n^{1/4}}$, $v(T_i) = i$ packs into K_{n+1}.

Surprisingly few results are known for special classes of tree families. It was proved already in [2] that Conjecture 5 holds when all the trees are stars and paths. In [7, 8] trees of small diameter are considered. Moreover, Fishburn [9] proved that it is at least possible to adequately match up the degrees of the trees $T_1, \dots, T_n$ appearing in Conjecture 5: If we add $n - i$ isolated vertices to the tree T_i and let $d_{i,1}, \dots, d_{i,n}$ denote the degree sequence of the resulting forest, then there are permutations $\pi_1, \dots, \pi_n$ such that $\sum_i d_{i,\pi_i(j)} = n - 1$ for all $j \in [n]$.

Our main result deals with almost perfect packings of bounded-degree trees into a complete graph.

Theorem 1. For any $\varepsilon > 0$ and any $\Delta \in \mathbb{N}$ there is an $n_0 \in \mathbb{N}$ such that for any $n \geq n_0$ the following holds. Any family of trees $\mathcal{T} = (T_i)_{i\in[k]}$ such that T_i has maximum degree at most Δ and order at most n for each $i \in [k]$, and $\sum_{i\in[k]} e(T_i) \leq \binom{n}{2}$ packs into $K_{(1+\varepsilon)n}$.

Thus, for trees of bounded degree, we give a common generalisation that proves the asymptotic correctness of Conjectures 5 and 6 without needing any further requirement than just the obvious upper bound on the total number of edges.

Outline of the proof. A very natural approach to pack the trees $T_1, \dots, T_k$ into $K_{(1+\varepsilon)n}$ is to use a *random embedding process*:

Start with $G = K_{(1+\varepsilon)n}$. Successively build a packing of the trees, edge by edge, starting with an arbitrary edge in an arbitrary tree and then following the

structure of the trees. In one step of this procedure, when we want to embed an edge xy of some tree T_i, with x already embedded to $h(x)$, choose a random neighbour $v \in V(G)$ of $h(x)$ which is not contained in the set $U_i \subset V(G)$ of T_i-images so far, and embed y to $h(y) := v$. After embedding xy, remove the edge uv from G and add v to U_i.

Clearly, this process produces a proper packing unless we get stuck, that is, unless the neighbourhood $N_G\big(h(x)\big)$ of $h(x)$ gets empty. But if, during the evolution, the host graph G always remains sufficiently quasirandom, then with high probability $N_G\big(h(x)\big)$ should not get empty. Unfortunately, however, graph processes like this are extremely difficult to analyse because of their dynamically evolving environment in each step. One way to bypass these difficulties is to use the *nibble method*:

Pack the trees in r rounds (with r constant). For this purpose, cut each tree T_i into small equally sized forests F_i^j with $j \in [r]$ and use in each round exactly one forest of each tree. In round j, for each i construct a *random homomorphism* from the forest F_i^j to G as follows. First, randomly embed some forest vertex x, then choose a random neighbour $v \in V(G)$ of $h(x)$ which is not contained in the *forbidden set* $U_i \subset V(G)$ of vertices used by T_i in previous rounds, then continue with the next vertex in F_i^j, following again the structure of T_i. After round j, delete all edges from G to which some forest edges were mapped in this round and add to U_i all images of vertices of F_i^j.

In other words, the difference between this approach and the random process described above is that the host graph G and the sets U_i are not updated after the embedding of each single vertex, but only at the end of each round.

Naturally, this procedure will not produce a proper packing of the trees: Firstly, it will create *vertex collisions*, that is, two vertices of some tree T_i are mapped to the same vertex of the host graph G. Secondly, there will be *edge collisions*, that is, two edges of different trees are mapped to the same host graph edge. However, since all forests F_i^j are small we expect that this will create only a small proportion of vertex and edge collisions in each round, and the updates at the end of each round guarantee that there are no collisions between rounds. Moreover, it turns out that vertex and edge collisions can be *corrected* at the end of the process.

The difficulty with this construction of random homomorphisms though is that it still leads to lots of small dependencies between embedded vertices, which we found difficult to control. (We remark that techniques recently developed by Barber and Long [10] allow to handle these dependencies; but we need further properties in our proof.) So instead we use the following construction of random homomorphisms in round j of the nibble approach described above, which we call *limping homomorphisms*:

For each i, call one of the colour classes of F_i^j the set of *primary vertices*, and the other the set of *secondary vertices*. Now first map all primary vertices randomly to vertices of $V(G) \setminus U_{i,s}$. Then map each secondary vertex randomly into the common $(G - U_{i,s})$-neighbourhood of the images of its forest neighbours – unless

this common neighbourhood is smaller than expected, in which case we simply *skip* this secondary vertex.

Observe that, if our host graph is quasirandom (and the forest has bounded degree), then most common neighbourhoods are big and hence few vertices will get skipped (and these we can also handle at the end). Of course in this random construction we still have dependencies. But since these occur only between vertices with distance at most 2 in the trees, we now can control them and prove that the host graph is quasirandom after each round and that we get few collisions.

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The number of real roots of a random polynomial

Van Vu

(joint work with Hoi Nguyen, Oanh Nguyen)

Consider a random polynomial $P_{n,\xi}(z) = \sum_{i=0}^{n} \xi_i x^i$ where ξ_i are iid copies of a real random variable ξ with mean zero. Let $N_{n,\xi}$ denote the number of real roots of $P_{n,\xi}$. In what follows the asymptotic notations are used under the assumption that $n \to \infty$.

Waring was the first to study roots of random polynomials as far back as 1782 (see, for instance, Todhunter's book on early history of probability [16, page 618]). Systematic studies of $N_{n,\xi}$ started in the 1930s. In 1932, Bloch and Pólya [1] considered the special case when ξ is uniformly distributed in $\{-1, 0, 1\}$ and established the upper bound

$$\mathbf{E}N_{n,\xi} = O(n^{1/2}).$$

Their method can be extended to other discrete distributions such as Bernoulli ($\xi = \pm 1$ with probability $1/2$). This bound is not sharp, and it was a considerable surprise when Littlewood and Offord showed that random polynomials actually have a remarkably small number of real zeroes. In a series of important papers [10, 11, 12], published between 1939 and 1945, they proved a strong bound

$$\frac{\log n}{\log\log\log n} \ll N_{n,\xi} \ll \log^2 n \tag{1}$$

with probability $1 - o(1)$, for many basic variables ξ (such as Bernoulli, Gaussian, and uniform on $[-1, 1]$).

During this time, in 1943, another fundamental result was achieved by Kac [8], who found an asymptotic estimate for $\mathbf{E}N_{n,\xi}$ in the case that ξ is standard real Gaussian $N(0, 1)$, showing

$$\mathbf{E}N_{n,N(0,1)} = \left(\frac{2}{\pi} + o(1)\right)\log n. \tag{2}$$

It took much effort to extend (2) to other distributions. Kac's method does provide a formula for $\mathbf{E}N_{n,\xi}$ for any ξ. However, this formula is quite hard to estimate when ξ is not Gaussian. In a subsequent paper [9], Kac managed to extend (2) to the case when ξ is uniform on $[-1, 1]$ and Stevens [15] extended it further to cover a large class of ξ having continuous and smooth distributions with certain regularity properties (see [15, page 457] for details). These papers relied on Kac's formula and the analytic properties of the distribution of ξ are essential. It took sometime until Erdős and Offord in 1956 [3] found a completely new approach to handle discrete distributions. Considering the case when ξ is Bernoulli, they proved that with probability $1 - o(\frac{1}{\sqrt{\log\log n}})$

$$N_{n,\xi} = \frac{2}{\pi}\log n + o(\log^{2/3} n \log\log n). \tag{3}$$

In late 1960s and early 1970s, Ibragimov and Maslova [4, 5] successfully refined Erdős-Offord method to handle any variable ξ with mean 0. They proved that for any ξ with mean zero which belong to the domain of attraction of the normal law,

$$\mathbf{E}N_{n,\xi} = \frac{2}{\pi}\log n + o(\log n). \tag{4}$$

The error term $o(\log n)$ is implicit in their papers. However, by following the proof (see the last bound in [4, page 247]) one can replace it by a more precise term $O(\log^{1/2} n \log\log n)$. For related results, see also [6, 7]. Few years later, Maslova [13, 14] showed that if ξ has mean zero and variance one and $\mathbb{P}(\xi = 0) = 0$, then the variance of $N_{n,\xi}$ is $(\frac{4}{\pi}(1 - \frac{2}{\pi}) + o(1))\log n$.

Another important development was made in 1995, when Edelman and Kostlan [2] introduced a new way to handle the Gaussian case and estimate $\mathbf{E}N_{n,N(0,1)}$. They proved the following stunningly precise formula

$$\mathbf{E}N_{n,N(0,1)} = \frac{2}{\pi}\log n + C + \frac{2}{\pi n} + O(\frac{1}{n^2}) \tag{5}$$

where $C \approx .625738072..$ is an explicit constant.

The approach used in [2] relies heavily on the fact that a random Gaussian vector distributes uniformly on the unit sphere and one cannot use it for other distributions. The true nature of the error term in $\mathbf{E}(N_{n,\xi})$ has not been known in general. It also seems very difficult to improve upon the order of magnitude of the error term in Ibragimov and Maslova's analysis.

In this paper, we provide a new method to estimate $\mathbf{E}N_{n,\xi}$. We succeed in deriving the following universal estimate.

Theorem 1. For any random variable ξ with mean 0 and variance 1 and bounded $(2+\epsilon)$-moment

$$\mathbf{E}N_{n,\xi} = \frac{2}{\pi}\log n + O_{\epsilon,\xi}(1).$$

To emphasize the dependence of the constant in O on the atom variable ξ, let us notice that if ξ is Bernoulli ($\xi = \pm 1$ with probability $1/2$), then the random polynomial $P_{n,\xi}$ does not have any real root in the interval $(-1/2, 1/2)$ with probability 1. On the other hand, in the case when ξ is gaussian the expectation of real roots in $(-1/2, 1/2)$ is $\frac{1}{\pi}\log 3 + o(n^{-17})$. Thus, it is reasonable to expect that the expectation in the Gaussian case exceeds that in the Bernoulli case by at least a positive constant. Our numerical experiment tends to agree with this.

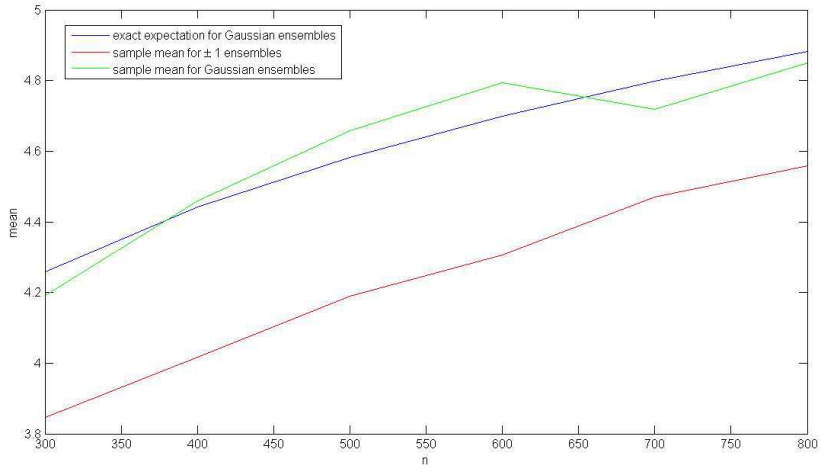

FIGURE 1. Sample means of the number of real roots for Gaussian and Bernoulli ensembles

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Cycle factors and renewal theory

Eyal Lubetzky

(joint work with Jeff Kahn, Nick Wormald)

An H-factor of a graph G is a collection of vertex-disjoint copies of the graph H covering all vertices of G. Thresholds for the existence of H-factors in random graphs have been extensively studied — from classical works in the 1960's (for instance, perfect matchings [5]) to recent ones (such as triangle-factors and the related "Shamir's problem" of matchings in hypergraphs [9]). Here we consider the following question on k-cycle factors in random regular graphs.

> *(Cycle factors.)* For which $k = k(n)$ does a random 3-regular graph on n vertices contain n/k vertex-disjoint k-cycles w.h.p.[1]?

[1] A sequence of events (A_n) is said to hold *with high probability* (w.h.p.) if $\mathbb{P}(A_n) \to 1$ as $n \to \infty$.

When $k = n$ this is the Hamiltonicity problem, which was finally answered in the affirmative in 1992 by Robinson and Wormald [12]. At the other extreme, for $k = O(1)$ it is known that the total number of k-cycles in G is asymptotically Poisson with bounded mean, and so there is no k-cycle factor w.h.p. (the total number of vertices on such cycles is uniformly bounded); moreover, the typical absence of a k-cycle factor extends to the range $k \ll \log n$, throughout which most vertices do not lie on k-cycles w.h.p. No results were known on intermediate k.

Somewhat surprisingly, a major role in our study of the above problem will be played by a question on tail probabilities of *renewal processes without replacement* — where the recurrence times from the classical setting of renewal processes, rather than being IID random variables, are drawn uniformly yet *without* replacement from a finite set. We now define this question formally.

Let $X = \{x_1, \ldots, x_m\}$ be a multiset of positive integers summing to n. Let (Y_i) be a sequence of IID uniform samples of the x_i's (recurrence times), and let $S_t = \sum_{i=1}^{t} Y_i$ be its partial sum sequence (the renewal process). By the classical Renewal Theorem [4], the probability that the partial sums "hit" some integer k, denoted $R_k = \mathbb{P}(k \in \{S_1, S_2, \ldots\})$, tends to m/n as $k \to \infty$ provided $\gcd(x_1, \ldots, x_m) = 1$ (see, e.g., [6]). Consider the following variant of R_k:

> *(Renewals without replacement.)* Let $X = \{x_1, \ldots, x_m\}$ be a multiset of positive integers summing to n, take a uniform permutation $\sigma \in \mathcal{S}_m$ and let $S_t = \sum_{i=1}^{t} x_{\sigma(i)}$. What is the probability $P_k = \mathbb{P}(k \in \{S_1, S_2, \ldots\})$ that the partial sums hit $k \in \mathbb{Z}$?

Our main result will hinge on sharp quantitative bounds for P_k (and a variant of it called Q_k), including asymptotic second order terms and correct exponential tails.

Revisiting the main problem on cycle factors in random regular graphs, it is interesting to compare the situation for the Erdős-Rényi random graph $\mathcal{G}(n, p)$, where for a given k one is interested in the threshold p_c at which the probability of a k-cycle factor is $\frac{1}{2}$ (say). Here it is often natural to expect that p_c coincides (asymptotically) with the threshold for the property that every vertex lies on a k-cycle. E.g., for k fixed, the latter has order $n^{-\frac{k-1}{k}}(\log n)^{\frac{1}{k}}$; the threshold for k-cycle factors is known [9] to have the same order, yet its asymptotics are unknown.

For the property that every vertex lies on a k-cycle in the random 3-regular graph, the threshold ("threshold" now referring to k) is at $k = (1 + o(1)) \log_2 n$.

Our main result settles the problem completely and shows that here the preceding intuition is not quite correct: the phase transition from no k-cycle factor to the existence of one occurs around $[1 - \frac{1}{2}\log_2 3]^{-1} \log_2 n \approx 4.82 \log_2 n$. Furthermore, we establish a 2-point concentration result (a single point for most values of n).

Theorem 1. Let G be a random 3-regular graph on n vertices and let

$$K_0(n) = \frac{1}{1 - \frac{1}{2}\log_2 3} \log_2(2n/e)\,. \tag{1}$$

If $k \geq K_0(n)$ is a divisor of n then G contains a k-cycle factor w.h.p., and on the other hand if $k \leq K_0(n) - \frac{\log^2 n}{n}$ then w.h.p. there is no k-cycle factor in G.

Moreover, the number of k-cycle factors in G, denoted by CF_k, satisfies

$$\frac{CF_k}{\mathbb{E}[CF_k]} \xrightarrow{\mathrm{d}} W = \prod_{j=3}^{\infty}(1+\delta_j)^{Z_j} e^{-\delta_j \mathbb{E}Z_j} \quad \text{as } n \to \infty \tag{2}$$

for any $k \geq K_0(n)$ dividing n, where $\delta_j = \frac{(-1)^j - 1}{2^j}$ and the Z_j's are IID Poisson($\frac{2^{j-1}}{j}$).

The proof of [12] that a random cubic graph is Hamiltonian introduced the *small subgraph conditioning method*, a twist on the second moment method: the second moment of H_n, the number of Hamilton cycles in that random graph, is unfortunately (just barely) too large, namely $\mathbb{E}H_n^2/(\mathbb{E}H_n)^2 \to c$ for fixed $c > 0$. The culprit is the set of small cycles (those with bounded length) in the graph, which in a sense blow up the variance by allowing local detours along a Hamilton cycle. Luckily — and quite mysteriously (in various situations this fails, e.g., when half the degrees are 3 and half are 4) — the second moment drops to $\epsilon(\mathbb{E}H_n)^2$ once we *condition on the joint cycle distribution* up to length $M(\epsilon)$, implying Hamiltonicity with high probability (see [13] and [8, §9.3] for more information).

As our result gives Hamiltonicity for the special case $k = n$, naturally we follow the framework of small subgraph conditioning, which was highly nontrivial already for a single cycle. Though far more delicate to carry out in our setting (as explained below), this method enjoys two byproducts (implicit in [12] and formalized in [7] (and for (b) below also in [11]); cf. [8, §9.5],[13, §4]): (a) it gives the limiting law of the variable (as in Eq. (2) above), and (b) it further implies *contiguity*.

Definition (Contiguity of distributions)**.** Let $\mathbb{P}_n$ and $\mathbb{Q}_n$ be two sequences of probability measures defined on the same measurable spaces $(\Omega_n, \mathcal{F}_n)$. We say that $\mathbb{P}_n$ and $\mathbb{Q}_n$ are contiguous, denoted $\mathbb{P}_n \approx \mathbb{Q}_n$, if for any sequence of events (A_n) one has $\lim_{n\to\infty} \mathbb{P}_n(A_n) = 1 \Leftrightarrow \lim_{n\to\infty} \mathbb{Q}_n(A_n) = 1$.

The first example of contiguity in our context ([7, 11]) was that $\mathcal{G}(n,3)$, the uniform distribution on 3-regular graphs on n vertices, is contiguous to the union of a uniform Hamilton cycle and a uniform perfect matching (conditioned on no multiple edges). Our work extends this result: roughly put, the next corollary says that one can distinguish with probability $1 - o(1)$ between $\mathcal{G}(n,3)$ and the union of a k-cycle factor and a perfect matching if and only if $k \geq K_0(n)$ from Eq. (1).

Corollary 2. Let $\mathcal{G}(n,3)$ be the uniform 3-regular graph on n vertices and, for $k \mid n$, let $\mathcal{G}(n,k,3)$ be the union of a uniform k-cycle factor and a uniform prefect matching, conditioned on no multiple edges. If $k \geq K_0(n)$ then $\mathcal{G}(n,3) \approx \mathcal{G}(n,k,3)$ whereas if $k \leq K_0(n) - \frac{\log^2 n}{n}$ then $\mathcal{G}(n,3) \not\approx \mathcal{G}(n,k,3)$.

The most challenging hurdles in the proofs of these results arise in the second moment calculation, already before the small subgraph conditioning enters the picture. Indeed, calculating the second moment of the number of Hamilton cycles amounts to understanding the typical intersection of two cycles (a collection of paths): a Hamilton cycle is formed simply by orienting and then ordering these.

However, for k-cycle factors, the common intersection of two such factors must be stitched into n/k cycles, and now one seeks only those permutations of the m

parts that form k cycles: letting $X = \{x_1, \dots, x_m\}$ be the set of path lengths, we see the connection to the above question on renewals without replacement, as we wish to hit all multiples of k with the partial sums of the permutation. It is further seen that very sharp error estimates are needed, up to the correct exponential error-term; e.g., when $k \asymp \log n$, we must repeatedly hit k for about $\exp(ck)$ times (to build n/k cycles), along which these errors accumulate.

As an example for the general estimates that we establish, recall that for a set $X = \{x_1, \dots, x_m\}$ of positive integers summing to n, we let P_k be the probability that $k \in \{S_1, S_2, \dots\}$ where $S_t = \sum_{i=1}^{t} x_{\sigma(i)}$ with $\sigma \in \mathcal{S}_m$ being uniform. Also, let Q_n be the probability that $k \in \{S_1, S_2, \dots\}$ when $\mathbb{P}(\sigma(1) = j) = x_j/n$ (size-biased sample) and $(\sigma(2), \dots, \sigma(m)$ is uniform in $\mathcal{S}_{m-1}$ over the remaining elements.

Example (geometric distribution)**.** If $x_1 = 0$ and $x_\ell/m \sim 2^{1-\ell}$ for $\ell \geq 2$, then for any $1 \ll k \ll m^{1/4}$ and any fixed $\epsilon > 0$ we have

$$\begin{aligned} R_k &= \tfrac{1}{3} + O\big((2-\epsilon)^{-k}\big) && \textit{(with replacement)}, \\ P_k &= \tfrac{1}{3} - \tfrac{2/27 - o(1)}{m} + O\big((2-\epsilon)^{-k}\big) && \textit{(without replacement)}, \\ Q_k &= \tfrac{1}{3} - o(1/m) + O\big((2-\epsilon)^{-k}\big) && \textit{(size-biased without replacement)}. \end{aligned}$$

Applications for the Erdős-Rényi random graph. An immediate corollary of Theorem 1 is that for $k \geq K_0(n) \approx 4.82 \log_2 n$ the threshold for a k-cycle factor in $\mathcal{G}(n,p)$ has order $\frac{\log n}{n}$ (the upper and lower bounds a factor of $2 - o(1)$ apart).

Another corollary is the "Comb Conjecture." A *comb of order* k, for some $k \mid n$, is a tree consisting of an (n/k)-vertex path P together with disjoint k-vertex paths beginning at the vertices of P. When $k = \sqrt{n}$ we will call this *the comb* and denote it Comb_n. The "Comb Conjecture" says that the threshold for the appearance of Comb_n in $\mathcal{G}(n,p)$ has order $\frac{\log n}{n}$ (the lower bound is obvious due to connectivity).

Consideration of the threshold for combs was suggested by the first author about 20 years ago as a test case for the more general conjecture (this was also proposed at that time by the first author, but, being a natural guess, is perhaps better regarded as folklore), that the same threshold statement holds for general (n-vertex) trees of bounded degree. The case $k = \sqrt{n}$ of this suggestion has come to be known as the "Comb Conjecture." The rationale for considering combs was the idea that they interpolated between instances for which the general conjecture was known to be true, namely Hamilton paths (for which the threshold $(1 + o(1))\frac{\log n}{n}$ was established in [1] and [3]) and trees with order n leaves (which are easily handled via Hall's Theorem). For related work on this topic, see, e.g., [2], which shows that, already at $p = O(\frac{1}{n})$, w.h.p. the random graph contains *every* bounded-degree tree on $(1-\epsilon)n$ vertices, with an implicit constant depending on ϵ and Δ; and [10], which shows that the threshold for any bounded-degree tree T is at most $n^{-1+o(1)}$, by observing that such trees have either many leaves or long paths of degree-2 vertices, and then deploying a separate strategy in each case. The comb is indeed an extremal example as it precisely balances between these two elements.

Theorem 3. For any $\epsilon > 0$ the Erdős-Rényi random graph $\mathcal{G}(n,p)$ with $p = (2+\epsilon)\frac{\log n}{n}$ contains a copy of Comb_n as a spanning subgraph w.h.p. In particular, the threshold for the appearance of Comb_n in $\mathcal{G}(n,p)$ is at $p \asymp \frac{\log n}{n}$.

More generally, we get to within a factor of $2 + o(1)$ of the threshold for containing the comb of order k for every $k \geq K_0(n)$. We separately treat the complementary range of k and conclude that for any $k = k(n)$ the threshold is $O(\frac{\log n}{n})$

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Permanent Hamiltonicity

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(joint work with Asaf Ferber, Benny Sudakov)

We present a general approach to a variety of extremal problems about counting and packing Hamilton cycles in dense undirected and oriented graphs. Our main results are listed below. First, some notation. For a graph G, let $\mathrm{reg}(G)$, resp. $\mathrm{reg}_{even}(G)$, be the maximum degree, resp. maximum even degree, of a spanning regular subgraph of G. An oriented graph is obtained by orienting the edges of a simple graph.

Theorem 1. For every $\varepsilon > 0$, there exists $n_0 = n(\varepsilon)$ so that for every graph G with $|V(G)| = n \geq n_0$ and $\delta(G) \geq (1/2 + \varepsilon)n$, G contains at least $(1 - \varepsilon)reg_{even}(G)/2$ edge-disjoint Hamilton cycles.

This result is clearly optimal asymptotically and establishes the approximate form of a conjecture of Kühn, Lapinskas and Osthus [6].

Theorem 2. Let n be a sufficiently large integer and let G be a graph on n vertices such that $\delta(G) \geq n/2$. Then the number of Hamilton cycles in G is at least $\left(\frac{\mathrm{reg}_{even}(G)}{e}\right)^n (1 - o(1))^n$.

This result substantially improves a counting result of Sárközy, Selkow and Szemerédi, who proved [7] that for all large enough n, every Dirac graph G on n vertices (a graph with minimum degree $\delta(G) \geq n/2$) contains at least $c^n n!$ Hamilton cycles, for some absolute constant $c > 0$. That result was obtained through the Regularity Lemma of Szemerédi. Our result is however inferior in general to a result of Cuckler and Kahn, who showed [3] that the number of Hamilton cycles (under the same assumptions) is in fact at least $\left(\frac{\delta(G)}{e}\right)^n (1 - o(1))^n$; that result was derived by analyzing and counting long self-avoiding random walks in Dirac graphs.

Theorem 3. For every $c > 3/8$ and every $\eta > 0$ there exist a positive constant $\varepsilon := \varepsilon(c, \eta) > 0$ and an integer $n_0 := n_0(c, \eta)$ such that the following holds. Suppose that:

(1) G is an oriented graph with $|V(G)| = n \geq n_0$, and
(2) all in- and out-degrees in G are in the range $[(c - \varepsilon)n, (c + \varepsilon)n]$.

Then $h(G) = \left(\frac{(c\pm\eta)n}{e}\right)^n$. In particular, if G is cn-regular, then $h(G) = \left(\frac{(c+o(1))n}{e}\right)^n$.

This extends substantially a result of Cuckler [2], who proved that every regular tournament on n vertices contains at least $\frac{n!}{(2+o(1))^n}$ Hamilton cycles, thus settling in turn an old conjecture of Thomassen [8].

The common underlying technique in the proofs of the above result is applying classical estimates on permanents of non-negative matrices: the upper bound, the so called Minc conjecture, established by Brégman [1], and the lower bound, the van der Waerden conjecture, settled by Egorychev [4] and by Falikman [5].

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Combinatorics: Local and local-to-global

Nathan Linial

Our subject is the "local geometry" of large combinatorial objects, the two main questions being: (i) Which local views are possible? and (ii) Which global properties can be inferred based on local data? This general framework applies very well to graphs, trees, tournaments, permutations, words, matrices and more. Here we introduce mostly the graph-theoretic terminology, but the reader should have no trouble extending the notions to other objects. Let $\mathcal{H}_k$ the set of all (isomorphism types of) k-vertex graphs. For a graph G and $H \in \mathcal{H}_k$, let $p(H,G)$ be the probability that if we sample at random a set S of k vertices in G, the subgraph of G induced on S is isomorphic to H. The probability distribution $\pi_k(G) = \{p(H,G)|H \in \mathcal{H}_k\}$ is called the *k-profile of G*. If this distribution is strictly positive, we say that G is *k-universal*. Here is a concrete statement of our main problems:

Problem 1. Which probability distributions on $\mathcal{H}_k$ can occur as the k-profile of a graph?

Problem 2. Given the probability distribution $\pi_k(G)$, which *global* conclusions can you draw about the graph G?

Our main interest is asymptotic, so we attempt to understand the set Δ_k of those $x \in \mathbb{R}^d$ for which the exists a sequence of graphs G_n of growing size such that $\lim_{n\to\infty} \pi_k(G_n) = x$, where $d = |\mathcal{H}_k|$. If $\pi_k(G_n)$ tends to a strictly positive vector, we say that G_n is *asymptotically k-universal*.

Obviously, these questions are very closely related to

- Extremal combinatorics.
- The theory of graph limits [11].
- The field of property testing.

In addition, these problems are motivated by a real practical need. In numerous areas of science, engineering and commerce there is a pressing need to analyze huge graphs, e.g. protein-protein interaction graphs, social networks and more. Analyzing the local profile of such a huge graph seems like a viable approach to the problem. Also, whereas there is substantial progress in understanding Δ_3 (e.g. [8, 9] and the references therein), our understanding of Δ_k for larger k is very limited.

Local profiles: There are structures such as tournaments for which the concept of a local profile is essentially identical to the one just described for graphs. In trees, it makes much more sense to consider only the distribution of k-vertex *subtrees*. This part of the theory has only recently been initiated [2]. Here is a nice feature of the limit sets for trees. The graph-theoretic limit sets Δ_k are complicated: By [13] they are not even semi-algebraic, and by [7] it is undecidable to discover their supporting hyperplanes. In contrast, the analogous limit sets for trees are *convex*. Here is how we define the k-profile of a permutation. Think of $\pi \in S_n$ as a list of the n numbers $1, \ldots, n$ in some order. A random set of k entries in this list yields an ordered k-tuple of distinct integers. We associate with it a permutation in S_k according to the relative ordering of these k integers. Consequently, the k-profile of π is a probability distribution on S_k.

For lack of the space here is a small sample of topics of which I spoke.

- **Local phenomena**
  - **Large homogeneous sets must exist** *"Ramsey"*
    - *Graphs:* $\hom(G) := \max(\alpha(G), \omega(G))$. By Ramsey's Theorem $\hom(G) \geq \frac{1}{2}\log_2 n$ for every n-vertex graph, while $2\log_2 n \geq \hom(G)$ for most graphs.
    - *Tournaments:* $\hom(T) := \mathrm{tran}(T)$, the largest transitive subtournament of T. For every n-vertex tournament $\hom(T) \geq \log_2 n$. There are tournaments with $2\log_2 n \geq \hom(G)$.
    - *Trees:* $\hom(T) := \max(s(T), p(T))$ where $s(T), p(T)$ is largest star, resp. longest path contained in T. For every n-vertex T there holds $\hom(T) \geq \frac{\log n}{\log\log n}$. The bound is tight.
    - *Permutations:* $\hom(\pi)$ is defined as the length of the longest monotone subsequence of $\pi \in S_n$. Erdős-Szekeres: $\hom(\pi) \geq \lceil \sqrt{n} \rceil$. The bound is tight.
  - **There are always many homogeneous constant-size sets** *"Goodman"* [6]
    - *Graphs:* In every graph at least $\frac{1}{4}$ of the triples are homogeneous. The bound is attained by $G(n, \frac{1}{2})$ graphs. The analogous statement for sets of four or more vertices is incorrect [14]. There is presently no conjecture as to the optimal construction for this graph parameter.
    - *Tournaments:* For every k the number of transitive k-vertex subtournaments is asymptotically minimal for random tournaments.
  - **Inducibility**
    - *Graphs:* We define the *inducibility* of a fixed graph H as $\limsup_{|G|\to\infty} p(H, G)$. We still know very little about this natural parameter. In particular there is not even a conjecture about the inducibility of the 4-vertex path P and the nature of the graphs for which $p(P, G)$ is largest.

        * *Permutations:* There is rich literature concerning the so-called *packing density* of permutations.
- **Global phenomena**
  - **Quasirandomness**
    * *Graphs:* The count of C_4's is maximized by random graphs. Graphs with that number of induced C_4's are quasirandom [4].
    * *Tournaments:* Similar phenomena are known [3].
    * *Permutations:* 4-uniformity implies quasirandomness [10].
  - **Universality vs. size of homogeneous sets**
    * *Graphs:* The Erdős-Hajnal Conjecture [5]: For every integer k there is a $\delta > 0$ such that every n-vertex graph G for which $\hom(G) < n^\delta$ is k-universal.
    * *Tournaments:* The Erdős-Hajnal Conjecture for tournaments is equivalent to its graph version [1].
    * *Trees:* We do not even know yet if for every k there is an asymptotically k-universal sequence of large trees [2].
  - **Universality vs. few small homogeneous sets?** [12]
    * *Graphs:* **Theorem:** There is a constant $\rho = 0.159....$ such that an n-vertex graph G with $p(K_3, G) < \rho - o_n(1)$ and $p(\overline{K_3}, G) < \rho - o_n(1)$ is 3-universal. The bound is tight. This does not imply 5-universality, the case of 4-universality is still open.
    * *Tournaments:* A tournament in which the frequency of transitive 4-vertex tournament is $< \frac{1}{2}$ is 4-universal. The bound is tight.

A key ingredient in many of these recent developments is the use of *flag algebras* [13].

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Ramsey numbers

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(joint work with David Conlon, Choongbum Lee, Benny Sudakov)

The Ramsey number $r(H;t)$ of a k-uniform hypergraph H is the minimum N such that every t-coloring of the complete k-uniform hypergraph on N vertices contains a monochromatic copy of H. In the case $t = 2$, we simply write $r(H) := r(H;t)$. In this survey, we discuss results around the following question.

Question 1. What properties of H roughly determines its Ramsey number?

We first discuss the case for graphs (when $k = 2$). Classical results of Erdős-Szekeres and Erdős give that the Ramsey number of the complete graph K_n on n vertices satisfies $2^{n/2} \le r(K_n) \le 2^{2n}$. There have been several small improvements on these bounds (see, for example, [2]). However, the exponents have remained largely unchanged for over sixty years. It is then a modest goal to try to determine $\log r(H)$ up to a constant factor for a graph H.

Not surprisingly, the area has stretched in different directions. One such direction that has become fundamental in its own right is that of looking at Ramsey numbers of various types of sparse graphs.

In particular, Burr and Erdős conjectured that for each Δ there is $c(\Delta)$ such that $r(H) \le c(\Delta)n$ for every graph H on n vertice with maximum degree Δ. This conjecture was proved by Chvátal, Rödl, Szemerédi and Trotter [1] in one of the earliest applications of Szemerédi's celebrated regularity lemma. Remarkably, this means that for graphs of fixed maximum degree the Ramsey number only has a linear dependence on the number of vertices. Unfortunately, because it uses the regularity lemma, the bounds that the original method gives on $c(\Delta)$ are a tower of 2s with exponential in Δ. The bound on $c(\Delta)$ was later improved by Eaton to $2^{2^{c\Delta}}$ using a variant of the regularity lemma. In 2000, Graham, Rödl, and Rucinski [14] proved, by a beautiful method which avoids any use of the regularity lemma, that there exists a constant c for which $c(\Delta) < 2^{c\Delta \log^2 \Delta}$. They also proved that $c(\Delta) > 2^{c\Delta}$, and even when restricted to bipartite graphs [15]. Conlon, Sudakov, and I [8] recently improved the upper bound to $c(\Delta) < 2^{c\Delta \log \Delta}$. We have the following conjecture.

Conjecture 7. We have $c(\Delta) = 2^{\Theta(\Delta)}$.

This conjecture is proved for graphs of constant chromatic number in [3], [12].

A graph is d-degenerate if every subgraph of it has minimum degree at most d. The degeneracy $d(H)$ of a graph H is the minimum d such that H is d-degenerate. The degeneracy of a graph is a good measure of its sparseness. Burr and Erdős conjectured that every d-degenerate graph H on n vertices satisfies $r(H) \leq c_d n$. Kostochka and Sudakov [17] proved a weaker bound of the form $r(H) \leq n^{1+o(1)}$. The estimate was improved by Fox and Sudakov [13] to $r(H) \leq 2^{c_d\sqrt{\log n}}n$.

We conjecture that $\log r(H) = \Theta(d(H) + \log n)$ holds for every graph H on n vertices. This conjecture would determine $\log r(H)$ up to a constant factor. Sudakov and I proved this conjecture for graphs of constant chromatic number in [13]. Some further results in this direction are also proved in [13].

Our understanding of hypergraph Ramsey numbers is considerably worse. The existence of hypergraph Ramsey numbers was proven by Ramsey in 1930, but no proper consideration of the values of these numbers was made until the paper by Erdős and Rado in 1952. To understand the growth of Ramsey numbers for hypergraphs, it is useful to introduce the tower function $t_i(x)$, which is defined by $t_1(x) = x$ and $t_{i+1}(x) = 2^{t_i(x)}$. Erdős and Rado showed that for H being the complete k-uniform hypergraph $K_n^{(k)}$ on n vertices, $r(H) \leq t_k(c_k n)$. In the other direction, Erdős and Hajnal proved that for the same H, $r(H) \geq t_{k-1}(c_k n^2)$. For fixed $t \geq 4$, they improved this to $r(H) = t_k(\Theta(n))$.

Since the first proof that bounded degree graphs have linear Ramsey number used Szemerédi's regularity lemma, it was natural to expect that this could be extended to hypergraphs using the hypergraph regularity method. Indeed, using this approach, linear bounds was extended to hypergraphs by several authors (Cooley, Fountoulakis, Kühn, and Osthus [9, 10]; Nagle, Olsen, Rödl, and Schacht [18]), with the result that we now have the following theorem.

Theorem 1. For every k and Δ there is $c(\Delta, k)$ such that every k-uniform hypergraph H on n vertices with maximum degree Δ satisfies $r(H) \leq c(\Delta, k)n$.

This improved upon a nearly linear bound of Kostochka and Rödl [16]. Extending the ideas of [16] with new embedding techniques, Conlon, Sudakov and I [5] gave a simplified proof of the linear bound on hypergraph Ramsey number numbers. Furthermore, we proved that $c(\Delta, k) \leq t_k(c_k\Delta)$ for $k \geq 4$, and $t_k(c\Delta \log \Delta)$ for $k = 3$, giving an essentially tight bound on $c(\Delta, k)$. This improved upon the previous Ackermann-type bounds that arise from using the hypergraph regularity lemma.

A hypergraph is d-degenerate if every subgraph has minimum degree at most d. The degeneracy $d(H)$ of a hypergraph H is the minimum d such that H is d-degenerate. The degeneracy of a hypergraph is an insufficient parameter for estimating bounding the hypergraph's Ramsey number. Indeed, Kostochka and Rödl [16] proved that for each n there is a 1-degenerate 4-uniform hypergraph H with $r(H) \geq 2^{cn^{1/3}}$.

In the other direction, there are quite dense hypergraphs with relatively small Ramsey number. Denote the complete balanced k-uniform s-partite hypergraph on n vertices by $T_{n,s}^{(k)}$. If n is a multiple of s, this hypergraph has $\binom{s}{3}(n/s)^3 > (1 -$

$3/s)\binom{n}{3}$ edges. Hence, as $s \to \infty$, the edge density approaches 1. Conlon, Fox, and Sudakov [7], solving a problem of Erdős and Hajnal, proved that $r(T^{(3)}_{n,s}) \leq 2^{c_s n^2}$, which is tight up to the factor c_s. Further, it is shown in [6] that there is $c = c_{s,k}$ such that $r(T^{(k)}_{n,s}) \leq 2^{n^c}$.

We recently proved in [4] that hypergraph Ramsey numbers are related to *ordered* Ramsey numbers of graphs. An ordered graph on n vertices is a graph with vertex set $[n]$. An ordered graph G on N vertices contains a copy of an ordered graph H on n vertices if there is a mapping $\phi : [n] \to [N]$ such that, for all $1 \leq i < j \leq n$, we have $\phi(i) < \phi(j)$ and $(\phi(i), \phi(j))$ is an edge of G if (i, j) is an edge of H. The ordered Ramsey number $r_<(H, t)$ of an ordered graph H is the minimum N such that every t-coloring of the ordered complete graph on N vertices contains a monochromatic copy of H. In [4], we prove various results about ordered Ramsey numbers of graphs, and show how to bound Ramsey numbers of hypergraphs using them.

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Chasing the k-SAT threshold

Amin Coja-Oghlan

Let $k \geq 3, n, m$ be integers and let $\mathbf{\Phi}$ denote the random k-SAT formula over n variables with m clauses. Since the early 1990s experimental work has indicated that there is a threshold for satisfiability [3, 10]. That is, for any $k \geq 3$ there is a critical clause/variable density r_k where the probability that the random formula is satisfiable drops from asymptotically 1 to asymptotically 0. This talk is about some recent progress in pinning down the threshold r_k.

The random k-SAT problem has been studied via non-rigorous statistical mechanics methods. Indeed, physicists have developed a systematic albeit non-rigorous approach for locating phase transitions in discrete structures called the *cavity method* [15]. The simpler version of the cavity method, called the *replica symmetric ansatz*, is associated with the *Belief Propagation* message passing algorithm. It predicts upper and lower bounds on the k-SAT threshold, namely [17]

$$r_c = 2^k \ln 2 - \frac{3}{2}\ln 2 - o_k(1) \leq r_k \leq 2^k \ln 2 - \ln 2/2 + o_k(1). \tag{1}$$

Here and throughout, $o_k(1)$ signifies a term that tends to 0 as k grows. The more powerful but also more intricate version of the cavity method, the *1-step replica symmetry breaking ansatz* ("1RSB"), is associated with the *Survey Propagation* message passing procedure [16]. The 1RSB cavity method is conjectured to predict the precise thresholds not only in random k-SAT, but also in a host of other problems. Specifically, the 1RSB prediction is [14]

$$r_k = 2^k \ln 2 - \frac{1+\ln 2}{2} + o_k(1). \tag{2}$$

There is a conceptual difference between the replica symmetric and the 1RSB cavity method. In fact, the lower bound in (1) is expected to mark a phase transition called *condensation*, at which the probabilistic nature of the problem changes [12].

Random k-SAT is more challenging than most of the other standard examples of random problems (such as NAESAT or graph coloring). This is because in satisfiability there is an asymmetry between the "spins", i.e., the Boolean values 'true' and 'false'. Indeed, think of generating a random formula $\mathbf{\Phi}$ and then sampling a random satisfying assignment σ. The local "shape" of $\mathbf{\Phi}$ holds significant clues as to the probability that a given variable x takes the value 'true' under σ. For instance, if x appears many more times more positively than negatively in the formula, then we should expect that the probability that x takes the value 'true' under σ is greater than $1/2$. By contrast, in the graph coloring

problem all the colors have the same "meaning". Similarly, the k-NAESAT ("Not-All-Equal-Satisfiability") problem, which asks for a satisfying assignment whose inverse assignment is also satisfying, is perfectly symmetric by its very definition.

Friedgut [8] proved the existence of a non-uniform satisfiability threshold r_k for any $k \geq 3$. However, his approach only establishes existence, without revealing the location of the k-SAT threshold. A three-line "first moment" calculation yields an upper bound on the k-SAT threshold, namely $r_k \leq 2^k \ln 2 - \ln 2/2 + o_k(1)$. A more sophisticated first moment argument [11] actually shows an upper bound of $r_k \leq 2^k \ln 2 - \frac{1}{2}(1 + \ln 2) + o_k(1)$, matching (2).

Establishing *lower* bounds on r_k is more challenging. Proving $r_k \geq 2^k \ln 2 - \frac{3}{2} \ln 2 - o_k(1)$, in [5] we turned the replica symmetric prediction (1) into a rigorous lower bound, the best prior one. The proof is based on the second moment method, whose use in the context of random k-SAT was pioneered by Achlioptas and Moore [1], Achlioptas and Peres [2] and Frieze and Wormald [9].

The contribution of this work is a more powerful "1RSB version" of the second moment method. While the second moment argument in [5] was based on the (replica symmetric) Belief Propagation message passing procedure, the present argument uses the mightier Survey Propagation method and the associated notion of "relaxed" satisfying assignments called *covers* [13, 15]. Based on this novel approach, we establish

Theorem 1. The k-SAT threshold satisfies $r_k = 2^k \ln 2 - \frac{1+\ln 2}{2} + o_k(1)$.

Theorem 1 is among the first rigorous theorems to verify a 1RSB prediction, and it is the very first such rigorous result in an asymmetric problem. In order to cope with asymmetry, the proof of Theorem 1 explicitly incorporates the Survey Propagation calculations into the proof.

In the symmetric random k-NAESAT, a similar result is known [4]. Moreover, independently of the present work, Ding, Sly and Sun [6, 7] used a second moment approach for a concept similar to "covers" of the present work to obtain tight results on the random regular NAESAT problem and the independent set problem in random regular graphs.

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Coloring graphs with excluded induced paths

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Given an integer $k \geq 3$, the problem of deciding whether a graph can be k-colored is known to be NP-complete [8]. It is thus natural to ask what restrictions can be imposed on the input graph to make the problem more tractable. One such type of restrictions would be excluding certain induced subgraphs. However, Kaminski and Lozin [7] proved the following (the *girth* of a graph is the length of the shortest cycle) :

Theorem 1. For every $g, k \geq 3$, the problem of k-coloring graphs with girth $\geq g$ is NP-complete.

Given graphs H and G, we say that G is *H-free* if no induced subgraph of G is isomorphic to H. It therefore follows that if a graph H has a cycle, then the problem of k-coloring H-free graphs is NP-complete for all $k \geq 3$. To see this, apply 1 with $g = |V(H)|$.

It is a well-known theorem of Holyer [5] that

Theorem 2. For every $k \geq 3$, the problem of k-edge-coloring is NP-complete.

2 implies that k-coloring claw-free graphs is NP-complete for $k \geq 3$ (the *claw* is the complete bipartite graph $K_{1,3}$).

Thus, as far as k-coloring H-free graphs is concerned, the only graphs H for which the complexity may be polynomial are the so called "linear forests". A *linear forest* is a graph each of whose components is a path. Recently, significant

progress has been made in the special case when H is a path. We denote by P_t the path on t vertices. Due to results of [1, 6] the complexity of the problem of k-coloring P_t-free graphs remains unknown only in the following cases:

- 3-coloring P_t-free graphs where $t \geq 7$, and
- 4-coloring P_6-free graphs.

Our main results are the following:

Theorem 3. [1, 2] The problem of 3-coloring P_7-free graphs can be solved in polynomial time.

and

Theorem 4. [3] The problem of 4-coloring graphs that are both P_6-free and C_5-free can be solved in polynomial time.

Given a graph G and a collection L of lists $L(v)$ for every vertex $v \in V(G)$, the *list-coloring problem* is the question of determining whether G has a proper vertex coloring c, such that $c(v) \in L(v)$ for every $v \in V(G)$. If such a coloring exists, we say that *the pair* (G, L) *is colorable*. In the proofs of both 3 and 4, the original problem with input graph G is replaced by polynomially many problem (G', L'), such that G has the desired coloring if and only if one of (G', L') does.

Our algorithms rely heavily on the following result [4]:

Theorem 5. Let G be a graph, and let L be such that $|L(v)| \leq 2$ for all $v \in V(G)$. Then there is a polynomial-time algorithm to check if the pair (G, L) is colorable.

Clearly, both 3 and 4 are obvious in the case when the input graph is perfect, and so we may assume that one of the obstructions to perfection is present in the input G. In the case of 3, this means that G contains an induced cycle of length five or seven, in 4—the complement of an induced cycle of length seven. Starting with such an obstruction, call it O, we classify the remaining vertices of G by their distance from O. In some cases, the graph G has a coloring if and only if a certain induced subgraph of G does, and otherwise one can deduce significant structural information about G. Since O has at most seven vertices, one can, in polynomial time, try all possible colorings of O. The idea now is to check whether any of these colorings extend to a coloring of G. To do so, we record the restrictions the coloring of O imposes on the remaining vertices of G, and thus reduce the original problem to polynomially many list-coloring problems. Finally, applying 5 and its generalizations, our results follow.

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Balls into bins via local search: (cover time and) maximum load

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(joint work with Thomas Sauerwald, Alexandre Stauffer, He Sun)

A simple procedure for allocating n balls into n bins is to place each ball into a bin chosen independently and uniformly at random. It is well know that the maximum load for this *1-choice process* (i.e., the maximum number of balls allocated to any single bin) is $\Theta\left(\frac{\log n}{\log\log n}\right)$ [3].

Motivated by load balancing problems in distributed systems, Bogdan et al. [1] introduced a natural allocation process called *local search allocation*. Consider that the bins are organized as the vertices of an undirected graph $G=(V,E)$ with $n=|V|$. At each time step a ball is "born" at a vertex chosen independently and uniformly at random. Starting from this vertex, the ball performs a local search in G, where it repeatedly moves to an adjacent vertex which is less loaded. We assume that ties are broken independently and uniformly at random. The local search ends at a local load minimum. After that, the next ball is born and the above procedure is repeated.

The main result in [1] establishes that when G is an expander graph with bounded maximum degree the maximum load after n balls have been allocated is $\Theta(\log\log n)$. Moreover, it was shown that the maximum load is $\Theta\Big(\big(\frac{\log n}{\log\log n}\big)^{\frac{1}{d+1}}\Big)$ on d-dimensional grids, and $\Theta(1)$ on regular graphs of degrees $\Omega(\log n)$.

We derive new upper and lower bounds for the maximum load on bounded degree graphs. In order to state our results, we introduce the following quantity that is related to the local neighborhood growth of G:

$$R = R(G) = \min\{r\in\mathbb{N}\colon |B_u^r|\cdot r\log r \geq \log n \text{ for all } u\in V\},$$

where B_u^r denotes the set of vertices within distance r from vertex u.

Our main result establishes that the maximum load after allocating n balls on any bounded degree graph G is $O(R)$ with high probability. This is tight if G is "sufficiently homogeneous", a technical condition that ensures that R is attained at sufficiently many vertices u. In particular, for any vertex-transitive graph the maximum load is $\Omega(R)$ with high probability.

As a corollary, we obtain that any connected bounded-degree graph has maximum load at most $O\big(\sqrt{\frac{\log n}{\log\log n}}\big)$ w.h.p., which is tight for paths and cycles. Finally, for any bounded-degree graph the maximum load is w.h.p. $\Omega(\log\log n)$.

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Online Ramsey Games in Random Graphs

Andreas Noever

Consider the following one-player game played on an initially empty graph on n vertices. In every turn we insert a new edge chosen uniformly at random among all non-edges of the graph. The player, henceforth called Painter, must then immediately color this edge with one of r available colors. Her objective is to avoid a monochromatic copy of some fixed graph F for as long as possible.

We call $N_0(F, r, n)$ a threshold function for the online F-avoidance game with r colors if for every $N \ll N_0(F, r, n)$ there exists a strategy for Painter that survives for N steps with high probability and if for every $N \gg N_0(r, n)$ every strategy fails to survive for N steps with high probability. Note that such a threshold function always exists [4, Lemma 7].

This game was first studied by Friedgut, Kohayakawa, Rödl, Ruciński and Tetali, who have shown in [3] that $N_0(K_3, 2, n) = n^{4/3}$ is a threshold for the online triangle-avoidance game with two colors. In 2009 Marciniszyn, Spöhel and Steger proved the following lower bound

Theorem 1 ([4])**.** Let F be a graph that is not a forest, and let $r \geq 1$. Then the online F-avoidance game with r colors has a threshold $N_0(F, r, n)$ that satisfies

$$N_0(F, r, n) \geq n^{2-1/\overline{m}_2^r(F)},$$

where $\overline{m}_r^2(F)$ is given by

$$\overline{m}_2^r(F) := \begin{cases} \max_{H \subseteq F} \frac{e_H}{v_H} & \text{if } r = 1, \\ \max_{H \subseteq F} \frac{e_H}{v_H - 2 + 1/\overline{m}_2^{r-1}(F)} & \text{if } r \geq 2. \end{cases}$$

In an accompanying paper they provide matching upper bounds in the two color case for a large class of graphs, which includes cycles and cliques:

Theorem 2 ([5])**.** Let F be a graph that is not a forest which has a subgraph $F_- \subset F$ with $e_F - 1$ edges satisfying

$$m_2(F_-) \leq \overline{m}_2^2(F).$$

Then the threshold for the online F-avoidance coloring game with two colors is

$$N_0(F, 2, n) = n^{2-1/\overline{m}_2^2(F)}.$$

They conjecture that a similar result is true for all $r \geq 3$.

For three or more colors no tight upper bounds are known. The corresponding offline triangle avoidance game (where Painter gets to see all N edges at once) has a threshold given by $N = n^{3/2}$ [6]. Clearly this upper bound also applies to the online game. In [2] Belfrage, Mütze and Spöhel connected the probabilistic one player game to a deterministic two player game. Here the edges are no longer presented in a random order but can be chosen by a second player called Builder. They show that if there exists a winning strategy for Builder which only creates subgraphs of density at most d, then $n^{2-1/d}$ is a threshold for the original probabilistic game.

This technique was used by Balogh and Butterfield in [1] to improve the upper bound for the online triangle avoidance game to $n^{3/2-c_r}$ for some constant $c_r > 0$. Thus the thresholds of the online and offline games differ. Still their upper bound of $n^{2/3-c_r}$ does not match the lower bound provided by Marciniszyn, Spöhel and Steger.

Our contribution is an upper bound for an arbitrary number of colors for a class of graphs which includes cycles and cliques. That is we show the following:

Theorem 3. Let F be a cycle or a clique, and let $r \geq 1$. Then the online F-avoidance game with r colors has a threshold $N_0(F, r, n)$ that satisfies

$$N_0(F, r, n) = n^{2-1/\overline{m}_2^r(F)}.$$

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Hamilton-generated flow-lattices of dense and random graphs: consequences of embedding spanning substructures

Peter Heinig

For a graph G let $\mathrm{Z}_1(G)$ denote the finitely-generated free abelian group of all integer flows on G in the usual sense, two flows being added edge-wise. In recent literature (e.g. [3, 6, 13]) $\mathrm{Z}_1(G)$ is called the *lattice of integer flows of* G. It is the same as the 1-dimensional cycle group of standard simplicial homology. This talk

was about conditions on G sufficient for $\mathrm{Z}_1(G)$ to admit a generating set consisting of integer flows all of which have supports equal to Hamilton-circuits of G.

The only such conditions known so far assumed G to be a Cayley graph on a finite abelian group [2, 12]. I recently proved sufficient conditions for Hamilton-generated cycle spaces over $\mathbb{F}_2$, both for graphs with high minimum-degree and for binomial random graphs. For the minimum-degree conditions, the proof uses the bandwidth theorem which was conjectured by Bollobás and Komlós and proved by Böttcher, Schacht and Taraz in [5]. For the conditions on random graphs, the proof uses a theorem of Kühn and Osthus from [11]. These theorems make it possible to guarantee spanning embeddings of preselected graphs (which, crucially, need not be quite as symmetrical as Cayley graphs) into the dense or random graphs, to then conclude that the host-graph has a Hamilton-generated cycle space, too.

The main concern of this talk was to point out that only with $\mathbb{Z}$-coefficients, i.e. for $\mathrm{Z}_1(G)$ (as opposed to the cycle space $\mathrm{Z}_1(G;\mathbb{F}_2)$ over $\mathbb{F}_2$) can one make full use of the information that the bandwidth-theorem provides: it allows one to *preselect* specific spanning substructures for which one can carry out explicit calculations of Smith Normal Forms of incidence matrices of sets of Hamilton-circuits. It is this specificity which allows one to guarantee not only some Hamilton-supported generating set, but a Hamilton-supported *basis* of $\mathrm{Z}_1(G)$, i.e. a *rank-sized* generating set. By contrast, the usual element-chasing arguments in which one expresses a given arbitrary generator in terms of Hamilton-supported generators seem to merely yield *some* such generating set, without control on the number of generators (in abelian groups, a generating system need not contain a basis).

Let us say that a graph G with n vertices and e edges

(D.1) is *Hamilton-generated* if and only if there exists an $(e-n+1)$-element set of Hamilton-circuits of G such that every cycle in G can be constructed as the symmetric difference of some of these Hamilton circuits,
(D.2) is *almost Hamilton-generated* if and only if in G there exist one circuit of length $n-1$ and $e-n$ Hamilton-circuits such that every cycle in G can be constructed as the symmetric difference of some of these circuits,
(D.3) has a *Hamilton-generated flow-lattice* if and only if there exists a generating set of $\mathrm{Z}_1(G)$ consisting of flows whose support is a Hamilton-circuit of G,
(D.4) has a *Hamilton-based flow-lattice* if and only if G has a Hamilton-generated flow-lattice in the sense of (D.3), with the generating set being required to have $(e-n+1)$ elements only.

Based on the embedding-theorems in [4, 5], I recently proved:

Theorem 1 (sufficient conditions for Hamilton-generatedness; cf. [8, Theorem 1])**.** For every $\gamma > 0$ there is $n_0 \in \mathbb{N}$ such that for every graph G with at least $n \geq n_0$ vertices, e edges and minimum-degree δ:

(I.1) if $\delta \geq (\frac{1}{2}+\gamma)n$ and n is odd, then G is Hamilton-generated,
(I.2) if $\delta \geq (\frac{1}{2}+\gamma)n$ and n is even, then G is almost Hamilton-generated,
(I.3) if $\delta \geq (\frac{1}{4}+\gamma)n$ and G is balanced bipartite, then G is Hamilton-generated,

(I.4) if in (I.1) and (I.2) the condition '$\delta \geq (\frac{1}{2}+\gamma)n$' is changed to '$\delta \geq \frac{2}{3}n$', then without further change to (I.1) or (I.2), it suffices to take $n_0 := 2 \cdot 10^8$.

(I.1) becomes false if '$\delta \geq (\frac{1}{2}+\gamma)n$' is changed to '$\delta \geq \lfloor \frac{1}{2}n \rfloor$ and G Hamilton-connected'.

For odd n, every n-vertex graph G with $\delta(G) \geq \lceil \frac{1}{2}n \rceil$ is Hamilton-connected, but Hamilton-connectedness by itself does not imply Hamilton-generatedness.

Besides strengthening the conclusion of Dirac's theorem on Hamilton-circuits, another goal is the following open conjecture of Bondy. It is plausible that the strategy from [8], i.e. using embedding-techniques to guarantee some preselected spanning substructures and draw conclusions about the host-graph, can lead to further progress on Conjecture 8 (Theorem 1 asymptotically proves an approximation to the special case $d = n/2$):

Conjecture 8 (Bondy 1979; [7, p. 246] [1, p. 12])**.** For every $d \in \mathbb{N}$ and every vertex-3-connected graph G with $|G| \geq 2d$ and $\delta(G) \geq d$, every cycle in G can be constructed as the symmetric difference of some circuits with length $\geq 2d-1$.

As to the conditions for random graphs, from [11] I deduced the following:

Theorem 2 ([9, Theorem 1])**.** If $\varepsilon > 0$ and $p \in [0,1]^{\mathbb{N}}$ with $p(n) \geq n^{-\frac{1}{2}+\varepsilon}$, then for $n \to \infty$ a random graph $G \sim G_{n,p}$ asymptotically almost surely is Hamilton-generated for odd n and almost Hamilton-generated for even n.

A lower bound on p sufficient for $G_{n,p}$ to be a.a.s. Hamilton-generated has been found; the result should be viewed against the backdrop that a.a.s. hamiltonicity of $G_{n,p}$ already holds for $p \geq (\log n + \log\log n + \omega(1))/n$ and that the property of being Hamilton-generated by itself does not imply minimum-degree ≥ 3 (but Theorem 3 shows that in random graphs it a.a.s. does):

Theorem 3 (Hamilton-generatedness of the cycle space is not guaranteed right from the onset of hamiltonicity of $G_{n,p}$; [9, Theorem 8])**.** If $p \in [0,1]^{\mathbb{N}}$ is such that with $G \sim G_{n,p}$ a.a.s. for odd n

(1) G is not a forest ,
(2) every cycle in G is a symmetric difference of Hamilton circuits ,

then for every constant $c > 0$ there is n_0 such that $p_n > \frac{\log n + 2\log\log n + c}{n}$ for all odd $n \geq n_0$.

By adapting the methods from [11] to more economical auxiliary spanning substructures (having edge-density of about 3/2) one might prove Theorem 3 with $-\frac{1}{2}$ replaced by $-\frac{2}{3}$. In view of some promising arguments using expansion- and robustness-properties of $G_{n,p}$ that I recently learned from T. Łuczak, and which for $\mathbb{F}_2$-coefficients might reach all the way to the lower bound from Theorem 3, apparently there is a point to that pursuit only for $\mathbb{Z}$-coefficients, i.e. for the flow-lattice $\mathrm{Z}_1(G)$. There, methods based on [10, 11] at the moment seem the only way to go, another example for the main concern of this talk.

As to the statements about flow-lattices, to save space the case of even n will not be discussed here (similar statements to the ones for $\mathbb{F}_2$-coefficients can be proved). Proofs for the following conjectural statements are currently being written up:

(C.1) Theorem 1.(I.1) and Theorem 1.(I.3) remain true if 'is Hamilton-generated' replaced by 'has a Hamilton-*based* flow-lattice',
(C.2) the statement about the explicit lower bound on n_0 in Theorem 1.(I.4) remains correct even after the conclusion in Theorem 1.(I.1) has been strengthened as in (C.1),
(C.3) Theorem 2 for odd n remains true with 'is Hamilton-generated' replaced by 'has a Hamilton-*based* flow-lattice' (and the p-condition unchanged),
(C.4) Theorem 2 remains true with '$p(n) \geq n^{-\frac{1}{2}+\varepsilon}$' replaced by '$p(n) \geq n^{-\frac{2}{3}+\varepsilon}$' and 'is Hamilton-generated' by 'has a Hamilton-*generated* flow-lattice'.

Whether (C.4) remains true with the conclusion 'has a Hamilton-*based* flow-lattice', or even in the doubly-strengthened form of having '$p(n) \geq n^{-\frac{1}{2}+\varepsilon}$' replaced with '$p(n) \geq \frac{\log n + 2\log\log n + \omega(1)}{n}$' and 'Hamilton-*generated*' replaced with 'Hamilton-*based*' seems less likely and, if true, to require new techniques to prove.

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Filtered geometric lattices and tropical Lefschetz theorems

Anders Björner
(joint work with Karim Adiprasito)

Let M denote a matroid on the ground set $[n] = \{1, 2, \cdots, n\}$. By a *weight* we mean a mapping $\omega : [n] \to \mathbb{R}$, and for subsets σ of $[n]$ we set $\omega(\sigma) = \sum_{x\in\sigma} \omega(x)$. A weight is *generic* if $\omega(\sigma) \neq 0$ for all proper subsets σ.

Let $\mathcal{L} = \widehat{\mathcal{L}} \setminus \{\widehat{0}, \widehat{1}\}$ be the proper part of the lattice of flats $\widehat{\mathcal{L}}$ of M, and let $t \in \mathbb{R}$. Then we use $\mathcal{L}^{>t}$ to denote the subset $\mathcal{L}^{>t} = \{\sigma \in \mathcal{L} \mid \omega(\sigma) > t\}$ with the induced partial order. Note that these posets are not lattices in general. The following result generalizes to subposets $\mathcal{L}^{>t}$ a well-known fact for complete geometric lattices $\mathcal{L}$ concerning the topological properties of order complexes.

Theorem. *Let $\widehat{\mathcal{L}}$ be the lattice of flats of a matroid of rank $r \geq 2$, and let ω denote any generic weight on its atoms. Let t be a real number with $t \leq \min\{0, \omega([n])\}$. Then $\mathcal{L}^{>t}$ is homotopy Cohen–Macaulay of dimension $r - 2$. In particular, it is topologically $(r - 3)$-connected.*

Of particular interest is when $\omega([n]) = 0$ and $t = 0$, in which case the geometric lattice splits into two parts $\mathcal{L}^{>0}$ and $\mathcal{L}^{<0}$, which are both homotopy Cohen-Macaulay and of the same dimension.

This theorem had been conjectured by Mikhalkin and Ziegler in 2008 because of its relevance for questions in tropical geometry. More precisely, it is needed in order to establish analogues for tropical varieties of the classical Lefschetz hyperplane section theorem for smooth complex algebraic varieties. The theorem translates to a crucial index estimate for the stratified Morse data at critical points of the tropical variety. It can itself be interpreted as a Lefschetz section theorem for matroids.

(The paper can be found at
http://arxiv.org/abs/1401.7301)

Morphing planar graphs

Penny Haxell
(joint work with Fidel Barrera-Cruz, Anna Lubiw)

Consider two straightline planar drawings Γ and Γ' of a planar triangulation G, in which the outer face is fixed. A *morph* between Γ and Γ' is a continuous family $\{\Gamma^t\}_{t\in[0,1]}$ of drawings of G, such that $\Gamma^0 = \Gamma$ and $\Gamma^1 = \Gamma'$. We call a morph between Γ and Γ' *planar* if Γ^t is a straightline planar drawing of G for all $t \in [0, 1]$. The existence of a planar morph from Γ to Γ' was proved by Cairns in 1944 [3]. We call a morph *linear* if each vertex moves from its position in Γ^0 to its position in Γ^1 along a line segment and at constant speed. It is easy to see that in general the linear morph from Γ to Γ' will not be planar.

Here we consider the algorithmic problem of finding a planar morph between two given drawings Γ and Γ' with fixed outer face. This problem arises in a

number of practical settings, for example in computer animation, motion planning and 3D image reconstruction. For various reasons it is desirable to find morphs in which each vertex trajectory is fairly simple. Thus we focus on the problem of constructing a planar morph consisting of a polynomial number of steps, in which each step is a planar linear morph.

One previous solution to the morphing problem involved Tutte's graph drawing algorithm [6] (Floater and Gotsman [4]), which gave good visual results but the vertex trajectories were not piecewise linear. Alamdari et al. [1] succeeded in solving the problem using a polynomial number of linear morphing steps, by modifying the approach of Cairns. Here we give a somewhat simpler solution that consists of two main parts. In the first part we show how to morph from an arbitrary drawing of a planar triangulation to a *weighted Schnyder drawing* in $O(n)$ linear planar morphing steps. A weighted Schnyder drawing is obtained from a Schnyder wood together with an assignment of positive weights to the interior faces. A Schnyder wood is a special type of colouring and orientation of the edges of a planar triangulation, that partitions the interior edges into three rooted directed trees. This notion was introduced in [5], where it is shown that every planar triangulation G admits a Schnyder wood, and that every Schnyder wood gives a straightline planar drawing of G. Our morphing scheme uses the *unidirectional morph* introduced in [2]. Then in the second part we show how to morph between any two weighted Schnyder drawings in $O(n^2)$ planar linear morphs.

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Enumerative problems in extremal combinatorics

Wojciech Samotij

(joint work with József Balogh, Robert Morris, and Lutz Warnke)

Given a fixed graph H, we say that a graph G is *H-free* if G contains no copy of H as a (not necessarily induced) subgraph. The study of H-free graphs is one of the cornerstones of extremal graph theory. The classical theorems of Turán [22] and Erdős and Stone [11] state that for every H, the largest number of edges in

an H-free graph with n vertices, denoted by $\mathrm{ex}(n,H)$, is approximately equal to the number of edges in the balanced complete $(\chi(H)-1)$-partite graph, that is,

$$\mathrm{ex}(n,H)=\left(1-\frac{1}{\chi(H)-1}\right)\binom{n}{2}+o(n^2).$$

Moreover, the stability theorem of Erdős and Simonovits [10, 21] asserts that every n-vertex H-free graph that has $\mathrm{ex}(n,H)-o(n^2)$ edges is "almost" $(\chi(H)-1)$-partite, that is, it can be made $(\chi(H)-1)$-partite by deleting from it some $o(n^2)$ edges. Let us mention here that analogous results are known in several other settings, e.g., hypergraphs (of uniformity higher than two) not containing a fixed subhypergraph, sum-free sets in Abelian groups, Sidon sets, or sets containing no arithmetic progression of a fixed length.

Whereas the aforementioned results determine what the *largest* H-free graphs look like, we are interested in the structure of a *typical* one. Let $\mathcal{F}_n(H)$ denote the family of all H-free graphs on the vertex set $\{1,\dots,n\}$. A famous result of Kolaitis, Prömel, and Rothschild [14] states that for every $r\geq 2$, almost all K_{r+1}-free graphs are r-partite. More precisely, with probability tending to 1 as $n\to\infty$, a uniform random element of $\mathcal{F}_n(K_{r+1})$ is an r-partite graph. It was later shown by Prömel and Steger [18] that the same conclusion holds if one replaces K_{r+1} by any graph H with $\chi(H)=r+1$ that contains an edge whose removal reduces the chromatic number of H. Many (almost) sharp results describing the structure of almost all graphs in $\mathcal{F}_n(H)$ for a general non-bipartite graph H were proved in a series of papers of Balogh, Bollobás, and Simonovits [2, 3, 4].

One limitation of the results mentioned above is the fact that they say nothing about sparse H-free graphs. This is simply because almost all n-vertex bipartite graphs have $\Omega(n^2)$ edges. Therefore, it is natural to ask for a more detailed description of a random H-free graph, which also takes into the account its density. To this end, given an integer m, let $\mathcal{F}_{n,m}(H)$ be the subfamily of $\mathcal{F}_n(H)$ containing only graphs with exactly m edges. We would like to know for which values of m a uniform random element of $\mathcal{F}_{n,m}(H)$ still resembles an extremal H-free graph. This question, motivated by the study of the evolution of random graphs, has been considered by many researchers.

The following standard argument shows that one cannot expect "nice" behaviour for all densities. Define

$$d_2(H)=\max\left\{\frac{e(K)-1}{v(K)-2}\colon K\subseteq H,\, v(K)\geq 3\right\}.$$

A first moment computation shows that if $m\ll n^{2-1/d_2(H)}$, then a.a.s., the random graph $G_{n,m}$ with n vertices and m edges may be made H-free by deleting from it some $o(m)$ edges. This in turn implies that $|\mathcal{F}_{n,m}(H)|\geq\binom{n^2/2-o(n^2)}{m}$. In particular, a typical element of $\mathcal{F}_{n,m}(H)$ has unbounded chromatic number, even after deleting from it any $o(m)$ edges.

On the other hand, if $m\gg n^{2-1/d_2(H)}$, then one can derive approximate structural description of a typical member of $\mathcal{F}_{n,m}(H)$ from a recent general theorem

of Balogh, Morris, and Samotij [5], proved independently by Saxton and Thomason [19] (cf. the work of Conlon and Gowers [9] and Schacht [20]), which in the context of H-free graphs has the following useful corollary.

Theorem 1 ([5, 19])**.** For every graph H and every positive ε, there exists a constant C such that the following holds. There are a family $\mathcal{S}$ consisting of n-vertex graphs with at most $Cn^{2-1/d_2(H)}$ edges and a map f from $\mathcal{S}$ to the family of n-vertex graphs that contain at most $\varepsilon n^{v(H)}$ copies of H with the following property: For every $G \in \mathcal{F}_n(H)$, there is an $S \in \mathcal{S}$ such that $S \subseteq G \subseteq f(S)$.

The above theorem essentially reduces questions about typical structure of H-free graphs to proving various "supersaturation" statements, i.e., sufficient conditions on a graph G which imply that G contains many copies of H. In the case when $\chi(H) > 2$, many such results are known and consequently Theorem 1 yields many interesting corollaries. For example, from a "robust" version of the Erdős-Simonovits stability theorem, which roughly says that any n-vertex graph with at least $\mathrm{ex}(n, H) - o(n^2)$ edges either contains $\Omega(n^{v(H)})$ copies of H or it is "almost" $(\chi(H) - 1)$-partite, one can easily deduce the following result: If $m \gg n^{2-1/d_2(H)}$, then almost every graph in $\mathcal{F}_{n,m}(H)$ may be made $(\chi(H) - 1)$-partite by deleting from it some $o(m)$ edges. Prior to [5, 19], this statement had been derived by Łuczak [15] from the so-called KŁR conjecture, which was proved only in [5, 19].

The case when $\chi(H) = 2$ is much more delicate and not much is known except when H is a cycle [12, 13, 16] or a complete bipartite graph [7, 8]. One of the main obstacles is the fact that the standard methods used for proving "supersaturation" statements for non-bipartite graphs completely fail in the bipartite case.

In the case when H is a clique, much more precise structural characterisation of typical H-free graphs is known. The following theorem was proved by Osthus, Prömel, and Taraz [17]. Let ε be an arbitrary positive constant and let

$$m_2 = m_2(n) = \frac{\sqrt{3}}{4} n^{3/2} \sqrt{\log n}.$$

First, if $m \ll n$, then almost all graphs in $\mathcal{F}_{n,m}(K_3)$ are bipartite. Second, if $n/2 \leq m \leq (1 - \varepsilon)m_2$, then almost all these graphs are not bipartite. Third, if $m \geq (1 + \varepsilon)m_2$, then again almost all of them are bipartite. Very recently, we generalised this result to cliques of arbitrary size.

Theorem 2 ([6])**.** For every $r \geq 3$, there exists a $d_r = d_r(n) = \Theta(n)$ such that the following holds for every $\varepsilon > 0$. Define

$$\theta_r = \frac{r-1}{2r} \cdot \left[r \cdot \left(\frac{2r+2}{r+2} \right)^{\frac{1}{r-1}} \right]^{\frac{2}{r+2}}$$

and

$$m_r = m_r(n) = \theta_r n^{2-\frac{2}{r+2}} (\log n)^{\frac{1}{\binom{r+1}{2}-1}}.$$

If $F_{n,m}$ is the uniformly chosen random element of $\mathcal{F}_{n,m}(K_{r+1})$, then

$$\lim_{n\to\infty} \Pr[F_{n,m} \text{ is } r\text{-partite}] = \begin{cases} 1, & \text{if } m \le (1-\varepsilon)d_r, \\ 0, & \text{if } (1+\varepsilon)d_r \le m \le (1-\varepsilon)m_r, \\ 1, & \text{if } m \ge (1+\varepsilon)m_r. \end{cases}$$

We remark that the existence of the first threshold and the function d_r in the above theorem follows directly from the fact that for every $r \ge 3$, the property of being r-colourable has a sharp threshold, as proved by Achlioptas and Friedgut [1]. Finally, let us say that the methods used to prove Theorem 2 seem to generalise to the case when $\chi(H) > 2$ and H contains a vertex whose deletion decreases the chromatic number. The case of a general non-bipartite graph H remains a challenging open problem.

Let us conclude by making several comments about our proof of Theorem 2. The existence of the first threshold is an easy consequence of the aforementioned result of Achlioptas and Friedgut [1] and thus we focus on establishing the existence of the second threshold at m_r. In order to get some intuition where the function m_r comes from, one needs to compare the number of graphs in $\mathcal{F}_{n,m}(K_{r+1})$ that are not r-partite but have exactly one monochromatic edge in some r-coloring to the number of r-partite graphs. Using a characterization of r-colorable graphs and a version of the FKG inequality for the hypergeometric distribution, we convert this intuition into a proof of the 0-statement. The main part of [6] is the proof of the second 1-statement, which consists of three parts. First, we use Theorem 1 to deduce that almost every $F_{n,m}$ is "approximately" r-partite for some "balanced" r-partition of the vertex set. Second, we use a version of Janson's inequality for the hypergeometric distribution to estimate the number of graphs in $\mathcal{F}_{n,m}(K_{r+1})$ that are not r-partite but are "very close" to being r-partite. Since m is very close to the threshold m_r, this needs to be done very carefully. Finally, we bound the number of graphs in $\mathcal{F}_{n,m}(K_{r+1})$ that are "approximately" r-partite but not "very close" to being r-partite. In this case, the bounds obtained from Janson's inequality are not sufficiently close to the truth and we need to resort to structural arguments. To aid this approach, we develop a new concentration inequality for the number of edges induced by a random subset in sparse uniform hypergraphs.

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The existence of designs

Peter Keevash

A *Steiner system* with parameters (n, q, r) is a set S of q-subsets of an n-set X, such that every r-subset of X belongs to exactly one element of S. The question of whether there is a Steiner system with given parameters is one of the oldest problems in combinatorics, dating back to work of Plücker (1835), Kirkman (1846) and Steiner (1853); see [42] for a historical account.

More generally, we say that a set S of q-subsets of an n-set X is a *design* with parameters (n, q, r, λ) if every r-subset of X belongs to exactly λ elements of S. There are some obvious necessary 'divisibility conditions' for the existence of such S, namely that $\binom{q-i}{r-i}$ divides $\lambda\binom{n-i}{r-i}$ for every $0 \leq i \leq r-1$ (fix any i-subset I of X and consider the sets in S that contain I). It is not known who first advanced the 'Existence Conjecture' that the divisibility conditions are also sufficient, apart from a finite number of exceptional n given fixed q, r and λ.

The case $r = 2$ has received particular attention because of its connections to statistics, under the name of 'balanced incomplete block designs'. We refer the reader to [5] for a summary of the large literature and applications of this field.

The Existence Conjecture for $r = 2$ was a long-standing open problem, eventually resolved by Wilson [43, 44, 45] in a series of papers that revolutionised Design Theory, and had a major impact in Combinatorics. In this talk, we announce a proof of the Existence Conjecture in general, via a new method, which we refer to as Randomised Algebraic Constructions.

In fact, we obtain a more general result on clique decompositions of hypergraphs that satisfy a certain pseudorandomness condition. To describe this we make the following definitions.

Definition 5. A *hypergraph* G consists of a vertex set $V(G)$ and an edge set $E(G)$, where each $e \in E(G)$ is a subset of $V(G)$. We identify G with $E(G)$. If every edge has size r we say that G is an *r-graph*. For $S \subseteq V(G)$, the *neighbourhood* $G(S)$ is the $(r - |S|)$-graph $\{f \subseteq V(G) \setminus S : f \cup S \in G\}$. For an r-graph H, an *H-decomposition* of G is a partition of $E(G)$ into subgraphs isomorphic to H. Let K_q^r be the complete r-graph on q vertices.

Note that a Steiner system with parameters (n, q, r) is equivalent to a K_q^r-decomposition of K_n^r. It is also equivalent to a perfect matching (a set of edges covering every vertex exactly once) in the auxiliary $\binom{q}{r}$-graph on $\binom{[n]}{r}$ (the r-subsets of $[n]$) with edge set $\{\binom{Q}{r} : Q \in \binom{[n]}{q}\}$. The next definition generalises the necessary divisibility conditions described above.

Definition 6. Suppose G is an r-graph. We say that G is *K_q^r-divisible* if $\binom{q-i}{r-i}$ divides $|G(e)|$ for any i-set $e \subseteq V(G)$, for all $0 \leq i \leq r$.

Next we formulate a simplified version of our quasirandomness condition. It is easy to see that it holds whp if $G = G^r(n, p)$ is the usual random r-graph and n is large given p, c and h.

Definition 7. Suppose G is an r-graph on n vertices. We say that G is *(c, h)-typical* if there is some $p > 0$ such that for any set A of $(r - 1)$-subsets of $V(G)$ with $|A| \leq h$ we have $|\cap_{S \in A} G(S)| = (1 \pm c)p^{|A|}n$.

Now we can state a simplified version of our main theorem.

Theorem 8. Let $1/n \ll c \ll d, 1/h \ll 1/q \leq 1/r$. Suppose G is a K_q^r-divisible (c, h)-typical r-graph on n vertices with $|G| > dn^r$. Then G has a K_q^r-decomposition.

Applying this with $G = K_n^r$, we deduce that for large n the divisibility conditions are sufficient for the existence of Steiner systems. Our method gives a randomised algorithm for constructing designs and also an estimate on the number of designs with given parameters.

Theorem 8 gives new results even in the graph case ($r = 2$); for example, it is easy to deduce that the standard random graph model $G(n, 1/2)$ whp has a partial triangle decomposition that covers all but $(1 + o(1))n/4$ edges: deleting a perfect matching on the set of vertices of odd degree and then at most two 4-cycles gives a graph satisfying the hypotheses of the theorem. This is the asymptotically best

possible leave, as whp there are $(1+o(1))n/2$ vertices of odd degree and any partial triangle decomposition must leave at least one edge uncovered at each vertex of odd degree.

We also note that if an r-graph G on n vertices satisfies $|G(S)| \geq (1-c)n$ for every $(r-1)$-subset S of $V(G)$ then it is (hc, h)-typical (with $p = 1$), so we also deduce a minimum $(r-1)$-degree version of the theorem, generalising Gustavsson's minimum degree version [15] of Wilson's theorem.

We make some brief comments here on our new method of Randomised Algebraic Constructions. The idea is to start by taking a random subset of a model for the problem that is algebraically defined. Here the result is a partial K_q^r-decomposition that covers a constant fraction of the edge set, and also carries a rich structure of possible local modifications. We treat this partial decomposition as a template for the final decomposition. By various applications of the nibble and greedy algorithms, we can choose another partial K_q^r-decomposition that covers all edges not in the template, which also spills over slightly into the template, so that some edges are covered twice.

To handle the 'spill', we express as it as an integral design that is the difference $D^+ - D^-$ of two partial K_q^r-decompositions of the underlying r-graph of the template. We apply local modifications to the template decomposition so that it includes D^+, then delete D^+ and replace it by D^-. The resulting partial decomposition of the template has a hole that exactly matches the spill, so fits together with the previous partial decomposition to provide a complete decomposition.

At this level of generality, our local modification method sounds somewhat similar to that of Gustavsson, and also to the 'absorbing method' of Rödl, Ruciński and Szemerédi [35]. However, Gustavsson's method is inherently an argument for graphs rather than hypergraphs, and the absorbing method relies on a dense set of 'absorbing configurations' that cannot be guaranteed in the sparse setting of the decomposition problem.

We also remark that our argument is by induction on r, and so we need to consider the general setting of typical simplicial complexes: even if we just want to prove the existence of Steiner (n, q, r)-systems, our proof uses the existence of K_q^s-decompositions of typical complexes with $s < r$. A final comment is that although our quasirandomness condition resembles those used in Hypergraph Regularity Theory, we make no use of this theory in our proof, so our constants could in principle be moderately effective (but we do not attempt to calculate them).

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Lagrangians of intersecting families

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(joint work with Liana Yepremyan)

Given an r-uniform hypergraph (*an r-graph*) on $[n]$ define a polynomial

$$p_G(x) = \sum_{I \in E(G)} \prod_{i \in I} x_i.$$

The *lagrangian* of G is defined by

$$\lambda(G) = r! \max_{x \geq 0, \|x\|_1 = 1} p_G(x).$$

For a family of r-graphs $\mathcal{F}$ we define $\lambda(\mathcal{F}) = \sup G \in \mathcal{F} \lambda(G)$ Lagrangians of graphs and hypergraphs have been an important tool in extremal graph theory since a beautiful paper of Motzkin and Straus [3]. If $\mathcal{F}$ is a hypergraph family then it is easy to see that

$$\lambda(G) \geq \sup_{G \in \mathcal{F}} |E(G)| / \binom{V(G)}{r}.$$

The equality holds in many important cases. A hypergraph G is *intersecting* if $A \cap B \neq \emptyset$ for all $A, B \in E(G)$. Our main result gives the exact value of the lagrangian for the family of intersecting r-graphs, confirming a conjecture of Hefetz and Keevash [2]

Theorem 1. Let $r \geq 4$ and let $\mathcal{F}$ be the family of intersecting r-graphs then

$$\lambda(\mathcal{F}) = \left(1 - \frac{1}{r}\right)^{r-1}.$$

Our proof relies on compression techniques, borrowing the ideas from the proof of the Complete Intersection Theorem of Ahlswede and Khachatrian [1].

We apply Theorem 1 to solve the following Turán type problem, proposed in [2]. Let $\mathcal{K}^r_{r,r}$ be the r-graph with the vertex set

$$V(\mathcal{K}^r_{r,r}) = \{x_i, y_i \mid 1 \leq i \leq r\} \cup \{z_{ijk} \mid 1 \leq i, j \leq r, 1 \leq k \leq r-2\},$$

and edge set

$$E(\mathcal{K}^r_{r,r}) = \{x_1, \ldots, x_r\} \cup \{y_1, \ldots, y_r\} \cup \{\{x_1, y_j, z_{ij1}, \ldots, z_{ij(r-2)}\} \mid 1 \leq i, j \leq r\}.$$

Let S^r_n be the r-graph on n vertices with parts A and B and $E(S^r_n) = \{I \subset A \cup B \ |I| = r, |I \cap A| = 1\}$, where the size of A is chosen to maximize the number of edges. It is easy to see that $\mathcal{K}^r_{r,r}$ is not a subgraph of S^r_n.

Theorem 2. There exists $n_0 = n_0(r)$ such that for every $n \geq n_0$ and every r-graph G on n vertices, not containing $\mathcal{K}^r_{r,r}$, one has $|E(G)| \leq |S^r_n|$.

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Maximizing the number of independent sets of a fixed size

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(joint work with Wenying Gan, Benny Sudakov)

Given a finite graph G, let $i_t(G)$ be the number of independent sets of size t in a graph, and let $i(G) = \sum_{t \geq 0} i_t(G)$ be the total number of independent sets. There are many extremal results on $i(G)$ and $i_t(G)$ over families of graphs with various degree restrictions. Kahn [6] and Zhao [8] studied the maximum number of independent sets in a d-regular graph. Relaxing the regularity constraint to a minimum degree condition, Galvin [5] conjectured that the number of independent sets in an n-vertex graph with minimum degree $\delta \leq \frac{n}{2}$ is maximized by a complete bipartite graph $K_{\delta, n-\delta}$. This conjecture was recently proved (in stronger form)

by Cutler and Radcliffe [3] for all n and δ, and they characterized the extremal graphs for $\delta > \frac{n}{2}$ as well.

One can further strengthen Galvin's conjecture by asking whether the extremal graphs also simultaneously maximize the number of independent sets of size t, for all t. This claim unfortunately is too strong, as there are easy counterexamples for $t = 2$. On the other hand, no such examples are known for $t \geq 3$. Moreover, in this case Engbers and Galvin [4] made the following conjecture.

Conjecture 9. For every $t \geq 3$ and $\delta \leq n/2$, the complete bipartite graph $K_{\delta,n-\delta}$ maximizes the number of independent sets of size t, over all n-vertex graphs with minimum degree at least δ.

Engbers and Galvin [4] proved this for $\delta = 2$ and $\delta = 3$, and for all $\delta > 3$, they proved it when $t \geq 2\delta+1$. Alexander, Cutler, and Mink [1] proved it for the entire range of t for bipartite graphs, but it appeared nontrivial to extend the result to general graphs. The first result for all graphs and all t was obtained by Law and McDiarmid [7], who proved the statement for $\delta \leq n^{1/3}/2$. This was improved by Alexander and Mink [2], who required that $\frac{(\delta+1)(\delta+2)}{3} \leq n$. Our work completely resolves this conjecture.

Theorem 1. Let $\delta \leq n/2$. For every $t \geq 3$, every n-vertex graph G with minimum degree at least δ satisfies $i_t(G) \leq i_t(K_{\delta,n-\delta})$, and when $t \leq \delta$, $K_{\delta,n-\delta}$ is the unique extremal graph.

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Combinatorical applications of the Subspace Theorem

József Solymosi
(joint work with Ryan Schwartz)

The Subspace Theorem is a powerful tool in number theory. Wolfgang M. Schmidt was the first to state and prove a variant of the Subspace Theorem in 1972 [11]. It has appeared in various forms and been adapted and improved over time. It's applications include diophantine approximation, results about integral points on algebraic curves and the construction of transcendental numbers. But its usefulness extends beyond the realms of number theory. Other applications of the Subspace Theorem include linear recurrence sequences and finite automata. In fact, these structures are closely related to each other and the construction of transcendental numbers. The Subspace Theorem also has a number of remarkable combinatorial applications. The purpose of this talk was to give a survey of some of these applications including sum-product estimates and bounds on unit distances. The presentation was presented from the point of view of a discrete mathematician. We stated a number of variants of the Subspace Theorem.

Before we state the Subspace Theorem we need some definitions. A *linear form* is an expression of the form $L(x) = a_1x_1 + a_2x_2 + \cdots + a_nx_n$ where $a_1, \ldots, a_n$ are constants and $x = (x_1, \ldots, x_n)$. A collection of linear forms is *linearly independent* if none of them can be expressed as a linear combination of the others. Given $x = (x_1, \ldots, x_n)$ we define the *maximum* norm of x:

$$\|x\| = \max(|x_1|, \ldots, |x_n|).$$

Theorem 1 (Subspace Theorem I)**.** Suppose we have n linearly independent linear forms $L_1, L_2, \ldots, L_n$ in n variables with algebraic coefficients. Given $\varepsilon > 0$, the non-zero integer points $x = (x_1, x_2, \ldots, x_n)$ satisfying

$$|L_1(x)L_2(x)\ldots L_n(x)| < \|x\|^{-\varepsilon}$$

lie in finitely many proper linear subspaces of $\mathbb{Q}^n$.

This generalises the Thue-Siegel-Roth Theorem on the approximation of algebraic numbers [10] to higher dimensions.

Theorem 1 has been extended in various directions by many authors including Schmidt himself, Schlickewei, Evertse, Amoroso and Viada. Analogues have been proved using p-adic norms and over arbitrary number fields and bounds on the number of subspaces required have been found. These bounds depend on the degree of the number field and the dimension. For some of these results and more information see [7], [8] and [2].

We presented some combinatorial applications. Here we sketch one of them related to Erdős Unit Distances Conjecture.

Using Mann's Theorem [9] Frank de Zeeuw and the authors were able to show the following theorem [13]. Two points in the plane are said to have *rational angle* if the angle that the line between these two points makes with the x-axis is a rational multiple of π.

Theorem 2. Let $\varepsilon > 0$. Given n points in the plane, the number of unit distances with rational angle between pairs of points is less than $n^{1+\varepsilon}$.

Using the Subspace Theorem in place of Mann's Theorem one can instead consider unit distances from a multiplicative group with rank not too large with respect to the number of points [12]. Note that a unit distance in the plane can be considered as a complex number of unit length. So all unit distances can be considered as coming from a subgroup of $\mathbb{C}^*$.

Theorem 3. Let $\varepsilon > 0$. There exist a positive integer n_0 and a constant $c > 0$ such that given $n > n_0$ points in the plane, the number of unit distances coming from a subgroup $\Gamma \subset \mathbb{C}^*$ with rank $r < c \log n$ is less than $n^{1+\varepsilon}$.

This is our first combinatorial application of the Subspace Theorem.

Performing a careful analysis of Erdős' lower bound construction one can show that all unit distances come from a group with rank at most $c \log n / \log \log n$ for some $c > 0$.

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Sharp threshold for van der Waerden's theorem

Mathias Schacht
(joint work with Ehud Friedgut, Hiệp Hàn, and Yury Person)

One of the main research directions in extremal combinatorics in the last two decades is the transfer of classical results of extremal graph theory, Ramsey theory, and combinatorial number theory to sparse (pseudo-)random settings. Prime examples of this are Ramsey's theorem for graphs and hypergraphs [3, 2, 1], van der Waerden's theorem (and more generally partition regular systems) [4] and Turán's and Szemerédi's theorems [5, 1]. In most of the above cases there is a "0-statement", and a "1-statement". The 0-statement claims that for some $p_0 = p_0(n)$ a random structure such as the random graph $G(n, p_0)$, or a random binomial subset of $\{1, 2, \dots, n\}$ with parameter p_0, asymptotically almost surely does not posses a given property, whereas if p_0 is replaced by p_1 (which usually differs from p_0 by a multiplicative constant), the property does hold *asymptotically almost surely*.

We wish to sharpen this result, in the case of a van der Waerden type 0-1 statement, and prove that there exist $p_0(n)$ and $p_1(n)$ as above, whose ratio tends to 1 as n tends to infinity. This is known as a *sharp threshold*.

Many important and interesting monotone graph and hypergraph properties are known to exhibit such a sharp threshold behavior (e.g. connectivity, non-k-colorability, and various Ramsey-type properties.) In some cases (such as connectivity) the precise threshold behavior is well understood, whereas in other cases the sharpness of the threshold is proven using an existence result. The guiding rule of thumb is that the only properties that do not exhibit a sharp threshold are those correlated with some "local" event, such as the appearance of a fixed subgraph. The main effort, therefore, when establishing the sharpness of a threshold, is to prove the seemingly obvious statement that the property in question is uncorrelated with any substantial local event.

For a subset $A \subseteq [n]$, or $A \subseteq \mathbb{Z}/n\mathbb{Z}$, we write $A \to (k\text{-AP})_r$ to denote the fact that no matter how one colors the elements of A with r colors there is always a monochromatic k-term arithmetic progression in A. Further, for any set X denote by X_p the binomial random subset formed by chosing every element of X independently with probability p.

Rödl and Ruciński studied Ramsey-type partition theorems for random discrete structures. In particular, those authors obtained the following result.

Theorem 1 (Rödl & Ruciński [3, 4])**.** For every k, $r \in \mathbb{N}$ there exist positive constants C_0 and C_1 such that

$$\lim_{n\to\infty} \Pr[[n]_p \to (k\text{-AP})_r] = \begin{cases} 0, & \text{if } p \le C_0 n^{-1/(1-k)}, \\ 1, & \text{if } p \ge C_1 n^{-1/(1-k)}. \end{cases}$$

We study random subsets of $\mathbb{Z}/n\mathbb{Z}$, since symmetry will play a crucial role in our proof. Note that the 1-statement in $[n]$ implies the corresponding 1-statement in $\mathbb{Z}/n\mathbb{Z}$, whereas for the 0-statement the implication is reversed. Due to this,

it seems that there is no easy reduction from a sharp threshold in one setting to the other. In the $\mathbb{Z}/n\mathbb{Z}$ setting, and the case of 2 colors, we are able to sharpen Theorem 1 and obtain a sharp threshold.

Theorem 2. For all $k \geq 3$ there exist positive constants c_0 and c_1, and a function $c(n)$, with $c_0 < c(n) < c_1$ such that for every $\varepsilon > 0$

$$\lim_{n\to\infty} \Pr[(\mathbb{Z}/n\mathbb{Z})_p \to (k\text{-AP})_2] = \begin{cases} 0, & \text{if } p \leq (1-\varepsilon)c(n)n^{-1/(k-1)}, \\ 1, & \text{if } p \geq (1+\varepsilon)c(n)n^{-1/(k-1)}. \end{cases}$$

Note that currently we are unable to prove this theorem for the case of more than two colors. we will expand a bit about this difficulty later.

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Independent sets in hypergraphs

Dhruv Mubayi

Let $K_4^{(3)}$ denote the complete 3-uniform hypergraph on 4 vertices. Ajtai, Erdős, Komlós, and Szemerédi (1981) asked if there is a function $\omega(d) \to \infty$ such that every 3-uniform, $K_4^{(3)}$-free hypergraph H with n vertices and average degree d has independence number at least $\frac{n}{d^{1/2}}\omega(d)$? Recently some evidence for this was provided by Cooper and the speaker who provided a positive answer if, in addition to $K_4^{(3)}$, the hypergraphs $C_3 = \{123, 345, 561\}$ and $F_5 = \{123, 124, 345\}$ are also excluded.

Frieze and the second author (2008) conjectured that the answer to the question of Ajtai et al. is yes and further conjectured that a much stronger statement holds even for chromatic number.

A hypergraph is c-sparse if every vertex subset S spans at most $c|S|^2$ edges. De Caen (1986) conjectured that for every positive c there is a function $\omega(n) \to \infty$ such that every c-sparse 3-uniform hypergraph has independence number at least $\omega(n)\sqrt{n}$.

We negatively answer the question of Ajtai et al. and disprove the conjectures of de Caen, Frieze and the second author, and many other variants. We also improve the lower bound of some hypergraph Ramsey numbers. In particular, for fixed $s \geq 4$, we prove that $r(P_s, t) = \Theta(t^2)$, where P_s is the tight 3-uniform path with s edges. Phelps and Rödl (1986) showed that $r(P_2, t) = \Theta(t^2/\log t)$. The order of magnitude of $r(P_3, t)$ remains open.

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Geometric representations of graphs

László Lovász

To represent a graph by a nice geometric picture is a natural goal in itself, but in addition it is an important tool in the study of various graph properties and in the design of graph algorithms. We survey several forms of interplay between graphs and geometry.

Rubber band representations. This form a representation was initiated by Tutte [15], who proved that if we replace the edges of a 3-connected planar graph by rubber bands, and nail down the nodes of a face to the vertices of a convex polygon, then in the equilibrium position we get an embedding of the graph in the plane. Rubber band representations have been used, among others, to characterize connectivity and to design connectivity algorithms.

Coin representations. The famous result of Koebe [6] representing a planar graph by touching discs in the plane has many generalizations and applications (see [14]). This theorem and the Riemann Mapping Theorem are mutually limiting cases of each other.

Nullspace representations. Let A be a (possibly weighted) adjacency matrix of a graph, and let λ be an eigenvalue. Write down a basis of the nullspace of A as column vectors, then the rows of the resulting matrix give an embedding of the graph in $\mathbb{R}^d$ (where d is the dimension of the nullspace), which has many nice properties, and plays an important role in the study of the Colin de Verdière number of the graph [2]. Van der Holst's Lemma [5] says that if λ is the second largest eigenvalue, then any hyperplane through the origin separates the graph into two connected pieces (with some well understood exceptions). This can be thought of as a discrete analogue of Courant's Nodal Theorem. For a 3-connected planar graph, appropriate weighting will give a 3-dimensional nullspace for the second largest eigenvalue, and the embedding obtained this way will represent the graph as the skeleton of a 3-polytope [10].

Orthogonal representations. In this case, we want to represent the nodes of a graph by vectors so that non-adjacent nodes should be represented by orthogonal vectors. One parameter to study is the smallest cone into which all the representing vectors can be squeezed in, which leads to the theory of the theta-function of a graph, estimates on the Shannon capacity in information theory, and algorithms for graph coloring (approximate in general, exact, among others, for perfect graphs). It is natural to ask for the smallest dimension in which such a representations exists; under appropriate non-degeneracy conditions, this can be

used to characterize connectivity, and to define an algebraic version of treewidth. Orthogonal representations are related to quantum information theory.

These applications motivate the task of describing the variety of all orthogonal representations in a given dimension. A meaningful question is to determine whether this variety can be decomposed into a finite number of subvarieties, and if so, what is the combinatorial significance of these. Lovász, Saks and Schrijver [12]) proved that in $\mathbb{R}^3$, a graph gives an irreducible variety if and only if it is $(n-3)$-connected. One new result can be reported here: in $\mathbb{R}^4$, a graph gives an irreducible variety if and only if it is $(n-4)$-connected, and different from the complement of the cube.

Distance respecting representations. We want to find a geometric representation such that the graph distance of any two nodes is as close to the euclidean (or ℓ_p, or other standard) distance between the representing vectors as possible. This is a basic problem in functional analysis, and many powerful results can be applied. But there are also some interesting graph-theoretic twists. Linial, London and Rabinovich [8] showed how a fundamental result of Bourgain [1] can be used to prove and improve the important approximate Max-Flow-Min-Cut Theorem of Leighton and Rao [7]. Feige [3] introduced the notion of volume-respecting embeddings, showed that a suitable version of Bourgain's method yields such an embedding, and used this to obtain polylogarithmic approximation algorithm for the bandwidth of a graph.

Representations by the adjacency matrix. Let us represent each node of a graph by the corresponding column of the square of the adjacency matrix. Lovász and Szegedy [13] showed that partitioning (clustering) the set of representing vectors into small diameter sets corresponds to a regularity partition of the graph (in the weak sense of Frieze and Kannan [4]). This provides efficient algorithms for finding weak regularity partitions, in particular in the property testing model (see [11]).

This incomplete list of examples suggests that there may be an emerging theory of geometric representations. A full theory may be far away, considering the variety of possibilities how the graph structure can be reflected by the geometry. Nevertheless, there seem to be some general ideas (non-degeneracy, duality, geometric clustering) that point to such a theory.

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Problem session

Among others, the following problems were presented.

Peter Allen: A k-uniform hypergraph is a hypergraph in which all edges have size k. A k-uniform tight path P_ℓ^k is given by an ordering, and a tight cycle C_ℓ^k of length ℓ by a cyclic ordering, of ℓ vertices with every k consecutive vertices in the ordering forming an edge.

What is the extremal number $ex(n; P_\ell^k)$? What is $\mathrm{ex}(n; C_\ell^k)$ when k divides ℓ?

The first question in the case of graphs, $k = 2$, is solved. If $\ell - 1$ divides n, then the extremal graph is a disjoint union of cliques on $\ell - 1$ vertices (Erdős and Gallai). This should be seen as coming from a combinatorial design with blocks of size $\ell - 1$ covering each 1-set exactly once. If $\ell - 1$ does not divide n, then the design does not exist, but an approximation does: namely one takes as many vertex-disjoint $(\ell - 1)$-cliques as possible and one smaller clique. This is extremal (Faudree and Schelp).

For graph cycles, the question is not so well understood. The design-based construction is far from optimal if ℓ is constant or grows very slowly; constructions giving provably good lower bounds on $ex(n; C_\ell)$ are not known for most constant even values of ℓ. But when ℓ is linear in n the question is again well understood, and the extremal constructions are similar: again one takes a maximum number

of $(\ell-1)$-cliques and if necessary one smaller clique, but this time with one vertex common to all cliques and no pair of cliques sharing any other vertex.

In k-uniform hypergraphs, for $k \geq 3$, only special cases of either problem are well understood. If ℓ is small then it seems likely that taking a design with blocks of size $\ell-1$ covering each $(k-1)$-set exactly once, and replacing the blocks with cliques, yields an extremal graph for P_ℓ^k. As with graphs these constructions are far from optimal for tight cycles.

For long tight paths and cycles, we can prove upper and lower bounds on the extremal number which differ by a small factor. With Julia Böttcher, Oliver Cooley and Richard Mycroft we can prove an approximate hypergraph version of the Erdős-Gallai theorem: for all $\varepsilon > 0$ and k, if n is sufficiently large, then every k-uniform n-vertex hypergraph with $(\alpha+\varepsilon)\binom{n}{k}$ edges contains C_ℓ^k for each ℓ divisible by k between $2k$ and αn inclusive. This matches asymptotically the formula one obtains from the design-based lower bounds. Unfortunately, these designs, or even approximations to them, do not exist for $k \geq 3$ and ℓ linear in n. The best lower bounds we know of valid for ℓ linear in n are obtained by fixing a set of $\frac{\ell}{k}-1$ vertices and drawing a hypergraph with edges present if and only if they contain a member of the fixed set (if ℓ is not too close to n) and by taking a clique on $\ell-1$ vertices together with every edge meeting the clique in at most $k-2$ vertices (if ℓ is close to n).

Noga Alon: For a graph $G=(V,E)$, let $\tau(G)$ denote the minimum number of pairwise edge disjoint complete bipartite subgraphs of G so that each edge of G belongs to exactly one of them.

Let $\alpha(G)$ denote the maximum size of an independent set of G. It is easy to see that for every graph G, $\tau(G) \leq n - \alpha(G)$. Erdős conjectured (see [2], and also [1] for a more general conjecture and some results supporting it) that for almost every graph G equality holds, that is, that for the random graph $G(n,0.5)$, $\tau(G) = n-\alpha(G)$ with probability that tends to 1 as n tends to infinity (*whp*, for short).

It turns out, however, that Erdős' conjecture for $G = G(n,0.5)$ is (slightly) incorrect. In fact for most values of n, and for $G=G(n,0.5)$, $\tau(G) \leq n-\alpha(G)-1$ whp, while for some exceptional values of n (that is, those values for which the size of $\alpha(G)$ is concentrated in two points, and not in 1 point), $\tau(G) \leq n-\alpha(G)-2$ with probability that is bounded away from 0. As far as we know it may be possible that for these values of n $\tau(G) = n-\alpha(G)$ with probability bounded away from 0 (but not with probability that tends to 1 as n grows).

We suggest the following alternative conjecture. Let $\beta(G)$ denote the largest number of vertices in an induced complete bipartite subgraph of G. It is easy to see that for every G, $\tau(G) \leq n-\beta(G)+1$. The revised conjecture is that for the random graph $G=G(n,0.5)$, $\tau(G) = n-\beta(G)+1$ whp.

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Maria Chudnovsky: Let G be a graph. A *stable set* in G is a set of vertices, all pairwise non-adjacent. Let $I_E(G)$ be the number of stable sets S in G with $|S|$ even, and $I_O(G)$ the number of stable sets S with $|S|$ odd. Write $I(G) = |I_E(G) - I_O(G)|$. We denote by $\chi(G)$ the chromatic number of G, and by $\omega(G)$ the size of the largest clique in G.

Kalai and Meshulam conjectured the following:

Theorem 1. For every $l \geq 0$ there exists k, such that if $I(H) \leq l$ for every induced subgraph H of G, then $\chi(G) \leq k$.

Observe that $I(K_n) = n - 1$, and thus a weaker statement is true:

Theorem 2. If G is a graph such that $I(H) \leq l$ for every induced subgraph H of G, then $\omega(G) \leq l + 1$.

Reinhard Diestel: A *k-block* in a graph G is a maximal set of at least k vertices such that no two of these can be separated in G by fewer than k other vertices. What properties of G force it to have a k-block? More generally, relate the *block number*

$$\beta(G) := \max\{\, k \mid G \text{ has a } k\text{-block} \,\}$$

to other graph invariants.

It would also be of interest to find weak conditions forcing a graph to contain either a k-block or a tangle of order k: what can we say about graphs that contain neither?

For related references, see the the references of the corresponding abstract on page 20.

Ehud Friedgut: A t-coset of S_n is a coset of the stabilizer of t fixed points. Is it true that every partition of S_n into t-cosets is a refinement of a partition into $(t-1)$-cosets?

Peter Heinig: Question 1. (Another strengthening of the conclusion of Dirac's theorem?) Is it true that for every odd $n \in \mathbb{N}$, every graph G on n vertices with minimum-degree at least $\lceil \frac{1}{2}n \rceil$ has the property that every cycle in G can be constructed as a symmetric difference of Hamilton circuits of G?

As to versions of this question for even n, by parity alone, one may only hope for every *even* cycle being generated by Hamilton circuits, and for this there is at least one infinite set of counterexamples: two cliques on $\frac{1}{2}n+1$ vertices intersecting in exactly two vertices. Here, the edge in the intersection is not contained in any Hamilton circuit, and there exist even cycles containing that edge. However, it seems likely that for even n, the set of all counterexamples to the above hope is easy to describe. Therefore, a more complete version of a conjectural strengthening of Dirac's theorem is the following: is it true that for every $n \in \mathbb{N}$ and for every n-vertex graph G with minimum-degree at least $\lceil \frac{1}{2}n \rceil$, the $\mathbb{F}_2$-linear span of the set of all the Hamilton circuits of G is as large as the parity of n by itself permits,

except when n is even and G in an infinite set of exceptional Dirac-graphs of rather low descriptional complexity?

(With the slightly stronger hypothesis of minimum-degree at least $(\frac{1}{2}+\gamma)n$ for arbitrarily small $\gamma > 0$, it was recently proved that the above questions have an affirmative answer for all n larger than some n_0 depending at most on the given γ. With this strengthened minimum-degree-condition there then do not exist any exceptional graphs in the case of even n.)

Question 2. (a structural reason for the above conjectural strengthening of Dirac's theorem?) Is it true that for every odd $n \geq 7$, every n-vertex graph with minimum-degree at least $\lceil \frac{1}{2}n \rceil$ contains a spanning copy of the graph obtained as follows: start with a square of an n-circuit, then delete every other edge on the periphery until you are left with exactly three consecutive degree-4-vertices. (This is then a graph with $n-3$ degree-3-vertices and 3 degree-4-vertices. An affirmative answer to Question 2 implies an affirmative answer to Question 1.)

June Huh:

(1) Is there a matroid whose chromatic (characteristic) polynomial is not the chromatic polynomial of a matroid which is representable over some field?
(2) Is there a matroid not representable over any field which is chromatically unique?
(3) How small can this matroid from (1) or (2) be?

Nathan Linial: I want to call attention to two old problems of which people are not necessarily sufficiently aware:

1. In the tensor product of graphs G and H there is an edge between (x, y) and (x', y') iff x, x' are neighbors in G and y, y' are neighbors in H. Clearly every coloring of G yields a coloring of the tensor product by referring only to the first coordinate. Likewise for H and the second coordinate. Hedetniemi's conjecture states that the chromatic number of the product equals $\min(\chi(G), \chi(H))$.

2. It is well known that there is exactly one infinite graph (the Rado Graph) G which is uniquely characterized by the *extension property*: Given two finite disjoint sets of vertices A and B there is a vertex x that is adjacent to every vertex in A and to none in B. In a finite $G(n, 1/2)$ graph this condition holds for all A and B of bounded cardinality. Model theorists were interested in similar phenomena in triangle-free graphs, in which case, of course, the set A must be independent. Indeed the infinite triangle-free analog of Rado graph is known to exist (this is easy). Gregory Cherlin has asked whether a finite version exists as well. We say that a triangle-free graph is k-existentially-complete (k-ec) if for every independent A and a disjoint B with $|A| + |B| \leq k$ there is an x that's a neighbor to all of A and none of B. Many examples of triangle-free 2-ec are known, some families with $k = 3$ are known as well, and none are known with a larger k. I conjecture that there is some constant r such that no triangle free graph can be r-ec.

Ron Peled: Question 1: Consider a uniformly chosen proper 3-coloring of the $n \times n$ grid graph. Prove that the probability that the north-east corner has the same color as the south-west corner goes to $1/3$ as n tends to infinity. This question is related to the square ice model (uniform 6-vertex model) and to the fluctuations of random graph homomorphisms from the grid graph to Z. See, e.g., `http://arxiv.org/abs/1005.4636`

Question 2: For a permutation σ in S_n, let $LIS(\sigma)$ denote the length of the longest increasing subsequence of σ. For each σ, fix an arbitrary increasing subsequence $S(\sigma)$ of σ having length $LIS(\sigma)$. Let $1 \leq t \leq n$ be a parameter. Define a matrix M whose rows are indexed by S_n and whose columns are indexed by permutations sigma with $LIS(\sigma) \geq n - t$ via the following definition: $M_{\pi,\sigma} := 1$ if π equals σ on all indices not in $S(\sigma)$ and $A_{\pi,\sigma} := 0$ otherwise. What is the determinant of A? Simulations indicate that it has a nice product structure featuring only prime factors of size at most n. For instance, for $n = 7$, $t = 2$ we get $7^{100} \cdot 5^{65} \cdot 3^{627} \cdot 2^{1150}$, but there is no conjecture for the exact formula the determinant takes. This question is related to counting the number of t-wise permutations, see Section 3.5 in `http://arxiv.org/abs/1302.4295` for more details.

Mathias Schacht: We say a subhypergraph B of an r-uniform hypergraph H is a *blow-up* of an r-uniform hypergraph F if it is isomorphic to a hypergraph which is obtained from F by replacing every vertex v of F by an independent set U_v and for every edge $e = \{v_1, \ldots, v_r\}$ of F we add a complete r-partite r-uniform hypergraph spanned by $U_{v_1}, \ldots, U_{v_r}$. We say a blow-up of F is a *t-blow-up* if $|U_v| \geq t$ for every $v \in V(F)$.

A classical result of Erdős [2] asserts that for sufficiently large n every r-uniform hypergraph on n vertices with cn^r edges contains a t-blow-up of $K_r^{(r)}$ (the r-uniform hypergraph consisting of a single edge) where

$$t \geq \gamma(\log n)^{\frac{1}{r-1}}$$

and $\gamma > 0$ only depends on c. Moreover, a random construction shows that this bound on t is asymptotically best possible. From Erdős' result it follows, that every n-vertex r-uniform hypergraph B containing at least cn^ℓ copies of the complete r-uniform hypergraph on ℓ vertices $K_\ell^{(r)}$ contains a t-blow-up of $K_\ell^{(r)}$ for some

$$t \geq \gamma(\log n)^{\frac{1}{\ell-1}}\,.$$

For graphs Nikiforov [3] obtained a much better bound by an elegant application of the Kövari-Sós-Turán theorem.

Theorem 3 (Nikiforov)**.** For every $\ell \geq 3$ and $c > 0$ there exist $\gamma > 0$ and n_0 such that every graph $G = (V, E)$ on $n \geq n_0$ vertices which contains cn^ℓ copies of K_ℓ contains a t-blow-up of K_ℓ for some $t \geq \gamma \log n$.

It would be interesting to investigate to which extend this result has generalizations for hypergraphs and the following question was suggested by several researchers.

Question 2. For which r-uniform hypergraphs F it holds that for every $c > 0$ there exists a $\gamma > 0$ such that for sufficiently large n every n-vertex r-uniform hypergraph H that contains $cn^{|V(F)|}$ copies of F also contains a t-blow-up of F with

$$t \geq \gamma(\log n)^{\frac{1}{r-1}} .$$

A few very special hypergraphs F were considered in [4]. Nikiforov and Reiher independently proved such a result for $K_4^{(3)-}$ (the 3-uniform hypergraph with three edges and four vertices). To our knowledge the question could have an affirmative answer for any hypergraph F. However, even the case $F = K_4^{(3)}$ is unresolved. We also put forward the following somewhat related question for graphs.

Question 3. Is it true that every n-vertex graph with $n^3/\exp(C\sqrt{\log n})$ triangles contains a t-blow-up of K_3 with $t \geq \gamma\sqrt{\log n}$ for some $\gamma = \gamma(C)$.

A closely related question already appeared in the work of Bollobás, Erdős, and Simonovits [1]. Owing to the supersaturation phenomenon it follows that every n-vertex r-uniform hypergraph with $(\pi(F)+\varepsilon)\binom{n}{r}$ contains at least $cn^{|V(F)|}$ copies of F, where $\pi(F)$ denotes the Turán-density of F and $c = c(\varepsilon)$. Consequently, an affirmative answer to Question 2 for every hypergraph F implies an affirmative answer to the following question from [1].

Conjecture 10 (Bolobás, Erdős & Simonovits)**.** For every r-uniform hypergraphs F and every $\varepsilon > 0$ there exists a $\gamma > 0$ such that for sufficiently large n every n-vertex r-uniform hypergraph H that contains at least $(\pi(F) + \varepsilon)\binom{n}{r}$ edges also contains a t-blow-up of F with

$$t \geq \gamma(\log n)^{\frac{1}{r-1}} .$$

Maybe for hypergraphs F for which the Turán-density $\pi(F)$ is known, Conjecture 10 is approachable.

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Van Vu: (1) Let $A(n, d)$ be the adjacency of the random regular graph $G(n, d)$, for $d > 2$. Then $A(n, d)$ is non-singular with probability tending to 1 as n tends to infinity.

(2) What is the limiting distribution of the permanent of a random ± 1 matrix?

(3) What is the limiting distribution of the determinant of a random symmetric ± 1 matrix?

For the determinant of a random ± 1 matrix it is known that the limiting distribution of its absolute value is log-normal.The same must hold for (3).

(4) Let M_n be a random ± 1 matrix of size n. I conjecture that with probability $1 - o(1)$, one cannot switch the signs of the diagonal entries to make the matrix singular.

(5) Let M_n be a random ± 1 matrix of size n. I conjecture that with probability $1 - o(1)$, M_n has at least two real eigenvalues. (The real conjecture is that M_n has $\Theta(\sqrt{n})$ real roots with high probability; this is known to be true for random Gaussian and random $(0, \pm 1)$ matrix.)

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