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Mini-Workshop: Quaternion Kähler Structures in Riemannian and Algebraic Geometry

Organised by
Anna Fino, Torino
Uwe Semmelmann, Stuttgart
Jaroslaw Wiśniewski, Warszawa
Frederik Witt, Münster

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Abstract. Metrics of special holonomy are of central interest in both Riemannian and complex algebraic geometry. We focus on an important classification problem of a particular type of special holonomy manifolds, namely compact quaternion-Kähler with positive scalar curvature (Salamon-LeBrun conjecture). In the language of algebraic geometry this corresponds to the classification of Fano contact manifolds. By bringing together leading experts in both fields this workshop pursued a two-fold goal: First, to revise old and to develop new strategies for proving the most central conjecture in the field of quaternionic Kähler geometry. Second, to introduce young researchers at PhD/PostDoc level to this interdisciplinary circle of ideas.

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Introduction by the Organisers

This mini-workshop was organised by Anna Fino (Università di Torino), Uwe Semmelmann (Universität Stuttgart), Jarosław Wiśniewski (Uniwersytet Warszawski) and Frederik Witt (Universität Münster). We had 16 participants (four of which at junior level) to discuss new and old approaches to the Salamon-LeBrun conjecture. Its Riemannian side asserts that any compact quaternion-Kähler with positive scalar curvature is necessarily symmetric. On the algebraic side this matches the conjecture that any Fano contact manifold is homogeneous.

To explain this link briefly, let us recall that metrics of special holonomy, such as quaternion-Kähler metrics, are of central interest in both Riemannian and complex algebraic geometry. This is evident in the case of Kähler and Calabi-Yau metrics where the metric induces a distinguished complex structure. The case of hyperkähler and quaternionic-Kähler manifolds is less clear. For hyperkähler metrics we have an S^2 worth family of complex structures which in general are not biholomorphic to each other, while for quaternionic-Kähler metrics there is in general no complex structure at all. However, a generalisation of the Atiyah-Hitchin-Singer twistor space construction in dimension four to higher dimensions associates with these metrics a well-defined complex manifold. For instance, if (M^{4k}, g) is a quaternionic-Kähler manifold of real dimension $4k$, then its associated twistor space Z is a complex contact manifold of complex dimension $2k+1$. Moreover, Z carries a positive Kähler-Einstein metric if M is compact with positive scalar curvature. In particular, Z is Fano.

Existence of special holonomy metrics, in particular non-symmetric ones, is one of the central issues of the theory. For instance, Berger's original classification of non-symmetric metrics included the case of Spin(9) which subsequently could be ruled out. Similarly, the only known examples of compact quaternion-Kähler manifolds with positive scalar curvature are symmetric. The Salamon-LeBrun conjecture (which for instance is true in dimension 8) asserts that these are the only examples. Translated into complex geometry via the twistor construction this boils down to show that any Fano contact manifold is homogeneous.

On the other hand, contact structures on complex projective manifolds are very rare. By results of Demailly, Kebekus, Peternell, Sommese and Wiśniewski, if a contact projective manifold admits a contact structure and its second Betti number is > 1 then the manifold in question is the projectivisation of the (co)tangent bundle over another projective manifold and the contact structure is the natural one. Thus, such manifolds seem to be exceptional and it is plausible to expect that the only known examples constitute the complete list of such manifolds.

The workshop started off with a couple of introductory lectures by Witt and Simon Salamon (who joined us via a video conference) on quaternionic-Kähler manifolds and the twistor construction, and by Wiśniewski on Fano contact manifolds. We then had more specialised lectures dealing with specific issues.

On the differential geometric side, Semmelmann and Weingart talked about representation theoretic methods for proving Weitzenböck formulæ and vanishing theorems. This linked also into Dessai's talk on quaternion-Kähler manifolds in dimension 12 where an application of the index theoretic and topological ideas successful in dimension 8 have failed so far. The talks by Swann, Cortés and Bielawski dealt with the construction of quaternion-Kähler metrics. Finally, Amann and Moroianu talked about important properties of quaternion-Kähler manifolds, namely formality and the non-existence of almost complex structures.

On the algebraic side, apart of the Wiśniewski's lecture which provided a general introduction to contact Fano manifolds, there were four lectures about specific algebraic geometry methods for studying such manifolds. Buczyński, Hwang and Kebekus lectures were about rational curves on contact Fano manifolds. Kebekus used families of minimal rational curves to recover the contact distribution. Hwang explained his results and expectations regarding the rôle of the variety of minimal rational tangents (or VMRT) of a contact manifold in the classification of such manifolds. On the other hand, Buczyński's talk was about the singularities of minimal rational curves on contact manifolds. The lecture by Campana concerned a different approach to the classification problem. Namely, Campana presented results by Clemens and Ran concerning generic semipositivity of sheaves of differential operators on Fano manifolds.

During the week we had intensive discussions on old and new approaches to this conjecture with stimulating exchanges between Riemannian and algebraic geometers during and after the talks. We also noted with pleasure the high commitment and very active presence of the young participants. As a result we are planning a sequel to this workshop in the near future. Finally we are happy to acknowledge the never failing support and professional handling by the entire Oberwolfach stuff.

Mini-Workshop: Quaternion Kähler Structures in Riemannian and Algebraic Geometry

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Abstracts

Introduction to quaternion-Kähler manifolds and the twistor space construction

Frederik Witt

This talk gave an introduction to quaternion-Kähler geometry and its associated twistor space in two sessions. Good references are [Bes87], [Bea07], [Sal82] and [Sal99].

Holonomy groups. With a Riemannian manifold (M^n, g) we can associate the so-called *holonomy group* which is a subgroup of $\mathrm{O}(n)$. In fact, a generic metric will always have full holonomy $\mathrm{O}(n)$. At the other extreme, triviality of the holonomy group implies flatness of the metric. In between are the metrics of *special holonomy.* It is a surprising and highly nontrivial fact that the list of possible holonomy groups for a simply-connected, complete and irreducible Riemannian manifold is rather simple. There are two cases to consider. If g is symmetric, then the holonomy group can be read off from Cartan's list of irreducible symmetric spaces. Otherwise, we can look it up in Berger's list [Ber55]:

dim	holonomy	geometry	algebra
$2m$	$\mathrm{U}(m)$	Kähler	$\mathbb{C}$
$2m$	$\mathrm{SU}(m)$	Calabi-Yau	$\mathbb{C}$
$4k$	$\mathrm{Sp}(k)\mathrm{Sp}(1)$	quaternion-Kähler	$\mathbb{H}$
$4k$	$\mathrm{Sp}(k)$	hyperkähler	$\mathbb{H}$
8	$\mathrm{Spin}(7)$	$\mathrm{Spin}(7)$	$\mathbb{O}$
7	G_2	G_2	$\mathbb{O}$

Loosely speaking the occurring geometries are tied to the three normed division algebras extending the reals, namely the complex numbers $\mathbb{C}$, the quaternions $\mathbb{H}$ and the octonians $\mathbb{O}$.

Quaternion-Kähler manifolds. By definition, a quaternion-Kähler manifold is a Riemannian manifold (M^{4k}, g) whose holonomy group is contained in $\mathrm{Sp}(k)\mathrm{Sp}(1)$ which is one of the possible holonomy groups of an irreducible Riemannian manifold. In particular, g is of special holonomy if g is not flat. At any rate, g must be *Einstein*, that is, its Ricci tensor must be a constant multiple of g, $\mathrm{Ric}_g = \lambda g$. If $\lambda \neq 0$, then g is irreducible and therefore, the holonomy group must be equal to $\mathrm{Sp}(k)\mathrm{Sp}(1)$.

Examples. (i) The projective quaternionic space $\mathbb{HP}^n = \mathrm{Sp(n+1)}/\big(\mathrm{Sp(n)} \times \mathrm{Sp(1)}\big)$

(ii) More generally, there is exactly one compact symmetric quaternion-Kähler manifold $G/K\mathrm{Sp}(1)$ for each compact simple Lie group G except SU(2), the so-called *Wolf spaces* [Wol65]. They are all positive, i.e. $\lambda > 0$. It is an open question if any positive compact quaternion-Kähler manifold is symmetric (*Salamon-LeBrun conjecture*).
(iii) Alekseevskiĭ proved that any homogeneous compact quaternion-Kähler manifold is symmetric [Ale68]. On the other hand he exhibited examples of non-symmetric homogeneous quaternion-Kähler manifolds [Ale75] (these have necessarily $\lambda < 0$).

Despite their name quaternion-Kähler manifolds are almost never Kähler. In fact, they are in general not even complex. Ultimately this stems from the following linear algebra fact. The standard action of $\mathrm{Sp}(k)$ on $\mathbb{R}^4 \cong \mathbb{H}^k$ commutes with the action of imaginary quaternions from the right, so that in particular, we have three complex structures i, j and k. However, these are permuted by the $\mathrm{Sp}(1)$ factor so that there is no distinct complex structure. In global terms this is reflected by the existence of a preferred rank three subbundle $\mathcal{E} \subset \mathrm{End}(TM)$ over a quaternion-Kähler manifold. Locally, $\mathcal{E}$ is spanned by three (almost) complex structures I, J and K which, however, have no global meaning in general.

Twistor spaces. If (M, g) is quaternion-Kähler, then the sphere bundle $\mathcal{Z}(M) = \mathbf{S}(\mathcal{E}) \to M$ of the endomorphism bundle $\mathcal{E} \to M$ is called the *twistor space* of (M, g). This is a generalisation of the twistor space of a selfdual 4-manifold constructed by Atiyah, Hitchin and Singer [AHS7].

Theorem (Bérard-Bergery [Bes87], Salamon [Sal82]). The twistor space $\mathcal{Z} \to M^{4k}$ carries a natural complex manifold structure of complex dimension $2k + 1$ such that

- the fibres are rational curves with normal bundle $2k\mathcal{O}(1)$.
- the usual antipodal map of $\mathbb{CP}^1$ induces a fibre preserving antiholomorphic involution $\mathcal{Z} \to \mathcal{Z}$.
- we have an exact sequence of holomorphic bundles

$$0 \to D \to T\mathcal{Z} \stackrel{\theta}{\to} L \to 0$$

  with $\theta \wedge (d\theta)^k \neq 0$, that is, θ defines a *contact structure* with $D = \ker\theta$. In particular,

$$L^{\otimes(k+1)} \cong K^{-1},$$

  where $K \to \mathcal{Z}$ is the canonical line bundle of $\mathcal{Z}$.

In the case of $\lambda > 0$, even more is true.

Theorem (LeBrun [LeB95], Salamon [Sal82]).

- If (M, g) is positive, then $\mathcal{Z}(M)$ carries a positive Kähler metric such that $\mathcal{Z} \to M^{4k}$ becomes a Riemannian submersion with totally geodesic fibres. In particular, $\mathcal{Z}(M)$ is a Fano contact manifold.

- Conversely, any $\mathcal{Z}(M)$ Fano contact manifold which admits a Kähler-Einstein metric is the twistor space of a positive quaternion-Kähler manifold.

Examples. (i) For the projective quaternionic space we find $\mathcal{Z}(\mathbb{HP}^k) = \mathbb{CP}^{2k+1}$ where the fibration is simply the Hopf fibration $\mathbb{C}^{2k+2} \setminus \{0\}/\mathbb{C}^* \to \mathbb{H}^{k+1} \setminus \{0\}/\mathbb{H}^*$. (ii) More generally, for the symmetric spaces $G/K\mathrm{Sp}(1)$ we find $\mathcal{Z}\big(G/K\mathrm{Sp}(1)\big) = G/K\mathrm{U}(1)$ where the fibration is induced by the inclusion $\mathrm{U}(1) \hookrightarrow \mathrm{Sp}(1)$.

In particular, a symmetric quaternion-Kähler space has a homogeneous twistor space. Further, if the symmetric space is compact, then its twistor space is a homogeneous Fano contact manifold with a positive Kähler-Einstein metric. More generally, it is conjectured that any Fano contact manifold (positive Kähler-Einstein or not) is homogeneous which, if it was true, would imply the Salamon-LeBrun conjecture (see for instance [Bea98] for some evidence in support of this conjecture).

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Introduction to contact Fano manifolds

Jarosław A. Wiśniewski

This talk was presented on the first day of the workshop. Its aim was to introduce the basic notions and techniques concerning Fano manifolds and contact complex structures which are in relation to the classification of positive quaternion Kähler manifolds. Excellent general references to this subject are [2] and [4].

Let X be a complex projective manifold with the (complex) tangent bundle T_X and the cotangent bundle Ω_X of holomorphic differentials. A line bundle L

over X is called ample if for some $m > 0$ the bundle $L^{\otimes m}$ is a pullback of the hyperplane section bundle over $\mathbf{P}^N$ for some embedding in the complex projective space $X \hookrightarrow \mathbf{P}^N$.

We say that X is a Fano manifold if the line bundle $\det T_X$ is ample. The classification of Fano manifolds is known in dimension ≤ 3. In general, it is known that in every dimension there is a finite number of deformation families of such manifolds.

The key result in the modern treatment of Fano manifolds is the following theorem by Mori, see [6]: For every point $x \in X$ there exists a rational curve $f : \mathbf{P}^1 \to X$, such that $x \in f(\mathbf{P}^1)$ and $0 < deg(f^* T_X) \leq dim X + 1$. Studying Fano manifolds via rational curves is the technique of choice for most algebraic geometers nowadays.

In particular, a Fano manifold X can be studied by using a variety of minimal rational tangents (or VMRT) which parametrizes tangent directions to minimal rational curves contained in X and passing through a sufficiently general point $x \in X$. In many instances the information about VMRT of X allows to recover the structure of X itself.

Let X be a complex manifold of dimension $2n + 1$ with a line bundle L. Let us consider a twisted holomorphic 1-form $\theta \in H^0(X, \Omega_X \otimes L)$. We say that θ is a contact form and X is a contact manifold if $\theta \wedge (d\theta)^n$ does not vanish anywhere. This implies that $L^{\otimes(n+1)} \cong \det T_X$.

If X is the twistor space of a positive quaternion Kähler manifold then it is a Fano contact manifold and, in addition, it admits Einstein metric so, in particular, its tangent bundle is stable.

If X is a contact Fano manifold of dimension $2n+1$ then, by [5] and [7], either X is the projectivized (co)-tangent bundle over $\mathbf{P}^{n+1}$ or its second Betti number is 1. In the latter case, by [3], either $X \cong \mathbf{P}^{2n+1}$ or L generates the group of line bundles over X (the Picard group of X).

Beauville proved, see [1], the following theorem: Suppose that X is a contact Fano manifold which admits Kähler–Einstein metric and L has enough (holomorphic) sections so that they generate first jets of L at a generic point of X or, equivalently, the rational map defined by these sections is generically finite-to-one. Then X is a rational homogeneous manifold. In fact, then X is isomorphic to the closed (minimal) orbit in the projectivization of the adjoint representation of a simple algebraic group.

Thus, in view of Beauville's result, we have a natural counterpart of the celebrated LeBrun-Salamon conjecture, [5], namely every contact Fano manifold is a rational homogeneous variety.

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Weitzenböck formulas on quaternionic Kähler manifolds

Uwe Semmelmann
(joint work with Gregor Weingart)

The aim of my talk was to show how Weitzenböck formulas on quaternionic Kähler manifolds can be used to prove the vanishing of certain characteristic numbers, e.g. indices of twisted Dirac operators.

Let (M^{4n}, g) be a complete quaternion Kähler manifold of positive scalar curvature. Then the holonomy condition $Hol(M, g) \subset Sp(n) \cdot Sp(1)$ implies a reduction of the frame bundle to a $Sp(n) \cdot Sp(1)$-principal bundle P. In the first part of my talk I explained how geometric vector bundles can be realized as vector bundles associated to P using the fundamental $Sp(n)$-representation E and the fundamental $Sp(1)$-representation H. Examples are the complexified tangent bundle $TM^{\mathbb{C}} \cong H \otimes E$ or the spinor bundle $S = \oplus_{i=0}^{n} R^{l,n-l}$ where the bundle $R^{l,d}$ is defined as $R^{l,d} = \mathrm{Sym}^l H \otimes \Lambda_0^d E$.

In the second part of my talk I introduced differential operators acting on sections of vector bundles $\pi(M)$ associated to P via a representation π. In particular the universal Laplace operator $\Delta_\pi := \nabla^*\nabla + q(R)$, where $q(R)$ is a certain curvature term, defined as endomorphism of $\pi(M)$. A first important property of Δ_π is the following: the Hodge Laplace operator $\Delta = dd^* + d^*d$ restricted to any parallel subbundle $\pi(M)$ of the form bundle $\Lambda^* T^* M$ is given by Δ_π. Moreover, on symmetric spaces G/K the operator Δ_π coincides with the Casimir operator of G. An other important class of operators are twisted Dirac operators $D_V : \Gamma(S \otimes V) \to \Gamma(S \otimes V)$. They are defined as a certain projections of the covariant derivative on $S \otimes V$.

In [1] we proved a formula relating Δ_π and the square of the twisted Dirac operator $D_{R^{l,d}}$ restricted to some irreducible representation $\pi \subset S \otimes R^{l,d}$:

$$\Delta_\pi \;=\; D^2_{R^{l,d}}\big|_\pi \;+\; \varphi(l,d)\,\mathrm{id}_\pi \;,$$

where $\varphi(l,d) = \frac{scal}{8n(n+2)}\,(l + d - n)\,(l - d + n + 2)$.

In the third part of my talk I explained several applications of this formula. The first application is an eigenvalue estimate of the operator Δ_π for representations

$\pi = \mathrm{Sym}^k H \otimes \Lambda_0^{a,b} E$. The result is:

$$\Delta_\pi \;\geq\; \lambda_\pi := \frac{scal}{8n(n+2)}\,(k+a-b)\,(2n+2+k-a-b)\;.$$

In particular, $\Delta_\pi > 0$ for all representations π different from $\Lambda_0^{a,a}E$. Since $\Lambda_0^{a,a}E \subset \Lambda^{ev}TM$ there are no harmonic forms in odd degree. Hence we obtain a new proof of the well known fact (due to S. Salamon) that the odd Betti numbers of a positive quaternionic Kähler manifold have to vanish.

As a second application I showed in my talk that the kernel of the twisted Dirac operator $D_{R^{l,d}}$ is given as a direct sum of certain minimal eigenspaces of the operator Δ_π. In particular we have the following expression for the index of the twisted Dirac operator $D_{R^{l,d}}$ (details can be found in [1]):

$$\mathrm{index}(D_{R^{l,d}}) \;=\; \sum (-1)^{|\pi|} \dim\ker(\Delta_\pi - \lambda_\pi)$$

where the sum goes over all representations $\pi \subset S \otimes R^{l,d}$, such that the number $\varphi(l,d)$ is maximal under all $\varphi(r,s)$ with $\pi \subset S \otimes R^{r,s}$. In this situation the twist bundle $R^{l,d}$ is called the maximal twist of the representation π. The sign $(-1)^{|\pi|}$ is also explicitly given. The minimal eigenvalue λ_π is this maximal number $\varphi(l,d)$.

An immediate consequence is the vanishing result $\mathrm{index}(D_{R^{l,d}}) = 0$, for $l+d < n$, first proved by C. LeBrun and S. Salamon. Indeed, all maximal twist bundles $R^{r,s}$ satisfy $r+s \geq n$. Thus there is no representation π with maximal twist $R^{l,d}$ with $l+d<n$ and the kernel of $D_{R^{l,d}}$ is zero.

As a new example I considered in my talk the special twist bundle $R^{l,d} = \mathrm{Sym}^{n+1}H \otimes E$. The representations π appearing in the sum above are $\pi = H \otimes E$ and $\pi = \mathrm{Sym}^2 H \otimes \Lambda_0^{1,1}E$. However it is well known that $\ker(\Delta_{HE} - \lambda_\pi)$ is different from zero only on the quaternionic projective space $\mathbb{H}P^n$. This is an analogue of the Lichnerowicz Obata Theorem for quaternionic Kähler manifolds. Hence on all quaternionic Kähler manifolds different from $\mathbb{H}P^n$ we have

$$\mathrm{index}(D_{R^{n+1,1}}) \;=\; -\dim\ker(\Delta_{\mathrm{Sym}^2 H \Lambda_0^{1,1} E} - \lambda_\pi) \;\leq\; 0\;,$$

where for both summands λ_π is the minimal eigenvalue is $\frac{(n+1)scal}{2n(n+2)}$.

The final example in my talk was the twist bundle $R^{l,d} = \mathrm{Sym}^{n+2r}H$. In this case the representation $\pi = \mathrm{Sym}^{2r}H$ is the only one appearing in the sum above. Thus we obtain the following expression for the Hilbert Polynomial $P(r)$ (cf. [2] for further details):

$$P(r) \;:=\; \mathrm{index}(D_{\mathrm{Sym}^{n+2r}H}) \;=\; -\dim\ker(\Delta_{\mathrm{Sym}^{2r}H} - \lambda_\pi)$$

The Hilbert polynomial is also given as a holomorphic Euler characteristic $P(r) = \chi(Z, \mathcal{O}(L^r))$. Here L is the contact line bundle on the twistor space Z. We note that the bundle L is related to the canonical bundle K via $L^{n+1} = K^*$. The minimal eigenvalue λ_π is given as $\lambda_\pi = \frac{r(n+1+r)scal}{2n(n+2)}$.

The Hilbert polynomial P is a polynomial of degree $2n+1$ with leading coefficient $\frac{\deg(Z)}{(2n+1)!}$, where $\deg(Z) = < c_1(L)^{2n+1}, [Z] >$.

At the end of my talk I discussed an upper bound of $P(r)$ obtained in [2]. There we proved

$$P(r) \;\leq\; P_{\mathbb{H}P^n}(r) \;=\; \dim(\mathrm{Sym}^{2r}(H \oplus E)) \;=\; \binom{2n+1+2r}{2n+1}$$

Immediate consequences are upper bounds for the dimension of the isometry group $\dim Iso(M,g)$, the quaternionic volume $v(M)$ and the related degree $\deg(Z)$. We find that $\dim Iso(M,g)$ and $v(M)$ are bounded from above by the corresponding value for $\mathbb{H}P^n$ and similarly that $\deg(Z) \leq \deg(\mathbb{C}P^{2n+1})$.

The idea of the proof is to identify the minimal eigenspace of $\Delta_{\mathrm{Sym}^{2r}H}$ with the kernel of a certain first order differential operator D_u^+ of finite type:

$$\ker(\Delta_{\mathrm{Sym}^{2r}H} - \lambda_\pi) \;=\; H^0(Z, \mathcal{O}(L^r)) \;=\; \ker D_u^+ \;.$$

This can be done using Weitzenböck formulas on quaternionic Kähler manifolds (modifying the approach of [3]). Finally one has to show that the maximal prolongation of D_u^+ is given by $\mathrm{Sym}^{2r}(H \oplus E)$ (cf. [2]).

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Rational curves on complex manifolds with complex contact structure

Stefan Kebekus

Let X be a complex projective manifold which carries a contact structure, given by a sequence

$$0 \to F \to T_X \to L \to 0.$$

Then, the results of [Dem02] and [KPSW00] show that X is either isomorphic to a projectivized (co-)tangent bundle of a complex manifold, or that X is Fano and $b_2(X) = 1$. This talk discusses the latter case where X is Fano of $b_2(X) = 1$. It is generally believed that these assumptions imply that X is homogeneous. Assuming that X is not isomorphic to the projective space, it follows from our previous work [Keb02b, Keb02a] that X can always be covered by "contact lines", that is, rational curves ℓ which intersect the line bundle L with multiplicity one and are tangent to the contact distribution F wherever they are smooth. Thus, it seems natural to consider the geometry of lines in greater detail.

We illustrate how the contact structure influences the deformation theory of these curves. As a result, we obtain that if $x \in X$ is a general point, then all contact lines through x are smooth, [Keb01]. In addition, we show that the "variety of minimal rational tangents", that is, the set of tangent directions to lines at x, generates the contact distribution at x. It follows that the contact structure on X

can be reconstructed from the rational curves on X, and is hence unique, [Keb01]. This answers a question of C. LeBrun [LeB95b, Question 11.3]. The result was previously obtained by C. LeBrun [LeB95a] if X is a twistor space. The talk briefly indicates how related methods apply to show that the tangent bundle of X is slope-stable, [Keb05].

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Special lines on contact Fano manifolds

Jarosław Buczyński

(joint work with Grzegorz Kapustka and Michał Kapustka)

We addresses the problem of classification of contact projective manifolds. A lot of work has already been done in this direction, see [1] for an overview and motivation. The main remaining task is to classify contact Fano manifolds, which are expected to be always homogeneous spaces.

Among the main tools to approach the problem is the theory of minimal rational curves. In the case of contact Fano manifold X of dimension $2n+1$ it amounts to study *contact lines*. A rational curve C with the normalisation $f\colon \mathbb{P}^1 \to C \subset X$ is a *contact line*, if $f^*K_X = \mathcal{O}_{\mathbb{P}^1}(-n-1)$. Here K_X denotes the canonical divisor of X. Unless $X \simeq \mathbb{P}^{2n+1}$, those contact lines exist, cover X and form a family of pure dimension $3n-1$.

For a general contact line with a parametrisation $f\colon \mathbb{P}^1 \to X$ the pullback of the tangent bundle TX has a certian standard splitting type, namely:

$$f^*TX \simeq \mathcal{O}^{\oplus(n+1)} \oplus \mathcal{O}(1)^{\oplus(n-1)} \oplus \mathcal{O}(2) = \mathcal{O}(0^{n+1}, 1^{n-1}, 2). \quad (1)$$

Contact lines satisfying (1) are called standard. Life would be much easier (but more borring), if all the contact lines were known to be standard. This happens for homogeneous contact manifolds (because the group acts transitively on the set of lines). But in general we know very little. Kebekus proved that any line through general point is smooth and standard. We streghten his results and show:

Let X be a contact Fano manifold. The dimension of the scheme parametrising singular contact lines on X is at most $2n-1$. The dimension of the scheme parametrising special contact lines (i.e. lines for which (1) does not hold) on X is at most $3n-3$, i.e. it has no divisorial components in the scheme parametrising all contact lines.

The second statement was claimed earlier by Kebekus in an equivalent form, see [4, Prop. 3.2]. However, his argument contains a gap, see [1, Rem. 3.2] for more details. Proposition 3.2 in [4] is one of the key steps in the proof of [4, Thm 3.1], which we repeat:

Let X be a contact Fano manifold of Picard rank 1. Let $\mathcal{H}$ be an irreducible component of the space parametrising lines on X. Suppose $x \in X$ is a general point, then the set $\mathcal{H}_x$ of lines from $\mathcal{H}$ passing through x is irreducible.

This research is among the first attempts to study minimal rational curves on a projective manifold X, without assuming they are general, or they pass through a general point. Alternatively, one may view our work as an initiation of the studies of minimal rational curves on projective varieties, singular in codimension 1.

Sketch of proofs and intermediate results. Kebekus [2, Prop. 3.3] proved that a contact line through a fixed general point is necessarily smooth. Our proof of the claim about singular lines is a modification of his argument. We prove that the tangent space to the locus swept by the singular lines is perpendicular (with respect to the contact structure) to the tangent directions of the singular contact lines. This implies that the dimension of the locus plus the dimension of the locus singular lines through a fixed point is at most $2n = \dim X - 1$, and the conclusion is straightforward.

The proof of the statement about non-standard lines is much more involved. It is centered about the concept of a *linear subspace* in the contact manifold, which generalises the notion of contact line to higher dimensions. Precisely, a subvariety $\Gamma \subset X$ is a linear subspace, if and only if the normalisation of Γ is a projective space and the restriction of L to Γ is a line bundle of degree 1.

We suppose by contradiction that there is a component $\mathcal{B}$ of the set of non-standard lines of dimension $3n-2$. Then, by results of Kebekus, there is a divisor B on X swept out by the lines from $\mathcal{B}$. The aim is to prove that B is dominated by a family of linear subspaces of dimension n and to obtain a contradiction with the following statement:

Suppose X is a contact Fano manifold with $b_2 = 1$, and suppose X is not isomorphic to a projective space. Furthermore, let $\pi\colon \mathcal{U}_{\mathcal{R}} \to \mathcal{R}$ be a family of linear subspaces of dimension n on X with the evaluation map $\xi\colon \mathcal{U}_{\mathcal{R}} \to X$, which is a morphism birational onto its image. Then $\dim \mathcal{U}_{\mathcal{R}} \leq 2n-1$, i.e., the closure of the image $\overline{\xi(\mathcal{U}_{\mathcal{R}})}$ has codimension at least 2 in X.

To construct the linear subspaces, we use the following characterisation of a projective space:

Suppose Γ is a projective variety with an ample line bundle L, such that a general pair of points $x, y \in \Gamma$ is connected by a single rational curve $f\colon \mathbb{P}^1 \to \Gamma$ of degree 1, i.e., $f^*L \simeq \mathcal{O}_{\mathbb{P}^1}(1)$, $x, y \in f(\mathbb{P}^1)$. Then Γ admits a normalisation $\mu\colon \mathbb{P}^k \to \Gamma$, where $k = \dim \Gamma$ and $\mu^*L = \mathcal{O}_{\mathbb{P}^k}(1)$.

This is a consequence of [3, Thm 3.6] and in our argument is used twice. In the first place we construct a large family of linear subspaces of dimension 2, next we bundle together the planes, to obtain a family of linear subspaces of dimension n. More precisely, the tangent spaces to B naturally determine a distribution G of rank 1, i.e., a line subbundle of TB defined on an open subset of $U \subset B$. Suppose $c \in \mathcal{B}$ is a general non-standard line and $C \subset X$ is the corresponding curve in X. We take the union of leaves of G through points of C, and we let $\overline{\Gamma_c}$ to be the Zariski closure of this union. Then we show that every two points in $\overline{\Gamma_c}$ are connected by a contact line. Thus the normalisation of $\overline{\Gamma_c}$ is a projective space $\mathbb{P}^k$. We carefully study the distribution G restricted to $\overline{\Gamma_c}$ and conclude using that the leaves of G actually are lines from $\mathcal{B}$. In particular, $\dim \overline{\Gamma_c} = 2$, and its normalisation is $\mathbb{P}^2$.

This construction also equips each $\mathbb{P}^2$ with a distinguished point y, and its image in X. We consider $Y \subset X$ to be the union of all distinguished points in X obtained by varying $c \in \mathcal{B}$. The critical step in the proof is the dimension count: we show $\dim Y = n$. The conclusion is that there is a lot of $\overline{\Gamma_c}$, with the same distinguished point y. On the other hand the locus P^y of these projective planes is always contained in the locus of lines through a fixed point y, which is known to have dimension at most n. We use these informations to show that general two points $x_1.x_2$ in P^y are contained in a single $\overline{\Gamma_c}$, whose distinguished point is y. The line in the $\mathbb{P}^2$ normalising $\overline{\Gamma_c}$ is the required line connecting x_1 with x_2. Thus P^y is normalised by a projective space, and its dimension is calculated to be n. This is the way to construct the family of linear subspaces of dimension n, whose locus is the divisor B.

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VMRT-structures on Fano contact manifolds

Jun-Muk Hwang

Given a Fano manifold X with $b_2(X) = 1$, we can find a minimal dominating family of rational curves on X and the associated VMRT $C_x \subset \mathbf{P}T_xX$ at a general point $x \in X$. We discuss the following two conjectures.

Conjecture 1 Let X be a Fano manifold with $b_2(X) = 1$ equipped with a contact structure $F \subset TX$. If C_x at a general point $x \in X$ is a connected nonlinear Legendrean submanifold of $\mathbf{P}F_x$, then X is homogeneous.

Conjecture 2 Let X be a Fano manifold with $b_2(X) = 1$. If C_x at a general point $x \in X$ is a nonlinear smooth hypersurface of $\mathbf{P}T_xX$, then X is homogeneous.

The two conjectures are parallel in the sense that hypersurfaces are nonlinear submanifolds of maximal dimension in projective space in Conjecture 2, while Legendrean submanifolds are isotropic submanifolds of maximal dimension in the contact distribution in Conjecture 1.

Conjecture 1 is important because it implies LeBrun-Salamon conjecture for positive quaternionic-Kähler manifolds by [5] and the supplementary result presented in J. Buczynski's talk in this workshop.

To prove Conjecture 1 and Conjecture 2, it suffices to show that $C_x \subset \mathbf{P}T_xX$ is a homogeneous projective subvariety by the result of [6]. This strategy, however, has been unsuccessful and little progress has been made in this direction.

We introduce a different approach, concentrating on the following weaker versions of the two conjectures.

Conjecture 1' Let X be a Fano manifold with $b_2(X) = 1$ equipped with a contact structure $F \subset TX$. If C_x at a general point $x \in X$ is a connected nonlinear Legendrean submanifold of $\mathbf{P}F_x$, then X is quasi-homogeneous.

Conjecture 2' Let X be a Fano manifold with $b_2(X) = 1$. If C_x at a general point $x \in X$ is a nonlinear smooth hypersurface of $\mathbf{P}T_xX$, then X is quasi-homogeneous.

By [1], Conjecture 1' still implies LeBrun-Salamon conjecture. By Kobayashi-Ochiai criterion, one can show that Conjecture 2' implies Conjecture 2. Thus these weaker versions are as good as the original conjectures.

We have an approach to these conjectures by viewing VMRT as a geometric structure on the Fano manifold. Then both Conjecture 1' and Conjecture 2' can be reduced to local homogeneity of this geometric structure by [4]. Using this approach, Conjecture 2' has been settled in [2] and [3]. This sheds some hope on Conjecture 1'.

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Quaternionic Kähler manifolds: symmetries and related geometries

Andrew Swann
(joint work with Óscar Maciá, Valencia)

Given a quaternionic Kähler manifold M^{4n}, there are associated geometries on the twistor space Z^{4n+2} and the hyperKähler cone or Swann bundle $\mathcal{U}^{4n+4}$. In the case of positive scalar curvature on M, the twistor space is a Kähler-Einstein manifold with a complex contact structure, and $\mathcal{U}$ is a hyperKähler manifold with an action of $SO(3)$ rotating the complex structures.

Symmetries of the geometries on base M lift to symmetries of the geometries on Z and $\mathcal{U}$. Notions of moment maps are defined on each of these manifolds, and reductions at level 0 correspond [2]. However, for one-dimensional group actions, we may reduce $\mathcal{U}$ at a non-zero level. Haydys [1] noted in the case that M has positive scalar curvature that such reductions are hyperKähler with a circle symmetry preserving one complex structures and rotating the other two. In addition, he showed how to invert the construction.

In work in progress, we show that these constructions can be interpreted in terms of the twist construction of [3], and note that they apply in arbitrary signature. More precisely, suppose (N, g, I, J, K) is hyperKähler, with an isometry X such that $L_X I = 0$ and $L_X J = K$.

A twist manifold is specified as follows: let $F \in \Omega^2(N)$ be closed two form such that there is an $a \in C^\infty(N)$ with $da = -X \lrcorner N$. Let P be one-dimensional principal bundle with connection form α and principal vector field Y. Lift X to $X' = \tilde{X} + aY$, with $\tilde{X} \in \mathcal{H} = \ker \alpha$ the horizontal lift. Then the twist manifold is defined to be $M = P/X'$. The assumptions ensure that X' commutes with Y, so M inherits a vector field. If β_N and β_M are differential forms on N and M, then we say that they are $\mathcal{H}$-related if their pull-backs to P agree on $\mathcal{H}$. The essential computational fact is that $d\beta_M$ is then $\mathcal{H}$-related to $d\beta_N - \frac{1}{a} F \wedge X \lrcorner \beta_N$.

We prove that for $\dim N \neq 4$, there is only a one-parameter family of metrics $\tilde{g} = fg + hg_{\mathbb{H}X}$ on N that twist to quaternionic Kähler metrics on some manifold M, and that the twisting form is uniquely determined as $F = dX^\flat + \omega_I$. This shows that all descriptions of the hK/qK correspondence in the literature coincide.

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QK-manifolds in dimension 12

Anand Dessai

All known examples of quaternionic Kähler manifolds of positive scalar curvature (positive QK-manifolds for short) are symmetric spaces and it has been conjectured that there aren't any other examples (LeBrun-Salamon conjecture). The main purpose of this talk was to indicate how one proves the conjecture in real dimension 8 and to describe some of the frustrating attempts to prove the conjecture in dimension 12.

In dimension $8, 12, 16$ one knows from the work of Salamon [7] that the group of isometries of a positive QK-manifold is at least of dimension $6, 5, 8$, respectively. Using these symmetries Poon-Salamon [6] and LeBrun-Salamon [5] proved the conjecture in dimension 8. In the talk I gave a sketch of the proof following closely the argument in [5].

Similar arguments have been used in higher dimension under additional hypothesis. For example, by the work of Simon Salamon [5] and Manuel Amann [1, 2] any positive QK-manifold of dimension ≤ 24 with forth Betti number b_4 equal to one is symmetric.

The conjecture is also known to be true in dimension 12 assuming the assertion that the $\hat{A}$-genus vanishes on simply connected manifolds with S^1-action and $b_2 = 0$ (Haydée and Rafael Herrera [4]). In [4] the authors gave an argument for the latter assertion. Unfortunately, their argument cannot be correct in this generality. In fact, as explained in the talk, it is possible to construct for any $n \geq 2$ simply connected $4n$-dimensional manifolds with S^1-action and $b_2 = 0$ which have nonvanishing $\hat{A}$-genus [3]. The construction is based on a simple argument from equivariant surgery.

Towards the end of the talk I tried to describe the state of the art concerning the conjecture in dimension 12.

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Rozansky–Witten Invariants for Quaternionic Kähler Manifolds

Gregor Weingart

Several attempts to prove the Salamon conjecture extend the approach successfully taken in quaternionic dimension $n = 2$, which studies the relation between characteristic numbers and the geometry of positive quaternionic Kähler manifolds. The powerful vanishing theorems for indices of twisted Dirac operators on quaternionic Kähler manifolds imply relations between these characteristic numbers, which in turn have repercussions for the geometry of the underlying manifold.

Extending the construction of Rozansky–Witten invariants from hyperkähler to quaternionic Kähler manifolds we can simplify this approach to the Salamon conjecture significantly, information on just four characteristic numbers is sufficient to conclude that a positive quaternionic Kähler manifold is a symmetric space. Consider in fact the first Pontryagin class u and the first and second Pontryagin classes u_1, u_2 of the virtual vector bundles HM and $HM \oplus EM$ respectively on a quaternionic Kähler manifold M of quaternionic dimension $n \geq 2$ and positive scalar curvature $\kappa > 0$. The L^2–norm of the covariant derivative $\nabla\mathfrak{R}$ of the curvature tensor $\mathfrak{R} \in \Gamma(\mathrm{Sym}^4 EM)$ can be calculated from the characteristic numbers

$$\langle\, u^n,\, [M]\,\rangle \qquad \langle\, u_1 u^{n-1},\, [M]\,\rangle \qquad \langle\, u_1^2 u^{n-2},\, [M]\,\rangle \qquad \langle\, u_2 u^{n-2},\, [M]\,\rangle$$

by means of the identity

$$\begin{aligned}\frac{1}{4\pi^3}\left(\frac{\kappa}{16\pi n(n+2)}\right)^{2n-3}&\left(\,\|\,\nabla\mathfrak{R}\,\|_{L^2}^2 \;+\; 720\int_M \rho_{\mathrm{rest}}\,(\,\Delta + \frac{\kappa}{2(n+2)}\,)\,\rho_{\mathrm{rest}}\,\mathrm{vol}\right)\\ =\;\; &\frac{1}{(2n-2)!}\,\langle\,(7u_1^2 - 4u_2)u^{n-2},\,[M]\,\rangle\\ &+\frac{2}{3}\,\frac{n+2}{(2n-1)!}\langle\, u_1 u^{n-1},\,[M]\,\rangle \;-\; 5\,\frac{2n+1}{(2n-1)!}\,\frac{\langle\, u_1 u^{n-1},\,[M]\,\rangle^2}{\langle\, u^n,\,[M]\,\rangle}\end{aligned}$$

where $\rho_{\mathrm{rest}} \in C^\infty(M)$ is an auxiliary function characterized by:

$$\left(\,\Delta \;+\; \frac{\kappa}{2\,(n+2)}\,\right)\rho_{\mathrm{rest}} \;\;=\;\; |\,\mathfrak{R}\,| \;-\; \frac{\|\,\mathfrak{R}\,\|_{L^2}}{\mathrm{Vol}\,M}$$

Cartan's characterization of symmetric spaces as manifolds M, whose curvature tensor R is covariantly constant in the sense $\nabla R = 0$, thus implies that a positive quaternionic Kähler manifold M is a symmetric space, if and only if the specified four characteristic numbers make the right hand side of the decisive identity vanish. Needless to say this characterization of the symmetric positive quaternionic Kähler manifolds can be verified directly for all Wolf spaces, the interesting question however is, whether it is possible to prove sufficiently many relations between characteristic numbers to prove the Salamon conjecture in this way.

Generic semipositivity of sheaves of differential operators on complex projective manifolds, after Clemens-Ran

Frédéric Campana

This talk is just a report, mainly aimed at the differential geometers participating to this workshop, on the paper of Clemens-Ran ([1]). The methods introduced in this paper might indeed prove useful in the study of contact Fano manifolds.

After recalling the notion of slope (relative to given ample line bundle H) of a coherent torsion-free sheaf E of rank r on a connected n-dimensional complex projective manifold X, we defined stability, semi-stability, minimal $\mu_{min}(E)$ and maximal slopes, as well as the Harder-Narasimhan filtration, stating finally the Mehta-Ramanathan theorem. The notions of generic postivity and generic semi-positivity (relative to H) were then defined.

It was then proved that the tangent bundle T_X of a Fano manifold with $b_2 = 1$ is generically positive (with respect to the ample generator off $Pic(X)$, for example), using the vanishing theorems of Akizuki-Nakano and Kodaira.

Theorem ([1]). Let X be complex projective, together with a given H. Let E be a generically semi-positive vector bundle (or torsionfree coherent sheaf) on X. Assume that T_X is generically positive. Let $D^m(E)$ be the sheaf of differential operators of order $m > 0$ sending germs of sections of E to germs of holomorphic sections on X. Then $\mu_{min}(D^m(E)) \geq m.b + \mu_{min}(E^*)$, where $b := min\{a, \frac{1}{2}.\mu_{min}(T_X)\} > 0$, with $a := min\{D.H^{n-1}\}$ is the minimal degree of an effective divisor with respect to H.

In other words, the positivity of T_X compensates the negativity of E^*, dual of E, if one takes differential operators of sufficiently high order. Notice that this results applies in particular to Fano manifolds with $b_2 = 1$, since TX is then generically positive.

This theorem permits to show (as in [1]) that if X is a Fano manifold with $b_2 = 1$, and T_X semi-stable, then $(-K_X)^n \leq (2n)^n$.

The proof is an immediate consequence of the preceding theorem and of the following two easy lemmas:

Lemma. Let E be a vector bundle on (any) X such that $D^m(E)$ is generically semi-positive. Then any non-zero-section of E vanishes at order at most m at any generic point x of X.

Lemma ('Fano method'). Let L be an ample line bundle on (any) X. Assume that for some $x_k \in X$ and any non-zero section s of kL, for any $k > 0$, the vanishing order of s at x_k is bounded by km, for some real constant $m > 0$. Then $(L^n)^{\frac{1}{n}} \leq m$.

The rest of the talk was devoted to briefly explain the 3 main steps of the proof of the above theorem: First step: $m = 1, E$ is a line-bundle. Second step: $m = 1, E$ is of arbitrary rank. The proof of this step is quite delicate, and involves in particular the use of the Narasimhan-Seshadri theorem asserting that a stable bundle E_C with $c_1(E_C) = 0$ on a curve C is unitary flat. Third step: $m > 1$, E arbitrary. One proceeds inductively on m, using the surjective morphism of (left) sheaves of coherent modules on X: $D^1(J^m(E)) \to D^{m+1}(E)$, where $J^m(E) = D^m(E)^*$ is the sheaf of m-jets of E.

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Quaternionic Kähler manifolds of negative scalar curvature

Vicente Cortés

The talk is focussed on a construction of explicit complete quaternionic Kähler metrics of negative scalar curvature. It is based on the result [10] that every complete projective special real manifold of dimension n defines a complete quaternionic Kähler manifold of dimension $4n + 8$. A *projective special real manifold* is a smooth hypersurface $\mathcal{H} \subset \mathbb{R}^{n+1}$, such that there exists a homogeneous cubic polynomial h on $\mathbb{R}^{n+1}$ with the property that $h = 1$ on $\mathcal{H}$ and $\partial^2 h < 0$ on $T\mathcal{H}$. It is called *complete*, if $-\partial^2 g$ induces a complete metric on $\mathcal{H}$. More generally [10], every complete projective special Kähler manifold of (real) dimension $2n$ defines a complete quaternionic Kähler manifold of dimension $4n + 4$.

These are global versions of constructions, known in the physics literature under the names q-map [3] and c-map [2], respectively. The quaternionic Kähler property of the resulting metrics follows from [2] and [7].

We give a new proof, which shows that a certain one-parameter deformation of the c-map metric is also quaternionic Kähler [14]. Our proof is based on a generalization [11, 14] of the HK/QK-correspondence of Andriy Haydys [6], which allows for the use of indefinite metrics in all steps of the construction with control of the signature of the resulting metric and the sign of its scalar curvature. Furthermore, we provide a new formula for the quaternionic Kähler metric in the HK/QK-correspondence explicit enough to be matched with the c-map metric (for the corresponding pseudo-hyper-Kähler initial data). We find, that the Hamiltonian constant in the HK/QK-correspondence corresponds precisely to the one-loop parameter of the quantum corrected c-map metric g^c, as described in [5, 9]. Our work on the HK/QK-correspondence relates nicely to work in progress by Andrew Swann et al. on twist constructions presented in this workshop and to recent work by Nigel Hitchin [12].

Complete projective special real manifolds are classified up to dimension 2 [13]. Under the q-map, these examples give rise to complete quaternionic Kähler manifolds of dimensions 8, 12, and 16 of cohomogeneity ≤ 2. Moreover, all known examples of non-symmetric homogeneous quaternionic Kähler manifolds can be obtained by the q-map [1, 3, 4].

Work in progress (joint with Marc Nardmann and Stefan Suhr) shows that a projective special real manifold $\mathcal{H} \subset \{h = 1\} \subset \mathbb{R}^{n+1}$ is complete if and only if $\mathcal{H}$ is a component of $\{h = 1\}$. This provides an effective tool for the construction of complete quaternionic Kähler metrics of negative scalar curvature by the q-map.

Finally, I would like to remark, that the twistor spaces of the quaternionic Kähler manifolds obtained from the q-map are complex contact manifolds encoded in a real homogeneous cubic polynomial. More generally, one can associate a complex contact manifold with any non-degenerate *complex* homogeneous cubic polynomial [8], as I learned from Jun-Muk Hwang during the workshop.

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Formality of Positive Quaternion Kähler Manifolds

Manuel Amann
(joint work with Vitali Kapovitch)

In this talk the focus lied on the algebraic topology of Positive Quaternion Kähler Manifolds M. I collected several results on what is known on the Betti numbers of such manifolds. The situation here seems a little ambiguous: On the one hand there are strong results like the vanishing of odd-degree Betti numbers (yielding a lower bound on the Euler characteristic) or classification results under the assumption of $b_2 = 1$ or $b_4 = 1$ (in certain dimensions), on the other hand no upper bound on the Euler characteristic is known—see [6], [2].

The talk was centred around stating and discussing the following

Conjecture. A Positive Quaternion Kähler Manifold is geometrically formal and rationally elliptic.

The weakest direct corollary one could draw out of either of the two parts of this conjecture is an upper bound for the Euler characteristic of M^n given by 2^n.

Let me give a short overview of parts of the discussion motivating this conjecture. A simply-connected closed manifold M is *rationally elliptic* if its rational homotopy groups $\pi_i(M) \otimes \mathbb{Q}$ vanish from some degree on. Prototypical examples of rationally elliptic spaces are simply-connected homogeneous spaces of compact Lie groups. Since Positive Quaternion Kähler manifolds seem to share some "structural similarities" with manifolds of positive sectional curvature, the conjecture that they are rationally elliptic might be considered a "quaternionic Bott conjecture". Moreover, I can show that in low dimensions rationally elliptic Positive Quaternion Kähler Manifolds are rational homology Wolf spaces—see [1].

A Riemannian manifold is *geometrically formal*, if the product of harmonic forms is harmonic again. Very few examples of geometrically formal manifolds are known, only slightly exceeding the class of symmetric spaces of compact type. However, manifolds with non-formal metrics can already be found amongst n-symmetric spaces—see [4], [5].

Thus, in particular, if the LeBrun–Salamon conjecture is true, so is the conjecture above. Moreover, a wild speculation only based on the lack of examples might still be that the class of spaces combining the properties of the conjecture might not be much larger than the class of symmetric spaces.

An obstruction to geometric formality is *formality*, a property which expresses that the rational homotopy type of the manifold can be derived from the knowledge of its rational cohomology algebra already. Since the odd Betti numbers of a Positive Quaternion Kähler manifold vanish, formality is also an obstruction to rational ellipticity.

However, in [3] we prove that this obstruction vanishes, i.e. that a Positive Quaternion Kähler manifold is a formal space—this is generally conjectured for manifolds of special holonomy. Using the twistor fibration and the formality of Kähler manifolds, this is a simple special case of a more general result from that article: In a fibration of simply-connected spaces of finite type with the fibre being rationally elliptic, formal and satisfying a generalised version of the Halperin conjecture, the formality of the base space is equivalent to the one of the total space.

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Almost complex structures on quaternion-Kähler manifolds

Andrei Moroianu

(joint work with Paul Gauduchon and Uwe Semmelmann)

It is a well-known fact that the quaternionic projective spaces $\mathbb{HP}^n$ have no almost complex structure. The proof goes back to F. Hirzebruch in 1953 for $n \geq 4$ (cf. [5]). The non-existence of almost-complex structure on $\mathbb{HP}^1 = \mathbb{S}^4$ had been established a few years earlier by Ch. Ehresmann [3] and H. Hopf [7]. According to Hirzebruch's lecture at the 1958 ICM [6], J. Milnor had in the meantime settled the remaining cases $n = 2$ and 3, but his proof has remained unpublished. Later on, W.S. Massey [10] gave an original proof of the non-existence of almost-complex structure on $\mathbb{HP}^n$, for any n, based on the explicit calculation of the ring $\mathrm{K}(X)$ and of the Chern character $ch(\mathrm{T}X)$ for $X = \mathbb{HP}^n$.

Quaternionic projective spaces are particular examples of *quaternion-Kähler manifolds.* These, we recall, are $4n$-dimensional Riemannian manifolds, whose holonomy is contained in $\mathrm{Sp}(n) \cdot \mathrm{Sp}(1) \subset \mathrm{SO}(4n)$, if $n > 1$, or, if $n = 1$, (oriented) Einstein, self-dual 4-dimensional Riemannian manifolds. In all dimensions $4n$, $n \geq 1$, quaternion-Kähler manifolds are Einstein and are called *of positive type* if their scalar curvature is positive. In this talk we only consider quaternion-Kähler of positive type and we implicitly assume that they are complete, hence compact.

For $n \geq 2$, the above definition of quaternion-Kähler manifolds is equivalent to the existence of locally defined almost complex structures I, J, K, satisfying the quaternion relations and spanning a global rank 3 sub-bundle $\mathrm{Q} \subset \mathrm{End}(\mathrm{T}M)$, which is preserved by the Levi-Civita connection. Almost complex structures on M which are sections of Q are called *compatible.* In [1], it is shown that quaternion-Kähler manifolds of positive type admit no *compatible* almost complex structure. In particular the natural complex structure of the complex Grassmannians $\mathrm{Gr}_2(\mathbb{C}^{n+2})$, which constitute a well-known class of quaternion-Kähler manifolds of positive type, is not compatible.

The main result of this talk is:

Theorem 1. ([4]) *Let M^{4n}, $n \geq 2$, be a compact quaternion-Kähler manifold of positive type, which is not isometric to the complex Grassmannian* $\mathrm{Gr}_2(\mathbb{C}^{n+2})$.

Then M^{4n} has no weak almost complex structure, in the sense that the tangent bundle $\mathrm{T}M$ *is not stably isomorphic to a complex vector bundle.*

Notice that the assumption $n \geq 2$ is necessary, since $\mathbb{HP}^1 = \mathbb{S}^4$ is weakly complex but not almost complex.

It is well known that the complexified tangent bundle of a quaternion-Kähler manifold M is given as $\mathrm{T}M^{\mathbb{C}} = E \otimes H$. Recall that a quaternion-Kähler manifold M^{4n} of positive type is spin if and only if either $M^{4n} = \mathbb{HP}^n$, or the quaternionic dimension n is even ([12, Proposition 2.3]). If this holds, the spinor bundle ΣM decomposes as the direct sum of $R^{p,q} := \mathrm{Sym}^p H \otimes \Lambda_0^q E$ over all positive integers p, q with $p+q=n$, cf. e.g. [8, Proposition 2.1]. Here $\Lambda_0^q E$ denotes the sub-bundle of $\Lambda^q E$ defined as the kernel of the contraction with the symplectic form of E. In particular, the twisted spin bundles $\Sigma^{\pm} M \otimes R^{p,q}$ are globally defined whenever $p+q+n$ is even. We then denote by $D_{R^{p,q}}$ be the (twisted) Dirac operator defined on sections of $\Sigma^{+} M \otimes R^{p,q}$ and by $ind(D_{R^{p,q}})$ the index of $D_{R^{p,q}}$.

Our argument crucially relies on the following result of C. LeBrun and S. Salamon [9, Theorem 5.1]:

$$ind(D_{R^{p,q}}) = \begin{cases} 0 & \text{for} \quad p+q<n \\ (-1)^q \left(b_{2q}(M) + b_{2q-2}(M)\right) & \text{for} \quad p+q=n\,, \end{cases} \tag{1}$$

where $b_i(M)$ denote the Betti numbers of M. Consider the twist bundle $V = \mathrm{Sym}^{n-2} H \otimes \mathrm{T}M^{\mathbb{C}}$ (it is here that the assumption $n \geq 2$ is needed). The Clebsch-Gordan decomposition yields

$$V = (\mathrm{Sym}^{n-1} H \otimes E) \oplus (\mathrm{Sym}^{n-3} H \otimes E).$$

The bundle $\Sigma M \otimes V$ is globally defined for all quaternionic dimensions n and we can therefore compute the index $ind(D_V)$ of the corresponding twisted Dirac operator by using (1). We thus obtain

$$\begin{aligned} ind(D_{\mathrm{Sym}^{n-2} H \otimes \mathrm{T}M^{\mathbb{C}}}) &= ind(D_{\mathrm{Sym}^{n-1} H \otimes E}) + ind(D_{\mathrm{Sym}^{n-3} H \otimes E}) \\ &= -\left(b_2(M) + b_0(M)\right). \end{aligned} \tag{2}$$

A key fact, cf. [9, Corollary 4.3], is that $b_2(M) = 0$ for all compact quaternion-Kähler manifold M of positive type other than the Grassmannians of complex 2-planes $\mathrm{Gr}_2(\mathbb{C}^{n+2})$, whereas $b_2(M) = 1$ if $M = \mathrm{Gr}_2(\mathbb{C}^{n+2})$, which, as already observed, has a natural complex structure. We now assume that M is different from $\mathrm{Gr}_2(\mathbb{C}^{n+2})$, so that $b_2(M) = 0$. The above index calculation then reads

$$ind(D_{\mathrm{Sym}^{n-2} H \otimes \mathrm{T}M^{\mathbb{C}}}) = -1\,. \tag{3}$$

Assume, for a contradiction, that M carries an almost complex structure. Then the tangent bundle $\mathrm{T}M$ is a complex vector bundle and its complexification splits into the sum of two complex sub-bundles $\mathrm{T}M^{\mathbb{C}} = \theta \oplus \theta^*$. For the components of the Chern character we have $ch_i(\theta^*) = (-1)^i ch_i(\theta)$. On the other hand, $ch(\mathrm{Sym}^{n-2} H)$ and $\hat{\mathrm{A}}(\mathrm{T}M)$ have non-zero components only in degree $4k$. Indeed, $\hat{\mathrm{A}}(\mathrm{T}M)$ is a polynomial in the Pontryagin classes of M and $\mathrm{Sym}^{n-2} H$ is a self-dual locally defined complex bundle. The Atiyah-Singer formula for twisted Dirac operators (cf. [2]) then yields

$$\begin{aligned} ind(D_{\mathrm{Sym}^{n-2}H\otimes \mathrm{T}M^{\mathbb{C}}) &= ch(\mathrm{Sym}^{n-2}H)ch(\mathrm{T}M^{\mathbb{C}})\hat{\mathrm{A}}(\mathrm{T}M)[M] \\ &= 2\, ch(\mathrm{Sym}^{n-2}H)ch(\theta)\hat{\mathrm{A}}(\mathrm{T}M)[M]. \end{aligned} \tag{4}$$

Notice that $ch(\mathrm{Sym}^{n-2}H)$ is well-defined in $H^*(M,\mathbb{Q})$, even if n is odd.

Now, $ch(\mathrm{Sym}^{n-2}H)ch(\theta)\hat{\mathrm{A}}(\mathrm{T}M)[M]$ is the index of the twisted Dirac operator $D_{\mathrm{Sym}^{n-2}H\otimes\theta}$ on the (globally defined) bundle $\Sigma M \otimes \mathrm{Sym}^{n-2}H \otimes \theta$ and thus has to be an integer. This implies that $ind(D_{\mathrm{Sym}^{n-2}H\otimes \mathrm{T}M^{\mathbb{C}}})$ is *even*, hence contradicts (3).

If the manifold is assumed to be weakly complex then there exists a trivial real vector bundle ϵ such that $\mathrm{T}M \oplus \epsilon$ is a complex vector bundle. By replacing $V = \mathrm{Sym}^{n-2}H \otimes \mathrm{T}M^{\mathbb{C}}$ with $V = \mathrm{Sym}^{n-2}H \otimes (\mathrm{T}M \oplus \epsilon)^{\mathbb{C}}$ in the above argument, this remains unchanged, as the extra term

$$ind(D_{\mathrm{Sym}^{n-2}H\otimes\epsilon^{\mathbb{C}}}) = rk(\epsilon)\, ind(D_{\mathrm{Sym}^{n-2}H})$$

in (2) is zero, again because of (1).

The above methods were also used in order to classify compact inner symmetric spaces and, more generally, equal rank compact homogeneous spaces with stably complex tangent bundle (see [4], and [11] for details).

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Construction of potentially new QK-metrics

Roger Bielawski

The talk is really about hyperkähler metrics. It is known that to any quaternion-Kähler manifold M one can associate canonically a hyperkähler manifold: the *Swann bundle* of M.

This work has been motivated by three sources. First and foremost, the work of Nash [2], who gave a new twistor construction of hyperkähler metrics on moduli spaces of $SU(2)$ magnetic monopoles. Second, the so-called generalised Legendre transform construction of hyperkähler metrics, due to Lindström and Roček [1], which often leads to curves of higher genus. Third, the well-known fact that the smooth locus of the Hilbert scheme of (local complete intersection) curves of degree d and genus g in $\mathbb{P}^3$ has, if nonempty, dimension $4d$. Because of author's $\mathbb{H}$-bias, the factor 4 seems to him to call for some sort of a quaternionic structure.

Recall that hypercomplex or hyperkähler manifolds arise as parameter spaces of rational lines in twistor spaces. The twistor space Z of a hypercomplex manifold M is diffeomorphic to $M \times S^2$, and is canonically a complex manifold. The projection π onto $S^2 \simeq \mathbb{CP}^1$ is holomorphic and the antipodal map on S^2 induces an antiholomorphic involution σ on Z. A point of M corresponds to a section of π, and the normal bundle of such a section splits as sum of $\mathcal{O}(1)$-s. M with its hypercomplex structure is recovered as a connected component of the moduli space of σ-invariant sections with such a normal bundle. To get a hyperkähler metric we need additional structure on Z.

We now consider higher degree (say d) curves C in Z, i.e. d-fold (flat) coverings of $\mathbb{P}^1$. It turns out that the parameter space M_d of σ-invariant curves C of degree d, the normal bundle of which tensored with $\pi^*\mathcal{O}(-2)$ has no cohomology, is again a hypercomplex manifold (of dimension $d \dim M$). Moreover, was M hyperkähler (resp. Swann bundle), then so is M_d.

The complex structures of M_d are those of (unramified covering of) open subset of the smooth locus of the Hilbert scheme of d points in M (with the corresponding complex structure).

As an example, if we start with the twistor space $\mathbb{P}^3 - \mathbb{P}^1$ of the flat $\mathbb{R}^4$, we shall obtain a hyperkähler structure on manifolds parameterising σ-invariant curves in $\mathbb{P}^3$ not intersecting a fixed line. Moreover, these manifolds are double covers of a Swann bundle. In the simplest case, that of twisted normal cubics, the resulting 12-dimensional metric is still flat, and the question arises what happens for other admissible values of genus and degree. Potentially more interesting is what happens if we start with the twistor space of the Swann bundle of compact quaternion-Kähler manifolds ($\dim \geq 8$).

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Other activities

In addition to the scheduled talks the mini-workshop involved several problem sessions and vivid discussions centering around the topics of the talks and the problem of classifiying positive quaternion Kähler manifolds in general. In addition, people used the time in order to collaborate in the evenings and to take profit of the gathering of experts from very different fields.

It is especially notable to have incorporated video conferences with Simon Salamon from Imperial College, London, who was also willing to present his views on quaternion Kähler geometry in a video talk. This did boost and enrich our insight into the subject heavily and we are both thankful to Simon Salamon as well as the MFO to have made this possible.

Reporter: Manuel Amann

Participants

Dr. Manuel Amann
Institut für Algebra u. Geometrie
Fakultät für Mathematik
KIT
Kaiserstr. 89-93
76133 Karlsruhe
GERMANY

Prof. Dr. Roger Bielawski
Institut für Mathematik
Universität Hannover
Welfengarten 1
30167 Hannover
GERMANY

Prof. Dr. Jaroslaw Buczynski
Institute of Mathematics of the
Polish Academy of Sciences
P.O. Box 21
00-956 Warszawa
POLAND

Prof. Dr. Frédéric Campana
Département de Mathématiques
Université de Nancy I
B.P. 239
54506 Vandoeuvre-les-Nancy Cedex
FRANCE

Prof. Dr. Vicente Cortés
Mathematik Department
Universität Hamburg
Schwerp.: Analysis & Diff. Geometrie
Bundesstr. 55
20146 Hamburg
GERMANY

Prof. Dr. Anand N. Dessai
Département de Mathématiques
Université de Fribourg
Perolles
Chemin du Musée 23
1700 Fribourg
SWITZERLAND

Marysia Donten-Bury
Institute of Mathematics
University of Warsaw
02-097 Warszawa
POLAND

Dr. Anna Maria Fino
Dipartimento di Matematica
Universita degli Studi di Torino
Via Carlo Alberto, 10
10123 Torino
ITALY

Prof. Dr. Jun-Muk Hwang
School of Mathematics
Korea Institute for Advanced Study
(KIAS)
87 Hoegiro, Dongdaemun-gu
Seoul 130-722
KOREA, REPUBLIC OF

Prof. Dr. Stefan Kebekus
Mathematisches Institut
Universität Freiburg
Eckerstr. 1
79104 Freiburg
GERMANY

Dr. Andrei Moroianu
Université de Versailles - St Quentin
UMR 8100 du CNRS
45, Ave. des Etats-Unis
78035 Versailles Cedex
FRANCE

Prof. Dr. Uwe Semmelmann
Institut für Geometrie & Topologie
Universität Stuttgart
70569 Stuttgart
GERMANY

Prof. Dr. Andrew F. Swann
Matematisk Institut
Aarhus Universitet
Ny Munkegade 118
8000 Aarhus C
DENMARK

Prof. Dr. Gregor Weingart
Instituto de Matemáticas
Universidad Nacional Autónoma de México
Avenida Universidad s/n
62210 Cuernavaca, Morelos
MEXICO

Prof. Dr. Jaroslaw Wisniewski
Instytut Matematyki
Uniwersytet Warszawski
ul. Banacha 2
02-097 Warszawa
POLAND

Dr. Frederik Witt
Mathematisches Institut
Universität Münster
Einsteinstr. 62
48149 Münster
GERMANY