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Langlands Correspondence and Constructive Galois Theory

Organised by
Michael Dettweiler, Bayreuth
Jochen Heinloth, Essen
Zhiwei Yun, Stanford

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Abstract. Recent progress in the Langlands programm provides a significant step towards the understanding of the arithmetic of global fields. The geometric Langlands program provides a systematic way to construct l-adic sheaves (resp. D-modules) on algebraic curves which subsumes the construction of classical sheaves, like rigid local systems, used in inverse Galois theory (by Belyi, Malle, Matzat, Thompson, Dettweiler, Reiter) for the construction of field extension of the rational function fields $\mathbb{F}_p(t)$ or $\mathbb{Q}(t)$ (recent work of Heinloth, Ngo, Yun and Yun). On the other hand, using Langlands correspondence for the field $\mathbb{Q}$, Khare, Larsen and Savin constructed many new automorphic representations which lead to new Galois realizations for classical and exceptional groups over $\mathbb{Q}$. It was the aim of the workshop, to bring together the experts working in the fields of Langlands correspondence and constructive Galois theory.

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Introduction by the Organisers

The workshop "Langlands Correspondence and Constructive Galois Theory" organized by Michael Dettweiler, Jochen Heinloth and Zhiwei Yun was held from February 2 to February 8, 2014. It was attended by 23 participants. The participants ranged from senior leaders in the field to young post-doctoral fellows and one PhD student. The range of expertise covered areas ranging from geometric Langlands correspondence to inverse Galois theory. The program contained 18 talks of 60 minutes.

Overview

Recently, techniques originating from the Langlands programme have been successfully applied to obtain uniform constructions of Galois representations and local systems with surprising properties.

One of the approaches originated from a construction of Gross who managed to find automorphic forms with very special properties. Interpreting these automorphic forms from the point of view of the geometric Langlands correspondence allowed to give an explicit description of the correspondig Hecke eigensheaves from which one can then obtain local systems that are cohomologically rigid and whose corresponding Galois representations have dense image in exceptional groups.

This was first obtained in the geometric situation, considering the function field of the projective line over a finite field by Ngo, Heinloth and Yun, building up on previous work of Katz and Frenkel, Gross. Zhiwei Yun surprisingly managed to find arithmetic examples by a similar procedure, which allowed him to solve longstanding open problems of constructing Galois extensions and motives of the rational numbers with Galois images in exceptional algebraic groups, generalizing previous work of Dettweiler, Katz and Reiter. The results of Yun lead to exciting new Galois realizations of exceptional finite simple groups. In the meanwhile, these Galois realizations have been reproven by Guralnick and Malle using the character theory of groups of Lie type.

It turns out that the geometric Langlands conjectures which led to the above constructions also indicate a geometric interpretation of the rigidity property of many classical local systems. Namely, it seems that rigidity of local systems should be reflected on the automorphic side of the Langlands correspondence by the phenomenon that the moduli stacks of bundles that support the automorphic sheaves are essentially 0 dimensional, which is a condition which is very easy to check computationally. Again, it is surprising that one can understand many classical rigid local systems from this point of view, again showing close relation of the classical results of Belyi, Katz and others to the geometric Langlands correspondence. This indicates more generally that the cases of the Langlands correspondence for which the moduli stacks have a simple geometric description should also have applications in constructive Galois theory.

Researchers attending the conference reported on the substantial progress (achieved within the last three years), discussed open problems, and exchanged methods and ideas. Most lectures were followed by lively discussions among participants.

The program can be highlighted as follows:

- Nick Katz' introductory lecture, motivating brilliantly the topic of the conference building the bridge between the area of classical constructions of rigid local systems and the new approach using Langlands correspondence
- The talk of Zhiwei Yun on the construction of rigid local systems using geometric Langlands correspondence, also providing a proof of a long standing

question of Serre on the existence of motives with an exceptional simple motivic Galois group.
- Gunter Malle's talk on the Galois realizations of finite exceptional groups of Lie type which were motivated by the work of Yun, illustrating the need of cooperation between both fields of research. Stefan Reiter's report on the construction of rigid G_2-local systems and associated $\mathcal{D}$-modules.
- The talks of Gordan Savin, Sara-Aria-de-Reina and Gabor Wiese on the construction of Galois groups using automorphic forms.
- Talks on progress in the geometric Langlands program by Yakov Varshavsky, Dac Tuan Ngo, Timo Richarz and Roman Fedorov.
- Talks of Lars Kindler and Hélène Esnault on $\mathcal{D}$-modules in positive characteristic, providing new insights into the structure of the étale fundamental groups and associated local systems in positive characteristics.
- Lectures on general properties and conjectures on Galois representations by Gebhard Böckle, Anna Cadoret, Pierre Dèbes and David Roberts.

Conculding, the framework of the conference was perfect for various mutual interactions between the participants, resulting in various new collaboriations between the participants. It became clear, that many open problems in the construction of rigid local systems can be attacked using geometric Langlands corespondence, resulting in the success this conference.

Workshop: Langlands Correspondence and Constructive Galois Theory

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Abstracts

Introduction

NICHOLAS M. KATZ

1. Why some people care about local systems-character sums over finite fields

Here is an example. We take an elliptic curve E over a finite field $\mathbb{F}_q$ of odd characteristic, and write its number of rational points as

$$\#E(\mathbb{F}_q) = q + 1 - a_q(E).$$

Then $a_q(E)$ is visibly an integer, and a main problem in the early 1930's, solved by Hasse, was to establish the estimate

$$|a_q(E)| \leq 2\sqrt{q}.$$

Write a Weierstrass equation $Y^2 = f(X)$ for E, with f a square free cubic, and denote by $\chi_2 : \mathbb{F}_q^\times \to \pm 1 \subset \mathbb{C}^\times$ the quadratic character. Define $\chi_2(0) := 0$. Then $a_q(E)$ is the character sum

$$a_q(E) = -\sum_{x \in \mathbb{F}_q} \chi_2(f(x)).$$

If we take the Legendre family of elliptic curves $Y^2 = X(X-1)(X-\lambda)$, then for each $\mathbb{F}_q$ of odd characteristic, and each $\lambda \in \mathbb{F}_q \setminus \{0, 1\}$, we have an elliptic curve $E_\lambda/\mathbb{F}_q$. Using Hasse's bound, we write

$$a_q(E_\lambda) = 2\sqrt{q}\cos(\theta_{q,\lambda})$$

for a unique angle $\theta_{q,\lambda}$ in the closed interval $[0, \pi]$. The question of how this angle varies was raised by Sato in the early 1960's, and taken up by Tate a bit later. If we fix a large q and vary λ, the Sato-Tate conjecture asserted that as q grows, these $q-2$ angles in $[0, \pi]$ become equidistributed for the probability measure $(2/\pi)\sin^2(\theta)d\theta$.

The Artin-Grothendieck ℓ-adic theory gives a (in this case a 2-adic !) local system, call it $\mathcal{F}$, on $\mathbb{P}^1 \setminus \{0, 1, \infty\}$ over $\mathbb{Z}[1/2]$ whose trace function at any point $(\mathbb{F}_q, \lambda)$ is $a_q(E_\lambda)$. By Hasse's estimate, this local system $\mathcal{F}$ is pure of weight one. In each odd characteristic p, we can separately form a half Tate twist $\mathcal{F}(1/2)$, whose trace function is now $2\cos(\theta_{q,\lambda})$, and which is pure of weight zero.

Deligne's general equidistribution theorem tells us that for any local system $\mathcal{G}$ which is pure of weight zero, we should look at the two algebraic groups $G_{geom} \subset G_{arith}$ (here over $\overline{\mathbb{Q}_2}$!) which are the Zariski closures of the images of π_1^{geom} and of π_1 of the base (here $\mathbb{P}^1 \setminus \{0, 1, \infty\}$ over $\mathbb{F}_p$, any odd p). The group G_{geom} has its identity component semisimple. If it happens that $G_{geom} = G_{arith}$, then we embed $\overline{\mathbb{Q}_2}$ into $\mathbb{C}$, and choose a maximal compact subgroup K of the complex Lie group $G_{arith}(\mathbb{C})$. Deligne tells us that the semisimplification of the image of each Frobenius class in $G_{arith}(\mathbb{C})$ is conjugate to an element of K, which is itself unique up to K-conjugacy. The theorem is then that as we take larger and larger $\mathbb{F}_q$, these Frobenius conjugacay classes in K become equidistributed for "Haar measure" on the space $K^\#$ of conjugacy classes in K.

In the case of the Legendre family, one shows that in each odd characteristic, one has $G_{geom} = G_{arith} = SL(2)$. Here K is the compact group $SU(2)$, its space of conjugacy classes is the interval $[0, \pi]$, its "Haar measure" is $(2/\pi)\sin^2(\theta)d\theta$, and for each $(\mathbb{F}_q, \lambda)$, its Frobenius conjugacy class is the angle $\theta_{q,\lambda}$. In other words, the Sato-Tate conjecture for the Legendre family holds.

In general, the upshot is that an "interesting" family of character sums will be the values of the trace function of some local system. With luck, Deligne's Weil II results will show that the local system in question is pure of weight zero, and we will be able to both calculate G_{geom} and to show that $G_{geom} = G_{arith}$. When all this works, we end up with an equidistribution theorem for the sums we were interested in.

2. Why other people care about local systems-the automorphic approach

In brief summary, automorphic considerations allow the construction of local systems which are pure of weight zero, and where one knows a priori that G_{geom} is some specified group (G_2, F_4, E_7, E_8 have all been obtained this way, cf. [HNY]) and that $G_{geom} = G_{arith}$. So the automorphic constructions lead to cases where one knows a Sato-Tate theorem for the trace function values (indeed for the Frobenius conjugacy classes) of the local systems in question. What one does NOT know in general is a concrete calculation of what the trace function is for these local systems.

3. Concluding summary

If we start with interesting sums, we need to calculate G_{geom} in order to get a Sato-Tate theorem. If we start on the automorphic side, we get G_{geom} for free, but we don't know what sums we are proving Sato-Tate for. Can the two sides come together?

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Galois groups over $\mathbb{Q}$ attached to automorphic representations

Gordan Savin

We gave an overview of results contained in the following two papers:

Functoriality and the inverse Galois problem (with C. Khare and M. Larsen), Compositio Math. **144** (2008) 541-564.

Functoriality and the inverse Galois problem II: groups of type B_n and G_2 (with C. Khare and M. Larsen), Ann. Fac. Sci. Toulouse Math. (the issue in honor of Khare's *Prix Fermat*) Vol XIX, no 1 (2010) 37-70.

These two papers contain an application of Langlands' functoriality principle to the classical problem: which finite groups, in particular which simple groups, appear as Galois groups over $\mathbb{Q}$? Let ℓ be a prime. Let G be an adjoint simple Lie group over the finite field $\mathbb{F}_\ell$ of type C_n, B_n or G_2. Let t be a positive integer. We show that there exists an integer k divisible by t such that $G(\mathbb{F}_{\ell^k})$ or its derived group appears as Galois groups over $\mathbb{Q}$. In particular, for each of the three Lie types and fixed ℓ we construct infinitely many Galois groups, however, we do not have a precise control of k. It is worth mentioning that, in contrast, the rigidity method (due to Thompson, Beily and Matzat) produces usually groups of Lie type as Galois groups over a (large) cyclotomic extension $\mathbb{Q}$. Only perhaps for $k = 1$ these Galois groups are over $\mathbb{Q}$. Our method is based on the following three steps:

- Construction of a cohomological, automorphic representation Π of $GL_m(\mathbb{A})$ with pre-described ramification as a lift of a cuspidal generic representation σ of $G = SO_{2n+1}$, Sp_{2n} and G_2 respectively.
- Attaching an m-dimensional ℓ-adic representation

  $$r_\Pi : Gal(\mathbb{Q}) \to GL_m(\bar{\mathbb{Q}}_\ell)$$

  to Π compatible with the local Langlands parametrization outside ℓ. This means that for every prime $v \neq \ell$ the restriction of r_Π to the decomposition group at v gives the Langlands parameter of Π_v.
- Reducing r_Π modulo ℓ. In this way we obtain a finite subgroup $\bar{\Gamma}$ of $GL_m(\bar{\mathbb{F}}_\ell)$. $\bar{\Gamma}$ can be controled by: (1) picking a (large) prime q so that it splits completely in every Galois extension of $\mathbb{Q}$ ramified at ℓ only and of the degree $\leq d$, for some sufficiently large d, and (2) by picking Π_q to be the lift (in the sense of the first step) of a tame supercuspidal representation σ_q.

The first step is based on constructing a generic automorphic representation σ of a classical group, with some desired local properties. We lift σ to a cuspidal representation Π of $GL_m(\mathbb{A})$ using a result of Cogdell, Kim, Piatetski-Shapiro and Shahidi, combined with the local parameterization by Jiang and Soudry. The lift from G_2 is based on the exceptional theta correspondence arising from the minimal representation of E_7. The second step is based on a result of Harris and Taylor. That result is subject to a condition that Π has a discrete series representation at one place. Since $GL_{2n+1}(\mathbb{Q}_q)$ has no self-dual tame supercuspidal representations and, in fact, no self-dual supercuspidal representations at all since q is odd, the representation Π_q is not a discrete series representation if $m = 2n+1$. This technical difficulty had originally prevented us from constructing Galois groups of type B_n and G_2 in the first paper. A way around this is to arrange Π_2 to be a ramified self-dual cuspidal representation of $GL_{2n+1}(\mathbb{Q}_2)$. Alternatively, one can use a generalization of the result of Harris-Tayor, due to Shin, where the condition that Π has a discrete series representation at one place has been removed. The third step relies on a paper of Larsen and Pink, a wast generalization to reductive groups of Dickson's classification of finite subgroups of $PGL_2(\bar{\mathbb{F}}_\ell)$.

Lightly ramified number fields (with an eye towards automorphic forms)

David P. Roberts

The talk summarizes my recent and ongoing work towards constructing number fields which are lightly ramified for their Galois group G in various senses. The light ramification makes these fields promising candidates for explicit matches with automorphic forms. The talk is organized by increasing Galois group and towards the beginning some explicit matches are discussed. Some of the fields presented:

$G = SL_2^{\pm}(\mathbb{F}_{11})$. Malle's M_{22} cover has an exceptional specialization point which yields an $PGL_2(11)$ field ramified at 2, 3, and 11 only. We present a degree 24 even polynomial lifting this field to the double cover $SL_2(\mathbb{F}_{11})$. The ramification is tame at 11 and we have found explicit corresponding classical modular forms in $S_4(24)$ and $S_8(24)$ whose mod 11 Galois representations agree with this lift. This example provides a very explicit illustration of Gross' theory of companion forms.

$G = SL_2(16).4$, $PGL_2(27).3$, $SL_2(5)^5.10$. We present lightly ramified fields in these cases which numerically match Hilbert modular forms. The first two matches form part of work with Dembélé and Diamond. They have helped to refine a conjecture relating ramification at the residual prime ℓ of Galois representations with Serre weights of automorphic forms. The last is particularly interesting because the field ramifies at 5 only, the form having been found by Démbélé, Greenberg, and Voight.

$G = W(E_7)^+$. Representing many polynomials giving mod ℓ Galois representations associated to Katz's rigid local systems, we present equations for five one-parameter families of number fields with Galois group $W(E_7)^+$. These families are numbers 58-62 on the Beukers-Heckman list, with the first one having equation

$$2^{18}(x^3+3x^2-3)^9 = t3^6x^3(3x+4)(x^2+6x+6)^{12}.$$

The five explicit polynomials are found in a simple uniform way by specializing Shioda's universal $W(E_7)^+$ polynomial. The corresponding curves in the t-x plane all have genus zero.

$G = G_2(2) = U_3(3).2$. There are newer theories of rigidity associated to more general algebraic groups. We present a two-parameter family with Galois group $G_2(2) = U_3(3).2$ and bad reduction at 2 and 3 only. Like the examples of the previous paragraph, this family comes in several ways from mod ℓ representations associated to motives, and these connections should facilitate future explicit connections with automorphic forms.

$G = 2.M_{11}.2$. The final example is a field with Galois group $2.M_{12}.2$ and bad reduction at 11 only. Gross remarks that an associated automorphic form might come from the embedding $M_{12} \subset E_7$. While the group is very large here, the fact that the automorphic form may be everywhere unramified reduces complexity.

Geometric interpretation and stability of characters of Deligne–Lusztig representations of *p*-adic groups

Yakov Varshavsky

In my talk I am going to explain a joint work with Roman Bezrukavnikov ([BV]). Our main goal is to give a geometric proof of the stability of L-packets of Deligne–Lusztig represention of p-adic groups, shown earlier in my joint work with David Kazhdan [KV].

To show the result, we reinterpret these characters in terms of homology of affine Springer fibres. After this is done, we deduce the stability from a theorem of Zhiwei Yun [Yun], which asserts that commuting actions of the affine Weyl group and the stabilizer are compatible.

Let us describe our results in more details. Let F be a local non-archimedean field, W_F the Weil group of F, l a prime number different from the characteristic of F, G an unramified connected reductive group over F and ${}^LG = \widehat{G} \rtimes W_F$ the Langlands dual group of G over $\overline{\mathbb{Q}_l}$.

Recall that a local Langlands conjecture predicts that for every Langlands parameter $\lambda : W_F \to {}^LG(\overline{\mathbb{Q}_l})$ should gives rise to certain finite set Π_λ of irreducible admissible representations of $G(F)$, called the L-packet. Moreover, it is believed that a certain explicit linear combination $\chi^{st}_\lambda = \sum_{\pi\in\Pi_\lambda} d_\lambda \chi(\pi)$ of characters of representations from Π_λ should be stable. In particular, one expects to have a map $\lambda \mapsto \chi^{st}_\lambda$ from the set of Langlands parameters to the space of stable generalized functions $D_{st}(G(F), \overline{\mathbb{Q}_l})$.

Though conjectural generalized functions χ^{st}_λ are predicted in some cases, classically there is no neither characterization of the map $\lambda \mapsto \chi^{st}_\lambda$ nor a general procedure, which for a given λ produces a generalizes function χ^{st}_λ. On the other hand, when F is of positive characteristic, one believes that should exists a "natural" map $\lambda \mapsto \chi^{st}_\lambda$ "via geometry".

Assume λ is tamely ramified and factors through ${}^LT \subset {}^LG$, for certain embedding of L-groups ${}^LT \hookrightarrow {}^LG$ for certain elliptic unramified torus T over F of the same absolute rank as G. In this case, the expected generalized function is known classically. The goal of my talk is to give a geometric description of the restriction of χ^{st}_λ to the set of regular semisimple compact elements $G^{rss,c}(F)$. Using this description we give an alternative geometric proof of the stability of χ^{st}_λ at least on the regular semisimple locus.

First we describe the classical construction. Recall that by the local Langlands for torus, $\lambda : W_F \to {}^L G$ corresponds to tamely ramified homomorphism $\theta = \theta_\lambda : T(F) \to \overline{\mathbb{Q}_l}^\times$. Let $\mathbb{F}_q$ be the residue field of F. Then T gives rise to a torus $\overline{T}$ over $\mathbb{F}_q$ and the restriction $\theta|_{T(\mathcal{O})} : T(\mathcal{O}) \to \overline{\mathbb{Q}_l}^\times$ factors through a character $\overline{\theta} : \overline{T}(\mathbb{F}_q) \to \overline{\mathbb{Q}_l}^\times$. On the other hand, an embedding ${}^L T \hookrightarrow {}^L G$ gives rise to an embedding of a maximal torus $a : T \hookrightarrow G$, defined uniquely up to a stable conjugacy.

Let $a_1 = a, a_2, \dots, a_n$ be a set of representatives of the set of conjugacy class of embeddings $T \hookrightarrow G$, which are stably conjugate to a. Since $a(T) \subset G$ is elliptic, for each such a_i, there exists a unique parahoric subgroup $G_{a_i} \subset G(F)$ such that $a_i(T(\mathcal{O})) \subset G_{a_i}$. We denote by L_{a_i} the corresponding "Levi subgroup", by $\overline{a}_i : \overline{T} \hookrightarrow L_{a_i}$ the map induced by a_i, by $\rho_{\overline{a}_i,\overline{\theta}}$ the virtual cuspidal Deligne-Lusztig representation of $L_{a_i}(\mathbb{F}_q)$, corresponding to a torus $\overline{a}_i(\overline{T}) \subset L_{a_i}$ and a character $\overline{\theta}$.

Next let $\rho_{a_i,\theta}$ be the unique representation of $G_{a_i} Z^0(F)$ whose restriction to G_{a_i} is the inflation of $\overline{\rho}_{\overline{a}_i,\overline{\theta}}$ and restriction to $Z^0(F)$ is equal to the restriction of θ, and let π_{a_i} be the induced representation $\mathrm{Ind}_{G_{a_i} Z^0(F)}^{G(F)}(\rho_{a_i})$. The π_{a_i} is known to be a virtual cuspidal representation, and we denote by $\pi^{st}_\lambda = \pi^{st}_{\lambda,G}$ to be the sum $\sum_i \pi_{a_i}$.

Now we assume that F is a local field of positive characteristic and describe our geometric construction of the restriction of χ^{st}_λ to $G^{rss,c}(F)$. By a theorem of Lang, a character $\overline{\theta} : \overline{T}(\mathbb{F}_q) \to \overline{\mathbb{F}_q}$ gives rise to a one-dimensional local system $\mathcal{L}_\theta$ on $\overline{T}$, equipped with a Weil structure. Recall that G is defined over $\mathbb{F}_q$, and let $\overline{T}_G$ be the abstract Cartan over $\mathbb{F}_q$, and W_G the Weyl group of G. Note that there is a natural W_G-conjugacy class of isomorphisms $\phi : \overline{T}_G \xrightarrow{\sim} \overline{T}$ (defined over $\overline{\mathbb{F}_q}$). In particular, we get a W_G-equivariant local system $\mathcal{E}_{\overline{\theta}} := \oplus_\varphi \varphi^*(\mathcal{L}_{\overline{\theta}})$ on $\overline{T}_G$, equipped with a Weil structure.

Now for each $\gamma \in G^{rss,c}(F)$, there is a natural projection $Spr_\gamma \to \overline{T}_G$, defined over $\mathbb{F}_q$. Since $\mathcal{E}_{\overline{\theta}}$ is a Weil sheaf, its pull back is a Weil sheaf on Spr_γ, which we again denote by $\mathcal{E}_{\overline{\theta}}$. Moreover, since $\mathcal{E}_{\overline{\theta}}$ was W_G-equivariant, its homology $V_{i,\gamma} := H_i(Spr_\gamma, \mathcal{E}_{\overline{\theta}})$ is equipped with an action of the extended affine Weyl group $\widetilde{W}_G := W_G \ltimes \Lambda$, where $\Lambda := X_*(\overline{T}_G)$ is a lattice of cocharacters of $\overline{T}_G$. Moreover, each $V_{i,\gamma}$ is a finitely generated $\widetilde{W}_G$ module, therefore the derived coinvariants $H_{*,\gamma} := H_*(\widetilde{W}_G, \oplus_i (-1)^i V_{i,\gamma})$ is virtual finite dimensional vector space, equipped with an action of $\mathrm{Gal}(\overline{\mathbb{F}_q}/\mathbb{F}_q)$.

The main result of this work asserts that stable character $\chi^{st}_\lambda(\gamma)$ equals the trace of the geometric Frobenius element $\mathrm{Tr}(\mathrm{Fr}, H_{*,\gamma})$. As a consequence we give an alternative geometric proof of the fact that the character χ^{st}_λ is stable on the set of regular semisimple elements.

The main technical tool in our arguments is the following result, which is essentially due to Zhiwei Yun [Yun]. More precisely, Yun proved the corresponding result for Lie algebras using global methods, while we deduce the group algebra version using the topological Jordan decomposition and quasi-lograithm maps.

To describe the result of Yun, we recall that the homology group $V_{i,\gamma}$ is also equipped with an action of the centralizer $G_\gamma(F)$, commuting with the action of $\widetilde{W}_G$. Moreover, this action factors through the group of connected components $\pi_0(G_\gamma) = X_*(G_\gamma)_{\Gamma_{F^{nr}}}$. Note also that there is a natural isomorphism $\Lambda \xrightarrow{\sim} X_*(G_\gamma)$, unique up to a W_G-conjugacy, hence a canonical homomorphism $p : \overline{\mathbb{Q}_l}[\Lambda]^{W_G} \to \overline{\mathbb{Q}_l}[\pi_0(G_\gamma)]$. Then the result we heavily used is an assertion that for each i there is a finite $\widetilde{W}_G \times G_\gamma$-equivariant filtration of $V_{i,\gamma}$ such that the induced action of $\overline{\mathbb{Q}_l}[\Lambda]^{W_G}$ on each graded piece is induced from the action of $\pi_0(G_\gamma)$ via homomorphism p.

Let us indicate how a theorem of Yun was used. Recall that Spr_γ is an ind-scheme such that the quotient $\pi_0(G_\gamma)\backslash Spr_\gamma$ is of finite type. Therefore each $V_{i,\gamma}$ is a finitely generated $\pi_0(G_\gamma)$ module. Using theorem of Yun we deduce that each $V_{i,\gamma}$ is a finitely generated $\overline{\mathbb{Q}_l}[\Lambda]^{W_G}$ module, hence $\widetilde{W}_G$-module. Therefore $H_{*,\gamma}$ is a virtual finite dimensional vector space, hence the trace $\mathrm{Tr}(\mathrm{Fr}, H_{*,\gamma})$ is defined. Secondly, Yun's theorem implies that $\pi_0(G_\gamma)$ trivially acts on on the space of $\widetilde{W}_G$-coinvariants $H_{*,\gamma}$, which immediately implies the stability of the function $\gamma \mapsto \mathrm{Tr}(\mathrm{Fr}, H_{*,\gamma})$.

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Rigid local systems from automorphic forms

Zhiwei Yun

In my talk, I gave a survey on some new construction of local systems on punctured lines from the point of automorphic representations. For more details we refer to [9].

Rigid automorphic data. Consider $\mathbb{P}^1_k - S$ where k is a finite field and $S \subset \mathbb{P}^1(k)$ is a finite set. Let $F = k(t)$ be the function field of $\mathbb{P}^1_k$ and let $\mathbb{A}_F$ be the ring of adèles of F. Let G be a semisimple split algebraic group over k. Let $\widehat{G}$ be the Langlands dual group of G over $\overline{\mathbb{Q}}_\ell$. We can talk about automorphic representations of $G(\mathbb{A}_F)$. We would like to impose local conditions on the automorphic representations π of the following kind:

(1) π_v is unramified for places v of $\mathbb{P}^1$ not in S;
(2) For each $v \in S$, π_v contains an eigenvector of a certain compact open subgroup $K_v \subset G(\mathcal{O}_v)$ on which it acts through a character $\chi_v : K_v \to \overline{\mathbb{Q}}_\ell^\times$.

(3) We require the pairs $\{K_v, \chi_v\}_{v\in S}$ to "come from geometry", hence make sense when we base change k to a finite field extension k'.

An automorphic datum $(K_v, \chi_v)_{v\in S}$ as above is said to be *rigid*, if there is a constant N such that for every finite extension k' of k (correspondingly $F' = k'(t)$), the number of automorphic representation π' of $G(\mathbb{A}_{F'})$ satisfying the obvious F'-analogues of the above conditions is bounded from above by N (and there is such a π' for some extension k').

When $G = \mathrm{PGL}_2$, we give two examples.

First, take $S = \{0, 1, \infty\}$, take K_v to be an Iwahori subgroup and require the characters χ_v to factor through the torus quotient of K_v. Under some genericity conditions on these characters, $(K_v, \chi_v)_{v\in S}$ is rigid.

Second, take $S = \{0, \infty\}$, take K_0 to be an Iwahori with χ_v trivial, while K_∞ to be the pro-unipotent radical of an Iwahori with a generic additive character χ_∞ on it. Then $(K_v, \chi_v)_{v\in S}$ is also rigid.

Constructing local systems. Using ideas from geometric Langlands correspondence, we are able to produce $\widehat{G}$-local systems over $\mathbb{P}^1_k - S$ from a rigid automorphic datum $(K_v, \chi_v)_{v\in S}$ as above. We can control the local monodromy of the resulting local system, which often implies that the global geometric monodromy is the whole $\widehat{G}$.

In the first example, taking $\chi_0 = \chi_1$ to be trivial and χ_∞ quadratic, the corresponding $\widehat{G} = \mathrm{SL}_2$-local system is the local system of ℓ-adic Tate modules of the Legendre family of elliptic curves $E_t : y^2 = x(x-1)(x-t)$, $t \in \mathbb{P}^1 - \{0, 1, \infty\}$.

In the second example, the $\widehat{G} = \mathrm{SL}_2$-local system we get is the Kloosterman sheaf introduced by Deligne [1]. Its Frobenius trace function is the classical Kloosterman sum

$$t \mapsto \sum_{xy=t, x,y\in k^\times} \psi(x+y).$$

Applications. In [4], we produce $\widehat{G}$-local systems over $\mathbb{P}^1_{\mathbb{F}_p} - \{0, \infty\}$ that are tame at 0 and wild at ∞. When $G = \mathrm{PGL}_2$ this is the second example above. The construction is uniform for all $\widehat{G}$, and they give examples of local systems with geometric monodromy groups of type E_7, E_8, F_4 and G_2 (the G_2 case was known by Katz [5]).

In [8], we produce tame $\widehat{G}$-local systems over $\mathbb{P}^1_k - \{0, 1, \infty\}$ where k is any prime field of characteristic not two, including $\mathbb{Q}$. When $G = \mathrm{PGL}_2$ it belongs to the first example above. For $\widehat{G} = E_7, E_8, F_4$ and G_2 we again get full geometric monodromy. Specializing to a general $\mathbb{Q}$-point, the local system gives a motive over $\mathbb{Q}$ with motivic Galois group is of type $\widehat{G}$, answering a question of Serre in [6]. Reduce the local system modulo ℓ we deduce that $E_8(\mathbb{F}_\ell), F_4(\mathbb{F}_\ell)$ and $G_2(\mathbb{F}_\ell)$ are Galois groups over $\mathbb{Q}$ for sufficiently large ℓ, solving special cases of the inverse Galois problem (in both situations the G_2 cases were known before our work, see [2], [3] and [7]).

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Galois realization of $E_8(p)$, $p \geq 7$

Gunter Malle

We reported on the proof of the theorem, obtained in joint work with Robert Guralnick (USC), that all finite simple groups $E_8(p)$, for $p \geq 7$ a prime, occur as Galois groups of infinitely many linearly disjoint extensions of the field of rational numbers $\mathbb{Q}$. This extends earlier results of the speaker (who showed the claim for a set of primes of density 16/30 in 1986) and of Z. Yun (who recently showed it by completely different methods for all primes larger than an unspecified bound, see the previous talk).

Our proof relies on the rigidity criterion of Belyi, Fried, Matzat and Thompson. We exhibit three rational conjugacy classes of $E_8(p)$, one of them consisting of regular unipotent elements, which we show to be rigid. This is proved by first carefully estimating the structure constant for that triple using results of Lusztig on characters of finite reductive groups. As a side result we obtain a remarkable symmetry between the character table of a finite reductive group when restricted to semisimple classes and semisimple characters, and that of its Langland's dual group.

We then show that any triple from the chosen conjugacy classes with product 1 must generate $E_8(p)$ by classifying the Lie primitive subgroups of $E_8(q)$ containing regular unipotent elements.

Our results also show that a triple of elements from the chosen classes can already be found in $E_8(S)$, where $S = \mathbb{Z}[1/30]$.

On the classification of orthogonally rigid G_2-local systems

Stefan Reiter

Let $\mathcal{L}$ be a $\mathbb{C}$-local system on $X := \mathbb{P}^1 \setminus \{x_1, \ldots, x_r, x_{r+1} = \infty\}$ of rank n and

$$\rho : \pi_1(X, x_0) = \langle \gamma_1, \ldots, \gamma_{r+1} \rangle \to \mathrm{GL}(\mathcal{L}_{x_0}) \cong \mathrm{GL}_n(\mathbb{C}), \gamma_i \mapsto \rho(\gamma_i) =: g_i$$

the corresponding monodromy representation. Thus $\prod_{i=1}^{r+1} g_i = 1$. We call $\mathcal{L}$ a G-local system if $\overline{\rho(\pi_1)} \subseteq G(\mathbb{C})$, where G is a linear algebraic group. Further, $\mathcal{L}$ is called irreducible if the monodromy representation is irreducible. An irreducible local system $\mathcal{L}$ of rank n is called GL_n-rigid if

$$-(r-1)n^2 + \sum_{i=1}^{r+1} \dim C_{\mathrm{GL}_n}(g_i) = \chi(\mathbb{P}^1, j_* End\mathcal{L}_{|X}) = 2.$$

In this case ρ is uniquely determined by its local monodromy (up to simultaneous conjugation) by a result of Deligne and Katz, cf. [1] . We call an irreducible local system $\mathcal{L}$ of rank n orthogonally rigid if $\mathcal{L}$ a O_n-local system and

$$-(r-1)\dim O_n + \sum_{i=1}^{r+1} \dim C_{\mathrm{O}_n}(g_i) = 0.$$

This a necessary condition that there are only finitely many equivalence classes of such monodromy representations with given local monodromy data. (In general such a local system is not uniquely determined by its local monodromy data).

By the work of N. Katz on the middle convolution functor MC_χ, all GL_n-rigid irreducible local systems $\mathcal{L}$ on the punctured line can be constructed by applying iteratively MC_χ and tensor products with rank-1-sheaves to a rank-1-sheaf. For orthogonally rigid local systems with G_2-monodromy we prove that there is a similar method of construction:

Theorem 1. *Let $\mathcal{L}$ be an orthogonally rigid $\mathbb{C}$-local system on a punctured projective line $\mathbb{P}^1 \setminus \{x_1, \ldots, x_{r+1}\}$ of rank 7 whose monodromy group is dense in the exceptional simple group G_2. If $\mathcal{L}$ has nontrivial local monodromy at $x_1, \ldots, x_{r+1}$, then $r = 2, 3$ and $\mathcal{L}$ can be constructed by applying iteratively a sequence of the following operations to a rank-1-system:*

- *Middle convolutions MC_χ, with varying χ.*
- *Tensor products with rank-1-local systems.*
- *Tensor operations like symmetric or alternating products.*
- *Pullbacks along rational functions.*

Especially, each such local system which has quasi-unipotent monodromy, i.e. all the eigenvalues of the local monodromy are roots of unity, is motivic, i.e., it arises from the variation of periods of a family of varieties over the punctured projective line.

The verification that the monodromy group is inside the group $G_2(\mathbb{C})$ cannot be decided by looking at the local monodromy data alone. Therefore we interpretate MC_χ at the level of differential operators. The differential operators which belong to the local systems of Thm. 1 under Riemann-Hilbert correspondence can be determined by the following criterion:

Theorem 2. *Let $L = \sum_{i=0}^{7} a_i(x)\partial^i \in \mathbb{C}(x)[\partial]$ be monic, i.e. $a_7(x) = 1$. If L is self adjoint then*

$$\begin{aligned} a_6(x) &= 0 \\ a_4(x) &= \frac{5}{2}\frac{d}{dx}a_5(x) \\ a_2(x) &= -\frac{5}{2}\frac{d^3}{dx^3}a_5(x) + \frac{3}{2}\frac{d}{dx}a_3(x) \\ a_0(x) &= \frac{1}{2}\frac{d}{dx}a_1(x) - \frac{1}{4}\frac{d^3}{dx^3}a_3(x) + \frac{1}{2}\frac{d^5}{dx^5}a_5(x) \end{aligned}$$

Moreover, if the above conditions hold and if further

$$a_3(x) = 3\frac{d^2}{dx^2}a_5(x) + \frac{1}{4}a_5(x)^2,$$

then the differential Galois group and the monodromy group of L is contained in $G_2(\mathbb{C})$.

This is a joint work with Michael Dettweiler [2].

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On the cohomology of stacks of G-shtukas

Ngo Dac Tuan

Let X be a smooth, projective, geometrically connected curve over a finite field $\mathbb{F}_q$ et let F be its function field. Let G a split reductive group over $\mathbb{F}_q$ equipped with a Borel pair (T, B). Denote by $\hat{G}$ the Langlands dual group of G over $\bar{\mathbb{Q}}_l$ with $l \neq \mathrm{char}(\mathbb{F}_q)$.

For every finite set (not necessarily non-empty) I, every irreducible representation W of $\hat{G}^I$ and every finite subscheme N of X, we define the stack $\mathrm{Cht}_{I,W,N}$ classifying G-shtukas associated to the pair (I, W) with level structure N. The stack $\mathrm{Cht}_{I,W,N}$ is a Deligne-Mumford stack, locally of finite type, equipped with a natural map

$$p : \mathrm{Cht}_{I,W,N} \to (X - N)^I.$$

For every rational dominant coweight μ of T^{ad}, we introduce the open substack $\mathrm{Cht}_{I,W,N}^{\leq\mu}$ of $\mathrm{Cht}_{I,W,N}$. It is well-known that the connected components of $\mathrm{Cht}_{I,W,N}^{\leq\mu}$ are quotients of a quasi-projective scheme over $(X - N)^I$ by a finite group. We define

$$H_{c,I,W,N}^{*,\leq\mu} = Rp_!(\mathrm{IC}(\mathrm{Cht}_{I,W,N}^{\leq\mu})) \in D_c^b((X - N)^I, \bar{\mathbb{Q}}_l)$$

where $\mathrm{IC}(\mathrm{Cht}^{\leq\mu}_{I,W,N})$ is the intersection sheaf of $\mathrm{Cht}^{\leq\mu}_{I,W,N}$. Then we define the cohomology of the stack $\mathrm{Cht}_{I,W,N}$ as follows:

$$H^*_{c,I,W,N} = \varinjlim H^{*,\leq\mu}_{c,I,W,N}.$$

Remark that it is necessary to introduce a lattice Ξ as in [8], but we ignore it in this note for simplicity.

The generic fiber $H^*_{c,I,W,N}{}_{|\overline{\eta^I}}$ is equipped with three actions: the action of the Hecke algebra $\mathcal{H}_N$ of level N, the action of the Galois group $\mathrm{Gal}(\overline{\eta^I}/\eta^I)$ and the actions of partial Frobenius. These actions commute with each other. However, the actions of the Hecke algebra and partial Frobenius do not preserve $H^{*,\leq\mu}_{c,I,W,N}$. The main goal of our talk is how to define the so-called essential cohomology $H^{*,\mathrm{ess}}_{c,I,W,N}{}_{|\overline{\eta^I}}$ in which the Langlands correspondence would be realized.

The first example is the case of the general linear group $G = \mathrm{GL}_r$ for $r \geq 2$. Let I be a set of two elements and W be the representation $V \boxtimes V^\vee$ where V is the standard representation of GL_r. The associated stack $\mathrm{Cht}_{I,W,N}$ is known as the stack classifying Drinfeld's shtukas of rank r. Drinfeld (for $r = 2$) [1, 2] and L. Lafforgue (for $r > 3$) [5, 6, 7] constructed $H^{*,\mathrm{ess}}_{c,I,W,N}{}_{|\overline{\eta^I}}$ and proved that

$$H^{*,\mathrm{ess}}_{c,I,W,N}{}_{|\overline{\eta^I}} = \bigoplus_{\pi} \pi^{K_N} \otimes \sigma(\pi) \otimes \sigma^\vee(\pi)$$

where the sum runs through cuspidal automorphic representations of $\mathrm{GL}_r(\mathbb{A}_F)$ (whose determinant is of finite order) and $\sigma(\pi) : \mathrm{Gal}(\bar{F}/F) \to \mathrm{GL}_r(\bar{\mathbb{Q}}_l)$ is the Galois representation associated to π.

The second example is due to Varshavsky [12]. We still work with the general linear group $G = \mathrm{GL}_r$ for $r \geq 2$. For any triple (I, W, N) as above, Varshavsky suggested a construction for $H^{*,\mathrm{ess}}_{c,I,W,N}{}_{|\overline{\eta^I}}$ and conjectured that

$$H^{*,\mathrm{ess}}_{c,I,W,N}{}_{|\overline{\eta^I}} = \bigoplus_{\pi} \pi^{K_N} \otimes W$$

where the sum runs through cuspidal automorphic representations of $\mathrm{GL}_r(\mathbb{A}_F)$ (whose determinant is of finite order) and the Galois action on W is given via the associated Galois representation $\sigma(\pi) : \mathrm{Gal}(\bar{F}/F) \to \mathrm{GL}_r(\bar{\mathbb{Q}}_l)$.

The third striking example is due to Vincent Lafforgue [8]. We take I to be the empty set and W to be the trivial representation. Then V. Lafforgue defined the so-called Hecke-finite cohomology $H^{0,\mathrm{Hf}}_{c,I,W,N}{}_{|\overline{\eta^I}}$. He proved that it is exactly $\mathcal{C}^{\mathrm{cusp}}_c(G(F)\backslash G(\mathbb{A}_F/K_N, \bar{\mathbb{Q}}_l)$, the space of $\mathbb{Q}_l$-valued cuspidal functions of compact support on $G(F)\backslash G(\mathbb{A}_F)/K_N$ and admits a canonical decomposition of this space indexed by Langlands parameters.

In this talk, we formulate a precise conjecture for $H^{*,\mathrm{ess}}_{c,I,W,N}{}_{|\overline{\eta^I}}$ for any split reductive group G over $\mathbb{F}_q$. The conjecture is based on our recent work [9, 10, 11] which gives a formula for the number of fixed points of $\mathrm{Cht}^{\leq\mu}_{I,W,N}$ under the composition of a Hecke action with a certain power of the Frobenius. It is inspired from the work of Kottwitz on certain Shimura varieties [3, 4].

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On modular forms in the inverse Galois problem

Gabor Wiese

For many finite groups the Inverse Galois Problem (IGP) can be approached through modular/automorphic Galois representations. This report is about the **ideas** and the **methods** that my coauthors and I have used so far, and their **limitations** (in my experience).

In this report I will mostly stick to the case of 2-dimensional Galois representations because it is technically much simpler and already exhibits essential features; occasionally I'll mention n-dimensional symplectic representations; details on that case can be found in Sara Arias-de-Reyna's report on our joint work with Dieulefait and Shin (see page 310 in this report).

Basics of the approach

The link between the IGP and Galois representations. Let $K/\mathbb{Q}$ be a finite Galois extension such that $G := \mathrm{Gal}(K/\mathbb{Q}) \subset \mathrm{GL}_n(\overline{\mathbb{F}}_\ell)$ is a subgroup. Then $G_\mathbb{Q} := \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \twoheadrightarrow \mathrm{Gal}(K/\mathbb{Q}) \hookrightarrow \mathrm{GL}_n(\overline{\mathbb{F}}_\ell)$ is an n-dimensional continuous Galois representation with image G. Conversely, given a Galois representation $\rho : G_\mathbb{Q} \to \mathrm{GL}_n(\overline{\mathbb{F}}_\ell)$ (all our Galois representations are assumed continuous), then $\mathrm{im}(\rho) \subset \mathrm{GL}_n(\overline{\mathbb{F}}_\ell)$ is the Galois group of the Galois extension $\overline{\mathbb{Q}}^{\ker(\rho)}/\mathbb{Q}$.

Source of Galois representations: abelian varieties. Let A be a GL_2-type abelian variety over $\mathbb{Q}$ of dimension d with multiplication by the number field $F/\mathbb{Q}$ (of degree d) with integer ring $\mathcal{O}_F$. Then for every prime ideal $\lambda \lhd \mathcal{O}_F$, the λ-adic Tate module of A gives rise to $\rho_{A,\lambda} : G_{\mathbb{Q}} \to \mathrm{GL}_2(\mathcal{O}_{F,\lambda})$. These representations are a special case of those presented next (due to work of Ribet and the proof of Serre's modularity conjecture).

Source of Galois representations: modular/automorphic forms. Let $f = \sum_{n=1}^{\infty} a_n e^{2\pi i n z}$ be a normalised Hecke eigenform of level N and weight k without CM (or, more generally, an automorphic representation of a certain type over $\mathbb{Q}$). The coefficients a_n are algebraic integers and $\mathbb{Q}_f = \mathbb{Q}(a_n \mid n \in \mathbb{N})$ is a number field, the **coefficient field** of f. Denote by $\mathbb{Z}_f$ its ring of integers. The eigenform f gives rise to a **compatible system of Galois representations**, that is, for every prime λ of $\mathbb{Q}_f$ a Galois representation $\rho_{f,\lambda} : G_{\mathbb{Q}} \to \mathrm{GL}_2(\mathbb{Z}_{f,\lambda})$ such that $\rho_{f,\lambda}$ is unramified outside $N\ell$ (where $(\ell) = \mathbb{Z} \cap \lambda$) and for all $p \nmid N\ell$ we have $\mathrm{Tr}(\rho_{f,\lambda}(\mathrm{Frob}_p)) = a_p$. All representations thus obtained are odd (determinant of complex conjugation equals -1).

Reduction and projectivisation. We consider the representations $\overline{\rho}_{f,\lambda} : G_{\mathbb{Q}} \xrightarrow{\rho_{f,\lambda}} \mathrm{GL}_2(\mathbb{Z}_{f,\lambda}) \twoheadrightarrow \mathrm{GL}_2(\mathbb{F}_{f,\lambda})$ and $\overline{\rho}^{\mathrm{proj}}_{f,\lambda} : G_{\mathbb{Q}} \xrightarrow{\overline{\rho}_{f,\lambda}} \mathrm{GL}_2(\mathbb{F}_{f,\lambda}) \twoheadrightarrow \mathrm{PGL}_2(\mathbb{F}_{f,\lambda})$, where $\mathbb{F}_{f,\lambda} = \mathbb{Z}_{f,\lambda}/\lambda$. In our research we focus on projective representations because the groups $\mathrm{PSL}_2(\mathbb{F}_{\ell^d})$ are simple for $\ell^d \geq 4$.
Main idea: By varying f and λ (and thus ℓ), realise as many finite subgroups of $\mathrm{PGL}_2(\overline{\mathbb{F}}_\ell)$ as possible.

Trust in the approach. If $\ell > 2$, the oddness of the representations leads to $\overline{\mathbb{Q}}^{\ker(\overline{\rho}^{\mathrm{proj}}_{f,\ell})}$ being totally imaginary. **The approach through modular Galois representations for the groups $\mathrm{PSL}_2(\mathbb{F}_{\ell^d})$ and $\mathrm{PGL}_2(\mathbb{F}_{\ell^d})$ to the IGP should in principle work** for the following reason: If $\mathrm{Gal}(K/\mathbb{Q}) \subset \mathrm{PGL}_2(\overline{\mathbb{F}}_\ell)$ is a finite (irreducible) subgroup and $K/\mathbb{Q}$ is totally imaginary (which is 'much more likely' than being totally real), then Serre's modularity conjecture implies that K can be obtained from some f and λ. In more general contexts, there are generalisations of Serre's modularity conjecture (however, unproved!) and I am inclined to believe that the approach is promising in more general contexts than just GL_2.

The two directions. We have so far explored two directions for the realisation of $\mathrm{PSL}_2(\mathbb{F}_{\ell^d})$ and $\mathrm{PSp}_n(\mathbb{F}_{\ell^d})$. **Vertical direction:** fix ℓ, let d run (results by me for PSL_2 [Wie08], generalised by Khare-Larsen-Savin for PSp_n [KLS08]); **horizontal direction:** fix d, let ℓ run (results by Dieulefait and me for PSL_2 [DW11] and by Arias-de-Reyna, Dieulefait, Shin and me for PSp_n [AdDSW13]).

Main challenges

In approaching the IGP through modular forms for specific groups, in my experience one is faced with two challenges:

(1) Control/predetermine the type of the image $\overline{\rho}^{\text{proj}}_{f,\lambda}(G_{\mathbb{Q}})$.
(2) Control/predetermine the coefficient field $\mathbb{Q}_f$.

Problem (2) appears harder to me.

Controlling the type of the images. By a classical theorem of Dickson, if $\overline{\rho}_{f,\lambda}$ is irreducible, then it is either induced from a lower dimensional representation (only possiblity: a character) or $\overline{\rho}^{\text{proj}}_{f,\lambda}(G_{\mathbb{Q}}) \in \{\mathrm{PSL}_2(\mathbb{F}_{\ell^d}), \mathrm{PGL}_2(\mathbb{F}_{\ell^d})\}$ for some d (we call this case **huge/big image**). Under the assumption of a transvection in the image, we have generalised this result to symplectic representations. In our applications we want to exclude reducibility and induction. One can expect a **generic huge image result** (for GL_2 this is classical work of Ribet; for other cases e.g. recent work of Larsen and Chin Yin Hui in this direction [HL13]).

Inner twists. If one has e.g. determined that $\overline{\rho}^{\text{proj}}_{f,\lambda}(G_{\mathbb{Q}})$ is huge, one still needs to compute which $d \in \mathbb{N}$ and which of the two cases $\mathrm{PSL}_2(\mathbb{F}_{\ell^d})$, $\mathrm{PGL}_2(\mathbb{F}_{\ell^d})$ occurs. The answer is given by **inner twists**. For GL_2 these are well-understood (with Dieulefait we exclude them by a good choice of f); for PSp_n we proved a generalisation allowing us to describe d by means of a number field, but, as to now we are unable to distinguish between the two cases.

Coefficient field

One knows that $\mathbb{Q}_f$ is either totally real or totally imaginary (depending on the nebentype of f). Moreover, $[\mathbb{Q}_f : \mathbb{Q}] \leq \dim S_k(N)$, where $S_k(N)$ is the space of cusp forms of level N and weight k. Furthermore, a result of Serre says that for any sequence $(N_n, k_n)_n$ such that $N_n + k_n$ tends to infinity, there is $f_n \in S_{k_n}(N_n)$ such that $[\mathbb{Q}_{f_n} : \mathbb{Q}]$ tends to infinity. However, to the best of my knowledge, almost **nothing is known about the arithmetic of the coefficient fields and the Galois groups of their normal closures over** $\mathbb{Q}$. In my experience, this is the **biggest obstacle** preventing us from obtaining very strong results on the IGP.

Almost complete control through Maeda's conjecture. A conjecture of Maeda gives us some control on the coefficient field by claiming that for any $f \in S_k(1)$ one has $[\mathbb{Q}_f : \mathbb{Q}] = \dim S_k(1) =: m_k$ and that the Galois group of the normal closure of $\mathbb{Q}_f$ over $\mathbb{Q}$ is S_{m_k}, the symmetric group. The conjecture has been numerically tested for quite high values of k, but to my knowledge a proof is out of sight at the moment and there's no generalisation to higher dimensions either. Assuming Maeda's conjecture I was able to prove in [Wie13] that for even d the groups $\mathrm{PSL}_2(\mathbb{F}_{\ell^d})$ occur as Galois groups over $\mathbb{Q}$ with only ℓ ramifying for all ℓ, except possibly a density-0 set. In a nutshell, for the proof I choose a sequence f_n of forms of level 1 such that $[\mathbb{Q}_{f_n} : \mathbb{Q}]$ strictly increases. That the Galois group is the symmetric group ensures two things: firstly, every $\mathbb{Q}_{f_n}$ possesses a degree-d prime; secondly, the fields $\mathbb{Q}_{f_n}$ and $\mathbb{Q}_{f_m}$ for $m \neq n$ are almost disjoint (in the sense that their intersection is at most quadratic) and thus the sets of primes of degree d in the two fields are almost independent, so that their density adds up to 1 when $n \to \infty$. This illustrates that **some control on the coefficient field promises strong results on the IGP**.

A conjecture of Coleman on GL_2-type abelian varieties. The modular form f corresponding to a GL_2-type abelian variety with multiplication by F has coefficient field $\mathbb{Q}_f = F$. However, I don't know of any method to construct a GL_2-type abelian variety with multiplication by a given field. Indeed, a conjecture attributed to Coleman (see [BFGR06]) predicts that for a given dimension, only finitely many number fields occur. In other words, for weight-2 modular forms in all levels, there are only finitely many $\mathbb{Q}_f$ of a given degree. Under the assumption of Coleman's conjecture, it is impossible to obtain $\mathrm{PSL}_2(\mathbb{F}_{\ell^2})$ for all ℓ from GL_2-type abelian surfaces because there will be a positive density set of ℓ that are split in all number fields of degree 2 that occur as multiplication fields. Although I don't know if there are finitely or infinitely many quadratic fields occuring as $\mathbb{Q}_f$ for f of arbitrary level and arbitrary weight, this nevertheless suggests to me that one should make use of modular forms of **arbitrary coefficient degrees** for approaching $\mathrm{PSL}_2(\mathbb{F}_{\ell^d})$ for fixed d (as we did when we assumed Maeda's conjecture).

Numerical data. Some very simple computer calculations for $p = 2$ during my PhD have very quickly revealed that all $\mathrm{PSL}_2(\mathbb{F}_{2^d})$ with $1 \le d \le 77$ occur over $\mathbb{Q}$. With Marcel Mohyla we plotted $\mathbb{F}_{f,\lambda}$ for small fixed weight and f having prime levels [MW11]. The computations suggest that the maximum and the average degrees (for f in $S_k(N)$ for N prime) of $\mathbb{F}_{f,\lambda}$ are roughly proportional to the dimension of $S_k(N)$.

The local 'bad primes' approach to the main challenges

We need to gain some control on the coefficient fields and in the absence of a generic huge image result, we also need to force huge image of the Galois representation. In all our work (like in that of Khare-Larsen-Savin [KLS08]), we approach this by choosing suitable inertial types, or in the language of abelian varieties, by choosing certain types of bad reduction. The basic idea appeared in the work of Khare-Wintenberger on Serre's modularity conjecture. More precisely, one chooses inertial types at some primes q guaranteeing that $\overline{\rho}_{f,\lambda}(I_q)$ contains certain elements (I_q denotes the inertia group at q) . For instance, if an element that is conjugate to $\left(\begin{smallmatrix}1&1\\0&1\end{smallmatrix}\right)$ is contained, the representation cannot be induced. In the n-dimensional symplectic case, we use this to obtain a transvection in the image, allowing us to apply our classification (see above). We also employ Khare-Larsen-Savin's generalisation of Khare-Wintenberger's good-dihedral primes. More precisely, for GL_2 we impose $\overline{\rho}_{f,\lambda}|G_{\mathbb{Q}_q} = \mathrm{Ind}_{\mathbb{Q}_{q^2}}^{\mathbb{Q}_q}(\alpha)$ where α is a character of $\mathbb{Q}_{q^2}^\times$ of prime order t not descending to $\mathbb{Q}_q^\times$. This has two uses: (1) As the representation is **irreducible** locally at q, so it is globally. (2) $\mathbb{Q}_f$ contains $\zeta_t + \zeta_t^{-1}$ (this follows from an explicit description of the induction). This **cyclotomic field in the coefficient field** can be exploited in two ways. (2a) By making t big, $[\mathbb{F}_{f,\lambda} : \mathbb{F}_\ell]$ becomes big. **This**

leads to the results in the vertical direction. (2b) Given d, by choosing t suitably, $\mathbb{Q}(\zeta_t + \zeta_t^{-1})$ contains prime ideals of degree d, thus $\mathbb{Q}_f$ **contains prime ideals of degree d, which makes the results in the horizontal direction work**. In the absence of any knowledge on the Galois closure of $\mathbb{Q}_f$ over $\mathbb{Q}$ in general, I do not know of any other way to guarantee that degree-d primes exist at all (we need them to realise $\mathrm{PSL}_2(\mathbb{F}_{\ell^d})$).
My feeling is that the cyclotomic field $\mathbb{Q}(\zeta_t+\zeta_t^{-1})$ only makes up a very small part of the coefficient field, i.e. that $[\mathbb{Q}_f : \mathbb{Q}]$ will be much bigger than $[\mathbb{Q}(\zeta_t+\zeta_t^{-1}) : \mathbb{Q}]$. Thus, in our results in the horizontal direction, for given d and f, we only obtain very small densities. Moreover, I cannot prove that by varying f for fixed d, the sets of primes of residue degree d are not contained in each other. Any information, for instance, on the ramification of $\mathbb{Q}_f$ changing with f or on the Galois group would probably enable us to obtain a big density by taking the union of the sets of degree-d primes for many f.

Constructing the relevant modular/automorphic forms

For finishing the approach, one must finally construct or show the existence of modular/automorphic forms having the required inertial types. For modular forms one can do this in quite a down-to-earth way by using level raising. This approach was taken in the work by Dieulefait and me. In the symplectic case, we exploit work of Shin, as well as level-lowering results of Barnet-Lamb, Gee, Geraghty and Taylor [BLGGT13]. Khare-Larsen-Savin [KLS08] use other automorphic techniques.

Conclusion

The presented approach to the IGP for many families of finite groups through automorphic representations seems in principle promising. In my opinion, the main obstacle is a poor understanding of the coefficient fields.
The approach has the advantage that it allows **full control on the ramification**. A disadvantage is that one does not obtain a regular realisation.

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Compatible systems of symplectic Galois representations and the inverse Galois problem

Sara Arias-de-Reyna

Together with L. Dieulefait, S.W. Shin and G. Wiese, we have completed a project on the realization of (projective) symplectic groups over finite fields as Galois groups over $\mathbb{Q}$, making use of the compatible systems of Galois representations attached to certain automorphic forms (cf. [1], [2], [3]).

As a motivation for our work, consider a principally polarized n-dimensional abelian variety A defined over $\mathbb{Q}$. Then, for all prime numbers ℓ, we can consider the ℓ-torsion Galois representation

$$\overline{\rho}_{A,\ell} : G_{\mathbb{Q}} \to \mathrm{GSp}(A[\ell], e_\ell) \simeq \mathrm{GSp}_{2n}(\mathbb{F}_\ell),$$

where $G_{\mathbb{Q}}$ denotes the absolute Galois group of $\mathbb{Q}$ and e_ℓ is the Weil pairing on $A[\ell]$. If $\overline{\rho}_{A,\ell}$ is surjective, we obtain a realization of $\mathrm{GSp}_{2n}(\mathbb{F}_\ell)$ as a Galois group over $\mathbb{Q}$. Choosing a suitable abelian variety (e.g. [7]), it can be proven that, for all sufficiently large ℓ, $\mathrm{GSp}_{2n}(\mathbb{F}_\ell)$ can be realized as the Galois group of an extension $K/\mathbb{Q}$. Moreover, $K/\mathbb{Q}$ ramifies only at ℓ and the primes dividing the conductor of A.

We could try to replace the field $\mathbb{F}_\ell$ by $\mathbb{F}_{\ell^d}$, for some fixed integer $d \geq 1$. This naturally leads us to consider compatible systems of symplectic Galois representations $\rho_\bullet = (\rho_\lambda)_\lambda$, where λ runs through the primes of a number field L, and

$$\rho_\lambda : G_{\mathbb{Q}} \to \mathrm{GSp}_{2n}(\overline{L}_\lambda),$$

where $\overline{L}_\lambda$ denotes an algebraic closure of the completion of L at the prime λ, and ℓ denotes the rational prime below λ. The result we obtain is the following.

Theorem 1 (A., Dieulefait, Shin, Wiese)**.** *Let $n, d \in \mathbb{N}$. There exists a positive density set $\mathcal{L}$ of rational primes such that, for every prime $\ell \in \mathcal{L}$, the group $\mathrm{PGSp}_{2n}(\mathbb{F}_{\ell^d})$ or $\mathrm{PSp}_{2n}(\mathbb{F}_{\ell^d})$ can be realized as a Galois group over $\mathbb{Q}$. The corresponding number field ramifies at most at ℓ and two more primes, which are independent of ℓ.*

This result generalizes to the n-dimensional setting the work of Dieulefait and Wiese on the realization of groups of the form $\mathrm{PGL}_2(\mathbb{F}_{\ell^d})$ and $\mathrm{PSL}_2(\mathbb{F}_{\ell^d})$ (see [6]). In the terminology introduced in Gabor Wiese's report on applications of modular Galois representations to the inverse Galois problem, the theorem presented above can be encompassed in the *horizontal direction*, complementing the results in the *vertical direction* due to Wiese in the 2-dimensional setting (cf. [10]) and Khare, Larsen and Savin for symplectic groups of arbitrary dimension (cf. [8]).

To prove this result, we need to address the following questions:

1: Find conditions ensuring that the image of the residual Galois representation $\overline{\rho}_\lambda : G_{\mathbb{Q}} \to \mathrm{GSp}_{2n}(\overline{\mathbb{F}}_\ell)$ is *huge*, i.e., contains the subgroup $\mathrm{Sp}_{2n}(\mathbb{F}_\ell)$ (note that if $\mathrm{Im}\overline{\rho}_\lambda$ is huge, then the projective image of $\overline{\rho}_\lambda$ is $\mathrm{PGSp}_{2n}(\mathbb{F}_{\ell^r})$ or $\mathrm{PSp}_{2n}(\mathbb{F}_{\ell^r})$ for some $r \in \mathbb{N}$).

A key observation is that the classification of the finite subgroups of $\mathrm{GSp}_{2n}(\overline{\mathbb{F}}_\ell)$ containing a transvection is quite simple.

Theorem 2. *Let $G \subset \mathrm{GSp}_{2n}(\overline{\mathbb{F}}_\ell)$ be a finite subgroup containing a transvection. Then G is either reducible, imprimitive, or it contains $\mathrm{Sp}_{2n}(\mathbb{F}_\ell)$.*

If $G = \mathrm{Im}\overline{\rho}_\lambda$, this theorem implies that $\overline{\rho}_\lambda$ is either reducible, induced from an open subgroup of $G_{\mathbb{Q}}$, or has huge image. We assume that $\mathrm{Im}\overline{\rho}_\lambda$ contains a transvection, and look for conditions ensuring that the other two possibilities cannot occur. The reducible case can be ruled out if the compatible system possesses a maximally induced place of order p, which is a generalization to the n-dimensional setting, due to Khare, Larsen and Savin (cf. [8]), of the notion of good-dihedral prime appearing in the work of Khare and Wintenberger on Serre's Modularity Conjecture. To rule out the induced case for ℓ sufficiently large, we need to assume some regularity condition for the restriction of $\overline{\rho}_\lambda$ to a decomposition group at ℓ.

2: Determine the smallest field $\mathbb{F}(\lambda)$ such that the image of the composition $\overline{\rho}_\lambda^{\mathrm{proj}}$ of $\overline{\rho}_\lambda$ with the projection $\mathrm{GSp}_{2n}(\overline{\mathbb{F}}_\ell) \to \mathrm{PGSp}_{2n}(\overline{\mathbb{F}}_\ell)$ can be defined over $\mathbb{F}(\lambda)$.

Assume that ρ_λ is (absolutely) residually irreducible. Then ρ_λ can be conjugated (in $\mathrm{GL}_{2n}(\overline{L}_\lambda)$) to take values in $\mathrm{GL}_{2n}(L_\lambda)$. Enlarging L if necessary, we may assume that $L/\mathbb{Q}$ is a Galois extension. The key ingredient to address this question is the notion of *inner twist*. Namely, a pair (γ, ε) consisting of an element $\gamma \in \mathrm{Gal}(L_\lambda/\mathbb{Q}_\ell)$ and a character $\varepsilon : G_{\mathbb{Q}} \to L_\lambda^\times$ is called an inner twist of ρ_λ if the representations ${}^\gamma\rho_\lambda$ and $\rho_\lambda \otimes \varepsilon$ are conjugated. Let $\Gamma_{\rho_\lambda} \subset \mathrm{Gal}(L_\lambda/\mathbb{Q}_\ell)$ be the subgroup of elements appearing in inner twists of ρ_λ, and $K_{\rho_\lambda} := L_\lambda^{\Gamma_{\rho_\lambda}}$. If $\rho_\bullet$ satisfies several conditions (e.g. huge residual image, bounded inertial weights), then for all except finitely many primes λ the residue field $\mathbb{F}(\lambda)$ of K_{ρ_λ} is the smallest field on which $\overline{\rho}_\lambda^{\mathrm{proj}}$ can be defined. Moreover, there exists a global field $K_{\rho_\bullet} \subset L$ such that K_{ρ_λ} is the completion of $K_{\rho_\bullet}$ at the prime below λ (except for finitely many λ).

3: Force the field $K_{\rho_\bullet}$ to contain as many primes λ of residue degree d as possible. Let p, q be two rational primes, let $\zeta_p \in \overline{\mathbb{Q}}$ be a primitive p-th root of unity, and let

$\xi_p = \sum_{i=0}^{2n-1} \zeta_p^{q^i}$. The key observation is that the presence of a maximally induced place of order p at a prime $\mathfrak{q}$ above q implies that the cyclotomic field $\mathbb{Q}(\xi_p)$ is contained in $K_{\rho_\bullet}$. This implies that, for all $d|\frac{p-1}{2n}$, there exists a positive density set of rational primes ℓ such that $K_{\rho_\bullet}$ contains a prime λ above ℓ of residue degree d.

Once these three points have been addressed, we can formulate sufficient conditions on a compatible system $\rho_\bullet$ of symplectic Galois representations ensuring that the projective image of the residual representation $\overline{\rho}_\lambda$ will equal $\mathrm{PGSp}_{2n}(\mathbb{F}_{\ell^d})$ or $\mathrm{PSp}_{2n}(\mathbb{F}_{\ell^d})$, where ℓ runs through a positive density set $\mathcal{L}$ of rational primes as λ runs through the primes of L.

4: Find some object giving rise to a compatible system satisfying all the conditions above.

We exploit the compatible systems of Galois representations attached to regular, algebraic, essentially self-dual, cuspidal automorphic representations π of $\mathrm{GL}_{2n}(\mathbb{A}_\mathbb{Q})$. An additional condition on π ensures that these Galois representations have symplectic images (cf. [5]). We have to specify local conditions at two auxiliary primes (one to obtain a transvection in the image of ρ_λ, the other to obtain a maximally induced place of order p). Equivalently (via the Local Langlands Correspondence) we need to specify the local components of π at two finite places. The results of Shin on equidistribution of local components at a fixed prime in the unitary dual with respect to the Plancherel measure (cf. [9]) ensure the existence of the desired π. We still have to take care of the fact that the transvection contained in the image of ρ_λ may become trivial when we reduce mod λ. To ensure that this can occur only at a density zero set of rational primes ℓ, we use a level-lowering argument based on results of [4] over imaginary quadratic fields.

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Affine Grassmannians and Geometric Satake Equivalences

Timo Richarz

I report on my recent work [5] on the geometric Satake equivalence in ramified cases from the point of view of Bruhat-Tits theory [3]. The main result may be seen as an extension of Zhu's work [6] from the case of tamely ramified groups to the case of general connected reductive groups. As a prerequisite I prove basic structure theorems on the geometry of twisted affine flag varieties introduced by Pappas and Rapoport [4].

Let k be an algebraically closed field and denote by $F = k(\!(t)\!)$ the Laurent power series local field with ring of integers $\mathcal{O} = k[\![t]\!]$. Let G be a connected reductive F-group and let $\mathcal{G}$ be a smooth affine model over $\mathcal{O}$ with geometrically connected fibers. Denote by $\mathcal{F}\ell_{\mathcal{G}}$ the separated ind-scheme of ind-finite type over k parametrizing $\mathcal{G}$-bundles on the formal disc together with a trivialization on the punctured disc. Then $\mathcal{F}\ell_{\mathcal{G}}$ is ind-projective if and only if $\mathcal{G}$ is parahoric in the sense of Bruhat-Tits, cf. [5, Theorem A]. In this case, $\mathcal{F}\ell_{\mathcal{G}}$ is a twisted affine flag variety, cf. [4]. Let $L^+\mathcal{G}$ be the twisted positive loop group associated with $\mathcal{G}$. Then $L^+\mathcal{G}$ acts on $\mathcal{F}\ell_{\mathcal{G}}$ from the left by changing the trivialization. Fix a prime ℓ different from the characteristic of k. Let $\mathrm{Sat}_{\mathcal{G}}$ be the category of $L^+\mathcal{G}$-equivariant $\bar{\mathbb{Q}}_\ell$-perverse sheaves on $\mathcal{F}\ell_{\mathcal{G}}$. Then $\mathrm{Sat}_{\mathcal{G}}$ is semi-simple if and only if $\mathcal{G}$ is special parahoric, cf. [5, Theorem A]. In this case, the simple objects of $\mathrm{Sat}_{\mathcal{G}}$ are as follows: Choose suitable $T \subset B \subset G$, a maximal torus contained in a Borel subgroup, cf. [5, Introduction] for further explanation. The absolute Galois group $\Gamma = \mathrm{Gal}(\bar{F}/F)$ acts on the cocharacters $X_*(T)$, and we let $X_*(T)_\Gamma$ be the coinvariants. To every $\mu \in X_*(T)_\Gamma$, one associates a k-point t^μ in $\mathcal{F}\ell_{\mathcal{G}}$ via the Kottwitz morphism. Let Y_μ be the reduced $L^+\mathcal{G}$-orbit closure of t^μ. The scheme Y_μ is a projective k-variety which is in general not smooth. The reduced locus of $\mathcal{F}\ell_{\mathcal{G}}$ has an ind-presentation

$$(\mathcal{F}\ell_{\mathcal{G}})_{\mathrm{red}} = \varinjlim_{\mu \in X_*(T)^+_\Gamma} Y_\mu,$$

where $X_*(T)^+_\Gamma$ is the image of the set of dominant cocharacters under the canonical projection $X_*(T) \to X_*(T)_\Gamma$. Then the simple objects of $\mathrm{Sat}_{\mathcal{G}}$ are the intersection complexes IC_μ of Y_μ, as μ ranges over $X_*(T)^+_\Gamma$.

Let $\hat{G}$ be the Langland's dual group formed over $\bar{\mathbb{Q}}_\ell$. Then the Galois group Γ acts on $\hat{G}$ via a finite quotient. We denote by $\hat{G}^\Gamma$ the group of invariants, a reductive group which is in general not connected. Note that $X_*(T)_\Gamma = X^*(\hat{T}^\Gamma)$, and that for every $\mu \in X^*(\hat{T}^\Gamma)^+$, there exists a unique irreducible representation of $\hat{G}^\Gamma$ of highest weight μ, cf. [5, Appendix A] for the definition of highest weight representations in this case.

Theorem 1. *Let $\mathcal{G}$ be special parahoric.*

i) The category $\mathrm{Sat}_{\mathcal{G}}$ is stable under the convolution product $\star$, and the pair $(\mathrm{Sat}_{\mathcal{G}}, \star)$ admits a unique structure of a symmetric monoidal category such that the global cohomology functor $\omega(-) = \oplus_{i\in\mathbb{Z}} H^i(\mathcal{F}\ell_{\mathcal{G}}, -)$ is symmetric monoidal.

ii) The functor ω is a faithful exact tensor functor, and induces via the Tannakian formalism an equivalence of tensor categories

$$(\mathrm{Sat}_{\mathcal{G}}, \star) \xrightarrow{\simeq} (\mathrm{Rep}_{\bar{\mathbb{Q}}_\ell}(\hat{G}^I), \otimes)$$

which is uniquely determined up to inner automorphisms by elements in $\hat{T}^\Gamma$ by the property that $\omega(\mathrm{IC}_\mu)$ is the irreducible representation of highest weight μ.

iii) Let $T \subset M \subset G$ be a Levi subgroup. Denote by $\mathcal{M}$ the flat closure of M in $\mathcal{G}$. Then $\mathcal{M}$ is special parahoric and there exists a tensor functor $C_{\mathcal{M}} : \mathrm{Sat}_{\mathcal{G}} \to \mathrm{Sat}_{\mathcal{M}}$ compatible with the fiber functors such that $C_{\mathcal{M}}$ induces under the equivalence in ii) the restriction functor of representations from $\hat{G}^\Gamma$ to $\hat{M}^\Gamma$.

Part i) and ii) of the Theorem are proven in [5, Theorem C] whereas part iii) will be adressed in subsequent work. The proof of iii) relies on Braden's hyperbolic localization theorem [2]. As a corollary I recover Arkhipov and Bezrukavnikovs' result [1, Theorem 4 b)] in the case of a general connected reductive group.

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Independence of ℓ-adic representations of geometric Galois groups

Gebhard Böckle
(joint work with W. Gajda, S. Petersen)

Let L be a set of prime numbers (which will usually be denoted by ℓ). Let K be any field and denote by $\Gamma_K = \mathrm{Gal}(K^{\mathrm{sep}}/K)$ its absolute Galois group, and by $p \geq 0$ its characteristic. In the following we consider families

$$\rho_\bullet = (\rho_\ell)_{\ell \in L} = (\rho_\ell \colon \Gamma_K \longrightarrow \mathrm{Aut}_{\mathbb{Q}_\ell}(V_\ell))_{\ell \in L}$$

of continuous representations of Γ_K on finite dimensional $\mathbb{Q}_\ell$-vector spaces V_ℓ, equipped with their ℓ-adic topology.

In [17], Serre calls such a family *independent* if the natural inclusion

$$\Big(\prod_{\ell \in L} \rho_\ell\Big)(\Gamma_K) \subseteq \prod_{\ell \in L} \big(\rho_\ell(\Gamma_K)\big) \tag{1}$$

is an isomorphism. The family $\rho_\bullet$ is called *almost independent* if there exists a finite field extension K'/K such that the family $\rho_\ell|_{\Gamma_{K'}}$ obtained by restriction to $\Gamma_{K'} \subset \Gamma_K$ is independent.

A sufficient condition for almost independence is that the subgroup in (1) is open in the ambient group. As simple examples show, the converse is not true. The independence of a family can also be formulated in terms of the linear independence of fields: denote for each ℓ by K_ℓ the fixed field $\overline{K}^{\mathrm{Ker}(\rho_\ell)}$; then $\rho_\bullet$ is independent if the span of any finite subset of $(K_\ell)_{\ell \in L}$ is linearly disjoint from any other field K_ℓ not in that subset. If one further defines K'_ℓ as the span of the fields K_ℓ with $\ell \neq \ell'$, then almost independence is equivalent to the field $\bigcap_{\ell \in L} K'_\ell$ being a finite extension of K; see [17, Thme 1'].

We now introduce the main example of a family for which the question of independence is of interest: Let X be a separated scheme of finite type over K, let q be an integer and let $? \in \{\varnothing, c\}$. Let L be the set of primes different from p. Then we define

$$\rho^q_{X,?,\ell} \colon \Gamma_K \to \mathrm{Aut}_{\mathbb{Q}_\ell}(H^q_?(X_{\overline{K}}, \mathbb{Q}_\ell))$$

as the representation that arises from the action of Γ_K by functoriality on étale cohomology or étale cohomology with supports $H^q_?(X_{\overline{K}}, \mathbb{Q}_\ell)$ for each $\ell \in L$.

For instance if X is an abelian variety over K, then $\rho^1_{X,\bullet}$ is the dual of the Galois representations of Γ_K on the ℓ-adic Tate module of A. In the special case where $X = E$ is an elliptic curve over the rational numbers $\mathbb{Q}$ (or more generally over any number field), and if E does not have complex multiplication, then around 1970 Serre showed in [16] that

$$\Big(\prod_{\ell \in L} \rho^1_{E,\ell}\Big)(\Gamma_\mathbb{Q}) \subseteq \prod_{\ell \in L} \mathrm{GL}_2(\mathbb{Z}_\ell) \tag{2}$$

is open. As we noted above this implies that $\rho^1_{E,\bullet}$ is almost independent. Already in the 1950's, Igusa in [9, 2] considered the case of a non-isotrivial elliptic curve over the field of rational functions $\mathbb{F}_p(t)$ over the field of p-elements $\mathbb{F}_p$. He showed that

$$\Big(\prod_{\ell\in L}\rho^1_{E,\ell}\Big)(\Gamma_{\overline{\mathbb{F}_p(t)}})\subseteq\prod_{\ell\in L}\mathrm{SL}_2(\mathbb{Z}_\ell)$$

is open. The direct analog of (2) is however not true: the determinants of the latter family, i.e., the family

$$\rho_\ell\colon\Gamma_{\mathbb{F}_p}\to\mathrm{Aut}_{\mathbb{Q}_\ell}(\mu_{\ell^\infty}(\overline{\mathbb{F}_p}))$$

is not almost independent.

To state the main results, we define $\rho_{X,\bullet}:=\bigoplus_{q\geq 0}\big(\rho^q_{X,\bullet}\oplus\rho^q_{X,c,\bullet}\big)$.

Theorem 1 (Serre, [17])**.** *If K is a number field, then $\rho_{X,\bullet}$ is almost independent.*

Theorem 2 (Gajda-Petersen, [7])**.** *If K is a finitely generated field over $\mathbb{Q}$, then $\rho_{X,\bullet}$ is almost independent.*

Theorem 3 (Böckle-Gajda-Petersen, indep. Cadoret-Tamagawa, [4, 5])**.** *Let $k\subset K$ be a subfield such that K/k is finitely generated. Then $\rho_{X,\bullet}|_{\Gamma_{\overline{k}K}}$ is almost independent.*

In the remainder, we discuss the proof of Theorem 3.

1: Some group theory. For positive integers c,d define the set $\mathrm{Jor}(d)$ of *d-Jordanian groups* as the set of finite groups H that possess a normal abelian subgroup A of index at most d. The subset of $\mathrm{Jor}(d)$ of those H such that A has order prime to ℓ is denoted $\mathrm{Jor}_\ell(d)$. Define also the set $\Sigma_\ell(c)$ as the set of profinite groups M that possess open normal subgroups $P\trianglelefteq I\trianglelefteq M$ such that P is a pro-ℓ group, I/P is finite abelian of order at most c and prime to ℓ and M/I is a product of finite simple characteristic ℓ Lie type groups. From [14] one then deduces:

Proposition 4. *There exists a constant $J(n)$ such that for all primes ℓ and all compact Hausdorff subquotients G_ℓ of $\mathrm{GL}_n(\overline{\mathbb{Q}_\ell})$ there exists a short exact sequence*

$$1\longrightarrow M_\ell\longrightarrow G_\ell\longrightarrow H_\ell\longrightarrow 1$$

such that $M\in\Sigma_\ell(2^n)$ and $H_\ell\in\mathrm{Jor}_\ell(J(n))$.

2: Strategy of proof. The proof now proceeds in the following steps:
i) Replace K by a finite extension K' such that for all ℓ the ramification of the restriction $\rho_\ell|_{\Gamma_{K'}}$ is at most pro-ℓ. Below we shall give more details on this.
ii) Use finiteness results for tame fundamental groups to find a finite extension K'' of K' such that after restriction to $\Gamma_{\overline{k}K''}$ the H_ℓ in Proposition 4 become trivial. This uses standard results from [8, 10] and [7] for $p=0$.
iii) Use the 2^n-bound in $\Sigma_\ell(2^n)$ in Proposition 4 and again the results used in ii), to reduce to the case where each M_ℓ is generated by all its pro-ℓ-Sylow subgroups.
iv) Define Ξ_ℓ as the set of all characteristic ℓ simple Lie type groups together with the cyclic group of order ℓ. Then after iii) all simple quotient of M_ℓ lie in Ξ_ℓ. Now from [1, 12] one knows that for all $\ell,\ell'\geq 5$, if $\ell\neq\ell'$, then $\Xi_\ell\cap\Xi_{\ell'}=\varnothing$. And the proof is complete.

3: On the proof of i). An immediate algebro-geometric proof of i) follows from a recent preprint by Orgogozo:

Proposition 5 ([15]). *Let X/K be as above. Then there exists a finite extension K' over $\overline{k}K$ and a smooth projective scheme $Y'/\overline{k}$ with function field K' containing a normal crossings divisor D' such that for all $\ell \neq p$ the following hold:*

- *the action of $\Gamma_{K'}$ on $H^*(X_{\overline{K}}, \mathbb{Q}_\ell)$ factors via the tame fundamental group $\pi_1^{\mathrm{tame}}(Y', D')$ along D';*
- *at all maximal points of D' the ramification on $H^*(X_{\overline{K}}, \mathbb{Q}_\ell)$ is unipotent.*

The proof in [4] is different, and may have other uses: The main point is to control the ramification of $\rho_{X,\bullet}$. After developing suitable notions of ramification bounds in families, a first simple reduction allows one to assume that k is finite or a number field, and K/k is finitely generated. A second reduction is based on de Jong's alteration technique [6] as applied in [3, Thm. 6.3.2]: there exists a finite extension K' of K and a finite family of smooth projective schemes Y_i over K' such that for all ℓ the semisimplification $\rho^{\mathrm{ss}}_{X,\bullet}$ is a direct summand of $\bigoplus_i \rho^{\mathrm{ss}}_{Y_i,\bullet}$.

Suppose from now on that k is finite. Let X/K be smooth, and choose a smooth scheme U over k with function field K and a smooth projective model $\mathcal{X}$ over U for $X \to \operatorname{Spec} K$. A modification of a result of Kerz-Schmidt-Wiesend, cf. [11], give the following criterion: $\rho_{X,\bullet}$ regarded as a representation of $\pi_1(U)$ (we suppress a base point) is at most pro-ℓ ramified of this holds for the pullback to any curve $C \hookrightarrow U$. To such pullbacks (after semisimplification) one can apply L. Lafforgue's theorem on the global Langlands correspondence for GL_n over function fields, [13]: If all ramification of one $\rho_{X,\ell_0}|_{\pi_1(C)}$ is unipotent at ramified places, then this extends to the entire compatible system. To guarantee unipotence at some (sufficiently large) ℓ_0, choose a lattice Λ stable under ρ_{X,ℓ_0} and replace K' by a finite extension K'' such that the image of $\Gamma_{K''}$ under ρ_{X,ℓ_0} is trivial mod $\ell_0\Lambda$. This implies unipotence of ramification after pullback to any curve C, and hence over K'' all ramification of $\rho_{X,\bullet}$ is pro-ℓ unipotent.

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Specialization of modulo ℓ Galois groups in 1-dimensional families

Anna Cadoret
(joint work with Akio Tamagawa)

Let k be a finitely generated field of characteristic $p \geq 0$ with absolute Galois group $\Gamma_k := \mathrm{Gal}(k^{sep}|k)$. Let X be a smooth, separated, geometrically connected scheme over k with generic point η and set of closed points $|X|$. Set $\overline{X} := X \times_k \overline{k}$. When X is a curve let $\overline{X}^{cpt}$ denote the smooth compactification of $\overline{X}$, write $\partial\overline{X} := \overline{X}^{cpt} \setminus \overline{X}$ for the divisor at infinity and g_X, γ_X for the genus and gonality of $\overline{X}^{cpt}$respectively. Recall that any $x \in |X|$ produces a quasi-splitting $\sigma_x : \Gamma_{k(x)} \hookrightarrow \pi_1(X)$ of the structural projection $\pi_1(X) \to \Gamma_k$ (here $k(x)$ denotes the residue field at x).
Let $r \in \mathbb{Z}_{\geq 1}$. Let L be an infinite set of primes, $p \notin L$ and for every $\ell \in L$, fix a field F_ℓ of characteristic ℓ and a discrete $F_\ell[\pi_1(X)]$-module H_ℓ with F_ℓ-rank $r_\ell \leq r$ that is, equivalently, a continuous group morphism $\rho_\ell : \pi_1(X) \to \mathrm{GL}(H_\ell) \simeq \mathrm{GL}_{r_\ell}(F_\ell)$. Set $G_\ell := \mathrm{im}(\rho_\ell)$, $\overline{G}_\ell := \rho_\ell(\pi_1(\overline{X}))$ and $G_{\ell,x} := \rho_\ell \circ \sigma_x(\Gamma_{k(x)})$. Note that $\overline{G}_\ell \lhd G_\ell$ and $[G_\ell : \overline{G}_\ell G_{\ell,x}] \leq [k(x) : k]$.

The problem we want to address is the description of the local Galois images $G_{\ell,x}$ as x varies in $|X|$. In general, given a family $\mathcal{F}_\ell$ (= the 'moduli data') of subgroups of G_ℓ *which does not contain* G_ℓ, one expects that the set $X(\mathcal{F}_\ell)$ of all $x \in |X|$ such that $G_{\ell,x}$ is contained in a groups of $\mathcal{F}_\ell$ is 'small'. This naturally yields to introduce the *abstract modular scheme* (AMS or AMC for short) associated with ρ_ℓ, $\mathcal{F}_\ell$

$$X_{\mathcal{F}_\ell}^{(\rho_\ell)} := \bigsqcup_{U \in \mathcal{F}_\ell} X_U \to X,$$

where $X_U \to X$ denotes the connected étale cover (defined over the finite extension k_U of k) corresponding to the open subgroup $\rho_\ell^{-1}(U) \subset \pi_1(X)$. It follows from the general formalism of Galois categories that

- $X_U \times_{k_U} \overline{k} \to \overline{X}$ is the connected étale cover corresponding to $\overline{U} := U \cap \overline{G}_\ell \subset \overline{G}_\ell$.
- (Moduli) $x \in |X|$ lifts to a $k(x)$-rational point on X_U if and only if $G_{\ell,x} \subset U$.

Thus $X(\mathcal{F}_\ell)$ is exactly the set of all $x \in |X|$ which lift to a $k(x)$-rational point on $X_{\mathcal{F}_\ell}$. This gives a diophantine reformulation of our original group representation-theoretic problem and already shows that $X(\mathcal{F}_\ell)$ is thin. But, of course, one expects much more to hold. For instance, depending on the situation, that for every integer $d \geq 1$ the set $X(\mathcal{F}_\ell)^{\leq d}$ of all $x \in X(\mathcal{F}_\ell)$ such that $[k(x) : k] \leq d$ is not Zariski-dense (hence finite when X is a curve), that $X(\mathcal{F}_\ell)$ is of bounded height, that $X(\mathcal{F}_\ell)$ is not Zariski-dense or even that $X(\mathcal{F}_\ell)$ is empty. In this work, we focus on the weakest of these finiteness properties, namely, we would like to find minimal conditions on the ρ_ℓ, $\mathcal{F}_\ell$, $\ell \in L$ which ensure that $X(\mathcal{F}_\ell)(k)$ is not Zariski-dense. But even this weakened problem seems out of reach in whole generality. However, for curves, one has the remarkable fact that the genus controls the finiteness of its set of rational points. More precisely, recall

Fact (Faltings ($p = 0$), Voloch ($p > 0$)): *Let k be a finitely generated field. Then there exists an integer $g(k) \geq 2$ such that for every curve C over k with $g_C \geq g(k)$ one has $|C(k)| < +\infty$.*

When X is a curve, which we will assume from now on, this reduces our original problem to determining under which conditions on the ρ_ℓ, $\mathcal{F}_\ell$, $\ell \in L$ one has

$$g_{X_{\mathcal{F}_\ell}} := \min\{g_{X_U} \mid U \in \mathcal{F}_\ell\} \to +\infty?$$

If one consider the family $\mathcal{F}_{\ell,tot}$ of all subgroups U of G_ℓ such that $\overline{G}_\ell \not\subset \overline{U}$ (that is $X(\mathcal{F}_\ell)(k)$ is the set of all $x \in X(k)$ such that $G_{\ell,x} \subsetneq G_\ell$), it may happen that there exists an integer $B \geq 1$ and infinitely many ℓ such that $1 < [\overline{G}_\ell : \overline{U}] \leq B$ for some $U \in \mathcal{F}_{\ell,tot}$. This is an obstruction to $g_{X_{\mathcal{F}_{\ell,tot}}} \to +\infty$. This obstruction disappears if one replaces $\mathcal{F}_{\ell,tot}$ with the set $\mathcal{F}_{\ell,+}$ of all subgroups U of G_ℓ such that $\overline{G}_\ell^+ \not\subset \overline{U}$. Here, given a subgroup $G \subset \mathrm{GL}(H_\ell)$, we write $G^+ \subset G$ for the subgroup generated by its ℓ-Sylow. Surprisingly, almost no information is lost when replacing $\mathcal{F}_{\ell,tot}$ with $\mathcal{F}_{\ell,+}$; this is a general property of bounded families of continuous F_ℓ-representation of $\pi_1(X)$ for X a curve over a *finitely generated field* k.

Theorem A: *Assume (T): For every $x \in \partial\overline{X}$ there exists an open subgroup U_x of the inertia group at x such that $p \not| \rho_\ell(U_x))$, $\ell \in L$. Then there exists an open subgroup $\Pi \subset \pi_1(X)$ such that $\rho_\ell(\Pi) = \rho_\ell(\Pi)^+$, $\ell \in L$.*

In particular, $[\overline{G}_\ell : \overline{G}_\ell^+]$ is bounded from above independently of ℓ. This yields to consider the AMC $X_{\ell,+} := X_{\mathcal{F}_{\ell,+}}$. Also, as one can always construct family $C_\ell \to C$ of connected étale covers with group $\mathbb{Z}/\ell$ and C_ℓ of genus 0, the following perfectness condition is necessary

(P): For every open subgroup $\Pi \subset \pi_1(\overline{X})$ there exists an integer $B_\Pi \geq 1$ such that $|\rho_\ell(\Pi)^{ab}| \leq B_\Pi$, $\ell \in L$.

When $p = 0$, $F_\ell = \mathbb{F}_\ell$ and assuming (P), one already knows (see below) that $\gamma_{X_{\ell,+}} := \min\{\gamma_{X_U} \mid U \in \mathcal{F}_\ell\} \to +\infty$. Thus, *when* $F_\ell = \mathbb{F}_\ell$, which we assume from now on unless otherwise mentioned, the following seems the best possible result.

(Main) Theorem: *Assume (T) and (P). Then* $g_{X_{\ell,+}} \to +\infty$.

Corollary: *Assume (T) and (P). Then there exists an integer* $B \geq 1$ *such that for* $\ell \gg 0$ *and all but finitely many* $x \in X(k)$ *one has* $[G_\ell : G_{\ell,x}] \leq B$ *(and if* $\overline{G}_\ell = \overline{G}_\ell^+$ *for* $\ell \gg 0$*, one can even take* $B = 1$*).*

The main Theorem and its Corollary apply to families of the form

$$\rho_\ell : \pi_1(X) \to \mathrm{GL}(\mathrm{H}(Y_{\overline{\eta}}, \mathbb{F}_\ell)), \ \ \ell \in L$$

for $Y \to X$ a smooth proper morphism. In that case, (T) and the boundedness condition follow from de Jong's alterations and the fact - due to Gabber - that $\mathrm{H}(Y_{\overline{\eta}}, \mathbb{Z}_\ell)$ is torsion-free for $\ell \gg 0$. After several reductions (including Theorem A, Nori-Serre's approximation theory and specialization of tame fundamental group), (P) essentially reduces to the Weil conjectures[1].
They also apply to specialization of first cohomology groups and a consequence of them is the following. For $x \in |X|$, consider the restriction map

$$res_x : V_\ell \hookrightarrow \mathrm{H}^1(\pi_1(X), H_\ell) \overset{res_x}{\to} \mathrm{H}^1(k(x), H_\ell),$$

where V_ℓ is a $\mathbb{F}_\ell$-subvector space with $\mathbb{F}_\ell$-rank $s_\ell \leq s$. Assume that the family $\pi_1(X) \to \mathrm{GL}(H_\ell)$, $\ell \in L$ is bounded, satisfies (T), (SS): H_ℓ is a semi-simple $\pi_1(\overline{X})$-module for $\ell \gg 0$ and (I): for every open subgroup $\Pi \subset \pi_1(\overline{X})$, one has $H_\ell^\Pi = 0$ for $\ell \gg 0$. Then, for $\ell \gg 0$ and all but finitely many $x \in X(k)$, the restriction map $res_x : V_\ell \to \mathrm{H}^1(k, H_\ell)$ is injective. In particular, if $A \to X$ is an abelian scheme such that $A_{\overline{\eta}}$ contains no non-trivial isotrivial abelian subvariety, for $\ell \gg 0$ and all but finitely many $x \in X(k)$, the restriction map $A(X)/\ell \overset{Kummer}{\hookrightarrow} \mathrm{H}^1(\pi_1(X), H_\ell) \overset{res_x}{\to} \mathrm{H}^1(k, H_\ell)$ is injective which, as observed by Serre, implies that $A(X) \hookrightarrow A_x(k)$ is injective as well. This is an extension to arbitrary characteristic $p \geq 0$ of the Néron-Silverman specialization theorem. More generally, one can apply this kind of argument to specialization of the reduction modulo ℓ of the first higher ℓ-adic Abel-Jacobi maps.

[1]More generally, when $k = \mathbb{F}_p$, our results also apply to reduction modulo ℓ of families $\pi_1(X) \to \mathrm{GL}_r(\overline{\mathbb{Q}}_\ell)$, $\ell \in L$ of continuous representations which are pure in the usual sense since, by a result of Chin, these descend to an extension $\mathbb{F}_{\ell^s}$ of $\mathbb{F}_\ell$ for a fixed integer $s \geq 1$.

The strategy of the proof of the main Theorem is to construct a 'universal tensor representation' in order to separate by lines groups in $\mathcal{F}_{\ell,+}$ from $\overline{G}_\ell^+$ for $\ell \gg 0$. This allows to construct an auxilliary bounded family $\tilde{\rho}_\ell : \pi_1(X) \to \mathrm{GL}(\tilde{T}_\ell)$, $\ell \in L$ of continuous $\mathbb{F}_\ell$-representations such that every connected component of $X_{\ell,+}$ dominates a connected component of the AMC $X_{\ell,0}^{\tilde{\rho}_\ell}$ associated to the family $\mathcal{F}_{\ell,0}$ of all stabilizer of lines in $\tilde{T}_\ell$. This reduces the problem to showing that $g_{X_{\ell,0}^{\tilde{\rho}_\ell}} \to +\infty$ which, due to the specific shape of the moduli problem encoded in $\mathcal{F}_{\ell,0}$, is doable. More precisely, the two main intermediate statements are the following.

Theorem B: *There exists a map $f : (\mathbb{Z}_{\geq 0})^{\oplus 2} \to \mathbb{Z}_{\geq 0}$ with finite support such that for $\ell \gg 0$ and every $U \in \mathcal{F}_{\ell,+}$ there exists a line $D \subset T^f(H_\ell) := \bigoplus_{m,n\geq 0}(H_\ell^{\oplus m} \otimes (H_\ell^\vee)^{\oplus n})^{\oplus f(m,n)}$ (depending on U, $\overline{G}_\ell^+$) with the property that $\overline{G}_\ell^+ D \neq D$ but $UD = D$.*

Theorem C: *Assume (T) and (I). Then $g_{X_{\ell,0}} \to +\infty$.*

To deduce the main Theorem from Theorem B and Theorem C, just set $\tilde{T}_\ell := T_\ell / T_\ell^{\overline{G}_\ell^+}$, where $T_\ell := T^f(H_\ell)$, $\ell \in L$. Then the family $\tilde{\rho}_\ell : \pi_1(X) \to \mathrm{GL}(\tilde{T}_\ell)$, $\ell \in L$ is bounded and satisfies (T) and (I) as soon as the family ρ_ℓ, $\ell \in L$ satisfies (T) and (P). From Theorem B, every connected component of $X_{\ell,+}$ dominates a connected component of $X_{\ell,0}^{\tilde{\rho}_\ell}$ and, from Theorem C, $g_{X_{\ell,0}^{\tilde{\rho}_\ell}} \to +\infty$.

Theorem B is a variant for finite subgroups of $\mathrm{GL}_r(\mathbb{F}_\ell)$ (r fixed, ℓ varying) of the classical Chevalley theorem for algebraic groups and, unsurprisingly, it relies on approximation theory. Approximation theory[2] associates to a subgroup G of $\mathrm{GL}_r(F_\ell)$ a connected algebraic subgroup $\tilde{G} \hookrightarrow \mathrm{GL}_{r,\mathbf{F}_\ell}$ ($\mathbf{F}_\ell \subset F_\ell$) - the algebraic enveloppe - whose properties reflect those of G and whose rational points approximate well G for $\ell \gg 0$. There are two approaches, one by Nori and Serre, which works only for $F_\ell = \mathbb{F}_\ell$ but is 'functorial' and one by Larsen and Pink, which works for arbitrary fields F_ℓ of characteristic ℓ but is 'not functorial'. The restriction of our results to $\mathbb{F}_\ell$-coefficients comes from the fact that we resort to the former[3], where $\tilde{G} \hookrightarrow \mathrm{GL}_{H_\ell}$ is defined as the algebraic subgroup generated by the one-parameter groups $\mathbb{A}^1_{\mathbb{F}_\ell} \to \mathrm{GL}_{H_\ell}$, $t \to \exp(t\log(g))$ for $g \in G$ of order ℓ. By construction $\tilde{G}$ is connected and generated by its unipotent elements and for $\ell \gg 0$ the following properties hold: (i) $\tilde{G}(\mathbb{F}_\ell)^+ = G^+$, (ii) $\tilde{G}(\mathbb{F}_\ell)/\tilde{G}(\mathbb{F}_\ell)^+$ is abelian of order $\leq 2^{r-1}$, (iii) there exists an abelian subgroup of prime-to-ℓ order $A \subset G$ such that G^+A is normal in G with $[G : G^+A] \leq \delta(r)$. To prove Theorem B, one considers a family $\mathrm{GL}_r \times \mathcal{N}_r \supset \mathcal{U}_r \to \mathcal{N}_r$ over $\mathbb{Z}[\frac{1}{r!}]$ parametrizing exponentially generated

[2] Here, we consider, again, an arbitrary field F_ℓ of characteristic ℓ

[3] Though it is possible that, resorting to much more elaborate group-theoretic arguments, our approach extends to arbitrary F_ℓ-coefficients *via* Larsen-Pink's approximation theory.

subgroups of GL_r and, by induction on dimension and the classical Chevalley theorem, one constructs a universal map $f : (\mathbb{Z}_{\geq 0})^{\oplus 2} \to \mathbb{Z}_{\geq 0}$ with the property that every exponentially generated subgroup of GL_{r,F_ℓ} ($\ell > r$) is the stabilizer of a line in $T^f(F_\ell^{\oplus r})$. By approximation theory (property (i) above), f separates - in the sense of Theorem B - U^+ from $\overline{G}_\ell^+$ for $U \in \mathcal{F}_{\ell,+}$ and $\ell \gg 0$. Then, by *ad-hoc* arguments (including properties (i), (iii) above), one adjusts f so that it satisfies exactly the conclusion of Theorem B.

To prove Theorem C, one proves first that, for the 'Galois closure' $\hat{X}_{\ell,0}$ of $X_{\ell,0} \to X$, the ratio $\lambda_{\hat{X}_{\ell,0}}$ ='genus/degree' is bounded from below by an absolute constant $K > 0$. Since the cover $\hat{X}_{\ell,0} \to X$ is Galois, Stichenoth's bound and the Riemann-Hurwitz formula show that this amounts to prove that $g_{\hat{X}_{\ell,0}} \leq 1$ which, in turn, reduces to a combination of group-theoretic arguments involving the classification of finite subgroups of automorphism groups of genus ≤ 1 curves, Theorem A and assumptions (T), (I). One then shows by Riemann-Hurwitz formula that $(\lambda_{\hat{X}_{\ell,0}} - \lambda_{X_{\ell,0}}) \to 0$. Here, the main difficulty is to control the length of the ramification filtration and the size of the ramification terms. Using assumption (T) and Theorem A, this eventually amounts to the following 'non-concentration' estimate: *there exists a sequence $\epsilon(\ell)$, $\ell \in L$ such that $\epsilon(\ell)\ln(\ell) \to 0$ and for every $\mathbb{F}_\ell$-vector subspace $N_\ell \subset H_\ell$ and $0 \neq v_\ell \in H_\ell$, if $\overline{G}_\ell^+ v_\ell \not\subset N_\ell$ then* $\frac{|\overline{G}_\ell^+ v_\ell \cap N_\ell|}{|\overline{G}_\ell^+ v_\ell|} \leq \epsilon(\ell)$, which, again, is proved using Nori's algebraic enveloppe.

To conclude, let us mention two further possible directions, still in the case where the base scheme X is a curve.

Arbitrary F_ℓ-coefficients: By an easy specialization argument, one can always assume that $F_\ell \subset \overline{\mathbb{F}}_\ell$. Using Theorem A and Larsen-Pink's approximation theory, one can reduces our main theorem for arbitrary F_ℓ-coefficients to a deep[4] group-theoretical result by Guralnick. However, we know no counter-example to the following conjectural statement:

Conjecture: *Let $\rho_\ell : \pi_1(X) \to \mathrm{GL}_r(\overline{\mathbb{F}}_\ell)$, $\ell \in L$ be a bounded family of continuous representations satisfying (T), (P). Then there exists an integer $s \geq 1$ such that $\rho_\ell|_{\pi_1(\overline{X})} : \pi_1(\overline{X}) \to \mathrm{GL}_r(\overline{\mathbb{F}}_\ell)$ is $\mathrm{GL}_r(\overline{\mathbb{F}}_\ell)$-conjugate to a representation with coefficients in $\mathbb{F}_{\ell^s}$, $\ell \in L$.*

This conjecture is in the spirit of the ℓ-independence conjectures/statements for families of automorphic representations but the compatibility condition (P) and the arithmetic input that the representations are not only representations of $\pi_1(\overline{X})$

[4]involving satellite theorems of the classification like Aschbasher's theorem for maximal subgroups of finite classical groups.

but also of $\pi_1(X)$ are weaker than the standard ℓ-independency and purity assumptions about the characteristic polynomials of Frobenii. One could try and tackle first this conjecture when k is finite and for $\overline{\mathbb{Q}}_\ell$-coefficients or replacing (P) by the assumption that for every $x \in |X|$ the characteristic polynomial of the Frobenius at x is the reduction modulo ℓ of a polynomial independent of ℓ, with coefficients in the completion of a finite extension of $\mathbb{Q}$ independent of x and pure.

Gonality: When $p = 0$ and $F_\ell = \mathbb{F}_\ell$, it was shown by Ellenberg, Hall and Kovalski that Theorem A combined with Cayley-Schreier graphs and complex-analytic technics implies that, under (P), $\gamma_{X_{\ell,+}} \to +\infty$. The generalization of this result to arbitrary F_ℓ-coefficients seems to be conditioned by the extension of the Cayley-Schreier graphs part of the proof (due to Pyber and Szabo). One can also ask for similar results when $p > 0$. Akio Tamagawa and I have obtained some partial positive results in this direction when $F_\ell = \mathbb{F}_\ell$ by purely algebraic methods[5].

Counting Galois extensions of $\mathbb{Q}$

Pierre Dèbes

Given a finite group G and a real number $y > 0$, there are only finitely many Galois extensions $E/\mathbb{Q}$ (inside a fixed algebraic closure $\overline{\mathbb{Q}}$ of $\mathbb{Q}$) of group G and discriminant $|d_E| \leq y$. We are interested in lower bounds for their number $N(G, y)$.

The Malle conjecture is a classical landmark in this context. It predicts that for some constant $a(G) \in]0, 1]$, specifically defined by Malle (depending only on G), and for all $\varepsilon > 0$,

$$(*) \qquad c_1(G)\, y^{a(G)} \leq N(G, y) \leq c_2(G, \varepsilon)\, y^{a(G)+\varepsilon} \qquad \text{for all } y > y_0(G, \varepsilon).$$

for some positive constants $c_1(G)$, $c_2(G, \varepsilon)$ and $y_0(G, \varepsilon)$ [6]. A more precise asymptotic for $N(G, Y)$ is even offered in [7], namely $N(G, y) \sim c(G)\, y^{a(G)}\, (\log(y))^{b(G)}$, for some other specified constant $b(G) \geq 0$, and an another constant $c(G) > 0$.

The lower bound in (*) is a strong statement; it implies in particular that G is the Galois group of at least one extension $E/\mathbb{Q}$, which is an open question for many groups – the so-called *Inverse Galois Problem.* Relying on the Shafarevich theorem solving the IGP for solvable groups, Klüners and Malle proved the conjecture (*) for nilpotent groups [4]. Klüners also established the lower bound for dihedral groups of order $2p$ with p an odd prime [3]. The more precise asymptotic for $N(G, y)$ is known for abelian groups.

There is another classical class of finite groups known to be Galois groups over $\mathbb{Q}$: those groups G which are *regular Galois groups* over $\mathbb{Q}$, *i.e.*, such that there exists a Galois extension $F/\mathbb{Q}(T)$ of group G with $F \cap \overline{\mathbb{Q}} = \mathbb{Q}$. In addition to abelian groups and dihedral groups, this class includes many non solvable groups, for example, all symmetric and alternating groups and many simple groups. We prove a lower bound like in (*) for all these groups.

[5]Gonality may decrease under specialization and there is *a priori* no hope to reduce the study of gonality when $p > 0$ to a characteristic 0 setting.

In this regular situation, Hilbert's irreducibility theorem classically produces "many" $t_0 \in \mathbb{Q}$ such that the corresponding specialized extensions $F_{t_0}/\mathbb{Q}$ are Galois extensions of group G. Beyond making more precise these "many $t_0 \in \mathbb{Q}$" and controlling the corresponding discriminants, our goal requires an even further step which is to show that many of these extensions are distinct. It is for this part that the *self-twisted cover*, a new tool that we construct, is used (see §).

Furthermore, in addition to being of group G and discriminant $\leq y$, the Galois extensions we count can be prescribed any unramified behavior at every large prime $p \leq \log(y)/\delta$ (for some $\delta > 1$). Malle had suggested that his estimates should hold with some local conditions [7, Remark 1.2]. Ours however have a set of primes growing with y, which provides noteworthy constraints on regular Galois groups, related to the Grunwald problem and the Tchebotarev theorem (§).

Main result. Given a finite group G, a finite set S of primes and for each $p \in S$, a subset $\mathcal{F}_p \subset G$ consisting of a non-empty union of conjugacy classes of G, the collection $\mathcal{F} = (\mathcal{F}_p)_{p \in S}$ is called a *Frobenius data* for G on S. The number of Galois extensions $E/\mathbb{Q}$ of group G, of discriminant $|d_E| \leq y$ and which are unramified with Frobenius $\mathrm{Frob}_p(E/\mathbb{Q}) \in \mathcal{F}_p$ ($p \in S$) is denoted by $N(G, y, \mathcal{F})$.

Theorem 1. *Let G be a regular Galois group over $\mathbb{Q}$, non trivial. There exists a constant $p_0(G)$ with the following property. For every $\delta > \delta(G)$, there exists $y_0(G,\delta) > 0$ such that for every $y > y_0(G,\delta)$ and every Frobenius data $\mathcal{F}_y$ on the set $\mathcal{S}_y = \{p_0(G) < p \leq \log(y)/\delta\}$, we have*

$$N(G, y, \mathcal{F}_y) \geq y^{\alpha(G,\delta)} \qquad \textit{with} \quad \alpha(G,\delta) = (1 - 1/|G|)/\delta.$$

The parameter $\delta(G)$ is the *minimal affine branching index* of regular realizations of G over $\mathbb{Q}$, *i.e.* the minimal degree of the discriminant $\Delta_P(T)$ of a polynomial $P \in \mathbb{Q}[T,Y]$, monic in Y, such that $\mathbb{Q}(T)[Y]/\langle P \rangle$ is a regular Galois extension of $\mathbb{Q}(T)$ of group G. If a regular realization $F/\mathbb{Q}(T)$ of G is given by a polynomial $P \in \mathbb{Q}[T,Y]$, monic in Y, then $\delta(G) < 2|G| \deg_T(P)$ and so one can take $\delta = 2|G| \deg_T(P)$ in theorem 1; the more intrinsic value $\delta = 3r\,|G|^3 \log|G|$ with r the branch point number of $F/\mathbb{Q}(T)$ can also be used. Our exponent $\alpha(G,\delta)$ can be shown to be bigger than or equal to Malle's exponent $a(G)$ and so our result is compatible with Malle's conjecture in this case.

Remark 2. Extending theorem 1 to number fields seems to present no major obstacles. As each finite group is known to be a regular Galois group over some suitable number field, the same can then be deduced for the lower bound part in the Malle conjecture: given any finite group, a lower bound like in (*) (appropriately generalized) holds over some suitable number field.

The Grunwald problem and the Tchebotarev theorem. Theorem 1 has some implication (already present in our previous work [1] with N. Ghazi) towards issues related to the Tchebotarev density theorem.

Definition 3. Given a real number $\ell \geq 0$, we say that a finite group G is of *Tchebotarev order* $\leq \ell$, which we write $\mathrm{tch}(G) \leq \ell$, if there exist real numbers

$m, \delta > 0$ such that for every $x > m$ and every *Frobenius data* $\mathcal{F}_x = (\mathcal{F}_p)_{m<p\leq x}$ for G, there exists *at least one* Galois extension $E/\mathbb{Q}$ of group G such that these two conditions hold:
1. for each $m < p \leq x$, $E/\mathbb{Q}$ is unramified and $\mathrm{Frob}_p(E/\mathbb{Q}) \in \mathcal{F}_p$,
2. $\log|d_E| \leq \delta x^{\ell}$.

Fix $\delta > \delta(G)$ and m suitably large. Theorem 1 for $y = e^{\delta x}$ with $x > m$ provides many extensions $E/\mathbb{Q}$ satisfying conditions of definition 3 with $\ell = 1$.

Corollary 4. *If a finite group G is a regular Galois group over $\mathbb{Q}$, then* $\mathrm{tch}(G) \leq 1$.

On the other hand there is a universal lower bound for $\mathrm{tch}(G)$: some famous estimates on the Tchebotarev theorem [5] show that, under the General Riemann Hypothesis, for every finite group G, we have

$$\mathrm{tch}(G) > (1/2) - \varepsilon, \text{ for every } \varepsilon > 0.$$

Corollary 4 raises the question of whether $\mathrm{tch}(G) > 1$ for some group G, in which case G could not be a regular Galois group over $\mathbb{Q}$. Such a group may not exist (if the RIGP is true), while at the other extreme it cannot be excluded at the moment that $\mathrm{tch}(G) = \infty$ for some group G. Even without questioning its being a Galois group over $\mathbb{Q}$: for example realizations $E_p/\mathbb{Q}$, totally split at p, could exist for all but finitely many primes p — a strong form of IGP —, but only with discriminants exceeding the bounds from definition 3.

Role of the self-twisted cover. Our method starts with a regular realization $F/\mathbb{Q}(T)$ of G. The extensions $E/\mathbb{Q}$ that we wish to produce are specializations $F_{t_0}/\mathbb{Q}$ at some integers t_0. A key tool is the twisting lemma from [1], which reduces the search of specializations of a given type to that of rational points on a certain twisted cover. We use it twice, first over $\mathbb{Q}_p$ as in [1], to construct specializations $F_{t_0}/\mathbb{Q}$ with a specified local behavior. A main ingredient for this first stage is the Lang-Weil estimate for the number of rational points on a curve over a finite field. We obtain many good specialisations $t_0 \in \mathbb{Z}$ and a lower bound for their number.

The next question is to bound the corresponding specializations $F_{t_0}/\mathbb{Q}$ that are distinct. First we reduce it to counting integral points of a given size on certain twisted covers. This is our second use of the twisting lemma, over $\mathbb{Q}$ this time. For the count of the integral points, we use a result of Walkowiak [8] based on a method of Heath-Brown [2]. However the bounds from [8] involve the height of the defining polynomials, which here depend on the specializations $F_{t_0}/\mathbb{Q}$. We have to control the dependence in t_0. This is where enters the self-twisted cover, which is a family of covers, depending only on the original extension $F/\mathbb{Q}(T)$ and which has all the twisted covers among its fibers. As a result, a bound of the form $c_1 t_0^{c_2}$ for the height of the polynomials follows with c_1 and c_2 depending on $F/\mathbb{Q}(T)$.

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Local-to-global extensions of $\mathcal{D}$-modules in positive characteristic

Lars Kindler

Let k be an algebraically closed field of positive characteristic p, and X a smooth k-variety. Let $\mathcal{D}_{X/k}$ denote the ring of differential operators, as defined in [EGA4, §16.]. Write $\mathrm{Strat}(X)$ for the category of $\mathcal{O}_X$-coherent $\mathcal{D}_{X/k}$-modules. Following Grothendieck ([Gro68]) such objects are called *stratified bundles*; they are automatically locally free as $\mathcal{O}_X$-modules. After choosing a base point, $\mathrm{Strat}(X)$ is a neutral Tannakian category over k. For every finite étale covering $f : Y \to X$, the vector bundle $f_*\mathcal{O}_X$ carries a natural $\mathcal{D}_{X/k}$-action, and if Y is Galois, then the monodromy group of the stratified bundle $f_*\mathcal{O}_Y$ is the constant k-group scheme attached to the Galois group of Y/X. Thus stratified bundles can be seen as natural generalizations of coverings of X.

One can generalize other notions related to $\mathcal{D}$-modules over $\mathbb{C}$ to this context. For example there is a notion of regular singularity for stratified bundles ([Gie75], [Kin12]), and for f as above, $f_*\mathcal{O}_Y$ is regular singular if and only if f is tamely ramified in the sense of [KS10].

In my talk I discuss stratified bundles on $\mathbb{G}_{m,k} = \mathbb{P}^1_k \setminus \{0, \infty\}$. In characteristic 0, the formal local variant of a flat connection on $\mathbb{C}(\!(t)\!)$ is the notion of a differential module; in our characteristic $p > 0$ context, a formal local variant of a stratified bundle also exists and is called *iterated differential module* in [MvdP03]. For brevity, we write $\mathrm{Strat}(k(\!(t)\!))$ for the category of such objects. A priori, there is no obvious choice for a neutralization of $\mathrm{Strat}(k(\!(t)\!))$. If we consider $\mathrm{Spec}(k(\!(t)\!)) \subset \mathbb{G}_{m,k}$ as the punctured disc around the origin, we obtain a restriction functor $\mathrm{Strat}(\mathbb{G}_{m,k}) \to \mathrm{Strat}(k(\!(t)\!))$. In [Kin13], I show that this functor is "split". More precisely, I define a full tannakian subcategory $\mathrm{Strat}^{\mathrm{special}}(\mathbb{G}_{m,k}) \subset \mathrm{Strat}(\mathbb{G}_{m,k})$ of *special* stratified bundles on $\mathbb{G}_m$, such that the restriction functor

$$\mathrm{Strat}^{\mathrm{special}}(\mathbb{G}_{m,k}) \xrightarrow{\cong} \mathrm{Strat}(k(\!(t)\!)) \tag{1}$$

is an equivalence. Amongst other applications, this allows us to neutralize the category $\mathrm{Strat}(k(\!(t)\!))$.

This is of course heavily inspired by the work of Katz: In [Kat87] he proves the exact analogue of (1) for flat connections on $\mathbb{G}_{m,\mathbb{C}}$, and in [Kat86] he proves

(1) with Strat replaced by Cov; the categories of étale coverings. In fact the equivalence (1) implies the corresponding result for coverings from [Kat86] as a "finite monodromy part".

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Cycles on modular varieties via geometric Satake

Xinwen Zhu

(joint work with Liang Xiao)

The Tate conjecture and the Langlands conjecture together predict that there should exist a large number of algebraic cycles on the special fibers of the modular varieties (i.e. Shimura varieties or the moduli of Shtukas).

Let us explain this in a relative simple but already new case. Namely, we consider Shimura varieties S attached to the unitary group $U(2,2)$ defined by a quadratic imaginary field E. This is a 4-fold over $\mathbb{Q}$. We write its middle dimensional (compactly supported) cohmology as

$$H^4_c(S_{\bar{\mathbb{Q}}}, \mathbb{Q}_\ell(2)) \simeq \bigoplus_\pi \pi_f^K \otimes W(\pi),$$

where π_f^K's are in a certain set of Hecke modules, with $W(\pi)$ its multiplicity space, equipped with the action of the absolute Galois group $G_{\mathbb{Q}}$ of $\mathbb{Q}$. Then in the *first approximation*, Langlands conjecture predicts that

$$W(\pi) = m(\pi)\rho_\pi,$$

where $m(\pi)$ is the multiplicity of π, and ρ_π is a 6-dimensional representation of $G_{\mathbb{Q}}$ of the form

$$\rho_\pi : G_{\mathbb{Q}} \xrightarrow{\mathrm{rec}(\pi)} \mathrm{GL}_4 \rtimes \mathrm{Gal}(E/\mathbb{Q}) \xrightarrow{\wedge^2 \mathrm{Std}} \mathrm{GL}_6,$$

where $\mathrm{rec}(\pi)$ is the Langlands parameter attached to π, and $(\wedge^2 \mathrm{std})$ is the second wedge of the standard representation of GL_4 (which canonically extends to a representation of $\mathrm{GL}_4 \rtimes \mathrm{Gal}(E/\mathbb{Q})$).

Now, we assume that p is a prime *inert* in E and S has a good reduction at p. Let us write $\bar{S}$ for its (geometric) special fiber. Let σ_p denote the Frobenius. Then it is easy to see that in this six-dimensional vector space, $\rho_\pi(\sigma_p^2)$ always fixes a two-dimensional subspace. In fact, we can assume $\mathrm{rec}(\pi)(\sigma_p) = t \times \sigma_p \subset \mathrm{GL}_4 \rtimes \mathrm{Gal}(E/\mathbb{Q})$, where $t \in \mathrm{GL}_4$ is a diagonal matrix. Then $\rho_\pi(\sigma_p^2)$ acts trivially on the weight subspaces of $(\wedge^2 \mathrm{std})$ of weights $\varepsilon_i + \varepsilon_{4-i}$, where ε_i form a standard basis of the weight lattice of GL_4. Note that if t is sufficiently, (i.e. the Hecke eigenvalue of π at p is not too special), the subspace fixed by $\rho_\pi(\sigma_p^2)$ is exactly two-dimensional.

Now according to the Tate conjecture, there should exist two families of cycles in $\bar{S}$, defined over $\mathbb{F}_{p^2}$, whose cycle classes span these two-dimensional spaces (for various π). It turns out that in this case the supersingular Newton stratum of $\bar{S}$ has a partition into two families of cycles, whose cycle classes realize the above predicted Tate classes. Note that the geometry of the supersingular Newton Stratum for $\bar{S}$ has recently studied by [HP].

The above analysis of the special example can be generalized. Also, as is seen, the span of these Frobenius-invariant cohomology classes are related to certain weight spaces of the representation of the Langlands dual group. So it is natural to ask whether the corresponding algebraic cycles as predicted by the Tate conjecture also relate to the Mirkovic-Vilonen (MV) cycles from the geometric Satake correspondence. This turns out to be the case. Our results are as follows:

When there should exist Tate cycles of the above type, we can partition the basic Newton stratum of the modular variety (Shimura varieties of Hodge type, or moduli of Shtukas) into several families $\{Z_\gamma\}$ of cycles. The families $\{\gamma\}$ are parameterized by certain MV cycles. In fact, we prove this by giving the partition of (the reduced subscheme of) certain Rapoport-Zink spaces and affine Deligne-Lusztig varieties.

It remains to prove that these cycle classes span the Tate classes as predicted by the Langlands conjecture.

In the function field case, using some idea similar to the recent work of V. Lafforgue [La] *(namely the fusion of Beilinson-Drinfeld (BD) Grassmannians), we can prove that in the generic case these cycle classes are linearly independent and therefore realize the expected Tate classes (at least in type A case).*

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Partial Fourier-Mukai transform for algebraically integrable systems

Roman Fedorov

This is a report on a joint ongoing project with Dima Arinkin. The celebrated Fourier-Mukai transform is an equivalence between the derived category of an abelian variety and that of the dual abelian variety. Recently there have been a lot of interest in Fourier-Mukai transforms for singular degenerations of abelian varieties, e.g., for Jacobians of singular curves. However, very little is known beyond the Jacobian case.

In a joint work with D. Arinkin we suggest a different setup. Let $\pi_X : X \to B$ be a flat morphism of smooth complex varieties with integral projective fibers. We also assume that X is symplectic and the smooth locus of each fiber is lagrangian (thus, we do not assume that the fibers are smooth). Let us call this data *an algebraically integrable system.* Assume also that we have a section $\zeta_X : B \to X$.

Let $X^{sm} \subset X$ be the locus, where π_X is smooth. Our first claim is that X^{sm} has a natural structure of a commuative group scheme over B such that the action of X^{sm} on itself by translations extends to an action $X^{sm} \times_B X \to X$.

Let $\mathrm{Pic}(X/B)$ be the relative Picard scheme. Let $\mathrm{Pic}^\tau(X/B) \subset \mathrm{Pic}(X/B)$ be the open subscheme classifying numerically trivial line bunldes (equivalently, the fiber of $\mathrm{Pic}^\tau(X/B)$ over $b \in B$ classifies line bundles whose image in the group of connected components of $\mathrm{Pic}(X_b)$ is torsion). The point is that one should view $\mathrm{Pic}^\tau(X/B)$ as the smooth part of the 'dual integrable system'.

We argue that the universal line bundle on $X \times_B \mathrm{Pic}^\tau(X/B)$ gives rise to a fully faithful functor

$$D^b(Qcoh(\mathrm{Pic}^\tau(X/B))) \to D^b(Qcoh(X)).$$

Here *Qcoh* stands for the category of quasi-coherent sheaves, D^b stands for the bounded derived category.

Applications to Hitchin systems are also discussed.

Étale fundamental group and local systems

Hélène Esnault

1. Introduction

For a smooth variety X defined over the field of complex numbers, the Grothendieck-Malcev theorem asserts that if the étale fundamental group $\pi_1^{ét}(X)$ of X is trivial, then so are all finite dimensional complex local systems, thus, via the Riemann-Hilbert correspondence, any regular singular $\mathcal{O}_X$-coherent $\mathcal{D}_X$-module is trivial.

If the ground field k, which we assume to be algebraicaly closed, has characteristic $p > 0$, the category of $\mathcal{O}_X$-coherent $\mathcal{D}_X$-modules is k-linear Tannakian, the objects are identified with F^∞-divised vector bundles $\{(E_0, E_1, \ldots, \sigma_0, \sigma_1, \ldots), \sigma_i : F^*E_{i+1} \xrightarrow{\cong} E_{i+1}\}$ (Katz, [Gie75]), the profinite completion of its Tannaka k-groupscheme $\pi_1^{\mathrm{strat}}(X)$ is identified $\pi_1^{\text{ét}}(X)$ (dos Santos, [dSan07]).

Gieseker (1975) ([Gie75]) conjectured that Grothendieck-Malcev's theorem remains true in this context if X is smooth projective. It has been answered in the positive

Theorem 1. [EsnMeh10] *Let X be a smooth projective variety over k. Then if $\pi_1^{\text{ét}}(X)$ is trivial, so are all $\mathcal{O}_X$-coherent $\mathcal{D}_X$-modules trivial.*

The proof makes use of geometry, boundedness (construction of moduli by Langer), and Hrushovsky's theorem on the Weil-Lang estimates.

On the other hand, in his thesis, L. Kindler defined the right notion of regular singularities at infinity, without assuming resolution of singularities, defining the quotient $\pi_1^{\mathrm{strat}}(X) \twoheadrightarrow \pi_1^{\mathrm{strat,rs}}(X)$, and he showed that the profinite completion of this surjection is $\pi_1^{\text{ét}}(X) \twoheadrightarrow \pi_1^{\text{ét,tame}}(X)$.

This enables one to pose the question:

Question 2. *U smooth quasi-projective over $k = \bar{k}$ of char. $p > 0$.*

i) $\pi_1^{\text{ét,tame}}(U) = 0 \Longrightarrow$ no non-trivial r.s. stratified bundle?
ii) $\pi_1^{\text{ét}}(U) = 0 \Longrightarrow$ no non-trivial stratified bundle?

On i): $\mathbb{A}^n$ OK (E-Kindler), and (Kindler) $\pi_1^{\text{tame, ab}}(U) = 0$ and no $\mathbb{Z}/p$-quotient von $\pi_1^{\mathrm{ab}}(X)$ implies no non-trivial stratified bundle E with $G(E)$ abelian. Here X is a good compactification (so we assume it exists, e.g. in dimension 2).

Our problem is to understand ii), as it in particular contains the question of understanding the geometry of U with $\pi_1^{\text{ét}}(U) = 0$.

From now on: joint work in progress with V. Srinivas.

2. First property of simply connected U

$$H^1(U, \mathbb{F}_p) = 0 \Longrightarrow H^0(U, \mathcal{O}_U) = k \tag{1}$$

so no such non-proper U in dim 1. Assume dim $X = 2$. Let $U \subset X$ be a good compactification, such that $D := X \setminus U$ is a NCD.

Lemma 3. *Assume $H^0(U, \mathcal{O}_U) = k$. Then $(D_i \cdot D_j) \leq 0$.*

Example 4. *Example with $H^0(U, \mathcal{O}_U) = k, (D_i \cdot D_j) < 0$, U has a normal compactification $U \hookrightarrow \bar{X}$ such that $\bar{X} \setminus U$ consists of normal finite quotient singularitiies, $\bar{X}$ has a desingularization $X \to \bar{X}$ such that $\pi_1^{\text{ét}}(X) = 0$, thus $\pi_1^{\text{ét}}(\bar{X}) = 0$, yet $\pi_1^{\text{ét}}(U)$ is infinite.*

$\bar{X}$ is the Kummer surface $\bar{X} = A/(\pm)$ with $p > 2$, where A is an abelian surface, $U = X_{\text{reg}}$. Then a desingularization $X \to \bar{X}$ is a $K3$, yet $U \times_{\bar{X}} A \subset A$ has complement of codimension 2, thus $\pi_1^{\text{ét}}(U \times_{\bar{X}} A) = \pi_1^{\text{ét}}(A)$ is infinite, thus $\pi_1^{\text{ét}}(U)$ is infinite.

Example 5. *Example with $H^0(U, \mathcal{O}_U) = k, (D_i \cdot D_j) \leq 0$ but not strictly, with $\pi_1^{\text{tame}}(U) = 0$ but we do not know $\pi_1^{\text{ét}}(U)$.*

Assume $k \supset \mathbb{F}_p(t)$, $p \neq 3$. Then X is the blow up of $\mathbb{P}^2$ in 9 general points on a genus 1 curve C, where general means that if D is the strict transform of D, thus $D^2 = 0$, one has $\mathcal{O}_D(D) \in \text{Pic}^0(D)$ not torsion. Then $U = X \setminus D$. Then $H^0(X, \mathcal{O}_X(nD)/\mathcal{O}_X) = 0$ while $H^0(U, \mathcal{O}_U) = \varinjlim_n H^0(X, \mathcal{O}_X(nD) = H^0(X, \mathcal{O}_X) = k$. Further one has $\mathbb{Z}/3 = \pi_1^{\text{ét,tame}}(\mathbb{P}^2 \setminus C) \twoheadrightarrow \pi_1^{\text{ét,tame}}(U)$ and the Kummer $3:1$ cover of $\mathbb{P}^2 \setminus C$ ramifies fully along C, thus along the exceptional lines in X, thus $\pi_1^{\text{ét,tame}}(U) = 0$. Finally, one has

$$H^1(U, \mathcal{O}_U) = \varinjlim_n H^1(X, \mathcal{O}_X(nD)/\mathcal{O}_X) = 0,$$

thus a fortiori $H^1(U, \mathbb{F}_p) = 0$.

Claim 6. *There is no surjective specialization map $\pi_1^{\text{ét}}(U) \to \pi_1^{\text{ét}}(U_{\bar{\mathbb{F}}})$.*

Proof. In a specialization, $H^0(U_{\bar{\mathbb{F}}}, \mathcal{O}) \neq \bar{\mathbb{F}}$, so $H^1(U_{\bar{\mathbb{F}}}, \mathbb{F}_p) \neq 0$. □

3. First non-trivial example for which Question 2 ii) has a positive answer

Y projective smooth over $k = \bar{k}$, $L = \mathcal{O}_X(\Delta)$ line bundle, Δ effective $\neq \emptyset$, $X = \mathbb{P}(\mathbb{I} \oplus L)$, $U = X \setminus \infty$-section so U/Y is the total space of the bundle L.

Proposition 7. *i) $\pi_1^{\text{ét}}(U) \to \pi_1^{\text{ét}}(Y)$ is an isomorphism and f^*, for $f : U \to Y$, induces an isomorphism on the category of stratified bundles.*

ii) If $\pi_1^{\text{ét}}(Y) = 0$, there are no non-trivial stratified bundles on U.

Remark 8. Note this example, for dim $Y = 1$, is of the kind $D^2 < 0$.

4. Second non-trivial example for which Question 2 ii) has a positive answer

Proposition 9. *Let C be a connected proper scheme over $k = \bar{k}$. Then the category of stratified bundles on it (i.e. F divided bundles, the Hom being the ones compatible with the relative structure) is Tannaka.*

Proposition 10. *F-divided bundles in the sense of Proposition 9 are modules with stratification in the sense of Saavedra.*

Theorem 11. *Let U be a smooth quasi-projective variety over $k = \bar{k}$, let $\iota : C \hookrightarrow U$ be a projective ample Cartier divisor. Then $\pi^{\mathrm{strat}}(C) \to \pi_1^{\mathrm{strat}}(U)$ is surjective.*

Proof makes use of Bost's most recent work on Lefschetz' theorems.

Theorem 12. *Let U be a quasi-projective surface over $k = \bar{k}$, let $\iota : C \to U$ be an ample curve such that $\pi_1^{\text{ét}}(C)$ is abelian. Then if $\pi_1^{\text{ét}}(U) = 0$, all stratified bundles on U are trivial.*

There are examples where the theorem applies.

5. General theorem

Theorem 13. *Let U be a quasi-projective surface over $\bar{\mathbb{F}}_p$, let $\iota : C \to U$ be an ample curve. Then if $\pi_1^{\text{ét}}(U) = 0$, all stratified bundles on U are trivial.*

The proof makes use of the following non-trivial theorem. Let $j : U \to X$ be a normal compactification.

Theorem 14. *Let $r \in \mathbb{N} \setminus \{0\}$ be given. Under the assumptions of the theorems, there are finitely many polynomials $\chi_i(m)$ in $\mathbb{Q}[m]$, $i \in I$, for any stratified bundle (E_n, σ_n), there is a n_0 such that for a for $n \geq n_0$, $\chi(X, j_* E_n(mC)) \in \{\chi_i(m), i \in I\}$.*

Analogy over $\mathbb{C}$:

Theorem 15 (Deligne, [Del14]). *Let U be a smooth variety over the field of complex numbers. Let $r \in \mathbb{N} \setminus \{0\}$ be given. Let $j : U \to X$ be a good compactification. Then the set of Chern classes of all Deligne's canonical extensions of E such that (E, ∇) is a regular singular connection on U, is finite.*

The main theorem could be generalized to a higher dimensional U such that there is a normal compactification X such that $X \setminus U$ has codimension ≥ 2 if one had a Lefschetz theorem for stratifications (thus in particular for the étale fundamental group), and could be generalized to any characteristic $p > 0$ field $k = \bar{k}$ if one had a surjective specialization theorem for the étale fundamental group.

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Reporter: Julian Tenzler

Participants

Dr. Sara Arias-de-Reyna
Université du Luxembourg
Unite de Recherche en Mathématiques
6, Rue Richard Coudenhove-Kalergi
1359 Luxembourg
LUXEMBOURG

Prof. Dr. Gebhard Böckle
Interdisziplinäres Zentrum
für Wissenschaftliches Rechnen
Universität Heidelberg
Im Neuenheimer Feld 368
69120 Heidelberg
GERMANY

Prof. Dr. Anna Cadoret
Centre de Mathématiques Laurent
Schwartz
École Polytechnique
91128 Palaiseau
FRANCE

Prof. Dr. Pierre Dèbes
U.F.R. de Mathématiques
Université de Lille 1
59655 Villeneuve d'Ascq Cedex
FRANCE

Dr. Michael Dettweiler
Mathematisches Institut
Universität Bayreuth
Postfach 101251
95447 Bayreuth
GERMANY

Viet Cuong Do
Université Henri Poincare, Nancy 1
Institut Elie Cartan
P.O. Box 239
54506 Vandoeuvre-les-Nancy
FRANCE

Prof. Dr. Dr. h.c. Hélène Esnault
FB Mathematik und Informatik
Freie Universität Berlin
Arnimallee 3
14195 Berlin
GERMANY

Prof. Dr. Roman Fedorov
Department of Mathematics
Kansas State University
Manhattan, KS 66506-2602
UNITED STATES

Dr. Nuno Freitas
Mathematisches Institut
Lehrstuhl für Computeralgebra
Universität Bayreuth
95440 Bayreuth
GERMANY

Jochen Heinloth
Fachbereich Mathematik
Universität Duisburg-Essen
Universitätsstr. 3
45117 Essen
GERMANY

Prof. Dr. Nicholas M. Katz
Department of Mathematics
Princeton University
Fine Hall
Washington Road
Princeton, NJ 08544-1000
UNITED STATES

Dr. Lars Kindler
Institut für Mathematik
Freie Universität Berlin
14195 Berlin
GERMANY

Prof. Dr. Gunter Malle
Fachbereich Mathematik
T.U. Kaiserslautern
67653 Kaiserslautern
GERMANY

Prof. Dr. Dac Tuan Ngo
Mathématiques
LAGA - CNRS: UMR 7539
Université Paris Nord (Paris XIII)
99, Avenue J.-B. Clement
93430 Villetaneuse
FRANCE

Dr. Stefan Reiter
Mathematisches Institut
Universität Bayreuth
Postfach 101251
95447 Bayreuth
GERMANY

Timo Richarz
Mathematisches Institut
Universität Bonn
53115 Bonn
GERMANY

Prof. Dr. David Roberts
Division of Science and Mathematics
University of Minnesota - Morris
Morris, MN 56267
UNITED STATES

Prof. Dr. Gordan Savin
Department of Mathematics
University of Utah
155 South 1400 East
Salt Lake City, UT 84112-0090
UNITED STATES

Julian Tenzler
Mathematisches Institut
Universität Bayreuth
Postfach 101251
95447 Bayreuth
GERMANY

Dr. Yakov Varshavsky
Einstein Institute of Mathematics
The Hebrew University
Givat Ram
91904 Jerusalem
ISRAEL

Prof. Dr. Gabor Wiese
Université de Luxembourg
Faculté des Sciences, de la Technologie
et de la Communication
6, rue Richard Coudenhove-Kalergi
1359 Luxembourg
LUXEMBOURG

Prof. Dr. Zhiwei Yun
Department of Mathematics
Stanford University
Building 380, Serra Mall
Stanford CA 94305-2125
UNITED STATES

Prof. Dr. Xinwen Zhu
Department of Mathematics
Northwestern University
2033 Sheridan Road
Evanston IL 60208-2730
UNITED STATES