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Representation Theory and Analysis of Reductive Groups: Spherical Spaces and Hecke Algebras

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Abstract. The workshop gave an overview of current research in the representation theory and analysis of reductive Lie groups and its relation to spherical varieties and Hecke algebras. The participants and the speakers represented an international blend of senior researchers and young scientists at the start of their career. Some particular topics covered in the 30 talks related to structure theory of spherical varieties, p-adic symmetric spaces, symmetry breaking operators, automorphic forms, and local Langlands correspondence.

Mathematics Subject Classification (2010): 22Exx, 20Cxx.

Introduction by the Organisers

The international conference *Representation Theory and Analysis of Reductive Groups: Spherical Spaces and Hecke Algebras*, organized by Bernhard Krötz (Paderborn), Eric M. Opdam (Amsterdam), Henrik Schlichtkrull (Copenhagen), and Peter Trapa (Salt Lake City) was held January 19th – January 25th, 2014. This conference brought together scientists from the separate, yet intimately related, fields of harmonic analysis of real Lie groups and representation theory of Hecke algebras. The meeting was attended by 50 participants, and a total of 30 lectures of length between 1/2 hour and 50 minutes were given. The participants and the speakers represented an international blend of senior researchers and young scientists at the start of their career. The meeting belongs to a long tradition of workshops around the theme of *Harmonische Analysis und Darstellungstheorie* *topologischer Gruppen*, but for most of the participants this was their first visit to Oberwolfach (and for some, even to Europe).

The workshop opened with a lecture of Friedrich Knop giving an overview of the structure of spherical spaces, first in the case of complex reductive group actions and then in the much more recently established real case. The theory in the real case is foundational for approaching a host of geometric and analytic questions with very far reaching applications. For example, the space of unramified Langlands parameters for a reductive p-adic group in a great number of cases (and, optimistically, all cases) arises from the structure theory of certain spherical spaces of Langlands parameters for real reductive groups. The theory covered in Knop's lectures should, therefore, provide insight into the intricate relationship between the representation theory of real and p-adic groups, something envisioned in rough form in the original work of Harish-Chandra and Langlands on the subject.

The talk of Friedrich Knop was followed by an equally impressing talk by Patrick Delorme covering a new and interesting approach to the harmonic analysis on p-adic reductive symmetric spaces. Previous results by Sakellaridis and Venkatesh were put in a geometric perspective which opens up a wide possibility for generalization.

All the selected speakers gave interesting talks of high quality in which recent research results were presented, and they were followed by vivid discussion among the participants.

Also the talks by the young participants were noteworthy for their presentation of new methods in the field. For example it was interesting to see in the talk by Benjamin Harris how microlocal techniques can provide some very basic information on representation of real reductive Lie groups.

The relationship between representation theory of real and p-adic groups was again on display in the talk of Kei Yuen Chan, one of the workshop's graduate student participants. His talk sought to transport ideas for the construction of discrete series representations of real reductive groups to the p-adic case via the theory of affine Hecke algebras. Chan's talk sparked a suggestion from a senior workshop participant, namely to use deformations in the affine Hecke algebra setting (a tool unavailable in the real case), which Chan is now implementing to great effect. The workshop provided an ideal setting to encourage this kind of interaction between junior and senior participants.

In the talk of Allen Moy it was shown that the sum of the depth zero Bernstein projectors for p-adic SL(2) is supported on the set of topologically unipotent elements (joint work with Howe). This beautiful result raises interesting questions for higher depth and more general groups. Moy's talk was dedicated to the memory of Paul Sally, who passed away a few weeks before the meeting. The early representation theory for p-adic SL(2;R) was to a large extend developed by Paul, whose passion for the topic is legendary.

Workshop: Representation Theory and Analysis of Reductive Groups: Spherical Spaces and Hecke Algebras

Table of Contents

Abstracts

The local structure theorem

Friedrich Knop
(joint work with B. Krötz, E. Sayag, H. Schlichtkrull)

Let G be a connected reductive group defined over $\mathbb{C}$.

Definition 1. The action of G on a variety X is *elementary* if the derived subgroup (G, G) acts trivially on X. In other words, the G-action on X factors through a torus action.

In [1], Brion, Luna, and Vust introduced a technique for reducing an arbitrary G-action to an elementary one.

Theorem 2 (Local Structure Theorem)**.** *Let X be a normal G-variety and let $Y \subseteq X$ be a non-empty G-invariant subvariety. Then there exists a parabolic subgroup $Q \subseteq G$, a Levi decomposition $Q = L \ltimes Q_u$ and an affine L-invariant subvariety $S \subseteq X$ such that:*

(1) The morphism

$$Q_u \times S = Q \times^L S \to X : [q, x] \mapsto qx$$

is an Q-equivariant isomorphism onto an open subset $X_0 \subseteq X$.

(2) The intersection $S \cap Y$ is non-empty and the action of L on $S \cap Y$ is elementary.

This theorem plays a fundamental rôle in describing the generic as well as the asymptotic geometry of X.

Idea of proof. We sketch a proof in a version first given in [2]. First, using an affine cone construction, one reduces to the case that X is affine. Let $B \subseteq G$ be a Borel subgroup. Then one observes the following elementary but fundamental

Lemma 3 (Key Lemma)**.** *The action of G on Y is elementary if and only if every B-semiinvariant regular function on Y is already G-semiinvariant.*

If Y is not yet elementary there is a B-semiinvariant rational function f_0 in X which is not a G-semiinvariant. By complete reducibility it is possible to extend the function f_0 to a B-semiinvariant f on X. Then Q, the stabilizer of the line $\mathbb{C}f$ through f, is a proper parabolic subgroup of G. Put $X_0 := \{x \in X \mid f(x) \neq 0\}$ and consider the morphism

$$m : X_0 \to (\operatorname{Lie} G)^* : x \mapsto \left[\xi \in \operatorname{Lie} G \mapsto \frac{\xi f(x)}{f(x)}\right]$$

Fix $x_0 \in X_0$ and consider its image $a := m(x_0) \in (\operatorname{Lie} G)^*$. Then one shows that the isotropy subgroup $L := G_a$ is a Levi complement of Q and that the image of m is contained in the orbit Ga. This way, one gets a Q-equivariant morphism

$m : X_0 \to Ga \cong Q/L$. Then $Q \times^L S \to X_0$ is an isomorphism with $S = m^{-1}(a)$. Now use induction on the L-variety S and its subvariety $S \cap Y$.

□

The principal goal of the talk was to present an extension of the Local Structure Theorem to varieties which are defined over $\mathbb{R}$. Assume from now on that G is defined over $\mathbb{R}$. Let $P \subseteq G$ be a minimal parabolic $\mathbb{R}$-subgroup.

Definition 4. The action of G on a real G-variety is called *real elementary* if all non-compact simple factors of G act trivially on X. Equivalently, all unipotent elements in $G(\mathbb{R})$ act as identitity on X. Another reformulation is: the action of G on X factors through a group $\overline{G}$ such that $\overline{G}(\mathbb{R})$ is compact modulo center.

In [3], we proved the following real version of the Local Structure Theorem:

Theorem 5 (Real Local Structure Theorem)**.** *Let X be a normal G-variety and let $Y \subseteq X$ be a G-invariant subvariety where everything is defined over $\mathbb{R}$. Assume moreover that the set $Y(\mathbb{R})$ of real points is Zariski dense in Y. Then there exists a parabolic subgroup $Q \subseteq G$, a Levi decomposition $Q = L \ltimes Q_u$ and an affine L-invariant subvariety $S \subseteq X$ where everything is defined over $\mathbb{R}$ such that:*

(1) The morphism

$$Q_u \times S = Q \times^L S \to X : [q, x] \mapsto qx$$

is an Q-equivariant isomorphism onto an open subset $X_0 \subseteq X$.

(2) The intersection $S \cap Y$ is non-empty and the action of L on $S \cap Y$ is real elementary.

The proof is pretty much the same as in the complex situation, except that the Key Lemma is less obvious.

Lemma 6 (Real Key Lemma)**.** *The action of G on Y is elementary if and only if every P-semiinvariant regular function on Y is already G-semiinvariant.*

We describe one application.

Definition 7. The variety X is *real spherical* if $P(\mathbb{R})$ has an open orbit in $X(\mathbb{R})$.

Corollary 8. *Let X be a real spherical variety. Then the number of orbits of $G(\mathbb{R})$ in $X(\mathbb{R})$ is finite.*

Another application from [4] is

Theorem 9 (Approximate Polar Decomposition)**.** *Let $H \subseteq G$ be a real subgroup such that $X := G/H$ is a real spherical variety. Let $K \subseteq G(\mathbb{R})$ be a maximal compact subgroup. Then there are finite subsets $F, F' \subseteq G(\mathbb{R})$ and a split $\mathbb{R}$-subtorus $A \subseteq G$ such that*

$$G(\mathbb{R}) = F' \cdot K \cdot A(\mathbb{R}) \cdot F \cdot H(\mathbb{R}).$$

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Neighborhoods at infinity and the Plancherel formula for a reductive *p*-adic symmetric space

Patrick Delorme

Yiannis Sakellaridis and Akshay Venkathesh have determined, when the group G is split and the field F is non archimedean of characteristic zero, the Plancherel formula for any spherical space X for G modulo the knowledge of the discrete spectrum.

The starting point is the determination of good neighborhoods at infinity of X/J, where J is a small compact open subgroup of G. These neighborhoods are related to "boundary degenerations" of X. The proof of their existence is made by using wonderful compactifications.

In this article we will show the existence of such neighborhoods assuming that F is of characteristic different from 2 and X is symmetric. In particular, one does not assume that G is split. Our main tools are the Cartan decomposition of Benoist and Oh, our previous definition of the constant term and asymptotic properties of Eisenstein integrals due to Nathalie Lagier .

Once the existence of these neighborhoods at infinity of X is established, the analog of the work of Sakellaridis and Venkatesh is straightforward and leads to the Plancherel formula for X.

Model intertwining operators for symmetric pairs

Jan Möllers
(joint work with Yoshiki Oshima, Bent Ørsted)

1. Motivation - multiplicity-one property

Let G be a real reductive group and $G' \subseteq G$ a closed subgroup. For two irreducible smooth admissible representations π and π' of G and G', respectively, we consider the multiplicities

$$m(\pi, \pi') = \dim \mathrm{Hom}_{G'}(\pi|_{G'}, \pi').$$

Recently there has been a considerable interest in upper bounds for these multiplicities (see e.g. [1, 5, 10] and references therein). In particular, in [10] the multiplicity-one property is proved for several pairs (G, G') of classical groups, i.e. for all irreducible representations π and π' (in a certain class) we have

$$m(\pi, \pi') \leq 1.$$

For several applications (e.g. the restriction of automorphic functions, see [2, 9]) these quantitative results have to be supplemented by more qualitative results, namely the construction of explicit non-zero operators in $\mathrm{Hom}_{G'}(\pi|_{G'}, \pi')$ if $m(\pi, \pi') = 1$ (also called *symmetry breaking operators* by Kobayashi [3]). We carry out this construction in the case where (G, G') a symmetric pair and π and π' are spherical principal series representations, induced from compatible parabolic subgroups of G and G', respectively. For details we refer the reader to [8].

2. Construction of model intertwining operators

2.1. **Principal series representations.** Let G be a real reductive group in the Harish-Chandra class and G' a proper symmetric subgroup of G. For parabolic subgroups $P = MAN \subseteq G$ and $P' = M'A'N' \subseteq G'$ satisfying certain compatibility conditions (see [8, conditions (G), (H) and (D)] for details) we consider the spherical principal series representations (smooth normalized parabolic induction):

$$\pi_\nu := \mathrm{Ind}_P^G(\mathbf{1} \otimes e^\nu \otimes \mathbf{1}), \qquad \nu \in \mathfrak{a}_\mathbb{C}^*,$$

$$\tau_{\nu'} := \mathrm{Ind}_{P'}^{G'}(\mathbf{1} \otimes e^{\nu'} \otimes \mathbf{1}), \qquad \nu' \in (\mathfrak{a}')_\mathbb{C}^*,$$

where $\mathfrak{a}$ and $\mathfrak{a}'$ denote the Lie algebras of A and A', respectively. We realize both representations in the compact picture. More precisely, π_ν will be realized on $C^\infty(X)$ with $X = K/(M \cap K)$ and $\tau_{\nu'}$ will be realized on $C^\infty(X')$ with $X' = K'/(M' \cap K')$. This has the advantage that the representation spaces do not depend on the parameters ν and ν'.

2.2. **Intertwining operators.** In [8, Theorem 3.3 (1) and (2)] we construct a family

$$A(\alpha, \beta) : C^\infty(X) \to C^\infty(X')$$

of continuous linear operators depending meromorphically on the parameters $\alpha, \beta \in \mathfrak{a}_\mathbb{C}^*$. We then show in [8, Theorem 3.3 (3)]:

Theorem. *Let (α, β) be a regular point for $A(\alpha, \beta)$. Then for certain $\nu = \nu(\alpha, \beta)$ and $\nu' = \nu'(\alpha, \beta)$ (see* [8, equation (3.4)]*) the operators $A(\alpha, \beta)$ belong to*

$$\mathrm{Hom}_{G'}(\pi_\nu|_{G'}, \tau_{\nu'}).$$

2.3. **Uniqueness for split rank one.** Let

$$(G, G', P) = (SU(1, n; \mathbb{F}), S(U(1, m; \mathbb{F}) \times U(n - m; \mathbb{F})), P_{\min})$$

with $\mathbb{F} = \mathbb{R}, \mathbb{C}, \mathbb{H}$ and $0 < m < n$ or $\mathbb{F} = \mathbb{O}$ and $n = m + 1 = 2$. Then $\mathfrak{a} = \mathfrak{a}' \cong \mathbb{R}$ and the map $(\alpha, \beta) \mapsto (\nu, \nu')$ is one-to-one.

Theorem. *For generic parameters ν, ν' we have*

$$\mathrm{Hom}_{G'}(\pi_\nu|_{G'}, \tau_{\nu'}) = \mathbb{C} \cdot A(\alpha, \beta).$$

For $m = n - 1$ and $\mathbb{F} = \mathbb{R}, \mathbb{C}$ this also follows from the general multiplicity-one property (see [1, 10]). Further, for $m = n - 1$ and $\mathbb{F} = \mathbb{R}$ Kobayashi–Speh [6] determine $\mathrm{Hom}_{G'}(\pi_\nu|_{G'}, \tau_{\nu'})$ explicity also for singular parameters ν, ν'.

3. Outlook - applications

We name a few applications:

3.1. **Branching laws for unitary representations.** The family $A(\alpha, \beta)$ of intertwining operators for $(G, G') = (SO(1, n), SO(1, m) \times SO(n - m))$ constructs both the discrete and the continuous part in the restriction of spherical complementary series of G to G' (joint work with Y.Oshima, see [9]).

3.2. **Singular integral operators on nilpotent groups.** Realization of π_ν and $\tau_{\nu'}$ on smooth function on $\overline{N}$ and $\overline{N}'$ induces a meromorphic family of operators

$$\tilde{A}(\alpha, \beta) : C_c^\infty(\overline{N}) \to C^\infty(\overline{N}').$$

A detailed study of the meromorphic nature of $\tilde{A}(\alpha, \beta)$ (location of the poles, calculation of the residues) was for $(G, G') = (SO(1, n), SO(1, n - 1))$ done by Kobayashi–Speh [6] and is of interest for other cases, too.

3.3. **Harmonic analysis on spherical spaces.** We have

$$\mathrm{Hom}_{G'}(\pi|_{G'}, \tau) \cong \mathrm{Hom}_{G'}(\pi|_{G'} \otimes \tilde{\tau}, \mathbb{C}),$$

where $\tilde{\tau}$ is the contragredient representation of τ. Therefore, each intertwining operator provides an embedding $\pi \otimes \tilde{\tau} \subseteq C^\infty((G \times G')/\Delta(G'))$ where $\Delta(G') = \{(g, g) : g \in G'\}$. The space $(G \times G')/\Delta(G')$ is spherical e.g. for the cases

$$(G, G') = (SO(p, q), SO(p, q - 1)), \qquad (G, G') = (SU(p, q), U(p, q - 1)).$$

For a complete classification of such spherical pairs see [4].

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Cusp forms for reductive symmetric spaces

Job J. Kuit
(joint work with Erik P. van den Ban, Henrik Schlichtkrull)

Just as in the theory of automorphic forms, cusp forms are of fundamental importance for the representation theory of reductive Lie groups. There are many analogies between the theory of automorphic forms and the representation theory of reductive symmetric spaces, a class of spaces which contains the reductive groups. Yet for reductive symmetric spaces the appropriate notion of cusp form is not yet well understood. The integrals one would like to use from an a priori point of view fail to converge in general. About ten years ago M. Flensted-Jensen suggested to use modified integrals to define cusp forms.

Harish-Chandra showed that the closure $\mathscr{C}_{\mathrm{ds}}(G)$ in $\mathscr{C}(G)$ of the direct sum of the discrete series representations can be described in an elegant and useful manner using integrals. For this, he considered certain subgroups of G, called parabolic subgroups. We write N_P for the unipotent radical of a parabolic subgroup P. Harish-Chandra showed that for every parabolic subgroup $P \neq G$ and all $g, g' \in G$, the integral

$$\int_{N_P} \phi(gng')\, dn \tag{1}$$

is absolutely convergent for all $\phi \in \mathscr{C}(G)$. In analogy to the theory of automorphic forms, he defined a cusp form to be a function $\phi \in \mathscr{C}(G)$ such that the integral (1) vanishes for all $g, g' \in G$ and every parabolic subgroup $P \neq G$. Harish-Chandra proved that the space of cusp forms $\mathscr{C}_{\mathrm{cusp}}(G)$ is equal to $\mathscr{C}_{\mathrm{ds}}(G)$. (See [6], in particular Section 18 and 27.)

In reaction to the determination of the Plancherel formula for reductive symmetric spaces in the work of P. Delorme ([4]) and E.P. van den Ban and H. Schlichtkrull ([2] and [3]), M. Flensted-Jensen gave a presentation during a conference in Oberwolfach in July 2000, in which he discussed joint work with F. Rouviere. (See [5].) In this lecture he raised the question whether the discrete series for a reductive symmetric space could be characterized as cusp forms for a suitable notion of cusp form. The first major problem one encounters is that the integrals that one would like to use to define cusp forms may not be convergent.

To be more precise, a reductive symmetric space is a homogeneous space G/H for a reductive Lie group G, where H is an open subgroup of the fixed point subgroup G^σ for an involution σ of G. Let G/H be a reductive symmetric space and let σ be the corresponding involution on G. Let P be a parabolic subgroup of G. For a suitable function ϕ on G/H we define the function $\mathcal{R}_P\phi$ on G to be given by

$$\mathcal{R}_P\phi(g) = \int_{N_P/(N_P\cap H)} \phi(gn\cdot H)\,dn. \tag{2}$$

The involution σ determines a special class of parabolic subgroups P of G, namely the parabolic subgroups P for which $\sigma(P)$ is a parabolic subgroup opposite to P. Such a parabolic subgroup is called a σ-parabolic subgroup. If one looks at the Plancherel decomposition, it seems natural to define a cusp form to be a function $\phi \in \mathscr{C}(G/H)$ such that $\mathcal{R}_P\phi = 0$ for every σ-parabolic subgroup $P \neq G$. However, in general the integrals (2) fail to converge for all $\phi \in \mathscr{C}(G/H)$ and σ-parabolics P. Moreover, the definition of a cusp form should somehow generalize the definition of Harish-Chandra for the Lie group G considered as a symmetric space and this definition would fail to do that.

To overcome these problems (at least in case G/H is split rank 1), we follow a suggesion of M. Flensted-Jensen not to use σ-parabolic subgroups but instead a different class of parabolic subgroups which we call H-compatible parabolic subgroups.

From now on we assume that the split rank of G/H is equal to 1. Let P be a minimal parabolic subgroup. Then P admits a Langlands decomposition $P = M_PA_PN_P$ such that A_P is σ-stable. P is called H-compatible if and only if $\dim(A_P\cap H)$ is minimal (i.e., $\dim(A_P/(A_P\cap H)) = 1$) and if the following holds. Let

$$\rho_{P,\mathfrak{h}} = \frac{1}{2}\sum_{\substack{\alpha\in\Sigma^+(\mathfrak{a}_P;P)\\ \sigma\alpha=\alpha}} \dim(\mathfrak{g}_\alpha)\alpha.$$

Then $\langle\alpha,\rho_{P,\mathfrak{h}}\rangle \geq 0$ for all $\alpha \in \Sigma^+(\mathfrak{a}_P;P)$. Using H-compatible parabolic subgroups in (2) to define cusp forms would give a generalization of the definition of Harish-Chandra.

Theorem 1. *Let P be H-compatible. Then the integral in (2) is absolutely convergent for every $\phi \in \mathscr{C}(G/H)$ and every $g \in G$.*

This result allows to define cusp forms for reductive symmetric spaces of split rank 1. We say that $\phi \in \mathscr{C}(G/H)$ is a cusp form if and only if $\mathcal{R}_P\phi = 0$ for every H-compatible parabolic subgroup P. We write $\mathscr{C}_{\mathrm{cusp}}(G/H)$ for the space of cusp forms and $\mathscr{C}_{\mathrm{ds}}(G/H)$ for the closure in $\mathscr{C}(G/H)$ of the direct sum of the discrete series representations of G/H.

Theorem 2. *Let K be a maximal compact subgroup of G.*

a. $\mathscr{C}_{\mathrm{cusp}}(G/H) \subseteq \mathscr{C}_{\mathrm{ds}}(G/H)$.

b. $\mathscr{C}_{\mathrm{cusp}}(G/H) = \mathscr{C}_{\mathrm{ds}}(G/H)$ *if and only if no discrete series representation has a non-zero K-fixed vector. If π is a discrete series representation with a non-zero K-fixed vector, then the subrepresention $\mathscr{C}_\pi(G/H)$ of $\mathscr{C}(G/H)$ corresponding to π is orthogonal (with respect to the $L^2(G/H)$-inner product) to $\mathscr{C}_{\mathrm{cusp}}(G/H)$. Moreover π occurs with multiplicity 1 in the Plancherel decomposition of $L^2(G/H)$.*

c. *Let ϕ be K-finite. Then $\phi \in \mathscr{C}_{\rm ds}(G/H)$ if and only if for every $g \in G$ the map*

$$A_P/(A_P \cap H) \ni a \mapsto a^{\rho_P - \rho_{P,\mathfrak{h}}} \mathcal{R}_P \phi(ga)$$

is of exponential polynomial type with non-zero exponents (where the zero function is considered to be of this form).

These results generalize to a large extend the results from [1] for real hyperbolic spaces.

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Quotients by conjugation action, cross-sections, singularities, and representation rings

Vladimir L. Popov

Below all algebraic varieties are taken over an algebraically closed field k. We use the standard notation and conventions of [2].

Let G be a connected semisimple algebraic group, $G \neq \{e\}$. Let $\pi_G \colon G \to G/\!\!/G$ be a categorical quotient for the conjugating action of G on itself, i.e., $G/\!\!/G$ is an affine variety and π_G is a dominant morphism such that $\pi_G^*\big(k[G/\!\!/G]\big)$ is the algebra $k[G]^G$ of class functions on G. In fact, π_G is surjective and, by [9], every fiber of π_G contains only finitely many conjugacy classes, one of which (called regular) is open and dense, and regular classes in general position are closed.

In 1965 R. Steinberg proved the following

Theorem 1 ([9])**.** *If G is simply connected, then*

(i) $G/\!\!/G \cong \mathbf{A}^r$*, where* $r := \operatorname{rk} G$;
(ii) *there is a section* $\sigma \colon G/\!\!/G \to G$ *of* π_G*, i.e., a morphism such that* $\pi_G \circ \sigma = \mathrm{id}_{G/\!\!/G}$;
(iii) *the cross-section* $\sigma(G/\!\!/G)$ *intersects every regular conjugacy class in* G.

A. Grothendieck in a letter of January 15, 1969 to J.-P. Serre [4, pp. 240–241] asked several questions related to Theorem 1. One of them concerns validity of (ii) and (iii) if G is not simply connected. Theorem 2 below answers this question.

Namely, let $\tau \colon \widehat{G} \to G$ be the universal covering of G, i.e., a central isogeny such that $\widehat{G}$ is a simply connected semisimple algebraic group.

Theorem 2 ([5]). *Let G be a connected semisimple algebraic group.*

(i) *The following properties are equivalent:*
  (a) *G contains a cross-section of π_G;*
  (b) *the isogeny τ is bijective.*

(ii) *If $\sigma\colon G/\!\!/G \to G$ is a section of π_G, then $\sigma(G/\!\!/G)$ intersects every regular conjugacy class and does not intersect other conjugacy classes.*

Remark 3. The isogeny τ is bijective if and only if it is either an isomorphism or purely inseparable (radical). The latter holds if and only if $\operatorname{char} k = p > 0$ and p divides the order of the fundamental group of G.

One can prove (see [5]) that if $\operatorname{char} k = 0$ and a surjective morphism $\alpha\colon X \to Y$ of irreducible varieties admits a section, then smoothness of X implies smoothness of Y. This shows that there is a connection between the existence of a cross-section in G and smoothness of $G/\!\!/G$. This, in turn, prompts to explore smoothness of $G/\!\!/G$. The answer is given by

Theorem 4 ([10, 7, 8, 5]). *The following properties are equivalent*:

(i) *$G/\!\!/G$ is smooth*;
(ii) *$G/\!\!/G$ is isomorphic to the affine space $\mathbf{A}^r$;*
(iii) *$G = G_1 \times \cdots \times G_s$ where every G_i is either a simply connected simple algebraic group or isomorphic to $\mathbf{SO}_{n_i}$ for an odd n_i.*

This criterion of smoothness of $G/\!\!/G$ may be interpreted as that of the existence of r generators of the algebra of class functions on G. It appears that it is possible to describe a minimal system of generators of this algebra and singularities of $G/\!\!/G$ in the general case. This also yields a minimal system of generators of the representation ring of G.

Namely, fix a choice of Borel subgroup B of G and maximal torus $T \subset B$. Let $\mathrm{X}(T)$ be the character lattice of T in additive notation, and let $\mathcal{C}$ be the Weyl chamber in $\mathrm{X}(T) \otimes_{\mathbf{Z}} \mathbf{R}$ determined by B.

The additive monoid $\mathcal{D} := \mathcal{C} \cap \mathrm{X}(T)$ is the set of highest weights of all simple algebraic G-modules (regarding T and B). This monoid is finitely generated and therefore determines the affine toric variety $Y_{\mathcal{C}} := \operatorname{Spec}(k \otimes_{\mathbf{Z}} \mathbf{Z}[\mathcal{D}])$ of the torus T, where $\mathbf{Z}[\mathcal{D}]$ is the semigroup ring of $\mathcal{D}$ over $\mathbf{Z}$.

Theorem 5 ([5]). *$G/\!\!/G$ and $Y_{\mathcal{C}}$ are isomorphic varieties.*

Corollary 6 ([5]). *There is an algebraic action of T on $G/\!\!/G$ with a dense open orbit.*

The next corollary answers the second Grothendieck's question from [4].

Corollary 7 ([5]). *$G/\!\!/G$ is a rational variety.*

If $\varpi \in \mathcal{D}$ and $\mathrm{E}(\varpi)$ is a simple G-module with ϖ as the highest weight, let $\mathrm{ch}_{\varpi} \in k[G]^G$ be the character of $\mathrm{E}(\varpi)$. Let $R(G)$ be the representation ring of G over $\mathbf{Z}$ and let $[\mathrm{E}(\varpi)]$ be the class of $\mathrm{E}(\varpi)$ in $R(G)$.

Theorem 8 ([5]). *The rings $R(G)$ and $\mathbf{Z}[\mathcal{D}]$ are isomorphic.*

Remark 9. In fact, in [5] are explicitly described the isomorphisms $G/\!\!/G \to Y_{\mathcal{C}}$ and $R(G) \to \mathbf{Z}[\mathcal{D}]$, and the action of T on $G/\!\!/G$ specified in Corollary 6.

The (finite) set $\mathcal{H} := \mathcal{D}_+ \setminus (\mathcal{D}_+ + \mathcal{D}_+)$, where $\mathcal{D}_+ := \mathcal{D} \setminus \{0\}$, is the unique minimal generating set (called the Hilbert basis) of $\mathcal{D}$.

Theorem 10 ([5])**.**

(i) *The cardinality of every generating set of $k[G]^G$ is not less than the cardinality of $\mathcal{H}$. The same holds for every generating set of $R(G)$.*
(ii) $\{[E(\varpi)] \mid \varpi \in \mathcal{H}\}$ *is a generating set of $R(G)$.*
(iii) $\{\mathrm{ch}_\varpi \mid \varpi \in \mathcal{H}\}$ *is a generating set of $k[G]^G$.*

Geometrically, $\mathcal{H}$ is the set of weight of T in the Zariski cotangent space of $G/\!\!/G$ at the fixed point of the action of T on $G/\!\!/G$ specified in Corollary 6.

If τ is separable, i.e., $G = \widehat{G}/Z$ for a subgroup Z of the center $\widehat{C}$ of $\widehat{G}$, then $G/\!\!/G$ may be identified with the quotient $\mathbf{A}^r/Z$ for an explicitly described action of Z on $\mathbf{A}^r$. This yields a complete description of $\mathcal{H}$ and the singularities of $G/\!\!/G$.

Namely, the action of $\widehat{C}$ on $\widehat{G}$ by translations descends to $\widehat{G}/\!\!/\widehat{G}$, the group $\widehat{C}$ becomes a subgroup of $\mathrm{Aut}\,\widehat{G}/\!\!/\widehat{G}$, and the following holds:

Theorem 11 ([5])**.** *There exists an isomorphism $\varphi\colon \widehat{G}/\!\!/\widehat{G} \to \mathbf{A}^r$ such that after the identification of $\widehat{G}/\!\!/\widehat{G}$ and $\mathbf{A}^r$ by means of φ*

(i) *$\widehat{C}$ becomes a subgroup of the standard r-dimensional diagonal torus*;
(ii) *$x_1^{m_1} \cdots x_r^{m_r} \in k[\mathbf{A}^r]^Z \Leftrightarrow m_1\varpi_1 + \cdots + m_r\varpi_r \in \mathcal{D}$, where $x_1, \ldots, x_r$ are the standard coordinate functions on $\mathbf{A}^r$ and $\varpi_1, \ldots, \varpi_r$ are the fundamental dominant with the Bourbaki numbering.*

The subgroup $\widehat{C}$ of the standard r-dimensional diagonal torus is explicitly described in [5] for every simple $\widehat{G}$.

Example 12. Let $\widehat{G}$ be of type E_7. Then $\widehat{C} = \{\mathrm{diag}(1, t, 1, 1, t, 1, t) \mid t^2 = 1\}$. Hence $\widehat{C} \neq \{e\} \Leftrightarrow |\widehat{C}| = 2 \Leftrightarrow \mathrm{char}\, k \neq 2$. Let $\mathrm{char}\, k \neq 2$ and $Z = \widehat{C}$ (i.e., G is adjoint). Then $k[G/\!\!/G] \cong k[\mathbf{A}^7]^{\widehat{C}}$ and the minimal generating system of monomials of this algebra consists of x_1, x_3, x_4, x_6 and the monomial of degree 2 in x_2, x_5, x_7. Hence $\mathcal{H} = \{\varpi_1, \varpi_3, \varpi_4, \varpi_6, 2\varpi_2, 2\varpi_5, 2\varpi_7, \varpi_2 + \varpi_5, \varpi_2 + \varpi_7, \varpi_5 + \varpi_7\}$ and $G/\!\!/G$ is isomorphic to $\mathbf{A}^4 \times Y$, where Y is the affine cone over the Veronese variety $\nu_2(\mathbf{P}^2)$.

The next theorem answers the third Grothendieck's question from [4].

Theorem 13 ([5])**.** *There is a rational section of π_G, i.e., a rational map*

$$\delta\colon G/\!\!/G \dashrightarrow G$$

such that $\pi_G \circ \delta = \mathrm{id}_{G/\!\!/G}$;

One can show that for $\mathrm{char}\, k = 0$ the existence of a rational section of π_G is equivalent to unirationality of $k(G)$ over $k(G)^G$. Theorem 13 then naturally leads to the Lüroth type problem: is $k(G)$ rational over $k(G)^G$? For simple G of all types but G_2 the answer was obtained in [3], and for type G_2 in [1]. This yields

Theorem 14. *Let $\mathrm{char}\, k = 0$ and let G be simple. Then $k(G)$ is rational over $k(G)^G$ if and only if G is of type* A, C, *or* G_2.

Rationality of $k(G)$ over $k(G)^G$ is intimately related to constructing counter-examples to the Gelfand–Kirillov conjecture; see [6].

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Symmetry breaking for real rank one orthogonal groups and the Gross Prasad conjectures. Examples and Conjectures

Birgit Speh

The Gross Prasad conjectures concern the existence of H equivariant operators between irreducible tempered representations of Vogan packets of orthogonal groups G and of Vogan packets of orthogonal subgroups H.

Let $G = SO(p,q)$ and let U_G be an irreducible tempered representation of G on a Hilbert space, Suppose that $H = SO(p-1,q) \subset G$ and that U_H is an irreducible unitary tempered representations of H. The superscript ∞ denotes the representation on the Freshet space of C^∞–vectors. It is known that

$$\dim \mathrm{Hom}_H(U_G^\infty \otimes U_H^\infty, \mathbb{C}) \leq 1$$

for all irreducible representations U_G, U_H for pairs (G,H) of orthogonal groups. (Sun-Zhu [3] , Aizenbud-Gourevitch over other fields)

We consider a Vogan packet $\mathcal{F}_G$ of tempered representations which contains the spherical representations of G_0=SO(n,1) and a Vogan packet $\mathcal{F}_H$ of tempered spherical representations of $H =$ SO(n-1,1). Contained in $\mathcal{F}_G \otimes \mathcal{F}_H$ is the set of relevant pairs $\mathcal{F}_{rel}$ of representations. The B.Gross and D.Prasad conjectured [1] that in each family $\mathcal{F}_{rel}$ there is exactly one pair U_G, U_H of representations so that

$$\dim \mathrm{Hom}_H(U_G^\infty \otimes U_H^\infty, \mathbb{C}) \leq 1$$

The results of [2] imply

Theorem 1. *Fix the packet $\mathcal{F}_G$ and $\mathcal{F}_H$. Then $\mathcal{F}_{rel}$ contains at least one pair U_G, U_H of representations in $\mathcal{F}_{rel}$ so that*

$$dim\ Hom_H(U_G^\infty \otimes U_H^\infty, \mathbb{C}) \leq 1$$

If $n < 6$ there is is exactly one such pair in one pair U_G, U_H in $\mathcal{F}_{rel}$.

Remark: The conjecture was proved by C. Moeglin and J-L Waldspurger for tempered representations over an non archimedean field. [3]

Some computations for small n suggest that it may be possible to extend the conjecture to families of other representations for example to the cohomologically induced representation with nontrivial cohomology [5]. Here it is important to find the right generalization of Vogan packets and relevant pairs.

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Regular Functions of Classical Nilpotent Orbits and Quantization

Kayue Daniel Wong

For any nilpotent orbit $\mathcal{O}$ in a complex semisimple Lie algebra $\mathfrak{g}$, it is known that the normality of its Zariski closure $\overline{\mathcal{O}}$ is related to the regular functions of $\mathcal{O}$. Namely, denoting the ring of regular functions of a variety X as $R(X)$, then

$$R(\overline{\mathcal{O}}) \subsetneq R(\mathcal{O}) \iff \overline{\mathcal{O}} \text{ } is\ NOT\ normal$$

On the other hand, if G is classical, Kraft and Procesi gave a combinatorial criterion (KP) on the normality of $\overline{\mathcal{O}}$. To relate these two conditions, we give an algorithm computing the multiplicities $[R(\mathcal{O}) : V_i]$ of some 'small' representations V_i appearing in $R(\mathcal{O})$ for special classical nilpotent orbits, and conclude that

$$(KP)\ fails \iff [R(\overline{\mathcal{O}}) : V_i] < [R(\mathcal{O}) : V_i]\ for\ some\ i \iff \overline{\mathcal{O}}\ is\ NOT\ normal.$$

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On irreducible representations of $\mathbf{GL_{2n}(F)}$ with a symplectic period

Arnab Mitra

In 1984, A.A. Klyachko [2] defined a model for irreducible representations of $\mathrm{GL}_n(\mathbb{F}_q)$ which was later introduced in the p-adic setting by M.J. Heumos and S. Rallis [1]. Among other things, they classified all irreducible unitary representations of $\mathrm{GL}_4(F)$, where F is a non archimedean local field. In a series of papers ([4], [5], [6]), O. Offen and E. Sayag resolved the complete picture for the unitary case. In particular, given any Klyachko model, they classified all irreducible unitary representations that embed in it.

It is a natural task now to classify all irreducible admissible representations embedding in a given Klyachko model. In this talk, we deal with the case of the symplectic model. In other words, we want to classify all irreducible representations of $\mathrm{GL}_{2n}(F)$ which have a non trivial $\mathrm{Sp}_{2n}(F)$-invariant linear functional. We give the answer in the small rank cases of $\mathrm{GL}_4(F)$ and $\mathrm{GL}_6(F)$.

Explicitly, following is the complete list of irreducible admissible representations θ of $\mathrm{GL}_4(F)$ with a symplectic period:
1) $\theta = Z([\sigma_2, \nu\sigma_2])$ where σ_2 is a cuspidal representation of $\mathrm{GL}_2(F)$.

2) $\theta = Z(\Delta_1, \Delta_2)$ where $\Delta_1 = [\chi_1\nu^{-1/2}, \chi_1\nu^{1/2}]$ and $\Delta_2 = [\chi_1\nu^{-3/2}, \chi_1\nu^{-1/2}]$. ($\chi_1$ is a character of $F^\times$)
3) $\theta =$ character of $\mathrm{GL}_4(F)$.
4) $\theta = \chi_2 \times \chi_2^{'}$ where $\chi_2, \chi_2^{'}$ are characters of $\mathrm{GL}_2(F)$.

Now we come to the case of $\mathrm{GL}_6(F)$. Following is the complete list of irreducible admissible representations in this case with a symplectic period:
1) The irreducible representations of the form $\chi_2 \times \theta_4$ where χ_2 is a character of $\mathrm{GL}_2(F)$ and θ_4 is an irreducible representation of $\mathrm{GL}_4(F)$ having a symplectic period.
2) $\chi_1 Z([1,\nu],[\nu,\nu^2],[\nu^2,\nu^3])$ where χ_1 is a character of $F^\times$.
3) Any character of $\mathrm{GL}_6(F)$.
4) $Z([\sigma_3, \nu\sigma_3])$ where σ_3 is a cuspidal representation of $\mathrm{GL}_3(F)$.
5) $\chi_1 Z([1,\nu],[\nu,\nu^4])$ and its contragredient where χ_1 is a character of $F^\times$.

In [3], E. Lapid and A. Minguez introduced a class of representations called the ladder representations. Call a representation of the form $Z([\nu^{a_1}\rho, \nu^{b_1}\rho], ..., [\nu^{a_t}\rho, \nu^{b_t}\rho])$ (or $Q([\nu^{a_1}\rho, \nu^{b_1}\rho], ..., [\nu^{a_t}\rho, \nu^{b_t}\rho])$), ladder if

$$a_1 < a_2 < ... < a_t \text{ and } b_1 < b_2 < ... < b_t.$$

Denote the set of all ladder representations by $\mathcal{L}'$. Further define,

$$\mathcal{L} = \{\pi_1 \times \cdots \times \pi_r |\ \pi_1, ..., \pi_r \in \mathcal{L}^{'} \text{ and the product is irreducible}\}.$$

Define a family of representations $\mathfrak{G}^{'}$ to consist of all representations of the form $Q(\Delta_1, ..., \Delta_{2r})$ in $\mathcal{L}'$ such that $Q(\Delta_{2i+1}, \Delta_{2i+2})$ is essentially Speh (for all $0 \le i \le r-1$). Let $\theta \in \mathfrak{G}^{'}$. Then we conjecture that θ has a symplectic period. In the same vein as $\mathcal{L}$, define $\mathfrak{G}$ to be the class of representations obtained by taking irreducible products of representations of $\mathfrak{G}^{'}$. We know that if $\theta_1, ..., \theta_d$ are distinguished then $\theta_1 \times \cdots \times \theta_d$ is distinguished. If the above conjecture is true, then this will further say that the class $\mathfrak{G}$ is distinguished.

We further conjecture that if $\pi_1, ..., \pi_r \in \mathcal{L}'$ such that $\pi_1 \times \cdots \times \pi_r$ is irreducible and distinguished then each π_i is distinguished. These conjectures are the subject of an ongoing joint project with O. Offen and E. Sayag. We can show that nothing outside $\mathfrak{G}^{'}$ is distinguished in the class $\mathcal{L}'$ so the above two conjectures give a complete classification of irreducible distinguished representations in the class $\mathcal{L}$.

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Approximation of L^2-invariants of locally symmetric spaces

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(joint work with Jonathan Pfaff)

L^2-invariants are spectral invariants associated to the universal covering $\widetilde{X}$ of a closed Riemannian manifold X. This is only of interest, if $\widetilde{X}$ is non-compact. L^2-invariants are the L^2-analogues of usual spectral invarinats of compact Riemannian manifolds. Examples are the L^2-Betti numbers, the Γ-index theorem of Atiyah and the L^2-analytic torsion. The whole subject started with Atiyah's paper [1]. Lott [3] and Mathai [6] introduced the L^2-analytic torsion. Lück [4] has studied L^2-invariants to a great extend.

Recall the definition of some of these invariants. By the Hodge-de Rham theorem, the p-the Betti number $b_p(X)$ of X equals the dimension of the space of harmonic p-forms $\mathcal{H}^p(X)$. The L^2-Betti number $b_p^{(2)}(X)$ is then defined as the von Neumann dimension of the space of $\mathcal{H}^p_{(2)}(\widetilde{X})$ of L^2 harmonic p-forms on $\widetilde{X}$. More precisely, let $F \subset \widetilde{X}$ be a fundamental domain for the action of $\Gamma := \pi_1(X, x_0)$ on $\widetilde{X}$. Let $P\colon L^2\Lambda^p(\widetilde{X}) \to \mathcal{H}^p_{(2)}(\widetilde{X})$ be the orthogonal projection. It is given by a smooth kernel $p(x, y)$. Then $b_p^{(2)}(X) := \int_F \operatorname{tr} p(x, x)\, dx$. Let $D\colon C^\infty(X, E) \to C^\infty(X, F)$ be an elliptic differential operator and let $\widetilde{D}$ be the lift of D to $\widetilde{X}$. Then the Γ-index $\operatorname{Ind}_\Gamma(\widetilde{D})$ is defined in a similar way using the von Neumann dimension of the kernel and cokernel of $\widetilde{D}$. The definition of the analytic torsion and its L^2 counterpart is more subtle. It involves a flat vector bundle $E \to X$, associated to a finite-dimensional representation $\rho\colon \Gamma \to \mathrm{GL}(V)$. Choose a Hermitian fibre metric h in E and let $\Delta_p(\rho)$ be the Laplace operator in the space of E-valued p-forms $\Lambda^p(X; E)$. The operator$\Delta_p(\rho)$ is essentially selfadjoint and non-negative. Its spectrum consisits of a sequence of eigenvalues $0 \le \lambda_1 \le \lambda_2 \le \cdots \to \infty$ of finite multiplicity. Let $b_p(\rho) := \dim\ker \Delta_p(\rho)$. Let $e^{-t\Delta_p(\rho)}$ be the heat operator. Then the zeta function $\zeta_p(s; \rho)$ of $\Delta_p(\rho)$ can be defined by

$$\zeta_p(s;\rho) := \frac{1}{\Gamma(s)} \int_0^\infty (\operatorname{Tr}(e^{-t\Delta_p(\rho)}) - b_p(\rho)) t^{s-1}\, dt, \quad \operatorname{Re}(s) > n/2,$$

where $n = \dim X$. Using the asymptotic expansion of $\operatorname{Tr}(e^{-t\Delta_p(\rho)})$ as $t \to 0$, it follows that $\zeta_p(s; \rho)$ admits a meromorphic continuation to whole complex plane and is holomorphic at $s = 0$. Then the Ray-Singer analytic torsion $T_X(\rho) \in \mathbb{R}^+$ is defined as

$$\log T_X(\rho) = \frac{1}{2}\sum_{p=1}^{n}(-1)^p p \frac{d}{ds}\zeta_p(s;\rho)\big|_{s=0}. \tag{1}$$

(see [11]). Let $\widetilde{E} \to \widetilde{X}$ be the lift of E to $\widetilde{X}$ and let $\widetilde{\Delta}_p(\rho)$ be the Laplacian on $\widetilde{E}$-valued p-forms on $\widetilde{X}$. Let $e^{-t\widetilde{\Delta}_p(\rho)}$ be the associated heat operator. It is an integral operator with smooth kernel $K_p^\rho(t,x,y)$. Then the Γ-trace of $e^{-t\widetilde{\Delta}_p(\rho)}$ is defined as

$$\mathrm{Tr}_\Gamma(e^{-t\widetilde{\Delta}_p(\rho)}) := \int_F \operatorname{tr} K_p^\rho(t,x,x)\,dx.$$

It can be shown that $\mathrm{Tr}_\Gamma(e^{-t\widetilde{\Delta}_p(\rho)})$ admits an asymptotic expansion as $t \to 0$ [3], [6]. To define the Mellin transform of $\mathrm{Tr}_\Gamma(e^{-t\widetilde{\Delta}_p(\rho)})$ one also has to know its behavior as $t \to \infty$. This leads to the definition of the Novikov-Shubin invariants. Let $\widetilde{\Delta}_p(\rho)'$ be the restriction of $\widetilde{\Delta}_p(\rho)$ to the orthogonal complement of the kernel of $\widetilde{\Delta}_p(\rho)$. Then the p-th Novikov-Shubin invariant is defined to be

$$\alpha_p(X,\rho) := \sup\left\{\beta\colon\ \mathrm{Tr}_\Gamma(e^{-t\widetilde{\Delta}_p(\rho)'}) = O(t^{-\frac{\beta}{2}}) \ \text{ as } \ t\to\infty\right\} \in [0,\infty].$$

If $\alpha_p(X,\rho) > 0$ for all $p = 0,\dots,n$, the L^2-torsion $T_X^{(2)}(\rho)$ can be defined by a formula analogues to (1) (see [3],[6]). For compact locally symmetric spaces $\Gamma\backslash G/K$ and the trivial representation ρ, L^2-invariants have been studied by Olbrich [8]. Especially, he has shown that the Novikov-Shubin invariants are all positive and therefore, the L^2 tosion is defined.

Now assume that Γ is residual finite, i.e., there exists a tower $\Gamma \supset \Gamma_1 \supset \Gamma_2 \supset \cdots$ of normal subgroups of finite index with trivial intersection. Let $X_k := \Gamma_k\backslash\widetilde{X}$. Let α be some spectral invariant. The approximation of L^2-invariants is the following problem: *Does* $\alpha(X_k)/[\Gamma\colon\Gamma_k]$ *converge and if so, what is the limit?*

A natural candidate is the corresponding L^2-invariant $\alpha^{(2)}$. This problem has been studied by Lück to a great extend. Especially, he has answered this question affirmatively for Betti numbers. By [5, Theorem 1], one has

$$\lim_{k\to\infty}\frac{b_p(X_k)}{[\Gamma:\Gamma_k]} = b_p^{(2)}(X).$$

The case of the analytic torsion has been studied in [2]. The representation has to be strongly acyclic. This means that there exists a uniform positive lower bound of the spectrum of all Laplacians $\Delta_{X_k,p}(\rho)$, $k\in\mathbb{N}$, $p = 0,\dots,n$. It is proved in [2] that such representations exists. They are obtained as restriction to Γ of irreducible finite-dimensional representations of G. Let $\Gamma\backslash G/K$ be a compact locally symmetric manifold. Let $\Gamma_k \le \Gamma$ be a sequence of uniform lattices in G and let $X_k = \Gamma_k\backslash\widetilde{X}$. Assume that $\mathrm{vol}(X_k)\to\infty$ as $k\to\infty$. Let $\rho\colon\Gamma\to\mathrm{GL}(V)$ be strongly acyclic. Then it is proved in [2] that

$$\lim_{k\to\infty}\frac{\log T_{X_k}(\rho)}{[\Gamma\colon\Gamma_k]} = \log T_X^{(2)}(\rho). \tag{2}$$

Since X is locally symmetric, one has $\log T_{X_k}(\rho) = \operatorname{vol}(\Gamma\backslash G/K) t^{(2)}_{\widetilde{X}}(\rho)$, where $t^{(2)}_{\widetilde{X}}(\rho)$ depends only on G and ρ. It can be computed by the Plancherel theorem. If $\operatorname{rank}_{\mathbb{C}}(G) - \operatorname{rank}_{\mathbb{C}}(K) = 1$, it can be shown that $t^{(2)}_{\widetilde{X}}(\rho) \neq 0$. Using the equality of analytic torsion and Reidemeister torsion, proved by Cheeger and myself, (2) was used in [2] to study the growth of torsion in the cohomology of uniform arithmetic lattices.

In [7], this result has been extended to the case of finite volume hyperbolic manifolds. Let $G = \mathrm{SO}_0(d,1)$ and $K = \mathrm{SO}(d)$ with $d > 1$. Then $\widetilde{X} = G/K$ is the hyperbolic space of dimension d. Let $\Gamma_0 \subset G$ be lattice (discrete subgroup of finite co-volume) and let $\Gamma_i \subset \Gamma_0$ be any finite-index torsion free normal subgroup. Let $X_i := \Gamma_i\backslash\widetilde{X}$. Given a finite-dimensional representation $\tau\colon G \to \mathrm{GL}(V)$, denote by $T_{X_i}(\tau)$ the analytic torsion of X_i with respect to $\tau|_{\Gamma}$. Let $\theta\colon G \to G$ be the Cartan involution. Let $\tau_\theta = \tau \circ \theta$. Then one of the main results of [7] is the following theorem.

Theorem. *Let Γ_0 be a lattice in G and let Γ_i, $i \in \mathbb{N}$, be a sequence of finite-index normal subgroups which is cusp uniform and such that each Γ_i, $i \geq 1$, is torsion-free. If $\lim_{i\to\infty}[\Gamma_0 : \Gamma_i] = \infty$ and if each $\gamma_0 \in \Gamma_0 - \{1\}$ only belongs to finitely many Γ_i, then for each τ with $\tau \neq \tau_\theta$ one has*

$$\lim_{i\to\infty} \frac{\log T_{X_i}(\tau)}{[\Gamma : \Gamma_i]} = t^{(2)}_{\widetilde{X}}(\tau) \operatorname{vol}(X_0). \tag{3}$$

The 3-dimensional case has been studied by J. Raimbault [9], [10]. We expect that as in [2], (3) will have applications to the growth of torsion in the cohomoloy of arithmetic subgroups of G. A challenging problem is to remove the condition on τ.

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Adelic action on periods of automorphic representations and special values of L-functions

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(joint work with Joseph Bernstein)

We are interested in invariant functionals defined on automorphic representations via period integrals. We consider the action of an adelic subgroup on such an invariant functional. We show that in certain cases this action gives rise to another period integral, and this corresponds to a known relation between an automorphic period and a special value of an appropriate L-function. However, even in some of the simplest cases, we find that the relation to L-functions is more puzzling.

0.1. **Periods and special values of L -functions.** Periods play a central role in the modern theory of automorphic functions. In particular, there are instances when periods of automorphic functions are related to L-functions. Such a relation goes back to the foundational work of E. Hecke [3], where he constructed the Hecke L-function on GL(2) as the period integral along the split torus in GL(2). This is the most basic of "period to L-function" relations. Another striking example was discovered by J.-L. Waldspurger [7] and connects the period along a non-split torus in GL(2) to the special value of an L-function of the appropriate base change lift. We also mention the vast generalization of the Waldspurger's result formulated as a conjecture by B. Gross and D. Prasad [2]. Consequently, the exact form of the Gross-Prasad period relation was conjectured by A. Ichino and T. Ikeda [4]. This led to other formulas relating normalized periods and L-functions (e.g., an analog for the Whittaker functional was considered in [5]). A general framework for period formulas in the context of Plancherel measures was recently proposed by Y. Sakellaridis and A. Venkatesh [6].

Our main aim is to try to reformulate (at least part of) the Ichino-Ikeda approach in terms of representation theory without appealing to L-functions directly (in fact the original paper [7] already contains the idea we are trying to expand). By doing so we are able to treat periods which, as it seems to us, do not fall in the familiar framework and for which a relation to L-functions is more puzzling. Namely, our construction leads to an Euler product with non-standard local factors which nevertheless could be regularized with the help of an appropriate L-function.

0.2. **Action on periods.** We are interested in the following setup. Let G be an algebraic (reductive) group over a global field k (in practice we will assume $k = \mathbb{Q}$ for simplicity), and let $H, F \subset G$ be two algebraic subgroups of G also defined over k (e.g., a split over k torus and an associated unipotent subgroup in $G = \mathrm{GL}(2)$). Let $G(\mathbb{A}), H(\mathbb{A}), F(\mathbb{A})$ be the corresponding adele groups, and we denote by $X_G = G(k)\backslash G(\mathbb{A}), X_H = H(k)\backslash H(\mathbb{A}), X_F = F(k)\backslash F(\mathbb{A})$ the corresponding automorphic quotient spaces. Let π be an automorphic representation of G (we will be vague at this point of what is required of π). We are interested in the period functional given by the integral $p_H(\phi) = \int_{X_H} \phi(h)dh$ over the $H(\mathbb{A})$-orbit $X_H \subset X_G$ of an automorphic function ϕ belonging to the space of the representation π (and similarly for the period p_F for X_F). This of course requires a choice of (invariant) measures and imposes certain restrictions on π and X_H. Assuming that all periods are defined, it is natural to ask if there is a relation between functionals p_H and p_F which are defined on the same automorphic representation π. Periods p_H and p_F define functionals on π, and one possibility would be to compute their correlation (i.e., the scalar product, if it is defined of course). In fact it is possible in many cases (see [1]), but we found it a little bit easier to make another comparison in terms of the action of adelic groups. Namely, we can try to integrate the functional p_H with respect to the action of the adelic group $F(\mathbb{A})$. Assuming that such an operation is well-defined, we would obtain an $F(\mathbb{A})$-invariant functional $\tilde{p}_F = \int_{f\in F(\mathbb{A})} \pi^\vee(f)p_H \; df$ on π (i.e., $\tilde{p}_F(v) = \int_{f\in F(\mathbb{A})}\int_{x\in X_H} v(xf)dxdf$ for any smooth vector v in the representation π). This does not identify such a functional in general, but in the case when $F(\mathbb{A})$ is a Gelfand subgroup of $G(\mathbb{A})$ (i.e., the space of $F(\mathbb{A})$-invariant functionals on π is at most one-dimensional), we should get a functional which is proportional to the period functional p_F (again, assuming the integration procedure makes sense!). What we found is that the above mentioned "classical" period to L-function formulas allows one to compute the coefficient of proportionality between $\tilde{p}_F$ and p_F in some cases. Moreover, we find the "L-functions free" formulation of this relation between periods even more interesting. Such a reformulation allows us to consider cases where the relation to L-functions is somewhat more mysterious.

0.2.1. *Idea of the construction.* We will work only with periods satisfying the local uniqueness property (and hence also satisfying global uniqueness). For a place $\mathfrak{p}$ of k, we consider local groups $G_\mathfrak{p}, H_\mathfrak{p}, F_\mathfrak{p}$ (i.e., a group of points over a local field $k_\mathfrak{p}$). For $H \subset G$ as above and irreducible representations $\pi = \hat{\otimes}\pi_\mathfrak{p}$ of $G(\mathbb{A})$ and $\sigma = \hat{\otimes}\sigma_\mathfrak{p}$ of $H(\mathbb{A})$, we consider the complex vector space $P(\pi,\sigma) = \mathrm{Hom}_{H(\mathbb{A})}(\pi,\sigma)$ and its local counterparts $P_\mathfrak{p}(\pi_\mathfrak{p},\sigma_\mathfrak{p}) = \mathrm{Hom}_{H_\mathfrak{p}}(\pi_\mathfrak{p},\sigma_\mathfrak{p})$ (in fact it would be more appropriate to consider spaces $\mathrm{Hom}_{H_\mathfrak{p}}(\pi_\mathfrak{p} \otimes \sigma_\mathfrak{p}^\vee, \mathbb{C})$, but in examples considered in this paper σ will be one-dimensional and hence these spaces are canonically isomorphic). We call a tuple $(G_\mathfrak{p}, \pi_\mathfrak{p}, H_\mathfrak{p}, \sigma_\mathfrak{p})$ local Gelfand data (or a multiplicity one tuple) if $\dim P_\mathfrak{p}(\pi_\mathfrak{p},\sigma_\mathfrak{p}) \leq 1$. In such a case we have $P(\pi,\sigma) = \hat{\otimes}P_\mathfrak{p}(\pi_\mathfrak{p},\sigma_\mathfrak{p})$, and the global space of maps is also at most one-dimensional. We call the tuple (G,π,H,σ) globally Gelfand.

Let (G,π,H,σ) and (G,π,F,τ) be two Gelfand tuples and $P(\pi,\sigma)$, $P(\pi,\tau)$ the corresponding *one-dimensional* complex vector spaces. Our goal is to construct a canonical map

$$I : P(\pi,\sigma) \to P(\pi,\tau) \; . \tag{1}$$

between these one-dimensional vector spaces in the presence of the corresponding *automorphic* periods. We do this in two steps.

First step is purely local. It is relatively easy to construct *local* maps $I_\mathfrak{p}$: $P_\mathfrak{p}(\pi_\mathfrak{p},\sigma_\mathfrak{p}) \to P_\mathfrak{p}(\pi_\mathfrak{p},\tau_\mathfrak{p})$ between local spaces using the integration along the subgroup $F_\mathfrak{p} \subset G_\mathfrak{p}$. Namely, if $\tau_\mathfrak{p}$ is a character, for a given vector $\xi_\mathfrak{p} \in P_\mathfrak{p}(\pi_\mathfrak{p},\sigma_\mathfrak{p})$ we define a vector $I_\mathfrak{p}(\xi_\mathfrak{p}) \in P_\mathfrak{p}(\pi_\mathfrak{p}, \tau_\mathfrak{p})$ by $I_\mathfrak{p}(\xi_\mathfrak{p}) := \int_{F_\mathfrak{p}} \tau_\mathfrak{p}^{-1}(f_\mathfrak{p})\pi_\mathfrak{p}^\vee(f_\mathfrak{p})(\xi_\mathfrak{p})df_\mathfrak{p}$, where $\pi_\mathfrak{p}^\vee$ denotes the natural representation of $G_\mathfrak{p}$ on $V^*_{\pi_\mathfrak{p}}$. The integral is understood in a weak sense. This means that for any smooth vector $v_\mathfrak{p} \in V_{\pi_\mathfrak{p}}$, we have $I_\mathfrak{p}(\xi_\mathfrak{p})(v_\mathfrak{p}) = \int_{F_\mathfrak{p}} \tau_\mathfrak{p}^{-1}(f_\mathfrak{p})\xi_\mathfrak{p}(\pi_\mathfrak{p}(f_\mathfrak{p})v_\mathfrak{p})df_\mathfrak{p}$. The last integral might be divergent, but in many cases could be evaluated by a standard procedure (usually involving analytic continuation). We stress that local maps are assumed to be defined *canonically* for *all* $\mathfrak{p}$ (i.e., the local map does not depend on any choices we have to make neither on parameters of local representations $\pi_\mathfrak{p}$, $\tau_\mathfrak{p}$, $\sigma_\mathfrak{p}$).

The next step is to “glue” local maps $I_\mathfrak{p}$ to a global map. This is a more subtle procedure. We construct the global map I by regularizing the tensor product $\otimes I_\mathfrak{p}$ of local maps with the help of appropriate weight factors. This is possible only for local maps which are coming from automorphic periods, and the weight factors are provided by the theory of automorphic L-functions. This construction of the map I (in certain cases) is the main observation of this paper.

Let us describe this process in more detail. Let ξ be a vector in $P(\pi, \sigma)$. We can write it as a product $\xi = \otimes_\mathfrak{p}\xi_\mathfrak{p}$, where $\xi_\mathfrak{p} \in P_\mathfrak{p}(\pi_\mathfrak{p}, \sigma_\mathfrak{p})$ and for almost all $\mathfrak{p}$ we have $\xi_\mathfrak{p}(e^0_\mathfrak{p}) = 1$ for the standard vector $e^0_\mathfrak{p} \in V_{\pi_\mathfrak{p}}$. Given a decomposable vector $v \in V_\pi$, we can write it in a form $v = \otimes_\mathfrak{p} v_\mathfrak{p}$, where $v_\mathfrak{p} = e^0_\mathfrak{p}$ for almost all $\mathfrak{p}$. Now we would like to set $I(v) = \prod_\mathfrak{p} d_\mathfrak{p}$, where $d_\mathfrak{p} := I_\mathfrak{p}(\xi_\mathfrak{p})(v_\mathfrak{p})$. This product is usually not convergent. But we can use the fact that outside of a finite number of places the coefficients $d_\mathfrak{p}$ can be explicitly computed using an unramified computation. The unramified factors do not depend on a choice of the vector v. The result of the unramified computation allows us to use the following regularization procedure. We find an appropriate automorphic L-function (or a ratio of several L-functions) with the partial Euler product $L_S(s) = \prod_{\mathfrak{p}\notin S} L_\mathfrak{p}(s)$ (here S is a finite set of primes and the Euler factors for all $\mathfrak{p} \notin S$ are some rational functions of $q_\mathfrak{p}^{-s}$) and find some complex number s_0 (usually in the region of the analytic continuation of $L(s)$ extension to which we will take for granted) such that if we replace for almost all $\mathfrak{p}$, coefficients $d_\mathfrak{p}$ by the normalized coefficients $d^0_\mathfrak{p} := d_\mathfrak{p} L_\mathfrak{p}(s_0)$ then the product $\prod_\mathfrak{p} d^0_\mathfrak{p}$ is absolutely convergent (this condition does not depend on a specific choice of the vector v). After this we define for a large enough finite set of (ramified) primes S,

$$I(v) := L_S(s_0)^{-1} \prod_{\mathfrak{p}\notin S} d^0_\mathfrak{p} \prod_{\mathfrak{p}\in S} d_\mathfrak{p} \ .$$

Here for $Re(s) \gg 1$, $L_S(s) = \prod_{\mathfrak{p}\notin S} L_\mathfrak{p}(s)$ is the partial L-function. It is clear that this procedure is well defined (at least after we fix the L-function and its Euler product expansion). We note that in some examples the unramified factor $d_\mathfrak{p}$ *does not* coincide with an Euler factor of a Langlands L-function.

Having constructed the map I we can ask what is the effect of it on automorphic periods. Namely, we can try to compare period functionals p_F and $\tilde{p}_F = I(p_H)$. The coefficient of proportionality (when defined) gives rise to an invariant of the automorphic representation π (for τ and σ fixed). When $\tilde{p}_F = p_F$ this invariant is equal to 1 (as we find in many classical examples using the “period to L-function” relation we mentioned above, e.g., for Hecke and Waldspurger periods). However, we find that sometimes the relation between $\tilde{p}_F$ and p_F is more complicated in terms of L-functions, and in particular functionals $\tilde{p}_F$ and p_F do *not* coincide. Hence it seems that sometimes this invariant is non-trivial and this leads to an interesting "period" invariant of an automorphic representation the nature of which is not clear to us.

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Hilbert series of Cauchy–Riemann filtrations

Benjamin Schwarz

Let (X,h) be a Kähler manifold, and let $\mathcal{E} \to X$ be a holomorphic vector bundle. Let $T^{1,0}$ and $T^{0,1}$ denote the holomorphic and anti-holomorphic tangent bundle on X. The composition of the $\overline{\partial}$-operator and the isomorphism h_* induced by the Riesz-isomorphism $(T^{0,1})^* \cong T^{1,0}$ given by h defines the *Cauchy–Riemann operator* $\bar{D}$,

$$C^\infty(X,\mathcal{E}) \xrightarrow{\ \overline{\partial}\ } C^\infty(X,\mathcal{E}\otimes(T^{0,1})^*) \xrightarrow{\ h_*\ } C^\infty(X,\mathcal{E}\otimes T^{1,0}), \qquad \bar{D} = h_* \circ \overline{\partial} \tag{1}$$

Since $\mathcal{E} \otimes T^{1,0}$ is again a holomorphic vector bundle, iterates of the Cauchy–Riemann operator are defined in the obvious way. By abuse of notation, we simply write

$$\bar{D}^m := \bar{D} \circ \cdots \circ \bar{D} : C^\infty(X,\mathcal{E}) \to C^\infty(X,\mathcal{E}\otimes(T^{1,0})^{\otimes m}).$$

The kernels $\mathcal{N}^m_{\mathcal{E}} := \ker \bar{D}^{m+1}$ of these higher order Cauchy–Riemann operators define the filtered vector space of so called *nearly holomorphic sections*,

$$\mathcal{N}_{\mathcal{E}} := \bigcup_{m\geq 0} \mathcal{N}^m_{\mathcal{E}} \quad \text{with} \quad \mathcal{N}^0_{\mathcal{E}} \subseteq \mathcal{N}^1_{\mathcal{E}} \subseteq \mathcal{N}^2_{\mathcal{E}} \subseteq \cdots .$$

We note that $\mathcal{N}_{\mathcal{E}}^0 = \mathcal{O}(X, \mathcal{E})$, the space of holomorphic sections. The concept of nearly holomorphic functions (associated to the trivial line bundle) was introduced by Shimura [10]. The corresponding Cauchy–Riemann operators have been generalized to the present form by Engliš and Peetre [2].

Even though Cauchy–Riemann operators and the associated nearly holomorphic sections are naturally defined in this general setting, they have been studied in detail almost exclusively in the case of Hermitian symmetric spaces. We report on progress in the study of Cauchy–Riemann operators and nearly holomorphic sections in the setting of generalized flag varieties. One of the first natural questions concerns the existence of non-trivial nearly holomorphic sections. On general compact Kähler manifolds, this is a highly non-trivial question. For flag varieties, representation theory provides an appropriate approach.

Let $X = G/P$ be the generalized flag manifold with G a complex simple simply-connected Lie group, and $P \subseteq G$ a parabolic subgroup, and let $\mathcal{E} = G \times^P E$ be the G-homogeneous holomorphic vector bundle corresponding a simple P-module E. We fix a maximal compact subgroup U in G and a U-invariant Kähler metric h on X. Then it is immediate from the definition that Cauchy–Riemann operators are U-equivariant. Moreover, since X is compact, it is known [9] that all $\mathcal{N}_{\mathcal{E}}^m$, $m \in \mathbb{N}$, are finite dimensional. It follows that each nearly holomorphic section is U-finite. We are able to show that nearly holomorphic sections exhaust all U-finite smooth sections.

Theorem 1. *Let $X = G/P$, and $\mathcal{E} = G \times^P E$ be the G-homogeneous vector bundle associated to a simple P-module E. Then,*

$$\mathcal{N}_{\mathcal{E}} = C^\infty(X, \mathcal{E})_{U-\mathit{finite}}.$$

In particular, $\mathcal{N}_{\mathcal{E}}$ is a dense subspace of $C^\infty(X, \mathcal{E})$ (with respect to uniform convergence).

We note that the right hand side of this relation is independent of the choice of the Kähler metric, however the filtration given by $\mathcal{N}_{\mathcal{E}}^\bullet$ depends on h. We call this the *Cauchy–Riemann filtration* of $C^\infty(X, \mathcal{E})_{U-\mathrm{finite}}$ associated to h. The second goal is to determine the corresponding Hilbert series,

$$\mathcal{H}(\mathcal{N}_{\mathcal{E}}^\bullet, q) := \sum_{m \geq 0} \mathrm{ch}(\mathcal{N}_{\mathcal{E}}^m / \mathcal{N}_{\mathcal{E}}^{m-1})\, q^m,$$

where $\mathrm{ch}(M)$ denotes the formal character of a U-module M. We first describe the result in the case that $P = B$ is a Borel subgroup of G. Choose a maximal torus $T \subseteq B$, and let $\Lambda \supseteq \Lambda^+$ denote the corresponding weight lattice and the set of dominant weights, respectively.

Theorem 2. *Let $X = G/B$ and $\mathcal{L}_\mu = G \times^B \mathbb{C}_\mu$ be the G-homogeneous line bundle associated to $\mu \in \Lambda$. If μ is dominant, then*

$$\mathcal{H}(\mathcal{N}_{\mathcal{L}_\mu^*}^\bullet, q) = \sum_{\lambda \in \Lambda^+} m_\lambda^\mu(q)\, \mathrm{ch}\, V_\lambda^*,$$

where $m_\lambda^\mu(q)$ is Lusztig's q-analog of Kostant's weight multiplicity formula.

Lusztig's polynomials $m^{\mu}_{\lambda}(q)$ are defined [7] in purely combinatorial terms by

$$m^{\mu}_{\lambda}(q) = \sum_{w\in W} \operatorname{sgn}(w)\, p_q(w * \lambda - \mu),$$

where W denotes the Weyl group, and p_q is the q-analog of Kostant's partition function, i.e., the coefficient of q^m in $p_q(\nu)$ is the number of ways to write $\nu \in \Lambda^+$ as the sum of precisely m positive roots. Lusztig's polynomials occur in various branches of representation theory. In the special case $\mu = 0$ (corresponding to the trivial line bundle), the polynomials $m^0_{\lambda}(q)$ were first constructed, independently, by Hesselink [5] and Peterson [8]. They discovered that the polynomials $m^0_{\lambda}(q)$ are the coefficients of the Hilbert series corresponding to the (graded) coordinate ring of the nilpotent cone in the Lie algebra $\mathfrak{g}$ of G. Prior to this, Kostant determined this Hilbert series in terms of generalized exponents by an investigation of G-harmonic polynomials on $\mathfrak{g}$, see [6]. For general μ, Lusztig and Kato proved that $m^{\mu}_{\lambda}(q)$ are closely related to certain Kaszdan–Lusztig polynomials, see [3], and hence encode deep combinatorial and geometric information.

We briefly indicate that for general parabolic $P \subseteq G$, Theorem 2 still holds, when $m^{\mu}_{\lambda}(q)$ is replaced by the parabolic version of Lusztig's polynomials, and if $\mathcal{E}_{\mu} = G \times^P E_{\mu}$ satisfies the following vanishing cohomology condition,

$$\text{(V)} \qquad H^k_{\text{alg}}((T^{1,0})^*, \pi^*\mathcal{E}^*_{\mu}) = 0 \quad \text{for} \quad k > 0.$$

Here, $\pi : (T^{1,0})^* \to X$ is the holomorphic cotangent bundle, and H^k_{alg} denotes cohomology in the algebraic category, so we consider $(T^{0,1})^*$ as complex algebraic variety. The vanishing condition (V) is essentially a condition imposed on the highest weight μ of the simple P-module E_{μ}, and (V) is known to hold for almost all dominant highest weights μ, see [1, 4]. We expect this to be true for all highest weights μ.

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On a conjecture of Sakellaridis-Venkatesh on the unitary spectrum of a spherical variety

Raul Gomez
(joint work with Wee Teck Gan, Chen-Bo Zhu)

In this talk, we will use the theory of dual pairs and θ-lifting of generalized Whittaker models to prove some cases of a conjecture announced in a recent preprint of Sakellaridis and Venkatesh. Furthermore, we will consider a family of examples that seem to suggest that the theory developed by Sakellaridis and Venkatesh may be extendable to a larger class of homogeneous spaces.

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Distributions on *p*-adic groups, finite under the action of the Bernstein center

Dmitry Gourevitch
(joint work with Avraham Aizenbud, Eitan Sayag)

Let $\mathbf{G}$ be a reductive group defined over a non-Archimedean local field F, let $G := \mathbf{G}(F)$ and let $\mathcal{S}(G)$ be the space of locally constant compactly supported functions on G. Let $\mathfrak{z} := \mathfrak{z}(G) := End_{G\times G}(\mathcal{S}(G))$ denote the Bernstein center (see [6]). The action of $\mathfrak{z}$ on $\mathcal{S}(G)$ gives rise to the dual action on the space of distributions $\mathcal{S}^*(G)$. In our recent work [2] we study $\mathfrak{z}$-finite distributions, i.e. distributions ξ such that $\dim \mathfrak{z}\xi < \infty$.

Our first result concerns the wave-front set of such distributions. For $x \in G$ let $WF_x(\xi)$ denote the intersection of the wave-front set of ξ with the cotangent space T^*_xG.

Theorem 1. *Suppose that F has characteristic zero. Let $\xi \in \mathcal{S}^*(G)$ be a $\mathfrak{z}$-finite distribution. Then for any $x \in G$ we have*

$$WF_x(\xi) \subset \mathcal{N} \tag{1}$$

where $\mathcal{N} \subset \mathfrak{g}^*$ is the nilpotent cone, and we identify $\mathfrak{g}$ with T_xG using the right action.

Let $H_1, H_2 \subset G$ be two (closed) subgroups and χ_i be characters of H_i. Consider the two-sided action of $H_1 \times H_2$ on G and let

$$\mathcal{I} := \mathcal{S}^*(G)^{(H_1 \times H_2, \chi_1 \times \chi_2)}$$

be the space of $(H_1 \times H_2, \chi_1 \times \chi_2)$-equivariant distributions on G.

We will require some conditions on the pairs (G, H_i). We will call pair (G, H) satisfying these conditions *a pair of finite type*. As we show, a spherical pair with finite multiplicities over a field of characteristic zero satisfies these conditions. It is conjectured that if $char F = 0$ then all spherical pairs have finite multiplicities. This is proven for many spherical pairs, including all symmetric pairs of reductive groups in [18, Theorem 5.1.5], and [7]. In what follows we assume that the pairs (G, H_i) are of finite type.

Theorem 2. *The space of $\mathfrak{z}(G)$-finite distributions in $\mathcal{I}$ is dense in $\mathcal{I}$.*

Important examples of $\mathfrak{z}$-finite distributions in $\mathcal{I}$ are $(H_1 \times H_2, \chi_1 \times \chi_2)$-spherical characters of admissible (finite length) representations. It turns out that those examples are exhaustive. Namely, we have the following proposition.

Proposition 3. *Any $\mathfrak{z}(G)$-finite distribution in $\mathcal{I}$ is an $(H_1 \times H_2, \chi_1 \times \chi_2)$-spherical character of some admissible representation of G.*

Together with Theorem 2 it implies

Corollary 4. *The space of $(H_1 \times H_2, \chi_1 \times \chi_2)$-spherical characters of admissible representations of G is dense in $\mathcal{I}$.*

In order to prove Proposition 3 we proved the following lemma:

Lemma 5. *Let (G, H) be a pair of finite type and χ be a character of H. Let $\xi \in \mathcal{S}^*(G)^{H,\chi}$ be an (H, χ)-equivariant $\mathfrak{z}(G)$-finite distribution. Then $V := \mathcal{S}(G) * \xi$ is an admissible representation of G.*

This lemma implies the following corollary:

Corollary 6. *Let $\xi \in \mathcal{S}^*(G)$ be a $\mathfrak{z}(G)$-finite distribution. Then $V := \mathcal{S}(G) * \xi * \mathcal{S}(G)$ is an admissible representation of $G \times G$.*

Theorem 1 provides a simple proof of the easy part of Harish-Chandra's regularity theorem [11, 14], namely the regularity of the character on the set of regular semi-simple elements. We generalize this result to the realm of spherical pairs. For that, we introduce the notion of $H_1 \times H_2$-cotoric elements and prove the following result.

Corollary 7. *Suppose that F has characteristic zero. Let $\xi \in \mathcal{I}$ be a $\mathfrak{z}$-finite distribution. Then ξ is smooth in the neighborhood of any $H_1 \times H_2$-cotoric element.*

This result generalizes the main result of [16, §5], since if $H_1 = H_2$ is a symmetric subgroup then the H-regular semisimple elements are cotoric.

1. Related results

The germ at the unit element of the character of an irreducible representation of G expands as a linear combination of Fourier transforms of invariant measures of nilpotent orbits. This was shown in [13] for $\mathbf{G} = \mathrm{GL}_n$ and in [10] for general $\mathbf{G}$. This cannot be naively generalized to the case of symmetric pairs, since the nilpotent orbital integrals are not defined for symmetric spaces in general. However, in [16, §7] it is shown that the germ at the unit element of a spherical character is a Fourier transform of a distribution supported on the nilpotent cone. Theorem 1 can be viewed as a version of these results, which gives less information but works in wider generality. Namely, it implies that the germ of any $\mathfrak{z}$-finite distribution, for example any spherical character of any spherical pair, is a Fourier transform of a distribution supported near the nilpotent cone.

Distributions coming from the representation theory are often $\mathfrak{z}$-finite. In the Archimedean case (where $\mathfrak{z}$ means the center of the universal enveloping algebra of the Lie algebra) this was widely exploited. For example it was used to prove the Harish-Chandra regularity theorem ([8, 9]), uniqueness of Whittaker models ([19]) and Kirillov's conjecture ([5]). Recently, it was used in [15] to prove uniqueness of Ginzburg-Rallis models and in [1] to show non-vanishing of Bessel-like functions. However, in the non-Archimedean case there were no tools that use finiteness of distributions under the Bernstein center. This work provides such a tool.

A classical result says that characters of admissible representations span a dense subspace of the space of conjugation-invariant distributions on G. One can view Corollary 6 as the relative counterpart of this result.

The analog of Theorem 2 for the Archimedean case is not known.

2. Ideas of our proofs

Sketch of the proof of Theorem 1. We first analyze the representation generated by ξ under the two sided action of the Hecke algebra $\mathcal{H}(G)$, which is admissible by Corollary 6. Then we use the theory of fuzzy balls (see [17, sections 4,5] and [2, Appendix A]), that produces, for any admissible representation, a large collection of elements in the Hecke algebra $\mathcal{H}(G)$ that annihilate it. Those elements will also annihilate ξ. In other words, for certain elements $e_B \in \mathcal{H}(G)$ we have the following vanishing of convolutions: $e_B * \xi = 0$.

Next we want to linearize this information. For this we use the exponentiation map and a proposition that says that in behaves as a homomorphism for certain pairs of functions. Unfortunately, this proposition is not directly applicable to the pair (e_B, ξ). However, we use the vanishing $e_B * \xi = 0$ to construct other vanishing convolutions, to which this proposition is applicable. Thus we get that certain convolutions on the Lie algebra vanish. Those vanishings imply the desired restriction on the wave front set.

Sketch of the proof of Theorem 2. Let us assume for simplicity that χ_i are trivial and H_i are unimodular. To prove Theorem 1 we first note that $\mathcal{I}$ is dual to the space $\mathcal{S}(G)_{H_1 \times H_2}$ of $(H_1 \times H_2)$-coinvariants of $\mathcal{S}(G)$. We can decompose $\mathcal{S}(G)$

to a direct sum with respect to Bernstein blocks. This leads to a decomposition of $\mathcal{S}(G)_{H_1 \times H_2}$. The finite type assumption implies that each summand is finitely generated over $\mathfrak{z}(G)$. Thus Artin-Rees Lemma and Hilbert's Nullstellensatz imply that the space of $\mathfrak{z}(G)$-finite functionals on those summands is dense in the space of arbitrary functionals.

3. Future applications

We believe that this work can be used in order to prove the following analog of Harish-Chandra's theorem [10, Theorem 3.1] on density of orbital integrals.

Conjecture. *Suppose that G is quasisplit, let B be its Borel subgroup, U be the nilradical of B, ψ be a non-degenerate character of U, $H \subset G$ be a reductive spherical subgroup and $X = G/H$. Let $\mathcal{O}$ be the union of all open B-orbits in G.*

Then the sum of the one-dimensional spaces $\mathcal{S}^*(Ux)^{U,\psi}$, where x ranges over $\mathcal{O}$, is dense in $\mathcal{S}^*(X)^{U,\psi}$.

In our work in progress [3] we prove a non-archimedean analog of [1], which we consider as a step towards this conjecture. Namely, we use Theorem 1 in order to prove that under certain conditions on H any $\mathfrak{z}$-finite distribution $\xi \in \mathcal{S}^*(X)^{U,\psi}$ which is supported in the complement to $\mathcal{O}$ vanishes.

In our work in progress [4] we prove that the set of cotoric elements is open and dense in G if H_1, H_2 are spherical subgroups. By Theorem 7 this implies that $H_1 \times H_2$- spherical characters are smooth almost everywhere. In fact, in [4] we show that the dimension of the variety

$$\mathfrak{S} = \{(x, v) \in G \times \mathcal{N} \mid \langle v, \mathfrak{h}_1 x\rangle = \langle v, \mathfrak{h}_1 x\rangle = 0\} \subset T^*G$$

equals the dimension of G. Theorem 1 implies that the wave-front set of any $H_1 \times H_2$- spherical character lies in $\mathfrak{S}$. Thus we obtain a certain version of holomonicity for spherical characters.

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Analysis on real spherical manifolds and their applications to Shintani functions and symmetry breaking operators

Toshiyuki Kobayashi

A complex manifold $X_{\mathbb{C}}$ with action of a complex reductive group $G_{\mathbb{C}}$ is called *spherical* if a Borel subgroup of $G_{\mathbb{C}}$ has an open orbit in $X_{\mathbb{C}}$. In the real setting, in search of a good framework for global analysis on homogeneous spaces which are broader than the usual (*e.g.* symmetric spaces), we proposed to call:

Definition 1 ([4, §2])**.** Let G be a real reductive Lie group. We say a smooth manifold X with G-action is *real spherical* if a minimal parabolic subgroup P of G has an open orbit in X.

The significance of this geometric property is the finite-multiplicity property in the regular representation of G on $C^\infty(X)$, which we discovered and proved by using the theory of hyperfunctions and regular singularities of a system of partial differential equations:

Fact 2 ([8, Theorems A and B])**.** *Let G be a real reductive linear Lie group, and H an algebraic subgroup. For an algebraic representation W of H, we form a G-equivariant vector bundle $\mathcal{W} := G \times_H W$ on G/H.*

1) *The following two conditions on the pair (G, H) are equivalent:*
   (i) *The homogeneous space G/H is real spherical.*
   (ii) $\dim \operatorname{Hom}_G(\pi^\infty, C^\infty(G/H, \mathcal{W}))\,(= \dim(\pi^{-\infty} \otimes W)^H) < \infty$ *for any smooth admissible representation π^∞ of G and for any algebraic representation W of H.*
2) *The following two conditions on the pair (G, H) are equivalent:*

(i) *The complexification $G_{\mathbb{C}}/H_{\mathbb{C}}$ is spherical.*
(ii) *There exists a constant $C > 0$ such that*

$$\dim \mathrm{Hom}_G(\pi^\infty, C^\infty(G/H, \mathcal{W}))\,(= \dim(\pi^{-\infty} \otimes W)^H) \le C \dim W$$

for any smooth irreducible admissible representation π^∞ of G and any algebraic representation W of H.

More precisely, a quantitative estimate for upper and lower bounds of the dimension was also given in [8].

Instead of smooth sections, we may also consider other function spaces such as the Hilbert space of square integrable functions. An earlier work for the construction of discrete series representations for some non-symmetric spherical homogeneous spaces can be found in [3].

The primary purpose of the talk was to explain some application of Fact 2 to the relationship among Shintani functions, branching problems, and real spherical varieties. For this, we fix some terminologies. Denote by $\mathfrak{g}$ the Lie algebra of G, and by $U(\mathfrak{g}_{\mathbb{C}})$ the universal enveloping algebra of the complexified Lie algebra $\mathfrak{g}_{\mathbb{C}} := \mathfrak{g} \otimes_{\mathbb{R}} \mathbb{C}$. For $X \in \mathfrak{g}$ and $f \in C^\infty(G)$, we set

$$(L_X f)(g) := \frac{d}{dt}|_{t=0} f(\exp(-tX)g), \quad (R_X f)(g) := \frac{d}{dt}|_{t=0} f(g\exp(tX)),$$

and extend these actions to those of $U(\mathfrak{g}_{\mathbb{C}})$. We denote by $\mathfrak{Z}_G$ the $\mathbb{C}$-algebra of G-invariant elements in $U(\mathfrak{g}_{\mathbb{C}})$. Let $\mathfrak{j}$ be a Cartan subalgebra of $\mathfrak{g}$. Then any $\lambda \in \mathfrak{j}_{\mathbb{C}}^\vee$ gives rise to a $\mathbb{C}$-algebra homomorphism $\chi_\lambda : \mathfrak{Z}_G \to \mathbb{C}$ via the Harish-Chandra isomorphism $\mathfrak{Z}_G \xrightarrow{\sim} S(\mathfrak{j}_{\mathbb{C}})^{W_G}$. The finite group W_G is the Weyl group of the root system $\Delta(\mathfrak{g}_{\mathbb{C}}, \mathfrak{j}_{\mathbb{C}})$ if G is connected.

Suppose that G' is an algebraic reductive subgroup. Analogous notation will be applied to G', *e.g.*, $\mathrm{Hom}_{\mathbb{C}\text{-alg}}(\mathfrak{Z}_{G'}, \mathbb{C}) \simeq (\mathfrak{j}'_{\mathbb{C}})^\vee / W_{G'}$, $\chi_\nu \leftrightarrow \nu$, where $\mathfrak{j}'$ is a Cartan subalgebra of the Lie algebra $\mathfrak{g}'$ of G'.

We take a maximal compact subgroup K of G such that $K' := K \cap G'$ is a maximal compact subgroup. Following Murase–Sugano [10], we call:

Definition 3 (Shintani function)**.** We say $f \in C^\infty(G)$ is a *Shintani function* of $(\mathfrak{Z}_G, \mathfrak{Z}_{G'})$-infinitesimal characters (λ, ν) if f satisfies the following three properties:

(1) $f(k'gk) = f(g)$ for any $k' \in K'$, $k \in K$.
(2) $R_u f = \chi_\lambda(u) f$ for any $u \in \mathfrak{Z}_G$.
(3) $L_v f = \chi_\nu(v) f$ for any $v \in \mathfrak{Z}_{G'}$.

We denote by $\mathrm{Sh}(\lambda, \nu)$ the space of Shintani functions of type (λ, ν).

For $G = G'$ and $\lambda = -\nu \mod W_G$, Shintani functions are nothing but Harish-Chandra's zonal spherical functions.

The following two theorems (Theorems 1 and 2) go back to [4], and the proof was given in [6] (and partly in [8]).

Theorem 1. *The following four conditions on a pair of real reductive algebraic groups $G \supset G'$ are equivalent:*

(i) (Shintani function) $\mathrm{Sh}(\lambda, \nu)$ *is finite-dimensional for any pair* (λ, ν) *of* $(\mathfrak{Z}_G, \mathfrak{Z}_{G'})$*-infinitesimal characters.*
(ii) (Symmetry breaking) $\mathrm{Hom}_{G'}(\pi^\infty, \tau^\infty)$ *is finite-dimensional for any pair* $(\pi^\infty, \tau^\infty)$ *of admissible smooth representations of* G *and* G'*.*
(iii) (Invariant bilinear form) $\mathrm{Hom}_{G'}(\pi^\infty \widehat{\otimes} \tau^\infty, \mathbb{C})$ *is finite-dimensional for any pair* $(\pi^\infty, \tau^\infty)$ *of admissible smooth representations of* G *and* G'*.*
(iv) (Geometric property (PP)) *The homogeneous space* $(G' \times G')/\Delta G'$ *is real spherical.*

The last geometric condition (PP) may be restated as follows. Let P' be a minimal parabolic subgroup of G'.

Definition 4. We say the pair (G, G') satisfies (PP) if one of the following five equivalent conditions are satisfied.

(PP1) $(G \times G')/\operatorname{diag} G'$ is real spherical as a $(G \times G')$-space.
(PP2) G/P' is real spherical as a G-space.
(PP3) G/P is real spherical as a G'-space.
(PP4) G has an open orbit in $G/P \times G/P'$ via the diagonal action.
(PP5) There are finitely many G-orbits in $G/P \times G/P'$ via the diagonal action.

The dimension of the Shintani space $\mathrm{Sh}(\lambda, \nu)$ depends on λ and ν in general. We give a characterization of the uniform boundedness property:

Theorem 2. *The following four conditions on a pair of real reductive algebraic groups* $G \supset G'$ *are equivalent:*

(i) (Shintani function) *There exists a constant* C *such that*

$$\dim_{\mathbb{C}} \mathrm{Sh}(\lambda, \nu) \leq C$$

for any pair (λ, ν) *of* $(\mathfrak{Z}_G, \mathfrak{Z}_{G'})$*-infinitesimal characters.*

(ii) (Symmetry breaking) *There exists a constant* C *such that*

$$\dim_{\mathbb{C}} \mathrm{Hom}_{G'}(\pi^\infty, \tau^\infty) \leq C$$

for any pair $(\pi^\infty, \tau^\infty)$ *of irreducible admissible smooth representations of* G *and* G'*.*

(iii) (Invariant bilinear form) *There exists a constant* C *such that*

$$\dim_{\mathbb{C}} \mathrm{Hom}_{G'}(\pi^\infty \widehat{\otimes} \tau^\infty, \mathbb{C}) \leq C$$

for any pair $(\pi^\infty, \tau^\infty)$ *of irreducible admissible smooth representations of* G *and* G'*.*

(iv) (Geometric property (BB)) *The homogeneous space* $(G_{\mathbb{C}} \times G'_{\mathbb{C}})/\Delta G'_{\mathbb{C}}$ *is spherical.*

Theorems 1 and 2 hold for general pairs of real reductive groups (G, G'). Among others, typical examples are obtained in the case where (G, G') are symmetric pairs.

Example 5 ([4, Example 2.8.6])**.** Let G be a simple Lie group. Then (i) and (ii) are equivalent:

(i) $(G \times G \times G)/\Delta G$ is real spherical ($\Leftrightarrow$ $(G \times G, \Delta G)$ satisfies (PP)).
(ii) G is compact or $\mathfrak{g} \simeq \mathfrak{o}(n,1)$.

Thus we could expect detailed analysis on invariant trilinear forms for $G = O(n,1)$. See [2] for an example of the research in this direction.

Example 6 ([6])**.** Let $(G, G') = (GL(n+1,\mathbb{F}), GL(n,\mathbb{F}) \times GL(1,\mathbb{F}))$.

1) For $\mathbb{F} = \mathbb{R}$, $\mathbb{C}$ or $\mathbb{H}$, $\dim \mathrm{Sh}(\lambda,\nu) < \infty$ for all λ and ν.
2) $\sup_\lambda \sup_\nu \dim \mathrm{Sh}(\lambda,\nu) < \infty$ for $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$, whereas $\sup_\lambda \sup_\nu \dim \mathrm{Sh}(\lambda,\nu) = \infty$ for $\mathbb{F} = \mathbb{H}$.

Under an additional assumption that (G, G') is a symmetric pair, we gave a complete classification of the pairs (G, G') satisfying (PP), or equivalently, satisfying any of (i), (ii) or (iii) in Theorem 1 (see [7]). The idea of the classification extends the idea of 'linearization' which was the key in the proof of Example 5, together with some further ideas such as constructing invariants for quivers.

In [9], we constructed and classified all the symmetry breaking operators for spherical principal series representations for $(G, G') = (O(n+1,1), O(n,1))$. In this connection, we explained in the talk how symmetry breaking operators of the restriction of smooth admissible representations yield Shintani functions of moderate growth, and presented an explicit dimension formula (*cf.* [6]).

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Tempered modules with non-zero Dirac cohomology for graded affine Hecke algebras

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Let $\Phi = (R, V, R^\vee, V^\vee)$ be a reduced root system and let W be its corresponding real reflection groups. Let $C(V^\vee)$ be the Clifford algebra associated to $V^\vee$ and a W-invariant bilinear form on $V^\vee$. The graded affine Hecke algebra $\mathbb{H}$ associated to Φ and a parameter function was a deformation of the skew group ring for the W-representation $V^\vee$. A Dirac-type element D was introduced by Barbasch-Ciubotaru-Trapa [2] in the space $\mathbb{H} \otimes C(V^\vee)$ for the study of unitary representations.

The original idea of Dirac cohomology was made by Vogan in real reductive groups to generalize the kernel of the Dirac operator to study all representations. In the situation of graded affine Hecke algebra, the Dirac cohomology $H^D(X)$ was similarly defined as

$$H^D(X) := \frac{\ker(D : X \otimes S \to X \otimes S)}{\mathrm{im}(D : X \otimes S \to X \otimes S) \cap \ker(D : X \otimes S \to X \otimes S)},$$

where X is an $\mathbb{H}$-module and S is a fixed choice of simple $C(V^\vee)$-modules. . Let $\widetilde{W}$ be the spin cover of the reflection group W. The Dirac cohomology $H^D(X)$ is endowed with a natural $\widetilde{W}$-representation structure. It was shown in [2] that if a module X has a non-zero Dirac cohomology, then the central characters of X is determined by the $\widetilde{W}$-representations in $H^D(X)$. This is analogous to a conjecture by Vogan for Harish-Chandra modules, which was later proven by Huang-Pandžić.

A tempered module X for $\mathbb{H}$ can be defined algebraically as a module, for which any weight γ of X satisfy the relation $\omega(\gamma) \leq 0$ for all fundamental coweights in $V^\vee$. For an equal positive parameter function k and a Weyl group W, tempered modules can be parametrized by the G-orbits on $\left\{(e, \phi) : e \in \mathcal{N}, \phi \in \widehat{A_0(e)}\right\}$, where $\mathcal{N}$ is the nilpotent cone in the relevant Lie algebra, G is the corresponding simply connected Lie group and $\widehat{A_0(e)}$ is the set of irreducible representations appearing in the Springer correspondence. Under this parametrization, it was proven in [2] that when e has a solvable centralizer, the corresponding tempered modules have non-zero Dirac cohomology. This result heavily relies on the computation in [1] by Ciubotaru, which also provide a more explicit $\widetilde{W}$-structure on the Dirac cohomology of those tempered modules.

Our goal is to study the Dirac cohomology of tempered modules in a general setting (i.e. arbitrary parameter functions and any real reflection groups). For tempered modules corresponding to nilpotent elements with a solvable centralizer, I introduced a combinatorial characterization on their central characters. Such characterization is in the nature of the one defined for residual points by Heckeman-Opdam [4]. In a general setting, a point satisfying such characterization is called solvable [3]. In particular, any residual points are solvable. We conjecture that any tempered module with a solvable central character has non-zero Dirac cohomology.

Apart from the equal parameter case for Weyl groups, there are some evidences in the case of non-crystallographic types [3].

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Towards Asymptotic Plancherel Formulas for Reductive Homogeneous Spaces

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(joint work with Hongyu He, Gestur Olafsson)

In [4], Kashiwara and Vergne gave an asymptotic description of induction problems for compact, connected Lie groups. Suppose $K_1 \subset K$ are compact, connected Lie groups, let $T_1 \subset T$ be maximal tori for K_1 and K, and let $\mathfrak{t}_1^*$, $\mathfrak{t}^*$ be the (real) duals of the corresponding Lie algebras. For each irreducible, unitary representation $\sigma \in \widehat{K}$, we may associate its infinitesimal character, λ_σ, which we may view as a W orbit in $i\mathfrak{t}^*$ where $W = N_K(T)/T$ is the Weyl group. If $\tau \in \widehat{K_1}$ is an irreducible representation of K_1, consider the set

$$\operatorname{supp}(\operatorname{Ind}_{K_1}^K \tau) = \{\sigma \in \widehat{K} |\ \operatorname{Hom}_K(\sigma, \operatorname{Ind}_{K_1}^K \tau) \neq \{0\}\}$$

and let

$$i\mathfrak{t}^*\text{-}\operatorname{supp}(\operatorname{Ind}_{K_1}^K \tau) = \bigcup_{\sigma \in \operatorname{supp}(\operatorname{Ind}_{K_1}^K \tau)} \lambda_\sigma$$

be the corresponding W invariant subset of $i\mathfrak{t}^*$. If $S \subset W$ is a subset of a finite dimensional vector space, then we define the asymptotic cone of S to be

$$\operatorname{AC}(S) = \{\xi \in W |\ \xi \in \Gamma \text{ an open cone} \Rightarrow \Gamma \cap S \text{ noncompact}\} \cup \{0\}.$$

Kashiwara-Vergne showed

$$\operatorname{AC}(i\mathfrak{t}^*\text{-}\operatorname{supp}(\operatorname{Ind}_{K_1}^K \tau)) = W \cdot i(\mathfrak{t}/\mathfrak{t}_1)^* \subset i\mathfrak{t}^*.$$

This result gives an elegant and simple asymptotic description of the induction problem.

It is worth noting that this statement requires a non-canonical choice of a maximal torus $T \subset K$. A more canonical way to do things is to view the infinitesimal character of $\sigma \in \widehat{K}$ as a coadjoint orbit $\mathcal{O}_\sigma \subset i\mathfrak{k}^*$ where $\mathfrak{k}$ is the Lie algebra of K.

This is the coadjoint K orbit consisting of all $\lambda_\sigma(T) \subset i\mathfrak{t}^*$ for all maximal tori T. Define the orbital support of $\operatorname{Ind}_{K_1}^K \tau$ to be

$$\mathcal{O}\text{-}\operatorname{supp}(\operatorname{Ind}_{K_1}^K \tau) = \bigcup_{\sigma \in \operatorname{supp} \operatorname{Ind}_{K_1}^K \tau} \mathcal{O}_\sigma.$$

Here is another way of stating the result of Kashiwara and Vergne.

One can also write the result of Kashiwara-Vergne as

$$\operatorname{AC}(\mathcal{O}\text{-}\operatorname{supp}(\operatorname{Ind}_{K_1}^K \tau)) = \operatorname{Ad}^*(K) \cdot i(\mathfrak{k}/\mathfrak{k}_1)^* \subset i\mathfrak{k}^*.$$

This statement has the advantage of not depending on a choice of a maximal torus $T \subset K$ and the disadvantage of being a bit harder to visualize (since one is dealing with unions of potentially high dimensional orbits for a compact Lie group instead of orbits of a finite group).

Either way, this is a nice result, and the purpose of the recent paper "Wave Front Sets of Reductive Lie Group Representations" that the author wrote jointly with Hongyu He and Gestur Ólafsson is to generalize these results to noncompact reductive Lie groups [3]. We give an example of one of our recent results.

Suppose G is a real, reductive algebraic group, and suppose $H \subset G$ is a closed, reductive algebraic subgroup. Let $\widehat{G}_{\text{temp}}$ denote the set of irreducible, tempered representations of G. Following Duflo and Rossmann ([2], [5], [6]), for each $\sigma \in \widehat{G}_{\text{temp}}$, we associate a finite union of coadjoint orbits $\mathcal{O}_\sigma \subset i\mathfrak{g}^*$. In the generic case, when σ has regular infinitesimal character, $\mathcal{O}_\sigma$ is a single coadjoint orbit.

Following Benoist and Kobayashi [1], we say that a homogeneous space G/H is *tempered* if the decomposition of the representation $L^2(G/H) = \operatorname{Ind}_H^G \mathbb{1}$ into irreducible representations contains only irreducible, tempered representations of G. In the recent preprint [1], Benoist and Kobayashi give a simple and computable necessary and sufficient condition for G/H to be tempered. If G/H is tempered, we define the orbital support of $L^2(G/H) = \operatorname{Ind}_H^G \mathbb{1}$ to be

$$\mathcal{O}\text{-}\operatorname{supp} L^2(G/H) = \bigcup_{\sigma \in \operatorname{supp} L^2(G/H)} \mathcal{O}_\sigma.$$

Suppose G is a real, reductive algebraic group, and suppose $H \subset G$ is a closed, reductive algebraic subgroup. If G/H is a tempered homogeneous space, then we have shown

$$\operatorname{AC}(\mathcal{O}\text{-}\operatorname{supp} L^2(G/H)) \supset \overline{\operatorname{Ad}^*(G) \cdot i(\mathfrak{g}/\mathfrak{h})^*}.$$

From Example 5.6 of [1], we see that if $G = \operatorname{SO}(p,q)$ and $H = \prod_{i=1}^r \operatorname{SO}(p_i, q_i)$ with $p = \sum_{i=1}^r p_i$, $q = \sum_{i=1}^r q_i$, and $2(p_i + q_i) \le p + q + 2$ whenever $p_i q_i \neq 0$, then G/H is tempered. One deduces from some elementary combinatorics that if in addition, $2p_i \le p+1$, $2q_i \le q+1$ for every i and $p+q > 2$, then

$$i\mathfrak{g}^* = \overline{\operatorname{Ad}^*(G) \cdot i(\mathfrak{g}/\mathfrak{h})^*}.$$

The above result now implies that $\operatorname{supp} L^2(G/H)$ is "asymptotically equivalent to" $\operatorname{supp} L^2(G)$. In particular, suppose p and q are not both odd and $\mathcal{F}$ is one of the $\binom{p+q}{p}$ families of discrete series of $G = \mathrm{SO}(p,q)$. Then

$$\mathrm{Hom}_G(\sigma, L^2(G/H)) \neq \{0\}$$

for infinitely many different $\sigma \in \mathcal{F}$. A particularly nice example is when $G = \mathrm{SO}(4n,2)$ and $H = \mathrm{SO}(n,1) \times \mathrm{SO}(n,1) \times \mathrm{SO}(2n)$. In this case, one deduces

$$\mathrm{Hom}_G(\sigma, L^2(G/H)) \neq \{0\}$$

for infinitely many distinct (possibly vector valued) holomorphic discrete series σ of G.

To end, we note that this result is in the spirit of the Kirillov-Kostant orbit method. Roughly it says *the orbit method is asymptotically at least half true for tempered homogeneous spaces.*

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Automorphisms of double affine Hecke algebras

Bogdan Ion

(joint work with S. Sahi)

Let R^{aff} be an irreducible (but not necessarily reduced) affine root system and let r be its affine type, which is indication whether R^{aff} is untwisted ($r = 1$), or obtained by folding an untwisted affine root system by an order r ($r = 2, 3$) automorphism, with the exception of the affine root system of type $A_{2n}^{(2)}$ which is obtained by folding by an order 2 automorphism but for which we assign $r = 1$.

To R^{aff} one can associate a double affine Hecke algebra H^{daff} (as explained in the work of Cherednik (for $r = 1$) and Macdonald) as the endomorphism algebra of the representation of affine Hecke algebra associated to R^{aff} induced from the trivial representation of its finite Hecke sub-algebra. To some extent one can think of H^{daff} as the Iwahori-Hecke algebra corresponding to a reductive group over a 2-dimensional local field [4]. The understanding of $\mathrm{Aut}(H^{\mathrm{daff}})$ is especially important in this context. For example, virtually all of Cherednik's work on the subject ultimately rests on such information.

For $r \in \{2,3\}$ consider the congruence group

$$\Gamma_1(r) = \{\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in SL(2,\mathbb{Z}) \mid a \equiv d \equiv 1 \ (\mathrm{mod}\ r),\ c \equiv 0 \ (\mathrm{mod}\ r)\}.$$

For uniformity we denote $\Gamma_1(1) = SL(2,\mathbb{Z})$. The group $\Gamma_1(r)$ has an abstract presentation as the group generated by two elements $\mathfrak{a}, \mathfrak{b}$, satisfying the r-lace braid relation (i.e. $\mathfrak{ab}\cdots = \mathfrak{ba}\cdots$ where there are $3,4,6$ factors on each side if $r = 1,2,3$, respectively), and the relation $(\mathfrak{ab})^3 = -I_2$ if $r = 1$, $(\mathfrak{ab})^2 = -I_2$ if $r = 2$, $(\mathfrak{ab})^3 = I_2$ if $r = 3$. Let $\widetilde{\Gamma}_1(r)$ be the universal central extension of $\Gamma_1(r)$. The group $\widetilde{\Gamma}_1(r)$ has an abstract presentation as the group generated by two elements, $\widetilde{\mathfrak{a}}, \widetilde{\mathfrak{b}}$, satisfying the r-lace braid relation. The center of $\widetilde{\Gamma}_1(r)$ is generated by element $\widetilde{\mathfrak{c}}$, defined as $(\mathfrak{ab})^3$ if $r = 1$, $(\mathfrak{ab})^2$ if $r = 2$, $(\mathfrak{ab})^3$ if $r = 3$.

I will explain the following result.

Theorem. There is faithful group morphism $\widehat{\Gamma}_1(r) \to \mathrm{Aut}(H^{\mathrm{daff}})$. This descends to a group morphism $\Gamma_1(r) \cong \widehat{\Gamma}_1(r)/\langle\ \widetilde{\mathfrak{c}}^2\ \rangle \to \mathrm{Out}(H^{\mathrm{daff}})$ which is faithful (if the finite Dynkin diagram associated to R^{aff} is A_n, $n \geq 2$, D_{2n+1}, $n \geq 2$, E_6) or has kernel $\pm I_2$ (otherwise).

For $r = 1$ such results were discovered by Cherednik. The same result is true if H^{daff} is replaced by the corresponding braid group, called the double affine braid group, and the Hecke algebra statement is a trivial consequence of the braid group statement. Our proof for the braid group statement relies on a new presentation of the braid group in terms of what we call *double affine Coxeter diagrams*.

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On quotients of spherical varieties by unipotent subgroups

Dmitry Timashev

In this talk, we discuss a problem of pushing down certain geometric properties of an algebraic variety equipped with a group action to the quotient variety. More precisely, let X be an affine complex algebraic variety equipped with an action of a linear algebraic group H. The concept of a quotient variety commonly used in the geometric invariant theory is that of a *categorical quotient* defined as $X/\!\!/H := \mathrm{Spec}\,\mathbb{C}[X]^H$, provided that the algebra $\mathbb{C}[X]^H$ of H-invariant regular functions on X is finitely generated. The idea behind this definition is that invariant functions on a variety acted on by a group should be thought of as functions on the orbit space. However, in general, this "categorical quotient space" can be quite far from a naïve "space of orbits". First of all, the categorical quotient may even not exist as an affine algebraic variety, i.e., the algebra $\mathbb{C}[X]^H$ is not necessarily finitely generated. Further, the natural quotient map $\pi_H : X \to X/\!\!/H$ (dual to the inclusion of algebras $\mathbb{C}[X]^H \subseteq \mathbb{C}[X]$) can be non-surjective and does not necessarily separate orbits. (For instance, if an orbit lies in the closure of another one, then these two orbits map to one and the same point of the quotient space.)

If H is reductive, then the situation with the categorical quotient is as good as possible: the quotient space always exists, the quotient map is surjective and separates closed orbits [2, 4.4]. Generally, one can sometimes reduce to this special case if H is contained in a bigger reductive group G acting on X. The following statement is a geometric version of the so-called Transfer Principle [1]:

Theorem. *If $G/\!\!/H$ exists, then $X/\!\!/H$ exists, too, and $X/\!\!/H \simeq (G/\!\!/H \times X)/\!\!/G$.*

Recall that H is called a *Grosshans subgroup* of G if G/H is quasiaffine and $\mathbb{C}[G/H]$ is finitely generated. In this case, $G/\!\!/H$ exists and contains G/H as a dense open G-orbit.

Now suppose that G is a connected reductive group and H is a unipotent normal subgroup of a Borel subgroup $B \subseteq G$. We assume additionally that $H \subset G$ is a Grosshans subgroup. (A long-standing conjecture of Popov–Pommerening would imply this assumption.) Then B/H acts naturally on $X/\!\!/H$.

Suppose that X is a *spherical* G-variety, i.e., B has a dense open orbit in X. A theorem of Brion and Vinberg says that B acts on X with finitely many orbits. A natural question raised by Panyushev (2012) was whether B/H still has finitely many orbits on $X/\!\!/H$. (A tricky point is that $\pi_H : X \to X/\!\!/H$ may be non-surjective, whence not all B/H-orbits on $X/\!\!/H$ necessarily come from B-orbits on X.)

This question is motivated by observing that algebraic varieties acted on by a solvable group with finitely many orbits have a nice stratification by locally closed "cells" isomorphic to $\mathbb{C} \times \cdots \times \mathbb{C} \times \mathbb{C}^\times \times \cdots \times \mathbb{C}^\times$ and therefore nice homological and intersection-theoretic properties.

In some easy cases, one can give an affirmative answer to Panyushev's question. Namely, if H is the unipotent radical of a parabolic subgroup $P \supseteq B$, then $X/\!\!/H$ is a spherical P/H-variety on which the Borel subgroup $B/H \subseteq P/H$ acts with finitely many orbits. Also, if X has a (necessarily unique) G-fixed point 0, then there is a contracting $\mathbb{C}^\times$-action on X commuting with G. Now if the fiber $\pi_H^{-1}\pi_H(0)$ has the proper dimension (equal to the dimension of general H-orbits in X), then π_H is surjective, hence B/H has finitely many orbits on $X/\!\!/H$.

An interesting case, on which we focus in this talk, is $H = [U, U]$, the derived subgroup of the unipotent part $U \subset B$. In this case, Panyushev conjectured that B/H always has finitely many orbits on $X/\!\!/H$. We disprove this conjecture. Our first main result is:

Theorem 1. *For a fixed reductive group G, the categorical quotient $X/\!\!/[U, U]$ of any affine spherical G-variety X has finitely many $B/[U, U]$-orbits if and only if all simple factors of G are of types $\mathsf{A}_1, \mathsf{B}_2, \mathsf{G}_2$.*

Since Theorem 1 is somewhat disappointing, because we would like to have an affirmative answer to Panyushev's question, it is natural to ask whether it can be obtained under some restrictions on X. The restrictions which we impose are of representation-theoretic nature. Recall from the representation theory of reductive groups that the irreducible representations of G are parameterized by their *highest weights*, which are the dominant weights of the root data of G. For any dominant weight λ, we consider the subdiagram of the Dynkin diagram of G whose nodes correspond to the fundamental weights occurring in the decomposition of λ with positive coefficients, and call it the *support* of λ.

Theorem 2. *Suppose that the supports of all highest weights of the irreducible G-modules occurring in $\mathbb{C}[X]$ do not contain subdiagrams of type A_2. Then $B/[U,U]$ acts on $X/\!\!/[U,U]$ with finitely many orbits.*

This theorem can be extended to quotients by intermediate subgroups $U \supseteq H \supseteq [U,U]$. It applies, for instance, to $X = GL_{2n}(\mathbb{C})/Sp_{2n}(\mathbb{C})$, the moduli space of symplectic structures on $\mathbb{C}^{2n}$.

We conclude by giving an outline of the proofs. Details can be found in [3].

The main idea in the study of the geometry of $X/\!\!/H$ is to use the Transfer Principle. It follows that all orbits of B/H on $X/\!\!/H$ come from the orbits of $G \times B/H$ on $G/\!\!/H \times X$. (Here B/H acts on $G/\!\!/H \times X$ by G-equivariant automorphisms via its action on $G/\!\!/H$ "from the right" extending the action on the dense orbit G/H.) Thus it is important to analyze $(G \times B/H)$-orbits on $G/\!\!/H$.

For $U \supseteq H \supseteq [U,U]$, we start with an explicit model of $G/\!\!/H$ as a certain orbit closure $\overline{Gv_0}$ in a G-module V and describe the B/H-action on $G/\!\!/H$ in terms of certain intertwining operators on V. From this description, we get a finite complete list of orbit representatives v_i for $\widehat{G} = G \times B/H$ acting on $G/\!\!/H \simeq \overline{Gv_0}$. Each orbit $\widehat{G}v_i$ is a union of G-orbits permuted by B/H transitively. We compute the stabilizers $H_i = G_{v_i}$ and $\widehat{H}_i = \widehat{G}_{v_i}$. It turns out that $\overline{Gv_i} \simeq G/\!\!/H_i$ for all i.

Now $X/\!\!/H \simeq (G/\!\!/H \times X)/\!\!/G$ is covered by the π_G-images of various subvarieties $\widehat{G}v_i \times X$, which are the B/H-spans of $\pi_G(Gv_i \times X) = \pi_G(\{v_i\} \times X)$. In its turn, each $\pi_G(\{v_i\} \times X)$ is a dense subset in the closed subvariety $(\overline{Gv_i} \times X)/\!\!/G \simeq X/\!\!/H_i$ of $(\overline{Gv_0} \times X)/\!\!/G \simeq X/\!\!/H$. The isotropy subgroup $\widehat{H}_i$ acts on this subvariety. This action factors through the projection $\widehat{H}_i \to G$, whose image we denote by N_i.

The problem now boils down to the study of N_i/H_i-orbits in $\pi_{H_i}(X) \subseteq X/\!\!/H_i$ for all i. If all simple factors of G are $\mathsf{A}_1, \mathsf{B}_2, \mathsf{G}_2$ or under the assumptions of Theorem 2, the number of orbits is finite. Otherwise, for sufficiently "big" X, one can point out i such that generic N_i/H_i-orbits in $X/\!\!/H_i$ constitute an infinite family. One verifies that these orbits are not glued together under the action of $B/[U,U]$. This concludes the proof of Theorem 1.

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Lie algebra homology of Schwartz spaces and comparison theorems

Gang Liu
(joint work with Avraham Aizenbud, Dmitry Gourevitch, Bernhard Krötz)

In our joint project, we start to develop the theoretical background for comparison theorems in homological representation theory attached to a real reductive group G.

The algebraic side of representation theory for G is encoded in the theory of Harish-Chandra modules V. These are modules for the Lie algebra $\mathrm{Lie}\,G$ of G with a compatible algebraic action of a fixed maximal compact subgroup K of G. According to Casselman-Wallach (see [3]) Harish-Chandra modules V can be naturally completed to smooth moderate growth modules V^∞ for the group G. Somewhat loosely speaking one might think of V, resp. V^∞, as the regular, resp. smooth, functions on some real algebraic variety.

Fix a subalgebra $\mathfrak{h} < \mathfrak{g}$. As $V \subset V^\infty$ we obtain natural maps

$$\Phi_p : \mathrm{H}_p(\mathfrak{h}, V) \to \mathrm{H}_p(\mathfrak{h}, V^\infty)\,.$$

Conjecture 1. (Comparison Conjecture) *If $\mathfrak{h}$ is a real spherical subalgebra, then Φ_p is an isomorphism for all p.*

Note that $\mathrm{H}_p(\mathfrak{h}, V)$ is finite dimensional. In case of $\mathfrak{h} = \mathfrak{n}$ is a maximal unipotent subalgebra then the Conjecture is the still not fully established Casselman comparison theorem (see [6]).

According to the Casselman subrepresentation theorem every smooth completion V^∞ is the quotient of the section module of an equivariant vector bundle $\mathcal{E} \to X$ where $X = G/P$ is a minimal flag variety. A first step towards the Comparison Conjecture is to understand the topological nature of the modules $\mathrm{H}_p(\mathfrak{h}, C^\infty(X, \mathcal{E}))$. In particular one needs to know whether these topological vector spaces are separated (Hausdorff). Under some restrictions on $\mathfrak{h}$ and on V, we will answer this question.

Let now H_1 be a Nash group (not necessarily reductive). Let X be a Nash manifold and $\mathcal{E}$ be a Nash vector bundle over X. Assume that H_1 acts equivariantly on X and $\mathfrak{E}$. Let $\mathcal{S}(X, \mathcal{E})$ be the space of Schwartz sections with respect to $\mathcal{E} \longrightarrow X$. Then $\mathcal{S}(X, \mathcal{E})$ is a *nuclear Fréchet* space and H_1 acts smoothly on $\mathcal{S}(X, \mathcal{E})$. Since the action of H_1 on $\mathcal{S}(X, \mathcal{E})$ is smooth, the space $\mathcal{S}(X, \mathcal{E})$ becomes a $\mathfrak{h}_1$-module. Equipped with the quotient topology, each homological space $\mathrm{H}_i(\mathfrak{h}_1, \mathcal{S}(X, \mathcal{E}))$ becomes a topological vector space. Our goal is to prove that all the $\mathrm{H}_i(\mathfrak{h}_1, \mathcal{S}(X, \mathcal{E}))$ are separated.

As a first step, we have:

Theorem 2. *Let H be a* ***normal*** *Nash sub-group of H_1 (with Lie algebra $\mathfrak{h}$). Assume that the following two conditions are satisfied: (1) The number of H_1-orbits in X is finite ($\sharp X/H_1 < \infty$). (2) The sub-group H and all the stabilizers H_x ($x \in X$) are homologically trivial (i.e. all their homology except H_0 vanish and $\mathrm{H}_0 = \mathbb{R}$; e.g. contractible). Then*

(1) $\mathrm{H}_i(\mathfrak{h}, \mathcal{S}(X, \mathcal{E}))$ are separated.
(2) The topological duality holds: $\mathrm{H}_i(\mathfrak{h}, \mathcal{S}(X, \mathcal{E})) \cong (\mathrm{H}^i(\mathfrak{h}, \mathcal{S}(X, \mathcal{E})'))'$.

This theorem has the following consequence:

Corollary 3. *Let G be a connected (linear) semisimple Lie group with Lie algebra $\mathfrak{g}$. Let $G = KAN$ be an Iwasawa decomposition. Let $\overline{\mathfrak{n}} \subseteq \mathfrak{h} \subseteq \mathfrak{a} + \overline{\mathfrak{n}}$. Let V be a Harish-Chandra module, such that V^∞ is a quotient of a Nash generalized principal series representation. Then we have*

$$\mathrm{H}_0(\mathfrak{h}, V) \cong \mathrm{H}_0(\mathfrak{h}, V^\infty).$$

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Equivariant real structures on spherical varieties

Dmitry Akhiezer

A real structure on a complex manifold X is an anti-holomorphic involutive diffeomorphism $\mu : X \to X$. Let X^μ be the fixed point set of μ. If $X^\mu \neq \emptyset$ then $X^\mu \subset X$ is a closed real submanifold of half-dimension. Suppose a complex Lie group G acts holomorphically on X and let $\sigma : G \to G$ be an anti-holomorphic automorphism of G. The fixed point subgroup of σ is a real form of G denoted by G^σ. A real structure $\mu : X \to X$ is said to be σ-equivariant if $\mu(g \cdot x) = \sigma(g) \cdot \mu(x)$ for all $g \in G, x \in X$. We are interested in such structures on complex homogeneous manifolds and their equivariant embeddings. In what follows, the group G is connected and semisimple. Let $\mathfrak{g}$ and $\mathfrak{g}_0$ be the Lie algebras of G and G^σ, respectively.

For any $\sigma : G \to G$ there is a unique automorphism class $\epsilon = \epsilon_\sigma \in \mathrm{Aut}(\mathfrak{g})/\mathrm{Int}(\mathfrak{g})$ acting on the Dynkin diagram in the following way, see [1]. Choose a maximally non-compact Cartan subalgebra in $\mathfrak{g}_0$. Then we have the set of compact simple roots $\Pi_\bullet$ and the set of non-compact simple roots $\Pi_\circ$. Denote by $W_\bullet$ the subgroup of the Weyl group W generated by the reflections in compact simple roots and let $w_\bullet \in W_\bullet$ be the element of maximal length with respect to these generators. Then $\epsilon(\alpha) = -w_\bullet(\alpha)$ if $\alpha \in \Pi_\bullet$ and $\epsilon(\alpha) = \omega(\alpha)$ if $\alpha \in \Pi_\circ$, where ω is the involutory self-map of $\Pi_\circ$ associated with σ. On the Satake diagram, ϵ interchanges the white circles connected by two-pointed arrows and permutes the black ones as the outer automorphism of order 2 for simple compact algebras of type $A_n (n \geq 2), D_n (n \text{ odd})$ and E_6, acting identically otherwise.

Theorem 1. Assume $\epsilon_\sigma = \mathrm{id}$. If $H \subset G$ is a spherical subgroup then $\sigma(H) = a \cdot H \cdot a^{-1}$ for some $a \in G$. The map $\mu_0 : G/H \to G/H,\ \mu_0(g \cdot H) := \sigma(g) \cdot a \cdot H$, is correctly defined, anti-holomorphic and σ-equivariant. Moreover, if H is self-normalizing then μ_0 is involutive, hence a σ-equivariant real structure on G/H, and such a structure is uniquely defined.

One can prove a similar theorem for wonderful varieties introduced by D.Luna. These are smooth equivariant completions of spherical homogeneous spaces, such that the complement to the open orbit is a finite union of r prime divisors with normal crossings and the orbit closures are precisely the partial intersections of these divisors. The number r is called the rank of a wonderful variety. It coincides with the rank defined in the theory of spherical varieties. If $H \subset G$ is a self-normalizing spherical subgroup then G/H has a wonderful embedding which is unique up to an isomorphism.

Theorem 2. Let H be a self-normalizing spherical subgroup of G and let X be the wonderful completion of G/H. If $\epsilon_\sigma = \mathrm{id}$ then there exists one and only one σ-equivariant real structure on X.

Example. Depending on n, there are at most two non-equivalent real structures on $\mathbb{P}_n$, namely, $\mu_1(z_1 : z_2 : \cdots : z_{n+1}) = (\overline{z_1} : \overline{z_2} : \cdots : \overline{z_{n+1}})$ and $\mu_2(z_1 : z_2 : \cdots : z_{n+1}) = (-\overline{z_2} : \overline{z_1} : \cdot : -\overline{z_{2d}} : \overline{z_{2d-1}})$ for $n = 2d - 1$. Let σ_1 and σ_2 be two anti-holomorphic involutions of $G = \mathrm{SL}(n+1, \mathbb{C})$ with $G^{\sigma_1} = \mathrm{SL}(n+1, \mathbb{R})$, $G^{\sigma_2} = \mathrm{SL}(d, \mathbb{H})$. Then μ_1 is σ_1-equivariant, μ_2 is σ_2-equivariant, and no real structure is σ-equivariant if σ defines a pseudo-unitary group $\mathrm{SU}(p, q)$, $p+q = n+1$.

Assume G^σ is a split real form of G. Since the Satake diagram has no black circles and no two-pointed arrows, $\epsilon_\sigma = \mathrm{id}$ by the definition of ϵ_σ. In this case the above theorems are joint results with S.Cupit-Foutou, see [2]. Furthermore, assume that X is a strict wonderful variety, i.e., the stabilizer of each point of X is self-normalizing. Then for G^σ split the total number of orbits of the connected component of G^σ on X^μ does not exceed $\sum_{k=0}^{r} \binom{r}{k} 2^k = 3^r$, see [2].

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Ext analogues of branching laws

Dipendra Prasad

Considering the restriction of representations of a group to one of its subgroups, say of $\mathrm{SO}_{n+1}(F)$ to $\mathrm{SO}_n(F)$ for a non-Archimedean local field F has been a very fruitful direction of research specially through its connections to questions on period integrals of automorphic representations, cf. [2] for the conjectural theory both locally and globally. The question for local fields amounts to understanding $\mathrm{Hom}_{\mathrm{SO}_n(F)}[\pi_1, \pi_2]$ for irreducible admissible representations π_1 of $\mathrm{SO}_{n+1}(F)$, and π_2 of $\mathrm{SO}_n(F)$. The first result proved about this is the multiplicity one property cf. [1], [4], which says that this space of intertwining operators is at most one dimensional. It may be mentioned that before the full multiplicity one theorem was proved, even finite dimensionality of the space was not known. A precise description of when $\mathrm{Hom}_{\mathrm{SO}_n(F)}[\pi_1, \pi_2] \neq 0$ has now become available in a series of works due to Waldspurger, and Moeglin-Waldspurger. There is also a recent series of papers by Raphael Beuzart-Plessis on similar questions for unitary groups.

Given the interest in the space $\mathrm{Hom}_{\mathrm{SO}_n(F)}[\pi_1, \pi_2]$, it is natural to consider the other related spaces $\mathrm{Ext}^i_{\mathrm{SO}_n(F)}[\pi_1, \pi_2]$, and in fact homological algebra methods suggest that the objects for which one might expect simplest answers are not these individual groups, but the alternating sum of their dimensions:

$$\mathrm{EP}[\pi_1, \pi_2] = \sum_{i=0}^{\infty} (-1)^i \dim \mathrm{Ext}^i_{\mathrm{SO}_n(F)}[\pi_1, \pi_2];$$

these hopefully more manageable objects –certainly more flexible– when coupled with vanishing of higher Ext's (when available) may give theorems about $\mathrm{Hom}_{\mathrm{SO}_n(F)}[\pi_1, \pi_2]$. We hasten to add that before we can define $\mathrm{EP}[\pi_1, \pi_2]$, we need to prove that $\mathrm{Ext}^i_{\mathrm{SO}_n(F)}[\pi_1, \pi_2]$ are finite dimensional for π_1 and π_2 finite length admissible representations of $\mathrm{SO}_{n+1}(F)$ and $\mathrm{SO}_n(F)$ respectively. Towards a proof of this finite dimensionality of Ext^i in this case, we only note here that unlike $\mathrm{Hom}_{\mathrm{SO}_n(F)}[\pi_1, \pi_2]$, where we will have no idea how to prove finite dimensionality if both π_1 and π_2 are cuspidal, exactly this case we can handle apriori, for $i > 0$, as almost by the very definition of cuspidal representations, they are both projective and injective objects in the category of smooth representations.

1. Conjectures and some results

We will make a precise formulation regarding the Ext-version of the branching laws from $\mathrm{GL}_{n+1}(F)$ to $\mathrm{GL}_n(F)$. First we recall the basic result in this context.

Theorem 1. *Given an irreducible generic representation π_1 of $\mathrm{GL}_{n+1}(F)$, and an irreducible generic representation π_2 of $\mathrm{GL}_n(F)$,*

$$\mathrm{Hom}_{\mathrm{GL}_n(F)}[\pi_1, \pi_2] = \mathbb{C}.$$

The following theorem which can be considered as the Euler-Poincare version of this theorem, proved in a much softer way using some results of Bernstein and

Zelevinsky regarding the structure of representations of $\mathrm{GL}_{n+1}(F)$ restricted to a mirabolic subgroup..

Theorem 2. *Let π_1 be an admissible representation of $\mathrm{GL}_{n+1}(F)$ of finite length, and π_2 an admissible representation of $\mathrm{GL}_n(F)$ of finite length. Then, $\mathrm{Ext}^i_{\mathrm{GL}_n(F)}[\pi_1, \pi_2]$ are finite dimensional vector spaces over $\mathbb{C}$, and*

$$\mathrm{EP}_{\mathrm{GL}_n(F)}[\pi_1, \pi_2] = \dim \mathrm{Wh}(\pi_1) \cdot \dim \mathrm{Wh}(\pi_2),$$

where $\mathrm{Wh}(\pi_1)$, resp. $\mathrm{Wh}(\pi_2)$, denotes the space of Whittaker models for π_1, resp. π_2, with respect to a fixed non-trivial character $\psi : F \to \mathbb{C}^\times$.

The following conjecture seems to be at the root of why the simple and general result of previous section on Euler-Poincare characteristic translates into a simple result about Hom spaces for generic representations. The author has not managed to prove it in any generality.

Conjecture 1. *Let π_1 be an irreducible generic representation of $\mathrm{GL}_{n+1}(F)$, and π_2 an irreducible generic representation π_2 of $\mathrm{GL}_n(F)$. Then,*

$$\mathrm{Ext}^i_{\mathrm{GL}_n(F)}[\pi_1, \pi_2] = 0,$$

for all $i > 0$.

We move on to branching from $\mathrm{SO}_{n+1}(F)$ to $\mathrm{SO}_n(F)$.

Theorem 3. *Let V be a quadratic space over the non-Archimedean local field F with W a quadratic subspace of codimension 1. Then for any irreducible admissible representation π of $\mathrm{SO}(V)$ and irreducible admissible representation σ of $\mathrm{SO}(W)$, $\mathrm{Ext}^i_{\mathrm{SO}(W)}[\pi, \sigma]$ are finite dimensional vector spaces over $\mathbb{C}$ for all $i \geq 0$.*

The following theorem is proved by Waldspurger.

Theorem 4. *Let V be a quadratic space over the non-Archimedean local field F with W a quadratic subspace of codimension 1. Then for any irreducible admissible representation σ of $\mathrm{SO}(V)$ and irreducible admissible representation σ' of $\mathrm{SO}(W)$,*

$$\sum_{T \in \mathcal{T}} |W(H,T)|^{-1} \int_{T(F)} c_\sigma(t) c_{\sigma'}(t) D^H(t) dt,$$

is a finite sum of absolutely convergent integrals on a certain set of Elliptic tori. (We refer to the papers of Waldspurger for the details on this as well as $c_\sigma(t)$.) If either σ is a supercuspidal representation of $\mathrm{SO}(V)$, and σ' is arbitrary irreducible admissible representation of $\mathrm{SO}(W)$, or both σ and σ' are tempered representations, then

$$\dim \mathrm{Hom}_{\mathrm{SO}(W)}[\sigma, \sigma'] = \sum_{T \in \mathcal{T}} |W(H,T)|^{-1} \int_{T(F)} c_\sigma(t) c_{\sigma'}(t) D^H(t) dt.$$

Given this theorem of Waldspurger, it is most natural to propose the following conjecture on Euler-Poincare pairing which we consider as an analogue of the Riemann-Roch theorem, together with a conjectured vanishing theorem in the style of Kodaira vanishing.

Conjecture 2. *Let V be a quadratic space over the non-Archimedean local field F with W a quadratic subspace of V of codimension 1 as before. Then for any irreducible admissible representation σ of $\mathrm{SO}(V)$ and irreducible admissible representation σ' of $\mathrm{SO}(W)$,*

(1)

$$\begin{aligned}\mathrm{EP}_{\mathrm{SO}(W)}[\sigma,\sigma'] &= \sum_i (-1)^i \dim \mathrm{Ext}^i_{\mathrm{SO}(W)}[\sigma,\sigma'] \\ &= \sum_{T\in\mathcal{T}} |W(H,T)|^{-1} \int_{T(F)} c_\sigma(t) c_{\sigma'}(t) D^H(t)dt.\end{aligned}$$

(2) If σ and σ' are irreducible tempered representations, then $\mathrm{Ext}^i_{\mathrm{SO}(W)}[\sigma,\sigma'] = 0$ for $i > 0$.

The following theorem is a reformulation of a result due to Schneider-Stuhler which we consider as an analogue of the Serre duality.

Theorem 5. *Let G be a reductive p-adic group, and π an irreducible, admissible representation of G. Let $d(\pi)$ be the largest integer $i \geq 0$ such that there is an irreducible, admissible representation π' of G with $\mathrm{Ext}^i_G[\pi,\pi']$ nonzero. Then,*

(1) There is a unique irreducible representation π' of G with $\mathrm{Ext}^{d(\pi)}_G[\pi,\pi'] \neq 0$.
(2) The representation π' in (1) is nothing but $D(\pi)$ where $D(\pi)$ is the Aubert-Zelevinsky involution of π.
(3) $\mathrm{Ext}^{d(\pi)}_G[\pi, D(\pi)] \cong \mathbb{C}$.
(4) For any smooth representation π' of G, the bilinear pairing

$$\mathrm{Ext}^i_G[\pi,\pi'] \times \mathrm{Ext}^j_G[\pi', D(\pi)] \to \mathrm{Ext}^{i+j=d(\pi)}_G[\pi, D(\pi)] \cong \mathbb{C},$$

is nondegenerate. (We remind the reader that the nondegeneracy of the pairing means, among other things, that if $\mathrm{Ext}^i_G[\pi,\pi'] = 0$ or $\mathrm{Ext}^j_G[\pi', D(\pi)] = 0$, then so is the other one.)

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On L-packets for p-adic symplectic groups

Shaun Stevens
(joint work with Corinne Blondel, Guy Henniart)

Let F be a p-adic field, with residue field of odd cardinality q and absolute value $|\cdot|$ with image $q^{\mathbb{Z}}$. Let π be an irreducible supercuspidal representation of $\mathrm{Sp}_{2n}(F)$, the symplectic group of rank n, let ρ be an irreducible cuspidal representation of $\mathrm{GL}_m(F)$ and put $m_\rho = m$. The unramified characters of $\mathrm{GL}_m(F)$ are all of the form $|\det(\cdot)|^s$, for some $s \in \mathbb{C}$, and we denote by $[\rho]$ the *inertial class of* ρ, that is the set of representations of the form $\rho|\det|^s$. We also denote by $t(\rho)$ the *twist number* of ρ, that is, the number of unramified characters χ such that $\rho\chi \simeq \rho$, which depends only on the inertial class $[\rho]$

Writing $G = \mathrm{GL}_{2(m+n)}(F)$, with standard parabolic subgroup P with Levi component $M = \mathrm{GL}_m(F) \times \mathrm{Sp}_{2n}(F)$, we denote by

$$I(\rho, \pi, s) := \mathrm{Ind}_{M,P}^G \, \rho|\det|^s \otimes \pi, \qquad s \in \mathbb{C},$$

the (normalized) parabolically induced representation. If none of the unramified twists $\rho|\det|^s$ are self-dual then $I(\rho, \pi, s)$ is always irreducible. Otherwise, replacing ρ by an unramified twist, we may assume ρ is self-dual; in fact there are exactly two (inequivalent) choices for such a representation – if one of them is ρ then the other is $\rho' := \rho|\det|^{\pi i/t_\rho \log(q)}$. In this case, a result of Silberger says that there is a (unique) non-negative *real* number s_ρ such that

$$\{s \in \mathbb{R} \mid I(\rho, \pi, s) \text{ is reducible}\} = \{\pm s_\rho\}.$$

If π is generic then $s_\rho \in \{0, \frac{1}{2}, 1\}$, by results of Shahidi; in general, results of Arthur and Mœglin imply that $s_\rho \in \frac{1}{2}\mathbb{Z}$.

Regarding π as fixed, and allowing ρ to vary over all self-dual cuspidal representations of all $\mathrm{GL}_m(F)$, the interesting reducibilities (which will characterize the Galois parameter of π) are those for which $s_\rho \geq 1$. More precisely, Mœglin defines a *reducibility set*

$$\mathrm{Red}(\pi) = \{(\rho, s_\rho) \mid s_\rho \geq 1\},$$

which is a finite set. Indeed, Mœglin also proves an equality (equivalent to)

$$\sum_{(\rho, s_\rho) \in \mathrm{Red}(\pi)} \lfloor s_\rho^2 \rfloor m_\rho = 2n + 1, \tag{1}$$

where the $2n+1$ comes from the fact that the Langlands dual group is $\mathrm{SO}_{2n+1}(F)$. Then, in order to determine the cuspidal representations in the L-packet of π, it suffices to find all cuspidal representations π' of $\mathrm{Sp}_{2n}(F)$ with $\mathrm{Red}(\pi') = \mathrm{Red}(\pi)$. (Of course there may also be discrete series representations in the L-packet of π; indeed, there will be some exactly when there exists $(\rho, s_\rho) \in \mathrm{Red}(\pi)$ with $s_\rho > 1$.)

The aim of our work is then to describe explicitly the set $\mathrm{Red}(\pi)$ in terms of the local data used to construct π in the work of Stevens (following Bushnell–Kutzko's constructions for $\mathrm{GL}_m(F)$). So far, our results only relate to the inertial reducibility *multiset*

$$\mathrm{IRed}(\pi) := \{([\rho], s_\rho) \mid (\rho, s_\rho) \in \mathrm{Red}(\pi)\}.$$

The method we use is to translate the problem to one on Hecke algebras, via Bushnell–Kutzko's theory of types and covers. This gives us an inclusion of Hecke algebras $\mathcal{H}_M \hookrightarrow \mathcal{H}_G$, with $\mathcal{H}_M = \mathbb{C}[Z^{\pm 1}]$ and $\mathcal{H}_G$ generated by T_0, T_1 satisfying quadratic relations of the form

$$(T_i - q^{r_i})(T_i + 1) = 0, \qquad r_i \geq 0.$$

Then a result of Blondel says that

$$\{s_\rho, s_{\rho'}\} = \left\{ \frac{|r_0 \pm r_1|}{2t_\rho} \right\},$$

where we recall that ρ' is the self-dual unramified twist of ρ (not equivalent to ρ) and t_ρ is the twist number of ρ. In principle, it is possible to distinguish s_ρ and $s_{\rho'}$ from the embedding of the Hecke algebras, but this requires the calculation of a subtle sign in the normalization of the embedding. It is for this reason that our results so far only concern $\mathrm{IRed}(\pi)$.

The parameters r_i (and thus the pair or reducibility points $\{s_\rho, s_{\rho'}\}$) are computable: the operator T_i generates a two-dimensional subalgebra of $\mathcal{H}_G$, which is isomorphic to the endomorphism algebra of a representation induced from a cuspidal representation of a maximal parabolic subgroup in a finite reductive group. The parameters for these can be computed from results of Lusztig, using the Jordan decomposition of characters to reduce to the case of unipotent representations (though care must be taken since the finite reductive groups which appear need not have connected centre, or even be connected). Hidden in these isomorphisms is a (tame) quadratic character, which is also quite subtle and not explicit.

The first result we arrive at through these considerations can be roughly described as follows: the endo-classes (with multiplicity) in ρ, for $(\rho, s_\rho) \in \mathrm{Red}(\pi)$, are exactly the ones that would be guessed from the construction of π. (See [1] for the definition of endo-class.) This can be interpreted as saying that all the cuspidal representations in an L-packet are constructed from equivalent semisimple characters (but possibly embedded differently in $\mathrm{Sp}_{2n}(F)$); that is, they have the same "wild part". (See [2] for the definition of semisimple character.) It would also be possible to interpret this as giving a bijection between "skew semisimple endo-classes" and the restrictions to wild inertia of elliptic galois parameters, compatible with the local Langlands correspondence.

We also have a more precise result, describing the inertial reducibility multiset $\mathrm{IRed}(\pi)$ in terms of the multiset for simpler cuspidal representations of smaller symplectic groups. (Here, "simpler" means that the semisimple character they contain is in fact simple.) Again, a tame quadratic character appears here, which can be described as the signature of a certain adjoint action. We are then able to recover the fact that $s_\rho \in \frac{1}{2}\mathbb{Z}$, as well as equation (1); this is also the case when F is of positive (odd) characteristic.

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A computation with Bernstein projectors of depth 0 for p-adic SL(2)

Allen Moy

We present the result of a computation on the support of the sum of the depth zero Bernstein projectors of the p-adic group SL(2). The computation is based on a conversation with Roger Howe in August 2013. The computation is elementary, but it and expected similar results for sums of fixed higher depth Bernstein projectors provides a useful expansion of the delta distribution δ_0 on the Lie algebra into an infinite sum of G-invariant distributions supported on the set of topologically nilpotent elements. The computation in particular relies on the SL(2) discrete series character table computed by Sally-Shalika in 1968.

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A Hecke algebra correspondence for the metaplectic group over $\mathbb{Q}_2$

Aaron Wood

Let W be a nondegenerate symplectic space over a p-adic field, $G = \widetilde{\mathrm{Sp}}(W)$ a two-fold central extension of the symplectic group $\underline{G} = \mathrm{Sp}(W)$ of type C_n, and ω the Weil representation of G.

If p is odd, then ω has a one-dimensional subspace upon which the inverse image of the Iwahori subgroup acts. This minimal type was used by Gan and Savin in [2] to establish an equivalence of categories between certain representations of G and certain other representations of split adjoint groups of type B_n. In this equivalence, the even Weil representation of G corresponds to the trivial representation of the orthogonal group. The correspondence is realized explicitly as an isomorphism between the Hecke algebra of this minimal type and the classical affine Hecke algebra of type B_n. For $p = 2$, there is no such luxury as a one-dimensional type for the Iwahori subgroup.

For $p = 2$, a compact subgroup J of the Iwahori group is described that has a one-dimensional type, and the corresponding Hecke algebra is computed. In the end, the result is the same as in the case of p odd; that is, the Hecke algebra is shown to be isomorphic to that of the split adjoint orthogonal group.

The description of the appropriate open compact subgroup stems from the following fact: in characteristic 2, the bilinear form attached to a symmetric quadratic form is also alternating. Hence, the finite split orthogonal group $O_{2n}(\mathbb{F}_2)$ may be realized as a subgroup of the finite symplectic group $\mathrm{Sp}_{2n}(\mathbb{F}_2)$. If B' is a Borel subgroup of $O_{2n}(\mathbb{F}_2)$ which sits in a Borel subgroup B of $\mathrm{Sp}_{2n}(\mathbb{F}_2)$, then the inverse image $\underline{J}$ of B' (under the projection map $\mathbb{Z}_2 \to \mathbb{F}_2$) is a subgroup of the Iwahori subgroup. Let J be the full inverse image in G of $\underline{J}$. In the $\widetilde{\mathrm{SL}}_2$ case, this J is conjugate to the congruence group $\Gamma_0(4)$.

For the SL_2 case, the line on which J acts is spanned by the characteristic function of $\mathbb{Z}_2$; the support of the associated Hecke algebra $\mathcal{H}$ is reduced to the set of J-double cosets parametrized by the affine Weyl group; and generators T_0 and T_1 for $\mathcal{H}$ are given which satisfy only the quadratic relations $(T_0 - 2)(T_0 + 1) = 0$ and $T_1^2 = 1$, giving the identification with the affine Hecke algebra of $\mathrm{PGL}_2(\mathbb{Q}_2)$.

For the general symplectic case, the line on which J acts is spanned by the characteristic function of the standard lattice; the support of the associated Hecke algebra $\mathcal{H}$ is again reduced to the set of J-double cosets parametrized by the affine Weyl group; and generators $T_0, \ldots, T_n$ for $\mathcal{H}$ are given which satisfy the quadratic relations

$$(T_i - 2)(T_i + 1) = 0 \quad \text{and} \quad T_n^2 = 1$$

and the braid relations of type C_n.

The affine Hecke algebra $\mathcal{H}'$ of the split adjoint orthogonal group has generators $t_0, \ldots, t_n$ and τ, where τ corresponds to the involution of the extended Dynkin diagram of type B_n, with $\tau^2 = 1$ and $\tau t_1 \tau = t_0$. The quadratic relations for the generators t_i are $(t_i - 2)(t_i + 1) = 0$. Since t_0 may be expressed in terms of t_1 and τ, it is not needed to define $\mathcal{H}'$ as an abstract algebra. The braid relation between τ and t_1 is $\tau t_1 \tau t_1 = t_1 \tau t_1 \tau$, so the isomorphism from $\mathcal{H}'$ to $\mathcal{H}$ is given by

$$t_n \mapsto T_0, \ \ldots \ t_1 \mapsto T_{n-1} \quad \text{and} \quad \tau \mapsto T_n.$$

Finally, this map preserves the trace and $*$ operations for Hilbert algebras, which ensures, by [1], that the induced Plancherel measures coincide in the correspondence of representations given by the Hecke algebra isomorphism.

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Conormal variety over a double flag variety and exotic nilpotent cone

Kyo Nishiyama

Let G be a connected reductive algebraic group over $\mathbb{C}$, and $\theta \in \mathrm{Aut}(G)$ be an involution. Let (G, K) be the corresponding symmetric pair, where $K = G^\theta$ is a subgroup of fixed points of the involution θ. We consider a *double flag variety* $X := K/Q \times G/P$, where P (resp. Q) is a parabolic subgroup of G (resp. K). The symmetric group K acts on X diagonally, and if there are finitely many K-orbits, X is called of *finite type*.

We are interested in the case where X is of finite type. We gave two useful conditions which implies finiteness of K-orbits in X in [2] (joint work with Hiroyuki Ochiai). Also, in the joint work with Xuhua He, Hiroyuki Ochiai and Yoshiki Oshima [6], we gave a complete classification of X of finite type when P is a Borel subgroup of G, or Q is a Borel subgroup of K. In the latter case, the classification is equivalent to that of K-spherical partial flag varieties G/P.

From now on, let us assume that $X = K/Q \times G/P$ is of finite type. Consider a moment map $\mu_X : T^*X \to \mathfrak{k}^*$ with respect to the action of K, and put $Y = \mu_X^{-1}(0)$. Y is called a *conormal variety*, and it is a union of conormal bundles over K-orbits (hence the name of Y).

$$Y = \bigcup\nolimits_{\mathbb{O}} \overline{T^*_{\mathbb{O}} X} \quad (\mathbb{O} \text{ moves over } K\text{-orbits in } X)$$

There exists an intermediate "moment map" denoted $\widetilde{\mu}$ from Y to an *exotic nilpotent cone* $K/Q \times \mathcal{N}(\mathfrak{k}^\perp)$. In very good cases, it is known that $K/Q \times \mathcal{N}(\mathfrak{k}^\perp)$ admits a finite number of K-orbits (e.g., the case of enhanced nilpotent cone ([1], [4]), or that of Kato's exotic nilcone ([5])). Thus we get $\widetilde{\mu}(\overline{T^*_{\mathbb{O}} X}) = \overline{\mathcal{O}}$ for a certain nilpotent K-orbit $\mathcal{O}$. In this way, we can define a map: $\Phi : X/K \ni \mathbb{O} \mapsto \mathcal{O} \in \big(K/Q \times \mathcal{N}(\mathfrak{k}^\perp)\big)/K$.

The following theorem is a recent result of the joint work with Lucas Fresse (being prepared for publication).

Theorem 3 (Fresse-N)**.** *Let* $(G, K) = (\mathrm{GL}_n, \mathrm{GL}_p \times \mathrm{GL}_q)$ *be a symmetric pair of type* AIII. Assume that Q is a mirabolic subgroup of K, and P is a maximal parabolic subgroup of G. Then the orbit correspondence

$$\Phi : X/K \ni \mathbb{O} \mapsto \mathcal{O} \in \big(K/Q \times \mathcal{N}(\mathfrak{k}^\perp)\big)/K$$

is generically one-to-one and it can be described explicitly in terms of 2-bipartitions in the sense of Johnson.

We can generalize this result in various ways. For example, in the case where P is not necessarily maximal or Q is a product of mirabolic subgroups, we can argue similarly, and we hope to obtain some kind of Robinson-Schensted correspondence (cf. [3]). Details will be reported elsewhere.

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Conservation relations for local theta correspondence

Chen-bo Zhu

(joint work with Binyong Sun)

The goal of this talk is to explain "conservation relations" for local theta correspondence, which was first conjectured by Kudla and Rallis in the mid 1990's, and proved in a recent work of the authors [8].

1. Dual pairs of type I: Fix a triple $(\mathrm{F}, \mathrm{D}, \epsilon)$ where $\epsilon = \pm 1$; F is a local field of characteristic zero; and D is either F, or a quadratic field extension of F, or a central division quaternion algebra over F.

Let U be an ϵ-Hermitian right D-vector space, and let $\mathrm{G}(U)$ be the isometry group of U. Let V be an ϵ-skew-Hermitian left D-vector space, and let $\mathrm{G}(V)$ be the isometry group of V. Following Howe, we call $(\mathrm{G}(U), \mathrm{G}(V))$ an irreducible dual pair of type I. We may form the modified Jacobi group

$$\bar{\mathrm{J}}(U, V) := (\bar{\mathrm{G}}(U) \times \bar{\mathrm{G}}(V)) \ltimes \mathrm{H}(U \otimes_{\mathrm{D}} V),$$

where

$$\bar{\mathrm{G}}(U) := \begin{cases} \widetilde{\mathrm{Sp}}(U), & \text{if } U \text{ is a symplectic space, namely } (\mathrm{D}, \epsilon) = (\mathrm{F}, -1); \\ \mathrm{G}(U), & \text{otherwise,} \end{cases}$$

and $\mathrm{H}(U \otimes_{\mathrm{D}} V)$ is the Heisenberg group associated to the symplectic space $U \otimes_{\mathrm{D}} V$.

2. Local theta correspondence: For the purpose of this talk, we assume that F is non-archimedean. Fix a non-trivial unitary character $\psi : \mathrm{F} \to \mathbb{C}^\times$. Recall that for any symplectic space W over F, there exists a unique (up to isomorphism) irreducible smooth representation of $\mathrm{H}(W)$ with central character ψ.

Definition: Let J be a totally disconnected, locally compact, Hausdorff topological group. Assume that J contains the Heisenberg group $\mathrm{H}(W)$ as a closed normal subgroup. A smooth representation of J is called a smooth oscillator representation if its restriction to $\mathrm{H}(W)$ is irreducible and has central character ψ.

By the well-known result of splitting metaplectic covers [4], there exists a genuine smooth oscillator representation of $\bar{\mathrm{J}}(U, V)$. Dixmier's version of Schur's

Lemma implies that smooth oscillator representations are unique up to twisting by characters. For a genuine smooth oscillator representation $\omega_{U,V}$ of $\bar{\mathrm{J}}(U,V)$, put

$$\mathcal{R}_{\omega_{U,V}}(U,V) := \{(\pi,\pi') \in \mathrm{Irr}(\bar{\mathrm{G}}(U)) \times \mathrm{Irr}(\bar{\mathrm{G}}(V)) \mid \mathrm{Hom}_{\bar{\mathrm{G}}(U)\times\bar{\mathrm{G}}(V)}(\omega_{U,V}, \pi\otimes\pi') \neq 0\},$$

$$\mathcal{R}_{\omega_{U,V}}(U) := \{\pi \in \mathrm{Irr}(\bar{\mathrm{G}}(U)) \mid \mathrm{Hom}_{\bar{\mathrm{G}}(U)}(\omega_{U,V}, \pi) \neq 0\},$$

and analogously for $\mathcal{R}_{\omega_{U,V}}(V)$.

Howe Duality Conjecture ([1]): The set $\mathcal{R}_{\omega_{U,V}}(U,V)$ is the graph of a bijection between $\mathcal{R}_{\omega_{U,V}}(U)$ and $\mathcal{R}_{\omega_{U,V}}(V)$.

Waldspurger [9] proves the above conjecture when F has odd residue characteristic. For the archimedean case, the corresponding result is due to Howe [2].

3. The generalized Witt group: Denote by $\widehat{\mathcal{W}}_0^+$ the commutative monoid of all isometry classes of ϵ-skew-Hermitian left D-vector spaces. Denote by $\mathbb{H}$ the hyperbolic plane in $\widehat{\mathcal{W}}_0^+$. A subset of $\widehat{\mathcal{W}}_0^+$ of the form

$$\mathbf{t} := \{V_0, V_0+\mathbb{H}, V_0+2\mathbb{H}, \cdots\}$$

is called a Witt tower in $\widehat{\mathcal{W}}_0^+$, where V_0 is an anisotropic element in $\widehat{\mathcal{W}}_0^+$.

Denote by $\mathcal{W}_0$ the set of all Witt towers in $\widehat{\mathcal{W}}_0^+$. This is a quotient set of $\widehat{\mathcal{W}}_0^+$, and the monoid structure on $\widehat{\mathcal{W}}_0^+$ descends to a monoid structure on $\mathcal{W}_0$. The resulting monoid $\mathcal{W}_0$ is a finite abelian group, which is called the Witt group of ϵ-skew-Hermitian left D-vector spaces.

Write $\mathrm{d}_{\mathrm{D},\epsilon}$ for the maximal dimension of an anisotropic element in $\widehat{\mathcal{W}}_0^+$:

$$\mathrm{d}_{\mathrm{D},\epsilon} := \max\{\dim V_0 \mid V_0 \text{ is an anisotropic element of } \widehat{\mathcal{W}}_0^+\}.$$

Proposition: One has that

$$(1) \quad \mathrm{d}_{\mathrm{D},\epsilon} = \begin{cases} 0, & \text{if } V \text{ is a symplectic space;} \\ 1, & \text{if } V \text{ is a quaternionic Hermitian space;} \\ 2, & \text{if } V \text{ is a Hermitian space or a skew-Hermition space;} \\ 3, & \text{if } V \text{ is a quaternionic skew-Hermitian space;} \\ 4, & \text{if } V \text{ is a symmetric bilinear space.} \end{cases}$$

Moreover, there exists a unique element $V^\circ \in \widehat{\mathcal{W}}_0^+$ which is anisotropic and has dimension $\mathrm{d}_{\mathrm{D},\epsilon}$; and every anisotropic element of $\widehat{\mathcal{W}}_0^+$ is isometrically isomorphic to a subspace of V°.

Denote by $\mathbf{t}_0^\circ$ the unique element of $\mathcal{W}_0$ of anisotropic degree $\mathrm{d}_{\mathrm{D},\epsilon}$ and call it the anti-split Witt tower in $\mathcal{W}$.

For a fixed $\pi \in \mathrm{Irr}(\bar{\mathrm{G}}(U))$, we shall consider occurrence of π in $\omega_{U,V}$, or the membership of π in $\mathcal{R}_{\omega_{U,V}}(U)$, as V vary. The latter set depends only on the restriction of $\omega_{U,V}$ to the subgroup

$$\bar{\mathrm{J}}_U(V) := \bar{\mathrm{G}}(U) \ltimes \mathrm{H}(U \otimes_{\mathrm{D}} V) \subset \bar{\mathrm{J}}(U,V).$$

Definition: An enhanced oscillator representation of $\bar{\mathrm{G}}(U)$ is a pair (V, ω), where V is an ϵ-skew-Hermitian left D-vector space, and ω is a genuine smooth oscillator representation of $\bar{\mathrm{J}}_U(V)$. Two enhanced oscillator representations (V_1, ω_1) and (V_2, ω_2) of $\bar{\mathrm{G}}(U)$ are said to be isomorphic if there is an isometric isomorphism $V_1 \cong V_2$ such that ω_1 is isomorphic to ω_2 with respect to the induced isomorphism $\bar{\mathrm{J}}_U(V_1) \cong \bar{\mathrm{J}}_U(V_2)$.

Denote by $\widehat{\mathcal{W}}_U^+$ the set of all isomorphism classes of enhanced oscillator representations of $\bar{\mathrm{G}}(U)$, which has a natural commutative monoid structure.

We may define notion of the hyperbolic plane $\mathbb{H}_U$ in $\widehat{\mathcal{W}}_U^+$, and thus notion of a Witt tower $\mathbf{t} \subset \widehat{\mathcal{W}}_U^+$:

$$\mathbf{t} = \{\sigma_{\mathbf{t}}, \sigma_{\mathbf{t}} + \mathbb{H}_U, \sigma_{\mathbf{t}} + 2\mathbb{H}_U, \cdots\},$$

where $\sigma_{\mathbf{t}}$ is "anisotropic". Write $\mathcal{W}_U$ for the set of all Witt towers in $\widehat{\mathcal{W}}_U^+$. Similar to the Witt group $\mathcal{W}_0$, the additive structure on $\widehat{\mathcal{W}}_U^+$ descends to an additive structure on $\mathcal{W}_U$ which makes $\mathcal{W}_U$ an abelian group. There is a short exact sequence

$$1 \to (\mathrm{G}(U))^* \to \mathcal{W}_U \to \mathcal{W}_0 \to 1.$$

Here $(\mathrm{G}(U))^*$ denotes the group of all characters on $\mathrm{G}(U)$.

4. First occurrence index and conservation relations

For a given $\sigma = (V, \omega) \in \widehat{\mathcal{W}}_U^+$, one seeks to determine the occurrence set

$$\mathcal{R}_\sigma := \{\pi \in \mathrm{Irr}(\bar{\mathrm{G}}(U)) \mid \mathrm{Hom}_{\bar{\mathrm{G}}(U)}(\omega, \pi) \neq 0\}.$$

Let $\pi \in \mathrm{Irr}(\bar{\mathrm{G}}(U))$ and let $\mathbf{t} \in \mathcal{W}_U$. Assume that π is genuine with respect to $\mathbf{t}$. There are two basic properties concerning occurrence:

- Occurrence in the so-called stable range:

$$\text{for all } \sigma \in \mathbf{t}, \text{ if } \operatorname{rank} \sigma \geq \dim U, \text{ then } \pi \in \mathcal{R}_\sigma.$$

- Kudla's persistence principle [3]:

$$\text{for all } \sigma_1, \sigma_2 \in \mathbf{t}, \text{ if } \dim \sigma_1 \leq \dim \sigma_2, \text{ then } \mathcal{R}_{\sigma_1} \subset \mathcal{R}_{\sigma_2}.$$

Define the first occurrence index

$$\mathrm{n}_{\mathbf{t}}(\pi) := \min\{\dim \sigma \mid \sigma \in \mathbf{t}, \pi \in \mathcal{R}_\sigma\}.$$

Generalizing the anti-split Witt tower $\mathbf{t}_0^\circ \in \mathcal{W}_0$, we define the anti-split Witt tower $\mathbf{t}_U^\circ \in \mathcal{W}_U$. When U is a symmetric bilinear space, $\mathbf{t}_U^\circ \in \mathcal{W}_U = (\mathrm{O}(U))^*$ is the sign character; when U is a symplectic space or a quaternionic Hermitian space (in which case $\mathrm{G}(U)$ is a perfect group), $\mathbf{t}_U^\circ \in \mathcal{W}_U = \mathcal{W}_0$ is identical to $\mathbf{t}_0^\circ$. The element $\mathbf{t}_U^\circ \in \mathcal{W}_U$ has anisotropic degree $\mathrm{d}_{\mathrm{D},\epsilon}$, and has order 2 unless U is the zero symmetric bilinear space (in this exceptional case the group $\mathcal{W}_U$ is trivial).

In the non-archimedean case, the conservation relations assert the following:

Theorem: Let $\mathbf{t}_1$ and $\mathbf{t}_2$ be two Witt towers in $\mathcal{W}_U$ with difference $\mathbf{t}_U^\circ$. Then for any $\pi \in \mathrm{Irr}(\bar{\mathrm{G}}(U))$ which is genuine with respect to $\mathbf{t}_1$ (and hence genuine with respect to $\mathbf{t}_2$), one has that

$$\mathrm{n}_{\mathbf{t}_1}(\pi) + \mathrm{n}_{\mathbf{t}_2}(\pi) = 2\dim U + \mathrm{d}_{\mathrm{D},\epsilon}.$$

Remarks: (a) For orthogonal-symplectic and unitary-unitary dual pairs, the conservation relations were conjectured by Kudla and Rallis in the mid 1990's. For quaternionic dual pairs, the conjectured statements appeared in an article of Gan-Tantono. (b) For π supercuspidal, the conservation relations were known by the work of Kudla and Rallis, and Minguez.

We comment on the ideas of proof of the main theorem. There are two equally important aspects of the conservation relations, which respectively assert the non-occurrence:

$$\mathrm{n}_{\mathbf{t}_1}(\pi) + \mathrm{n}_{-\mathbf{t}_2}(\pi^\vee) \geq 2\dim U + \mathrm{d}_{\mathrm{D},\epsilon}, \tag{2}$$

and the occurrence:

$$\mathrm{n}_{\mathbf{t}_1}(\pi) + \mathrm{n}_{\mathbf{t}_2}(\pi) \leq 2\dim U + \mathrm{d}_{\mathrm{D},\epsilon}. \tag{3}$$

Assuming both (2) and (3), we then have

$$\begin{cases} \mathrm{n}_{\mathbf{t}_1}(\pi) + \mathrm{n}_{-\mathbf{t}_2}(\pi^\vee) & \geq\ 2\dim U + \mathrm{d}_{\mathrm{D},\epsilon}; \\ \mathrm{n}_{\mathbf{t}_2}(\pi) + \mathrm{n}_{-\mathbf{t}_1}(\pi^\vee) & \geq\ 2\dim U + \mathrm{d}_{\mathrm{D},\epsilon}; \\ \mathrm{n}_{\mathbf{t}_1}(\pi) + \mathrm{n}_{\mathbf{t}_2}(\pi) & \leq\ 2\dim U + \mathrm{d}_{\mathrm{D},\epsilon}; \\ \mathrm{n}_{-\mathbf{t}_1}(\pi^\vee) + \mathrm{n}_{-\mathbf{t}_2}(\pi^\vee) & \leq\ 2\dim U + \mathrm{d}_{\mathrm{D},\epsilon}. \end{cases}$$

This forces all the above inequalities to be equalities!

For the non-occurrence, the key phenomenon is the late occurrence of the trivial representation 1_U in the anti-split Witt tower $\mathbf{t}_U^\circ$:

$$\mathrm{n}_{\mathbf{t}_U^\circ}(1_U) \geq 2\dim U + \mathrm{d}_{\mathrm{D},\epsilon}. \tag{4}$$

This asserts the vanishing of certain vector valued $\bar{\mathrm{G}}(U)$-invariant distributions. We show (4) following the method of Rallis [6, 7]. For the occurrence, we use the doubling method, theory of local zeta integrals, and critically the known structure of degenerate principle series, which are in fact all part of the foundational work of Kudla and Rallis [5].

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Limit multiplicities for congruence subgroups of arithmetic lattices

Tobias Finis
(joint work with Erez Lapid, Werner Müller)

The limit multiplicity problem, which goes back to DeGeorge and Wallach, concerns the asymptotic behavior of the spectra of lattices Γ (discrete subgroups of finite covolume $\mathrm{vol}(\Gamma\backslash G)$) in a fixed semisimple Lie group G in the situation where $\mathrm{vol}(\Gamma\backslash G) \to \infty$. In a great number of cases, the normalized discrete spectra μ_Γ converge then to the Plancherel measure μ_{pl} of the group G.

For uniform lattices Γ (lattices for which the quotient $\Gamma\backslash G$ is compact), general results on this problem have been known for some time. The case of normal towers, i.e., of descending sequences of finite index normal subgroups of a given uniform lattice with trivial intersection, was completely resolved by Delorme [4]. Recently, limit multiplicity has been shown for much more general sequences of uniform lattices [1, 2].

In the case of non-compact quotients $\Gamma\backslash G$, where the spectrum also contains a continuous part, much less is known. In a recent joint preprint of the author with E. Lapid and W. Müller [9], this case has been analyzed in a rather general setup. An extension of these results will appear in a forthcoming paper of E. Lapid and the author. (See [9, §1] for previous results in the literature.) The new approach is based on a careful study of the spectral side of Arthur's trace formula in the recent form given in [5, 7]. The results are unconditional only for the groups $\mathrm{GL}(n)$ and $\mathrm{SL}(n)$, but in the general case a substantial reduction of the problem is obtained.

To give more details, let G be a connected linear semisimple Lie group with a fixed choice of Haar measure. Since the group G is of type I, we can write unitary representations of G on separable Hilbert spaces as direct integrals (with multiplicities) over the unitary dual $\Pi(G)$, the set of isomorphism classes of irreducible unitary representations of G with the Fell topology. The regular representation of $G \times G$ on $L^2(G)$ decomposes as the direct integral of the tensor products $\pi \otimes \pi^*$ against the *Plancherel measure* μ_{pl} on $\Pi(G)$. The support of the Plancherel measure is called the *tempered dual* $\Pi(G)_{\mathrm{temp}} \subset \Pi(G)$. The Plancherel measure and the tempered dual are well understood, mainly by the work of Harish-Chandra.

By definition, a Jordan measurable subset of $\Pi(G)_{\mathrm{temp}}$ is a bounded set A such that $\mu_{\mathrm{pl}}(\bar{A} - A^\circ) = 0$. We say that a collection $\mathfrak{M}$ of Borel measures μ on $\Pi(G)$ has the *limit multiplicity property* if the following two conditions are satisfied:

(1) For any Jordan measurable Borel set $A \subset \Pi(G)_{\mathrm{temp}}$ we have[1]

$$\mu(A) \to \mu_{\mathrm{pl}}(A), \quad \mu \in \mathfrak{M}.$$

(2) For any bounded Borel set $A \subset \Pi(G) \setminus \Pi(G)_{\mathrm{temp}}$ we have

$$\mu(A) \to 0, \quad \mu \in \mathfrak{M}.$$

We apply this setup to the regular representation R_Γ of G on $L^2(\Gamma\backslash G)$ for a lattice Γ in G. For any $\pi \in \Pi(G)$ let $m_\Gamma(\pi)$ be the multiplicity of π in $L^2(\Gamma\backslash G)$. These multiplicities are known to be finite, at least if either G has no compact factors or if Γ is arithmetic. We define the discrete spectral measure on $\Pi(G)$ with respect to Γ by

$$\mu_\Gamma = \frac{1}{\mathrm{vol}(\Gamma\backslash G)} \sum_{\pi\in\Pi(G)} m_\Gamma(\pi)\delta_\pi,$$

where δ_π is the Dirac measure at π.

In [1, 2] it is shown (in particular) that the collection of measures μ_Γ, where Γ ranges over the collection of all congruence subgroups of a given *uniform* lattice not containing non-trivial central elements, has the limit multiplicity property. An independent proof for this will be contained in work in preparation of E. Lapid and the author (see [6]). These proofs are purely group-theoretic and do not involve automorphic representations.

For the non-cocompact lattices $\mathrm{SL}(n, \mathfrak{o}_F) \subset \mathrm{SL}(n, F \otimes \mathbb{R})$, where F is a number field, we can show the following:

Theorem 1. *Let F be a number field. The collection of measures μ_Γ, where Γ runs over all congruence subgroups of $\mathrm{SL}(n, \mathfrak{o}_F)$ not containing non-trivial central elements, has the limit multiplicity property.*

Note that for $n \geq 3$ and F not totally complex, every finite index subgroup of $\mathrm{SL}(n, \mathfrak{o}_F)$ is in fact a congruence subgroup.

For general arithmetic lattices, our results reduce the corresponding statement to two conjectural properties of intertwining operators (denoted by (TWN) and (BD) in [9]). It seems very likely that the theorem can be generalized to the lattices $\mathrm{SL}(m, \mathfrak{o}_D)$, where D is a division algebra with center F. Other cases (classical groups) seem accessible. Unlike the results of [1, 2] for uniform lattices, our current proof does not cover more general sequences of lattices, where infinitely many distinct commensurability classes are allowed, although it might be possible to deal with this situation by making the dependence of all parameters on D and F explicit.

As mentioned already above, the proof of Theorem 1 is based on Arthur's trace formula [3], an elaborate extension of the Selberg trace formula to the non-cocompact case. One needs to control both its geometric and its spectral side. It is the spectral side which poses genuinely new problems. The contribution from

[1] Here convergence means that for any $\varepsilon > 0$ the set of all $\mu \in \mathfrak{M}$ with $|\mu(A) - \mu_{\mathrm{pl}}(A)| \geq \varepsilon$ is finite.

the continuous spectrum to R_Γ involves generalized logarithmic derivatives of intertwining operators. In [5, 7] it was shown that these can be rewritten in terms of usual logarithmic derivatives $A^{-1}(s)A'(s)$ of operator-valued meromorphic functions $A(s)$ of one variable. Each such operator can be decomposed as a product of a scalar normalizing factor and a tensor product (over all places of the ground field F) of locally defined normalized intertwining operators, whose matrix coefficients are essentially rational functions. The necessary control of the scalar factors can for $\mathrm{GL}(n)$ and $\mathrm{SL}(n)$ be deduced from the theory of Rankin-Selberg L-functions (i.e., the control over their poles and the explicit functional equation). Regarding the local operators, the fact that only first derivatives occur implies that we only need to bound the degrees of their matrix coefficients in terms of the level of the congruence subgroup, which was achieved (at least for $\mathrm{GL}(n)$ and $\mathrm{SL}(n)$) in [8].

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The local Langlands correspondence for inner forms of SL_n

Maarten Solleveld

(joint work with Anne-Marie Aubert, Paul Baum, Roger Plymen)

Let F be a local non-archimedean field. A fundamental result in representation theory is the proof of the local Langlands correspondence (LLC) for $GL_n(F)$. It provides a canonical bijection between

- a set of Langlands parameters for $GL_n(F)$;
- the space of irreducible admissible complex representations of $GL_n(F)$.

From this one can derive the LLC for some other groups, in particular for $SL_n(F)$ and for the inner forms of $GL_n(F)$.

In this talk we establish the LLC for a more difficult class of groups, namely the inner forms of $SL_n(F)$. Every such group looks like $SL_m(D)$, where D is a division algebra with centre F and $\dim_F(D) = (n/m)^2$. The complications are mainly caused by the L-packets, which (in contrast to for $GL_n(F)$) need not be singletons. To parametrize all L-packets we adjust the classical setup in two ways:

- We consider all inner forms of $SL_n(F)$ simultaneously, parametrizing them by isomorphism classes of central simple F-algebras $M_m(D)$.
- We enhance every Langlands parameter ϕ with an irreducible representation ρ of a finite group $\mathcal{S}_\phi$, a certain central extension of the geometric R-group R_ϕ.

Our main result reads:

Theorem [1]
There exists a bijection

$$\left\{ \begin{array}{c} \text{enhanced Langlands} \\ \text{parameters for } SL_n(F) \end{array} \right\} \longleftrightarrow \left\{ (G,\pi) \middle| \begin{array}{c} G \text{ an inner form of } SL_n(F), \\ \pi \in \mathrm{Irr}(G) \end{array} \right\}$$
$$(\phi,\rho) \mapsto \big(SL_m(D_\rho), \pi(\phi,\rho)\big)$$

Here $SL_m(D_\rho)$ and $M_m(D_\rho)$ are determined by their Kottwitz parameter, which is essentially the restriction of ρ to $\ker(\mathcal{S}_\phi \to R_\phi)$.

Over p-adic fields, this result relies on a beautiful paper of Hiraga and Saito [2].

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Galois Cohomology of Real Groups

Jeffrey Adams

Suppose G is a connected, reductive algebraic group defined over a local field F, and let Γ be the absolute Galois group. The Galois cohomology groups $H^i(\Gamma, G)$ are important invariants associated to G. For example if G is the symmetry group of a form (symplectic, orthogonal, Hermitian, etc.) then $H^1(\Gamma, G)$ is closely related to the classification of such forms. The F-forms of G which are "inner" to the given one are parametrized by $H^1(\Gamma, G_{\mathrm{ad}})$, and these are well understood.

Now take $F = \mathbb{R}$. A real form of G is simply an antiholomorphic involution of $G = G(\mathbb{C})$, with equivalence given by conjugation by inner automorphisms of G (this is a slight variant of the usual definition, which allows conjugation by all of $\mathrm{Aut}(G)$). Now the real case is special for several reasons, the first of which is that we can classify real forms by their Cartan involution: this is a holomorphic involution of $G(\mathbb{C})$, whose fixed points on $G(\mathbb{R})$ is a maximal compact subgroup of $G(\mathbb{R})$.

This gives rise to two parallel approaches to real forms: the Galois (antiholomorphic) and θ (holomorphic) pictures. The equivalence of these amounts to an isomorphism $H^1(\Gamma, G_{\mathrm{ad}}) \simeq H^1_\theta(\mathbb{Z}_2, G_{\mathrm{ad}})$ Here $H^1_\theta(\mathbb{Z}_2, *)$ is the Galois cohomology of the group $\mathbb{Z}_2 = \mathbb{Z}/2\mathbb{Z}$, with the nontrivial element acting by θ.

The equivalence of the two pictures amounts to an isomorphism $H^1_\sigma(\Gamma, G_{\mathrm{ad}}) \simeq H^1_\theta(\mathbb{Z}_2, G_{\mathrm{ad}})$, where we have written $H^1_\sigma(\Gamma, G)$ to indicate the action by the antiholomorphic involution σ. Our main result is that this isomorphism holds in general [1].

Theorem Suppose σ and θ are antiholomorphic and holmorphic respectively, defining the same real form of G as above. Then there is a canonical isomorphism $H^1_\sigma(\Gamma, G) \simeq H^1_\theta(\mathbb{Z}_2, G)$.

We give several applications. In particular we compute $H^1(\Gamma, G)$ for any simple, simply connected real group G. In the p-adic case $H^1(\Gamma, G) = 1$ for such a group (Kneser's theorem), while in the real case these groups were not known in some cases.

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Convexity theorems for semisimple symmetric spaces

Erik P. van den Ban
(joint work with Dana Balibanu)

We present a new convexity theorem for a semisimple symmetric space G/H, and an application to harmonic analysis on such a space.

Here G is a connected semisimple Lie group with finite center, and H the connected component of the group of fixed points of an involution σ. We write $\mathfrak{q} \subset \mathfrak{g}$ for the minus -1 eigenspace of the derived involution.

Let K be a σ-stable maximal compact subgroup of G, θ the associated infinitesimal Cartan involution and $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ the associated Cartan decomposition of $\mathfrak{g}$. Moreover, let $\mathfrak{a}_{\mathrm{q}}$ be a maximal abelian subspace of $\mathfrak{p} \cap \mathfrak{q}$ and $G = KAN$ an Iwasawa decomposition such that $A \supset \exp(\mathfrak{a}_q)$. The given Iwasawa decomposition determines an Iwasawa projection $\mathfrak{H} : G \to \mathfrak{a}$. The convexity theorem describes the image $\mathfrak{H}(aH)$ under this map for any given $a \in A$.

The convexity theorem generalizes the well-known non-linear convexity theorem of Kostant [1], which arises if σ is a Cartan-involution, in which case $K = H$. It also generalizes the convexity theorem of [2] which deals with the case that the set $\Sigma(\mathfrak{n})$ of $\mathfrak{a}$-roots in $\mathfrak{n}$ satisfies

$$\alpha \in \Sigma(\mathfrak{n}) \cap \sigma\Sigma(\mathfrak{n}) \Longrightarrow \sigma\alpha = \alpha. \qquad (*)$$

Equivalently, this means that the minimal parabolic subgroup $Z_G(\mathfrak{a})N$ is contained in a minimal $\sigma\theta$-stable parabolic subgroup of G. The present convexity theorem is valid without any restriction on $\Sigma(\mathfrak{n})$.

Let $\mathrm{pr}_{\mathrm{q}} : \mathfrak{a} \to \mathfrak{a}_{\mathrm{q}}$ denote the projection along $\mathfrak{a} \cap \mathfrak{h}$.

Theorem. *The image $\mathfrak{H}(aH) \subset \mathfrak{a}$ is invariant under translation by $\mathfrak{a} \cap \mathfrak{h}$. Furthermore,*

$$\mathrm{pr}_{\mathrm{q}} \circ \mathfrak{H}(aH) = \mathrm{conv}(\mathrm{W}_{\mathrm{K}\cap\mathrm{H}}(\mathfrak{a}_{\mathrm{q}}) \cdot \mathrm{pr}_{\mathrm{q}}(\log a)) + \Gamma(N).$$

Here conv indicates that the convex hull is taken, $W_{K\cap H}(\mathfrak{a}_{\mathrm{q}})$ denotes the Weyl group $N_{K\cap H}(\mathfrak{a}_{\mathrm{q}}/Z_{K\cap H}(\mathfrak{a}_{\mathrm{q}})$ and $\Gamma(N)$ denotes the convex polyhedral cone generated by $\mathrm{pr}_{\mathrm{q}} H_\alpha$, for α in the set

$$\Sigma(N)_- := \{\alpha \in \Sigma(N) \cap -\sigma\Sigma(N) \mid \sigma\alpha = -\alpha \Longrightarrow \mathfrak{g}_\alpha \not\subset \ker(\sigma\theta - I)\}.$$

The map $(a, h) \mapsto \mathfrak{H}(aH)$ appears in the definition of Eisenstein integrals for G/H obtained by induction from the minimal parabolic subgroup $Z_G(\mathfrak{a})N$, even if (*) is not satisfied; this is work in progress of the author with Job Kuit. The convexity theorem allows to determine regions in which the Eisenstein integral depends holomorphically on the continuous spectral parameter.

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