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New Trends in Teichmüller Theory and Mapping Class Groups

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Abstract. The program "New Trends in Teichmüller Theory and Mapping Class Groups" brought together people working in various aspects of the field and beyond. The focus was on the recent developments that include higher Teichmüller theory, the relation with three-manifolds, mapping class groups, dynamical aspects of the Weil-Petersson geodesic flow, and the relation with physics. The goal of bringing together researchers in these various areas, including young PhDs, and promoting interaction and collaboration between them was attained.

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Introduction by the Organisers

Teichmüller theory is a broad and important field of research, and it can be considered from various point of views (algebraic geometry, hyperbolic geometry, complex analysis, uniformization theory and partial differential equations). This stems from the fact that Teichmüller space can be seen as a space of equivalence classes of marked conformal structures, or of marked hyperbolic metrics, or of complex algebraic curves, or of representations of the fundamental group of a surface into the Lie group $\mathrm{PSL}(2,\mathbb{R})$. There are also other aspects. The theory has a wide

range of applications in low-dimensional topology, algebraic topology, representations of discrete groups in Lie groups, symplectic geometry, topological quantum field theory, string theory, etc.

The workshop *New Trends in Teichmüller Theory and Mapping Class Groups*, organized by Shigeyuki Morita (Tokyo), Athanase Papadopoulos (Strasbourg), Robert C. Penner (Aarhus and Caltech) and Anna Wienhard (Heidelberg) was attended by 55 participants from all over Europe, the Americas and several countries in Asia. The participants included world specialists of the subjects and also young researchers, comprising more than 10 PhD students and several post-docs. The discussions and the talks concerned several aspects of Teichmüller theory, that included complex geometry in one and in several variables (Riemann surfaces and uniformization and families of Riemann surfaces), the study of symmetric spaces and the analogies with Teichmüller theory, the metric theory (Teichmüller, Weil-Petersson and Thurston), complex projective structures on surfaces, 3-manifolds and their invariants, mapping class groups (representation theory, factorizations, quasihomomorphisms, Johnson-Morita theory, characteristic classes), relation with theoretical physics (chord diagrams and random matrices), representations of surface groups in character varieties in arbitrary semisimple Lie groups modulo conjugacy (higher Teichmüller theory), dynamics (Teichmüller and Weil-Petersson geodesic flows) and the symplectic geometry of moduli spaces. There was also an open problem session.

Several new collaborations were started during that workshop.

Workshop: New Trends in Teichmüller Theory and Mapping Class Groups

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Intertwining Hitchin Representations

Norbert A'Campo

Let S_g be a closed surface of genus $g \geq 2$ and Hit_g^n the Hitchin component in the representation space

$$\{\rho : \pi = \pi_1(S_g) \to \mathrm{PSL}(n,\mathbb{R})\} \Big/ \mathrm{PSL}(n,\mathbb{R}) \ , \ n \geq 2.$$

We want to construct a mapping class group equivariant intertwiner $I_g^n : Hit_g^n \to Hit_g^2 = \mathcal{T}_g$.

(1) Let $CR_{\mathbb{P}^1} : \left(\mathbb{P}^1\right)^4 - \{diag\} \to \mathbb{R}$ be the cross ratio function. A geometric definition is given as follows: for X, Y, a, b 1-dimensional subspaces, think of $\mathbb{R}^2$ as product $X \times Y$ and the $a : X \to Y$, $b : X \to Y$ as maps with graphs a, b. $\lambda = CR_{\mathbb{P}^1}(X, Y, a, b)$ is the stretching factor of $b^{-1} \circ a : X \to X$.

(2) F. Labourie constructed for $\rho \in Hit_g^n$ a curve $\lambda_\rho : \partial\pi \to \mathbb{P}^n(\mathbb{R})$ with image L_ρ, which is a $C^{1+\alpha}$-submanifold homeomorphic to $\mathbb{P}^1$. The group π acts by projective motions $\rho(j), j \in \pi$.

(3) From L_ρ with π-action we construct a CR_{L_ρ} 4-points function on L_ρ as follows: Let X, Y, a, b in that cyclic order on L_ρ and let $j \in \pi$ be with fixpoints f, F on L_ρ. Let u be a $C^{1+\alpha}$ coordinate on L_ρ centered at F. We put $X_n = j^n(X), a_n = j^n(a), \ldots$ and

$$CR_{L_\rho}(X, Y, a, b) = \lim_{n\to\infty} \frac{(u(b_n) - u(X_n))\,(u(Y_n) - u(a_n))}{(u(a_n) - u(X_n))\,(u(Y_n) - u(b_n))}.$$

Claim: CR_{L_ρ} is a 4-point, π-invariant, cross ratio function, that is congruent by a homeomorphism to $CR_{\mathbb{P}^1}$.

(4) From the data L_ρ, CR_{L_ρ} we get a space

$$H_\rho^2 := \{\sigma : L_\rho \to L_\rho \mid \sigma \text{ involution, continuous, without fixpoints}\}.$$

The space H_ρ^2 has the geometry of the hyperbolic plane. π acts on H_ρ^2 producing a Riemann surface H_ρ^2/π, so we get an intertwiner $I_g^n : Hit_g^n \to H_g^2 = \mathcal{T}_g$. The proof of the claim uses the minimality of the action of π on pairs of distinct points of L_ρ.

Complex analytic properties of deformation spaces of Kleinian groups

Hiroshige Shiga

Let G_0 be a finitely generated non-elementary Kleinian group. We assume that the region of discontinuity $\Omega(G_0)$ is non-empty. We say that a quasiconformal map $w : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ is G_0-equivariant if there exists a $\rho_w \in \mathrm{Hom}(G_0, \mathrm{PSL}(2,\mathbb{C}))$ such that

$$w \circ \gamma = \rho_w(\gamma) \circ w \quad (\gamma \in G_0).$$

Two such maps w_1, w_2 are *equivalent* if there is $A \in \mathrm{PSL}(2,\mathbb{C})$ such that

$$\rho_{w_1}(\gamma) = A \circ \rho_{w_2}(\gamma) \circ A^{-1}$$

hold for all $\gamma \in G_0$. We denote by $[w]$ the equivalence class of w. The set of all equivalence classes is denoted by $D(G_0)$, and it is called *the (quasiconformal) deformation space of* G_0. Kra-Maskit [4] showed that $D(G_0)$ admits a natural complex structure and it is holomorphically convex. In this talk, we first show that $D(G_0)$ is H^∞-convex if every connected component of $\Omega(G_0)$ is simply connected. On the other hand, if some connected component is not simply connected, then we can show that $D(G_0)$ is not H^∞-convex.

Since $D(G_0)$ is defined by quasiconformal maps, the Teichmüller distance is defined on $D(G_0)$. We show that on $D(G_0)$, the Teichmüller distance is equal to the Kobayashi distance, which is a generalization of a theorem of H. Royden (cf. [3]).

References

[1] C. J. Earle, I. Kra and S. L. Krushkal', Holomorphic motions and Teichmüller spaces, Trans. Amer. Math. Soc. **343** (1994), 927–948.
[2] R. C. Gunning, Introduction to holomorphic functions of several variables, Volume I, Wadsworth & Brooks/Cole, 1990.
[3] Y. Imayoshi and M. Taniguchi, *Introduction to Teichmüller Spaces*, Springer-Tokyo 1992.
[4] I. Kra and B. Maskit, The deformation space of a Kleinian group, Amer. J. Math. **103** (1981), 1065–1102.

Spines of Teichmüller spaces and symmetric spaces

Lizhen Ji

Let $\mathcal{T}_g$ be the Teichmüller space of compact Riemann surfaces of genus $g \geq 1$, Mod_g the mapping class group of a compact oriented surface of genus g. Then Mod_g acts on $\mathcal{T}_g$ holomorphically and properly, and the quotient $\mathrm{Mod}_g \backslash \mathcal{T}_g$ is the moduli space $\mathcal{M}_g$ of compact Riemann surfaces of genus g.

Let G be a noncompact semisimple Lie group, $K \subset G$ a maximal compact subgroup. Then the quotient space $X = G/K$ with a G-invariant Riemannian metric is a symmetric space of noncompact type, and hence it is simply connected and nonpositively curved. Let $\Gamma \subset G$ be an arithmetic subgroup. Then Γ acts isometrically and properly on X. The quotient $\Gamma \backslash X$ is a locally symmetric space of finite volume.

There is a lot of similarities between these two group actions: $(\mathrm{Mod}_g, \mathcal{T}_g)$, (Γ, X). See [1] for an overview. In this talk, I discussed one more connection between them from the perspective of classifying spaces.

Given a discrete group Γ, there are several spaces associated with it: $B\Gamma$ is a CW-complex which is uniquely determined up to homotopy equivalence such that $\pi_1(B\Gamma) = \Gamma$, and $\pi_i(B\Gamma) = \{1\}$ for $i \geq 2$. The universal covering space $E\Gamma$ of $B\Gamma$ is characterized by the conditions: (1) $E\Gamma$ is a Γ-CW-complex such that Γ acts properly and fixed point freely on it, (2) $E\Gamma$ is contractible.

The Γ-principal bundle $E\Gamma \to B\Gamma$ classifies Γ-principal bundles. The classifying space $B\Gamma$ can be used to compute cohomology groups of Γ: $H^i(\Gamma, \mathbb{Z}) \cong H^i(B\Gamma, \mathbb{Z})$, and it is also a crucial ingredient of the Novikov conjectures and the Baum-Connes conjecture, which compute the global groups such as the algebraic K-groups and the surgery groups of the group ring $\mathbb{Z}\Gamma$.

For these purposes, it is important to find small and explicit models of $B\Gamma$ and of $E\Gamma$. For example, we hope to construct *models of $B\Gamma$ which are finite dimensional or even are finite CW-complexes.*

The Milnor construction gives an infinite-dimensional model of $E\Gamma$ and hence of $B\Gamma$. It is known that (1) *for every model of $B\Gamma$,* $\dim B\Gamma \geq \mathrm{cd}(\Gamma)$, the cohomological dimension of Γ, (2) *if Γ contains nontrivial torsion elements, then* $\mathrm{cd}(\Gamma) = +\infty$*, and hence every model of $B\Gamma$ is infinite dimensional.*

On the other hand, many natural groups contain torsion elements: the mapping class groups of surfaces $\mathrm{Mod}_{g,n}$, the outer automorphism group $\mathrm{Out}(F_n)$ of the free group F_n, arithmetic groups such as $\mathrm{SL}(n, \mathbb{Z})$ and $\mathrm{Sp}(2n, \mathbb{Z})$, etc. For such groups, a more convenient space is a *universal space for proper actions* of Γ, denoted by $\underline{E}\Gamma$, which is characterized by the following conditions: (1) Γ acts properly on $\underline{E}\Gamma$, (2) for every finite subgroup $F \subset \Gamma$, the set of fixed points $(\underline{E}\Gamma)^F$ is nonempty and contractible.

When Γ is torsion-free, then $\underline{E}\Gamma$ is reduced to $E\Gamma$. Similarly, $\underline{E}\Gamma$ and its quotient $\Gamma\backslash\underline{E}\Gamma$ are useful for the computation of cohomology groups of Γ and the Novikov conjectures and Baum-Connes conjecture for Γ. It is known that for every model of $\underline{E}\Gamma$, $\dim \underline{E}\Gamma \geq \mathrm{vcd}(\Gamma)$, where $\mathrm{vcd}(\Gamma) = \mathrm{cd}(\Gamma')$, Γ' being a torsion-free subgroup of Γ of finite index, and it is independent of the choice of Γ'.

When $\mathrm{vcd}(\Gamma) < +\infty$, we hope to obtain explicit models of $\underline{E}\Gamma$ which are cocompact with respect to Γ and has dimension as small as possible. For general Γ, it is a difficult problem.

Proposition 1. *(1) For $\Gamma = \mathrm{Mod}_g$, $\mathcal{T}_g$ is a finite dimensional model of $\underline{E}\Gamma$. (2) For an arithmetic subgroup of G as above, the symmetric space X is a finite dimensional model of $\underline{E}\Gamma$.*

The positive solution to the Nielsen realization is needed for the first statement, and the Cartan fixed point theorem for the second one. It is known that the quotient $\Gamma\backslash\mathcal{T}_g$ is noncompact since compact Riemann surfaces can degenerate and become singular. Similarly, for many natural arithmetic subgroups such as $\mathrm{SL}(n, \mathbb{Z})$, $\Gamma\backslash X$ is noncompact. It is known that $\Gamma\backslash X$ *is noncompact if and only if*

the $\mathbb{Q}$-rank of Γ (or rather the associated linear algebraic group whose real locus is equal to G and which defines the commensurable class of Γ) *is positive.*

There are two possible approaches to the above problems: (1) *Construct equivariant partial compactificatons $\overline{\mathcal{T}_g}$ with compact quotient under* Mod_g *such that the inclusion* $\mathrm{Mod}_g\backslash\mathcal{T}_g \hookrightarrow \mathrm{Mod}_g\backslash\overline{\mathcal{T}_g}$ *is an homotopy equivalence.* (2) *Construct equivariant deformation retracts S of $\mathcal{T}_g$ such that* $\mathrm{Mod}_g\backslash S$ *is compact and of dimension as small as possible. Such a subspace S is called a* **spine** *of $\mathcal{T}_g$.*

Similarly, we can try to carry out both constructions for symmetric spaces. Let Γ be an arithmetic subgroup acting on a symmetric space $X = G/K$ as above. Borel and Serre *constructed the partial Borel-Serre compactification* $\overline{X}^{BS}$ and *proved that the inclusion* $\Gamma\backslash X \hookrightarrow \Gamma\backslash\overline{X}^{BS}$ *is a homotopy equivalence and* $\mathrm{vcd}(\Gamma) = \dim X - \mathbb{Q}\text{−rank}(\Gamma)$.

It was later proved by Ji that $\overline{X}^{BS}$ is a cocompact model of $\underline{E}\Gamma$. (By construction, Γ acts properly on $\overline{X}^{BS}$. The point is to prove the contractibility of the fixed point set in $\overline{X}^{BS}$ of every finite subgroup of Γ.)

For the Teichmüller sapce $\mathcal{T}_g$, Harvey outlined a construction of $\overline{\mathcal{T}_g}^{BS}$, an analogue of $\overline{X}^{BS}$, which is a real analytic manifold with corners and introduced the notion of curve complex of a surface to play the role of Tits buildings for symmetric spaces. No detail has appeared, and Ivanov constructed a C^∞-analogue. But it has not been proved that $\overline{\mathcal{T}_g}^{BS}$ is a model of $\underline{E}\mathrm{Mod}_g$.

On the other hand, Ji and Wolpert proved that the thick part $\mathcal{T}_g(\varepsilon)$ of the Teichmüller space $\mathcal{T}_g$ is a *spine* of $\mathcal{T}_g$, where ε is a sufficiently small positive number. When $g \geq 2$, every compact Riemann surface Σ_g admits a unique hyperbolic metric which is conformal to the complex structure. (By definition, the thick part $\mathcal{T}_g(\varepsilon)$ consists of hyperbolic surfaces in $\mathcal{T}_g$ which do not contain geodesics of length less than ε.)

Harer proved that $\mathrm{vcd}(\mathrm{Mod}_g) = 4g-5$. Since $4g-5 < 6g-6 = \dim \mathcal{T}_g$ for $g \geq 2$, $\mathcal{T}_g(\varepsilon)$ is far from being of the optimal dimension. One folklore **open problem** is: *Construct a spine of $\mathcal{T}_g$ of dimension $4g-5$.* This is the number 1 problem in a list of open problems by Bridson and Vogtmann in 2006.

Similarly, for a nonuniform arithmetic subgroup Γ of G, one **open problem** is: *Construct a spine of the symmetric space X with respect to Γ of dimension equal to* $\mathrm{vcd}(\Gamma) = \dim X - \mathbb{Q}\text{−rank}(\Gamma)$.

In this talk, we presented the following results:

Theorem 2 ([2])**.** *The Teichmüller space $\mathcal{T}_g$ admits a spine S of codimension 1 and a spine S' of codimension 2 which are intrinsically defined in terms of the hyperbolic geometry of the surfaces in $\mathcal{T}_g$.*

A geodesic in a hyperbolic surface Σ_g of the shortest length is called a systole of the surface. The spine S consists of all hyperbolic surfaces in $\mathcal{T}_g$ which admit at least two systoles and two of them intersect. It is clear that S is invariant under Mod_g, and the collar theorem for hyperbolic surfaces implies that it has a compact quotient under Mod_g.

The spine S'' consists of those hyperbolic surfaces which admit at least three systoles and at least two of them intersect.

In 1985, Thurston circulated a preprint titled "*Spines of Teichmüller spaces*". He proposed a candidate for a spine consisting of hyperbolic surfaces whose systoles fill the surfaces, i.e., every closed geodesic of the hyperbolic surfaces intersects one of the systoles.

For Teichmüller space $\mathcal{T}_{g,n}$ of compact Riemann surfaces with n punctures, $n \geq 1$, spines of optimal dimension are known. The reason is that $\mathcal{T}_{g,n}$ admits an ideal triangulation, and the sub-complex of the dual complex obtained by removing the cells whose closures meet the ideal simplices gives the desired spine.

Theorem 3 ([3]). *When $Q-\mathrm{rank}(\Gamma) \leq 2$, X admits a spine of optimal dimension.*

The history of spines for symmetric spaces is as follows (see [1] for references):

(1) When $\Gamma = \mathrm{SL}(2,\mathbb{Z})$, $X = \mathbb{H}^2$, Serre observed a spine given by a trivalent tree.
(2) When $\Gamma = \mathrm{SL}(3,\mathbb{Z})$, Soule constructed a spine of optimal dimension using Minkowski reduction in his thesis.
(3) Later, for $\Gamma = \mathrm{SL}(n,\mathbb{Z})$, Lannes and Soule constructed the well-rounded deformation retract of the space of positive definite quadratic forms, or equivalently lattices in $\mathbb{R}^n$. The deformation retraction is also intrinsic.
(4) Ash generalized the above results of Soule to linear symmetric spaces (self-adjoint cones and their homothety sections) and symmetric spaces associated with $G = GL(n,A), SL(n,A)$, where A is a division algebra.
(5) Mendoza constructed explicit spines for some arithmetic groups acting on the three-dimensional hyperbolic space.
(6) When $\mathbb{Q}-\mathrm{rank}(\Gamma) = 1$, spines of codimension 1 were constructed by Yasaki,
(7) MacPherson and McConell constructed a spine of codimension 2 of the Siegel upper-half of degree 2 for *torsion-free finite index subgroups* $\Gamma \subset \mathrm{Sp}(4,\mathbb{Z})$. This was the only known examples of *nonlinear* symmetric spaces of higher rank which admit such optimal spines.

Our result in Theorem 2 provides many new examples of high rank symmetric space with optimal spines.

The proofs of both results were motivated by the well-rounded retracts of lattices. When $\mathbb{Q}-\mathrm{rank}(\Gamma) \geq 2$, we have a natural candidate for a spine of optimal dimension $\mathrm{vcd}(\Gamma)$. On the other hand, for the Teichmüller space $\mathcal{T}_g$, there is no known candidate for a spine of optimal dimension. The spine proposed by Thurston seems to be far away from the optimal dimension. One natural *open problem* is to propose good spines of $\mathcal{T}_g$.

References

[1] L. Ji, A tale of two groups: arithmetic groups and mapping class groups. *Handbook of Teichmüller theory*. Volume III, 157–295, IRMA Lect. Math. Theor. Phys., 17, Eur. Math. Soc., Zürich, 2012.
[2] L. Ji, Equivariant deformation retracts of Teichmüller spaces, to appear, Enseign. Math. 2014.
[3] L. Ji, Spines of symmetric spaces, in preparation.

Holonomy fibers of complex projective structures

Subhojoy Gupta
(joint work with Shinpei Baba, Caltech)

Let S be a closed oriented surface of genus $g \geq 2$. A *complex projective structure* on S is a geometric structure modeled on $\mathbb{C}\mathrm{P}^1$, namely it is a maximal atlas of charts to $\mathbb{C}\mathrm{P}^1$ with transition maps in $\mathrm{PSL}_2(\mathbb{C}) = \mathrm{Aut}(\mathbb{C}\mathrm{P}^1)$. Its holonomy (or monodromy) determines a representation $\rho : \pi_1(S) \to \mathrm{PSL}_2(\mathbb{C})$, and defines the *holonomy map*

$$\mathrm{hol} : \mathcal{P} \to \chi$$

from the space $\mathcal{P}$ of all marked projective structures on S to χ, the $\mathrm{PSL}_2(\mathbb{C})$-character variety of $\pi_1(S)$. The image of hol consists of the representations in χ which are non-elementary and lift to representations of $\pi_1(S)$ to $SL_2(\mathbb{C})$ (see [4]).

A basic question is to understand the *holonomy fiber* $\mathrm{hol}^{-1}(\rho) =: \mathcal{P}_\rho$ in $\mathcal{P}$ ([5, 12, 4]). In particular [10, p 274] asked what the projection of $\mathcal{P}_\rho$ to the Teichmüller space $\mathcal{T}$ looks like, where the projection $p : \mathcal{P} \to \mathcal{T}$ is given by considering the underlying conformal structures of the projective structures. We can consider its further projection into the moduli space $\mathcal{M}$ of Riemann surfaces, the quotient of $\mathcal{T}$ by the action of the mapping class group of S:

$$\mathcal{P} \xrightarrow{p} \mathcal{T} \xrightarrow{\pi} \mathcal{M}.$$

In our talk we discuss our new result in [3] that proves that for any $\rho \in \mathrm{hol}(\mathcal{P})$, the holonomy fiber $\mathcal{P}_\rho$ projects to a dense set in moduli space $\mathcal{M}$.

The proof uses "grafting" deformations of complex projective structures, obtained by inserting projective annuli along geodesic multicurves on a hyperbolic structure on S, an operation that extends to *measured laminations* by taking limits. This results in the geometric parametrization

$$\mathcal{P} \cong \mathcal{T} \times \mathcal{ML}$$

due to Thurston (see [11, 13, 15]), where $\mathcal{ML}$ is the set of all measured laminations on S.

As we graft a hyperbolic surface, by scaling the transversal measure on the lamination, we obtain a ray of projective structures in $\mathcal{P}$ that descends to a *grafting ray* in $\mathcal{T}$. The results of [8] and [9] yield the strong asymptoticity of these grafting

rays to Teichmüller geodesic rays. By the ergodicity of the Teichmüller geodesic flow ([14, 16]), this implies that almost every grafting ray further projects to a dense set in $\mathcal{M}$.

On the other hand, grafting a projective surface C along *admissible* multicurves weighted by integer-multiples of 2π preserves the holonomy ([6], see also [1], [2]). Our new result is obtained by approximating dense grafting rays by such 2π-grafts along admissible loops on some $C \in \mathcal{P}_\rho$. The proof introduces a piecewise Euclidean/hyperbolic metric on projective surfaces that modifies the "Thurston metric" (*cf.* [13]) and involves the construction of "almost-isometric" maps with respect to these metrics. The arguments involve three-dimensional hyperbolic geometry, and in particular the geometry of the ρ-equivariant locally-convex pleated plane in $\mathbb{H}^3$ associated to a complex projective structure.

As described in [7], the holonomy map hol : $\mathcal{P} \to \chi$ gives a "resolution" of the mapping class group action on χ. Namely, the mapping class group action is hol-equivariant and its action on $\mathcal{P}$ is discrete. It would thus be interesting to see if the holonomy fibers $\mathcal{P}_\rho$ tell us about the action on χ.

References

[1] Baba, Shinpei, *Complex Projective Structures with Schottky Holonomy*, Geom. Funct. Anal., **22** 2 (2012), 267–310.

[2] Baba, Shinpei, *2π-graftings and complex projective structures I*, Submitted, arxiv:1011.5051.

[3] Baba, Shinpei and Gupta, Subhojoy, *Holonomy map fibers of $\mathbb{C}\mathrm{P}^1$-structures in moduli space*, Submitted, arxiv:1402.5445.

[4] Gallo, Daniel and Kapovich, Michael and Marden, Albert, *The monodromy groups of Schwarzian equations on closed Riemann surfaces*, Ann. of Math. (2) **151** 2 (2000), 625–704.

[5] Goldman, William M., *Discontinuous Groups and the Euler Class*, Ph.D. thesis, University of California, Berkeley (1980).

[6] Goldman, William M., *Projective structures with Fuchsian holonomy*, J. Differential Geom. **25** 3 (1987), 297–326.

[7] Goldman, William M., *The complex-symplectic geometry of $\mathrm{SL}(2,\mathbb{C})$-characters over surfaces*, In "Algebraic groups and arithmetic", 375–407, Tata Inst. Fund. Res., Mumbai (2004).

[8] Gupta, Subhojoy, *Asymptoticity of grafting and Teichmüller rays I , Geom. Topol. (to appear)* (2014).

[9] Gupta, Subhojoy, *Asymptoticity of grafting and Teichmüller rays II , Geom. Dedicata. (to appear)* (2014).

[10] Hubbard, John H., *The monodromy of projective structures.*, In "Riemann surfaces and related topics: Proceedings of the 1978 Stony Brook Conference (State Univ. New York", Ann. of Math. Stud. , **97** Princeton Univ. Press, (1981), 257–275.

[11] Kamishima, Yoshinobu and Tan, Ser P., *Deformation spaces on geometric structures*, In "Aspects of low-dimensional manifolds", Adv. Stud. Pure Math. **20** , Kinokuniya Toky (1992), 263–299.

[12] Kapovich, Michael, *On monodromy of complex projective structures*, Invent. Math. **119** 2 (1995), 243–265.

[13] Kulkarni, Ravi S. and Pinkall, Ulrich, *A canonical metric for Möbius structures and its applications*, Math. Z. **216** No. 1 (1994), 89–129.

[14] Masur, Howard, *Interval exchange transformations and measured foliations*, Ann. of Math. (2), **115** 1 (1982), 169–200.

[15] Tanigawa, Harumi, *Grafting, harmonic maps and projective structures on surfaces*, J. Differential Geom.**47** 3 (1997), 399–419.
[16] Veech, William A., *Gauss measures for transformations on the space of interval exchange maps*, Ann. of Math. (2) **115** 1 (1982), 201–242.

Complex projective surfaces bounding 3-manifolds

Steve Kerckhoff

A complex projective structure on a closed surface S of genus $g \geq 2$ determines a developing map from the universal cover of S to $\mathbb{C}P^1 \cong S^2$ and a holonomy representation $\rho : \pi_1(S) \to \mathcal{PSL}(2,\mathbb{C})$. It also determines a conformal structure on S, which, given a marking, determines a point in the Teichmuller space $\mathcal{T}(S)$. The map $p : \mathcal{P}(S) \to \mathcal{T}(S)$ of the space of complex projective structures to the Teichmuller space gives $\mathcal{P}(S)$ the structure of an affine bundle over $\mathcal{T}(S)$. The translation group of the fiber over a point $X \in T(S)$ equals the vector space of holomorphic quadratic differentials on X; the difference between two points in the same fiber is the Schwarzian derivative of the conformal map between the two projective structures.

The (Zariski) tangent space of $\mathcal{P}(S)$ is equal to $H^1(S;\mathfrak{g}_\rho)$, the first cohomology group of S with coefficients in the Lie algebra $\mathfrak{g}$ of $\mathcal{PSL}(2,\mathbb{C})$, twisted by the adjoint action induced from the holonomy representation ρ. There is a complex-valued, skew-symmetric pairing on the tangent space using the cup product on cohomology and the Killing form on the coefficients. This determines a (holomorphic) symplectic structure on $\mathcal{P}(S)$ which is due to Goldman([1]). Unless stated otherwise, all statements concerning a symplectic structure will refer to this one. The purpose of this talk is to provide a unified derivation of a number of known results about the symplectic geometry of $\mathcal{P}(S)$.

An infinitesimal change of complex projective structure (i.e., a vector tangent to $\mathcal{P}(S)$) determines an infinitesimal change in the developing map, which can be viewed as a vector field on the the universal cover $\tilde{S}$ of S with values in the tangent bundle of $\mathbb{C}P^1$ (pulled back by the developing map). We will refer to this simply as a "vector field" and denote it by v. Below we describe how to lift it naturally to a section s of the $\mathfrak{g}_\rho$ bundle over $\tilde{S}$ (also pulled back via the developing map). Taking the derivative of s determines a 1-form ds on $\tilde{S}$, with values in the $\mathfrak{g}_\rho$ bundle, that descends to S. It is a (deRham) representative of the cohomology class corresponding to the infinitesimal deformation.

To define the section s we identify the Lie algebra with the vector space of quadratic polynomial vector fields on S^2 and then, at each point $w \in S^2$ in the image of the developing map, take the quadratic vector field that best approximates v. Specifically, we choose $s(w) = v(w) + v_z(w)(z-w) + \frac{1}{2}v_{zz}(w)(z-w)^2$. Then, an elementary computation proves the following:

Let $\alpha, \beta \in H^1(S;\mathfrak{g}_\rho)$ be two infinitesimal deformations of a complex projective structure on S. Suppose they determine vector fields v and w, 1-forms ds and $d\hat{s}$,

respectively. Then the symplectic pairing $< \alpha, \beta >$ equals

$$\int_S v_{\bar{z}} w_{zzz} - w_{\bar{z}} v_{zzz}.$$

As an immediate corollary, we see that the fibers of the bundle $p : \mathcal{P}(S) \to \mathcal{T}(S)$ are *lagrangian* subspaces. This was first proved by Kawai ([2]).

One source of complex projective structures comes from any convex, cocompact hyperbolic 3-manifold M with non-empty boundary. For simplicity, we assume that the boundary components are incompressible. The holonomy group of the hyperbolic structure acts properly discontinuously on an open subset of the sphere at infinity with quotient equal to a (possibly disconnected) surface. Since the action is by elements of $\mathcal{PSL}(2, \mathbb{C})$, the surface inherits a complex projective structure. The quasi-conformal theory of Ahlfors and Bers implies that the space of convex cocompact hyperbolic structures on M is diffeomorphic to the Teichmuller space of the boundary; the map is defined by taking the induced conformal structure coming from the sphere at infinity. Thus, M determines a section $\sigma_M : \mathcal{T}(\partial M) \to \mathcal{P}(\partial M)$ of the affine bundle.

Restricting an infinitesimal deformation of the hyperbolic structure on M to an infinitesimal deformation of the projective structure on its boundary induces a map $i^* : H^1(M; \mathfrak{g}_\rho) \to H^1(\partial M; \mathfrak{g}_\rho)$. Using the long exact sequence for the cohomology of a manifold with boundary and Poincare duality, one can easily show that the image of i^* is half-dimensional and self-annihilating. Thus the image of the section σ_M is Langrangian for any M. In particular, if v and w are vector fields corresponding to deformations of M that are restricted to $S = \partial M$ we obtain $\int_S v_{\bar{z}} w_{zzz} - w_{\bar{z}} v_{zzz} = 0$. This formula was first derived by McMullen in [4], where it was called "Kleinian reciprocity." The techniques were completely different; no symplectic structure was utilized.

A special case occurs for $M \cong S \times I$; such structures are called "quasi-Fuchsian" since they are quasi-conformal deformations of Fuchsian groups, viewed as acting on 3-dimensional hyperbolic space. The space of such structures is diffeomorphic to a product of two copies of $\mathcal{T}(S)$. Fixing the conformal structure of one of the surfaces determines a slice, called a "Bers slice", which is holomorphically equivalent to $\mathcal{T}(S)$. The conformal surfaces of the varying end also have complex projective structures. An application of the formula above immediately implies that the image of the resulting section of $p : \mathcal{P}(S) \to \mathcal{T}(S)$ is lagrangian. A similar statement holds for any "generalized Bers slice", where the conformal structures of all but one of the components of ∂M are held fixed. These results first appeared in [2] and [3], respectively.

Finally, any section of the affine bundle turns it into a vector bundle, using the image of the section as the zero-section. This identifies $\mathcal{P}(S)$ with the bundle of holomorphic quadratic differentials over $\mathcal{T}(S)$, which, in turn, is identified with the cotangent bundle of $\mathcal{T}(S)$. If the section is holomorphic, these identifications are holomorphic. A cotangent bundle has a canonical symplectic form which we can compare with that coming from the Goldman pairing. Using the fact that the fibers and the image of the section are lagrangian, it is not difficult to show that,

for the various sections described above, the two symplectic forms are equal (up to a multiplicative constant). Again, similar results can be found in [2] and [3].

References

[1] W. Goldman, *The symplectic nature of fundamental groups of surfaces*, Adv. in Math. **54** (1984), 200-225.
[2] S. Kawai, *The symplectic nature of the space of projective connections on Riemann surfaces*, Math. Annalen **305** (1996), 161-182.
[3] B. Loustau, *La geometrie symplectique de l'espace des structures projectives complexes sur une surface*, Doctoral thesis (UPS Toulouse) (2011).
[4] C. McMullen, *The moduli space of Riemann surfaces is Kahler hyperbolic*, Ann. of Math (2) **151** (2000), 327-357.

Subgroups of mapping class groups associated to Heegaard splittings and their actions on projective lamination spaces

Ken'ichi Ohshika

We consider a Heegaard splitting of a 3-manifold $M = H_1 \cup_S H_2$. Any mapping class of $H_j (j = 1, 2)$ can be regarded as a mapping class of S by restricting it to the boundary. We consider a subgroup of the mapping class group $MCG(H_j)$ consisting of all classes represented by homeomorphisms homotopic to the identity in H_j, and denote it by $MCG^0(H_j)$. We let G_1 and G_2 be the subgroups of the mapping class group $MCG(S)$ corresponding to $MCG^0(H_1)$ and $MCG^0(H_2)$ respectively. We are interested in the group generated by G_1 and G_2 in $MCG(S)$, which we denote by $G := \langle G_1, G_2 \rangle$. It is Minsky that first took interest in this group G. He raised some problems on this group G, which can be found in the list of problems of Heegaard splittings edited by Gordon [2].

The sets of meridians for two handlebodies H_1 and H_2 define a subset Δ_1 and Δ_2 of the curve graph $C(S)$ of the splitting surface S. The distance between Δ_1 and Δ_2 is called the Hempel distance of the decomposition.

The first thing we are interested in is the algebraic structure of G. The following theorem is an answer to one of the questions posed by Minsky.

Theorem 1 (Bowditch-Ohshika-Sakuma [1])**.** *If the Hempel distance is large enough, then* $G = G_1 * G_2$.

To prove this theorem, we use the Gromov hyperbolicity of the curve graph and the acylindiricity of the action of the mapping class group on the curve complex which was proved by Bowditch.

Since the mapping class group acts on the projective lamination space $\mathcal{PL}(S)$, its subgroup $G = \langle G_1, G_2 \rangle$ also acts on it. The second thing we are interested in is to study the dynamics of this action: for instance we should like to know if there is a region of discontinuity for this action. In [1], we have shown that indeed this action has a non-empty region of discontinuity. Here we shall give a more concrete way to construct a region of discontinuity than the argument in [1].

Definition 2. For a Heegaard splitting $M = H_1 \cup_S H_2$, for $j = 1, 2$, we let Δ'_j be the set of weighted disjoint unions of meridians in H_j, which is regarded as a subset of the measured lamination space $\mathcal{ML}(S)$. We set Δ to be $G(\Delta'_1 \cup \Delta'_2)$. Then we define a subset U of $\mathcal{ML}(S)$ to be $U = \{\lambda \in \mathcal{ML}(S) \mid i(\lambda, \mu) > 0 \text{ for all } \mu \in \overline{\Delta}\}$, where the over line denotes the closure in $\mathcal{ML}(S)$. We also define PU to be the projection of U into the projective lamination space $\mathcal{PML}(S)$.

It is easy to see that PU is an open set in $\mathcal{PML}(S)$.

We have proved the following for Heegaard splittings with combinatorial bounded geometry. Here we shall not give a detailed definition of combinatorial boundedness of Heegaard splittings $M = H_1 \cup_S H_2$, but just mention that this corresponds to the condition that there is a positive lower bound for the injectivity radii of the hyperbolic 3-manifolds M. (Note that it is know that M is hyperbolic if the Hempel distance is greater than 2.)

Theorem 3 (Lecuire-Ohshika-Sakuma)**.** *For any positive constant D, there exists K such that any Heegaar splitting $M = H_1 \cup_S M_2$ with D-combinatorially bounded geometry and Hempel distance $\geq K$ has the following property.*

(1) PU is non-emtpy.
(2) G acts properly discontinuously on PU.
(3) PU is almost maximal: for any open set V in $\mathcal{PML}(S)$ containing PU on which G acts properly discontinuously, the Lebesgue measure of $V \setminus PU$ is 0.

References

[1] K. Ohshika and M. Sakuma, Subgroups of mapping class groups related to Heegaard splittings and bridge decompositions, arXiv:1308.0888
[2] C. Gordon, *Problems*, Workshop on Heegaard Splittings, 401–411, Geom. Topol. Monogr. **12**, Geom. Topol. Publ., Coventry, 2007.

Arbitrarily long factorizations in surface mapping class groups

Mustafa Korkmaz
(joint work with Elif Dalyan, Mehmetcik Pamuk)

Let Σ_g^n denote a compact connected oriented surface of genus g with $n \geq 1$ boundary components $\delta_1, \delta_2, \ldots, \delta_n$. The mapping class group $\mathrm{Mod}(\Sigma_g^n)$ of the surface Σ_g^n is the group of isotopy classes of orientation-preserving self-diffeomorphisms of Σ_g^n. Diffeomorphisms and isotopies are assumed to fix each point of the boundary.

By the results of Giroux [2] and Thurston-Winkelnkemper [5], every open book decomposition (Σ_g^n, Φ), where $\Phi \in \mathrm{Mod}(\Sigma_g^n)$, of a closed oriented 3-manifold M admits a compatible contact structure and all contact structures on compact 3-manifolds come from open book decompositions. If the monodromy Φ of the open book can be written as a product of positive Dehn twists, then the contact structure is Stein fillable. Writing Φ as a product of positive Dehn twists provides a Stein filling of the contact 3-manifold M via Lefschetz fibrations.

In this work we consider the following question: *Is the number of positive Dehn twists in a factorization of the boundary multitwist $t_{\delta_1}t_{\delta_2}\cdots t_{\delta_n}$ bounded?* For a simple closed curve a, the Dehn twist about a is denoted by t_a. Any such factorization of $t_{\delta_1}t_{\delta_2}\cdots t_{\delta_n}$ describes a Lefschetz fibration with n disjoint sections of self-intersection -1. Thus, this question is related to Lefschetz fibrations.

Baykur and Van Horn-Morris [1] proved that for $g \geq 8$ the boundary multitwist $t_{\delta_1}t_{\delta_2}$ in $\mathrm{Mod}(\Sigma_g^2)$ can be written as a product of arbitrarily large number of positive Dehn twists about nonseparating simple closed curves. In this work, we prove that the same conclusion can be drawn for all $g \geq 3$. For $g = 2$, this statement is not true anymore. Our main result is the following theorem.

Theorem. *Let a be a nonseparating simple closed curve on a surface Σ_g^2 of genus g with two boundary components δ_1 and δ_2. In the mapping class group $\mathrm{Mod}(\Sigma_g^2)$, the multitwist*

(i) $t_{\delta_1}t_{\delta_2}t_a$ for $g = 2$,
(ii) $t_{\delta_1}t_{\delta_2}$ for $g \geq 3$

can be written as a product of arbitrarily large number of positive Dehn twists about nonseparating simple closed curves.

By capping off one of the boundary components, we obtain the following immediate corollary for surfaces with one boundary component.

Corollary 1. *Let Σ_g^1 be a compact connected oriented surface of genus g with one boundary component δ. In the mapping class group $\mathrm{Mod}(\Sigma_g^1)$, the element*

(i) t_δ^2 for $g = 2$,
(ii) t_δ for $g \geq 3$

can be written as a product of arbitrarily large number of positive Dehn twists about nonseparating simple closed curves.

We note that in the mapping class group $\mathrm{Mod}(\Sigma_g)$ of a closed orientable surface Σ_g the identity element can be written as a product of positive Dehn twists about nonseparating simple closed curves. It follows that every element in $\mathrm{Mod}(\Sigma_g)$ can be expressed as a product of arbitrarily large number of nonseparating positive Dehn twists. However, in case $n \geq 1$, the identity element of $\mathrm{Mod}(\Sigma_g^n)$ admits no nontrivial factorization into a product of positive Dehn twists.

A factorization of the multitwist $t_{\delta_1}t_{\delta_2}\cdots t_{\delta_n}$ into a product of positive Dehn twists of the form

$$t_{\delta_1}t_{\delta_2}\cdots t_{\delta_n} = \prod_{i=1}^{r} t_{a_i}$$

in the group $\mathrm{Mod}(\Sigma_g^n)$ describes a genus-g Lefschetz fibration $X_g(r) \to S^2$ with n disjoint sections such that the self-intersection of each section is -1. The Euler characteristic of the total space $X_g(r)$ is $\chi(X_g(r)) = 2(2-2g)+r$.

The following corollary is an improvement of [1, Theorem 1.2].

Corollary 2. *For every $g \geq 3$, there is a family of genus-g Lefschetz fibrations $X_g(r) \to S^2$ with two disjoint sections of self-intersection -1 such that the set*

$\{\chi(X_g(r))\}$ of Euler characteristics is unbounded. The same conclusion holds true for genus–2 Lefschetz fibrations but this time with two disjoint sections of self-intersection -2.

Given a genus–g Lefschetz fibration $f : X \to S^2$ with a section σ and with a regular fiber Σ, the complement of a regular neighborhood of the union $\Sigma \cup \sigma$ is a Stein filling of its boundary M equipped with the induced tight contact structure ([4]). It was conjectured in [4] that the set

$$\mathcal{C}_{(M,\xi)} = \{\chi(X) \mid X \text{ is a Stein filling of } (M,\xi)\}$$

is finite. In [1], it was shown that this conjecture is false. Our theorem provides more counterexamples to this conjecture.

Corollary 3. *For every $g \geq 2$, there is a contact 3-manifold (M_g, ξ_g) admitting infinitely many pairwise non-diffeomorphic Stein fillings such that the set $\mathcal{C}_{(M_g,\xi_g)}$ is unbounded.*

We remark that Kaloti [3] showed that if a contact 3-manifold (M,ξ) can be supported by a planar open book, then $\mathcal{C}_{(M,\xi)}$ must be finite. Hence, the contact structure supported by the open book with monodromy $t_{\delta_1} t_{\delta_2}$, $g \geq 3$, cannot be supported by a planar open book.

Here is the idea of the proof of our theorem.

Suppose that $g \geq 2$ and $n \geq 1$. Let c_i, $1 \leq i \leq 4$, be nonseparating simple closed curves on Σ_g^n forming a chain. That is, c_i intersects c_{i+1} transversely once for $i = 1, 2, 3$, and c_i does not intersect c_j if $|i - j| > 1$. Let d and e be the boundary components of a regular neighborhood of $c_1 \cup c_2 \cup c_3$, so that $d \cup e$ bounds a surface of genus one. Let x be any nonseparating simple closed curve on Σ_g^n intersecting c_3 and d transversely only once. Let

$$T = (t_{c_1} t_{c_2} t_{c_3})^2 t_{c_2} t_{c_1} t_{c_3} t_{c_2}.$$

Then, in the mapping class group $\mathrm{Mod}(\Sigma_g^n)$, for any positive integer m, we write

$$\begin{aligned} \phi &= t_{c_4} t_{c_3} t_{c_2} t_{c_1} t_{c_1} t_{c_2} t_{c_3} t_{c_4} t_x t_d t_{c_3} t_x \\ &= (t_{c_3}^{-m} t_e^m t_{c_4} t_{c_3} t_{c_2} t_{c_1} t_{c_1} t_{c_2} t_{c_3} t_{c_4} t_x t_d t_{c_3} t_x t_e^{-m} t_{c_3}^m) T^m \\ &= t_{c_4'} t_{c_3'} t_{c_2'} t_{c_1'} t_{c_1'} t_{c_2'} t_{c_3'} t_{c_4'} t_{x'} t_{d'} t_{c_3'} t_{x'} T^m, \end{aligned}$$

where $c_i' = t_{c_3}^{-m} t_e^m(c_i)$ etc. In particular, for any positive integer m the element ϕ may be written as a product of $12 + 10m$ positive Dehn twists about nonseparating simple closed curves.

Given an element f of $\mathrm{Mod}(\Sigma_g^n)$, if one can factor f as $f = \phi h$ where h is a product of positive Dehn twists, then f can be written as a product of arbitrarily large number of positive Dehn twists. We use this idea to prove our theorem.

References

[1] R. İ. Baykur and J. Van Horn-Morris, *Topological complexity of symplectic 4-manifolds and Stein fillings*, arXiv:1212.1699.

[2] E. Giroux, *Géométrie de contact: de la dimension trois vers les dimensions supérieures,[Contact geometry: from dimension three to higher dimensions]*, Proceedings of the International Congress of Mathematicians, Vol. II, Higher Education Press, Beijing, 2002, 405–414.
[3] A. Kaloti, *Stein fillings of planar open books*, arXiv:1311.0208.
[4] B. Ozbagci and A. I. Stipsicz, *Contact 3-manifolds with infinitely many Stein fillings* **132** (2003), 1549–1558 .
[5] W. P. Thurston and H. Winkelnkemper, *On the existence of contact forms*, Proc. Amer. Math. Soc. **52** (1975), 345–347

On Weil-Petersson Funk metric on Teichmüller spaces

Sumio Yamada

1. The Funk metric in $\mathbb{R}^d$ and its representations

Let Ω be an open bounded convex subset in a Euclidean space $(\mathbb{R}^d, d)$ where d is the standard Euclidean metric. We set the presentation in [2] as our reference for the Funk and Hilbert metrics of Ω, and we also refer to the first part of the paper [8].

There are three different descriptions of the Funk metric. The first one is the original definition:

$$F_1(x,y) = \log \frac{d(x,b(x,y))}{d(y,b(x,y))},$$

where for $x \neq y$ in Ω, the point $b(x,y)$ is the intersection of the boundary $\partial\Omega$ with the Euclidean ray $\{x + t\xi_{xy} : t > 0\}$ from x though y and where ξ_{xy} is the unit tangent vector in $\mathbb{R}^d$ pointing from x to y. When $x = y$, we set $F(x,y) = 0$. The second description is the variational interpretation of the above value using the geometry of supporting hyperplanes;

$$F_2(x,y) = \sup_{\pi\in\mathcal{P}} \log \frac{d(x,\pi)}{d(y,\pi)},$$

where $\mathcal{P}$ is the set of all supporting hyperplanes of Ω. This is given in [8].

Finally, the Finsler structure $p_{\Omega,x}(\xi)$ is given by the following function (the Minkowski functional) on vectors ξ in each tangent space to Ω at x:

$$p_{\Omega,x}(\xi) = \sup_{\pi\in\mathcal{P}} \frac{\langle \nu_\pi(x), \xi\rangle}{d(x,\pi)},$$

where ν_π is the unit vector in $T_x\Omega$ perpendicular to, and directed toward π. This is a weak norm on each tangent space which is defined so that the Funk distance is described as the infimum of length of curves:

$$F_3(x,y) = \inf_\sigma \int_a^b p_{\Omega,\sigma(t)}(\dot\sigma(t))\, dt,$$

the infimum being taken over all the piecewise C^1 curves with $\sigma(a) = x$ and $\sigma(b) = y$.

For any convex domain $\Omega \subset \mathbb{R}^d$, the three quantities $F_1(x,y)$, $F_2(x,y)$, $F_3(x,y)$ are all equal to each other, and we set

$$F(x,y) := F_1(x,y) = F_2(x,y) = F_3(x,y)$$

for every x and y in Ω.

2. Weil-Petersson geometry

Let Σ_g be a closed topological surface of genus larger than one. We assume that Σ_g is equipped with some hyperbolic metric.

The Weil-Petersson metric on the Teichmüller space is the L^2 metric on the surface Σ for deformation tensors of the hyperbolic metric G;

$$\langle h_1, h_2 \rangle_{\mathrm{WP}} = \int_\Sigma \langle h_1(x), h_2(x) \rangle_{G(x)} \, d\mu_G(x)$$

where the tangency condition for the tensors h_1, h_2 are traceless and divergence-free with respect to G, which preserves the constant curvature condition as well as the perpendicularity to the diffeomorphism fibers. We denote by $d(x,y)$ the Weil-Petersson distance between the points x and y.

The Weil-Petersson completion $\overline{\mathcal{T}}$, a space of Cauchy sequences in $(\mathcal{T}, d)$, consists of the original Teichmüller space $\mathcal{T}$ as well as the bordification points of $\mathcal{T}$ so that Σ is allowed to have nodes, which are geometrically interpreted as simple closed geodesics of zero hyperbolic length. The completed space $\overline{\mathcal{T}}$ (also identified as augmented Teichmüller space by Bers and Abikoff) has the stratification

$$\overline{\mathcal{T}} = \cup_{\sigma \in C(\mathcal{S})} \mathcal{T}_\sigma$$

where the original Teichmüller space $\mathcal{T}$ is expressed as $\mathcal{T}_\emptyset$, and where $C(\mathcal{S})$ is the complex of curves.

We showed in [7] that this stratification is very much compatible with the Weil-Petersson geometry. Namely for each collection $\sigma \in C(\mathcal{S})$, each boundary Teichmüller space $\mathcal{T}_\sigma$ is a Weil-Petersson geodesically convex subset of $\overline{\mathcal{T}}$. Here geodesic convexity means that given a pair of points in $\mathcal{T}_\sigma$, there is a distance-realizing Weil-Petersson geodesic segment connecting them lying entirely in $\mathcal{T}_\sigma$. The non-positive curvature implies the uniqueness of the geodesics.

In [8], a new space was introduced which can be viewed as a Weil-Petersson geodesic completion, called the Teichmüller-Coxeter complex $D(\overline{\mathcal{T}}, \iota)$. The space is a development of the original space $\overline{\mathcal{T}}$ by a Coxeter group generated by reflections across the frontier stratum $\{\overline{\mathcal{T}}_\sigma\}$. It was shown by Wolpert [4] that two intersecting strata of the same dimension meet at a right angle (in the sense of the Alexandrov angle between Weil-Petersson geodesics,) making the development $D(\overline{\mathcal{T}}, \iota)$ a so-called cubical complex [1]. This feature then is used to show that the development is also a CAT(0) space.

In this setting, for each σ with $|\sigma| = 1$, one can consider a *half space*, namely the set H_σ in $D(\overline{\mathcal{T}}, \iota)$, containing $\overline{\mathcal{T}}$ and bounded by $D(\overline{\mathcal{T}}_\sigma, \iota)$. We note the fact obtained by Wolpert [4] that the Weil-Petersson metric completion $\overline{\mathcal{T}}$ is the

closure of the convex hull of the vertex set $\{\overline{\mathcal{T}}_\theta \mid |\theta| = 3g-3\}$, which suggests an interpretation of the Teichmüller space as a simplex.

We can summarize the above discussion as

$$\overline{\mathcal{T}} = \cap_{\sigma\in\mathcal{S}} H_\sigma \quad \text{with} \quad \partial\overline{\mathcal{T}} \subset \cup_\sigma D(\overline{\mathcal{T}}_\sigma, \iota)$$

where every boundary point $b \in \partial\overline{\mathcal{T}}$ belongs to $D(\overline{\mathcal{T}}_\sigma, \iota)$ for some σ in $\mathcal{S}$. Each half space H_σ is bounded by the "supporting hyperplane" $D(\overline{\mathcal{T}}_\sigma, \iota)$.

2.1. **The Weil-Petersson Funk metric F_2.** We now transcribe the Euclidean Funk geometry as well as its compatible Finsler structure in the previous section to the Weil-Petersson setting. We exhibited three equivalent ways of writing down the Funk distance, which we called F_1, F_2 and F_3. In the Weil-Petersson setting, these definitions a-priori differ from each other, and they are related by inequalities.

We define the Weil-Petersson Funk metric F_2 on $\mathcal{T}$ as

$$F_2(x,y) = \sup_{\sigma\in\mathcal{S}} \log \frac{d(x,\overline{\mathcal{T}}_\sigma)}{d(y,\overline{\mathcal{T}}_\sigma)},$$

where d is the Weil-Petersson distance defined on $\overline{\mathcal{T}}$.

We claim the following result concerning the three metrics, which are the Weil-Petersson analogues of F_1, F_2 and F_3 we have seen in the Euclidean setting.

Theorem 1. [9] *The three weak metrics are related by the foliowing inequalities*

$$F_1(x,y) \le F_2(x,y) \le F_3(x,y)$$

for x, y in $\mathcal{T}$, and there are pairs of points (x,y) for which the inequalities are strict.

As a consequence of the comparison, we obtain the following statement, which gives an interesting contrast with the other Funk type metrics, the Teichmüller metric and the Thurston metric, which are both Finsler.

Corollary 2. [9] *The Weil-Petersson Funk metric F_2 is not Finsler.*

References

[1] M.Davis. *The Geometry and Topology of Coxeter Groups.* Princeton Univ.Press, 2007.

[2] A. Papadopoulos and M. Troyanov, Weak Finsler structures and the Funk weak metric. *Proc.Camb.Philos.Soc.* 147 (2009) 419–437.

[3] S. Wolpert. Geometry of the Weil-Petersson completion of Teichmueller space. Surveys in Differential Geometry VIII: Papers in Honor of Calabi, Lawson, Siu and Uhlenbeck, editor S. T. Yau, International Press, (2003).

[4] S.Wolpert, Behavior of geodesic-length functions on Teichmüller space. *J.Diff.Geom.* 79 (2008), 277–334.

[5] S. Wolpert. Behavior of geodesic-length functions on Teichmüller space, J. Diff. Geom. **79** (2008),277–334.

[6] S.Yamada. Weil-Petersson convexity of the energy functional on classical and universal Teichmüller spaces. J.Diff.Geom. **51** (1999) 35-96.

[7] S.Yamada. On the geometry of Weil-Petersson completion of Teichmüller spaces. Math.Res.Let. **11** (2004) 327-344.

[8] S.Yamada. Weil-Petersson geometry of Teichmüller-Coxeter complex and its finite rank property. Geometria Dedicata, **145** (2010) 43-63.

[9] H. Miyachi, K. Ohshika and S.Yamada. Weil-Petersson Funk geometry of the Teichmüller spaces. In preparation (2014).

Chord diagrams, random matrices, and topological recursion

Piotr Sułkowski

In this summary the Hermitian matrix model with potential $V(x) = x^2/2 - stx/(1-tx)$ is introduced and its properties are discussed, following [1]. The partition function of this model enumerates linear chord diagrams of fixed genus with specified numbers of backbones generated by s and chords generated by t. This partition function is computed using the formalism of the topological recursion. The corresponding enumeration of chord diagrams – or more precisely some simple transform of those – gives the number of cells in Riemann's moduli spaces for surfaces with boundaries. These numbers have also other applications – for example, they provide the number of RNA complexes of a given topology. We recall that another matrix model, with logarithmic potential, computes Euler characteristic of moduli spaces, as shown by Penner in [2]. The model which we introduce here provides yet another example of how powerful a description of moduli spaces by random matrices is.

We recall that a chord diagram, which we assume to be connected, is comprised of a collection of $n \geq 0$ semi-circles (called chords) lying in the upper half plane, whose endpoints lie at distinct interior points of $b \geq 1$ pairwise disjoint, oriented and labeled intervals (called backbones) lying in the real line $\mathbb{R} \subset \mathbb{C}$. A chord diagram naturally determines an oriented and connected surface, which is characterized up to homeomorphism by its genus $g \geq 0$ and number $r \geq 1$ of boundary components. The Euler characteristic of this surface is $b - n = 2 - 2g - r$.

Let $c_{g,b}(n)$ denote the number of isomorphism classes of chord diagrams of genus g with n chords on b labeled backbones. We will show how to determine recursively the generating functions

$$C_{g,b}(z) = \sum_{n \geq 0} c_{g,b}(n)\ z^n, \text{ for } g \geq 0, \tag{1}$$

using the topological recursion [3, 4] of a Hermitian one-matrix model

$$Z = \int DH\ e^{-N \mathrm{tr} V(H)} = \exp\left(-N^2 s + \sum_{g=0}^{\infty} N^{2-2g} F_g\right), \tag{2}$$

where N denotes size of matrices, for a particular potential

$$V(x) = \frac{x^2}{2} - \frac{stx}{1-tx}. \tag{3}$$

The crucial fact is the statement that the free energy in genus g of this model encodes $C_{g,b}(t^2)$ via

$$F_g(s,t) = const + \sum_{b \geq 1} \frac{s^b}{b!}\ C_{g,b}(t^2), \text{ for } g \geq 0, \tag{4}$$

where the constant terms reproduce the Gaussian free energies given by $\frac{B_{2g}}{2g(2g-2)}$, where B_{2g} denote Bernoulli numbers. The extra factor $b!$ arises because $C_{g,b}(n)$ counts chord diagrams with labeled backbones as opposed to unlabeled in the topological recursion.

Therefore, the problem of enumerating cells in Riemann's bordered moduli spaces reduces to the problem of performing the matrix integral and determining free energies F_g in (2). To find free energies one should solve the so-called loop equations of the matrix model, which are equations satisfied by certain multi-linear correlators $W_n^{(g)}(p_1,\dots,p_n)$ in this model. The leading order equation among those identities specifies a so-called spectral curve, i.e., an algebraic curve which characterizes distribution of eigenvalues in the matrix model in the $N\to\infty$ limit. It also turns out that all correlators $W_n^{(g)}(p_1,\dots,p_n)$ and loop equations they satisfy can be encoded entirely in terms of this spectral curve. These loop equations can be solved in a recursive way [3], and in this manner, free energies F_g (for $g\geq 2$) are completely determined by correlators $W_1^{(g)}(p)$. Therefore, the spectral curve can also be regarded as the initial condition for this recursion. This entire procedure requires just the knowledge of the spectral curve (and a universal form of the solution to loop equations), and no other details of a matrix model from which this curve was derived. An important achievement of Eynard and Orantin [4] was to realize that one can use the recursive solution of loop equations to assign correlators $W_n^{(g)}(p_1,\dots,p_n)$ and F_g to an arbitrary algebraic curve, not necessarily of matrix model origin. On the other hand, it is guaranteed that F_g computed for the spectral curve of a matrix model reproduce the free energies.

In order to solve the matrix model (2) with the potential (3) we can therefore use the formalism described above. This has indeed been done in [1], and the main steps of this solution are as follows. First, we need to determine the spectral curve of the model (2). This can be done by the analysis of a distribution of eigenvalues in the large N limit. Because the potential (3) is a deformation of the quadratic function, it has a single minimum, and in the equilibrium configuration eigenvalues spread around this minimum. For large N the eigenvalues are distributed along an interval with end-points a and b, which defines a cut in a certain auxiliary complex plane. Such a one-cut solution defines the corresponding spectral curve which has genus zero, and it turns out to be given by the following algebraic equation for two complex variables x and y

$$4y^2(tx-1)^4=(x-a)(x-b)\Big(\big(tx-1+\frac{(a+b)t}{4}\big)^2+\gamma\Big)^2, \tag{5}$$

where

$$\gamma=-\frac{(at+bt)\big((at)^2+(bt)^2+14(at+bt-abt^2)-16\big)}{16(at+bt-2)}.$$

While the end-points of the cut a and b cannot be given in a closed form, it can be found that they are determined by the following system of equations

$$(6)\qquad \begin{cases} 0 = a+b+\frac{st(at+bt-2)}{\left((at-1)(bt-1)\right)^{3/2}}, \\ 16 = (a-b)^2+\frac{4s\left((2-\frac{(a+b)t}{2})(at+bt-2)+2abt^2-3t(a+b)+4\right)}{\left((at-1)(bt-1)\right)^{3/2}}. \end{cases}$$

From the knowledge of the curve (5) and the formalism of the topological recursion we can now determine F_g for $g \geq 2$ (F_0 and F_1 must be determined separately, independently of the topological recursion, for details see [1]). In particular we get the following exact result for the free energy at genus 2:

$$\begin{aligned} F_2 = &-\frac{t^4(1-\sigma)^2}{240\delta^4(1-\delta-4\sigma+3\sigma^2)^5(1+\delta-4\sigma+3\sigma^2)^5}\times \\ &\times\Big(160\delta^4(1-3\sigma)^4(1-\sigma)^6-80\delta^2(1-3\sigma)^6(1-\sigma)^8 \\ &+16(1-3\sigma)^8(1-\sigma)^{10}+\delta^{10}(-16+219\sigma-462\sigma^2+252\sigma^3) \\ &+10\delta^6(1-3\sigma)^2(1-\sigma)^4(-16-126\sigma-423\sigma^2+2286\sigma^3-2862\sigma^4+1134\sigma^5) \\ &+5\delta^8(1-\sigma)^2(16+189\sigma-2970\sigma^2+9549\sigma^3-11286\sigma^4+4536\sigma^5)\Big) \end{aligned}$$

where $\sigma = ((at+bt)/2$ and $\delta = (at-bt)/2$. We also obtain an exact result for the free energy F_3 which is yet more complicated, and its precise form is given in [1]. Expanding these results in the form given in (4), and using the perturbative expansion of a and b in s which follows from (6), we can determine appropriate generating functions $C_{g,b}(z)$. For example, expansion of the above F_2 in powers of s determines generating functions $C_{2,b}(z)$ for all $b.$, such as

$$\begin{aligned} C_{2,4}(z) &= \frac{144z^7}{(1-4z)^{13}}(38675+620648z+2087808z^2 \\ &\quad +1569328z^3+134208z^4), \\ C_{2,5}(z) &= \frac{144z^8}{(1-4z)^{\frac{31}{2}}}(2543625+62424520z+375044396z^2 \\ &\quad +671666053z^3+314761848z^4+18335696z^5), \end{aligned}$$

This procedure can be continued in an algorithmic manner, and with sufficient computational power one can determine exact form of F_g for any g, and so the corresponding $C_{g,b}(z)$, and finally all $c_{g,b}(n)$.

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References

[1] J. Andersen, L. Chekhov, R. Penner, C. Reidys, P. Sułkowski, *Topological recursion for chord diagrams, RNA complexes, and cells in moduli spaces*, Nucl. Phys. B866 (2013) 414.

[2] R. C. Penner, *Perturbative series and the moduli space of Riemann surfaces,* J. Diff. Geom. 27 (1988) 35-53.

[3] L. Chekhov, B. Eynard, *Hermitean matrix model free energy: Feynman graph technique for all genera*, JHEP 0603 (2006) 014.

[4] B. Eynard and N. Orantin, *Invariants of algebraic curves and topological expansion* Comm. Number Th. and Phys. 1 (2007) 347-452.

Combinatorial methods on actions on character varieties

Sara Maloni

(joint work with Frederic Palesi, Ser Peow Tan)

In his PhD thesis [6], McShane established the following remarkable identity for lengths of simple closed geodesics on a once-punctured torus $S_{1,1}$ with a complete, finite area hyperbolic structure:

$$\sum_{\gamma} \frac{1}{1+\exp(l(\gamma))} = \frac{1}{2}, \tag{1}$$

where γ varies over all simple closed geodesics on $S_{1,1}$, and $l(\gamma)$ is the hyperbolic length of γ under the given hyperbolic structure on $S_{1,1}$. This result was later generalized to (general) hyperbolic surfaces with cusps by McShane himself [7], to hyperbolic surfaces with cusps and/or geodesic boundary components by Mirzakhani [9], and to hyperbolic surfaces with cusps, geodesic boundary and/or conical singularities, as well as to classical Schottky groups by Tan, Wong and Zhang in [12], [14].

On the other hand, Bowditch in [1] gave an alternative proof of (1) via Markoff maps, and extended it in [3] to type-preserving representations of the once-punctured torus group into $\mathrm{SL}(2,\mathbb{C})$ satisfying certain conditions which we call here the BQ–conditions (Bowditch's Q–conditions). He also obtained in [2] a variation of (1) which applies to hyperbolic once-punctured torus bundles. In [13] Tan, Wong and Zhang also further extended Bowditch's results to representations of the once-punctured torus group into $\mathrm{SL}(2,\mathbb{C})$ which are not type-preserving, that is, where the commutator is not parabolic, and also to representations which are fixed by an Anosov element of the mapping class group and which satisfy a relative version of the Bowditch's Q–conditions. They also showed that the BQ-conditions defined an open subset of the character variety on which the mapping class group of the punctured torus acted properly discontinuously.

The above papers provided much of the motivation for this talk, in particular, the identities obtained were in many cases valid for the moduli spaces of hyperbolic structures, so invariant under the action of the mapping class group, and in the case of cone structures, they could be interpreted as identities valid for certain subsets of the character variety which were invariant under the action of the

mapping class group, even though the representations in the subset may be non-discrete or non-faithful. This leads naturally to the question of whether there were interesting subsets of the character varieties on which the mapping class group acts properly discontinuously, but which consists of more than just discrete, faithful representations, as explored in the punctured torus case in [13].

In this talk we will consider representations of the free group on three generators $F_3 = \langle \alpha, \beta, \gamma, \delta : \alpha\beta\gamma\delta = I \rangle$ into $\mathrm{SL}(2, \mathbb{C})$. We adopt the viewpoint that F_3 is the fundamental group of the four-holed sphere S, with $\alpha, \beta, \gamma, \delta$ identified with ∂S, and study the natural action of $\mathcal{MCG}(S)$, the mapping class group of S on the character variety

$$\mathcal{X} := \mathrm{Hom}(F_3, \mathrm{SL}(2, \mathbb{C}))//\mathrm{SL}(2, \mathbb{C}),$$

where we take the quotient in the sense of geometric invariant theory. If $\theta \in \mathcal{MCG}(S)$ and $[\rho] \in \mathcal{X}$, this action is given by

$$\theta([\rho]) = [\rho \circ (\theta_*)^{-1}],$$

where $\theta_* \colon \pi_1(S) \longrightarrow \pi_1(S)$ is the map associated to θ in homotopy. We are interested in the dynamics of this action, in particular, on the relative character varieties $\mathcal{X}_{(a,b,c,d)}$, which is the set of representations for which the traces of the boundary curves are fixed.

We describe the following result.

Theorem A. *There exists a domain of discontinuity for the action of $\mathcal{MCG}(S)$ on $\mathcal{X}_{(a,b,c,d)}$, that is, an open $\mathcal{MCG}(S)$–invariant subset $D \subset \mathcal{X}_{(a,b,c,d)}$ on which $\mathcal{MCG}(S)$ acts properly discontinuously.*

Remark 0.1. As already observed by several other authors in related situations (see Goldman [5], Tan–Wong–Zhang [13] and Minsky [8]), our domain of discontinuity contains the interior of the discrete and faithful characters, but also characters which may not be discrete or faithful. For example, when the boundary traces are in $(-2, 2)$ we can produce representations that are non-discrete, but are nevertheless in the domain of discontinuity.

This set is described by two conditions, much in the spirit of [3] and [13], and is given as follows. If $\mathcal{S}$ denotes the set of free homotopy classes of essential, non-peripheral simple closed curves on S, then the conditions for $[\rho]$ to be in D are

(i) $\mathrm{tr}\ \rho(\gamma) \notin [-2, 2]$ for all $\gamma \in \mathcal{S}$; and
(ii) $|\mathrm{tr}\ \rho(\gamma)| < K$ for only finitely many $\gamma \in \mathcal{S}$, where $K > 0$ is a fixed constant that depends only on a, b, c, d.

Furthermore, the set of γ satisfying condition (ii) above satisfy a quasi-convexity property, equivalently, is connected when represented as the subset of the complementary regions of a properly embedded binary tree. This property is particularly important when writing a computer program to draw slices of the domain of discontinuity.

Of particular interest is the set of real characters, which consists of representations in $\mathrm{SL}(2, \mathbb{R})$ or $\mathrm{SU}(2)$. In the latter case, Goldman [4] proved ergodicity

of the mapping class group action for all orientable hyperbolizable surfaces, with respect to the invariant measure induced by the natural symplectic structure on the moduli space. (This was generalized by Palesi in the non-orientable case in [10]). On the other hand, in the $\mathrm{SL}(2,\mathbb{R})$ case the dynamics is much richer and less understood. For example, when S_g is a closed surface of genus $g \geq 2$, Goldman conjectured that the action of $\mathcal{MCG}(S_g)$ on the components of $\mathcal{X}(S_g)$ with non-maximal Euler class is ergodic. An approach towards a proof of this would be to use a cut-and-paste argument involving pieces homeomorphic to one-holed tori and four-holed spheres. While the case of the one-holed torus was completely described by Goldman in [5], we obtain partial results in the four-holed sphere case here. In fact, an important corollary of our analysis is the following:

Theorem B. *In the real case, for all boundary datas, except a dimension one subset, there is a non-empty open domain of discontinuity for the action of* $\mathcal{MCG}(S)$ *on the relative* $\mathrm{SL}(2,\mathbb{R})$*–character variety.*

This implies that there are representations in these components for which all essential simple closed curves on S have hyperbolic representatives, even though these representations may not be discrete and faithful. There are also some surprises here, in particular, certain slices of the real character variety satisfying some general condition always have non-empty intersection with the domain of discontinuity.

References

[1] Brian H. Bowditch. A proof of McShane's identity via Markoff triples. *Bulletin of the London Mathematical Society*, 28(1):73–78, 1996.

[2] Brian H. Bowditch. A variation of McShane's identity for once-punctured torus bundles. *Topology*, 36(2):325–334, 1997.

[3] Brian H. Bowditch. Markoff triples and quasi-Fuchsian groups. *Proc. London Math. Soc.*, 77(3):697–736, 1998.

[4] William M. Goldman. Ergodic theory on moduli spaces. *Annals of mathematics*, 146(3):475–507, 1997.

[5] William M. Goldman. The modular group action on real SL(2)-characters of a one-holed torus. *Geometry and Topology*, 7(1):443–486, 2003.

[6] Greg McShane. *A remarkable identity for lengths of curves*. PhD thesis, University of Warwick, 1991.

[7] Greg McShane. Simple geodesics and a series constant over Teichmüller space. *Inventiones mathematicae*, 132(3):607–632, 1998.

[8] Yair N. Minsky. The classification of punctured-torus groups. *Ann. of Math. (2)*, 149(2):559–626, 1999.

[9] Maryam Mirzakhani. Simple geodesics and Weil-Petersson volumes of moduli spaces of bordered Riemann surfaces. *Inventiones mathematicae*, 167(1):179–222, 2007.

[10] Frederic Palesi. Ergodic actions of mapping class groups on moduli spaces of representations of non-orientable surfaces. *Geometriae Dedicata*, 151(1):107–140, 2011.

[11] Makoto Sakuma. Variations of McShane's identity for the Riley slice and 2-bridge links. *Surikaisekikenkyusho Kokyuroku*, 1104:103–108, 1999.

[12] Ser Peow Tan, Yan Loi Wong, and Ying Zhang. Generalizations of McShane's identity to hyperbolic cone-surfaces. *Journal of Differential Geometry*, 72(1):73–112, 2006.

[13] Ser Peow Tan, Yan Loi Wong, and Ying Zhang. Generalized Markoff maps and McShane's identity. *Adv. Math.*, 217(2):761–813, 2008.

[14] Ser Peow Tan, Yan Loi Wong, and Ying Zhang. McShane's identity for classical Schottky groups. *Pacific Journal of Math*, 237(1):183–200, 2008.

Penner coordinates for closed surfaces

Rinat Kashaev

Let S be a closed oriented surface of genus $g > 1$, and let

$$R_k \subset \mathrm{Hom}(\pi_1, PSL(2,\mathbb{R})), \quad \pi_1 \equiv \pi_1(S, x_0), \tag{1}$$

be the connected component of representations of Euler number $k \in \mathbb{Z}$. According to the result of Goldman [1], the component R_{2-2g} corresponds to discrete faithful representations, so that one has a principal $PSL(2,\mathbb{R})$-fibre bundle over the Teichmüller space $\mathcal{T} \equiv \mathcal{T}(S)$

$$p\colon R_{2-2g} \to \mathcal{T}. \tag{2}$$

Denoting by Ω the space of all horocycles in the hyperbolic plane $\mathbb{H}^2$, we consider the associated fibre bundle

$$\phi\colon \widetilde{\mathcal{T}} \to \mathcal{T}, \quad \widetilde{\mathcal{T}} \equiv R_{2-2g} \times_{PSL(2,\mathbb{R})} \Omega, \tag{3}$$

as a substitute for Penner's decorated Teichmüller space [3, 4] in the case of closed surfaces. We define the λ-distance

$$\lambda\colon \Omega \times \Omega \to \mathbb{R}_{\geq 0} \tag{4}$$

as follows. If $h, h' \in \Omega$ are based on distinct points of $\partial\mathbb{H}^2$, then $\lambda(h, h')$ is the hyperbolic length of the horocyclic segment between tangent points of a horocycle tangent simultaneously to both h and h', and we define $\lambda(h, h') = 0$ if h and h' are based on one and the same point of $\partial\mathbb{H}^2$.

To any $\alpha \in \pi_1 \setminus \{1\}$, we associate a function

$$\lambda_\alpha\colon \widetilde{\mathcal{T}} \to \mathbb{R}_{\geq 0}, \quad [\rho, h] \mapsto \lambda(\rho(\alpha)h, h). \tag{5}$$

The set $\lambda_\alpha^{-1}(0)$ is a sub-bundle of $\widetilde{\mathcal{T}}$ with the fibers homemorphic to $\mathbb{R} \sqcup \mathbb{R}$. Moreover, one has

$$\alpha \neq \beta \Rightarrow \lambda_\alpha^{-1}(0) \cap \lambda_\beta^{-1}(0) = \emptyset. \tag{6}$$

For any subset $A \subset \pi_1 \setminus \{1\}$, we associate the subset

$$\widetilde{\mathcal{T}}_A \equiv \cap_{\alpha \in A} \lambda_\alpha^{-1}(\mathbb{R}_{>0}) \tag{7}$$

together with a function

$$J_A\colon \widetilde{\mathcal{T}}_A \to \mathbb{R}_{>0}^A, \quad J_A(x)(\alpha) = \lambda_\alpha(x), \quad \forall x \in \widetilde{\mathcal{T}}_A,\ \forall \alpha \in A. \tag{8}$$

In what follows, for any cellular complex X, we will denote by X_i the set of its i-dimensional cells.

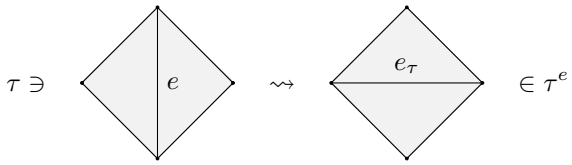
We define a *triangulation* of (S, x_0) as a cellular decomposition with only one vertex at x_0 and where all 2-cells are triangles. We denote by $\Delta \equiv \Delta(S, x_0)$ the set of all triangulations of (S, x_0).

For any $\tau \in \Delta$, to any edge $e \in \tau_1$ there correspond two mutually inverse elements $\gamma^{\pm 1} \in \pi_1$. By abuse of notation, we identify e with any of the functions $\lambda_{\gamma^{\pm 1}}$:

$$e \equiv \lambda_\gamma = \lambda_{\gamma^{-1}} \colon \widetilde{\mathcal{T}} \to \mathbb{R}_{>0}. \tag{9}$$

For any $\tau \in \Delta$ and $e \in \tau_1$, we denote by τ^e the triangulation obtained by the diagonal flip at e, with the flipped edge being denoted as e_τ:

(10)



It is easily shown that for any $\tau \in \Delta$, one has a finite covering

$$\widetilde{\mathcal{T}} = \widetilde{\mathcal{T}}_{\tau_1} \cup (\cup_{e \in \tau_1} \widetilde{\mathcal{T}}_{\tau_1^e}). \tag{11}$$

Our first result gives a realization of $\widetilde{\mathcal{T}}_{\tau_1}$ as an algebraic subset of co-dimension one in $\mathbb{R}^{\tau_1}_{>0}$. Namely, to any pair (τ, t) with $\tau \in \Delta$ and $t \in \tau_2$, we associate a function

$$\psi_{\tau,t} \equiv \sum_{t' \in \tau_2} \epsilon_t(t') \frac{a^2 + b^2 + c^2}{abc} \colon \mathbb{R}^{\tau_1}_{>0} \to \mathbb{R}, \tag{12}$$

where a, b, c are the three sides of t', while the function

$$\epsilon_t \colon \tau_2 \to \{-1, 1\} \tag{13}$$

takes the value -1 on t and the value 1 on all other triangles. We remark that

$$t \neq t' \Rightarrow \psi^{-1}_{\tau,t}(0) \cap \psi^{-1}_{\tau,t'}(0) = \emptyset. \tag{14}$$

We also define

$$\psi_\tau \equiv \prod_{t \in \tau_2} \psi_{\tau,t}. \tag{15}$$

Theorem 1. *For any $\tau \in \Delta$, the map $J_{\tau_1} \colon \widetilde{\mathcal{T}}_{\tau_1} \to \mathbb{R}^{\tau_1}_{>0}$ is an embedding with the image $\psi^{-1}_\tau(0) = \sqcup_{t \in \tau_2} \psi^{-1}_{\tau,t}(0)$.*

Remark 1. The transition functions $J_{\tau_1} \circ J^{-1}_{\tau_1^e}$ on the overlaps $\widetilde{\mathcal{T}}_{\tau_1} \cap \widetilde{\mathcal{T}}_{\tau_1^e}$ are given by the signed Ptolemy transformation of [2] (Proposition 4) with the sign function being given by (13).

Our second result gives explicit coordinatization of the sub-bundles $\lambda^{-1}_\alpha(0)$ together with the explicit $\mathbb{R}_{>0}$-action along the fibers. The result follows.

Let $\mathcal{S} \equiv \mathcal{S}(S)$ be the set of homotopy classes of essential simple closed curves in S, and $\Delta^\alpha \subset \Delta$ the set of triangulations of the form τ^α with τ having an edge representing α. From (6), it is easily seen that

$$\lambda^{-1}_\alpha(0) \subset \widetilde{\mathcal{T}}_{\tau_1}, \quad \forall \tau \in \Delta^\alpha. \tag{16}$$

For $\alpha \in \mathcal{S}$, let

$$\ell_\alpha \colon \mathcal{T} \to \mathbb{R}_{>0} \tag{17}$$

be the hyperbolic length of the geodesic in the homotopy class of α. Any $\tau \in \Delta^\alpha$ has a distinguished edge α_τ. Let τ_α be the quadrilateral having α_τ as its diagonal.

Theorem 2. *Let $\alpha \in \mathcal{S}$ and $\tau \in \Delta^\alpha$. Then*

(i): *one has the inclusion $J_{\tau_1}(\lambda_\alpha^{-1}(0)) \subset \cup_{t\in(\tau_\alpha)_2}\psi_{\tau,t}^{-1}(0)$;*

(ii): *for any $t \in (\tau_\alpha)_2$, the map*

$$L_{\alpha,\tau,t} \colon \widetilde{\mathcal{T}}(\alpha,t) \equiv \lambda_\alpha^{-1}(0) \cap (\psi_{\tau,t}\circ J_{\tau_1})^{-1}(0) \to \mathbb{R}_{>0}\times \mathbb{R}_{>0}^{\tau_1\setminus t_1} \qquad m \mapsto (\ell_\alpha(\phi(m)), J_{\tau_1\setminus t_1}(m)) \tag{18}$$

is a homeomorphism;

(iii): *For any $d \in \mathbb{R}_{>0}$ one has the following equivalence*

$$\phi(m) = \phi(m') \Leftrightarrow \exists\, c \in \mathbb{R}_{>0} \colon J_{\tau_1\setminus t_1}(m) = c\, J_{\tau_1\setminus t_1}(m'), \qquad \forall m, m' \in (\ell_\alpha\circ\phi)^{-1}(d) \cap \widetilde{\mathcal{T}}(\alpha,t). \tag{19}$$

Remark 2. The space $\widetilde{\mathcal{T}}(\alpha,t)$ in Theorem 2 is a connected component of $\lambda_\alpha^{-1}(0)$. It can also be singled out by fixing an orientation on α, and considering only the classes $[\rho,h]$, with h based on the attracting fixed point of $\rho(\alpha)$.

References

[1] William M. Goldman. Topological components of spaces of representations. *Invent. Math.*, 93(3):557–607, 1988.

[2] R. M. Kashaev. Coordinates for the moduli space of flat PSL$(2,\mathbb{R})$-connections. *Math. Res. Lett.*, 12(1):23–36, 2005.

[3] R. C. Penner. The decorated Teichmüller space of punctured surfaces. *Comm. Math. Phys.*, 113(2):299–339, 1987.

[4] R. C. Penner. *Decorated Teichmüller Theory*, volume 1 of *The QGM Master Class Series*. EMS Publishing House, Zürich, 2012.

Quantum representations of mapping class groups and asymptotics in Teichmüller space

Jørgen Ellegaard Andersen

Let Σ be a closed oriented surface of genus $g \geq 2$. We consider the quantum representations of the Witten-Reshetikin-Turaev Topology Quantum Field Theory [W, RT1, RT2]

$$\rho^{(k)} : \Gamma_\Sigma \to \mathbb{P}\mathrm{Aut}\left(\mathbb{P}Z^{(k)}(\Sigma)\right)$$

where Γ_Σ is the mapping class group of Σ and $\mathbb{P}Z^{(k)}(\Sigma)$ is the projectivization of the finite dimensional vector the WRT-TQFT associates to Σ for the quantum group $U_q(sl(2,\mathbb{C}))$, $q = \exp(2\pi i/(k+2))$.

The geometric construction of $\rho^{(k)}$ proceeds via geometric quantization of the moduli space of flat $SU(2)$-connections on Σ: Let

$$\mathcal{M} = \mathrm{Hom}\left(\pi_1(\Sigma), SU(2)\right)/SU(2)$$

be the moduli space of flat $SU(2)$-connections on the surface Σ. This moduli space has the Goldman symplectic form ω and a unique prequantum bundle $(L, \langle\ \rangle, \nabla)$ with

$$F_\nabla = -i\omega.$$

To quantize we consider a complex structure σ, which is a point in Teichmüller space $\mathcal{T}_\Sigma$ of Σ. $\sigma \in \mathcal{T}_\Sigma$ induces a complex structure on I_σ s.t. $(\mathcal{M}, \omega, I_\sigma) = \mathcal{M}_\sigma$ is Kähler and we define a holomorphic bundle $H^{(k)}$ over $\mathcal{T}_\Sigma$ given by

$$H^{(k)}_\sigma = H^0(\mathcal{M}_\sigma, L^k).$$

This bundle has a natural Γ_Σ-invariant connection $\boldsymbol{\nabla}^H$ constructed by Hitchin [H] who showed that this Hitchin connection is projectively flat.
By combining a theorem of Laszlo [L] with a theorem of this author and Ueno [AU1, AU2, AU3, AU4] we get that

$$\mathbb{P}Z^{(k)}(\Sigma) = \text{covariant constant sections of } \left(\mathbb{P}H^{(k)}, \boldsymbol{\nabla}^H\right).$$

By using the theory of Toeplitz operators we prove [A1]

Theorem (Asymptotic faithfulness)**.**

$$\bigcap_k \ker\rho^{(k)} = \begin{cases} \{1, H\} & g = 2 \\ \{1\} & g > 2 \end{cases}$$

where H is hyperelliptic involution.

By analysing the asymptotics of the connection ∇^H we get that it extends to a connection with log-singularity over augmented Teichmüller space $\widetilde{\mathcal{T}}_\Sigma$. Considering points in $\widetilde{\mathcal{T}}_\Sigma$ corresponding to pairs of pants decompositions of the surface Σ we then explicitly construct a unitary structure on $H^{(k)}$ which is preserved by ∇^H and the action of Γ_Σ. This means

$$\rho : \Gamma_\Sigma \longrightarrow \overline{\bigoplus_{k+2 \text{ prime}} \mathrm{End}_0\left(\mathbb{P}Z^{(k)}(\Sigma)\right)}$$

is a unitary Hilbert space representation. We prove it has an almost fixed vector, but Roberts has proved it has no fixed vector. Hence we arrive at the following theorem [A2]

Theorem. Γ_Σ *does not have Kazhdan's property T.*

References

[A1] J.E. Andersen, "Asymptotic faithfulness of the quantum $SU(n)$ representations of the mapping class groups". Annals of Mathematics, **163** (2006), 347–368.

[A2] J.E. Andersen, "Mapping Class Groups do not have Kazhdan's Property (T)", arXiv:0706.2184, pp. 21.

[AU1] J.E. Andersen & K. Ueno, "Geometric construction of modular functors from conformal field theory", Journal of Knot theory and its Ramifications. **16** 2 (2007), 127–202.

[AU2] J.E. Andersen & K. Ueno, "Abelian Conformal Field theories and Determinant Bundles", International Journal of Mathematics. **18**, (2007) 919–993.

[AU3] J.E. Andersen & K. Ueno, "Modular functors are determined by their genus zero data", Quantum Topology **3** 3/4 (2012) 255–291.

[AU4] J.E. Andersen & K. Ueno, "Construction of the Reshetikhin-Turaev TQFT from conformal field theory", arXiv:1110.5027, pp. 39.

[H] N. Hitchin, *Flat connections and geometric quantization*, Comm.Math.Phys., **131** (1990) 347–380.

[L] Y. Laszlo, *Hitchin's and WZW connections are the same*, J. Diff. Geom. **49** (1998), no. 3, 547–576.

[RT1] N. Reshetikhin & V. Turaev, *Ribbon graphs and their invariants derived fron quantum groups*, Comm. Math. Phys. **127** (1990), 1–26.

[RT2] N. Reshetikhin & V. Turaev, *Invariants of* 3*-manifolds via link polynomials and quantum groups*, Invent. Math. **103** (1991), 547–597.

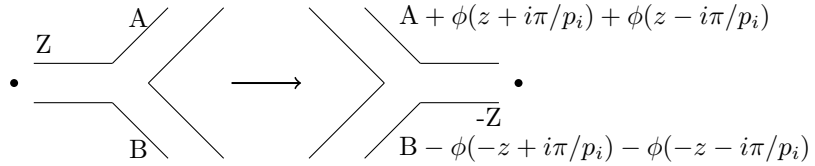
[W] E. Witten, *Quantum field theory and the Jones polynomial*, Commun. Math. Phys **121** (1989) 351–98.

Teichmüller spaces of orbifold Riemann surfaces and related algebras

Leonid Chekhov

We consider the combinatorial description in terms of fat graphs of decorated Teichmüller spaces $\mathcal{T}_{g,s,r}$ of Riemann surfaces $\Sigma_{g,s,r}$ of genus g with $s>0$ holes and with $r\geq 0$ orbifold points having orders p_i, $i=1,\dots,r$, where p_i are integers greater or equal two. $\Sigma_{g,s,r}$ is obtained using the Poincaré uniformization from the upper half-plane $\mathbb{H}^2$ endowed with the hyperbolic metric: $\Sigma_{g,s,r}=\mathbb{H}^2/\Delta_{g,s,r}$ where $\Delta_{g,s,r}\subset \mathrm{PSL}(2,\mathbb{R})$ is a discretely acting subgroup whose conjugacy classes are hyperbolic (parabolic in case of punctures) except exactly s elliptic classes. The spine (fat graph) of this surface is a fat graph of genus g containing exactly s faces, exactly r one-valent vertices, $6g-6+3s+2r$ edges and $4g-4+2s+r$ 3-valent vertices. We decorate all edges with real numbers z_α. The set $\{z_\alpha\}_{\alpha=1}^{6g-6+3s+2r}\in\mathbb{R}^{6g-6+3s+2r}$ which we identify with $\mathcal{T}_{g,s,r}$. We use the 1-1 correspondences between sets of closed geodesics on $\Sigma_{g,s,r}$, conjugacy classes of hyperbolic elements of $\Delta_{g,s,r}$ and closed paths not homeomorphic to orbifold points to construct geodesic functions G in accordance with rules: when passing through any edge we set $X_z=\begin{pmatrix}0 & -\mathrm{e}^{z/2}\\ \mathrm{e}^{-z/2} & 0\end{pmatrix}$, when turning left/right at 3-valent vertices we set $L,R=\begin{pmatrix}0 & 1\\ -1 & -1\end{pmatrix}$ or $\begin{pmatrix}1 & 1\\ -1 & 0\end{pmatrix}$, when going clockwise around $\mathbb{Z}_{p_i}$-orbifold points we set $F_{\omega_i}=\begin{pmatrix}0 & 1\\ -1 & -\omega_i\end{pmatrix}$, $\omega_i=2\cos\frac{\pi}{p_i}$, when going

twice $(-)F^2_{\omega_i}$ etc. $G = \mathrm{tr}(X_{z_n} L X_{z_{n-1}} \cdot \ldots \cdot X_{z_i} F_{\omega_i} X_{z_i} \cdot \ldots \cdot L X_{z_2} R X_{z_1} L)$ are traces of products of 2x2 matrices. We describe flip morphisms: the new one related to flipping a pending edge is



$$\phi = \log(1 + \mathrm{e}^z)$$

and we have the following theorem establishing a "completeness" of this description: for any $\Sigma_{g,s,r}$ we have a (nonunique) set of $\{z_\alpha\}$ (for any spine $\Gamma_{g,s,r}$) and for any $\{z_\alpha\}, \Gamma_{g,s,r}$ we have $\Sigma_{g,s,r}$. We introduce the Poisson bracket, quantize it and obtain quantum $\mathcal{MCG}$ transformations for which it suffices to replace ϕ by

$$\phi^\hbar(z) = -\frac{\pi\hbar}{2} \int \frac{\mathrm{e}^{-ipx}}{\sinh(\pi p)} \frac{\mathrm{d}p}{\sinh(\pi\hbar p)}.$$

We study the obtained quantum geodesic algebras and show that they satisfy the quantum skein relations.

Moments of the boundary hitting function for geodesic flow

Martin Bridgeman
(joint work with Ser Peow Tan)

We consider finite volume hyperbolic manifold with non-empty totally geodesic boundary. We consider the distribution of the time for the geodesic flow to hit the boundary and derive a formula for the moments of the associated random variable in terms of the orthospectrum. We show that the the first two moments correspond to two cases of known identities for the orthospectrum. We further obtain an explicit formula in terms of the trilogarithm functions for the average time for the geodesic flow to hit the boundary in the surface case, using the third moment.

On quasihomomorphisms with noncommutative targets

Koji Fujiwara
(joint work with Misha Kapovich)

This is a talk on the paper [4]. Let G be a group and H be a topological group equipped with a proper left-invariant metric d (e.g., a finitely-generated group, equipped with a word metric). A map $f : G \to H$ is called a *quasihomomorphism* if there exists a constant C so that

$$d(f(xy), f(x)f(y)) \leq C$$

for all $x, y \in G$. In the case when H is discrete (and in this paper we limit ourselves only to this class of groups), f is a quasihomomorphism if and only if the set of *defects* of f

$$D(f) = \{f(y)^{-1} f(x)^{-1} f(xy) : x, y \in G\}$$

is finite. A quasihomomorphism with values in $\mathbb{Z}$ is called a *quasimorphism*.

There is a substantial literature on constructing quasimorphisms, going back to the work of R. Brooks on free groups. For example, see [3] for hyperbolic groups.

We explain why it is so hard to construct quasihomomorphisms to noncommutative groups which are neither homomorphisms, nor come from quasihomomorphisms with commutative targets, provided that H is a discrete group. Our main theorem is:

Theorem 1. *Every quasihomomorphism $f : G \to H$ is* constructible. *Namely, there exists a finite-index subgroup $G_o < G$, a subgroup $H_o < H$, an abelian subgroup $A < H_o$ central in H_o, and a quasihomomorphism $f_o : G_o \to H_o$ within finite distance from $f|G_o$ so that:*

The projection of f_o to $G_o \to Q = H_o/A$ is a homomorphism.

Definition 1 (almost homomorphism)**.** *Suppose that a map $f : G \to H$ between groups has the property that $f(G)$ is contained in a subgroup $J < H$, J contains a finite normal subgroup $K < J$, so that the projection $\bar{f} : G \to \bar{J} = J/K$ is a homomorphism. We then will refer to f as an* almost homomorphism*, it is a homomorphism modulo a finite normal subgroup (in the range of f).*

Here are some sample applications.

Theorem 2. *1. Suppose that H is a torsion-free hyperbolic group. Then (for an arbitrary group G) every unbounded quasihomomorphism $f : G \to H$ is either a homomorphism or a quasihomomorphism to a cyclic subgroup of H.*

2. Suppose that H is a general hyperbolic group. Then for every unbounded quasihomomorphism $f : G \to H$ either the image of f is contained in an elementary subgroup of H or f is an almost homomorphism.

Corollary 1. *Suppose that Γ is an irreducible lattice in a semisimple Lie group of real rank ≥ 2. Then every quasihomomorphism $f : \Gamma \to H$, with hyperbolic target group H, is bounded.*

This sharpens the main result of Ozawa in [5], where he proves it only for lattices in $\mathrm{SL}(n, K)$. Our proof is different.

Theorem 3. *Suppose that Γ is a higher rank irreducible lattice. Then every quasihomomorphism of Γ to a mapping class group of a compact surface, $Map(\Sigma)$, has finite image.*

The conclusion was known for homomorphisms (cf. [1]), and we use that fact and a result by Burger-Monod from [2].

References

[1] M. Bestvina, K. Fujiwara, *Bounded cohomology of subgroups of mapping class groups*, Geometry and Topology 6 (2002), p. 69–89.

[2] M. Burger and N. Monod, *Bounded cohomology of lattices in higher rank Lie groups*, J. Eur. Math. Soc. 1 (1999), no. 2, p. 199–235.

[3] D. B. A. Epstein and K. Fujiwara, *The second bounded cohomology of word-hyperbolic groups*, Topology 36 (1997), no. 6, 1275-1289.

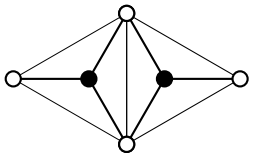
[4] K. Fujiwara, M. Kapovich. *On quasihomomorphisms with noncommutative targets*, arXiv:1312.7407.

[5] N. Ozawa, *Quasi-homomorphism rigidity with non-commutative targets*, J. Reine Angew. Math. 655 (2011), p. 89–104.

Combinatorics of integrable systems.

Vladimir Fock

In the first part of the talk we give an interpretation of cluster coordinates on $SL(N)$ character varieties and in particular sharing coordinates on Teichmüller space as a connection on a bipartite graph. For example, given a triangulated surface, replace each triangle by a Mercedes graphs as shown on the figure. Given an $PSL(2)$ local system on the surface with punctures in the vertices of the triangulation, choose a 1-dimensional subspace invariant under the monodromy about each puncture and associate it to the corresponding white vertex. Then associate to every black vertex the kernel of the map $V_a \oplus V_b \oplus V_c \to \mathbb{C}^2$, where V_a, V_b, V_c are the subspaces associated to the corresponding vertices. Together with natural maps associated to edges of the graph, this construction gives an Abelian local system on the graph. This construction can be easily generalized to $SL(N)$ local systems giving more complicated graphs attached to each triangles.



In the second part of the talk we describe following Goncharov and Kenyon [1] that the space of Abelian local systems on certain bipartite graphs Γ on a torus has a natural integrable system structure. We also study its properties following [2] and [3].

If the graph is embedded into a surface, the space of connections on the graph is fibered over the space $\mathcal{X}_\Gamma = \{(x_1, \dots, x_n)\}$ of monodromies around faces with fiber isomorphic to the space of cohomology of the surface with coefficient in the multiplicative group.

On every bipartite graph Γ there exists a *Kasteleyn orientation* — a marking of edges by ± 1 such that the number of sides counted with signs of every face be 2 modulo 4. For a given Abelian local system define a Dirac operator $\mathfrak{D}$ acting from the direct sum of spaces attached to black vertices to the direct sum of spaces attached to the white ones, which is just a direct sum of maps corresponding to edges with signs given by the Kasteleyn orientation. If the numbers of white an

black vertices coincide the Dirac operator can be nondegenerate on the space of graph connections outside of a divisor.

The space $\mathcal{X}_\Gamma$ possesses a canonical Poisson bracket given by $x_i, x_j = \varepsilon_{ij} x_i x_j$ with ε_{ij} determined by the combinatorics of the graph as the number of common edges of the faces i and j counted with signs determined by the orientations of the edges from black to white vertex.

If the surface is a torus, the fiber over every point $x \in \mathcal{X}_\Gamma$ is two-dimensional and the degeneration locus of the Dirac operator is an algebraic curve in it. This curve Σ is the zero locus of a Laurent polynomial of two variables $\sum_{(ij)\in\Delta} H_{ij}(x)\lambda^i\mu^j = \det \mathfrak{D}$ with a Newton polygon Δ fixed by the graph up to a shift and the action of $SL(2,\mathbb{Z})$. The coefficients of this polynomial (after normalization of three of them to one) give a full collection of commuting Hamiltonians.

If the curve is smooth the kernel of the Dirac operator defines a line bundle on Σ of degree $g-1$, where g is the genus of the curve, which is equal to the number of integer points strictly inside the polygon Δ.

It turns out that the map associating to a point of $\mathcal{X}_\Gamma$ an algebraic curve and a line bundle on it can be inverted. Namely for a planar algebraic curve Σ given by a Laurent polynomial with a Newton polygon Δ and a bipartite graph Γ corresponding to the same polygon one can explicitly describe the inverse image of Σ under the action map. This inverse image is isomorphic to the Jacobian of the curve $Jac(\Sigma)$ and our aim now is to make this isomorphism explicit.

The main observation is that the graph Γ can be (almost canonically) embedded into the spectral curve Σ in such a way that every connected component of the complement to Γ in Σ is a punctured disc. Denote by $\check{F}$ the set of such discs and for any $\alpha \in \check{F}$ denote by $\boldsymbol{w}_\alpha$ a point of the universal cover of the Picard variety $Pic^1(\Sigma)$ corresponding to the puncture of the disc and by $D\alpha$ the disc itself. Let W be the universal cover of the Jacobian of Σ. Fix a Lagrangian lattice $L \in W$ in the kernel of the projection $W \to Jac(\Sigma)$.

Associate to every face i an element $\boldsymbol{z}_i \in W$ that obeys the following rule. For any two faces i and j and any path γ_{ij} on the torus from i to j we have

$$\boldsymbol{z}_j - \boldsymbol{z}_i = \sum_{\alpha\in\check{F}} \langle \gamma_{ij}, \partial D_\alpha \rangle \boldsymbol{w}_\alpha \text{ modulo } L,$$

Such association is defined uniquely up to a shift $\boldsymbol{z}_i \to \boldsymbol{z}_i + \boldsymbol{l}_i + \boldsymbol{t}$ with $\boldsymbol{t} \in W$ and $\boldsymbol{l}_i \in L$.

Choose an odd nonsingular theta-characteristics $q \in Pic^{g-1}(\Sigma)$. Then the monodromies x_i around the faces given by

$$x_i = \prod_j \left(\frac{\theta_q(\boldsymbol{z}_j - \boldsymbol{z}_i)}{\theta_q(\boldsymbol{z}_j + \boldsymbol{t})} \right)^{\varepsilon_{ij}}$$

for any $\boldsymbol{t} \in W$ runs over the inverse Lagrangian torus, which is the common level set of the Hamiltonians. Here θ_q is the Riemann theta function on W periodic with respect to the lattice L.

The proof the formula is based on the Fay's triple secant formula.

The space $\mathcal{X}_\Gamma$ can be considered as a chart of a cluster x-variety. Changing the graph Γ without changing the Newton polygon Δ amounts to the cluster transformations of the chart. In particular the space $\mathcal{X}_\Gamma$ admits an action of an Abelian group G_Δ of birational transformations commuting with the Hamiltonian flows. This group turns out to be a subgroup to the group of divisors of the curve Σ of degree zero, supported at infinity modulo principal divisors. The rank of this group is equal to the number of corners of the Newton polygon Δ.

References

[1] A.B. Goncharov and R. Kenyon, *Dimers and cluster integrable systems*, `arXiv:1107.5588`.
[2] V. V. Fock and A. Marshakov, *Loop groups, Clusters, Dimers and Integrable systems*, *arXiv:1401.1606*.
[3] V. V. Fock, *Inverse spectral problem for GK integrable systems*, to appear.

Computations in formal symplectic geometry and characteristic classes of moduli spaces

Takuya Sakasai
(joint work with Shigeyuki Morita, Masaaki Suzuki)

1. Homology of a positive graded Lie algebra

Let $\mathfrak{g} = \oplus_{k=0}^{\infty}\mathfrak{g}(k)$ be a graded Lie algebra over $\mathbb{Q}$ and let $\mathfrak{g}^+ = \oplus_{k=1}^{\infty}\mathfrak{g}(k)$ be its ideal consisting of all the elements of $\mathfrak{g}$ with *positive* gradings. We assume that each piece $\mathfrak{g}(k)$ is finite dimensional for all k. Then the chain complex $C_*(\mathfrak{g})$ of $\mathfrak{g}$ splits into the direct sum

$$C_*(\mathfrak{g}) = \bigoplus_{w=0}^{\infty} C_*^{(w)}(\mathfrak{g})$$

of *finite dimensional* subcomplexes $C_*^{(w)}(\mathfrak{g}) = \oplus_{i=0}^{w} C_i^{(w)}(\mathfrak{g})$ where

$$C_i^{(w)}(\mathfrak{g}) = \bigoplus_{\substack{i_0+i_1+\cdots+i_w=i \\ i_1+2i_2+\cdots+wi_w=w}} \wedge^{i_0}(\mathfrak{g}(0)) \otimes \wedge^{i_1}(\mathfrak{g}(1)) \otimes \cdots \otimes \wedge^{i_w}(\mathfrak{g}(w))$$

so that $C_i^{(w)}(\mathfrak{g}) = 0$ for $i > w + \frac{1}{2}d(d-1)$ ($d = \dim \mathfrak{g}(0)$). This gives a bigrading to the homology group $H_*(\mathfrak{g})$ and we decompose it as

$$H_i(\mathfrak{g}) = \bigoplus_{w=0}^{\infty} H_i(\mathfrak{g})_w$$

where $H_i(\mathfrak{g})_w = H_i(C_*^{(w)}(\mathfrak{g}))$ is called the *weight* w-part of $H_i(\mathfrak{g})$.

2. Symplectic derivation Lie algebras

Let Σ_g be a closed connected oriented surface of genus $g \geq 1$. The first rational homology group $H_1(\Sigma_g;\mathbb{Q})$ has a non-degenerate and skew-symmetric form

$$\mu : H_1(\Sigma_g;\mathbb{Q}) \otimes H_1(\Sigma_g;\mathbb{Q}) \longrightarrow \mathbb{Q},$$

which is called the intersection pairing. By using this pairing, we can identify $H_1(\Sigma_{g,1};\mathbb{Q})$ with the dual space $H^1(\Sigma_{g,1};\mathbb{Q})$ and we denote them by H.

Let $\mathrm{Sp}(H)$ be the symplectic group, that is, the group of automorphisms of H preserving μ. This group $\mathrm{Sp}(H) \simeq \mathrm{Sp}(2g,\mathbb{Q})$ can be regarded as the group of automorphisms of H preserving the symplectic element

$$\omega_0 \in (H \otimes H)^{\mathrm{Sp}} \simeq \mathbb{Q}.$$

Here $(H \otimes H)^{\mathrm{Sp}}$ is the invariant subspace of $H \otimes H$ under the diagonal action of $\mathrm{Sp}(H)$ on $H \otimes H$.

Definition 2.1. We define symplectic derivation Lie algebras for three cases:

(1) $\mathfrak{c}_g^+ = \{$positive "Sp-derivations" of the free *commutative* algebra on $H\}$
$$= \left\{ \begin{array}{l} \text{Hamiltonian } \mathbb{Q}\text{-polynomial vector fields} \\ \text{on } H \otimes \mathbb{R} \cong \mathbb{R}^{2g} \text{ without constant and linear terms.} \end{array} \right\},$$

(2) $\mathfrak{h}_g^+ = \{$positive Sp-derivations of the free *Lie* algebra on $H\}$,

(3) $\mathfrak{a}_g^+ = \{$positive Sp-derivations of the free *associative* algebra on $H\}$.

Then we take direct limits with respect to g:

$$\mathfrak{c}_\infty^+ = \lim_{g\to\infty} \mathfrak{c}_g^+, \quad \mathfrak{h}_\infty^+ = \lim_{g\to\infty} \mathfrak{h}_g^+, \quad \mathfrak{a}_\infty^+ = \lim_{g\to\infty} \mathfrak{a}_g^+.$$

By the representation theory of $\mathrm{Sp}(H)$, the corresponding Sp-invariant chain complexes $C_*^{(w)}(\mathfrak{c}_\infty^+)^{\mathrm{Sp}}$, $C_*^{(w)}(\mathfrak{h}_\infty^+)^{\mathrm{Sp}}$ and $C_*^{(w)}(\mathfrak{a}_\infty^+)^{\mathrm{Sp}}$ are all *finite* dimensional for each weight w.

3. Main theorems

Kontsevich [5, 6] related the homology of the Lie algebras $\mathfrak{h}_\infty^+$, $\mathfrak{a}_\infty^+$ to the cohomology of outer automorphism groups $\mathrm{Out}\, F_n$ of free groups and mapping class groups of punctured surfaces. Geometrically, these cohomology are also interpreted as those of moduli spaces of metric graphs and punctured Riemann surfaces. The homology of $\mathfrak{c}_\infty^+$ also have topological meanings.

Our first result is on the abelianization of $\mathfrak{a}_\infty^+$.

Theorem 3.1 ([7])**.** $H_1(\mathfrak{a}_\infty^+)^{\mathrm{Sp}} = 0$.

As a corollary, we see that the "top" dimensional rational cohomology groups of moduli spaces of closed and once-punctured Riemann surfaces vanish.

Next, we consider the Euler characteristics of the above three complexes.

Theorem 3.2 ([8])**.** *The Euler characteristics e of the complexes $C_*^{(w)}(\mathfrak{c}_\infty^+)^{\mathrm{Sp}}$, $C_*^{(w)}(\mathfrak{h}_\infty^+)^{\mathrm{Sp}}$ and $C_*^{(w)}(\mathfrak{a}_\infty^+)^{\mathrm{Sp}}$ in low weights are given by the following:*

(1) $e(H_*(\mathfrak{c}_\infty^+)_w^{\mathrm{Sp}}) = 1, 2, 3, 6, 8, 14, 20, 32, 44, 68 \qquad (w = 2, 4, \ldots, 20)$,

(2) $e(H_*(\mathfrak{h}_\infty^+)_w^{\mathrm{Sp}}) = 1, 2, 4, 6, 10, 16, 23, 13, -96 \qquad (w = 2, 4, \ldots, 18)$,
(3) $e(H_*(\mathfrak{a}_\infty^+)_w^{\mathrm{Sp}}) = 2, 5, 12, 24, 50, 100, 188, 347 \qquad (w = 2, 4, \ldots, 16)$.

Applying Kontsevich's theorem to (2), we obtain the *integral* Euler characteristics of $\mathrm{Out}\, F_n$.

Corollary 3.3. *The integral Euler characteristics* e *of* $\mathrm{Out}\, F_n$ *for* $n = 2, 3, \ldots, 10$ *are*

$$e(\mathrm{Out}\, F_n) = 1, 1, 2, 1, 2, 1, 1, -21, -124.$$

In particular, there exist at least $1, 22, 125$ *odd dimensional non-trivial* $\mathbb{Q}$*-homology classes of* $\mathrm{Out}\, F_n$ *for* $n = 8, 9, 10$.

Remark 3.4. Hatcher-Vogtmann in [4] ($n \leq 5$) and Ohashi in [9] ($n = 6$) determined $H_*(\mathrm{Out}\, F_n; \mathbb{Q})$. Besides, Galatius [1] showed that $H_k(\mathrm{Out}\, F_n; \mathbb{Q}) = 0$ for $0 < k \ll n$. However, it had not been known whether non-trivial odd dimensional $\mathbb{Q}$-homology classes of $\mathrm{Out}\, F_n$ exist or not. Note that Gray [3] computed that $H_{12}(\mathrm{Out}\, F_8; \mathbb{Q})$ is non-trivial. This result with our computation $e(\mathrm{Out}\, F_8) = 1$ show that there exists at least one odd dimensional non-trivial $\mathbb{Q}$-homology class of $\mathrm{Out}\, F_8$.

Remark 3.5. As for the Euler characteristics of the associative case $\mathfrak{a}_\infty^+$, we can use Gorsky's formula [2]. By implementing it as a Mathematica program, we computed $e(H_*(\mathfrak{a}_\infty^+)_w^{\mathrm{Sp}})$ up to $w = 500$. In the talk, an expectation concerning the asymptotic behavior of these values was mentioned.

References

[1] S. Galatius, *Stable homology of automorphism groups of free groups*, Ann. Math 173 (2011), 705–768.
[2] E. Gorsky, *The equivariant Euler characteristic of moduli space of curves*, Adv. Math. 250 (2014), 588–595.
[3] J. Gray, *On the homology of automorphism groups of free groups*, Ph.D. thesis, University of Tennessee, 2011.
[4] A. Hatcher, K. Vogtmann, *Rational homology of* $\mathrm{Aut}(F_n)$, Math. Res. Lett. 5 (1998), 759–780.
[5] M. Kontsevich, *Formal (non)commutative symplectic geometry*, in: "The Gel'fand Mathematical Seminars, 1990–1992", Birkhäuser, Boston (1993), 173–187.
[6] M. Kontsevich, *Feynman diagrams and low-dimensional topology*, in: "First European Congress of Mathematics, Vol. II (Paris, 1992)", Progr. Math. 120, Birkhäuser, Basel (1994), 97–121.
[7] S. Morita, T. Sakasai, M. Suzuki, *Abelianizations of derivation Lie algebras of the free associative algebra and the free Lie algebra*, Duke Mathematical Journal 162, Number 5 (2013), 965–1002.
[8] S. Morita, T. Sakasai, M. Suzuki, *Computations in formal symplectic geometry and characteristic classes of muduli spaces*, to appear in Quantum Topology.
[9] R. Ohashi, *The rational homology group of* $\mathrm{Out}(F_n)$ *for* $n \leq 6$, Experiment. Math. 17 (2008), 167–179.

The Goldman-Turaev Lie bialgebra and the Johnson homomorphisms

Yusuke Kuno
(joint work with Nariya Kawazumi)

Consider a compact oriented surface Σ of genus $g > 0$ with one boundary component. Taking a basepoint on the boundary we let $\pi = \pi_1(\Sigma)$. We denote by $\hat{\pi}$ the set of free homotopy classes of loops on Σ. The $\mathbb{Q}$-vector space $\mathbb{Q}\hat{\pi}$ is a Lie algebra with respect to the *Goldman bracket* [1]. We can make its completion $\widehat{\mathbb{Q}\hat{\pi}}$ by using the augmentation ideal of the group ring $\mathbb{Q}\pi$.

On the other hand, let $H = H_1(\Sigma;\mathbb{Q})$ be the first homology of Σ, and consider the completed tensor algebra $\widehat{T} = \prod_{m=0}^{\infty} H^{\otimes m}$. By the intersection form on Σ, we can identify H with its dual $H^* = \mathrm{Hom}(H,\mathbb{Q})$. Let $\omega \in H^{\otimes 2}$ be the 2-tensor corresponding to $1_H \in \mathrm{Hom}(H,H) \cong H^{\otimes 2}$. We define $\mathrm{Der}_\omega(\widehat{T})$ to be the set of (continuous) derivations on $\widehat{T}$ annihilating ω. This is called the *Lie algebra of symplectic derivations* [8].

Theorem 1 ([4][5])**.** *There exists a Lie algebra homomorphism $\lambda_\theta\colon \mathbb{Q}\hat{\pi} \to \mathrm{Der}_\omega(\widehat{T})$. Moreover, λ_θ induces an isomorphism $\widehat{\mathbb{Q}\hat{\pi}} \cong \mathrm{Der}_\omega(\widehat{T})$.*

Key ingredients of this theorem are the following.

(1) There is a map $\sigma\colon \mathbb{Q}\hat{\pi} \to \mathrm{Der}_\partial(\mathbb{Q}\pi)$ defined in a way similar to the definition of the Goldman bracket. Here the target is the set of derivations on $\mathbb{Q}\pi$ annihilating the boundary loop $\partial \in \pi$. Moreover, σ induces an isomorphism $\widehat{\mathbb{Q}\hat{\pi}} \cong \mathrm{Der}_\partial(\widehat{\mathbb{Q}\pi})$. Here $\widehat{\mathbb{Q}\pi}$ is the completion of the group ring with respect to the augmentation ideal.

(2) A *symplectic expansion* [9] is a map $\theta\colon \pi \to \widehat{T}$ satisfying some conditions. Such a map θ induces an isomorphism $\widehat{\mathbb{Q}\pi} \cong \widehat{T}$ of complete Hopf algebras sending ∂ to $\exp(\omega)$. The map λ_θ is given by $\lambda_\theta(\alpha)(u) = \theta(\sigma(\alpha)\theta^{-1}(u))$ for $\alpha \in \mathbb{Q}\hat{\pi}$ and $u \in \widehat{T}$.

Our proof uses the (co)homology theory of Hopf algebras. Massuyeau and Turaev [10] gave a different proof. We have an analogue of Theorem 1 and the isomorphism $\widehat{\mathbb{Q}\hat{\pi}} \cong \mathrm{Der}_\partial(\widehat{\mathbb{Q}\pi})$ for any compact oriented surface with *non-empty boundary*. For details see [7].

Let $1 \in \hat{\pi}$ be the class of a constant loop. The *Turaev cobracket* [13] is a map $\delta\colon \mathbb{Q}\hat{\pi}/\mathbb{Q}1 \to (\mathbb{Q}\hat{\pi}/\mathbb{Q}1)^{\otimes 2}$ defined by using the self intersection of loops. The $\mathbb{Q}$-vector space $\mathbb{Q}\hat{\pi}/\mathbb{Q}1$ is an *involutive Lie bialgebra* with respect to the Goldman bracket and the Turaev cobracket. The map δ extends naturally to the completion $\widehat{\mathbb{Q}\hat{\pi}}$.

Now the isomorphism λ_θ induces a Lie cobracket $\delta^\theta := (\lambda_\theta \hat{\otimes} \lambda_\theta) \circ \delta \circ \lambda_\theta^{-1}$ on $\mathrm{Der}_\omega(\widehat{T})$. To see δ^θ, we consider the embedding

$$\mathrm{Der}_\omega(\widehat{T}) \hookrightarrow \mathrm{Hom}(H,\widehat{T}) \cong H \otimes \widehat{T} = \widehat{T}_{\geq 1}, \quad D \mapsto D|_H.$$

Then $\mathrm{Der}_\omega(\widehat{T})$ is identified as a $\mathbb{Q}$-vector space with the space of cyclic invariant tensors. For $X_1,\dots,X_m \in H$, we set $N(X_1\cdots X_m) = \sum_i X_i\cdots X_m X_1\cdots X_{i-1} \in$

$H^{\otimes m}$, and define $\delta^{\mathrm{alg}}\colon \mathrm{Der}_\omega(\widehat{T}) \to (\mathrm{Der}_\omega(\widehat{T}))^{\otimes 2}$ by

$$\begin{aligned}&\delta^{\mathrm{alg}}(N(X_1\cdots X_m))\\ &:=\sum_{i<j}(X_i\cdot X_j)\left\{\begin{array}{l} N(X_{j+1}\cdots X_m X_1\cdots X_{i-1})\otimes N(X_{i+1}\cdots X_{j-1})\\ -N(X_{i+1}\cdots X_{j-1})\otimes N(X_{j+1}\cdots X_m X_1\cdots X_{i-1})\end{array}\right\}.\end{aligned}$$

This is a homogeneous $\mathbb{Q}$-linear map of degree -2.

Theorem 2 ([6] [11])**.** *We have*

$$\delta^\theta = \delta^{\mathrm{alg}} + \delta^\theta_{(1)} + \delta^\theta_{(2)} + \cdots .$$

Here, $\delta^\theta_{(n)}$ is a homogeneous $\mathbb{Q}$-linear map of degree n.

The proof uses a tensorial description of the *homotopy intersection form* by Massuyeau and Turaev [10]. In general, $\delta^\theta_{(n)}$ *does* depend on the choice of θ.

The *Johnson homomorphisms* were introduced by Johnson [2][3] and were elaborated later by Morita [12]. They are important tools to study the algebraic structure of the mapping class group and the Torelli group. By using the operation σ, in particular the isomorphism $\widehat{\mathbb{Q}\hat{\pi}} \cong \mathrm{Der}_\partial(\widehat{\mathbb{Q}\pi})$, we can embed the Torelli group of Σ into $\widehat{\mathbb{Q}\hat{\pi}}$. From this embedding we can recover the Johnson homomorphisms. From the fact that any diffeomorphism of the surface preserves the self intersection of curves, we can show that the image of our embedding is contained in the kernel of the Turaev cobracket. This gives a geometric constraint for the image of the Johnson homomorphisms. For more details see [7].

References

[1] W. M. Goldman, *Invariant functions on Lie groups and Hamiltonian flows of surface groups representations*, Invent. Math. **85** (1986), 263–302.

[2] D. Johnson, *An abelian quotient of the mapping class group $\mathcal{I}_g$*, Math. Ann. **249** (1980), 225–242.

[3] D. Johnson, *A survey of the Torelli group*, In: Low Dimensional Topology, Contemp. Math. **20**, AMS Providence 1983, 165–179.

[4] N. Kawazumi and Y. Kuno, *The logarithms of Dehn twists*, to appear in Quantum topology

[5] N. Kawazumi and Y. Kuno, *Groupoid-theoretical methods in the mapping class group of surfaces*, preprint, arXiv:1109.6479v2

[6] N. Kawazumi and Y. Kuno, *Intersections of curves on surfaces and their applications to mapping class groups*, preprint, arXiv:1112.3841v2

[7] N. Kawazumi and Y. Kuno, *The Goldman-Turaev Lie bialgebra and the Johnson homomorphisms*, preprint, arXiv:1304.1885

[8] M. Kontsevich, *Formal (non)-commutative symplectic geometry*, In: The Gelfand Mathematical Seminars, 1990-1992, Birkhauser, Boston (1993), 173–187.

[9] G. Massuyeau *Infinitesimal Morita homomorphisms and the tree-level of the LMO invariant*, Bull. Soc. Math. France **140** (2012), 101–161.

[10] G. Massuyeau and V. G. Turaev, *Fox pairings and generalized Dehn twists*, to appear in: Ann. Inst. Fourier, available at arXiv:1109.5248

[11] G. Massuyeau and V. G. Turaev, in preparation.

[12] S. Morita, *Abelian quotients of subgroups of the mapping class group of surfaces*, Duke Math. J. **70** (1993), 699–726.

[13] V. G. Turaev, *Skein quantization of Poisson algebras of loops on surfaces*, Ann. sci. Ecole Norm. Sup. (4) **24** (1991), 635–704.

Asymptotic Behaviors of Some Rays in Hitchin Components

Qiongling Li
(joint work with Brian Collier)

For a closed, connected, oriented surface S of genus $g \geq 2$, consider the space of group homomorphisms $\rho : \pi_1(S) \to G$ from the fundamental group $\pi_1(S)$ to a reductive Lie group G. Through the nonabelian Hodge correspondence, the representation variety

$$\mathcal{R}(\pi_1, G) = Hom(\pi_1(S), G)//G \cong \mathcal{M}_{Higgs}(G)$$

is diffeomorphic to the moduli space of semistable G-Higgs bundles.

In [2], Hitchin used Higgs bundles to show that this component of the character variety $\mathcal{R}(\pi_1, PSL(n, \mathbb{R}))$ is an open cell of complex dimension $(n^2 - 1)(g - 1)$. This component is called the Hitchin component. Furthermore, if we fix a Riemann surface structure Σ on S, with canonical bundle K, then $Hit_n(S)$ is parameterized by the space $\bigoplus_{j=2}^{n} (\Sigma, K^j)$ of holomorphic differentials.

For a stable $SL(n, \mathbb{C})$-Higgs bundle (E, ϕ), there is a unique hermitian metric h on E, with Chern connection A_h, solving the Higgs bundle equations

$$F_{A_h} + [\phi, \phi^{*_h}] = 0,$$

where ϕ^{*_h} denotes the the hermitian adjoint. Hitchin proved this for $n = 2$ and later Simpson [5] proved it for general n. Such a solution (A_h, ϕ) gives rise to a flat connection $A + \phi + \phi^{*_h}$.

Given a semistable Higgs bundle (E, ϕ), consider the family of semistable Higgs bundles $(E, t\phi)$, where $t \in \mathbb{C}$. Solving the Higgs bundle equations yields a family of harmonic metrics h_t on E and thus a family of flat connections ∇_t with corresponding representations ρ_t. For $P, Q \in \widetilde{\Sigma}$, let $T_{P,Q}(t)$ be the parallel transport matrices of the family of flat connections. In a recent arxiv paper [3], Katzarkov, Noll, Pandit, and Simpson asked the following question:

Question 1. What is the asymptotic behavior of ρ_t and $T_{P,Q}(t)$ as $t \to \infty$?

This is a difficult problem, as it involves asymptotically solving the Higgs bundle equations. In this paper we restrict to the following situation

- (E, ϕ) is in the Hitchin component
- $t \in \mathbb{R}$
- $\phi = \tilde{e}_1 + q_n e_{n-1}$ and $\phi = \tilde{e}_1 + q_{n-1} e_{n-2}$.

Instead of $t\phi$, we use tq_n and tq_{n-1}, which are equivalent to $t\phi$ after a gauge transformation.

We restrict our attention to our $\phi = \tilde{e}_1 + q_n e_{n-1}$ and $\phi = \tilde{e}_1 + q_{n-1} e_{n-2}$ because of the following theorem.

Theorem 2. *For $k = n$ and $k = n-1$ the Higgs fields are $\phi = \tilde{e}_1 + q_n e_{n-1}$ and $\phi = \tilde{e}_1 + q_{n-1} e_{n-2}$, and the harmonic metric splits as*

$$h_1 \oplus h_2 \oplus \cdots \oplus h_2^{-1} \oplus h_1^{-1}$$

on the line bundles

$$K^{\frac{n-1}{2}} \oplus K^{\frac{n-3}{2}} \oplus \cdots \oplus K^{-\frac{n-3}{2}} \oplus K^{-\frac{n-1}{2}}$$

For $k = n$, this was proven by Baraglia [1] in which he called cyclic case.

For $n = 3$ cyclic case, Loftin [4] considered the above questions in a slightly different setting. We try to generalize his method to apply to general n case.

The above theorem significantly simplifies the Higgs bundle equations from n^2 equations to $\lfloor \frac{n}{2} \rfloor$ equations. We first obtain estimates of the solution metric h_t of the Higgs bundle equations as $t \to \infty$.

Theorem 3. *(n=3 cyclic case by J.Loftin) For every point $p \in \Sigma$, as $t \to \infty$ we have*

1. For $(\Sigma, (0, \ldots, 0, tq_n)) \in Hit_n(S)$

$$h_t = \begin{pmatrix} t^{\frac{n-1}{2}}|q_n|^{\frac{n-1}{n}} & & & & \\ & t^{\frac{n-3}{n}}|q_n|^{\frac{n-3}{n}} & & & \\ & & \ddots & & \\ & & & t^{-\frac{n-3}{n}}|q_n|^{-\frac{n-3}{n}} & \\ & & & & t^{-\frac{n-1}{n}}|q_n|^{-\frac{n-1}{n}} \end{pmatrix} (1+O(t^{-\frac{2}{n}}))$$

2. For $(\Sigma, (0, \ldots, 0, tq_{n-1}, 0)) \in Hit_n(S)$

$$h_t = \begin{pmatrix} t|q_{n-1}| & & & & \\ & (2t)^{\frac{n-3}{n-1}}|q_{n-1}|^{\frac{n-3}{n-1}} & & & \\ & & \ddots & & \\ & & & (2t)^{-\frac{n-3}{n-1}}|q_{n-1}|^{-\frac{n-3}{n-1}} & \\ & & & & t^{-1}|q_n|^{-1} \end{pmatrix} (1+O(t^{-\frac{2}{n-1}}))$$

After obtaining asymptotic estimates of the solution metric together with error estimates, we make use of all the estimates to integrate the ODE defined by the flat connection. This yields an estimate of the parallel transport matrices $T_{P,Q}(t)$ as $t \to \infty$.

Theorem 4. *(n=3 cyclic case by J.Loftin) Let $P \in \widetilde{\Sigma}$ be away from the zeroes of q_n (or q_{n-1}), choose a neighborhood centered at P and coordinate z so that $q_n = dz^n$ (or $q_{n-1} = dz^{n-1}$). For any Q in the neighborhood $Q = Le^{i\theta}$, then as $t \to \infty$*

$$T_{P,Q}(t) = (Id + R)T \begin{pmatrix} e^{-Lt^{\frac{1}{n}}\mu_1} & & & \\ & e^{-Lt^{\frac{1}{n}}\mu_2} & & \\ & & \ddots & \\ & & & e^{-Lt^{\frac{1}{n}}\mu_n} \end{pmatrix} T^{-1}$$

where

1. *For the case* $\phi = \tilde{e}_1 + e_{n-1}q_n$*, the error term* $R \sim O(t^{-\frac{1}{n}})$*,* T *is a constant unitary matrix and the set* $\{\mu_1, \ldots, \mu_n\}$ *is the set* $\{2cos(\theta + \frac{2\pi j}{n})\}$*.*
2. *For the case* $\phi = \tilde{e}_1 + e_{n-2}q_{n-1}$*, the error term* $R \sim O(t^{-\frac{1}{n-1}})$*,* T *is a constant unitary matrix and there is one* $\mu_i = 0$ *and other* μ_j*'s are the roots of a degree* $n-1$ *polynomial depending on* θ*.*

Given a family of hermitian metrics h_t on E, select a positively oriented unitary frame $\{e_j\}$ over a base point $p \in \tilde{\Sigma}$. Parallel transportation using the family of flat connections gives a global setion of the unitary frame bundle which we will denote $\{e_j\}$. Define a family of ρ_t-equivariant maps $\{f_t\}$ by

$$\begin{aligned} \tilde{\Sigma} &\xrightarrow{f_t} SL(n,\mathbb{R})/SO(n,\mathbb{R}) \\ q &\longmapsto \{h_t(e_j(q), e_j(q))\}. \end{aligned}$$

We have the following corollary concerning the maps f_t.

Corollary 5. *For the family of rays* $(\Sigma, 0, \ldots, 0, tq_n)$*,* $(\Sigma, 0, \ldots, 0, tq_{n-1}, 0) \in Hit_n(S)$ *and for any* p *away from the zeros of* q_n *or* q_{n-1}*, there exists a neighborhood* U *of* p *so that the* ρ_t*-equivariant maps* $f_t : \tilde{\Sigma} \to SL(n,\mathbb{R})/SO(n,\mathbb{R})$ *satisfies* $f_t(U)$ *is mapped, aymptotically, into a flat of the symmetric space.*

Our approach is inspired by Loftin's work in [4] where he studied the asymptotic holonomy of convex real projective structures on the surface S along a ray (Σ, tq_3). We apply the maximum principal many times to the system of $\lfloor \frac{n}{2} \rfloor$ fully coupled nonlinear elliptic equations to obtain estimates of the solution metric as $t \to \infty$ along with error estimates. We then obtain the estimates of the first derivative of the solution metric using standard PDE techniques from the error estimates. In the end, we make use of all the estimates to integrate the ODE defined by the flat connection.

References

[1] D. Baraglia, *Cyclic Higgs bundles and the affine Toda equations*, eprint arXiv:1011.6421, (2010).

[2] N. Hitchin, *Lie groups and Teichmüller space*, Topology. An International Journal of Mathematics, **31**(1992), 449–473.

[3] L. Katzarkov, A. Noll, P. Pandit and C. Simpson, *Harmonic Maps to Buildings and Singular Perturbation Theory*, eprint arXiv:1311.7101v1, (2013).

[4] J. Loftin, *Flat metrics, cubic differentials and limits of projective holonomies*, Geometriae Dedicata, **128**(2007), 97–106.

[5] C. Simpson, *Higgs bundles and local systems*, Inst. Hautes Études Sci. Publ. Math., **75**(1992), 5–95.

Dynamics of Proper Actions on Lie Groups

Olivier Guichard
(joint work with François Guéritaud, Fanny Kassel, Anna Wienhard)

The purpose of the present work is to gain a better understanding of a certain class of groups acting properly discontinuously on a real rank one Lie group G by left and right multiplication. A precise statement is the following theorem.

Theorem 1. *Let Γ be a finitely generated subgroup of $G \times G$ the product of two copies of a real rank one semisimple Lie group G. Then the following statements are equivalent :*

(1) The injection of Γ into $G \times G$ is a quasi-isometric embedding and the action of Γ onto G by left and right multiplication is properly discontinuous.

(2) (Up to finite index and up to exchanging the 2 factors in $G \times G$) the group Γ is a graph $\{(\gamma, j(\gamma)) \,/\, \gamma \in \Gamma_0\}$ where $\Gamma_0 < G$ is a convex-cocompact subgroup and $j : \Gamma_0 \to G$ is a representation that is uniformly dominated by the injection of Γ_0 into G.

Let us start by defining the notions appearing in that theorem.

• The injection of Γ into $G \times G$ is a *quasi-isometric embedding* if there are positive constants k and C such that

$$\forall \gamma \in \Gamma, \quad k\ell_\Gamma(\gamma) - C \leq \ell_{G\times G}(\gamma) \leq k^{-1}\ell_\Gamma(\gamma) + C$$

where ℓ_Γ is a word length on Γ and $\ell_{G\times G}(\gamma)$ is the distance from γ to the identity element coming from a left invariant Riemannian metric on $G \times G$.

• The group $G \times G$ acts on G by *left and right multiplication* : $(g, h) \cdot x = gxh^{-1}$ for any $(g, h) \in G \times G$ and any $x \in G$.

• A semisimple Lie group G is of *real rank one* if and only if its symmetric $\mathcal{X}$ is negatively curved. For any g in G we will denote by $\lambda(g) \in \mathbf{R}_{\geq 0}$ its *translation length* on $\mathcal{X}$: $\lambda(g) = \inf_{x\in\mathcal{X}} d_\mathcal{X}(x, g \cdot x)$.

• A subgroup Γ_0 of G is said *convex cocompact* if there is a closed Γ_0-invariant convex C of $\mathcal{X}$ and Γ_0 acts on C is properly discontinuously and cocompactly.

• The morphism $j : \Gamma_0 \to G$ is *uniformly dominated* by the injection of Γ_0 into G if there is $\kappa \in (0, 1)$ such that $\lambda(j(\gamma)) \leq \kappa\lambda(\gamma)$ for any $\gamma \in \Gamma_0$ of infinite order.

The fact that subgroups Γ of $G \times G$ acting properly on G are always graphs over discrete subgroups of G (up to finite index) is a theorem of F. Kassel [Kas08]. Conjecturally every such action should have the above domination property, in fact a more general conjecture for proper actions on homogeneous spaces can be found in [KK12]. The additional hypothesis that Γ is quasi-isometrically embedded in $G \times G$ implies that Γ_0 is also quasi-isometrically embedded in G and this is known to be equivalent to Γ_0 being a convex-cocompact subgroup of G, see [Bow95].

The key step in order to obtain the conclusion that j is uniformly dominated by the inclusion of Γ_0 is to prove that the group Γ is Anosov in the sense of Labourie [Lab06]. A precise statement for the case when the real rank one Lie group G is $\mathrm{SO}(1, n)$ is the following theorem. Let us observe first that $\mathrm{SO}(1, n) \times \mathrm{SO}(1, n)$ is naturally a subgroup of $\mathrm{SO}(n + 1, n + 1)$.

Theorem 2. *Let Γ be a finitely generated subgroup of $\mathrm{SO}(1,n)\times\mathrm{SO}(1,n)$ then the following properties are equivalent*

(1) The injection of Γ into $\mathrm{SO}(1,n)\times\mathrm{SO}(1,n)$ is a quasi-isometric embedding and the left-right action of Γ onto $\mathrm{SO}(1,n)$ by is properly discontinuous.

(2) The group Γ is word hyperbolic and the injection of Γ into $\mathrm{SO}(n+1,n+1)$ is $\mathcal{F}_0$-Anosov.

The contraction properties involved in the defintion of Anosov representations imply the sought for domination. The more general definition of Anosov representations given in [GW12] is with respect to any partial flag variety G/P of the Lie group G. Here we restrict to the case $G=\mathrm{SO}(n+1,n+1)$ and $G/P=\mathcal{F}_0=\mathbb{P}(q=0)$ is the projectivization of the null cone of q the quadratic form of signature $(n+1,n+1)$. Hence elements of $\mathcal{F}_0$ are isotropic lines. The original definition of Anosov representations makes use of the flow space of the word hyperbolic group Γ and is therefore quite elaborate. The above theorem relies on the following characterization of Anosov representations that makes only use on the boundary at infinity $\partial_\infty\Gamma$.

Theorem 3. *Let Γ be a word hyperbolic group, $\partial_\infty\Gamma$ be its boundary at infinity and $\rho:\Gamma\to\mathrm{SO}(n+1,n+1)$ be a representation.*

Then the representation ρ is $\mathcal{F}_0$-Anosov if and only if there exists a continuous and ρ-equivariant map $\xi:\partial_\infty\Gamma\to\mathcal{F}_0$ satisfying the following conditions:

- *ξ is* transverse*: for any t and $t'\neq t$ in $\partial_\infty\Gamma$ the lines $l=\xi(t)$ and $l'=\xi(t')$ are transverse (i.e. $l'\not\subset l^{\perp_q}$).*
- *ξ is* dynamics-preserving*: for any non-torsion element γ of Γ, if t^+_γ denotes its attracting fixed point in $\partial_\infty\Gamma$, the element $\xi(t^+_\gamma)$ is the attracting fixed point of $\rho(\gamma)$ in $\mathcal{F}_0$. In particular the action of $\rho(\gamma)$ is contracting on the tangent space $T_{\xi(t^+_\gamma)}\mathcal{F}_0$, we denote by $e^{-\alpha(\rho(\gamma))}$ its contraction factor.*
- *There is a positive constant $c>0$ such that, for any non-torsion element γ of Γ, $\alpha(\rho(\gamma))\geq c\ell_\infty(\gamma)$ where $\ell_\infty(\gamma)=\lim\frac{1}{n}\ell_\Gamma(\gamma^n)$ is the stable length of γ (and ℓ_Γ is the word length in Γ).*

The above results hold in a wider generality, we refer to [GGKW] for a more complete discussion.

References

[Bow95] Brian H. Bowditch, *Geometrical finiteness with variable negative curvature*, Duke Math. J. **77** (1995), no. 1, 229–274.

[GGKW] François Guéritaud, Olivier Guichard, Fanny Kassel and Anna Wienhard, *Anosov representations and proper actions*, in preparation.

[GW12] Olivier Guichard and Anna Wienhard, *Anosov representations: domains of discontinuity and applications*, Invent. Math. **190** (2012), no. 2, 357–438.

[Kas08] Fanny Kassel, *Proper actions on corank-one reductive homogeneous spaces*, J. Lie Theory **18** (2008), 961–978.

[KK12] Fanny Kassel and Toshiyuki Kobayashi, *Poincaré series for non-Riemannian locally symmetric spaces*, arXiv:1209.4075v2, 2012.

[Lab06] François Labourie, *Anosov flows, surface groups and curves in projective space*, Invent. Math. **165** (2006), no. 1, 51–114.

The Johnson-Morita theory for the rings of Fricke characters of free groups

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(joint work with Eri Hatakenaka)

In the 1980s, Dennis Johnson established a remarkable method to investigate the group structure of mapping class groups of surfaces in a series of his works. In particular, he [5] constructed a homomorphism τ to determine the abelianization of the Torelli group. Today, his homomorphism τ is called the first Johnson homomorphism, and it is generalized to those of higher degrees. Over the last two decades, good progress was made in the study of the Johnson homomorphisms of the mapping class group through the works of many authors including Morita [7], Hain [2] and others.

As is well known, the mapping class group of a compact oriented surface with one boundary component can be embedded into the automorphism group of a free group by a classical work of Dehn and Nielsen. The definition of the Johnson homomorphisms can be naturally generalized to the automorphism group of a free group. Let F_n be a free group of rank n, H the abelianization of F_n, and $\mathrm{Aut}\, F_n$ the automorphism group of F_n. The kernel of the homomorphism $\mathrm{Aut}\, F_n \to \mathrm{GL}(n,\mathbb{Z})$ induced from the action of $\mathrm{Aut}\, F_n$ on H, is called the IA-automorphism group of F_n, and is denoted by IA_n. The group IA_n is a free group analogue of the Torelli group. In 1965, Andreadakis [1] intorduced a central filtration

$$\mathrm{IA}_n = \mathcal{A}_n(1) \supset \mathcal{A}_n(2) \supset \cdots$$

of IA_n. He showed that each graded quotient $\mathrm{gr}^k(\mathcal{A}_n) := \mathcal{A}_n(k)/\mathcal{A}_n(k+1)$ is a free abelian group of finite rank, and that this filtration has the trivial intersection. We call the above filtration the Andreadakis-Johnson filtration of $\mathrm{Aut}\, F_n$. Johnson studied this kind of filtration for the mapping class groups of surfaces in 1980s. The conjugation action of $\mathrm{Aut}\, F_n$ on each $\mathrm{gr}^k(\mathcal{A}_n)$ induces that of the general linear group $\mathrm{GL}(n,\mathbb{Z})$. Let

$$\tau_k : \mathrm{gr}^k(\mathcal{A}_n) \to H^* \otimes_{\mathbb{Z}} \mathcal{L}_n(k+1)$$

be a homomorphism defined by

$$\sigma \pmod{\mathcal{A}_n(k+1)} \mapsto \Big[x \pmod{\Gamma_n(k+1)} \mapsto x^{-1}x^{\sigma} \pmod{\Gamma_n(k+2)} \Big].$$

Each τ_k is a $\mathrm{GL}(n,\mathbb{Z})$-equvariant injective homomorphism, and is called the k-th Johnson homomorphism of $\mathrm{Aut}\, F_n$. These τ_k are powerful and usuful tools to investigate the graded quotients $\mathrm{gr}^k(\mathcal{A}_n)$.

In this talk, we study a Fricke character analogue of the Andreadakis-Johnson filtration and the Johnson homomorphisms of $\mathrm{Aut}\, F_n$. Let $R(F_n)$ be the set of all $\mathrm{SL}(2,\mathbb{C})$-representations of F_n, and $\mathcal{F}(n,\mathbb{C})$ the set of all complex-valued functions on $R(F_n)$. Then $\mathcal{F}(n,\mathbb{C})$ naturally has a $\mathbb{C}$-algebra structure, and $\mathrm{Aut}\, F_n$ naturally

acts on $\mathcal{F}(n,\mathbb{C})$ from the right. For any $x \in F_n$, define $\operatorname{tr} x \in \mathcal{F}(n,\mathbb{C})$ to be

$$(\operatorname{tr} x)(\rho) := \operatorname{tr} \rho(x), \quad \rho \in R(F_n).$$

Here "tr" in the right hand side means the usual trace of 2×2 matrix $\rho(x)$. The element $\operatorname{tr} x$ is called the Fricke character of $x \in F_n$. The action of $\sigma \in \operatorname{Aut} F_n$ on $\operatorname{tr} x$ is given by $\operatorname{tr} x^\sigma$.

Classically, Fricke characters were introduced by Fricke to study the classification of the Riemann surfaces. In this talk, we concentrate ourselves on purely algebraic properties of the Fricke characters. Let $\mathfrak{X}_{\mathbb{Q}}(F_n)$ be the $\mathbb{Q}$-subalgebra of $\mathcal{F}(n,\mathbb{C})$ generated by all $\operatorname{tr} x$ for $x \in F_n$. We call $\mathfrak{X}_{\mathbb{Q}}(F_n)$ the ring of Fricke characters of F_n over $\mathbb{Q}$. Horowitz [3] showed that $\mathfrak{X}_{\mathbb{Q}}(F_n)$ is finitely generated by

$$\{\operatorname{tr} x_{i_1} \cdots x_{i_l} \mid 1 \le l \le 3,\ 1 \le i_1 < i_2 < \cdots < i_l \le n\}.$$

Here we consider an $\operatorname{Aut} F_n$-invariant ideal

$$J := (\operatorname{tr}' x_{i_1} \cdots x_{i_l} \,|\, 1 \le l \le 3,\ 1 \le i_1 < i_2 < \cdots < i_l \le n) \subset \mathfrak{X}_{\mathbb{Q}}(F_n)$$

where $\operatorname{tr}' x := \operatorname{tr} x - 2$ for any $x \in F_n$. Then, we have a descending filtration

$$J \supset J^2 \supset J^3 \supset \cdots,$$

and each graded quotient $\operatorname{gr}^k(J) := J^k/J^{k+1}$ is an $\operatorname{Aut} F_n$-invariant finite dimensional $\mathbb{Q}$-vector space. In general, however, by combinatorial complexities, it is quite difficult to give a basis of $\operatorname{gr}^k(J)$. In our previous paper [4], we explicitly give bases of $\operatorname{gr}^k(J)$ for $k = 1$ and 2.

Now, for any $k \ge 1$, let $\mathcal{E}_n(k)$ be the kernel of the homomorphism $\operatorname{Aut} F_n \to \operatorname{Aut}(J/J^{k+1})$ induced from the action of $\operatorname{Aut} F_n$ on J/J^{k+1}. Then the groups $\mathcal{E}_n(k)$ define a descending filtration

$$\mathcal{E}_n(1) \supset \mathcal{E}_n(2) \supset \cdots \supset \mathcal{E}_n(k) \supset \cdots$$

of $\operatorname{Aut} F_n$. This is a Fricke character analogue of the Andreadakis-Johnson filtration. In [4], we showed

Theorem 1 (Hatakenaka-S.)**.** *For $n \ge 3$,*

(1) $[\mathcal{E}_G(k), \mathcal{E}_G(l)] \subset \mathcal{E}_G(k+l)$ *for any* $k, l \ge 1$.
(2) $\mathcal{E}_n(1) = \operatorname{Inn} F_n \cdot \mathcal{A}_n(2)$.
(3) $\mathcal{A}_n(2k) \subset \mathcal{E}_n(k)$ *for any* $k \ge 1$.

Here $\operatorname{Inn} F_n$ is the inner automorphism group of F_n. In order to study the structure of the graded quotients $\operatorname{gr}^k(\mathcal{E}_n) := \mathcal{E}_n(k)/\mathcal{E}_n(k+1)$ of the above filtration, we [4] introduced a homomorphism

$$\eta_k : \operatorname{gr}^k(\mathcal{E}_n) \to \operatorname{Hom}_{\mathbb{Q}}(\operatorname{gr}^1(J), \operatorname{gr}^{k+1}(J))$$

defined by

$$\sigma \pmod{\mathcal{E}_n(k+1)} \mapsto \Big[f \pmod{J^2} \mapsto f^\sigma - f \pmod{J^{k+1}} \Big].$$

The homomorphism η_k is a Fricke character analogue of the Johnson homomorphism τ_k. In [4], we showed that each η_k is $\mathrm{Aut}\, F_n/\mathcal{E}_n(1)$-equivariant injective homomorphism. This implies that each of $\mathrm{gr}^k(\mathcal{E}_n)$ is torsion-free, and that $\dim_{\mathbb{Q}}(\mathrm{gr}^k(\mathcal{E}_n)\otimes_{\mathbb{Z}}\mathbb{Q})<\infty$. It seems a basic problem to determine the structure of $\mathrm{Im}(\eta_k)$.

On the other hand, another our interest is what kind of properties η_k and τ_k share with. In this talk, we consider the extendability of the first Johnson homomorphisms to the mapping class group and the automorphism group of a free group. In [8], Morita showed that the first Johnson homomorphism can be uniquely extended to the mapping class group as a crossed homomorphism up to coboundary. Similar result for $\mathrm{Aut}\, F_n$ was obtained by Kawazumi [6]. As a Fricke character analogue of these works, we [9] obtained

Theorem 2 (S.)**.** *For $n\geq 3$, the homomorphism η_1 can be extended to $\mathrm{Aut}\, F_n$ as a crossed homomorphism.*

At the present stage, the uniqueness of the extension of η_1 is not known since we can not comput the twisted first cohomology group of $\mathrm{Aut}\, F_n$ with coefficients in $\mathrm{Hom}_{\mathbb{Q}}(\mathrm{gr}^1(J),\mathrm{gr}^2(J))$ due to the combinatorial complexity.

References

[1] S. Andreadakis, *On the automorphisms of free groups and free nilpotent groups*, Proc. London Math. Soc. (3) **15** (1965), 239-268.

[2] R. Hain, *Infinitesimal presentations of the Torelli group*, J. of the Amer. Math. Soc. **10** (1997), 597–651.

[3] R. Horowitz, *Characters of free groups represented in the two-dimensional special linear group*, Comm. on Pure and Applied Math., Vol. **XXV** (1972), 635–649.

[4] E. Hatakenaka and T. Satoh, *On the graded quotients of the ring of Fricke characters of a free group*, preprint, `arXiv:1206.1500`.

[5] D. Johnson, *The structure of the Torelli group III: The abelianization of $\mathcal{I}$*, Topology **24** (1985), 127–144.

[6] N. Kawazumi, *Cohomological aspects of Magnus expansions*, preprint, `arXiv:math.GT/0505497`.

[7] S. Morita, *Abelian quotients of subgroups of the mapping class group of surfaces*, Duke Math. J., **70** (1993), 699–726.

[8] S. Morita, *The extension of Johnson's homomorphism from the Torelli group to the mapping class group*, Invent. math. **111** (1993), 197–224.

[9] T. Satoh, *The Johnson-Morita theory for the ring of Fricke characters of free groups*, preprint.

A discrete uniformization theorem for polyhedral surfaces

Feng Luo

(joint work with David Gu, Jian Sun, Tianqi Wu)

We introduce a notion of discrete conformality for polyhedral metrics on compact surfaces. Given a closed surface S with a finite non-empty subset of points V, a (Euclidean) polyhedral metric on (S,V), to be called a *PL metric* on (S,V),

is a flat cone metric on (S, V) with cone points in V. The discrete curvature k of a PL metric on (S, V) is the function on V sending a vertex to 2π less the cone angle at the vertex. It is well known that the Gauss-Bonnet formula holds for PL metrics, i.e., $\sum_{v\in V} k(v) = 2\pi\chi(S)$. Each polyhedral metric d on (S, V) has a Delaunay triangulation $\mathcal{T}_d$ which is a geodesic triangulation with vertices V so that for each edge, the sum of two opposite angles facing e is at most π. Suppose d and d' are two PL metrics on (S, V). We say they are *discrete conformal* if there is a sequence of PL metrics $d_1 = d, d_2, ..., d_n = d'$ and a sequence of triangulations $\mathcal{T}_1$, $\mathcal{T}_2$, ..., $\mathcal{T}_n$ of (S, V) (with vertex sets $V(\mathcal{T}_i) = V$) so that (1) each $\mathcal{T}_i$ is Delaunay in d_i; (2) if $\mathcal{T}_i \neq \mathcal{T}_{i+1}$, then there is an isometry h_i from (S, d_i) to (S, d_{i+1}) which is homotopic to the identity map on (S, V); and (3) if $\mathcal{T}_i = \mathcal{T}_{i+1}$, there is a function $x_i : V \to \mathbf{R}_{>0}$ so that for each edge $e = vv'$ in $\mathcal{T}_i$, the lengths $l_{d_i}(vv')$ and $l_{d_{i+1}}(vv')$ of e in d_i and d_{i+1} are related by

$$l_{d_{i+1}}(vv') = l_{d_i}(vv')x_i(v)x_i(v').$$

Our main theorems are,

Theorem 1. *Given two PL metrics on a closed marked surface (S, V), there exists an algorithm to decide if they are discrete conformal.*

Theorem 2. *Given any PL metric d on a closed marked surface (S, V) and any $k^* : V \to (-\infty, 2\pi)$ so that $\sum_{v\in V} k^*(v) = 2\pi\chi(S)$, there exists a PL metric d^*, unique up to scaling, so that*

(1) d^* is discrete conformal to d and,

(2) the discrete curvature of d^* is k^*.

Furthermore, d^* can be found by a finite dimensional variational principle.

Theorem 1 is proved by using the work of W. Thurston and L. Mosher on ideal triangulations of surfaces and the work of R. Penner on decorated Teichmüller spaces.

To prove Theorem 2, let recall that the PL Teichmüller space $T_{pl}(S, V)$ is defined to be the set of all Teichmüller equivalence classes of PL metrics on (S,V). Here two PL metrics are Teichmüller equivalent if they are isometric by an isometry homotopic to the identity in (S, V). The discrete conformality is an equivalence relation on $T_{pl}(S, V)$. For each point $[d] \in T_{pl}(S, V)$, let $CD([d])$ be the set of all metrics in $T_{pl}(S, V)$ which are discrete conformal to $[d]$. Let $K : T_{pl}(S, V) \to (-\infty, 2\pi)^V$ be the discrete curvature map sending a metric to its discrete curvature. Theorem 2 can be stated as saying that the restriction of the discrete curvature map K to the discrete conformal class $CD([d])$ is a bijection from $CD([d])/\mathbb{R}_{>0}$ to $(-\infty, 2\pi)^V \cap \{x \in \mathbb{R}^V \,|\, \sum_v x(v) = 2\pi\chi(S)\ \}$ where $\mathbb{R}_{>0}$ acts on PL metrics by scaling. To achieve this, we proceed in two steps. In the first step, we show that the discrete conformal class $DC([d])$ is naturally a Euclidean space. This is achieved by producing a C^1 smooth diffeomorphism A from PL Teichmuller space $T_{pl}(S, V)$ to Penner's decorated Teichmuller space $T(S - V) \times \mathbb{R}^V_{>0}$ so that two PL metrics d and d' are discrete conformal if and only if the projections of $A(d)$ and $A(d')$ to the Teichmuller space $T(S - V)$ are the same. The map A

is constructed in a piecewise smooth manner on the natural cell decompositions of $T_{pl}(S,V)$ and $T(S-V)\times\mathbb{R}^V_{>0}$. These cell decompositions are derived from the Delaunay triangulations of the underlying spaces by the work of Penner and Rivin. In the second step, we exam the restriction of discrete curvature map on the $DC([d])$. By step 1, $DC([d])$ is naturally a Euclidean space. Using a variational principle developed by Luo in 2004, we show that the discrete curvature map on $DC([d])/\mathbb{R}_{>0}$ is the gradient of a strictly convex function. Thus, it is an injective map. On the other hand, by analyzing the degeneration of discrete conformality of triangles and using a result of Akiyoshi, we show that the image $K(DC([d]))$ is closed in $Y=(-\infty,2\pi)^V\cap\{x\in\mathbb{R}^V|\sum_v x(v)=2\pi\chi(S)\}$. Since both $DC([d])/\mathbb{R}_{>0}$ and Y are connected manifolds of the same dimension, we conclude that $K|$ is a homeomorphism and thus prove theorem 2.

There are several open problems related to the discrete conformality. First, we do not know how to prove theorem 2 for non-compact surfaces. Second, we conjecture that discrete conformality converges to the classical conformality when triangulations become finer and finer. The numerical evidences to this conjecture are very strong. However, a rigorous proof is still lacking.

References

[1] D. Gu, F. Luo, J. Sun, T. Wu, *A discrete uniformization theorem for polyhedral surfaces*, arXiv:1309.4175.

Problem Session

Problems compiled by Daniele Alessandrini, session chaired by Norbert A'Campo

Problem 1. (Norbert A'Campo) Given an immersed curve in $\mathbb{R}^2$ with transverse self-intersection, we can count the number of self-intersections. Denote by $N(g)$ the number of curves as above with exactly g self-intersections, up to isotopy. Give an interpretation to the power series:

$$\sum_{g\geq 0} N(g)z^g$$

This problem is related with matrix models and to the problem of counting cells in $M_{g,1}$, the moduli space of Riemann surfaces with one marked point.

Problem 2. (Athanase Papadopoulos) Understand the relation between the Galois group and the Teichmüller space. The absolute Galois group, $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ acts on the set of isotopy classes of finite planar trees. This is almost in bijection with the set of complex polynomials in 1 variable having 0 and 1 as their only critical values, given such a polynomial p the planar tree is given by $p^{-1}([0,1])$. The coefficients of such polynomials lie in some number field, hence the absolute Galois group acts on the set of polynomial by changing the coefficients. This action is faithful. The problem is to understand the orbits at the level of planar trees. This is in relation with problem 1.

Problem 3. (Ursula Hamenstädt) A Hurwitz curve is a Riemann surface with maximal symmetry, for example the one obtained from the triangle surface with angles $\pi/2, \pi/3, \pi/7$. The Galois group acts on the collection of Hurwitz curves. Robert Kucharczyk proved that this action is faithful. What is the smallest genus of a Riemann surface such that the action sends it to one that is not isomorphic?

Problem 4. (Lizhen Ji)

(1) Does the Teichmüller space $T_{g,n}$ admit a complete $CAT(0)$-metric invariant under the mapping class group MCG?
(2) Does the outer space O_n admit a complete $CAT(0)$-metric invariant under $\mathrm{Out}(F_n)$?

Note that $T_{g,n}/MCG$ and $O_n/\mathrm{Out}(F_n)$ are not compact, so the questions don't imply that the groups MCG and $\mathrm{Out}(F_n)$ are $CAT(0)$, which is false.

Problem 5. (Lizhen Ji) Consider the Teichmüller space $T_{g,n}$ with the Teichmüller metric. The moduli space $M_{g,n} = T_{g,n}/MCG$ has finite volume. How special is this volume? Is it algebraic or transcendental? One may conjecture that it is transcendental, and some special value of an L-function.

Similar question for the first eigenvalue of the Laplacian for the Teichmüller metric, $\lambda_1(T_{g,n})$. McMullen proved that it is strictly positive, but how special is it?

In Thurston's list of open problems he asks how special are the volumes of closed hyperbolic 3-manifolds.

Problem 6. (Lizhen Ji) What is the number of integral points on the moduli space?

Problem 7. (Sumio Yamada) Thurston's stretch map between two hyperbolic surfaces minimises the Lipschitz constant of maps in its homotopy class. Is there a distance on the target surface that makes the stretch map harmonic? This question is motivated by the analogy with the Teichmüller map, minimising the quasiconformal constant of maps in its homotopy class. This map is harmonic with reference to the flat metric with singularities induced by the Teichmüller quadratic differential.

Problem 8. (Michael Wolf) The Hitchin component $\mathrm{Hit}(n)$ is a connected component of the variety of representations of the fundamental group of a closed surface in $\mathrm{PSL}(n,\mathbb{R})$ up to conjugation. Hitchin proved that given a fixed Riemann surface structure X on the surface, $\mathrm{Hit}(n)$ is parametrised by the space $\bigoplus_{i=2}^{n} H^0(X, K^i)$, where K is the canonical bundle of X. Labourie asked the question whether it is possible to parametrise $\mathrm{Hit}(n)$ with the space $\bigcup_{X\in T_g} \bigoplus_{i=3}^{n} H^0(X, K^i)$. Labourie and Loftin proved this for $n = 3$ using affine spheres and the Monge-Ampere equation.

Problem 9. (Hiroshige Shiga) Let X be a finite type Riemann surface with negative Euler characteristic, and consider a holomorphic map $\varphi : X \to M_g$. This map can be lifted to a holomorphic map $\phi : \mathbb{H}^2 \to T_g$, with monodromy

$\rho : \pi_1(X) \to \mathrm{Mod}_g$. A result of rigidity by Imayoshi and Shiga says that given two such holomorphic maps φ_1, φ_2, if their monodromies agree, then $\varphi_1 = \varphi_2$.

Is this again true if we consider the representation ρ' that is the composition of the monodromy ρ with the representation $\mathrm{Mod}_g \to \mathrm{Sp}(2g, \mathbb{Z})$? More explicitely, if two holomorphic maps φ_1, φ_2 satisfy $\rho'_1 = \rho'_2$, is it true that $\varphi_1 = \varphi_2$?

If this is true, we can have an effective bound on the number of holomorphic maps $X \to M_g$.

Problem 10. (Ursula Hamenstädt) What is the smallest genus of a closed surface that can be mapped to M_g in such a way that the associated monodromy is injective?

Problem 11. (Mustafa Korkmaz)

(1) Is the mapping class group linear?
(2) Let S be a surface with genus $g \geq 3$ and boundary components $\delta_1, \ldots, \delta_n$. Denote by τ_γ the Dehn twist around γ. Is it possible to write $\tau_{\delta_1} \cdots \tau_{\delta_n}$ as a product of positive Dehn twists about non-separating simple closed curves? This is true for $n \leq 4g + 4$, but what about the general case?
(3) If $g \geq 3$ and $n \geq 3$, is it possible to write $\tau_{\delta_1} \cdots \tau_{\delta_n}$ as a product of an arbitrarily large number of positive Dehn twists about non-separating simple closed curves? The motivation comes from the theory of Stein filling contact 3-manifolds.
(4) Can we understand $H_3(\mathrm{Mod}_g, \mathbb{Z})$? For the moment we can only understand $H_3(\mathrm{Mod}_g, \mathbb{Q})$.
(5) Is the Torelli group finitely presented?
(6) Is there a finite index subgroup of Mod_g that contains the Torelli group?

Problem 12. (Gregor Masbaum) The group $\mathrm{Mod}_{g,n}$ acts on Teichmüller space and on the curve complex. Consider the normal subgroup $t(k)$ that is generated by all the k-th powers of Dehn twists. Is it possible to find some nice space with a good action of $\mathrm{Mod}_{g,n}/t(k)$? This is interesting for the theory of quantum representations, where we can construct representations with kernel containing $t(k)$. This problem can be solved for $\mathrm{Mod}_{1,1}$, is it possible to generalise? (Leonid O. Chekhov provided additional insight about the properties of $\mathrm{Mod}_{1,1}$ and its relations with Kontsevich's matrix models, asking if this can also be generalised).

Problem 13. (Jørgen E. Andersen) Let $M = \mathrm{Hom}(\pi_1(S), \mathrm{SU}(2))/\mathrm{SU}(2)$ or $M = \mathrm{Hom}(\pi_1(S), \mathrm{SL}(2, \mathbb{C}))//\mathrm{SL}(2, \mathbb{C})$, and let M' be the subset corresponding to irreducible representations. The mapping class group Γ acts on M and M'. Let U be one of the rings $\mathcal{O}(M), L^2(M), C_c^\infty(M), C^\infty(M')$. What is $H^1(\Gamma, U)$? The motivation for this question is to understand if there is a unique *-product on the set of functions that are invariants under Γ.

Problem 14. (Nariya Kawazumi) Let S be a compact connected oriented surface. Let $\hat{\pi}(S)$ denote the set of homotopy classes of curves in S, and $\mathbb{R}\hat{\pi}(S)$ denote the Goldman Lie algebra. A theorem states that if S is closed, then the center of $\mathbb{R}\hat{\pi}(S)$ is the subspace generated by 1. What happens for surfaces with boundary?

Is it true that the center is the linear span of 1 and of all the powers of boundary curves?

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