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Representation Theory of Quivers and Finite Dimensional Algebras

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Abstract. Methods and results from the representation theory of quivers and finite dimensional algebras have led to many interactions with other areas of mathematics. Such areas include the theory of Lie algebras and quantum groups, commutative algebra, algebraic geometry and topology, and in particular the new theory of cluster algebras. The aim of this workshop was to further develop such interactions and to stimulate progress in the representation theory of algebras.

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Introduction by the Organisers

The representation theory of quivers is probably one of the most fruitful parts of modern representation theory because of its various links to other mathematical subjects. This has been the reason for devoting a substantial part of this Oberwolfach meeting to problems that can be formulated and solved involving quivers and their representations. The interaction with neighbouring mathematical subjects like geometry, topology, and combinatorics is one of the traditions of such Oberwolfach meetings; it continues to be a source of inspiration. The rapid progress in the theory of cluster algebras has been the subject of another Oberwolfach workshop, just a couple of months before this one. There were 27 lectures given at the meeting, and what follows is a quick survey of their main themes.

Representations of quivers. In two beautiful talks, Emmanual Letellier and Fernando Rodriguez-Villegas spoke about different parts of their proof with Tamás Hausel of Kac's positivity conjecture for the polynomial counting the number of absolutely indecomposable representations of a quiver. This completed work that was seemingly already being contemplated when the three collaborators spoke at the last workshop in this series in 2011.

There is no natural variety parameterizing isomorphism classes of indecomposable representations of a quiver; instead one is led to consider moduli spaces of semistable representations. If the components of the dimension vector are coprime, these moduli spaces are smooth projective varieties, and Markus Reineke has in the past computed their cohomology. In his talk at the workshop, he spoke about a generalization of this result to intersection cohomology in the non-coprime case. He also discussed a series of intriguing conjectures concerning the existence of small resolutions of the moduli spaces and affine pavings.

Besides geometry, there is fascinating combinatorics appearing in representations of quivers. Hugh Thomas spoke about a natural 1-1 correspondence between cofinite quotient-closed subcategories of the category of representations of a quiver and elements of the corresponding Weyl group, while Lutz Hille spoke about how the number of tilting modules for a Dynkin quiver can be computed in terms of the volume of suitable polytopes.

Module varieties. Geometric ideas can be applied to algebras, not just quivers, by considering module varieties. For example one might aim to classify the irreducible components of the module varieties for an algebra, and Ryan Kinser spoke about this classification in an explicit example. Another construction is the so-called quiver Grassmannian of submodules of a given module, and Andrew Hubery spoke about a version of the Krull-Remak-Schmidt theorem which applies in this setting. Claus Michael Ringel defined certain varieties, Auslander varieties, which classify morphisms, according to Maurice Auslander's theory of morphisms determined by objects, and he showed that for a controlled wild algebra, any projective variety can arise as an Auslander variety. Jerzy Weyman spoke about local cohomology of determinantal varieties, which of course arise from the very simplest of quivers, with two vertices and one arrow. Of indirect interest, Michel Brion gave a beautiful talk about linearization of line bundles.

Derived categories and tilting. Much of the recent progress in representation theory of algebras is formulated in terms of derived categories. In fact, the derived category of an algebra captures a wealth of homological information and is an interesting invariant in its own right. Tilting modules and complexes provide us with derived equivalences. A special class of tilting modules arises from tilting mutations. Lidia Angeleri Hügel discussed these in her talk on support τ-tilting modules. Mutations for Brauer graph algebras were the subject of Robert Marsh's talk.

Derived equivalences provide a powerful link between representation theory and algebraic geometry. The derived categories of non-commutative rational projective nodal curve were described in Igor Burban's talk, providing geometric models for a broad class of derived tame algebras. Michel van den Bergh discussed a conjecture of Orlov which describes functors between derived categories of coherent sheaves on smooth projective varieties as Fourier-Mukai functors. He gave an example to illustrate that one needs to assume the functors to be fully faithful. Given an algebra, there is a characteristic map from Hochschild cohomology to the center of the derived category. Its kernel carries a lot of information, and Srikanth Iyengar's talk about the annihilation of cohomology discussused this for noetherian algebras, connecting this to the generation of module categories.

A first instance of tilting theory are the classical BGP reflection functors. The talk of Jan Stovicek discussed these functors in the context of monoidal triangulated categories using Grothendieck's theory of derivators. The talk of Hiroyuki Minamoto carried Gabriel topologies and the completion of modules into the world of differential graded categories.

Algebras defined by homological properties. Auslander algebras are the endomorphism algebras of additive generators in the module category, and play an important role in Auslander-Reiten theory. A related class of algebras was discussed in a lecture by Steffen Koenig (joint work with Fang). He introduced gendo-symmetric algebras, which are the endomorphism algebras of generator-cogenerators over symmetric algebras. He explained their interesting properties, in particular, existence of comultiplications. Important examples are given by Schur algebras and blocks of BGG-categories. They are also examples of quasi-hereditary algebras, an important class of algebras which behave nicely in a homological viewpoint. The notion of an exact Borel subalgebra of a quasi-hereditary algebra is natural in this context. In a lecture by Julian Külshammer (joint work with Koenig and Ovsienko), it is shown that, up to Morita equivalence, any quasi-hereditary algebra has an exact Borel subalgebra. The proof uses A_∞-categories and classical boxes.

The notion of support varieties in modular representation theory of finite groups was generalized to algebras satisfying the 'Fg condition' by using Hochschild cohomology rings. Therefore it is important to know if an algebra A satisfies the Fg condition. In a lecture by Øyvind Solberg (joint work with Psaroudakis and Skartstærhagen), it was shown that, under certain conditions, A satisfies Fg condition if and only if so does its subalgebra eAe corresponding to an idempotent $e \in A$.

The periodicity of finite dimensional algebras with respect to the syzygy is one of the classical subjects in representation theory. An important open problem asks whether A is periodic over its enveloping algebra if all simple A-modules are periodic. This was discussed in a lecture by Andrzej Skowronski (joint work with Bialkowski and Erdmann), and a positive answer was given for algebras of polynomial growth.

Higher preprojective algebras and d-hereditary algebras. Preprojective algebras are one of the fundamental classes of algebras in representation theory. Recently higher preprojective algebras $\Pi(A)$ of algebras A of finite global dimension were introduced, and played important roles in cluster theory, non-commutative algebraic geometry, and higher dimensional Auslander-Reiten theory.

In a lecture by Claire Amiot (joint work with Iyama, Reiten and Oppermann), she explained that higher preprojective algebras of d-hereditary algebras, which are important in higher dimensional Auslander-Reiten theory, satisfy very nice homological properties. In fact they are characterized by certain $(d+1)$-Calabi-Yau properties. Moreover she gave sufficient conditions for a given graded algebras to be a higher preprojective algebra of some algebra.

In a lecture by Ragnar-Olaf Buchweitz (joint work with Hille), he explained that algebraic geometry provides an ample supply of d-representation infinite algebras, a class of d-hereditary algebras. He showed that the endomorphism algebra of a tilting sheaf on a smooth projective variety X of dimension d is d-representation infinite if it has global dimension d. Moreover in this case, its higher preprojective algebra is derived equivalent to the total space of the canonical bundle on X. An example from the second Hirzebruch surface was explained.

Another source of d-representation infinite algebras is given by Geigle-Lenzing projective spaces, which are generalization of weighted projective lines. This was explained in an informal lecture by Osamu Iyama (joint work with Herschend, Minamoto and Oppermann).

Quantum groups and cluster algebras. Cluster algebras are certain commutative algebras whose generators and relations are constructed recursively. They were introduced by Fomin and Zelevinsky, motivated by the study of Kashiwara/Lusztig's canonical bases in quantum groups and the closely related notion of total positivity in algebraic groups. The talk of Bernard Leclerc discussed a cluster algebra structure for the cooordinate rings arising from strata of flag varieties. Philipp Lampe explained in his talk when a cluster algebra is a unique factorization domain and described its divisor class group. The talk of Fan Qin described quantum groups via Grothendieck groups of perverse sheaves on cyclic quiver varieties. The study of canonical bases via actions of Lie algebras and quantum groups on representations of Brauer algebras was explained in Catharina Stroppel's talk.

The format of the workshop has been a combination of introductory survey lectures and more specialized talks on recent progress. In addition there was plenty of time for informal discussions. Thus the workshop provided an ideal atmosphere for fruitful interaction and exchange of ideas. It is a pleasure to thank the administration and the staff of the Oberwolfach Institute for their efficient support and hospitality.

Workshop: Representation Theory of Quivers and Finite Dimensional Algebras

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Abstracts

Preprojective algebras and Calabi-Yau duality

Claire Amiot
(joint work with O. Iyama, I. Reiten and S. Oppermann)

The properties of the preprojective algebra are very different whether the associated quiver is of Dynkin type or not. However in both cases, one can construct from it a triangulated category of Calabi-Yau dimension 2. In this note we explain the generalizations of this fact in the context of higher preprojective algebra, and we give some homological properties that characterize preprojective algebras.

1. Classical case

Let $k = \overline{k}$ be an algebraically closed field. Let Q be a finite quiver. The double quiver $\overline{Q}$ of Q is defined from Q by adding for each arrow $a \in Q_1$ an arrow $\overline{a}$ in the opposite direction. The *preprojective algebra* of Q is defined by

$$\Pi_Q := k\overline{Q}/\langle \prod_{a\in Q_1} [a,\overline{a}]\rangle.$$

This notion has been defined by Gelfand and Ponomarev in [9].

Example 1. Let Q be the following quiver $1 \circlearrowright x$. Then we have $kQ \cong k[x]$. The preprojective algebra of Q is presented by the quiver $\overline{x} \circlearrowleft 1 \circlearrowright x$ with the relation $x\overline{x} - \overline{x}x = 0$. That is $\Pi_Q \cong k[x,\overline{x}]$.

Example 2. Let Q be the quiver $1 \xrightarrow{a} 2$. Then the preprojective algebra of Q is presented by the quiver $1 \underset{\overline{a}}{\overset{a}{\rightleftarrows}} 2$ with the relations $a\overline{a} = \overline{a}a = 0$.

To preprojective algebras we can associate some triangulated categories with a duality of Calabi-Yau type.

Definition 3 (Kontsevich)**.** Let $\mathcal{T}$ be a k-linear triangulated category with finite dimensional Hom spaces. The category $\mathcal{T}$ is said to be *d-Calabi-Yau* if for all objects X, Y in $\mathcal{T}$, there exists an isomorphism $\mathsf{Hom}_{\mathcal{T}}(X,Y) \cong D\mathsf{Hom}_{\mathcal{T}}(Y, X[d])$ functorial in X and Y.

The link between preprojective algebras and Calabi-Yau categories is given by the following classical result.

Theorem 4. [8, 6, 4] *Let Q be a finite quiver without oriented cycles. Then*

- *if Q is Dynkin, the preprojective algebra Π_Q is finite dimensional selfinjective and the stable category $\underline{\mathsf{mod}}\,\Pi_Q$ is 2-Calabi-Yau.*
- *if Q is not Dynkin, then Π_Q is infinite dimensional and the bounded derived category $\mathcal{D}^{\mathrm{b}}(\mathsf{fd}\,\Pi_Q)$ of finite dimensional Π_Q-modules is 2-Calabi-Yau.*

The aim of this note is to explain how this result generalizes to higher preprojective algebras.

2. Higher preprojective algebras

Definition 5 (Iyama, Keller)**.** Let Λ be a k-algebra of global dimension d. The ($(d+1)$-) preprojective algebra of Λ is the tensor algebra $\Pi(\Lambda) := \mathsf{T}_\Lambda \mathsf{Ext}^d_{\Lambda^{\mathrm{e}}}(\Lambda, \Lambda^{\mathrm{e}})$ where $\Lambda^{\mathrm{e}} = \Lambda^{\mathrm{op}} \otimes_k \Lambda$ is the envelopping algebra.

As a tensor algebra, the preprojective algebra of Λ is naturally equipped with a structure of positively graded algebra, with Λ as degree 0 subalgebra.

Remark 6. Ringel showed in [15] (see also [5]) that there is an isomorphism $\Pi_Q \cong \Pi(kQ)$.

Example 7. If Λ is the the polynomial algebra in d variables, it has global dimension d and its associated preprojective algebra is isomorphic to the polynomial algebra in $d+1$ variables.

From now on Λ we only consider the case where Λ is finite dimensional. We denote by $\mathbb{S}_d^{-1} = - \overset{\mathbf{L}}{\otimes}_\Lambda \mathbf{R}\mathsf{Hom}_{\Lambda^{\mathrm{e}}}(\Lambda, \Lambda^{\mathrm{e}})[d]$ the autoequivalence of the derived category $\mathcal{D}^{\mathrm{b}}(\Lambda)$. We define the following subcategories of $\mathcal{D}^{\mathrm{b}}(\Lambda)$:

$$\mathcal{U} = \mathsf{add}\{\mathbb{S}_d^{-p}\Lambda,\ p \in \mathbb{Z}\} \quad \text{and} \quad \mathcal{U}^+ = \mathsf{add}\{\mathbb{S}_d^{-p}\Lambda,\ p \in \mathbb{N}\}.$$

Definition 8. [11, 10] Let Λ be a finite dimensional algebra of global dimension d. Then

- Λ is said *d-representation finite* (d-RF) if $\mathcal{U} = \mathcal{U}[d]$.
- Λ is said *d-representation infinite* (d-RI) if $\mathcal{U}^+ \subset \quad \mathrm{mod}\,\Lambda$.

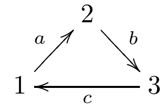
Since the algebra Λ has global dimension d, we have an isomorphism between $\mathrm{H}^0(\mathbb{S}_d^{-p}(\Lambda))$ and $\mathsf{Ext}^d_{\Lambda^{\mathrm{e}}}(\Lambda, \Lambda^{\mathrm{e}})^{\otimes p}$ which is the homogeneous part of degree p of $\Pi(\Lambda)$. Therefore one verifies that if Λ is d-RF, then $\Pi(\Lambda)$ is finite dimensional, and if Λ is d-RI, then $\Pi(\Lambda)$ is infinite dimensional.

For $d = 1$, the functor $\mathbb{S}_1^{-1}$ is isomorphic to τ^{-1} the inverse of the Auslander-Reiten translation of the derived category $\mathcal{D}^{\mathrm{b}}(kQ)$. Then using Gabriel's theorem it is immediate to check the following equivalences:

$$kQ \text{ is 1-RF } \Leftrightarrow Q \text{ is Dynkin } \Leftrightarrow kQ \text{ is representation-finite}$$

$$kQ \text{ is 1-RI } \Leftrightarrow Q \text{ is non Dynkin } \Leftrightarrow kQ \text{ is representation-infinite}.$$

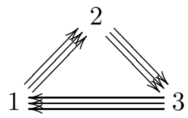
Example 9. Let Λ be the algebra presented by the quiver $1 \xrightarrow{a} 2 \xrightarrow{b} 3$ with the relation $ba = 0$. Then Λ is 2-RF. The preprojective algebra of Λ is presented by the following quiver with relations:



$ba = cb = ac = 0.$

There is a systematic way of constructing the preprojective algebra of an algebra of global dimension 2. We refer to [13] for a description of the precise construction.

Example 10. Let Λ be the algebra presented by the quiver $1 \overset{x}{\underset{z}{\overset{y}{\rightrightarrows}}} 2 \overset{x}{\underset{z}{\overset{y}{\rightrightarrows}}} 3$ with the commutativity relations. Then Λ is 2-RI. The preprojective algebra of Λ is presented by the following quiver with the commutativity relations:



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3. Results

3.1. d**-RI case.** The first result can be understood as a generalization of the non Dynkin case of Theorem 4. Furthermore it gives a homological characterization for the preprojective algebras of d-RI algebras.

Theorem 11. [13, 10, 14, 2] *Let* $\Gamma = \bigoplus_{i\geq 0}\Gamma_i$ *be a graded algebra with finite dimensional degree zero part* $\Lambda := \Gamma_0$*. Then the following are equivalent*

(1) Λ *is d-RI, has global dimension d and* $\Gamma \cong \Pi(\Lambda)$ *as graded algebras;*
(2) Γ *is* (1)*-twisted bimodule* $(d+1)$*-Calabi-Yau.*

Property (2) is an algebraic (and graded) enhancement of the $(d+1)$-Calabi-Yau property for the category $\mathcal{D}^{\mathrm{b}}(\mathsf{fd}\,\Gamma)$. This is the bimodule property that must be satisfied by Γ to ensure that $\mathcal{D}^{\mathrm{b}}(\mathsf{fd}\,\Gamma)$ is $(d+1)$-Calabi-Yau. It is given as follows:

$$(2) \Leftrightarrow \Gamma \in \mathsf{per}\,\Gamma^{\mathrm{e}} \text{ and } \quad \mathbf{R}\mathsf{Hom}_{\Gamma^{\mathrm{e}}}(\Gamma,\Gamma^{\mathrm{e}})[d+1] \cong \Gamma(1) \text{ in } \mathcal{D}(\mathsf{gr}\,\Gamma^{\mathrm{e}}).$$

Here $\Gamma(1)$ is the graded bimodule whose degree p part is Γ_{p+1}.

Theorem 11 is stated in [10, Thm 4.35]. The implication (1) $\Rightarrow$ (2) follows from [13, Thm 4.8], while (2) $\Rightarrow$ (1) was shown indenpendently in [14, Thm 4.8] and [2, Thm 3.4].

3.2. d**-RF case.** The next result is the d-RF analogue of Theorem 11. It can be seen as a generalization of the Dynkin case of Theorem 4, and gives a homological characterization of the preprojective algebras of d-RF algebras.

Theorem 12. [7, 3] *Let* $\Gamma = \bigoplus_{i\geq 0}\Gamma_i$ *be a finite dimensional graded algebra. Denote by* Λ *its degree zero part. Then the following are equivalent*

(1) Λ *is d-RF, has global dimension d and* $\Gamma \cong \Pi(\Lambda)$ *as graded algebras;*
(2) Γ *is selfinjective and* (1)*-twisted stably bimodule* $(d+1)$*-Calabi-Yau.*

Here again, property (2) is an algebraic (and graded) enhancement of the $(d+1)$-Calabi-Yau property for the category $\underline{\mathsf{mod}}\,\Gamma$. This is the bimodule property that must be satisfied by Γ to ensure that $\underline{\mathsf{mod}}\,\Gamma$ is $(d+1)$-Calabi-Yau. It is given as follows:

$$(2) \Leftrightarrow \mathsf{Hom}_{\Gamma^{\mathrm{e}}}(\Gamma,\Gamma^{\mathrm{e}})[d+2] \cong \Gamma(1) \text{ in } \underline{\mathsf{gr}}\,\Gamma^{\mathrm{e}}.$$

The implication (1) $\Rightarrow$ (2) is shown in [7, Thm 3.2], while the implication (2) $\Rightarrow$ (1) is shown in [3, Thm 3.1].

3.3. **Beyond the RF/RI cases.** Most of algebras of global dimension $d \geq 2$ are neither d-RF nor d-RI. So one could ask how Theorems 11 and 12 can be extended to general preprojective algebras. Here we focus on the case where the preprojective algebra is finite dimensional.

In general the finite dimensional preprojective algebras are not selfinjective but their behaviour is still similar to the one of the preprojective algebras of d-RF algebras. In the case $d = 2$, it is shown in [1] that when $\Pi(\Lambda)$ is finite dimensional, it is the endomorphism ring of a cluster-tilting object in a certain 2-Calabi-Yau category. Keller and Reiten proved that such algebras are Gorenstein (that is $\text{projdim} D\Pi = \text{injdim}\Pi < \infty$). Hence the correct analogue Calabi-Yau triangulated category is given by the stable category of maximal Cohen-Macaulay Π-modules. Indeed they proved in [12] that the category $\underline{\mathsf{CM}}\,\Pi(\Lambda)$ is 3-Calabi-Yau.

These results were the motivation for the following characterization of finite dimensional preprojective algebras.

Theorem 13. [3] *Let $\Gamma = \bigoplus_{i\geq 0}\Gamma_i$ be a (non trivially) graded finite dimensional algebra. Denote by Λ its degree zero part. Assume that*

(a) *Γ is Gorenstein of dimension $\leq d-1$;*
(b) *there is an isomorphism $\mathbf{R}\mathsf{Hom}_{\Gamma^e}(\Gamma,\Gamma^e)[d{+}2] \cong \Gamma(1)$ in $\mathcal{D}^b(\mathsf{gr}\,\Gamma^e)/\mathsf{per}\,\mathsf{gr}\,\Gamma^e$.*
(c) *$\mathsf{Ext}^i_{\Gamma^e}(\Gamma,\Gamma^e(j)) = 0$ for any $i \geq 1$ and any $j \leq -1$.*

Then Λ has global dimension d and $\Gamma \cong \Pi(\Lambda)$ as graded algebras.

Here property (b) is again an algebraic (and graded) enhancement of the $(d{+}1)$-Calabi-Yau property of the category $\underline{\mathsf{CM}}\,\Gamma$.

One also shows in [3] that these properties are satsified by finite dimensional preprojective algebras in the case $d = 2$ and $d = 3$ using the description of $\Pi(\Lambda)$ in term of quivers with relations.

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Large support τ-tilting modules

Lidia Angeleri Hügel
(joint work with Frederik Marks and Jorge Vitória)

Support τ-tilting modules are finitely generated modules over a finite dimensional algebra Λ that were introduced by Aihara, Iyama and Reiten in [1] in order to provide an appropriate setting for tilting mutation. They correspond bijectively to two-term silting complexes of finitely generated projective Λ-modules, as well as to certain t-structures in $\mathcal{D}(\text{Mod}\Lambda)$, see [1, Theorem 3.2] and [2, Theorem 4.10].

In this talk, we consider the analog of a support τ-tilting module in the category Mod$A$ of all modules over an arbitrary ring A, and we show that the bijections above hold also in this more general context.

We say that a module $T \in \text{Mod}A$ is *silting* if it admits a projective presentation $P_{-1} \xrightarrow{\sigma} P_0 \to T \to 0$ such that the class of all A-modules X for which the map $\text{Hom}_A(\sigma, X)$ is surjective coincides with the class Gen$T$ of all modules generated by T.

It turns out that a finitely generated module over a finite dimensional algebra is silting if and only if it is support τ-tilting. Tilting modules (possibly not finitely generated) over an arbitrary ring A are further examples of silting modules.

A two-term *silting complex* (called semitilting in [6]) is a complex of projective A-modules $P_{-1} \xrightarrow{\sigma} P_0$ that generates the unbounded derived category $\mathcal{D}(\text{Mod}A)$ as a triangulated category and satisfies $\text{Hom}_{\mathcal{D}(\text{Mod}A)}(\sigma, \sigma^{(I)}[1]) = 0$ for all sets I. This is equivalent to $T = H^0(\sigma)$ being a silting module.

By [3, Proposition 3.2] every complex σ generates a t-structure $(\mathcal{U}_\sigma, \mathcal{V}_\sigma)$ in $\mathcal{D}(\text{Mod}A)$, where $\mathcal{U}_\sigma$ is the smallest suspended subcategory of $\mathcal{D}(\text{Mod}A)$ that is closed under coproducts and contains σ. When σ is a silting complex and $T = H^0(\sigma)$ is the corresponding silting module, $(\mathcal{U}_\sigma, \mathcal{V}_\sigma)$ equals the t-structure associated as in [5] to the torsion pair $(\text{Gen}T, T^o)$ in Mod$A$, and it also coincides with the t-structure studied in [4, Theorem 2.10].

We discuss the assignments $\sigma \mapsto H^0(\sigma)$ and $\sigma \mapsto \mathcal{U}_\sigma$ and show that they yield bijections between two-term silting complexes, silting A-modules, and certain t-structures in $\mathcal{D}(\text{Mod}A)$.

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Linearization of line bundles

Michel Brion

Linearization of line bundles in the presence of algebraic group actions is a basic notion of geometric invariant theory, introduced by Mumford in [5]. Given a variety X over an algebraically closed field k, an action of an algebraic group G on X, and a line bundle L on X, a G-linearization of L is an action of G on the total space of L which lifts the G-action on X and which is linear on fibers (i.e., the map $L_x \to L_{g\cdot x}$ is linear for any $g \in G$ and $x \in X$).

If L is G-linearized, then so are its tensor powers $L^{\otimes n}$, where $n \in \mathbb{Z}$, and hence G acts linearly on the spaces of global sections $\Gamma(X, L^{\otimes n})$ and on the graded algebra $\bigoplus_n \Gamma(X, L^{\otimes n})$; this is the starting point of the construction of quotients in geometric invariant theory. Also, when X is the total space of a principal G-bundle $X \to Y$, the G-linearized line bundles on X are exactly the pull-backs of line bundles on Y. On an arbitrary G-variety X, the G-linearized line bundles can be identified with the line bundles on the quotient stack $[X/G]$.

The talk discussed the existence of G-linearizations of a given line bundle (or of some tensor power) on a G-variety X. When G is connected linear and X is normal, it is known that some positive power $L^{\otimes n}$ is G-linearizable (see Mumford [5] for X complete, and Sumihiro [6, 7] for the general case; see also [4, 3] for an exposition of Sumihiro's theorem in characteristic 0, and further developments).

This result does not extend to non-normal varieties, a classical example being the rational nodal curve X obtained from the projective line $\mathbb{P}^1$ by identifying 0 and ∞: the natural action of the multiplicative group $\mathbb{G}_m$ on $\mathbb{P}^1$ yields an action on X, and the $\mathbb{G}_m$-linearizable line bundles on X are those of degree 0. To see this, consider the normalization map $\eta : \mathbb{P}^1 \to X$. Then for any line bundle L on X, the pull-back $\eta^*(L)$ is equipped with an isomorphism $\eta^*(L)_0 \cong \eta^*(L)_\infty$; if L is $\mathbb{G}_m$-linearized, then this isomorphism is equivariant for the linear actions of $\mathbb{G}_m$ on both fibers. On the other hand, one checks that the weights of the $\mathbb{G}_m$-actions on $\mathcal{O}_{\mathbb{P}^1}(n)_0$ and $\mathcal{O}_{\mathbb{P}^1}(n)_\infty$ differ by n, for any linearization of $\mathcal{O}_{\mathbb{P}^1}(n)$.

For the rational cuspidal curve X equipped with the action of the additive group $\mathbb{G}_a$, the linearizable line bundles on X are still those of degree 0, if k has characteristic 0. But in prime characteristic p, the linearizable line bundles are those of degree divisible by p. For example, the morphism

$$\mathbb{P}^1 \longrightarrow \mathbb{P}^{p-1}, \quad [x : y] \longmapsto [x^p : x^{p-2}y : \cdots : xy^{p-1} : y^p]$$

factors through an immersion $X \to \mathbb{P}^{p-1}$ which is $\mathbb{G}_a$-equivariant for the action on $\mathbb{P}^{p-1}$ via $t \cdot (x, y) = (x, y + tx)$. Thus, the pull-back to X of $\mathcal{O}_{\mathbb{P}^{p-1}}(1)$ is a $\mathbb{G}_a$-linearized line bundle of degree p.

Returning to the rational nodal curve X, one shows that every line bundle on X admits a linearization after pull-back under the $\mathbb{Z}$-cover $\pi : Y \to X$, where Y is an chain of projective lines $(L_n)_{n\in\mathbb{Z}}$, and each L_n is glued to L_{n-1} and L_{n+1} at distinct marked points. Here $\mathbb{Z}$ acts on Y by translations, and $\mathbb{G}_m$ acts by its natural action on each L_n fixing both marked points.

The following result yields a common generalization of Sumihiro's theorem and the above examples:

Theorem. Let X be a variety equipped with an action of a connected linear algebraic group G. Denote by $\widehat{G}$ the group of multiplicative characters of G (this is a free abelian group of finite rank). Assume either that $\mathrm{char}(k) = p > 0$, or that $\mathrm{char}(k) = 0$ and X is seminormal. Then there exists a $\widehat{G}$-cover $\pi : Y \to X$ and a positive integer n such that $\pi^*(L^{\otimes n})$ is G-linearizable for any line bundle L on X.

Recall that a variety X (over an algebraically closed field of characteristic 0) is said to be seminormal if every finite bijective morphism $f : X' \to X$ is an isomorphism. For example, nodal curves and, more generally, divisors with normal crossings are seminormal, but cuspidal curves are not.

In the case where X is complete and G is a torus, the above theorem has been obtained by Alexeev (see [1, Thm. 4.3.1]) in the process of the construction of certain moduli spaces. His proof is based on the existence of the Picard scheme Pic_X and hence does not extend to the setting of non-complete varieties.

Our starting point for proving the above theorem is the exact sequence (implicit in [7])

$$\mathrm{Pic}^G(X) \xrightarrow{\varphi} \mathrm{Pic}(X) \xrightarrow{\alpha^* - p_2^*} \mathrm{Pic}(G \times X),$$

where $\mathrm{Pic}^G(X)$ denotes the group of isomorphism classes of G-linearized line bundles on X, and φ the forgetful map; $\alpha : G \times X \to X$ stands for the G-action, and $p_2 : G \times X \to X$ for the projection. Using the fact that α and p_2 coincide on $\{e_G\} \times X$, it easily follows that the sequence

$$\mathrm{Pic}^G(X) \xrightarrow{\varphi} \mathrm{Pic}(X) \xrightarrow{\alpha^*} \mathrm{Pic}(G \times X)/p_2^* \,\mathrm{Pic}(X)$$

is exact as well.

Since the variety G is rational, the first projection $p_1 : G \times X \to G$ yields an isomorphism $\mathrm{Pic}(G) \xrightarrow{\cong} \mathrm{Pic}(G \times X)/p_2^* \,\mathrm{Pic}(X)$ when X is normal. As $\mathrm{Pic}(G)$ is finite, this gives back Sumihiro's theorem.

For a not necessarily normal variety X, we rely on results and methods from algebraic K-theory, taken from an article of Weibel (see [8]). We construct an injective map

$$c : H^1_{\text{ét}}(X, \widehat{G}) \longrightarrow \operatorname{Pic}(G \times X)/p_2^* \operatorname{Pic}(X),$$

where the left-hand side denotes the first étale cohomology group with coefficients in the character group of G; recall that this cohomology group classifies $\widehat{G}$-covers of X. When X is seminormal, we show that the map

$$p_1^* \times c : \operatorname{Pic}(G) \times H^1_{\text{ét}}(X, \widehat{G}) \longrightarrow \operatorname{Pic}(G \times X)/p_2^* \operatorname{Pic}(X)$$

is an isomorphism; this implies the above theorem.

We refer to the preprint [2] for details and further developments, over an arbitrary base field.

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Higher Representation–Infinite Algebras from Geometry

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(joint work with Lutz Hille)

Recently, Iyama and collaborators introduced and studied *n–hereditary artinian algebras* over a field K; see [1, 2, 3, 6, 7, 9]. When ring-indecomposable these fall into two classes, the *n–representation–finite* and *n–representation–infinite* algebras, with *τ_n–finite* algebras closely related. Algebraic Geometry, classical or non-commutative, provides an ample supply of n–representation–infinite, even n–representation–tame algebras, as well as of τ_n–finite algebras.

Definition 1. Let Λ be a finite dimensional K–algebra of finite global dimension $g = \operatorname{gldim} \Lambda$. With $\mathbb{D}(-) = \operatorname{Hom}_K(-, K)$ the dual of a K–vector space, the *Serre auto–equivalence* on $D^b(\Lambda)$ is given by $\mathbb{S} \cong - \otimes^{\mathbb{L}}_{\Lambda} \mathbb{D}\Lambda$. With $\mathbb{S}_n = \mathbb{S} \circ [-n]$ the indicated translate for some integer n, the algebra Λ is

(a) *higher (or g–)representation infinite* if $H^j(\mathbb{S}_g^{-i}(\Lambda)) = 0$ for $j \neq 0$ and $i \geqslant 0$;

(b) τ_n–finite, for some integer $n \geqslant g$, if $H^0(\mathbb{S}_n^{-i}(\Lambda)) = 0$ for $i \gg 0$.

As concerns the role of τ_n–finiteness one has the following result.

Proposition 2 (cf. [9, Lemma 5.6]). *Assume the finite dimensional K–algebra Λ of finite global dimension is* not *τ_g–finite for $g = \operatorname{gldim}\Lambda$. If Γ is a finite dimensional K–algebra that is derived equivalent to Λ then either*

(a) $\operatorname{gldim}\Gamma > g = \operatorname{gldim}\Lambda$, *in which case Γ is τ_n–finite for any $n \geqslant \operatorname{gldim}\Gamma$, or*
(b) $\operatorname{gldim}\Gamma = g = \operatorname{gldim}\Lambda$ *and Γ is not τ_g–finite along with Λ.*

In other words, finite dimensional K–algebras of finite global dimension g that are not τ_g–finite are precisely the ones of minimal *global dimension within their derived equivalence class.*

In the context of classical Algebraic Geometry, the following holds for varieties over an algebraically closed field K.

Theorem 3. *Let X be a smooth proper K–variety of dimension $d = \dim X$ with $\pi\colon \omega_X \to X$ its canonical affine line bundle. Assume X carries a classical* [8] *tilting object $T \in D^b(\mathsf{Coh}\, X)$.*

The necessarily finite dimensional K–algebra $\Lambda = \operatorname{End}_{\mathcal{O}_X}(T)$ is of finite global dimension with $d \leqslant g = \operatorname{gldim}\Lambda < \infty$, cf. [4, 15], *and it is τ_n–finite whenever $n \geqslant \max\{g, d+1\}$.*

Further, the following statements are equivalent.

(1) *The K–algebra Λ is n–*representation–infinite *for some $n \geqslant 0$, and in that case necessarily $n = d$ and $d = \operatorname{gldim}\Lambda$.*
(2) *The pullback π^*T is a* tilting object *on the total space of the canonical affine line bundle. Equivalently, $\operatorname{Ext}^j_X(T, T \otimes_{\mathcal{O}_X} \omega_X^{-i}) = 0$ for all integers $j \neq 0 < i$.*
(3) *The global dimension of Λ equals d, the minimal possible value, and T is sheaf–like in that $\operatorname{Ext}^j_X(T, T \otimes_{\mathcal{O}_X} \omega_X^{-i}) = 0$ for all integers $j \notin [0, d]$ and all $i \in \mathbb{Z}$, equivalently, for all $i > 0$. This last vanishing condition is indeed satisfied as soon as T is equivalent to an $\mathcal{O}_X$–module under some auto–equivalence of the derived category of X.*

*Further, if these equivalent conditions hold then the $(d+1)$–*preprojective algebra *of Λ is isomorphic to the endomorphism ring of the pulled back tilting object,*

$$\Pi_\Lambda := \mathbb{T}_\Lambda\left(\operatorname{Ext}^{\dim X}_\Lambda(\mathbb{D}\Lambda, \Lambda)\right) \cong \operatorname{End}_\omega(\pi^*T)\,,$$

where $\mathbb{T}_\Lambda(-)$ denotes the tensor algebra over a Λ–bimodule.

In particular, [2, 10], *these naturally graded K–algebras are finitely generated in degrees 0 and 1, bimodule Calabi-Yau, of Hochschild dimension $\dim X + 1$, and of Gorenstein parameter 1, with $D(\Pi_\Lambda) \cong D(\mathbf{QCoh}(\omega_X))$.*

For Fano varieties, one can strengthen condition (3) and deduce additional information about the higher preprojective algebra.

Corollary 4. *With notation as in the Theorem above, if K is of characteristic zero and X is* Fano, *that is, ω_X^{-1} is* ample, *then the conditions* (1) *through* (3) *are as well equivalent to*

(3′) *The global dimension of Λ equals d, the minimal possible value, and the perfect complex $\mathbb{R}\mathcal{E}nd_X(T)$ is an $\mathcal{O}_X$–module in that $\mathcal{H}^j(\mathbb{R}\mathcal{E}nd_X(T)) = 0$ for $j \neq 0$.*

Furthermore, cf. [16, Prop.7.2]*, if these equivalent conditions are satisfied then the higher preprojective algebra* Π_Λ *is a maximal Cohen–Macaulay order over its Noetherian centre, so that* Λ *is* higher representation–tame [7, 6.10]. *That centre contains the anticanonical ring* $R = \oplus_{i\geqslant 0} H^0(X, \omega_X^{-i})$*, a Gorenstein ring with isolated singularity, and* Π_Λ *provides a* noncommutative crepant resolution *of the singularity defined by* R.

5. For various examples to which this Corollary applies see e.g. [4]. For toroidal non-Fano surfaces to which the Theorem applies, see [14]. Of particular note here is the second Hirzebruch surface $X = F_2$, for which King [11] first exhibited a tilting object whose endomorphism algebra turned out to be 2–representation–infinite and whose preprojective algebra is the twisted group algebra $\mathbb{C}[x, y, z] * \mu_4$, where the cyclic group μ_4 acts as $\frac{1}{4}(1, 1, 2)$ and the anti-canonical ring $R \cong \mathbb{C}[x, y, z]^{\mu_4}$ has non-isolated singularities along the z–axis.

To extend these results to the realm of Noncommutative Algebraic Geometry, we employ a suggestion essentially due to Bondal [5].

Definition 6. A K–linear, Ext–finite triangulated category $\mathcal{T}$ with Serre functor $\mathbb{S}$ is *geometric of dimension* d if for any objects F, G from $\mathcal{T}$ there exists an integer $a = a(F, G)$ such that

$$\mathrm{Ext}^j_{\mathcal{T}}(\mathbb{S}^i_d F, G) = 0 \quad \text{for} \quad j \notin [-a, a + d], i \in \mathbb{Z}\,.$$

If $\mathcal{T}$ admits a strong generator, then d, if it exists, is uniquely determined by $\mathcal{T}$.

We call an object F from $\mathcal{T}$ *sheaf–like* if one can take $a(F, F) = 0$.

The Theorem above then extends as follows.

Theorem 7. *Let* $\mathcal{T}$ *be a* K*–linear,* Ext*–finite triangulated category with Serre functor* $\mathbb{S}$ *that is geometric of dimension* d*. If* T *is a tilting object in* $\mathcal{T}$ *with endomorphism algebra* Λ*, then* Λ *is finite dimensional over* K *of finite global dimension with* $d \leqslant g = \mathrm{gldim}\,\Lambda < \infty$ *and it is* τ_n*–finite whenever* $n \geqslant \max\{g, d + 1\}$.

Further, the following statements are equivalent.

(1) *The* K*–algebra* Λ *is* n–representation–infinite *for some* $n \geqslant 0$*, and in that case necessarily* $n = d$ *and* $d = \mathrm{gldim}\,\Lambda$.
(2) *One has* $\mathrm{Ext}^j_{\mathcal{T}}(\mathbb{S}^i_d T, T) = 0$ *for all integers* $j \neq 0 < i$.
(3) *The global dimension of* Λ *equals* d*, the minimal possible value, and* T *is sheaf–like.*

If the equivalent conditions are satisfied, then $\mathrm{Hom}_{\mathcal{T}}(\mathbb{S}^i_d T, T) \cong \mathrm{Hom}_{\mathcal{T}}(\mathbb{S}_d T, T)^{\otimes_\Lambda i}$ *and*

$$\Pi_\Lambda \cong \bigoplus_{i\geqslant 0} \mathrm{Hom}_{\mathcal{T}}(\mathbb{S}^i_d T, T) \cong \mathbb{T}_\Lambda \left(\mathrm{Hom}_{\mathcal{T}}(\mathbb{S}_d T, T)\right)\,.$$

Results of Minamoto [12] and Minamoto–Mori [13] provide for a partial converse:

Theorem 8. *If the* K*–algebra* Λ *is higher representation–infinite and its higher preprojective algebra* Π_Λ *is* graded coherent*, then* $D^b(\Lambda)$ *is geometric of dimension* $d = \mathrm{gldim}\,\Lambda$.

It is conjectured that the higher preprojective algebra of a higher representation–infinite algebra Λ is always graded coherent. If true, higher representation–infinite algebras were thus exactly the ones whose derived category is geometric of dimension $d = \operatorname{gldim} \Lambda$, the maximal possible geometric dimension.

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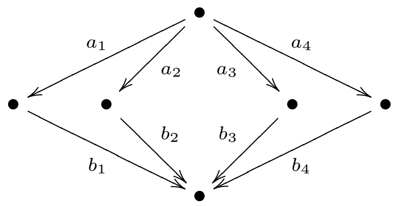
Non–commutative nodal curves and finite dimensional algebras

Igor Burban

(joint work with Yuriy Drozd and Volodymyr Gavran)

Let $\Bbbk$ be an algebraically closed field. Remind that the canonical tubular algebra $C_\lambda = C\big((2,2,2,2),\lambda\big)$ of Ringel is the path algebra of the following quiver



subject to the relations $b_1a_1 + b_2a_2 + b_4a_4 = 0$, $\lambda b_1a_1 + b_2a_2 + b_3a_3 = 0$, where $\lambda \in \Bbbk \setminus \{0,1\}$, see [7]. According to Geigle and Lenzing [6], C_λ is derived equivalent to the category of coherent sheaves on the tubular weighted projective line $\mathbb{X}\big((2,2,2,2),\lambda\big)$. The latter category is equivalent (in the abelian sense) to the category of $\mathbb{Z}_2$–equivariant coherent sheaves on the elliptic curve $E_\lambda = V\big(zy^2 - x(x-z)(x-\lambda z)\big)$ with respect to the involution $(x:y:z) \mapsto (x:-y:z)$.

One may ask whether the degenerate tubular algebra C_0 also admits a geometric interpretation. It is natural to expect a link with $\mathbb{Z}_2$–equivariant coherent sheaves on the nodal cubic curve $E_0 = V\big(zy^2 - x^2(x-z)\big)$. However, $D^b\big(\mathsf{Coh}^{\mathbb{Z}_2}(E_0)\big)$ and C_0 can not be derived equivalent since the first category has infinite global dimension and the second has global dimension two.

The goal of our work is to give a geometric interpretation of a broad class of finite dimensional algebras based on the notion of *non-commutative nodal curves*. Nodal rings are non-commutative analogues of the A_1–singularity $\Bbbk[\![u,v]\!]/(uv)$. They first appeared in a work of Drozd [3], where their tameness was proven. In a work of Burban and Drozd [2] their derived tameness was established. Derived category of coherent sheaves on a (commutative) nodal rational projective curve was studied by Burban and Drozd [1], a non-commutative version of this theory was developed by Drozd and Voloshyn [4, 5].

The following result gives a geometric interpretation of the degenerate tubular algebra C_0.

Proposition. Let $0 = (0:1), \infty = (1:0)$ and $* = (1:1)$ be three points on the projective line $\mathbb{P}^1$. Consider the nodal curve $\mathbb{E} = (\mathbb{P}^1, \mathcal{A})$, where $\mathcal{A}$ is the following sheaf of orders

$$\mathcal{A} = \begin{pmatrix} \mathcal{O} & \mathcal{I}_* & \mathcal{I}_* \\ \mathcal{I}_0 & \mathcal{O} & \mathcal{I}_{0*} \\ \mathcal{I}_\infty & \mathcal{I}_{\infty *} & \mathcal{O} \end{pmatrix}.$$

Then the following results are true:

- There exists a derived equivalence $D^b\big(\mathsf{Coh}(\mathbb{E})\big) \simeq D^b\big(C_0 - \mathsf{mod}\big)$.
- There exists an exact fully faithful functor $\mathsf{Perf}^{\mathbb{Z}_2}(E_0) \hookrightarrow D^b\big(\mathsf{Coh}(\mathbb{E})\big)$.

Our main result is the following

Theorem. Let $\mathbb{X}$ be a (non-commutative) rational projective nodal curve. Then there exists another non-commutative nodal projective curve $\widetilde{\mathbb{X}}$ satisfying the following properties:

- The category $\mathsf{Coh}(\widetilde{\mathbb{X}})$ has global dimension two.
- There exists a fully faithful exact functor $\mathsf{Perf}(\mathbb{X}) \hookrightarrow D^b\big(\mathsf{Coh}(\widetilde{\mathbb{X}})\big)$.
- The curves $\mathbb{X}$ and $\widetilde{\mathbb{X}}$ have the same (derived) representation type.
- There exists a derived equivalence $D^b\big(\mathsf{Coh}(\widetilde{\mathbb{X}})\big) \simeq D^b\big(\Lambda_{\mathbb{X}} - \mathsf{mod}\big)$, where $\Lambda_{\mathbb{X}}$ is a finite dimensional algebra of global dimension two.

In these terms we in particular get a geometric realization of a broad class of derived tame algebras which includes other degenerate tubular algebras, certain classes of gentle, skew-gentle, supercanonical and generalized canonical algebras as well as some new examples of derived tame algebras.

In a similar way one can show that any non-commutative rational projective curve can be categorically resolved by a finite dimensional algebra of finite global dimension.

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On the number of tilting modules for Dynkin quivers via polytopes

Lutz Hille

Let Q be any Dynkin quiver, where we also allow non-simply laced ones. We are interested in the number of tilting modules $t(Q)$ and the number of cluster tilting modules $t^{\text{clus}}(Q)$ for the corresponding path algebra of Q. For non-simply laced quivers we can define the path algebra over a field K that has a field extension L/K of order two (for type $\mathbb{B}$, $\mathbb{C}$, and $\mathbb{F}$) or of degree three (for type $\mathbb{G}_2$).

The principal aim of this note is to define certain polytopes, so that the volume of these polytopes coincides with the numbers above. All our polytopes are lattice polytopes in the root lattice corresponding to the finite root system associated to the quiver Q. However, our polytopes need not to be convex, they are unions of a finite number of simplices, whose vertices are lattice points (either a root or zero) and these simplices intersect only at a common face. We also note, that we define $\mathrm{vol}(\Delta) = 1$ for any simplex Δ whose vertices form an affine integral basis. Thus our volume is $n!$ times the standard euclidean volume. The advantage of this definition (that is just a normalization) is that any lattice polytope has an integral volume.

Example. Let Q be of type $\mathbb{A}$, then the number of tilting modules $t(\mathbb{A}_n)$ is well-known. We can define a polytope $C(\mathbb{A}_n) = P(\mathbb{A}_n)$ just as the convex hull of all roots $\Phi(\mathbb{A}_n)$. Inside, we consider the convex hull $P^+(\mathbb{A}_n)$ of the all positive roots $\Phi^+(\mathbb{A}_n)$ and zero and we also consider the convex hull $P^{\mathrm{clus}}(\mathbb{A}_n)$ of all positive roots and the negative simple roots. Then, the number of tilting modules is just the integral volume $t(\mathbb{A}_n) = \mathrm{vol}\, P^+(\mathbb{A}_n)$ and the number of cluster tilting modules is just the integral volume $t^{\mathrm{clus}}(\mathbb{A}_n) = \mathrm{vol}\, P^{\mathrm{clus}}(\mathbb{A}_n)$. In this case we get the Catalan numbers for both problems

$$t(\mathbb{A}_n) = \sum_{i=1}^{n} t(\mathbb{A}_n - \{i\}) = \sum t(\mathbb{A}_j)t(\mathbb{A}_{n-j-1}) = (2n)!/(n+1)n!n! = t^{\mathrm{clus}}(\mathbb{A}_{n-1}).$$

We explain this example in more detail at the end.

1. The polytopes

Let Q be any quiver of Dynkin type with n vertices, including the species. To fix notation we denote by $\mathcal{T}(Q)$ the set of isomorphism classes of basic tilting modules $T = \oplus_{i=1}^{n} T(i)$, where we can identify $\mathcal{T}(Q)$ also with a subset $\Phi(T)$ (with n elements) of the positive roots $\Phi(Q)^+$. With this notation, we identify any tilting module just with the set $\Phi(T) = \{\underline{\dim}\, T(1), \ldots, \underline{\dim}\, T(n)\}$. Note that all vectors $\underline{\dim}\, T(i)$ are positive roots. Moreover, the cone $\sigma(T)$ generated by $\Phi(T)$ is generated by an integral basis and the various cones $\sigma(T)$ and $\sigma(R)$ for any two tilting modules $T \not\simeq R$ intersect only in the boundary (Theorem 1).

Using this two facts, we define $\overline{\sigma}(T)$ to be the convex hull of

$$\{0, \underline{\dim}\, T(1), \ldots, \underline{\dim}\, T(n)\}.$$

That is the intersection of the cone $\sigma(T)$ with the half space containing zero and whose boundary contains the dimension vectors $\underline{\dim}\, T(i)$. For any support tilting module $T = \oplus_{i=1}^{t} T(i) \in \mathcal{T}^{\mathrm{clus}}(Q)$ with support S we define

$$\sigma(T) = \mathrm{cone}\{\underline{\dim}\, T(1), \ldots, T(t), -e(i) \mid i \notin S\}.$$

It is the cone over the dimension vectors $\underline{\dim}\, T(i)$ and the negative standard basis vectors corresponding to the complement of the support. Eventually, we define also cones associated to any subset I of Q_0 and any tilting module supported in I and its complement by taking the product of the associated cones. Here we denote by $Q(I)$ the sub quiver supported in I and by $Q \setminus I$ the sub quiver supported in the complement of I.

Using this definition, we can define three polytopes (that are in general not convex, see Theorem 3):

$$P^+(Q) = \bigcup_{T\in\mathcal{T}(Q))} \overline{\sigma}(T);\ P^{\mathrm{clus}}(Q) = \bigcup_{T\in\mathcal{T}^{supp}(Q)} ;\ P(Q) = \bigcup_{T\in\mathcal{T}(Q(I))\times\mathcal{T}(Q\setminus I)} \overline{\sigma}(T).$$

Note that $P^+(Q)$ has zero as a vertex and is contained both in $P(Q)$ and $P^{\mathrm{clus}}(Q)$. Moreover, $P(Q)$ and $P^{\mathrm{clus}}(Q)$ have zero as an internal lattice point and are reflexive (Theorem 1, see also [1]). Eventually, all three are lattice polytopes. In [5] we want to compare $P(Q)$ with the convex hull $C(\Phi)$ of all roots, $P^+(Q)$ with the convex hull $C^+(\Phi)$ of zero and all positive roots, and $P^{\mathrm{clus}}(Q)$ with the convex hull $C^{\mathrm{clus}}(\Phi)$ of the positive roots with the negative simple roots.

Theorem 1. *The elements $\underline{\dim}\, T(i)$ for any tilting module $T = \oplus T(i)$ form an integral affine basis of the root lattice of Φ. Moreover, the cones σ_T and σ_R for two different tilting modules T and R intersect in a common proper face. In particular,* $\mathrm{vol}\,\overline{\sigma}_T + \mathrm{vol}\,\overline{\sigma}_R = \mathrm{vol}(\overline{\sigma}_T \cup \overline{\sigma}_R)$.

2. First properties of the polytopes

We first note that the polytope $C(\Phi)$ is a convex lattice polytope with zero as an internal lattice point. It comes equipped with an action of the Weyl group, and after a choice of an orientation, with an action of the corresponding Coxeter transformation, that is a $\mathbb{Z}/h$–action, where h is the Coxeter number. Note that our polytopes come with a lattice structure, however, we can chose different euclidean metrics realizing our volume.

Theorem 2. *The polytope $C(\Phi)$ coincides with $P(Q)$ precisely when Q is of type $\mathbb{A}$ or $\mathbb{C}$ (including $\mathbb{B}_2 = \mathbb{C}_2$).*

Theorem 3. *The following conditions are equivalent:*
a) The polytope $P(Q)$ is convex.
b) The polytope $P^{\mathrm{clus}}(Q)$ is convex.
c) The polytope $P^+(Q)$ is convex.
d) The quiver Q is of type $\mathbb{A}$ or $\mathbb{C}$ (including $\mathbb{B}_2 = \mathbb{C}_2$).
e) In the Auslander-Reiten quiver of the category of representations of Q there are at most two middle terms in any Auslander-reiten sequence.

The polytopes $P(\Phi)$ is reflexive in the sense, that all facets (faces of codimension one) have lattice distance one from zero. This property does not hold for the polytopes $C(\Phi)$ in general, a classification can be found in [5].

3. Recursion formulas

From the definition of the polytopes we get several natural recursion formulas. In this formulas we have a correction term $\mathrm{cor}(Q, n)$ that depends only on the Hasse diagram of the sincere roots. The situation is very simple in type $\mathbb{A}$ since we have just one sincere root. Also for type $\mathbb{B}$ and $\mathbb{C}$ it is rather easy, since the sincere roots form a simplex of dimension $n - 1$ of volume one. For type $\mathbb{D}$ we get a simplex of dimension $n-3$. The situation gets complicated only for type $\mathbb{E}$ and $\mathbb{F}$, the remaining case $\mathbb{G}_2$ can be done again easily. Our starting point is to relate the volumes of the various polytopes in the first theorem. Then in a second theorem we obtain a recursion formula for the volume based on induction via smaller quivers (similar to the recursion of the Catalan numbers for type $\mathbb{A}$). We also expect to get a third recursion relating the volume of the polytope $P(Q)$ with the convex hull $C(\Phi)$ of the roots.

Theorem 4. *For any Dynkin quiver Q we denote its root system with ϕ. Then we obtain for the number of tilting modules*

$$t(\Phi) = \operatorname{vol} P^{+}(Q); \quad t^{\text{clus}}(\Phi) = \operatorname{vol} P^{\text{clus}}(Q) = \sum_{I \subset Q_0} \operatorname{vol} P^{+}(Q(I)).$$

Theorem 5. *For any Dynkin quiver Q we obtain for the number of tilting modules*

$$\operatorname{vol} P(Q) = \sum \operatorname{vol} P^{+}(Q(I)) \operatorname{vol} P^{+}(Q \setminus I)$$

Note that the number of cluster tilting modules was computed in [3]. Thus we can, by recursion, compute the other numbers as well. A recursion formula for the quivers of type $\mathbb{A}$, $\mathbb{B}$, $\mathbb{C}$, and $\mathbb{D}$ uses the structure of the sincere roots. Other recursion formulas can be found in [5] together with the actual values.

Theorem 6. *For any quiver exists a function $\operatorname{cor}(Q, n)$ that is a polynomial in n, only depending on the type (and not on the orientation of Q) so that*

$$\operatorname{vol} P^{+}(Q) = \sum_{i \in Q_0} \operatorname{vol} P^{+}(Q \setminus \{i\}) + \operatorname{cor}(Q, n).$$

4. Quivers of type $\mathbb{A}$

In the particular case of a quiver of type $\mathbb{A}$ we derive the well-known Catalan numbers for the number of tilting modules in two different ways. First of all we can use the first recursion formula, that reduces the computation to smaller quivers of type $\mathbb{A}$. Moreover, we can just compute the convex hull of all roots and compute the volume. This follows from the last result.

Theorem 7. *For any quiver Q of type $\mathbb{A}$ the polynomial $\operatorname{cor}(\mathbb{A}, n)$ is just zero. In particular, we obtain the well-known formula for the Catalan number*

$$\operatorname{vol} P^{+}(\mathbb{A}) = \sum_{i \in Q_0} \operatorname{vol} P^{+}(Q \setminus \{i\}).$$

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A geometric version of the Krull-Remak-Schmidt Theorem

Andrew Hubery

Given a finitely-generated algebra A over an algebraically-closed field K, there is a natural affine scheme mod_A^d parameterising the possible A-module structures on K^d. (More precisely, for a commutative K-algebra R, we define $\mathrm{mod}_A^d(R)$ to be the set of K-algebra homomorphisms $A \to \mathbb{M}_d(R)$.) The group scheme GL_d acts by base-change, and the orbits are in bijection with the isomorphism classes of A-modules.

Even though there will usually be infinitely many pairwise non-isomorphic modules for a given dimension d, there will be only finitely many irreducible components of mod_A^d, so one can hope to classify these. In fact, analogously to the Krull-Remak-Schmidt Theorem for modules, every irreducible component is uniquely (up to reordering) a 'direct sum' of indecomposable irreducible components (ones containing a dense subset of points corresponding to indecomposable modules).

Explicitly, whenever $d = \sum_i d_i$, the direct sum for modules yields a morphism

$$\Theta\colon \mathrm{GL}_d \times \mathrm{mod}_A^{d_1} \times \cdots \times \mathrm{mod}_A^{d_r} \to \mathrm{mod}_A^d.$$

Then every irreducible component of mod_A^d is of the form

$$\mathcal{C}_1 \oplus \cdots \oplus \mathcal{C}_r := \overline{\Theta(\mathrm{GL}_d \times \mathcal{C}_1 \times \cdots \times \mathcal{C}_r)}$$

for some decomposition $d = \sum_i d_i$ and some indecomposable irreducible components $\mathcal{C}_i \subset \mathrm{mod}_A^{d_i}$.

The question then becomes: determine when the direct sum of two irreducible components is again an irreducible component. This was answered by Crawley-Boevey and Schröer [1] in terms of the vanishing of extension groups generically on the irreducible components.

We show that this type of result holds true quite generally for schemes arising from representation theory, including for example $\mathrm{Gr}_A\binom{M}{d}$, the Grassmannian of d-dimensional submodules of a given module M. In this case the direct sum map becomes

$$\Theta\colon \mathrm{Aut}_A(M \oplus N) \times \mathrm{Gr}_A\binom{M}{d} \times \mathrm{Gr}_A\binom{N}{e} \to \mathrm{Gr}_A\binom{M \oplus N}{d+e}$$

and an irreducible component is indecomposable if it contains a dense subset of points $(U \subset M)$ corresponding to indecomposable modules for the algebra $\begin{pmatrix} A & A \\ 0 & A \end{pmatrix}$.

The proof relies on two properties. First one gives a homological interpretation for the cokernel of the differential of the orbit map (so the fibres of the conormal bundle to the orbit). This then allows one to show that the direct sum construction always yields a separable morphism. Together these reduce the problem to deciding whether the morphism is surjective on tangent spaces, which then becomes a vanishing condition in homology.

In fact, we prove the following general result regarding separability.

Theorem. *Let $H \leq G$ be smooth group schemes over K, Y a G-scheme and $X \subset Y$ an irreducible H-stable subscheme. For $x \in X(K)$ we have the orbit map $H \to X$, $h \mapsto h \cdot x$, yielding the commutative diagram*

$$\begin{array}{ccc} H & \longrightarrow & X \\ \downarrow & & \downarrow \\ G & \longrightarrow & Y \end{array}$$

Taking differentials we obtain

$$\begin{array}{ccccccc} T_1H & \longrightarrow & T_xX & \longrightarrow & N_{X,x} & \longrightarrow & 0 \\ \downarrow & & \downarrow & & \downarrow{\scriptstyle \theta_x} & & \\ T_1G & \longrightarrow & T_xY & \longrightarrow & N_{Y,x} & \longrightarrow & 0 \end{array}$$

If for all x in a dense subset of X we have that

(1) $\mathrm{Stab}_H(x)$ and $\mathrm{Stab}_G(x)$ are smooth and connected, and

(2) $(G \cdot x) \cap X$ decomposes as a finite union of H-orbits,

then $\Theta\colon G \times X \to Y$ is separable if and only if θ_x is injective on a dense subset of X.

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Annihilation of cohomology and strong generation of module categories

Srikanth B. Iyengar

(joint work with Ryo Takahashi)

The goal of my talk was to present some results from [7], the central theme of which is that the two topics that make up its title are intimately related. Inklings of this can be found in the literature, both on annihilators of cohomology, notably work of Popescu and Roczen [10] from 1990, and on generators for module categories that is of more recent vintage; principally the articles of Dao and Takahashi [4], and Aihara and Takahashi [1]. To set the stage for describing this relationship we consider a noetherian ring Λ that is finitely generated as a module over its center, Λ^{c}. We call such a Λ a *noether algebra*. For any non-negative integer n, the elements of Λ^{c} that annihilate $\mathrm{Ext}^n_\Lambda(M,N)$, for all M and N in $\mathrm{mod}\,\Lambda$, form ideal that we denote $\mathsf{ca}^n(\Lambda)$. It is not difficult to see that one gets a tower of ideals

$\cdots \subseteq \mathsf{ca}^n(\Lambda) \subseteq \mathsf{ca}^{n+1}(\Lambda) \subseteq \cdots$, so their union, $\mathsf{ca}(\Lambda)$, is also an ideal of Λ^{c}, that we call the *cohomology annihilator* of Λ. As Λ is noetherian there exists an integer s such that $\mathsf{ca}(\Lambda) = \mathsf{ca}^s(\Lambda)$.

The questions that interest us are: How big (in any measure of size, for example, the dimension of the closed subset of $\operatorname{Spec}\Lambda^{\mathsf{c}}$ it determines) is $\mathsf{ca}(\Lambda)$? Does it contain non-zerodivisors? What is the least integer s as above? Not every ring has a non-zero cohomology annihilator ideal. Indeed, consider the *singular locus* of Λ, that is to say, the subset

$$\operatorname{Sing}\Lambda := \{\mathfrak{p} \in \operatorname{Spec}\Lambda^{\mathsf{c}} \mid \operatorname{gldim}\Lambda_{\mathfrak{p}} \text{ is finite}\}\,.$$

Here gldim denotes global dimension. It is an elementary exercise to check this is contained in the closed subset of $\operatorname{Spec}\Lambda^{\mathsf{c}}$, with the Zariski topology, defined by $\mathcal{V}(\mathsf{ca}(\Lambda))$, the set of prime ideals of Λ^{c} containing $\mathsf{ca}(\Lambda)$. This means that when the ideal $\mathsf{ca}(\Lambda)$ contains non-nilpotent elements, $\operatorname{Sing}\Lambda$ is contained in a proper closed subset of $\operatorname{Spec}\Lambda^{\mathsf{c}}$. However, there are even commutative noetherian rings for which this is not the case; the first examples were constructed by Nagata [9]; see also Hochster [6].

On the other hand, for any M in $\operatorname{mod}\Lambda$ and integer $n \geq 1$ there is an equality

$$\operatorname{ann}_{\Lambda^{\mathsf{c}}} \operatorname{Ext}^n_\Lambda(M, \Omega^n M) = \operatorname{ann}_{\Lambda^{\mathsf{c}}} \operatorname{Ext}^{\geqslant n}_\Lambda(M, \operatorname{mod}\Lambda) \tag{1}$$

where $\Omega^n M$ denotes an nth syzygy module of M as an Λ-module. Observe that $\mathsf{ca}^n(\Lambda)$ is the intersection of the ideals on the right, as M varies over $\operatorname{mod}\Lambda$. This leads one to the following definition: We say that a module $G \in \operatorname{mod}\Lambda$ is a *strong generator* for $\operatorname{mod}\Lambda$ if there exists integers s and n such that for each $M \in \operatorname{mod}\Lambda$, there is Λ-module W and a filtration

$$\{0\} = Z_0 \subseteq Z_1 \subseteq \cdots \subseteq Z_n = Z \quad \text{where } Z = W \bigoplus \Omega^s M,$$

with Z_{i+1}/Z_i is in $\operatorname{add} G$, for each i. We require also that G contains Λ as a direct summand, so that G is a generator in the usual sense of the word.

This definition should be compared with that of a generator of a triangulated category, introduced by Bondal and Van den Bergh [3]. Unlike for triangulated categories, there are various possible notions of "generation" for module categories, stemming from the fact that, in the module category, kernels and co-kernels are not interchangeable; some of this is clarified in [7].

Given (1) it is not hard to prove that if $\operatorname{mod}\Lambda$ has a strong generator, with parameter s as above, and $d = \sup\{\operatorname{gldim}\Lambda_{\mathfrak{p}} \mid \mathfrak{p} \notin \operatorname{Sing}\Lambda\}$ is finite, then

$$\mathcal{V}(\mathsf{ca}(\Lambda)) = \mathcal{V}(\mathsf{ca}^{s+d+1}(\Lambda)) = \operatorname{Sing}\Lambda\,, \tag{2}$$

In particular, $\operatorname{Sing}\Lambda$ is a closed subset of $\operatorname{Spec}\Lambda^{\mathsf{c}}$, in the Zariski topology. One can also describe the cohomology annihilator ideal, at least up to radical, in terms of the generator of $\operatorname{mod}\Lambda$.

Going in the other direction, we prove that when R is a commutative noetherian ring of finite Krull dimension and there exists an integer s such that $\mathsf{ca}^s(R/\mathfrak{p}) \neq 0$ for each prime ideal $\mathfrak{p}$ in R, then $\operatorname{mod} R$ has a strong generator.

This results shifts the focus to finding non-zero cohomology annihilators, and one of our main tools for this is the noether different introduced by Auslander and Goldman [2], under the name "homological different". Building on these results we prove that when R is a commutative noetherian ring that is either a finitely generated algebra over a field or an equicharacteristic excellent local ring, then $\operatorname{mod} R$ has a strong generator and

$$\mathcal{V}(\mathsf{ca}(R)) = \mathcal{V}(\mathsf{ca}^{2d+1}(R)) = \operatorname{Sing} R\,, \quad \text{where } d = \dim R.$$

The part of the statement above dealing with generators extends results of Dao and Takahashi [4], who proved it for affine algebras and for complete local rings, assuming in addition that the coefficient field is perfect. Any generator for $\operatorname{mod} R$ gives one for its bounded derived category so these results extend work of Aihara and Takahashi [1] and Rouquier [11]—see also Keller and Van den Bergh [8]—on the existence of generators for bounded derived categories.

On the other hand, the identification of the singular locus with the closed subset defined by the cohomology annihilator is related to results of Wang [12, 13], that in turn extend earlier work of Dieterich [5], Popescu and Roczen [10], and Yoshino [14] on Brauer-Thrall conjectures for maximal Cohen-Macaulay modules over Cohen-Macaulay rings. The interested reader will find a detailed comparison in [7].

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Irreducible components of some module varieties via the Springer resolution

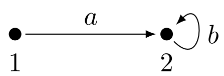
Ryan Kinser
(joint work with Jerzy Weyman)

Let Λ be a finite dimensional algebra over an algebraically closed field K. We assume Λ is basic so that $\Lambda \simeq KQ/I$, where Q is a quiver and I and ideal of relations. This way we can identify representations of Λ with representations of Q which satisfy the relations I. Then the module variety $\mathrm{mod}(\Lambda, \mathbf{d})$ parametrizes representations of Λ of dimension vector $\mathbf{d}$, along with a fixed basis.

In general, $\mathrm{mod}(\Lambda, \mathbf{d})$ can have several irreducible components. Crawley-Boevey and Schröer have shown that the components admit a geometric Krull-Schmidt type decomposition [1], but in general there is no straightforward way to index or even determine the number of irreducible components. In fact, there are not even many examples in the literature of algebras Λ such that the irreducible components of $\mathrm{mod}(\Lambda, \mathbf{d})$ are understood for all $\mathbf{d}$.

This report concerns work in progress on using the Springer resolution [2] of the variety of nilpotent matrices to study irreducible components when Q has a loop. We illustrate the idea with a small example that allows explicit indexing of all components for all $\mathbf{d}$ by this technique.

Let Λ be an algebra given by a quiver with relations of the following form:

(1) 

$b^n = b^m a = 0, \quad m < n \in \mathbb{N}.$

The algebra is representation finite for $(m, n) \leq (3, 3)$ or $(m, n) \leq (2, 5)$; it is tame for $(m, n) = (2, 6)$, and wild otherwise. When the action of a is injective, a representation V can be thought of as nilpotent linear operator on a vector space V_2, along with a choice of (not necessarily stable) subspace V_1. In a fixed irreducible component $C \subseteq \mathrm{mod}(\Lambda, \mathbf{d})$, there is an open subset such that the action of b has constant Jordan type λ, where $\lambda = (\lambda_1, \ldots, \lambda_n)$ is a partition of $\mathbf{d}_2$ such that

$$\lambda_i = \dim \ker b^i - \dim \ker b^{i-1}$$

on this open subset. More generally, for each partition λ, consider the locus $C(\lambda)^\circ \subset \mathrm{mod}(\Lambda, \mathbf{d})$ of points (A, B) where B is of type λ as above, and denote its closure by $C(\lambda)$. Thus we have $\mathrm{mod}(\Lambda, \mathbf{d}) = \bigcup_\lambda C(\lambda)$. It is easy to see that if $C(\mu) \subseteq C(\lambda)$, then $\mu \leq \lambda$ in reverse dominance order. But the following example illustrates that the converse is not true.

Example 1. Take Λ with relations $b^3 = b^2 a = 0$ and $\mathbf{d} = (3, 3)$. The partition $\lambda = (1, 1, 1)$ corresponding to one Jordan block of size 3 gives an irreducible component because this Jordan type is maximal, and $\mathrm{rank}\, A \leq 2$ on this component. But $\mu = (2, 1)$ also gives an irreducible component because it is maximal among partitions where $\mathrm{rank}\, A$ can equal 3. □

Let $\mathrm{Flag}(\lambda)$ be the corresponding flag variety

$$\mathrm{Flag}(\lambda) := \{0 = F_0 \subset F_1 \subset \cdots \subset F_{n-1} \subset F_n = V_2 \mid \dim F_i = \sum_{k=1}^{i} \lambda_k\},$$

and consider the incidence locus $Z \subseteq \mathrm{Hom}_K(V_1, V_2) \times \mathrm{End}_K(V_2) \times \mathrm{Flag}(\lambda)$ defined by triples $(A, B, F_\bullet)$ satisfying

$$B(F_i) \subseteq F_{i-1}, \quad A(V_1) \subseteq F_m.$$

The projection $p_\lambda \colon Z \to \mathrm{Flag}(\lambda)$ is a vector bundle. We also have a projection

$$q_\lambda \colon Z \to \mathrm{Hom}_K(V_1, V_2) \times \mathrm{End}_K(V_2)$$

with the image of q_λ contained in $\mathrm{mod}(\lambda, \mathbf{d})$.

Proposition 2. The subvariety $C(\lambda)$ is irreducible, so there is at most one irreducible component of $\mathrm{mod}(\Lambda, \mathbf{d})$ of a given generic Jordan type. Furthermore, the map q_λ gives a resolution of singularities of $C(\lambda)$.

Thus, to determine the irreducible components of $\mathrm{mod}(\Lambda, \mathbf{d})$, it is enough to determine the λ for which $C(\lambda)$ is maximal. In this example and others, we can describe the poset structure of $\{C(\lambda)\}_\lambda$ under the containment order, and explicitly determine its maximal elements. We illustrate the idea (without proofs) for the above example.

Define a poset $P := P(m, n, \mathbf{d})$ as follows. As a set, P consists of the partitions of $\mathbf{d}_2$ into at most n parts. Consider the function $r \colon P \to \mathbb{N}$ given by

$$r(\lambda) = \min \left\{ d_1, \sum_{k=1}^{m} \lambda_i \right\}. \tag{2}$$

Note that the second entry in the min function is the generic dimension of $\ker b^m$ in $C(\lambda)$, and thus $r(\lambda)$ is the generic rank of the action of a in $C(\lambda)$.

The partial order $\lessdot$ is given by $\mu \lessdot \lambda$ when both of the following hold:

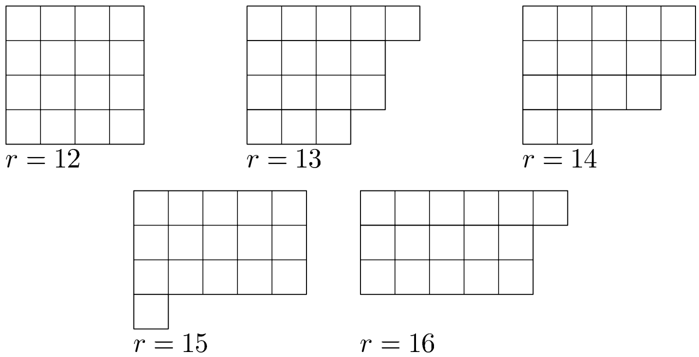
(1) $\mu \leq \lambda$ in reverse dominance order;
(2) $r(\mu) \leq r(\lambda)$.

We have proven that $C(\mu) \subseteq C(\lambda)$ if and only if $\mu \lessdot \lambda$, so this poset encodes the combinatorics of the Jordan stratification of $\mathrm{mod}(\Lambda, \mathbf{d})$. Then using combinatorial methods, we can get the following.

Proposition 3. Let Λ be an algebra of the form (1), with arbitrary parameters m, n, and let $\mathbf{d}$ be a dimension vector for Λ. Then the irreducible components of $\mathrm{mod}(\Lambda, \mathbf{d})$ are indexed by possible ranks of the action of b; more precisely, for each integer r in a certain range there is a unique irreducible component, and r is the generic rank of the action of b on that component.

We have a closed formula for the range of possible ranks, but it is somewhat unbecoming and omitted from this informal report. The proof also yields an explicit description of the partitions λ which are maximal in P.

Example 4. Consider the algebra Λ with parameters $m = 3$ and $n = 4$, so Λ is of wild representation type. For the dimension vector $\mathbf{d} = (16, 16)$, there are 5 irreducible components of $\text{mod}(\Lambda, \mathbf{d})$, with the following generic Jordan types (represented by Young diagrams).



If instead we take $\mathbf{d} = (14, 16)$, there are only 3 irreducible components, namely those with $r \leq 14$ (the first line above). □

This method can be applied to many similar algebras, often yielding tractable combinatorial problems to index and count the irreducible components.

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Gendo-symmetric algebras and dominant dimension

Steffen Koenig
(joint work with Ming Fang)

An algebra A over a field k is called *gendo-symmetric* if and only if there exists a symmetric algebra B and a B-module X such that $A \simeq \text{End}_B(B \oplus X)$. Under the classical Morita-Tachikawa correspondence, which sends a pair (Λ, M) with M a generator-cogenerator, to $\text{End}_\Lambda(M)$, which is an algebra of dominant dimension at least two, the gendo-symmetric algebras correspond exactly to the pairs (Λ, M) with Λ being symmetric; here, $M = B \oplus X$.

Denote by D the duality $\text{Hom}_k(-, k)$.

Theorem. ([2]) The following statements are equivalent:
(1) A is gendo-symmetric,
(2) $\text{Hom}_A(D(A), A) \simeq A$ as $A - A-$bimodules,
(3) $D(A) \otimes_A D(A) \simeq D(A)$ as $A - A-$bimodules,

(4) A has dominant dimension at least 2 and $D(Ae) \simeq eA$ as $eAe-A-$bimodules, where Ae is basic faithful projective-injective.

Kerner and Yamagata [4] have generalised this result to a characterisation of Morita algebras; these are corresponding to self-injective algebras under the Morita-Tachikawa correspondence.

Examples of gendo-symmetric algebras include the classical Schur algebras $S(n,r)$ (for $n \geq r$) of general linear groups GL_n and the blocks of the BGG-category $\mathcal{O}$ of semisimple complex Lie algebras. The double centraliser properties guaranteed by these algebras having dominant dimension at least two are classical Schur-Weyl duality and Soergel's structure theorem for $\mathcal{O}$, respectively ([5]). In [1] it has been shown that the dominant dimension of A controls the quality of the Schur functor $e \cdot -$ in a precise sense.

The property of being gendo-symmetric is Morita invariant [3]. More surprisingly, current work of W.Hu and M.Fang shows that under derived equivalences between Morita algebras the derived equivalence class of the underlying self-injective algebras is preserved.

Using property (3) a multiplication can be constructed on $D(A)$, when A is gendo-symmetric.

Theorem. ([3]) A gendo-symmetric algebra A has a comultiplication, which is a coassociative A-bimodule homomorphism. There exists a counit if and only if A is symmetric.

This extends a known characterisation of Frobenius algebras, due to L.Abrams.

Using the comultiplication, a cobar resolution can be constructed, from which the dominant dimension of A can be read off. Conversely, an algebra A admitting a comultiplication sharing the properties of the above comultiplication must be gendo-symmetric ([3]).

In recent work by M.Fang, the comultiplication of Schur algebras (with arbitrary parameters) has been determined on integral level, in terms of combinatorial objects called (bi)permanents. Fang has shown that over a field k the map corresponding to the one in (3) above is an isomorphism if either the characteristic of k is zero or $r \leq n(char(k)-1)$, and that this happens if and only if the dual of the Schur algebra equals a certain coefficient space of truncated symmetric powers, the socalled Doty coalgebra.

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Quasi-hereditary algebras, A_∞-categories and boxes

Julian Külshammer

(joint work with Steffen Koenig and Sergiy Ovsienko)

Highest weight categories are abundant in algebraic Lie theory as well as in representation theory of finite dimensional algebras. They also arise in geometry from exceptional collections. A highest weight category with a finite number of simples is precisely the module category of a quasi-hereditary algebra, unique up to Morita equivalence.

A prototypical example in algebraic Lie theory is Bernstein-Gelfand-Gelfand category $\mathcal{O}$ associated to a finite dimensional semisimple complex Lie algebra $\mathfrak{g}$. Here the standard modules Δ are the Verma modules, which are induced from a distinguished Lie subalgebra, the Borel subalgebra. Motivated by this example in [4] Koenig introduced the notion of an exact Borel subalgebra of a quasi-hereditary algebra having analogous properties. In particular a PBW theorem holds. He proved that the finite dimensional algebras associated to the blocks of Bernstein-Gelfand-Gelfand category $\mathcal{O}$ have such subalgebras and that there are examples of quasi-hereditary algebras not having exact Borel subalgebras. Despite intensive efforts, existence for other important classes of quasi-hereditary algebras like Schur algebras or algebras of global dimension two could not be proven.

Although not every quasi-hereditary algebra has an exact Borel subalgebra in joint work with Koenig and Ovsienko in [5] we showed that to every quasi-hereditary algebra A we can associate a Morita equivalent algebra R having an exact Borel subalgebra. Even better, we can construct R from a box $\mathfrak{B} = (B, W)$, the representation theoretic analogue of a differential graded category. Here, B is an algebra (which will be the exact Borel subalgebra of R) and W is a B-coalgebra. The category $\mathcal{F}(\Delta)$ can then be realised as the category of representations $\operatorname{mod}\mathfrak{B}$ of the box $\mathfrak{B}$. This category can also be seen as the Kleisli category of the corresponding comonad $W \otimes_B -$ (see e.g. [2]). The algebra R can then be obtained via $R = \operatorname{End}_{\mathfrak{B}}(B)$.

To prove this, we use a theorem of Keller and Lefèvre-Hasegawa [3] stating that the Yoneda algebra $\operatorname{Ext}^*(\Delta, \Delta)$, regarded as an A_∞-category, carries enough information to reconstruct the category of filtered modules $\mathcal{F}(\Delta)$ as so called twisted modules. Translating this result to the language of boxes gives what we call a directed box. Structure theory of boxes by Burt and Butler [1] can then be used to prove that B is indeed a Borel subalgebra of R.

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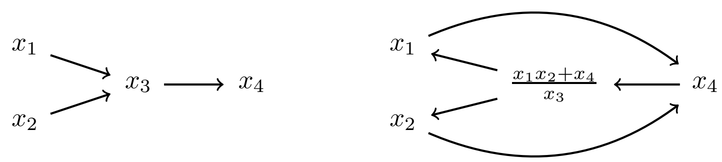
The divisor class group of a cluster algebra

Philipp Lampe

We wish to discuss the effects of cluster structures on a given algebra. Especially, we address the following questions: When is it a unique factorization domain? What is its divisor class group?

1. Fomin-Zelevinsky's cluster algebras

1.1. **Introduction.** Let Q be a quiver with n vertices. A method to study representations of Q are *Fomin-Zelevinsky's cluster algebras*: Let $\mathbf{x} = (x_1, x_2, \ldots, x_n)$ be a sequence of algebraically independent variables over a base field K of characteristic 0 which constitute an initial cluster. All variables obtained from the initial cluster by a sequence of mutations generate a *cluster algebra* $\mathcal{A}(\mathbf{x}, Q) \subseteq K[x_1^{\pm 1}, x_2^{\pm 1}, \ldots, x_n^{\pm 1}]$. An example of a mutation is the following:



1.2. **The lower bound.** Assume that the quiver Q is acyclic. Berenstein-Fomin-Zelevinsky [1] have proved that in this case the cluster algebra $\mathcal{A}(\mathbf{x}, Q)$ is already generated by the cluster variables which we obtain from the initial cluster $(\mathbf{x}, Q)$ by a single mutation. Moreover, the authors construct an isomorphism

$$\mathcal{A}(\mathbf{x}, Q) \cong K[X_i, X_i' \colon 1 \leq i \leq n]/(X_i X_i' - f_i \colon 1 \leq i \leq n)$$

where $f_i = \prod_{j \to i} X_j + \prod_{i \to k} X_k$ is the initial exchange polynomial. Especially, acyclic cluster algebras are finitely generated and hence noetherian.

1.3. **Freezing.** Sometimes we freeze certain vertices of the quiver to obtain two kinds of vertices – *mutable* and *frozen* vertices. Sequences of mutations at mutable vertices yield a smaller set $\mathcal{C}$ of cluster variables. In this case, we define the cluster algebra (with invertible coefficients) to be generated by $\mathcal{C}$ and the inverses of the frozen variables.

1.4. **Cluster algebras of finite type.** Fomin-Zelevinsky [3] have classified cluster algebras with only finitely many cluster variables by finite type root systems. More precisely, a (connected) cluster algebra is of finite type if and only if the mutation class of Q contains a quiver whose mutable part is an orientation of a Dynkin diagram of type A, D or E.

2. When is a cluster algebra a unique factorization domain?

2.1. **Unique factorization.** In this context, Geiß-Leclerc-Schröer [4] have proved two useful results. First, every cluster variable is an irreducible element. Second, if the quiver Q contains no frozen vertices, then we can describe the group of units as $\mathcal{A}(\mathbf{x}, Q)^\times = k^\times$. Moreover, the authors give two criteria:

(1) Assume that $f_i = f_j$ for some $i \neq j$. Then $\mathcal{A}(\mathbf{x}, Q)$ is not a unique factorization domain, a non-unique factorization is $x_i x_i' = x_j x_j'$ and the cluster variables x_i and x_j are not prime.
(2) Assume that the polynomial f_i is reducible in the ring $K[X_1, X_2, \ldots, X_n]$. Then $\mathcal{A}(\mathbf{x}, Q)$ is not a unique factorization domain and the cluster variable x_i is not prime.

For example, the cluster algebra attached to the quiver $1 \to 2 \to 3$ of type A_3 is not a unique factorization domain, because the initial exchange polynomials f_1 and f_3 both equal to $1 + X_2$. For another example, the cluster algebra attached to the Kronecker quiver is not a unique factorization domain over $K = \mathbb{C}$, because the polynomial $1 + X_1^2$ is reducible.

2.2. **Classification in finite type.** Assume that Q is an orientation of a Dynkin diagram of type A, D, E. In [5] we prove that the cluster algebra $\mathcal{A}(\mathbf{x}, Q)$ is a unique factorization domain if and only if it is of type A_n with $n \neq 3$ or of type E.

3. The divisor class group

3.1. **Divisor class group of a cluster algebra.** With an integrally closed domain A we associate its *divisor class group* $\mathrm{Cl}(A)$. A fundamental theorem asserts that A is a unique factorization domain if and only if $\mathrm{Cl}(A) = 0$.

Theorem. Suppose that Q is an acyclic quiver with a mutable vertex i. We construct a new quiver Q' by freezing the vertex i. If the cluster variable x_i is a prime element in the cluster algebra $\mathcal{A}(\mathbf{x}, Q)$, then $\mathrm{Cl}(\mathcal{A}(\mathbf{x}, Q)) \cong \mathrm{Cl}(\mathcal{A}(\mathbf{x}, Q'))$, where we consider cluster algebras with invertible coefficients.

As an immediate consequence we obtain the following necessary and sufficient and computer-checkable criterion: Let $(\mathbf{x}, Q)$ be an acyclic seed. Then the cluster algebra $\mathcal{A}(\mathbf{x})$ is a unique factorization domain if and only if all initial cluster variables $x_i \in \mathbf{x}$ are prime elements.

Using the criterion, we can reprove the classification result for Dynkin quivers and classify which cluster algebras attached to extended Dynkin $\tilde{A}, \tilde{D}, \tilde{E}$ diagrams are unique factorization domains.

Further examples of divisor class groups (over $K = \mathbb{Q}$) are $\mathrm{CL}(\mathcal{A}(\mathbf{x}, B)) \cong \mathbb{Z}$ for an exchange matrix B of type B_2, A_3 or D_n with $n \geq 5$. For the Kronecker quiver we have $\mathrm{CL}(\mathcal{A}(\mathbf{x}, Q)) \cong \mathbb{Z}^2$. We conjecture that $\mathrm{CL}(\mathcal{A}(\mathbf{x}, B))$ is always torsion-free.

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Cluster structures on strata of flag varieties

Bernard Leclerc

The talk was based on the recent preprint [10]. Let G be a simple, simply connected and simply laced algebraic group over $\mathbb{C}$. We fix a maximal torus H, a Borel subgroup B containing H, and we denote by B^- the Borel subgroup opposite to B with respect to H. Let N be the unipotent radical of B. Let $W = \mathrm{Norm}_G(H)/H$ be the Weyl group, with length function $w \mapsto \ell(w)$ and longest element w_0.

We consider the flag variety $X = B^-\backslash G$, and we denote by $\pi : G \to X$ the natural projection $\pi(g) := B^- g$. The Bruhat decomposition

$$G = \bigsqcup_{w\in W} B^- w B^-$$

projects to the Schubert decomposition

$$X = \bigsqcup_{w\in W} C_w, \tag{1}$$

where $C_w = \pi(B^- w B^-)$ is the Schubert cell attached to w, isomorphic to $\mathbb{C}^{\ell(w)}$. We may also consider the Birkhoff decomposition

$$G = \bigsqcup_{w\in W} B^- w B,$$

which projects to the opposite Schubert decomposition

$$X = \bigsqcup_{w\in W} C^w, \tag{2}$$

where $C^w = \pi(B^- w B)$ is the opposite Schubert cell attached to w, isomorphic to $\mathbb{C}^{\ell(w_0)-\ell(w)}$. The intersection

$$\mathcal{R}_{v,w} := C^v \cap C_w$$

has been considered by Kazhdan and Lusztig [8] in relation with the cohomological interpretation of the Kazhdan-Lusztig polynomials. One shows [8, 2] that $\mathcal{R}_{v,w}$ is non-empty if and only if $v \le w$ in the Bruhat order of W, and it is a smooth irreducible locally closed subset of C_w of dimension $\ell(w) - \ell(v)$. More recently, $\mathcal{R}_{v,w}$ has sometimes been called an open Richardson variety [9], because its closure in X is known as a Richardson variety [12].

Intersecting the decompositions (1) and (2) of X, we thus get a finer stratification

$$X = \bigsqcup_{v \le w} \mathcal{R}_{v,w}. \tag{3}$$

However, in contrast with (1) or (2), the strata $\mathcal{R}_{v,w}$ of (3) are not isomorphic to affine spaces.

The stratification (3) was used by Lusztig for studying the totally nonnegative part $X_{\ge 0}$ of the flag variety X. He conjectured [11] that the intersections of the complex strata of (3) with $X_{\ge 0}$ give a cell decomposition of $X_{\ge 0}$, and this was proved by Rietsch [13]. On the other hand, Goodearl and Yakimov [7] have given an interpretation of (3) in a Poisson geometric setting. They showed that the strata of (3) coincide with the H-orbits of the symplectic leaves of the standard Poisson structure on X.

The fact that the strata $\mathcal{R}_{v,w}$ occur in these two settings naturally leads to the following question: is there a cluster algebra structure (in the sense of Fomin and Zelevinsky [3]) on the coordinate ring $\mathbb{C}[\mathcal{R}_{v,w}]$, which is compatible with total positivity and Poisson geometry? More precisely, a cluster algebra structure on $\mathbb{C}[\mathcal{R}_{v,w}]$ provides the complex variety $\mathcal{R}_{v,w}$ with a positive atlas, and we would like each of the charts of this atlas to give us a description of $\mathcal{R}_{v,w} \cap X_{\ge 0}$. Moreover, if we denote by $\{\cdot,\cdot\}$ the Poisson bracket on $\mathbb{C}[\mathcal{R}_{v,w}]$, we would like to have, for every cluster $x = \{x_1, \ldots, x_n\} \subset \mathbb{C}[\mathcal{R}_{v,w}]$, the compatibility relations

$$\{x_i, x_j\} = \omega_{ij} x_i x_j, \qquad (1 \le i < j \le n),$$

for some appropriate $\omega_{ij} \in \mathbb{Z}$, see [6, §4.1].

To construct a cluster structure for $\mathcal{R}_{v,w}$, we follow the approach of our joint papers with C. Geiss and J. Schröer [4, 5], in which similar questions were studied for the unipotent cells N^w of G (or more generally, of a symmetric Kac-Moody group). The main feature of this approach is categorification. More precisely, let Λ be the preprojective algebra associated with G. To the variety N^w, we have attached a certain Frobenius subcategory $\mathcal{C}_w$ of $\mathrm{mod}(\Lambda)$ having a cluster structure in the sense of Buan-Iyama-Reiten-Scott [1]. This means that $\mathcal{C}_w$ is endowed with a family of cluster-tilting objects, which are related to each other by a procedure of mutation. Using Lusztig's Lagrangian construction of the enveloping algebra of $\mathfrak{n} = \mathrm{Lie}(N)$, we have constructed a cluster character φ from $\mathrm{mod}(\Lambda)$ to $\mathbb{C}[N]$. The subspace spanned by the image of the restriction of φ to $\mathcal{C}_w$ is a certain subring of $\mathbb{C}[N]$, and we have shown that, after localizing with respect to the multiplicative subset given by the images of the projective objects of $\mathcal{C}_w$, we obtain a ring isomorphic to $\mathbb{C}[N^w]$. Moreover, the images under φ of the cluster-tilting objects of $\mathcal{C}_w$ endow this ring with the structure of a cluster algebra.

The subcategory $\mathcal{C}_w$ is a torsion class (it is closed under extensions and factor modules). Let $\mathcal{C}^w$ be the corresponding torsion-free class, so that $(\mathcal{C}_w, \mathcal{C}^w)$ is a torsion pair. We introduce the following subcategory

$$\mathcal{C}_{v,w} := \mathcal{C}^v \cap \mathcal{C}_w, \qquad (v \leq w \in W).$$

This is a Frobenius stably 2-Calabi-Yau category, and one can check that it has a cluster structure. The main result is then:

Theorem. *For every $v, w \in W$ with $v \leq w$, the coordinate ring $\mathbb{C}[\mathcal{R}_{v,w}]$ contains a cluster subalgebra $R_{v,w}$ coming from the category $\mathcal{C}_{v,w}$. The number of cluster variables in a cluster of $R_{v,w}$ (including the frozen ones) is equal to* $\dim(\mathcal{R}_{v,w})$.

For a more detailed formulation of this result, see [10, Theorem 4.5].

If the category $\mathcal{C}_{v,w}$ has finitely many indecomposable objects, then the cluster algebra $R_{v,w}$ has finite cluster type, and one can check that $R_{v,w} = \mathbb{C}[\mathcal{R}_{v,w}]$. It is also easy to prove that $R_{v,w} = \mathbb{C}[\mathcal{R}_{v,w}]$ if $w = v'v$ with $\ell(w) = \ell(v') + \ell(v)$. We conjecture that this equality remains true in general. We also believe that $R_{v,w}$ is compatible with total positivity and Poisson geometry, see an extended example in [10, §7]. We hope to come back to these questions in a forthcoming publication.

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Positivity for Kac polynomials and DT-invariants of quivers

Emmanuel Letellier
(joint work with T. Hausel and F. Rodriguez-Villegas)

In [3] we prove positivity of the coefficients of two families of well-known polynomials, namely Kac polynomials that counts the number of isomorphism classes of absolutely indecomposable representations of quivers over finite fields and certain motivic DT-invariants introduced by Kontsevich and Soibelman [5]. We give a uniform proof of the positivity for these two kind of polynomials by giving a cohomological interpretation of the coefficients (in terms of quiver varieties). In this talk we reformulate our main theorem in terms of representation theory of finite Lie groups.

Let $\Gamma = (I, \Omega)$ be a finite quiver with $I = \{1, \dots, r\}$ and let $\mathbf{v} = (v_1, \dots, v_r) \in \mathbb{N}^I$ be a dimension vector. We then put

$$\mathrm{Rep}(\Gamma, \mathbf{v}) = \bigoplus_{i \to j \in \Omega} \mathrm{Mat}_{v_j, v_i}(\mathbb{F}_q), \qquad GL_{\mathbf{v}} = \prod_{i=1}^{r} \mathrm{GL}_{v_i}(\mathbb{F}_q).$$

The group $\mathrm{GL}_{\mathbf{v}}$ acts then naturally on $\mathrm{Rep}(\Gamma, \mathbf{v})$ as

$$(g \cdot \varphi)_{i \to j} = g_j \, \varphi_{i \to j} \, g_i^{-1}.$$

Consider the associated permutation representation $\rho_\Gamma : \mathrm{GL}_{\mathbf{v}} \to \mathrm{GL}\left(\mathbb{C}[\mathrm{Rep}(\Gamma, \mathbf{v})]\right)$.

For an irreducible complex character χ of $\mathrm{GL}_{\mathbf{v}}$, denote by $P_{\Gamma,\chi}(q)$ the multiplicity of χ in ρ_Γ. It is given by the inner product formula

$$P_{\Gamma,\chi}(q) = \langle \rho_\Gamma, \chi \rangle_{\mathrm{GL}_{\mathbf{v}}} := \frac{1}{|\mathrm{GL}_{\mathbf{v}}|} \sum_{g \in \mathrm{GL}_{\mathbf{v}}} \#\{\varphi \in \mathrm{Rep}(\Gamma, \mathbf{v}) \,|\, g \cdot \varphi = \varphi\} \; \chi(g).$$

We are interested in understanding the multiplicities $P_{\Gamma,\chi}(q)$. The character table of $\mathrm{GL}_n(\mathbb{F}_q)$ is known since Green in 1955. The building blocks of the character table are the so-called *unipotent characters* (the irreducible characters of GL_n can be all constructed by induction from the unipotent characters of Levi subgroups of the parabolic subgroups of GL_n). The unipotent characters of GL_n are naturally parametrized by the set of partitions $\mathcal{P}_n$ of n in such a way that the partition (n^1) of n corresponds to the trivial character and the partition (1^n) corresponds to the so-called Steinberg character.

The unipotent characters of $\mathrm{GL}_{\mathbf{v}}$ are then parametrized by the set of multipartitions $\mathcal{P}_{\mathbf{v}} := \mathcal{P}_{v_1} \times \mathcal{P}_{v_2} \times \cdots \times \mathcal{P}_{v_r}$. For a $\mu \in \mathcal{P}_{\mathbf{v}}$, we denote by $\mathcal{U}_\mu$ the corresponding unipotent character of $\mathrm{GL}_{\mathbf{v}}$.

We now consider a twisted version of unipotent characters. Fix once for all a linear character $\alpha : \mathbb{F}_q^\times \to \mathbb{C}^\times$ such that the subgroup of linear characters of $\mathbb{F}_q^\times$ generated by α is of cardinality $\sum_{i=1}^r v_i$. We then consider the following irreducible characters of $\mathrm{GL}_{\mathbf{v}}$

$$\mathcal{U}_\mu^\alpha := (\alpha \circ \det) \cdot \mathcal{U}_\mu, \qquad \mu \in \mathcal{P}_{\mathbf{v}}.$$

Write $(1^{\mathbf{v}})$ for the multi-partition $((1^{v_1}), \ldots, (1^{v_r}))$. We have the following proposition.

Proposition 1. (1) $P_{\Gamma,(\alpha\circ\det)}(q)$ equals the Kac polynomial of $(\Gamma, \mathbf{v})$ that counts the number of isomorphism classes of absolutely indecomposable representations of $(\Gamma, \mathbf{v})$ over $\mathbb{F}_q$.
(2) $P_{\Gamma,\mathcal{U}^\alpha_{(1^{\mathbf{v}})}}(q)$ equals a certain motivic DT-invariant for the double quiver $\overline{\Gamma}$ with zero potential as defined by Kontsevich and Soibelman.

The first assertion is almost straightforward and the second one is hidden in [3].

In [3], we define from any multi-partition $\mu \in \mathcal{P}_{\mathbf{v}}$ a quiver Γ_μ equipped with a dimension vector $\mathbf{v}_\mu$ that "extend" the pair $(\Gamma, \mathbf{v})$.

We are now ready to state a reformulation of the main theorem of [3].

Theorem 2. *(Hausel, Letellier, Rodriguez-Villegas, 2012) Let $\mu \in \mathcal{P}_{\mathbf{v}}$.*
(1) The polynomial $P_{\Gamma,\mathcal{U}^\alpha_\mu}(q)$ does not depend on the choice of α and has non-negative integer coefficients.
(2) $P_{\Gamma,\mathcal{U}^\alpha_\mu}(q)$ is non-zero if and only if $\mathbf{v}_\mu$ is a root of Γ_μ. Moreover, $P_{\Gamma,\mathcal{U}^\alpha_\mu}(q) = 1$ if and only if $\mathbf{v}_\mu$ is a real root.
(3) If non-zero, the polynomial $P_{\Gamma,\mathcal{U}^\alpha_\mu}(q)$ is monic of degree

$$1 - \frac{1}{2}{}^t\mathbf{v}_\mu C_\mu \mathbf{v}_\mu,$$

where C_μ is the Cartan matrix of Γ_μ.

The theorem implies the positivity of the coefficients of Kac polynomials (this was conjectured by Kac [4] and proved by Crawley-Boevey and van den Bergh for indivisible $\mathbf{v}$ [1]). It also implies the positivity of the coefficients of DT-invariants which was conjectured by Kontsevich and Soibelman [5] and first proved by Efimov [2]. The proof of Efimov is completely different from our proof.

We prove our main theorem by showing that, up to some explicit power of q, the polynomial $P_{\Gamma,\mathcal{U}^\alpha_\mu}(q)$ is the graded multiplicities of the irreducible character $\chi^{\mu'}$ of $S_{v_1} \times \cdots \times S_{v_r}$ (with S_n the symmetric group in n letters) in the compactly supported cohomology of a generic quiver variety attached to $(\Gamma_{(1^{\mathbf{v}})}, \mathbf{v}_{(1^{\mathbf{v}})})$ (such a quiver variety exists as $\mathbf{v}_{(1^{\mathbf{v}})}$ is indivisible).

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Brauer graph algebras and mutation

Robert J. Marsh
(joint work with Sibylle Schroll)

This was a talk on joint work from [5]. Let k be an algebraically closed field. Let Γ be a connected finite graph with at least one edge, together with a cyclic ordering of the edges around each vertex. Such an ordering can be obtained from a local embedding of Γ into the plane, using the orientation of the plane. A quiver Q_Γ can be associated to Γ with vertices v_a, a an edge of Γ, and arrows $v_a \to v_b$ whenever b is the successor of a in the cyclic ordering around a vertex of Γ, except if the vertex is a leaf.

We consider also relations R_Γ on Q_Γ, defined as follows. Let a be an edge of Γ with endpoints i, j. Let C_i (respectively, C_j) be the cycle in Q_Γ corresponding to following the cyclic ordering in Γ around i (respectively j), starting and ending at v_a. Firstly, the composition of the last arrow of C_i with the first arrow of C_j is zero, and the composition of the last arrow of C_j with the first arrow of C_i is zero. If i, j are not leaves in Γ, we take the relation $C_i = C_j$. If i is not a leaf and j is a leaf in Γ, we take the relation $\alpha C_i = 0$, where α is the first arrow of C_i.

The *Brauer graph algebra* associated to the above data is $B_\Gamma = \frac{kQ_\Gamma}{(R_\Gamma)}$. If Γ consists of a single edge, B_Γ is defined to be the algebra $k[x]/(x^2)$. The definition is extended to a disconnected graph Γ in a natural way. Further details concerning Brauer graph algebras can be found, for example, in [1, 2, 7].

Theorem 1. [2, 3.5]. *Let Γ and Γ' be Brauer graphs which are the same apart from a local move as shown in Figure 1. Then the corresponding Brauer graph algebras B_Γ and $B_{\Gamma'}$ are derived equivalent.*

Let (S, M) be a compact oriented surface with non-empty set of marked points M. We assume that (S, M) is not a sphere with 1 or 2 marked points and that each connected component of S contains at least one element of M. Let $\mathcal{T}$ be a

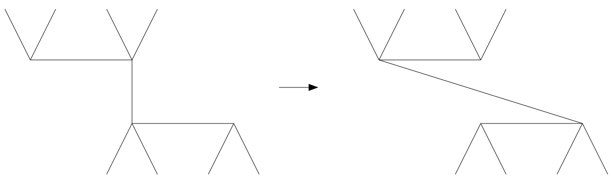

Figure 1. The Kauer move

triangulation of (S, M). Then $\mathcal{T}$ can be regarded as a Brauer graph, with cyclic ordering induced by the orientation of S. Then we have the following, by applying Theorem 1.

Proposition 2. The Brauer graph algebra $B_\mathcal{T}$ of a triangulation $\mathcal{T}$ of (S, M) does not depend on the choice of $\mathcal{T}$ up to derived equivalence.

Any Brauer graph, in the above sense, can be regarded as a ribbon graph. Such a graph can be embedded into a surface in such a way that the complement is a disjoint union of disks (a filling embedding), unique up to homeomorphism (see [3]). A similar result holds in the case of surface with boundary.

Proposition 3. Let (S, M) be a marked oriented surface with boundary in which every element of M lies on the boundary of S. Let $\iota : \Gamma \to S$ be a filling embedding of a ribbon graph Γ with boundary into S such that the set of images of the vertices of Γ coincides with M. Then applying the Kauer move to Γ at an edge e in the case where $\iota(e)$ is not on the boundary coincides with twisting $\iota(e)$ with respect to the set of remaining edges of $\iota(\Gamma)$ which are not on the boundary in the sense of [4, §3].

Given a triangulation $\mathcal{T}$ of a disk with n marked points on its boundary, we can consider the dual tree $\mathcal{T}^*$ whose vertices correspond to the triangles in $\mathcal{T}$ together with one vertex for each boundary edge of $\mathcal{T}$. Two vertices of the first kind are linked by an edge if and only if the corresponding triangles are adjacent. A vertex corresponding to a triangle in $\mathcal{T}$ and a vertex corresponding to a boundary edge are linked if and only if the boundary edge bounds the triangle.

Regarding $\mathcal{T}^*$ as a Brauer graph, the Kauer move (or triangle flip) applied to an internal edge of $\mathcal{T}$ induces a dual move on $\mathcal{T}^*$. By [6] we must have that $B_{\mathcal{T}^*}$ and $B_{(\mathcal{T}')^*}$ are derived equivalent. The following gives an explicit tilting complex giving this derived equivalence.

Theorem 4. *Let $\mathcal{T}, \mathcal{T}'$ be triangulations of a disk with n marked points, with $\mathcal{T}'$ obtained from $\mathcal{T}$ by applying a Kauer move at an internal edge of $\mathcal{T}$. Let a denote the edge in $\mathcal{T}^*$ dual to this edge. Let b, c be the edges in $\mathcal{T}^*$ following a in the clockwise ordering at its endpoints. Let $T = \oplus_i T_i$, where i ranges over all edges of $\mathcal{T}^*$, be defined by*

$$T_i = \begin{cases} P_i \to P_a & i = b, c; \\ P_i \to 0 & \begin{array}{l}\textit{if } i \textit{ lies in the connected component of } \mathcal{T}^* \setminus \{b\} \textit{ (or} \\ \textit{of } \mathcal{T}^* \setminus \{c\}\textit{) not containing } a;\end{array} \\ 0 \to P_i & \textit{otherwise,} \end{cases}$$

were P_i denotes the indecomposable projective $B_{\mathcal{T}^*}$-module associated to i. Then T is an Okuyama-Rickard tilting complex inducing a derived equivalence between $B_{\mathcal{T}^*}$ and $B_{(\mathcal{T}')^*}$.

A generalization to m-angulations of a disk also holds. We note, however, that this result does not generalize to the case of a punctured disk. It also does not hold if $\mathcal{T}^*$ is replaced by the dual graph of $\mathcal{T}$.

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Derived Gabriel topology and dg-completions.

Hiroyuki Minamoto

Gabriel topology is a special class of linear topology on rings, which plays an important role in the theory of localization of (not necessary commutative) rings[7]. Several evidences including derived completion theorem which is explained below, have suggested that there should be a corresponding notion for dg-algebras. In this talk I will introduce a notion of Gabriel topology on dg-algebras, derived Gabriel topology, and show its basic properties.

In the same way as the definition of derived Gabriel topology on a dg-algebra, we give the definition of topological dg-modules over a dg-algebra equipped with derived Gabriel topology. An important example of topology on dg-modules is the finite topology on the bi-dual module $M^{**} := \mathbb{R}\mathrm{Hom}_E(\mathrm{Hom}_A(M, J), J)$ of a dg-module M by another dg-module J.

We show that M^{**} equipped with the finite topology is the completion of M equipped with J-adic topology. From the view point of derived Gabriel topology this is a derived version of the results of J. Lambek [3, 4]. However our formulation is new: the derived bi-duality module M^{**} is quasi-isomorphic to the homotopy limit of a certain tautological diagram. This is a simple observation, which seems to be true in wider context. However this provide us a fundamental understanding of derived bi-duality functor.

We give applications. 1. we give a generalization and an intuitive proof of derived completion theorem which asserts that the completion of commutative ring satisfying some conditions is obtained as a derived bi-commutator [1, 2]. 2. We prove that every smashing localization of dg-category is obtained as a derived bi-commutator of some pure injective module. This is a derived version of the classical results in localization theory of ordinary rings. (A part of this talk was based on the preprint [5].)

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Quantum groups via cyclic quiver varieties

Fan Qin

For any given symmetric Cartan datum, let $\mathfrak{g}$ be the associated Kac-Moody Lie algebra and $U_t(\mathfrak{g})$ the corresponding quantized enveloping algebra. There have been several different approaches to the categorical realizations of $U_t(\mathfrak{g})$. In this talk, we present a new geometric approach based on certain cyclic quiver varieties.

The earliest and best developed theories are categorifications of a one-half quantum group. Notice that $U_t(\mathfrak{g})$ has the triangular decomposition $U_t(\mathfrak{g}) = U_t(\mathfrak{n}^+) \otimes U_t(\mathfrak{h}) \otimes U_t(\mathfrak{n}^-)$. Let Q denote a quiver associated with $\mathfrak{g}$ which has no oriented cycles. For any field k, let kQ denote the path algebra associated with Q.

In 1990, Ringel showed in [9] that the positive (resp. negative) one-half quantum group $U_t(\mathfrak{n}^+)$ (resp. $U_t(\mathfrak{n}^-)$) can be realized as a subalgebra of the *Hall algebra* of the abelian category $\mathbb{F}_qQ-$mod, where $\mathbb{F}_q$ is any finite field.

On the other hand, Lusztig has given a geometric construction of $U_t(\mathfrak{n}^+)$, cf. [7] [8], by considering the Grothendieck ring arising from certain perverse sheaves over the varieties of $\mathbb{C}Q-$mod. This geometric approach is very powerful. In particular, he obtained a positive basis[1] of $U_t(\mathfrak{n}^+)$, which is called *the canonical basis*, cf. also [4] for the *crystal basis*.

Remark 1. Assume that $\mathfrak{g}$ if of type $\mathbb{ADE}$. Hernandez and Leclerc showed that the varieties of $\mathbb{C}Q-$mod are isomorphic to some graded quiver varieties, cf. [2].

Recall that the quantum group $U_t(\mathfrak{g})$ is generated by the Chevalley generators E_i, F_i, K_i, K_{-i}, subject to $K_iK_{-i} = 1$ and other relations. We can remove the relation $K_iK_{-i} = 1$ and denote the corresponding algebra by $\tilde{U}_t(\mathfrak{g})$. It is a variant of the whole quantum group $U_t(\mathfrak{g})$ and has the triangle decomposition $U_t(\mathfrak{n}^+) \otimes \tilde{U}_t(\mathfrak{h}) \otimes U_t(\mathfrak{n}^-)$. The whole quantum group $U_t(\mathfrak{g})$ is obtained from $\tilde{U}_t(\mathfrak{g})$ by a reduction at the Cartan part $\tilde{U}_t(\mathfrak{h})$. This algebra naturally appears in the work of Bridgeland [1].

There have been various attempts to make a Hall algebra construction of the whole quantum group, cf. for example [3], [11], [12] [16]. The complete result was

[1] By a positive basis, we mean a basis whose structure constants are non-negative.

obtained in the recent work of Bridgeland [1], where the localization of $\tilde{U}_{\sqrt{q}}(\mathfrak{g})$ at the Cartan part $\tilde{U}_{\sqrt{q}}(\mathfrak{h})$ is realized as the localization of the Hall algebra of the 2-periodic complexes of projective $\mathbb{F}_q Q-$ mod at the contractible complexes.

Remark 2. By the work of Keller and Scherotzke [6, Theorem 2.7], graded quiver varieties are closely related to derived categories. The 2-periodic analog of derived categories are those used by Bridgeland. Therefore, we would like to consider the "2-periodic version" of the graded quiver varieties. These should be some cyclic quiver varieties.

Our main result is the following theorem.

Theorem 3 (Main result)**.** Assume the Lie algebra $\mathfrak{g}$ is of Dynkin type $\mathbb{A}$, $\mathbb{D}$, $\mathbb{E}$. Let h be the Coxeter number of $\mathfrak{g}$ and choose $q = e^{\frac{2\pi i}{2h}} \in \mathbb{C}^*$. Then by choosing special dimension vectors w, we can establish an algebra isomorphism

$$\oplus_w K_0^*(\mathcal{M}_0^q(w)) \otimes \mathbb{Q}(\sqrt{t}) \xrightarrow{\sim} \tilde{U}_t(\mathfrak{g}) \otimes \mathbb{Q}(\sqrt{t}),$$

where $K_0^*(\mathcal{M}_0^q(w))$ denotes the dual of the Grothendieck group of certain perverse sheaves on the cyclic quiver variety $\mathcal{M}_0^q(w)$.

In particular, we obtain a positive basis of $\tilde{U}_t(\mathfrak{g})$, which, up to rescaling by powers of $\sqrt{t}$, contains the dual canonical basis of a one-half quantum group $U_t(\mathfrak{n}^+)$ with respect to Lusztig's bilinear form.

Remark 4. In the work of Bridgeland, the Cartan part $\tilde{U}_t(\mathfrak{h})$ is generated by certain complexes homotopic to zero, which are redundant information in the study of the corresponding triangulated category.

In our work, we construct the Cartan part $\tilde{U}_t(\mathfrak{h})$ from certain strata of cyclic quiver varieties. The analog of these strata for graded quiver varieties provides redundant information in the study of quantum affine algebras [10].

Finally, it should be mentioned that the quiver Hecke algebras provide categorification of the modified quantum group $\dot{U}_t(\mathfrak{g})$, which is a different variant of the whole quantum group $U_t(\mathfrak{g})$. This approach was studied by Khovanov, Lauda, Rouquier, and for any symmetrizable type by Ben Webster, cf. [5] [13] [14] [15].

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Intersection cohomology of quiver moduli

Markus Reineke
(joint work with S. Meinhardt)

Let Q be an acyclic quiver with set of vertices Q_0 and Euler form $\langle \_, \_ \rangle$ on $\mathbf{Z}Q_0$. Let Θ be a stability, that is, a functional on $\mathbf{Q}Q_0$. This induces a slope function $\mu : \Lambda^+ \to \mathbf{Q}$ on the set $\Lambda^+ = \mathbf{N}Q_0 \setminus 0$ of non-zero dimension vectors; for a fixed $\mu \in \mathbf{Q}$, let Λ^+_μ be the set of dimension vectors of slope μ.

The slope of a non-zero complex representation V of Q is defined as $\mu(V) = \mu(\underline{\dim} V)$. The representation V is called semistable (resp. stable) if $\mu(U) \leq \mu(V)$ (resp. $\mu(U) < \mu(V)$) for every non-zero proper subrepresentation $U \subset V$. The semistable representations of a fixed slope $\mu \in \mathbf{Q}$ form a full abelian subcategory of $\mathrm{mod}\,\mathbf{C}Q$ whose simple objects are precisely the stables of slope μ. Its semisimple objects are called polystable of slope μ, these are thus direct sums of stables of the same slope.

By [2], for every $d \in \Lambda^+$, there exists a projective complex variety $M_d^{\mathrm{pst}}(Q)$ parametrizing isomorphism classes of polystable representations of dimension vector d. In case there exists a stable representation of dimension vector d, this is an irreducible variety of dimension $1 - \langle d, d \rangle$. If d is Θ-coprime, in the sense that $\mu(e) \neq \mu(d)$ for all non-zero $e \lneqq d$, then M_d^{pst} is a smooth variety; otherwise, it is typically singular.

It is an interesting problem to compute topological and geometric invariants of these moduli spaces. In the Θ-coprime case, the following was proved in [5]:

Theorem 1: For $d \in \Lambda^+$, define

$$P_d(q) = \sum_{d^*} (-1)^{s-1} q^{-\sum_{k\leq l}\langle d^l, d^k\rangle} \prod_{k=1}^{s} \prod_{i\in Q_0} \prod_{j=1}^{d_i^k} (1-q^{-j})^{-1} \in \mathbf{Q}(q),$$

where the sum ranges over all decompositions $d = d^1 + \ldots + d^s$ such that $d^k \neq 0$ for all k and $\mu(d^1 + \ldots + d^k) > \mu(d)$ for all $k < s$. If d is Θ-coprime, the Poincaré polynomial in singular cohomology of $M_d^{\mathrm{pst}}(Q)$ (with rational coefficients) is given by

$$\sum_{i\in\mathbf{Z}} \dim H^i(M_d^{\mathrm{pst}}(Q), \mathbf{Q}) q^{i/2} = (q-1) \cdot P_d(q).$$

For general d, and thus singular $M_d^{\mathrm{pst}}(Q)$, it is desirable to compute the Poincaré polynomial in intersection cohomology (again with rational coefficients). Under a mild genericity assumption on Θ, this turns out to be possible, again using the rational functions $P_d(q)$ defined in the previous theorem.

For fixed $\mu \in \mathbf{Q}$, consider the formal power series ring $\mathbf{Q}(q^{\frac{1}{2}})[[\Lambda_\mu^+]]$, and define an operator Log on series with constant term 1 by $\mathrm{Log}(1 - q^i X^d) = -q^i X^d$ and $\mathrm{Log}(P \cdot Q) = \mathrm{Log}(P) + \mathrm{Log}(Q)$.

Using methods from motivic Donaldson-Thomas theory [3] and an application of the Beilinson-Bernstein-Deligne-Gabber decomposition theorem applied to the Hilbert-Chow map [1] to $M_d^{\mathrm{pst}}(Q)$, the following is proved in [4]:

Theorem: Assume that the restriction of the Euler form $\langle \_, \_ \rangle$ to Λ_μ^+ is symmetric, and that there exist stable representations of the dimension vectors of slope μ. Then

$$\sum_{d\in\Lambda_\mu^+} (-q^{\frac{1}{2}})^{\langle d,d\rangle} \sum_{i\in\mathbf{Z}} \dim \mathrm{IH}^i(M_d^{\mathrm{pst}}(Q), \mathbf{Q}) q^{i/2} X^d =$$

$$= (q-1)\mathrm{Log}(1 + \sum_{d\in\Lambda_\mu^+} (-q^{\frac{1}{2}})^{\langle d,d\rangle} P_d(q) x^d).$$

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Auslander varieties for wild algebras

Claus Michael Ringel

1. Quiver Grassmannians and Auslander varieties.

Let k be an algebraically closed field and Λ a finite-dimensional k-algebra. Let $\mathrm{mod}\,\Lambda$ be the category of left Λ-modules of finite length (we call them just modules). A *dimension vector* $\mathbf{d}$ for Λ is a function defined on the set of isomorphism classes of simple modules S with non-negative integral values $\mathbf{d}_S$. If M is a module, its dimension vector $\mathbf{dim}\,M$ attaches to the simple module S the Jordan-Hölder multiplicity $(\mathbf{dim}\,M)_S = [M : S]$.

Given a module M, let $\mathbb{G}_\mathbf{e}M$ be the set of all submodules of M with dimension vector $\mathbf{e}$, this is called a quiver Grassmannian, it is always a projective variety. If we denote by $\mathcal{S}M$ be the set of the submodules of M, then $\mathcal{S}M$ is the disjoint union of (finitely many) subsets $\mathbb{G}_\mathbf{e}M$. Note that $\mathcal{S}M$ is a lattice with respect to intersection and sum, and the subsets $\mathbb{G}_\mathbf{e}M$ consist of pairwise incomparable elements.

If C, Y are modules, then we consider $\mathrm{Hom}(C, Y)$ as a $\Gamma(C)$-module, where $\Gamma(C) = \mathrm{End}(C)^{\mathrm{op}}$. The easiest way to define the Auslander varieties for Λ is to say that they are just the quiver Grassmannians $\mathbb{G}_\mathbf{e}\,\mathrm{Hom}(C, Y)$. This is the fast track definition, but it conceales the relevance of the Auslander varieties.

In order to provide the motivation, we have to outline Auslander's theory of C-determination of morphisms, developed already in 1974 (see [1], and also [4]). We assume now only that Λ is an artin algebra. The aim of Auslander's theory is to describe the global directedness of the category $\mathrm{mod}\,\Lambda$.

Let Y be a module. Let $\bigcup_X \mathrm{Hom}(X, Y)$ be the class of all morphisms ending in Y. We define a preorder $\preceq$ on this class as follows: Given $f : X \to Y$ and $f' : X' \to Y$, we write $f \preceq f'$ provided there is a morphism $h : X \to X'$ with $f = f'h$. As usual, such a preorder defines an equivalence relation by saying that f, f' are *right equivalent* provided we have both $f \preceq f'$ and $f' \preceq f$.

The object studied by Auslander is the **set** $[\to Y\rangle$ **of right equivalence classes of maps ending in** Y (it should be stressed that it is a set, not only a class). Using the preorder $\preceq$, this set $[\to Y\rangle$ is a poset, even a lattice (for the joins, one uses direct sums, for the meets, one uses pullbacks). The map $0 \to Y$ is the zero element of $[\to Y\rangle$, the identity map $Y \to Y$ is its unit element.

Recall that a map $f : X \to Y$ is said to be *right minimal* provided any direct summand X' of X with $f(X') = 0$ is equal to zero. Every right equivalence class in $[\to Y\rangle$ contains a right minimal morphism.

Let $f : X \to Y$ be a morphism and C a module. Then f is said to be *right C-determined* provided the following condition is satisfied: given any morphism $f' : X' \to Y$ such that $f'\phi$ factors through f for all $\phi : C \to X'$, then f' itself factors through f. We denote by ${}^C[\to Y\rangle$ the subset of $[\to Y\rangle$ of sll right equivalence classes of right C-determined morphisms.

Here are the main assertions of Auslander:

(1) *The set* $[\to Y\rangle$ *is the union of the subsets* ${}^{C}[\to Y\rangle$*.* If C, C' are modules, both ${}^{C}[\to Y\rangle$ and ${}^{C'}[\to Y\rangle$ are contained in ${}^{C\oplus C'}[\to Y\rangle$, thus we deal with a filtered union. The essential assertion is that any morphism is right determined by some module.

(2) Let C, Y be modules. *There is a lattice isomorphism*

$$\eta_{CY} : {}^{C}[\to Y\rangle \longrightarrow \mathcal{S}\operatorname{Hom}(C, Y)$$

defined as follows: if $f : X \to Y$, then $\eta_{CY}(f)$ is the image of $\operatorname{Hom}(C, f) : \operatorname{Hom}(C, X) \to \operatorname{Hom}(C, Y)$. The essential assertion is again the surjectivity of η_{CY}, thus to say that any $\Gamma(C)$-submodule of $\operatorname{Hom}(C, Y)$ is of the form $\Im\operatorname{Hom}(C, f)$. The isomorphisms η_{CY} are called the **Auslander bijections.**

The isomorphism η_{CY} allows to shift properties from $\mathcal{S}\operatorname{Hom}(C, Y)$ to ${}^{C}[\to Y\rangle$. Many properties of submodule lattices are known, all can be transfered via η_{CY} to ${}^{C}[\to Y\rangle$. It is a modular lattice (thus ${}^{C}[\to Y\rangle$ is a modular lattice): The modules M we are dealing with have finite length, we denote the length of M by $|M|$. The Jordan-Hölder theorem asserts that all composition series have the same length and given two composition series, there is a bijection between the composition factors. Via the transfer, we have a corresponding Jordan-Hölder theorem for ${}^{C}[\to Y\rangle$: given a right C-determined map f ending in Y, we can define its C-length $|f|_C = |\operatorname{Hom}(C, Y)| - |\eta_{CY}(f)|$. The C-length of f can also be defined directly, looking at suitable factorizations of f. Given a factorization $f = f'h$, where f, f' are right C-determined maps ending in Y with $|f|_C = |f'|_C + 1$, then $\eta_{CY}(f) \subset \eta_{CY}(f')$ and the factor $\eta_{CY}(f')/\eta_{CY}(f)$ is a simple $\Gamma(C)$-module. Thus, the Jordan-Hölder theorem for ${}^{C}[\to Y\rangle$ allows to attach to any right C-determined map its C-dimension vector.

Let us return to the case where Λ is a finite-dimensional k-algebra and k is an algebraically closed field. If C, Y are modules, we use the Auslander bijection $\eta_{CY} : {}^{C}[\to Y\rangle \longrightarrow \mathcal{S}\operatorname{Hom}(C, Y)$. Given a dimension vector $\mathbf{e}$ for $\Gamma(C)$, the elements of the Auslander variety $\mathbb{G}_{\mathbf{e}}\operatorname{Hom}(C, Y)$. correspond under η_{CY} to the right equivalence classes of maps ending in Y with C-dimension vector $\mathbf{e}$.

2. (Controlled) wildness

According to Drozd, any finite dimensional k-algebra is either tame or wild (and most algebras are wild). It has been conjectured that wild algebras are actually controlled wild (as defined below). A proof of this conjecture has been announced by Drozd [2] in 2007, but apparently it has not yet been published.

Let rad be the radical of $\operatorname{mod}\Lambda$, this is the ideal generated by all non-invertible maps between indecomposable modules. If $\mathcal{U}$ is a collection of objects of $\operatorname{mod}\Lambda$, we denote by $\operatorname{add}\mathcal{U}$ the closure under direct sums and direct summands. For every pair X, Y of modules, $\operatorname{Hom}(X, \mathcal{U}, Y)$ denotes the subgroup of $\operatorname{Hom}(X, Y)$ given by the maps $X \to Y$ which factor through a module in $\operatorname{add}\mathcal{U}$.

The algebra Λ is said to be *controlled wild* provided for any finite-dimensional k-algebra Γ (or, equivalently, just for the algebra $\Gamma = k[T_1, T_2, T_3]/(T_1, T_2.T_3)^2$) there is an exact embedding functor $F : \operatorname{mod}\Gamma \to \operatorname{mod}\Lambda$ and a full subcategory $\mathcal{U}$

of $\operatorname{mod}\Lambda$ (called the *control class*) such that for all Γ-modules X, Y, the subgroup $\operatorname{Hom}(FX,\mathcal{U},FY)$ is contained in $\operatorname{rad}(FX,FY)$ and

$$\operatorname{Hom}(FX,FY) = F\operatorname{Hom}(X,Y) \oplus \operatorname{Hom}(FX,\mathcal{U},FY).$$

3. Quiver Grassmannians

A recent paper of Reineke [3] asserts: *Every projective variety is a quiver Grassmannian $\mathbb{G}_{\mathbf{e}}M$ for a module M with endomorphism ring k.*

Let us outline a construction. Let $\mathcal{V}$ be a projective variety, say a closed subset of the projective space $\mathbb{P}^n$, defined by the vanishing of homogeneous polynomials $f_1,\dots,f_m$ of degree 2. Let Δ be the quiver with 3 vertices a,b,c, with $n+1$ arrows $b \to a$ labeled $x_0,\dots,x_n$ as well as $n+1$ arrows $c \to b$, also labeled $x_0,\dots,x_n$. The path algebra of Δ with all possible relations $x_ix_j = x_jx_i$ is called the Beilinson algebra B. Let Λ be the factor algebra of B taking the elements $f_1,\dots,f_m$ as additional relations (considered as linear combinations of paths of length 2). Let I be the indecomposable injective B-module corresponding to the vertex a, and take $\mathbf{e} = (1,1,1)$. Now $\mathbb{G}_{\mathbf{e}}I$ is the set of all serial submodules of I of length 3 (a module is serial, provided it has a unique composition series). There is an obvious identification of this set $\mathbb{G}_{\mathbf{e}}I$ with $\mathbb{P}^n$. Let M be the indecomposable injective Λ-module corresponding to the vertex a. Then M is a submodule of I. Also, a submodule W of I is a submodule of M if and only if W is a Λ-module. Thus the serial submodules W of M of length 3 correspond just to the elements of $\mathcal{V}$. One may say that this construction is really tautological.

Here are some remarks on the history: The 2-page paper by Reineke attracted a lot of interest, see for example blogs by L. Le Bruyn and by J. Baez. The construction given above was presented by M. Van den Bergh in Le Bruyn's blog, but actually, it is much older: it has been used before by B. Huisgen-Zimmermann (1998) and L. Hille (1996) dealing with related problems.

There are controlled wild algebras Λ such that not every projective variety can be realized as a quiver Grassmannian of a Λ-module.

As an example, take $\Lambda = k[T_1,T_2,T_3]/(T_1,T_2,T_3)^2$. One can show that $\mathbb{G}_iM$ is rationally connected, for every module M and any $0 \le i \le \dim M$.

4. Auslander varieties

Theorem. *Let Λ be a finite-dimensional k-algebra which is controlled wild. Let $\mathcal{V}$ be any projective variety. Then there are Λ-modules C, Y and a dimension vector $\mathbf{e}$ for $\Gamma(C)$ such that $\mathbb{G}_{\mathbf{e}}\operatorname{Hom}(C,Y)$ is of the form $\mathcal{V}$.*

Outline of proof. Let $\mathcal{V}$ be a projective variety. There is a finite-dimensional algebra Γ, a Γ-module M and a dimension vector $\mathbf{g}$ for Γ such that $\mathbb{G}_{\mathbf{g}}M$ is of the form $\mathcal{V}$, as we have seen in section 3. Since Λ is controlled wild, there is a controlled embedding $F : \operatorname{mod}\Gamma \to \operatorname{mod}\Lambda$, say with control class $\mathcal{U}$. Let $G = F({}_{\Gamma}\Gamma)$ and $Y = F(M)$. There is $U \in \operatorname{add}\mathcal{U}$ such that $\operatorname{Hom}(G,U,G) = \operatorname{Hom}(G,\mathcal{U},G)$ and $\operatorname{Hom}(G,U,Y) = \operatorname{Hom}(G,\mathcal{U},Y)$. Let $C = G \oplus U$ and $R = \operatorname{End}(C)^{\mathrm{op}}$. Let e_G be the projection of C onto G with kernel U and $e = e_U$ the projection of C onto U with kernel G, both e_G, e_U considered as elements of R. Note that

$$R = F(\mathrm{Hom}(\Gamma,\Gamma)) \oplus \mathrm{Hom}(G\oplus U, \mathcal{U}, G\oplus U) \oplus \mathrm{Hom}(G,U) \oplus \mathrm{Hom}(U,G) \oplus \mathrm{Hom}(U,U),$$

and

$$ReR = \mathrm{Hom}(G\oplus U, \mathcal{U}, G\oplus U) \oplus \mathrm{Hom}(G,U) \oplus \mathrm{Hom}(U,G) \oplus \mathrm{Hom}(U,U).$$

It follows that the map $\gamma \mapsto F(\gamma) \in e_G R e_G$ yields an isomorphism $\Gamma \to R/ReR$. Consider the R-module

$$N = \mathrm{Hom}(G\oplus U, Y) = \mathrm{Hom}(\Gamma, M) \oplus \mathrm{Hom}(G\oplus 0, U, Y) \oplus \mathrm{Hom}(0\oplus U, Y).$$

If we multiply N with the element $e = e_U \in R$, we obtain $eN = \mathrm{Hom}(0\oplus U, Y)$, thus

$$ReN = R\,\mathrm{Hom}(0\oplus U, Y) = \mathrm{Hom}(G\oplus 0, U, Y) \oplus \mathrm{Hom}(0\oplus U, Y).$$

This shows that N/ReN is canonically isomorphic to $F\,\mathrm{Hom}(\Gamma, M)$ as an R-module. Of course, these modules are annihilated by e, thus they are R/ReR-modules and as we know $R/ReR = \Gamma$, thus $\mathcal{S}_\Gamma(N/ReN)$ can be identified with $\mathcal{S}_R M$.

Let $\mathbf{c} = \mathbf{dim}\, ReN$. If $\mathbf{g}$ is a dimension vector and W belongs to $\mathbb{G}_{\mathbf{g}+\mathbf{c}}N$, then $W \supseteq ReN$, and W/ReN is an element of $\mathbb{G}_{\mathbf{g}}(N/ReN)$. As a consequence, the varieties $\mathbb{G}_{\mathbf{g}+\mathbf{c}}N$ and $\mathbb{G}_{\mathbf{g}}(N/ReN) = \mathbb{G}_{\mathbf{g}}M = \mathcal{V}$ can be identified.

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Positivity of Kac polynomials

Fernando Rodriguez Villegas

(joint work with Emmanuel Letelier and Tamas Hausel)

This talk was a companion to that of Emmanuel Letellier also at this workshop. Its goal was to give some context for our completion of the proof of Kac's conjecture [8] on the positivity of the coefficients of the Kac polynomial. This polynomial counts the number of absolutely indecomposable representations of a quiver of fixed dimension over a finite field. Our proof extends the work of Crawley-Boevey and van der Bergh [2] who proved the conjecture in the case that the dimension vector is indivisible. For quivers with at least one loop at every vertex Mozgovoy [10] proved Kac's conjecture for any dimension vector (via Efimov's proof [3] of a conjecture of Kontsevich–Soibelman [9]).

We fix a k-tuple of partitions $\mu = (\mu^1, \ldots, \mu^k)$ of positive integer $n = |\mu^i|$. Associated to this data we have two spaces of representations.

Multiplicative case

Let Σ be a Riemann surface of genus g and $S \subseteq \Sigma$ a set of k points in Σ, which we will call *punctures*. Pick semi-simple conjugacy classes $C_1, \ldots, C_k \subseteq \mathrm{GL}_n(\mathbb{C})$, where the multiplicity of the eigenvalues of C_i (in some ordering) are given by μ^i. Consider the space $\mathcal{M}_\mu$ of representations of the fundamental group $\pi_1(\Sigma \setminus S, \cdot)$ of dimension n, where a small oriented loop around the i-th puncture is mapped to C_i. With the appropriate notion of quotient this *character variety* is given by

$$\mathcal{M}_\mu = \mathrm{Hom}_C(\pi_1(\Sigma \setminus S, \cdot), \mathrm{GL}_n(\mathbb{C}))//\mathrm{GL}_n(\mathbb{C})$$

(the sufix C indicating the required condition at the punctures), where GL_n acts by conjugation. The space depends on the choice of eigenvalues of C_i but most of what we have to say is independent of this choice as long as it is *generic* (a notion that can be made very precise but we skip in this short note).

By using the standard presentation of π_1 we may identify $\mathcal{M}_\mu$ with the solutions to the matrix equation

$$(1) \qquad (x_1, y_1) \cdots (x_g, y_g) z_1 \cdots z_k = I_n, \qquad z_i \in C_i,$$

where $(x, y) := xyx^{-1}y^{-1}$ and I_n is the $n \times n$ identity matrix, up to the diagonal action of GL_n by conjugation in each variable.

Additive case

Let Q be the quiver consisting of one central node at which we attach g loops and k arbitrary long legs. To μ we associate a dimension vector for Q, also denote by μ, defined as follow. We put n at the central node and moving away from it along the i-th leg, we put $n - \mu^i_1, n - \mu^i_1 - \mu^i_2, \ldots$. Eventually we reach all zeroes and hence we are really considering the dimension vector of a finite quiver. It is just easier, in terms of notation, not to make this explicit.

We choose semi-simple adjoint orbits $\mathcal{O}_1, \ldots, \mathcal{O}_k$ in $\mathrm{gl}_n(\mathbb{C})$ whose eigenvalues have multiplicities given by μ^i. Assuming for the time being that μ is indivisible this choice can be made *generic*. (The same comments as in the multiplicative case apply here.)

Then by work of Crawley-Boevey [1] an appropriate preimage of the moment map of the double quiver is parametrized by solutions to the equation

$$(2) \qquad [x_1, y_1] + \cdots + [x_g, y_g] + z_1 + \cdots + z_k = 0, \qquad z_i \in \mathcal{O}_i,$$

where $[x, y] := xy - yx$. The corresponding quotient by the diagonal action of GL_n by conjugation in each variable yields a *Nakajima quiver variety* $\mathcal{Q}_\mu$.

Under the genericity assumption both $\mathcal{M}_\mu$ and $\mathcal{Q}_\mu$ are smooth affine varieties; if non-empty they have dimension

$$d_\mu := (2g-2+k)n^2 - \sum_{i,j}\left(\mu_i^j\right)^2 + 2.$$

For example, if $g=0, k=4, n=2, \mu^1=\mu^2=(1^2)$ then $d_\mu = 2$. Here each class C_i consists of diagonazible matrices with distinct eigenvalues. It is a classical fact going back to Fricke that $\mathcal{M}_\mu$ is a smooth projective cubic surface with a triangle of lines removed. One can also check that $\mathcal{Q}_\mu$ is in this case a smooth projective cubic surface with three lines meeting at a point removed.

One of our first goals is to count points of the varieties $\mathcal{M}_\mu$ and $\mathcal{Q}_\mu$ over finite fields $\mathbb{F}_q$. For this we use a Fourier decomposition of the function that gives the number of solutions to the left hand side of the equations (1) and (2) equal to a fixed element of $\mathrm{GL}_n(\mathbb{F}_q)$ or $\mathrm{gl}_n(\mathbb{F}_q)$ respectively.

From [2] we know that $\#\mathcal{Q}_\mu(\mathbb{F}_q) = q^{\frac{1}{2}d_\mu}A_\mu(q)$, where A_μ is the Kac polynomial of Q with dimension vector μ. It turns out that also $\#\mathcal{M}_\mu(\mathbb{F}_q)$ is a polynomial in q. Our calculation of the number of points can be encoded in the following generating function.

Given a partition λ we define

$$\mathcal{H}_\lambda(z,w) := \prod_{s\in\lambda} \frac{\left(z^{2a(s)+1} - w^{2l(s)+1}\right)^{2g}}{\left(z^{2a(s)+2} - w^{2l(s)}\right)\left(z^{2a(s)} - w^{2l(s)+2}\right)},$$

where the product is over all cells s of λ with $a(s)$ and $l(s)$ its arm and leg length, respectively. Let

$$\Omega(\mathbf{x}_1,\ldots,\mathbf{x}_k; z,w) := \sum_\lambda \mathcal{H}_\lambda(z,w) \prod_{i=1}^k \tilde{H}_\lambda(\mathbf{x}_i; z^2, w^2),$$

where $\mathbf{x}_1,\ldots,\mathbf{x}_k$ are k sets of infinitely many independent variables and $\tilde{H}_\lambda$ are the Macdonald polynomials. Finally, define $\mathbb{H}_\mu(z,w)$ (a priori rational functions of z,w) by the expansion

$$(z^2-1)(1-w^2)\operatorname{Log}\Omega(z,w) = \sum_\mu \mathbb{H}_\mu(z,w)m_\mu(\mathbf{x}), \tag{3}$$

where $\mathbf{x} = (\mathbf{x}_1,\ldots,\mathbf{x}_k)$ and $m_\mu(\mathbf{x}) := m_{\mu^1}(\mathbf{x}_1)\cdots m_{\mu^k}(\mathbf{x}_k)$ are the monomial symmetric functions.

The main corollary of our calculation is that for the multiplicative case

$$\#\mathcal{M}_\mu(\mathbb{F}_q) = q^{\frac{1}{2}d_\mu}\mathbb{H}_\mu(\sqrt{q}, 1/\sqrt{q})$$

and for the additive case

$$\#\mathcal{Q}_\mu(\mathbb{F}_q) = q^{\frac{1}{2}d_\mu}\mathbb{H}_\mu(0, \sqrt{q}).$$

We conjecture that the $\mathbb{H}_\mu(z,w)$ are polynomials of z,w that encode the mixed Hodge numbers of $\mathcal{M}_\mu$.

For our running example $\mathbb{H}_\mu(z,w) = z^2 + 4 + w^2$ and it is easy to check that indeed

$$\#\mathcal{M}_\mu(\mathbb{F}_q) = q^2 + 4q + 1, \qquad A_\mu(q) = q + 4.$$

Note that the dimension vector μ in this case gives the basic positive imaginary root of the affine $\tilde{D}_4$ diagram. It is also not difficult to check that the mixed Hodge numbers of $\mathcal{M}_\mu$ match the conjectured ones.

With the above result in mind we may compare the specialization $\Omega(\mathbf{x}_1,\ldots,\mathbf{x}_k;0,\sqrt{q})$ with Hua's general formula [7] for Kac's polynomials. We discover that the terms in his formula summed over partitions attached to the nodes of a leg gets replaced in Ω by a single Hall-Littlewood polynomial $\tilde{H}_\lambda(\mathbf{x};0,q)$.

It turns out that this process can in fact be applied to an arbitrary quiver. Given a quiver Q and a dimension vector v we do the following. We extend Q by adding long legs to each of its nodes. If v_i is the dimension at the i-th node we put dimensions $v_i - 1, v_i - 2, \ldots$ along the nodes of the attached leg. The new dimension vector is now indivisible and we may construct the corresponding generic Nakajima quiver variety $\mathcal{Q}_v$ of dimension d_v.

The Weyl group of the extended quiver contains a subgroup isomorphic to a product of symmetric groups $S_v := S_{v_1} \times \cdots$ coming from reflections at the nodes of the added legs. The group S_v acts on the cohomology of $\mathcal{Q}_v$ and the Kac polynomial A_v of the original quiver Q with dimension vector v can be recovered by looking at the sign isotypical component of this action.

More precisely, we have

$$A_v(q) = \sum_j \dim\left(H_c^{2j}(\mathcal{Q}_v;\mathbb{C})_\epsilon\right) q^{j-d_v},$$

where $\epsilon := \epsilon_1 \times \cdots$ and ϵ_i is the sign character on the i-th factor S_{v_i} of S_v. This clearly shows that the coefficients of A_v are non-negative.

As a bonus we discover that the analogous expression for the trivial character gives the DT-invariants of Q defined by Kontsevich and Soibelman [9]. They also, hence, have non-negative coefficients, a result first proved by Efimov [3].

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Periodic algebras of polynomial growth

Andrzej Skowroński
(joint work with Jerzy Białkowski and Karin Erdmann)

Let K be an algebraically closed field. By an algebra we mean an associative, finite-dimensional K-algebra with identity, and we denote by $\operatorname{mod} A$ the category of finite-dimensional right A-modules. An algebra A is called *self-injective* if A_A is an injective module in $\operatorname{mod} A$, or equivalently, the projective modules in $\operatorname{mod} A$ are injective. Any Frobenius algebra, and in particular any symmetric algebra is self-injective. An algebra is said to be *tame* if, for each positive integer d, all but finitely many (up to isomorphism) d-dimensional indecomposable modules in $\operatorname{mod} A$ come in a finite number of one-parameter families. The algebra A is said to be of *polynomial growth* if there exists a positive integer m such that, for each positive integer d, there are at most d^m such one-parameter families.

Let A be an algebra. Given a module M in $\operatorname{mod} A$, its *syzygy* is defined to be the kernel $\Omega_A(M)$ of a minimal projective cover $P_A(M) \to M$ of M in $\operatorname{mod} A$. A module M in $\operatorname{mod} A$ is said to be *periodic* if $\Omega_A^n(M) \cong M$ for some $n \geq 1$, and if so the minimal such n is called the period of M. An algebra A is said to be *periodic* if it is periodic as a module over the enveloping algebra $A^e = A^{\mathrm{op}} \otimes_K A$, or equivalently, as an A-A-bimodule. Periodicity of algebras is invariant under derived equivalence. Moreover, if A is a periodic algebra of period m, then for any indecomposable non-projective module M in $\operatorname{mod} A$ the syzygy $\Omega_A^m(M)$ is isomorphic to M. We also note that any periodic algebra A is self-injective. In fact, if all simple modules in $\operatorname{mod} A$ are periodic, then A is also self-injective (see [27]). It is conjectured that the following should be true:

$$\boxed{\text{simple modules in } \operatorname{mod} A \text{ are periodic}} \Rightarrow \boxed{A \text{ is a periodic algebra}}$$

This is known as the *periodicity conjecture*, and is an exciting open problem. It holds for example for blocks of group algebras of finite groups. Green, Snashall and Solberg proved in [19] that if all simple modules in $\operatorname{mod} A$ are periodic then some syzygy $\Omega_{A^e}^m(A)$ is isomorphic to a twisted bimodule ${}_1A_\sigma$, where σ is some K-algebra automorphism of A. However is not clear at all if such an automorphism σ has finite order.

Another exciting open problem is to describe all periodic algebras up to Morita equivalence and derived equivalence. A prominent class of periodic algebras is formed by the *preprojective algebras of Dynkin* [3, 18], or more generally the *deformed preprojective algebras of generalized Dynkin type* [7, 17]. We mention that the preprojective algebras of Dynkin type are the stable Auslander algebras of the categories of maximal Cohen-Macaulay modules of the Kleinian 2-dimensional hypersurface singularities [1, 2], and are mostly of wild representation type [16]. An interesting open problem is to find out if the stable Auslander algebra of any hypersurface singularity of finite Cohen-Macaulay type is periodic (see [17, Problem 8]). This was recently confirmed in [8, 9] for simple plane curve singularities of Dynkin type $\mathbb{A}_n$.

Motivated by blocks of group algebras, Erdmann introduced in [13] the *algebras of quaternion type*, which are tame algebras of non-polynomial growth. Applying the derived equivalence classification of all algebras of quaternion type by Holm [20], Erdmann and Skowroński proved in [16] that all these algebras are periodic of period 4. In particular, one obtains that a finite group G is periodic with respect to the group cohomology $H^*(G, \mathbb{Z})$ if and only if all blocks with nontrivial defect groups of the group algebras KG of G over all algebraically closed fields K are periodic. We also note that by a result of Swan [28] the periodic groups can be characterized as all finite groups acting freely on finite CW-complexes homotopically equivalent to spheres.

During the talk we have discussed the structure of periodic algebras of polynomial growth. In [14, 15] Erdmann, Holm and Snashall proved that all self-injective algebras of finite representation type and Dynkin type $\mathbb{A}_n$ are periodic. Later Brenner, Butler and King showed in [11] that all trivial extension algebras of the path algebras of Dynkin quivers are also periodic. Finally, Dugas proved in [12] that an arbitrary self-injective indecomposable algebra of finite representation type, not simple, is periodic. We mention that by general theory (see [26]) every basic, indecomposable, not simple, self-injective algebra A of finite representation type is socle equivalent to an orbit algebra $\widehat{B}/G$, where $\widehat{B}$ is the repetitive category of a tilted algebra B of Dynkin type and G is an admissible infinite cyclic automorphism group of $\widehat{B}$. Recall that two basic self-injective algebras A_1, A_2 are *socle equivalent* if the quotient algebras $A_1/\operatorname{soc}(A_1)$ and $A_2/\operatorname{soc}(A_2)$ are isomorphic.

We have presented the following description of the representation-infinite periodic algebras of polynomial growth, being the main result of our recent paper [10].

Theorem 1. *Let A be a basic, indecomposable, representation-infinite self-injective algebra of polynomial growth over an algebraically closed field K. Then the following statements are equivalent:*

- *(i) All simple modules in* $\operatorname{mod} A$ *are periodic;*
- *(ii) A is a periodic algebra;*
- *(iii) A is socle equivalent to an orbit algebra $\widehat{B}/G$ for a tubular algebra B and an admissible infinite cyclic automorphism group G of the repetitive category $\widehat{B}$ of B;*

(iv) A is socle equivalent to an algebra $(\mathrm{T}(B)^{(r)})^H$ *of invariants of an r-fold trivial extension algebra* $\mathrm{T}(B)^{(r)}$ *of a tubular algebra B with respect to free action of a finite automorphism group H of* $\mathrm{T}(B)^{(r)}$.

Moreover, if K is of characteristic different from 2 *and* 3*, we may replace in* (iii) *and* (iv) *"socle equivalent" by "isomorphic".*

Recall that the tubular algebras form a prominent class of algebras of polynomial growth whose representation theory has been described by Ringel [24]. We mention that, for any tubular algebra B, $\mathrm{gl.dim}\, B = 2$ and the Grothendieck group $K_0(B)$ of B has rank 6, 8, 9, or 10. We refer also to [4, 5, 6, 21, 22, 25, 26] for the structure and representation theory of self-injective algebras occurring in the above theorem.

By general theory the basic, indecomposable, self-injective algebras split into two classes: the standard algebras which admit simply connected Galois covering, and the remaining non-standard algebras. Then it follows from Theorem 1 that the standard periodic representation-infinite algebras of polynomial growth are exactly the orbit algebras $\widehat{B}/G$ of the repetitive categories $\widehat{B}$ of tubular algebras B with respect to actions of admissible infinite cyclic automorphism groups G of $\widehat{B}$, and the non-standard periodic representation-infinite algebras of polynomial growth are socle deformations of the corresponding periodic standard algebras. Moreover, the non-standard periodic representation-infinite algebras of polynomial growth occur only in characteristic 2 and 3 (see [6, 26]), and every such an algebra Λ is a geometric socle deformation of exactly one periodic representation-infinite standard algebra Λ' of polynomial growth, called the standard form of Λ. We have the following information on the dimensions of the low Hochschild cohomologies $HH^n(\Lambda)$ and $HH^n(\Lambda')$, $n = 0, 1, 2$, of these algebras.

Theorem 2. *Let* Λ *be a basic, indecomposable, non-standard, representation-infinite periodic algebra of polynomial growth over an algebraically closed field K, and* Λ' *the standard form of* Λ*. Then the following inequalities hold:*

(i) $\dim_K HH^0(\Lambda) = \dim_K HH^0(\Lambda')$;

(ii) $\dim_K HH^1(\Lambda) < \dim_K HH^1(\Lambda')$;

(iii) $\dim_K HH^2(\Lambda) < \dim_K HH^2(\Lambda')$.

We note that the dimensions of the Hochschild cohomologies are known to be invariants of the derived equivalences of algebras [23].

As an application of Theorems 1 and 2 and the theory of self-injective algebras of polynomial growth we obtain also the following general fact.

Theorem 3. *Let A be a standard self-injective algebra and* Λ *be a basic, indecomposable, non-standard, periodic, representation-infinite algebra of polynomial growth. Then A and* Λ *are not derived equivalent.*

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Gorenstein categories, singular equivalences and Finite generation of cohomology rings in recollements

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(joint work with Chrysostomos Psaroudakis, Øystein Skartsæterhagen)

For a finite dimensional algebra Λ over an algebraically closed field k, if the Hochschild cohomology ring $\mathrm{HH}^*(\Lambda)$ of Λ is noetherian and the Yoneda algebra of the direct sum of all simple Λ-modules up to isomorphism is a finitely generated $\mathrm{HH}^*(\Lambda)$-module, it is shown by Erdmann-Holloway-Snashall-Solberg-Taillefer [2] that this is sufficient to ensure the existence of a useful theory of support varieties for finitely generated Λ-modules. Call this condition Fg. By [2] an algebra Λ satisfying Fg must be Gorenstein, but any Gorenstein algebra doesn't satisfy Fg. It is open for which Gorenstein algebras Fg hold. The speaker has defined and considered support varieties of objects in $\mathscr{D}^b(\operatorname{mod}\Lambda)$ and showed that perfect complexes, $\operatorname{perf}(\Lambda)$, have trivial support varieties (see [4]). Hence the support theory really only says something about the quotient $\mathscr{D}^b(\operatorname{mod}\Lambda)/\operatorname{perf}(\Lambda)$ – the singularity category of Λ.

The motivation for this work was to understand how the property Fg behaves under taking the endomorphism ring of a projective Λ-module. A long the way we also investigate the intermediate notions of Gorensteinness and singularity categories. In particular, this means to compare Λ and $a\Lambda a$ for an idempotent a in Λ. To this end we introduce the following concept. Given a functor $f\colon \mathscr{B} \longrightarrow \mathscr{C}$ between abelian categories, the functor f is called an **eventually homological isomorphism** if there is an integer t such that for every pair of objects B and B' in $\mathscr{B}$, and every $j > t$, there is an isomorphism

$$\operatorname{Ext}^j_{\mathscr{B}}(B, B') \cong \operatorname{Ext}^j_{\mathscr{C}}(f(B), f(B')).$$

of abelian groups. Then the main results for artin algebras can be summarized as follows.

Main Theorem. *Let Λ be an artin algebra over a commutative ring k and let a be an idempotent element of Λ. Let e be the functor $a-\colon \operatorname{mod}\Lambda \longrightarrow \operatorname{mod} a\Lambda a$ given by multiplication by a. Consider the following conditions*:

$$(\alpha)\ \ \mathsf{id}_\Lambda\left(\frac{\Lambda/\langle a\rangle}{\operatorname{rad}\Lambda/\langle a\rangle}\right) < \infty \qquad\qquad (\beta)\ \ \mathsf{pd}_{a\Lambda a}\, a\Lambda < \infty$$

$$(\gamma)\ \ \mathsf{pd}_\Lambda\left(\frac{\Lambda/\langle a\rangle}{\operatorname{rad}\Lambda/\langle a\rangle}\right) < \infty \qquad\qquad (\delta)\ \ \mathsf{pd}_{(a\Lambda a)^{\mathrm{op}}}\, \Lambda a < \infty$$

Then the following hold.

(i) The following are equivalent:
  (a) (α) and (β) hold.
  (b) (γ) and (δ) hold.

(c) The functor e is an eventually homological isomorphism.

(ii) The functor e induces a singular equivalence between Λ *and* $a\Lambda a$ *if and only if the conditions* (β) *and* (γ) *hold.*

(iii) Assume that e is an eventually homological isomorphism. Then Λ *is Gorenstein if and only if* $a\Lambda a$ *is Gorenstein.*

(iv) Assume that e is an eventually homological isomorphism. Assume also that k is a field and that $(\Lambda/\operatorname{rad}\Lambda)\otimes_k(\Lambda^{\mathrm{op}}/\operatorname{rad}\Lambda^{\mathrm{op}})$ *is a semisimple* Λ^{e}*-module (for instance, this is true if k is algebraically closed). Then* Λ *satisfies* Fg *if and only if* $a\Lambda a$ *satisfies* Fg.

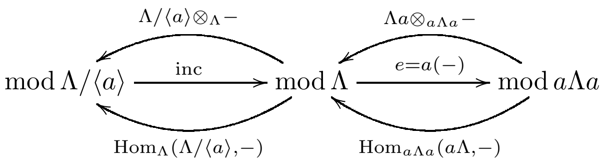
Remark. (1) The conditions (α) and (β) are independent. However, if (α_1) means that the dimension in (α) is less or equal to 1, then (α_1) implies (δ) (and similarly (γ_1) implies (β)).

(2) There is related work by Xiao-Wu Chen [1] for the statement in (ii). He shows, assuming condition (β), that e induces a singular equivalence between Λ and $a\Lambda a$ if and only if (γ) holds.

(3) Hiroshi Nagase have shown the statements in (iii) and (iv) under the assumption that $\langle a\rangle$ is a stratifying ideal and $\mathsf{pd}_{\Lambda^{\mathrm{e}}}\Lambda/\langle a\rangle<\infty$ (see [3]). One can show that these assumptions imply that e is an eventually homological isomorphism. In addition, there are examples which satisfy our conditions and where $\langle a\rangle$ is not a stratifying ideal.

(4) There is an example of an algebra Λ and an idempotent a, where Λ and $a\Lambda a$ are singular equivalent with $a\Lambda a$ satisfying Fg and Λ not Gorenstein. In particular Λ does not satisfy Fg. Hence, the singularity categories do not carry information about the property Fg.

(5) The underlying recollement for the above theorem is the following



The majority of results presented in the above theorem can be generalized to a recollement of abelian categories. See arXiv: 1402.1588 for further details.

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Abstract tilting theory using Grothendieck derivators

Jan Šťovíček
(joint work with Moritz Groth)

May [11] studied in 2001 the interaction between the triangulated and monoidal structure on a category, suggesting a collection of compatibility axioms which are widely satisfied. In other words, he tried to establish the axiomatics of tensor triangulated categories. A paper by Keller and Neeman [10] published a year later suggested that some of these complicated looking axioms have a lot to do with classical results of representation theory:

(1) the derived equivalence $D^b(\mathrm{mod}\, kD_4) \xrightarrow{\sim} D^b(\mathrm{mod}\, k\square)$, where k is a field and $k\square$ is the category algebra of a stand alone commutative square, and
(2) Happel's description [9] of the bounded derived category of $D^b(\mathrm{mod}\, kD_4)$.

They illustrated this connection on a class of suitably bounded derived categories of suitable k-linear abelian categories.

In a joint project with Moritz Groth [6, 7], we show that such a connection holds in any reasonable monoidal triangulated category, including for instance the stable homotopy category of topological spectra. The idea is to use derivators [4], which provide a sort of minimal enhancement of triangulated categories allowing for a well behaved calculus of homotopy limits and colimits. Along these lines we are also able to find analogues of other classical derived equivalences present in any suitable stable homotopy theory, demonstrating again how useful representation theory of finite dimensional algebras is in organizing facts about seemingly distant settings such as the homotopy theory of topological spectra.

As mentioned, the central concepts is that of a derivator. The main idea is present in work of Alex Heller, Bernhard Keller, Jens Franke and in a massive manuscript by Alexander Grothendieck [8]. If $\mathcal{A}$ is an abelian category, the philosophy is to consider not only its derived category $D(A)$, but also all the derived categories $D(\mathcal{A}^I)$, where $I \in \mathcal{C}at$ is a small category, and a whole lot of triangulated functors and transformations among them. The motivation is easy. If $\mathcal{A}$ is a cocomplete category and $I \in \mathcal{C}at$ a small category, we have the adjunction

$$\mathcal{A}^I \underset{\mathrm{const}}{\overset{\mathrm{colim}}{\rightleftarrows}} \mathcal{A},$$

where the right adjoint (below) sends each $X \in \mathcal{A}$ to a constant diagram and the left adjoint (above) is the colimit functor. If $\mathcal{A}$ is nice enough (a Grothendieck category for instance), we can take the corresponding diagram of total derived functors

$$D(\mathcal{A})^I \xleftarrow{\mathrm{diag}} D(\mathcal{A}^I) \underset{\mathrm{const}}{\overset{\mathrm{hocolim}}{\rightleftarrows}} D(\mathcal{A}).$$

The *homotopy colimit* functor hocolim is the left adjoint to the (exact) constant diagram functor, and diag: $D(\mathcal{A}^I) \to D(\mathcal{A})^I$ is the obvious comparison functor. The moral from the story is that:

(1) The right object to compute a homotopy colimit from is an object of $D(\mathcal{A}^I)$.
(2) If we are given an I-shaped diagram of objects of $D(\mathcal{A})$ instead, our success with obtaining a homotopy colimit depends on whether we can find a preimage of the diagram under the functor diag.

Pursuing this idea a little further leads to the definition of a *prederivator* as a strict 2-functor $\mathscr{D}\colon \mathcal{C}at^{\mathrm{op}} \to CAT$, where $\mathcal{C}at$ is the 2-category of small categories and CAT is (ignoring any complaints of set theoretic nature at this point) the 2-category of all categories:

$$\mathscr{D}\colon \quad I \underset{g}{\overset{f}{\underset{\Downarrow \eta}{\rightrightarrows}}} J \qquad \longmapsto \qquad \mathscr{D}(I) \underset{g^*}{\overset{f^*}{\underset{\Downarrow \eta^*}{\leftleftarrows}}} \mathscr{D}(J)$$

A *derivator* is a prederivator satisfying a list of axioms [4] providing us with a calculus of homotopy Kan extensions (so in particular homotopy limits and colimits).

A standard algebraic example is as follows: If k is a field, we put $\mathscr{D}_k(I) = D(\operatorname{Mod} kI)$ and if $u\colon I \to J$ is a functor in $\mathcal{C}at$, then $u^*\colon D(\operatorname{Mod} kJ) \to D(\operatorname{Mod} kI)$ is the forgetful functor. In this context we can say that representation theory of finite dimensional algebras provides us with a very detailed understanding of such derivators $\mathscr{D}_k$. In a similar way we can obtain derivators in the context of topological spaces or spectra, and in fact one obtains one from any Quillen model category [2, 4].

In our context we are interested only in stable homotopy theory, i.e. enhancements of triangulated categories, which means that we impose further axioms on $\mathscr{D}$. A derivator is *pointed* if $\mathscr{D}(I)$ is always a pointed category. In such a case, we can on each $\mathscr{D}(I)$ define an abstract suspension endofunctor $\Sigma\colon \mathscr{D}(I) \to \mathscr{D}(I)$. A derivator is *stable* if all the suspension functors are equivalences. This holds for all $\mathscr{D}_k$ as above, but also for the derivator of topological spectra, for instance. There is another technical condition on $\mathscr{D}$ giving it an adjective *strong*, which we shall not discuss here but which is satisfied in all examples. Now we have the following theorem:

Theorem 1 (Maltsiniotis, Groth)**.** *Let $\mathscr{D}$ be a strong stable derivator. Then $\mathscr{D}(I)$ carries a canonical triangulated structure for each $I \in \mathcal{C}at$ and the functor $u^*\colon \mathscr{D}(J) \to \mathscr{D}(I)$ is triangulated for each $u\colon I \to J$ in $\mathcal{C}at$.*

Note that the fact that $\mathscr{D}(I)$ are additive categories is *not* a part of the definition—it is a consequence of the axioms.

If Q is a quiver and $\mathscr{D}$ a derivator, then $\mathscr{D}(Q)$ stands for $\mathscr{D}$ applied on the (non-additive!) path category of Q. By a result of Happel [9], we know that for two finite oriented trees Q, Q' we have that $D^b(\operatorname{mod} kQ)$ and $D^b(\operatorname{mod} kQ')$ are equivalent if and only if Q and Q' differ only by orientation of arrows. We have obtained in [6, 7] a generalization of this result valid for *any* stable derivator:

Theorem 2. *Let Q and Q' be finite oriented trees. If Q and Q' differ only by orientation, then for every stable derivator $\mathscr{D}$ we have a triangle equivalence*

$$\mathscr{D}(Q) \xrightarrow{\sim} \mathscr{D}(Q').$$

Our strategy is to construct abstract versions of BGP reflection functors [1], which we do by considering a suitable semiorthogonal decomposition of a diagram category $\mathscr{D}(I)$ for a certain I containing the path categories of both Q and Q' as full subcategories. Along the way, we also study the concept of a homotopical epimorphism, which is an analogue of homological epimorphisms of Geigle and Lenzing [3] in our setting.

By similar means we also obtain the following equivalence [6]:

Theorem 3. *For every stable derivator $\mathscr{D}$ we have a triangle equivalence between $\mathscr{D}(Q)$ and $\mathscr{D}(\square)$, where Q is a Dynkin quiver of type D_4 (with any fixed orientation) and $\square$ is a stand alone commutative square.*

Now if we have a strong stable monoidal derivator $(\mathscr{D}, \otimes)$, which is an enhancement of a monoidal triangulated category [5], and we tensor together two maps $f\colon X \to Y$ and $g\colon X' \to Y'$ in the base category of $\mathscr{D}$, we get a commutative square $f \otimes g$. Lifting this square to an object of $\mathscr{D}(\square) \sim \mathscr{D}(D_4)$ and establishing an abstract version of what we know about the derived category of D_4 from [9], we obtain in [6] a conceptual explanation and extension of May's axioms [11] mentioned at the beginning.

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Quantum symmetric pairs and categorification

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(joint work with Michael Ehrig)

Many recent results in representation theory describe the categories of representations by actions of some Lie algebras or quantum groups. For instance, there is an action of the Lie algebra $\hat{\mathfrak{sl}}_p$ on the direct sum of the Grothendieck groups of the categories of $k[S_d]$-modules, for $d \geq 0$, where k is a field of characteristic p and S_d is the symmetric group of order $d!$. The action of the Lie algebra generators e_i and f_i come from exact functors between modules (i-induction and i-restriction). In case the characteristic is zero, we have an action of $\mathfrak{gl}_\infty$ and the Lie algebra generators e_i and f_i correspond to induction and restriction along the natural embedding of S_n into S_{n+1} followed by the projection onto some eigenspace for x_i, the i-th Jucys-Murphy element. This point of view goes back to 1932, to Young's orthogonal form, see e.g. [B] and references therein. As indicated in [B], it generalizes to higher level cyclotomic Hecke algebras, and is best done in terms of their graded versions introduced by Khovanov and Lauda in [KL]. The talk is motivated by the question if such a picture also works for Brauer algebras, and the question if there is some geometry which could provide the grading.

Brauer algebras. As the symmetric group, the classical Brauer algebra (introduced by Brauer in 1937) arises as a centralizer algebra. The Brauer algebra $Br_d(\mp n)$ is the centralizer of the natural action of the orthogonal (respectively symplectic) Lie algebra of rank n on the d-fold tensor product $V^{\otimes d}$ of its natural representation V in case n is large enough in comparision to d. As a centralizer of a reductive Lie algebra action, the Brauer algebra is semisimple. Moreover, there are analogues of the Jucys-Murphy elements and therefore of restriction and induction functors, [CDM].

Explicitly, the algebra is the unital algebra generated by s_i, e_i, $1 \leq i \leq d-1$ modulo the relations

$$\begin{gathered} s_a^2 = 1, \quad s_a s_b = s_b s_a, \quad s_c s_{c+1} s_c = s_{c+1} s_c s_{c+1}, \\ (e_a)^2 = \delta e_a \quad e_c e_{c+1} e_c = e_{c+1} \quad e_{c+1} e_c e_{c+1} = e_c \\ s_a e_a = e_a = e_a s_a, \quad s_c e_{c+1} e_c = s_{c+1} e_c \quad s_{c+1} e_c e_{c+1} = s_c e_{c+1} \end{gathered} \tag{1}$$

where $\delta = \mp n$. Note that it makes sense to consider Brauer algebras depending on an arbitrary complex number δ. Generically, the algebra is still semisimple, but for fixed d there are a finite number of intergal values where it is not. In any case however it appears naturally as a centralizer of some Lie superalgebra action:

Let W be a complex super vector space of superdimension $(m|n)$ and $\mathfrak{gl}(m|n) = \mathfrak{gl}(W)$ be the corresponding general linear superalgebra. We fix a supersymmetric bilinear form $(-,-)$ on W and consider $\mathfrak{osp}(m|n)$, the (Lie) superalgebra consisting of those endomorphisms in $\mathfrak{gl}(W)$ that respect the supersymmetric bilinear form. Explicitly this means that a homogeneous element in $X \in \mathfrak{osp}(W)$ has to satisfy the equality $\langle Xv, w\rangle + (-1)^{|X|\cdot|v|} \langle v, Xw\rangle = 0$, where $|\cdot|$ denotes the degree.

Theorem 1 ([ES2]). *Let $\delta = m-n$. Then there is a natural right action of* $\mathrm{Br}_d(\delta)$ *on $W^{\otimes d}$ commuting with the action of $\mathfrak{osp}(V)$. If $d < 2(m+n)$ the commutant is exactly the Brauer algebra.*

Whereas the existence of the action is an easy calculation, the second part is not obvious, but can be reduced to the case of mixed tensor products for $\mathfrak{gl}(W)$ and the action of the corresponding walled Brauer categories wBr, [BS], via the following diagram (in case $m = 2m'$ and $n = 2n'$ are even):

$$\begin{array}{ccc}
\mathrm{End}_{\mathfrak{osp}(2m'|2n')}\left(V^{\otimes d}\right) & \hookrightarrow & \mathrm{End}_{\mathfrak{gl}(m'|n')}\left(V^{\otimes d}\right) \\
\uparrow & \nearrow & \uparrow \\
\mathrm{Br}_d(m-n) & \hookrightarrow & \mathrm{wBr}_d(m'-n')
\end{array}$$

The action of the i-induction and i-restriction functors can be described most naturally via an action of a specialisation of a *quantum symmetric pair*. Let $\mathfrak{g}$ be a reductive Lie algebra and σ a Lie algebra automorphism of $\mathfrak{g}$. Then the fixed point Lie algebra $\mathfrak{g}^\sigma$ is again reductive and we have an inclusion of the corresponding universal envelopping (Hopf) algebras. This fails when passing to the quantum group $\mathcal{U}_q(\mathfrak{g})$. Gail Letzter [L] however showed that there is a coidal subalgebra H inside $\mathcal{U}_q(\mathfrak{g})$ which specializes to our universal enveloping algebra of fixed points.

We consider the special case of the symmetric pair $(\mathfrak{gl}_{2m}, \mathfrak{gl}_m \times \mathfrak{gl}_m)$. Then the fixed point Lie algebra comes along with the special generators $B_i = E_i + \sigma(E_i)$ and $B_{-i} = F_i + \sigma(F_i)$ for $0 \leq i \leq n-1$ constructed from the standard generators of $\mathfrak{gl}_{2m}$. These generators are not the standard generators of $\mathfrak{gl}_m \times \mathfrak{gl}_m$, but a good choice for quantization, [L].

Theorem 2 ([ES]). *The i-induction and i-restriction functors give rise to an action of the fixed point Lie algebra on the categories of* Br_d*-modules.*

The elements B_i for $i \neq 0$ generate a Lie subalgebra isomorphic to $\mathfrak{gl}_m$ inside $\mathfrak{gl}_{2m}$. If V_a is the natural $\mathfrak{gl}_a$-module then $V_{2m} \cong V_m \oplus V_m$ when we restrict to $\mathfrak{gl}_m$. We obtain a diagram as follows

$$\begin{array}{ccccc}
\mathfrak{gl}_{2m} & \curvearrowright & \bigoplus_\alpha \bigwedge^{\alpha_1} V_{2m} \otimes \bigwedge^{\alpha_2} V_{2m} \otimes \cdots \otimes \bigwedge^{\alpha_r} V_{2m} & \curvearrowleft & \mathfrak{gl}_r \\
\cup & & & & \cap \\
\mathfrak{gl}_m \times \mathfrak{gl}_m & \curvearrowright & \bigwedge(n,m,r) & \curvearrowleft & \mathfrak{gl}_r \times \mathfrak{gl}_r \\
\cup & & & & \cap \\
\mathfrak{gl}_m & \curvearrowright & \bigoplus_\gamma \bigwedge^{\gamma_1} V_m \otimes \bigwedge^{\gamma_2} V_m \otimes \cdots \otimes \bigwedge^{\gamma_{2r}} V_m & \curvearrowleft & \mathfrak{gl}_{2r}
\end{array}$$

where α (respectively γ runs through all compositions of some fixed integer n with r (respectively $2r$) parts. The first row indicates the classical skew Howe duality which restricts to the skew Howe duality in the third row. Restricting the actions to the fixed point Lie algebras one obtains the skew Howe dual pair shown in the middle row. The corresponding coideal subalgebra $H(m)$ inside $\mathcal{U}_q(\mathfrak{gl}_{2m})$ can be generated by certain quantized versions of the B_i, $-m < i < m$ such that the B_i for $i \neq 0$ generate a subalgebra isomorphic to $\mathcal{U}_q(\mathfrak{gl}m)$, , see [ES] for details.

Theorem 3 ([ES])**.** *The above diagram lifts to the quantum group level. In particular, the images of the actions of $H(m)$ and $H(r)$ are each others commutant.*

The first and last row has a well-known categorification in terms of parabolic category $\mathcal{O}$'s for $\mathfrak{gl}(n)$ where the type of parabolic is determined by α. The middle row can be categorified using type D, see [ES]:

Theorem 4. (1) *The quantum version of the representation $\bigwedge(n, m, r)$ can be identified with the complexified Grothendieck group of the direct sum of certain blocks of parabolic category $\mathcal{O}$ for $\mathfrak{so}(2n)$, where the parabolic is of type A given by the composition α. The singularity of the block is determined by the weight spaces of the representation.*

(2) *Graded versions of translation functors and derived Zuckerman functors give rise to the two commuting actions of the coideal subalgebras.*

(3) *The category is Koszul selfdual and Koszul duality swaps the two coideal algebra actions.*

(4) *The duality on category $\mathcal{O}$ induces a bar involution on the Grothendieck group and the classes of the simple modules give rise to a canonical basis in the sense of Lusztig.*

The theorem is the first instance of a categorification of actions of coideal subalgebras. The existence of a canonical basis in this context is new and surprising. Note that the combinatorics of this basis is determined by Kazdhan-Lusztig polynomials of type D.

Graded cyclotomic VW-algebras and Brauer algebras. There is an interesting application: Let $\mathfrak{g} = \mathfrak{so}(2n)$ and let $\mathfrak{p}$ be a maximal parabolic subalgebra of type A and let ω_0 be the fundamental weight corresponding to the remaining vertex in the Dynkin diagram of $\mathfrak{g}$. Let $\delta \in \mathbb{Z}_{\geq 0}$ and consider the parabolic Verma module $M^{\mathfrak{p}}(\underline{\delta})$ of highest weight $\underline{\delta} = \delta\omega_0$. Finally let V be the natural $\mathfrak{g}$-module.

Theorem 5 ([ES])**.** *The endomorphism ring $E := \mathrm{End}(M^{\mathfrak{p}}(\underline{\delta}) \otimes V^{\otimes d})$ has a natural grading which turns it into a quasi-hereditary Koszul algebra.*

In fact, the endomorphism ring can be identified with a cyclotomic VW-algebra (or degenerate BMW-algebra) studied in [AMR]. In analogy to the walled Brauer algebra case, [BS], one can find an idempotent e such that eEe is isomorphic to the Brauer algebra $\mathrm{Br}_d(\delta)$. In particular, the Brauer algebra inherits a grading which turns it, in case $\delta \neq 0$ into a graded quasi-hereditary Koszul algebra.

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Quotient-closed subcategories of quiver representations (plus. . .)

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(joint work with Steffen Oppermann and Idun Reiten)

Let Q be a quiver without oriented cycles, and k an algebraically closed field. We study the quotient-closed subcategories of kQ-mod. (Throughout, our subcategories are full, additive subcategories, closed under direct summands.) When Q is Dynkin, we give a bijection between the elements of W, the Weyl group associated to Q, and the quotient-closed subcategories of kQ-mod. When Q is non-Dynkin, we give a bijection between W and the cofinite quotient-closed subcategories; here "cofinite" means that they contain all except finitely many of the indecomposables of kQ-mod. The same results hold when kQ is replaced by a hereditary algebra finite-dimensional over a finite field. The results stated here are contained in [4].

Reminder on Weyl groups associated to quivers. Number the vertices of Q from 1 to n arbitrarily.

Recall that the Weyl group associated to Q has generators s_i for $1 \leq i \leq n$ subject to the conditions:

* $s_i^2 = e$,
* $s_i s_j s_i = s_j s_i s_j$ if i and j are connected by a single arrow,
* $s_i s_j = s_j s_i$ if i and j are not connected.

The generators s_i are called *simple reflections*. An element of W will have many different expressions as a product of simple reflections. An expression for w of minimal length is called *reduced*, and this length is called the *length* of w.

Q is Dynkin if and only if W is finite. In this case, there is a unique longest element of W, denoted w_0.

The Dynkin case. We fix an algebraically closed ground field k. We will work with categories of right modules.

Write out the AR quiver for kQ-mod. Associate to the modules in the τ^- orbit of P_i, the simple reflection s_i. If we read the resulting simple reflections from left to right, we obtain a reduced word for w_0. Here, by "left to right", we mean "in any order such that if there is an arrow from one symbol to another, the former is read before the latter". For convenience, we fix a particular reading order, and we refer to the word obtained using that order as the AR-word for w_0.

For X a subcategory of kQ-mod, define $f(X)$ to be the element of W obtained by reading (in the AR-word order) the reflections which correspond to the indecomposables which do not belong to X.

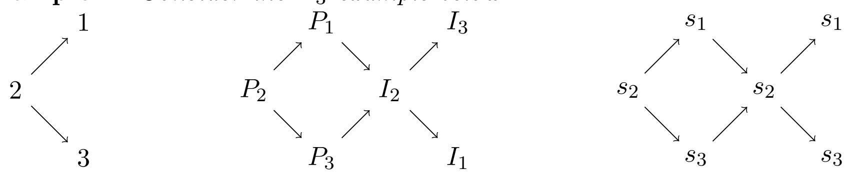
A *subword* of the AR-word is any word obtained from the AR-word by erasing some of its symbols. For w in W, we can consider the collection of all subwords of the AR-word which form a reduced word for w. Among these, the *leftmost* subword for w is the reduced subword which is lexicographically leftmost (i.e., its first letter is as far to the left as possible among all reduced subwords, its second letter is as far to the left as possible among all reduced subwords with the leftmost first letter, and so on).

For $w \in W$, define $g(w)$ to be the subcategory of kQ-mod which is the additive hull of the indecomposables not in the leftmost subword for w of the AR-word.

The following theorem was conjectured by Drew Armstrong and the speaker in 2007, and finally established in [4]:

Theorem 1. *For Q Dynkin, f and g are well-defined inverse bijections between W and the quotient-closed subcategories of kQ-mod.*

Example 1. *Consider the A_3 example below:*



We can choose the AR-word to be $s_2s_1s_3s_2s_1s_3$. (We could also have swapped the relative positions of either of the consecutive pairs s_1, s_3). Now, if X is the additive hull of P_1 and I_1, then $f(X) = s_2s_3s_2s_1$. If we take $w = s_3s_2s_3 = s_2s_3s_2$, then we find that the leftmost subword uses the letters in positions 1, 3, and 4 of the AR-word, so $g(w)$ is the additive hull of P_1, I_3, and I_1.

The non-Dynkin case. When Q is non-Dynkin, it turns out that the appropriate class of subcategories are those which are cofinite and quotient-closed. A subcategory is called *cofinite* if it contains all but at most finitely many indecomposable objects of the ambient category.

It is a simple lemma that a cofinite quotient-closed category necessarily includes all non-preprojective indecomposables. We can therefore restrict our attention to the preprojective component. There, we can carry out a procedure completely analogous to the one we have already discussed in the Dynkin case. To each indecomposable in the τ^- orbit of P_i, associate the simple reflection s_i. Fix a left-to-right reading order, and define the AR-word to be the resulting (infinite) word. For X any cofinite subcategory containing all non-preprojective indecomposables, we define $f(X)$ to be the product of the simple reflections corresponding to the indecomposables not in X, in the AR-word order.

For $w \in W$, we can again speak of the leftmost subword for w in the AR-word, and we define $g(w)$ to be the additive hull of the indecomposables not corresponding to simple reflections in the leftmost word for w, together with all the non-preprojective indecomposables.

Now, we have the following theorem, generalizing Theorem 1:

Theorem 2. *For arbitrary acyclic Q, the maps f and g are inverse bijections between W and the cofinite quotient-closed subcategories of kQ-mod.*

Connection to preprojective algebras and comment on proofs. In order to discuss the proofs, it is necessary to introduce another family of objects in bijection to W. Let Λ be the preprojective algebra corresponding to Q, let e_i be the idempotent corresponding to vertex i, and let $I_i = \Lambda(1 - e_i)\Lambda$.

For $w \in W$, take a reduced word $s_{i_1} \dots s_{i_r}$ for w, and define I_w to be $I_{s_{i_r}} \dots I_{s_{i_1}}$. (Note the reversal of order of the product, for consistency with conventions already established in the literature.) Then I_w is well-defined (i.e., does not depend on the choice of reduced word for w) [3, 1].

For M a Λ-module, we write M_{kQ} for the kQ-module structure on M induced by the inclusion of kQ into Λ.

We prove the following proposition, as part of an induction which also includes the theorems above:

Proposition 1. *For $w \in W$, the subcatgory $f(w)$ can also be obtained by taking the finitely generated kQ-modules in the additive hull of $(I_w)_{kQ}$.*

The case of a hereditary algebra finite-dimensional over a finite field. Theorem 2 also holds in this case.

Fix a finite field $\mathbb{F}_q$. Let Q be a quiver and σ an automorphism of Q. Define an automorphism F_σ of $\overline{\mathbb{F}}_q Q$ by

$$F_\sigma(\sum_i c_i p_i) = c_i^q \sigma(p_i)$$

where the p_i are paths in Q, and the c_i are coefficients in $\overline{\mathbb{F}}_q$. It is well-known (see for example [2]) that any hereditary algebra H finite-dimensional over $\mathbb{F}_q$ can be realized as the fixed subalgebra $(\overline{\mathbb{F}}_q Q)^{F_\sigma}$, for some Q and σ.

The category of representations of H is then equivalent to the category of F-stable representations of $\overline{\mathbb{F}}_q Q$, where an F-stable representation is a representation together with some additional data. By forgetting the additional data, from a representation of H, we obtain a representation of $\overline{\mathbb{F}}_q Q$. This allows us to relate quotient-closed subcategories of H-mod to a subset of the quotient-closed subcategories of $\overline{\mathbb{F}}_q Q$-mod. These then correspond to a subset of the elements of W_Q, which are exactly those in a subgroup of isomorphic to the Weyl group W_H.

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Scalar extensions of derived categories and non-Fourier-Mukai functors

Michel Van den Bergh
(joint work with Alice Rizzardo)

We report on the article [5]. For simplicity we assume below that k is an algebraically closed field of characteristic zero. Orlov's famous representability theorem [2, Thm 2.2] asserts that any fully faithful functor between the derived categories of coherent sheaves on smooth projective varieties is a Fourier-Mukai functor. Lunts and Orlov proved the following natural extension to quasi-coherent sheaves of Orlov's theorem:

Proposition ([1, Corollary 9.13(2)]). *Let X/k be a projective scheme such that $\mathcal{O}_X$ has no zero dimensional torsion and let Y be a quasi-compact separated scheme. Then every fully faithful exact functor $\Psi : \mathrm{Perf}(X) \to D(\mathrm{Qcoh}(Y))$ is isomorphic to the restriction of a Fourier-Mukai functor associated to an object in $D(\mathrm{Qcoh}(X \times Y))$.*

We have shown that this extension is false if we drop the condition that Ψ is fully faithful even in the case that X, Y are smooth and projective, suggesting that Orlov's original theorem may also be false without the full faithfulness hypothesis (for some positive results see [3]).

Our arguments are based on the properties of scalar extensions of derived categories whose investigation was started by Pawel Sosna and the first author. If $\mathfrak{A}$ is a k-linear category and L/k is a field extension then we denote by $\mathfrak{A}_L$ the category of L-objects in $\mathfrak{A}$, i.e. pairs (M, ρ) where $M \in \mathrm{Ob}(\mathfrak{A})$ and $\rho : L \to \mathfrak{A}(M, M)$ is a k-algebra morphism. In [4] the first author studied the obvious forgetful functor

$$F : D^b(\mathrm{Qcoh}(X_L)) \to D^b(\mathrm{Qcoh}(X))_L$$

She proved an essential surjectivity result for $\mathrm{trdeg}\, L/k \leq 2$. Unfortunately it turns out one cannot go beyond that:

Proposition. *Let X/k be a smooth connected projective variety which is not a point, a projective line or an elliptic curve. Then there exists a finitely generated field extension L/k of transcendence degree 3 together with an object $Z \in D^b(\mathrm{Qcoh}(X))_L$ which is not in the essential image of F.*

Very roughly speaking the object Z is constructed as the generic object in a generic family of indecomposable coherent sheaves. This explains why a few types of varieties are excluded: their moduli spaces are too small.

A counter example to the above mentioned result by Lunts and Orlov when dropping the full faithfulness hypothesis may now be obtained as follows.

Proposition. *Let X, Y be smooth projective schemes. Let $i_\eta : \eta \to X$ be the generic point of X and let $L = k(\eta)$ be the function field of X. Assume that $D^b(\mathrm{Qcoh}(Y))_L$ contains an object Z which is not in the essential image of $D^b(\mathrm{Qcoh}(Y)_L)$ (for example furnished by the previous proposition). Define Ψ as the composition*

$$\mathrm{Perf}(X) \xrightarrow{i_\eta^*} D(L) \xrightarrow{L \mapsto Z} D(\mathrm{Qcoh}(Y)) \; .$$

Then Ψ is not the restriction of a Fourier-Mukai functor.

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Computing local cohomology of determinantal varieties

Jerzy Weyman

(joint work with Claudiu Raicu and Emily Witt)

Given positive integers $m \geq n$ and a field $\mathbf{K}$ of characteristic zero, we consider the space $\mathbf{K}^{m\times n}$ of $m \times n$ matrices, and the ring S of polynomial functions on this space. For each $p = 1, \cdots, n$ we can define an ideal $I_p \subset S$ generated by the polynomial functions in S that compute the $p \times p$ minors of the matrices in $\mathbf{K}^{m\times n}$. The goal is to describe for each p the local cohomology modules $H^\bullet_{I_p}(S)$ of S with support in the ideal I_p. The case $p = n$ was analyzed in [2]. The general case was proved in [1].

There is a natural action of the group $\mathbf{GL}_m \times \mathbf{GL}_n$ on $\mathbf{K}^{m\times n}$ and hence on S, and this action preserves each of the ideals I_p. This makes $H^\bullet_{I_p}(S)$ into $\mathbf{GL}_m \times \mathbf{GL}_n$–representations, and our results describe the characters of these representations explicitly. Our methods also allow us to determine explicitly the characters of all the modules $\mathrm{Ext}^\bullet_S(S/I, S)$, where I is an ideal of S generated by an irreducible $\mathbf{GL}_m \times \mathbf{GL}_n$–subrepresentation of S, and in particular determine the regularity of such ideals. It is an interesting problem to determine the minimal free resolutions of such ideals I, which is unfortunately only answered in a small number of cases.

We write F (resp. G), for $\mathbf{K}$–vector spaces of dimension m (resp. n), thinking of $F^* \otimes G^*$ as the space $\mathbf{K}^{m\times n}$ of $m \times n$ matrices, and of $S = Sym(F \otimes G)$ as the ring of polynomial functions on this space. S is graded by degree, with the space of linear forms $F \otimes G$ sitting in degree 1. The *Cauchy formula* [3, Cor. 2.3.3]

$$S = \bigoplus_{\mathbf{x}=(x_1\geq\cdots\geq x_n\geq 0)} S_{\mathbf{x}}F \otimes S_{\mathbf{x}}G$$

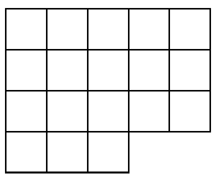
describes the decomposition of S into a sum of irreducible $\mathbf{GL}(F) \times \mathbf{GL}(G)$–representations, indexed by partitions $\mathbf{x}$ with at most n parts ($S_{\mathbf{x}}$ denotes the *Schur functor* associated to $\mathbf{x}$). We denote by $I_{\mathbf{x}}$ the ideal generated by $S_{\mathbf{x}}F\otimes S_{\mathbf{x}}G$. If we write (1^p) for the partition $\mathbf{x}$ with $x_1 = \cdots = x_p = 1$, $x_i = 0$ for $i > 0$, then $I_{(1^p)}$ is just the ideal I_p of $p \times p$ minors mentioned above. Our first result gives an explicit formula for the regularity of the ideals $I_{\mathbf{x}}$:

Theorem on Regularity of Equivariant Ideals. *For a partition $\mathbf{x}$ with at most n parts, letting $x_{n+1} = -1$ we have the following formula for the regularity of the ideal $I_{\mathbf{x}}$:*

$$\mathrm{reg}(\mathrm{I}_{\mathbf{x}}) = \max_{\substack{p=1,\cdots,n\\ x_p>x_{p+1}}} (\mathrm{n}\cdot(\mathrm{x_p}-\mathrm{p}) + \mathrm{p}^2 + 2\cdot(\mathrm{p}-1)\cdot(\mathrm{n}-\mathrm{p})).$$

In particular, the only ideals $I_{\mathbf{x}}$ which have a linear resolution are those for which $x_1 = \cdots = x_n$ (i.e. powers $I_n^{x_1}$ of the ideal I_n of maximal minors) or $x_1 - 1 = x_2 = \cdots = x_n$ (i.e. $I_n^{x_1-1}\cdot I_1$).

The theorem above is a consequence of the explicit description of the modules $\mathrm{Ext}^\bullet_S(S/I_{\mathbf{x}}, S)$ which we also obtain. This description is somewhat involved, so we avoid stating it. A key point is that the modules $\mathrm{Ext}^\bullet_S(S/I_{\mathbf{x}}, S)$ *grow* as we append new columns to the *end* of the partition $\mathbf{x}$. More precisely, we can identify a partition $\mathbf{x}$ with its pictorial realization as a Young diagram of left-justified rows of boxes, having x_i boxes in the i-th row: for example, $\mathbf{x} = (5, 5, 5, 3)$ corresponds to



and adding two columns of size 2 and three columns of size 1 to the end of $\mathbf{x}$ yields $\mathbf{y} = (10, 7, 5, 3)$.

Theorem on the Growth of Ext Modules. *Let $d \geq 0$ and consider partitions $\mathbf{x}, \mathbf{y}$, where $\mathbf{x}$ consists of the first d columns of $\mathbf{y}$, i.e. $x_i = \min(y_i, d)$ for all $i = 1, \dots, n$. The natural quotient map $S/I_{\mathbf{y}} \twoheadrightarrow S/I_{\mathbf{x}}$ induces injective maps*

$$\mathrm{Ext}^i_S(S/I_{\mathbf{x}}, S) \longrightarrow \mathrm{Ext}^i_S(S/I_{\mathbf{y}}, S),$$

for all $i = 0, 1, \dots, m \cdot n$.

The naive generalization of the statement above fails: if $\mathbf{y}$ is a partition containing $\mathbf{x}$ (i.e. $y_i \geq x_i$ for all i), then it is not always the case that the induced maps $\mathrm{Ext}^i_S(S/I_{\mathbf{x}}, S) \to \mathrm{Ext}^i_S(S/I_{\mathbf{y}}, S)$ are injective. A general partition $\mathbf{x}$ has the property that most modules $\mathrm{Ext}^i_S(S/I_{\mathbf{x}}, S)$ are non-zero, but it is always contained in some partition $\mathbf{y}$ with $y_1 = \cdots = y_n$. For such a $\mathbf{y}$, all but n of the modules $\mathrm{Ext}^i_S(S/I_{\mathbf{y}}, S)$ will vanish.

We write $\mathfrak{R}$ for the representation ring of the group $\mathbf{GL}(F) \times \mathbf{GL}(G)$. Given a $\mathbb{Z}$–graded S–module M, admitting an action of $\mathbf{GL}(F) \times \mathbf{GL}(G)$ compatible with the natural one on S, we define its character $\chi_M(z)$ to be the element in the Laurent power series ring $\mathfrak{R}((z))$ given by

$$\chi_M(z) = \sum_{i \in \mathbb{Z}} [M_i] \cdot z^i,$$

where $[M_i]$ denotes the class in $\mathfrak{R}$ of the $\mathbf{GL}(F) \times \mathbf{GL}(G)$–representation M_i. We will often work with bi-graded modules M_i^j, where the second grading (in j) is a cohomological one, and $M_\bullet^j \neq 0$ only for finitely many values of j: for us they will be either Ext-modules, or local cohomology modules. We define the character of such M to be the element $\chi_M(z, w) \in \mathfrak{R}((z))[w]$ given by

$$\chi_M(z, w) = \sum_{i,j \in \mathbb{Z}} [M_i^j] \cdot z^i \cdot w^j.$$

We will refer to an r–tuple $\lambda = (\lambda_1, \cdots, \lambda_r) \in \mathbb{Z}^r$ (for $r = m$ or n) as a weight. We say that λ is dominant if $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_r$, and denote by $\mathbb{Z}^r_{dom}$ the set of dominant weights. Note that a partition is just a dominant weight with non-negative entries. We will usually use the notation $\mathbf{x}, \mathbf{y}, \underline{z}$ etc. to refer to partitions indexing the subrepresentations of S, and λ, μ etc. to denote the weights describing the characters of other equivariant modules (Ext-modules or local cohomology modules).

For $\lambda \in \mathbb{Z}^n_{dom}$ and $0 \leq s \leq n$, we define

$$\lambda(s) = (\lambda_1, \cdots, \lambda_s, \underbrace{s-n, \cdots, s-n}_{m-n}, \lambda_{s+1} + (m-n), \cdots, \lambda_n + (m-n)) \in \mathbb{Z}^m. \tag{1}$$

Note that this notation differs slightly from the one in [2], in that $\lambda(s)$ is what used to be called $\lambda(n-s)$. For $0 \leq s \leq n$, we define (with the convention $\lambda_0 = \infty$, $\lambda_{n+1} = -\infty$)

$$h_s(z) = \sum_{\substack{\lambda \in \mathbb{Z}^n_{dom} \\ \lambda_s \geq s-n \\ \lambda_{s+1} \leq s-m}} [S_{\lambda(s)} F \otimes S_\lambda G] \cdot z^{|\lambda|}. \tag{2}$$

We write $H_p(z,w)$ for the character of the doubly-graded module $H^{\bullet}_{I_p}(S)$. In [2] we prove that for $m > n$

$$H_n(z,w) = \sum_{s=0}^{n-1} h_s(z) \cdot w^{(n-s)\cdot(m-n)+1},$$

and it is easy to see that the same formula holds for $m = n$ (in this case, the only non-zero local cohomology module is $H^1_{I_n}(S) = S_{\det}/S$, where det denotes the determinant of the generic $n \times n$ matrix, and S_{det} is the localization of S at det).

We write $p(a,b;c)$ for the number of partitions of weight c contained in an $a \times b$ rectangle. We define the Gauss polynomial $\binom{a+b}{b}$ to be the generating function for the sequence $p(a,b;c)_{c\geq 0}$:

$$\binom{a+b}{b}(w) = \sum_{c\geq 0} p(a,b;c)\cdot w^c = \sum_{b\geq t_1\geq t_2\geq\cdots\geq t_a\geq 0} w^{t_1+\cdots+t_a}. \tag{3}$$

The main result of [2] calculates the local cohomology modules of S with support in determinantal ideals.

Theorem on Local Cohomology with Support in Determinantal Ideals. *With the above notation, we have for each* $p = 1, \cdots, n$,

$$H_p(z,w) = \sum_{s=0}^{p-1} h_s(z)\cdot w^{(n-p+1)^2+(n-s)\cdot(m-n)} \cdot \binom{n-s-1}{p-s-1}(w^2).$$

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