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Nonlinear Evolution Equations: Analysis and Numerics

Organised by
Marlis Hochbruck, Karlsruhe
Serge Nicaise, Valenciennes
Alexander Ostermann, Innsbruck
Roland Schnaubelt, Karlsruhe

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Abstract. The workshop was devoted to the analytical and numerical investigation of nonlinear evolution equations. The main aim was to stimulate a closer interaction between experts in analytical and numerical methods for areas such as wave and Schrödinger equations or the Navier–Stokes equations and fluid dynamics.

Mathematics Subject Classification (2010): 35xx, 65xx.

Introduction by the Organisers

The qualitative theory of nonlinear evolution equations is an important tool for studying the dynamical behavior of systems in science and technology. A thorough understanding of the complex behavior of such systems requires a detailed analytical and numerical investigation of the underlying partial differential equations. Here one is interested in regularity and asymptotic properties of solutions as well as in efficient numerical approximations of the solutions which preserve their qualitative properties on a large time scale. Currently, driven by the challenging mathematical difficulties and supported by the influx of techniques from many branches of analysis and numerics, innovative and sophisticated methods have been developed throughout this area of mathematics.

This workshop has focused on recent developments in the qualitative theory of nonlinear evolution equations, both analytical and numerical, between which there has been an increasing interaction in recent years. One of the main goals of this conference was to bring together international experts with different backgrounds in analysis and numerics to stimulate the transfer of ideas, results, and techniques

among them. To this aim, the speakers reported on new trends in the fundamental classes of evolution equations, such as

- wave equations,
- Schrödinger equations,
- Maxwell's equations,
- Navier–Stokes equations and fluid dynamics,
- reaction-advection-diffusion equations,
- stochastic evolution equations.

The numerical study of evolution equations relies on efficient methods for time integration which allow for convergence results not depending on spatial discretization parameters (such as mesh size or number of basis functions). For the rigorous derivation of finite time error bounds and the study of geometric properties, one has to develop a framework for the discretization that captures the essential properties of the analytic problem. This makes a thorough understanding of the partial differential equation itself indispensable. Therefore, the various approaches in this field are closely linked with the analytic theory of evolution equations: splitting and approximation schemes can be formulated and treated within semigroup theory, exponential integrators are directly connected to functional calculi, and spectral methods heavily employ Fourier analysis. The proof of rigorous error bounds typically relies on regularity theory. This close relationship also plays a crucial role for geometric integrators which are designed to capture the qualitative properties of solutions (such as long term behavior or positivity).

In this spirit, the talks treated the long time behavior of fluid interaction models, geometric evolution equations and wave maps. The role of Strichartz estimates in dispersive problems was explored, and maximal regularity for deterministic and stochastic parabolic evolution equations was investigated. Space-time discretizations as well as semidiscretizations of evolution equations were discussed. Techniques from asymptotic analysis were exploited to construct and analyze multiscale problems. (Non-)Linear semigroup theory and Sobolev space theory were shown to be an enable the analysis of splitting methods and exponential integrators for Korteweg–de Vries and nonlinear Schrödinger equations. Convolution quadrature was successfully used for the numerical discretization of problems on unbounded domains. The presentations and the fruitful discussions demonstrated the far-reaching potential of further exchanges between analytical and numerical approaches to nonlinear evolution equations.

Workshop: Nonlinear Evolution Equations: Analysis and Numerics

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Abstracts

Well-posedness and stability for the volume-preserving mean curvature flow with a dynamic contact angle

Helmut Abels
(joint work with Harald Garcke, Lars Müller)

Let $\Omega \subseteq \mathbb{R}^3$ be an open, non-empty, connected domain with smooth boundary $\partial\Omega$. Furthermore, let $\Gamma(t) \subseteq \Omega$ be a connected, smooth hypersurface with boundary $\partial\Gamma(t)$ such that $\Gamma(t) \cup \partial\Gamma(t)$ is compact and $\emptyset \neq \partial\Gamma(t) \subseteq \partial\Omega$. $V(t) \subseteq \Omega$ denotes the region between $\Gamma(t)$ and $\partial\Omega$ and $D(t)$ shall be defined as $D(t) := \partial V(t) \cap \partial\Omega$. In particular, we have $\partial D(t) = \partial\Gamma(t)$. For a point $p \in \Gamma$ we denote the exterior normal to Γ in p by $n_\Gamma(p)$, where the term "exterior" should be understood with respect to V. Analogously, for the normal $n_{\partial\Omega}(p)$ for $p \in \partial\Omega$, which coincides with $n_D(p)$ if $p \in D \subseteq \partial\Omega$.

We consider the motion of the evolving hypersurface $\Gamma = (\Gamma(t))_{t\in I}$ with boundary curve $\partial\Gamma(t) = \partial D(t)$ driven by the volume-preserving mean curvature flow with dynamic contact angle

$$V_\Gamma = H_\Gamma - \bar{H}_\Gamma, \tag{1}$$

$$v_{\partial D}(t) = a + b\kappa_{\partial D}(t) + \cos(\alpha(t)). \tag{2}$$

where V_Γ is the normal velocity, H_Γ is the mean curvature (sum of principal curvatures), $\bar{H}_\Gamma$ is the mean value of the mean curvature, $v_{\partial D}$ is the normal boundary velocity of the contact curve, $\kappa_{\partial D}$ is its geodesic curvature with respect to $\partial\Omega$, $\cos(\alpha(t)) = n_{\Gamma(t)} \cdot n_{D(t)}$ describes the contact angle of $\Gamma(t)$ and $D(t)$. Here $a, b \in \mathbb{R}$ are constants with $b > 0$.

The volume preserving mean curvature without boundary contact (i.e., $\partial\Gamma = \emptyset$) was first before e.g by Huisken [5] and Escher and Simonett [4]. The flow arises as a gradient flow of the following energy

$$E(\Gamma) := \int_\Gamma 1\, dA - a\int_D 1\, dA + b\int_{\partial\Gamma} 1\, ds,$$

with respect to $L^2(\Gamma(t)) \times L^2(\partial\Gamma(t))$ under the constraint that the volume enclosed by $\Gamma(t)$ and $D(t)$ is constant. The three parts of the energy are

(1) The surface area functional of $\Gamma(t)$.
(2) A (de-)wetting penalty proportional to a.
(3) A line energy proportional to $b > 0$, penalizing long boundary curves.

In order to prove well-posedness locally in time, let $\Gamma^* \subset \Omega$ be a smooth "reference" manifold. We parametrize $\Gamma(t)$ with respect to Γ^*. To this end let $\Psi\colon \Gamma^* \times (-\epsilon_0, \epsilon_0) \to \Omega$ be such that

$$\Psi(q, w) := q + wn_{\Gamma^*}(q) + t(q, w)T(q),$$

where $T\colon \Gamma^* \longrightarrow \mathbb{R}^3$ is a suitable tangential vector field and t is a "tangential correction close to $\partial\Gamma^*$" such that

$$q + wn_{\Gamma^*}(q) + \tilde{t}(q,w)n_{\partial\Gamma^*}(q) \in \partial\Omega \qquad \forall\, w \in (-\epsilon_0,\epsilon_0), q \in \partial\Gamma^*,$$

cf. Vogel '00. For $\rho\colon [0,\infty)\times\Gamma^* \to (-\epsilon_0,\epsilon_0)$ we set $\Gamma_\rho(t) := \mathrm{Im}(\Psi(\cdot,\rho(t,\cdot))$.

With the aid of this parametrization the following theorem was proved in [1].

Theorem 1 (Well-Posedness Locally in Time)
Let $4 < p < \infty$. Then there exist $\epsilon > 0$, $T > 0$ such that for each $\rho_0 \in W_p^{2-\frac{2}{p}}(\Gamma^*)$ with $\rho_0|_{\partial\Gamma^*} \in W_p^{3-\frac{3}{p}}(\partial\Gamma^*)$ and $\|\rho_0\|_{W_p^{2-\frac{2}{p}}} + \|\rho_0|_{\partial\Gamma^*}\|_{W_p^{3-\frac{3}{p}}} < \epsilon$ there exists a unique solution

$$\rho \in W_p^1(0,T;L_p(\Gamma^*)) \cap L^p(0,T;W_p^2(\Gamma^*)) =: E_1$$
$$\rho|_{\partial\Gamma^*} \in W_p^{\frac{3}{2}-\frac{1}{2p}}(0,T;L_p(\partial\Gamma^*)) \cap L^p(0,T;W_p^{3-\frac{1}{p}}(\partial\Gamma^*)) =: E_2$$

of the system

$$\begin{aligned} V_\Gamma(\rho(t)) &= H_\Gamma(\rho(t)) - \bar{H}_\Gamma(\rho(t)) && \text{in } [0,T]\times\Gamma^*,\\ v_{\partial D}(\rho(t)) &= a + b\kappa_{\partial D}(\rho(t)) + n_\Gamma(\rho(t))\cdot n_D(\rho(t)) && \text{on } [0,T]\times\partial\Gamma^*,\\ \rho(0) &= \rho_0 && \text{in } \Gamma^*. \end{aligned}$$

Here $V_\Gamma(\rho(t)) = V_{\Gamma_{\rho(t)}}$, $H_\Gamma(\rho(t)) = H_{\Gamma_{\rho(t)}}$, and similarly for the other quantities.

The theorem is proved by linearizing the quasi-linear evolution equation for ρ, applying the results of [3] on maximal L^p-regularity for parabolic systems with dynamic boundary conditions as well as a contraction argument.

Next we consider stability a spherical cap $\Gamma^* = \partial B_R(x) \cap \mathbb{R}^3_+$ in $\Omega = \mathbb{R}^3_+$. Then $H_\Gamma - \overline{H}_\Gamma = 0$ and

$$0 = a + b\kappa_{\partial D} + n_\Gamma \cdot n_D \qquad \text{on } \partial\Gamma$$

is equivalent to

$$\cos(\alpha) = \frac{b}{r} - a.$$

In [2] the following stability result is proved:

Theorem 2 (Stability of Spherical Caps)
Let $a > -1$, $b > 0$ and $4 < p < \infty$. Moreover, assume Γ^* to be a stationary spherical cap with radius R^* and contact angle α^* that satisfies $b > -\frac{1}{3}R^*\sin(\alpha^*)^2\cos(\alpha^*)$. Then $\rho \equiv 0$ is stable in

$$X_\gamma := \left\{ \rho \in W_p^{2-\frac{2}{p}}(\Gamma^*) \,\middle|\, \rho|_{\partial\Gamma^*} \in W_p^{3-\frac{3}{p}}(\partial\Gamma^*) \right\}$$

and there exists $\delta > 0$ such that the unique solution $\rho(t)$ of the (1)-(2) with initial value $\rho_0 \in X_\gamma$ satisfying $\|\rho_0\|_{W_p^{2-\frac{2}{p}}(\Gamma^*)} + \|\rho_0|_{\partial\Gamma^*}\|_{W_p^{3-\frac{3}{p}}(\partial\Gamma^*)} < \delta$ exists on $\mathbb{R}^+$ and converges at an exponential rate to some ρ_∞, which parametrizes a stationary spherical cap as well.

A similar results was obtained by Escher and Simonett [4] without boundary contact. Since suitable translations and dilations of stationary spherical caps are again stationary spherical caps, the kernel of the linearized operator is three-dimensional and the usual principle of linearized stability cannot be used. The proof of the theorem is based on a generalized principle of linearized stability proved in [6] and a careful analysis of the spectrum of the linearized operator.

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Implicit–explicit BDF methods for parabolic equations

Georgios Akrivis

(joint work with Christian Lubich)

Let $T > 0, u^0 \in H$, and consider the abstract initial value problem for a possibly nonlinear parabolic equation

$$\begin{cases} u'(t) + A(t)u(t) = B(t, u(t)), \quad 0 < t < T, \\ u(0) = u^0, \end{cases} \tag{1}$$

in a usual triple of separable complex Hilbert spaces $V \subset H = H' \subset V'$, with V densely and continuously embedded in H. Here $A(t) : V \to V'$ are linear operators while the operators $B(t, \cdot) : V \to V'$ may be nonlinear. We denote by $(\cdot, \cdot)$ both the inner product in H and the duality pairing between V' and V, and by $|\cdot|$ and $\|\cdot\|$ the norms in H and V, respectively. The dual norm in V' is denoted by $\|\cdot\|_\star$.

For $q = 1, \dots, 5$, consider the implicit $q-$step BDF method (α, β) and the explicit $q-$step method (α, γ) described by the polynomials α, β and γ,

$$\begin{cases} \alpha(\zeta) = \displaystyle\sum_{j=1}^{q} \frac{1}{j} \zeta^{q-j} (\zeta - 1)^j = \sum_{i=0}^{q} \alpha_i \zeta^i, \quad \beta(\zeta) = \zeta^q, \\ \gamma(\zeta) = \beta(\zeta) - \beta_q (\zeta - 1)^q = \zeta^q - (\zeta - 1)^q = \displaystyle\sum_{i=0}^{q-1} \gamma_i \zeta^i. \end{cases} \tag{2}$$

The order of the $q-$step BDF method is q. For a given α, the scheme (α, γ) is the unique explicit $q-$step scheme of order q; the order of all other explicit $q-$step schemes $(\alpha, \tilde{\gamma})$ is at most $q-1$.

Let $N \in \mathbb{N}, N \geq q$, and consider a uniform partition $t^n := nk, n = 0, \ldots, N$, of the interval $[0, T]$, with time step $k := T/N$. Assuming we are given starting approximations $U^0, \ldots, U^{q-1} \in V$, we discretize (1) in time by the $q-$step implicit–explicit $(\alpha, \beta, \gamma)-$scheme, i.e., we define approximations $U^m \in V$ to the nodal values $u^m := u(t^m)$ of the exact solution as follows

$$\sum_{i=0}^{q} \alpha_i U^{n+i} + kA(t^{n+q})U^{n+q} = k \sum_{i=0}^{q-1} \gamma_i B(t^{n+i}, U^{n+i}), \tag{3}$$

$n = 0, \ldots, N-q$. The scheme (3) is referred to as the implicit–explicit $q-$step BDF method. The unknown U^{n+q} appears only on the left-hand side of (3); therefore, to advance in time, we only need to solve one linear equation, which reduces to a linear system if we discretize also in space, at each time level.

Natural conditions for the parabolicity of the abstract equation in (1) are uniform *coercivity* and *boundedness* of the operators $A(t) : V \to V'$, i.e.,

$$\mathrm{Re}\,(A(t)v, v) \geq \kappa(t)\|v\|^2 \quad \forall v \in V, \tag{4}$$

and

$$\|A(t)v\|_\star \leq \nu(t)\|v\| \quad \forall v \in V, \tag{5}$$

respectively, with two smooth positive functions $\kappa, \nu : [0, T] \to \mathbb{R}$.

Furthermore, we assume, that $B(t, \cdot)$ satisfies the following local Lipschitz condition in a ball $\mathcal{B}_{u(t)} := \{v \in V : \|v - u(t)\| \leq 1\}$, centred at the value $u(t)$ of the solution u at time t, and, for simplicity, defined here in terms of the norm of V,

$$\|B(t, v) - B(t, \tilde{v})\|_\star \leq \tilde{\lambda}(t)\|v - \tilde{v}\| + \tilde{\mu}|v - \tilde{v}| \quad \forall v, \tilde{v} \in \mathcal{B}_{u(t)}, \tag{6}$$

for all $t \in [0, T]$, with a smooth nonnegative function $\tilde{\lambda} : [0, T] \to \mathbb{R}$ and an arbitrary constant $\tilde{\mu}$.

Using (4) and (5), existence and uniqueness of the approximations $U^q, \ldots, U^N$ can be easily established by the Lax–Milgram lemma.

Since we are interested in stability properties of the method (3), besides the approximations $U^n \in \mathcal{B}_{u(t^n)}$, we consider also implicit–explicit BDF approximations $V^n \in \mathcal{B}_{u(t^n)}$ such that

$$\sum_{i=0}^{q} \alpha_i V^{n+i} + kA(t^{n+q})V^{n+q} = k \sum_{i=0}^{q-1} \gamma_i B(t^{n+i}, V^{n+i}), \tag{7}$$

$n = 0, \ldots, N - q$.

Theorem. *Assume* (4), (5) *and* (6), *and let* $\eta_1 := \eta_2 := 0, \eta_3 := 0.0836, \eta_4 := 0.2878$ *and* $\eta_5 = 0.8160$. *Then, under the stability condition*

$$\forall t \in [0, T] \quad \kappa(t) - \eta_q \nu(t) - (2^q - 1)(1 + \eta_q)\tilde{\lambda}(t) \geq \rho > 0, \tag{8}$$

the implicit–explicit $q-$step BDF method (3) *is locally stable in the sense that, with $\vartheta^m := U^m - V^m$, for k sufficiently small,*

$$c_q|\vartheta^n|^2 + \frac{1}{2}\rho k \sum_{\ell=q}^{n} \|\vartheta^\ell\|^2 \le C \sum_{j=0}^{q-1} \big(|\vartheta^j|^2 + k\|\vartheta^j\|^2\big), \tag{9}$$

for $n = q, \dots, N$, with c_q a positive constant depending only on q, and C a constant independent of ρ, k, n and the approximations. □

We write the sufficient stability condition (8) in the form

$$\forall t \in [0,T] \quad \eta_q \nu(t) + (2^q - 1)(1+\eta_q)\tilde{\lambda}(t) < \kappa(t) \tag{10}$$

and will comment on the constants η_q and $(2^q - 1)(1 + \eta_q)$ as well as on the functions $\nu(t), \tilde{\lambda}(t)$ and $\kappa(t)$.

Concerning the coefficient of $\tilde{\lambda}(t)$, it follows from the analysis in [3], where the case of time-independent, self-adjoint and positive definite operator $A(t)$ is considered, that $(2^q-1)(1+\eta_q)$ can be at most improved to 2^q-1. Furthermore, a straightforward application of the von Neumann criterion shows that the coefficient η_q of $\nu(t)$ in (10) can be at most improved to $\tilde{\eta}_q$ with $\tilde{\eta}_3 = 0.0692, \tilde{\eta}_4 = 0.2865, \tilde{\eta}_5 = 0.6139$.

As far as the functions $\nu(t), \tilde{\lambda}(t)$ and $\kappa(t)$ are concerned, we notice that they depend on the choice of the norm of V. A way to show stability under more favourable stability conditions, is to use time-dependent norms: We decompose the operators $A(t)$ in their self-adjoint and anti-self-adjoint parts $A_s(t)$ and $A_a(t)$, respectively, $A_s(t) := [A(t) + A(t)^\star]/2, A_a(t) := [A(t) - A(t)^\star]/2$, and introduce in V the time-dependent norm $\|\cdot\|_t$ by $\|v\|_t := (A_s(t)v, v)^{1/2}, v \in V$. We denote by $\|\cdot\|_{\star,t}$ the corresponding dual norm on V',

$$\forall v \in V' \quad \|v\|_{\star,t} := \sup_{w \in V\setminus\{0\}} \frac{|(v,w)|}{\|w\|_t} = \sup_{\substack{w \in V \\ \|w\|_t = 1}} |(v,w)|.$$

It follows easily from (4) and (5) that the norms $\|\cdot\|_t$ and $\|\cdot\|$ are equivalent.

We now denote by $\lambda_a(t) : [0,T] \to [1,\infty)$ a smooth function such that

$$\|A(t)v\|_{\star,t} \le \lambda_a(t)\|v\|_t \quad \forall v \in V. \tag{11}$$

An obvious consequence of (4) and (5) is that (11) is valid with $\lambda_a(t) = \lambda(t) := \nu(t)/\kappa(t)$. In general, however, (11) may be satisfied with $\lambda_a(t)$ much smaller than $\lambda(t)$. In the case of self-adjoint and positive definite operators $A(t)$, the estimate (11) holds as an equality with $\lambda_a(t) = 1$. We assume also that $A_s(t)$ satisfies a mild Lipschitz condition, with respect to t, namely

$$\|\big(A_s(t) - A_s(\tilde{t})\big)v\|_\star \le L|t - \tilde{t}|\,\|v\| \quad \forall t, \tilde{t} \in [0,T] \ \ \forall v \in V. \tag{12}$$

Furthermore, in analogy to (6), we assume that the operators B satisfy the local Lipschitz condition

$$\|B(t,v) - B(t,\tilde{v})\|_{\star,t} \le \tilde{\lambda}_b(t)\|v - \tilde{v}\|_t + \tilde{\mu}_b|v - \tilde{v}| \quad \forall v, \tilde{v} \in \mathcal{B}_{u(t)}, \tag{13}$$

for all $t \in [0,T]$, with a smooth nonnegative function $\tilde{\lambda}_b : [0,T] \to \mathbb{R}$ and an arbitrary constant $\tilde{\mu}_b$.

Then, the implicit–explicit $q-$step BDF method (3) is locally stable under the condition

$$\forall t \in [0,T] \quad \eta_q \lambda_a(t) + (2^q - 1)(1+\eta_q)\tilde{\lambda}_b(t) < 1. \tag{14}$$

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Maximal regularity for non-autonomous forms

Wolfgang Arendt

Let V, H be Hilbert spaces such that V is continuously and densely embedded in H and let

$$\mathfrak{a}\colon [0,T] \times V \times V \to \mathbb{R}$$

be a *bounded coercive non-autonomous form*; i.e., $\mathfrak{a}(t,.,.)$ is bilinear for all $t \in [0,T]$, $\mathfrak{a}(.,v,w)$ is measurable for all $v, w \in V$ and there exist constants $M \geq 0$, $\alpha > 0$ such that

$$|\mathfrak{a}(t,v,w)| \leq M\|v\|_V\|w\|_V, \quad \mathfrak{a}(t,v,v) \geq \alpha\|v\|_V^2$$

for all $t \in [0,T]$ and all $v, w \in V$. Consider the operator $\mathcal{A}(t) \in \mathcal{L}(V, V')$ defined by

$$\langle \mathcal{A}(t)v, w\rangle = \mathfrak{a}(t,v,w).$$

We discuss well-posedness and regularity of the Cauchy problem

$$(CP)\begin{cases} u'(t) + \mathcal{A}(t)u(t) = f(t) \\ u(0) = u_0. \end{cases}$$

The mixed Sobolev space

$$MR(V,V') := H^1(0,T;V') \cap L^2(0,T;V)$$

is continuously embedded into $C([0,T];H)$. It is the right space if we are interested in solutions with values in V' with maximal regularity. In fact, the following classical result by Lions establishes well-posedness where all three terms in the equation $u' + \mathcal{A}(.)u(.) = f$ are in $L^2(0,T;V')$.

Theorem 1. *Given $f \in L^2(0,T;V')$ and $u_0 \in H$ there exists a unique solution $u \in MR(V,V')$ of (CP).*

Several different proofs of this theorem exist in the literature: there is an elegant proof via a Riesz-representation theorem due to Lions, a proof by Galerkin-approximation and a recent interesting proof using Riemann sums due to Laasri [6].

However, for applying such results to boundary value problems and to non-linear problems more regularity is desirable. In fact, the operator one is really interested in is the part $A(t)$ of $\mathcal{A}(t)$ in H given by $D(A(t)) := \{v \in V : \mathcal{A}(t) \in H\}$, $A(t)v := \mathcal{A}(t)v$. Thus it is natural to suppose that $f \in L^2(0,T;H)$ and one is led to ask the following.

Question. Under which conditions is Lions' solution in $H^1(0,T;H)$?

This would imply that $u(t) \in D(A(t))$ and

$$u'(t) + A(t)u(t) = f(t) \quad \text{a.e.}$$

Thus u would be a solution with *maximal regularity in H*, i.e. all three functions u', $A(.)u(.)$, f appearing in the evolution equation are in $L^2(0,T;H)$.

At first, we consider the initial value $u_0 = 0$. Then in the autonomous case for each $f \in L^2(0,T;H)$ Lions' solution u for $u_0 = 0$ is indeed in $H^1(0,T;H)$. It was for longtime an open problem whether this remains true in the non-autonomous case. Only very recent Dominik Dier [5] gave a counterexample.

Counterexample. There exists a form (which is piecewise constant) and $f \in L^2(0,T;H)$ such that the solution u of Lions' Theorem with initial value $u_0 = 0$ is not in $H^1(0,T;H)$.

A reason for the failure of H-maximal regularity is the possible failure of the *square root property*, i.e. $D(A(t)+\omega)^{1/2})$ may be different from V. In the symmetric case, one always has the square root property, and, at least if the form is time-independent, V is the exact space of initial values for which the solution is in $H^1(0,T;H)$. The problem whether this remains true also in the non-autonomous case is still open. It is asked exactly in the following form by Lions [7, p. 64].

Problem (Lions). Assume that the form is symmetric, i.e. $\mathfrak{a}(t,v,w) = \mathfrak{a}(t,w,v)$ for all $v,w \in V$, $t \in [0,T]$. Does (CP) have maximal regularity in H?

Some answers under diverse additional assumptions are given in the book [7] by Lions. In a recent paper we find a positive answer if the form is Lipschitz in time. We also allow a multiplicative perturbation which is essential for non-linear problems.

Theorem 2 ([3])**.** *Assume that the form $\mathfrak{a}$ is symmetric and $\mathfrak{a}(.,v,w)\colon [0,T] \to \mathbb{R}$ is Lipschitz continuous for all $v,w \in V$. Let $B\colon [0,T] \to \mathcal{L}(H)$ be bounded and weakly measurable such that*

$$\beta\|g\|_H^2 \le \langle B(t)g, g\rangle \le \frac{1}{\beta}\|g\|_H^2$$

for all $t \in [0,T]$, $g \in H$ and some $\beta > 0$. Then given $u_0 \in V$, $f \in L^2(0,T;H)$ there exists a unique function $u \in H^1(0,T;H)$ such that $u(t) \in D(A(t))$ a.e. and

$$u'(t) + B(t)A(t)u = f(t) \quad a.e.,$$
$$u(0) = u_0.$$

Moreover, $u \in C([0,T];V)$.

As application global solutions to a quasilinear problem of the following form

$$u'(t) + m(t,x,u,\nabla u)\Delta u(t) = f(t)$$
$$u(0) = u_0$$
$$\partial_\nu u(t) + \beta(t,x)u(t)|_{\partial\Omega} = 0$$

are given in [3], thus imposing non-autonomous Robin boundary conditions.

Other results based on the technique of maximal regularity in H, or more generally in L^p for non-autonomous and non-linear problems are given in the talk, we refer also to [2], [1].

We also mention that meanwhile Dier [4] gave a positive answer to Lions' problem if the form satisfies merely a BV-condition:

Theorem 3 ([4])**.** *Assume that $\mathfrak{a}$ is symmetric and that an increasing function $g\colon [0,T] \to \mathbb{R}$ exists such that*

$$|\mathfrak{a}(t,v,w) - \mathfrak{a}(s,v,w)| \leq (g(t) - g(s))\|v\|_V \|w\|_V$$

for all $0 \leq s \leq t \leq T$ and all $v,w \in V$. Then for all $u_0 \in V$, $f \in L^2(0,T;H)$ Lions' solution u of (CP) is in $H^1(0,T;H)$.

Finally, we mention that for Hölder continuous $\mathfrak{a}(.,v,w)$ of Hölder index $\beta > 1/2$ (and a not necessarily symmetric form $\mathfrak{a}$) it is shown by Ouhabaz and Spina [8] that the solution u of (CP) is in $H^1(0,T;H)$ if $u_0 = 0$ and $f \in L^2(0,T;H)$.

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A new splitting scheme for motion by anisotropic mean curvature

Eric Bonnetier
(joint work with Elie Bretin, Antonin Chambolle)

Let $\phi : \mathbb{R}^n \longrightarrow [0, +\infty[$ denote a strictly convex function with regularity $\mathcal{C}^2(\mathbb{R}^n)$, that satisfies

$$\phi(t\xi) = |t|\phi(\xi), \quad \lambda|\xi| \leq \phi(\xi) \leq \lambda^{-1}|\xi| \qquad \xi \in \mathbb{R}^n, t \in \mathbb{R},$$

for some $\lambda > 0$, and let $\phi^o : \mathbb{R}^n \longrightarrow [0, +\infty[$ denote its dual function defined by

$$\phi^o(\xi^*) \quad = \quad \sup\{\xi \cdot \xi^*, \quad \phi(\xi) \leq 1\}.$$

Let $E \subset \mathbb{R}^n, n \geq 2$ be an open bounded domain, defined by the level sets of a function u, so that $\partial E = \{x \in \mathbb{R}^n, u(x) = 0\}$. In the context of Finsler geometry [1], ϕ^o represents an anisotropic surface energy density, $n_\phi = \phi^o(\nabla u)$ is called the Cahn-Hoffman vector field, and $\kappa_\phi = \mathrm{div}(n_\phi)$ the ϕ-curvature. We also define the anisotropic Laplacian by $\Delta_\phi = \mathrm{div}(\phi^o_\xi(\nabla u)\phi^o(\nabla u))$. The evolution by anisotropic mean curvature from the set $E(0) = E$ is a mapping $t \longrightarrow E(t)$ such that, at each time t, the normal velocity at a point $x \in \partial E(t)$ satisfies

$$V_n \quad = \quad -\kappa_\phi n_\phi.$$

The corresponding level-set formulation amounts to finding a viscosity solution of the nonlinear PDE

$$\partial_t u \quad = \quad \phi^o(\nabla u)\phi^o_{\xi\xi}(\nabla u) : \nabla^2 u, \tag{1}$$

for which existence and uniqueness have been proved in [6, 5].

Numerical methods for approximating (1) include finite difference methods based on the level-set formulation, finite element approximations of the anisotropic Allen-Cahn equation

$$\partial_t u \quad = \quad \Delta_\phi u - \varepsilon^{-2} W'(u),$$

where $W(s) = \frac{s^2(1-s)^2}{2}$ is a double-well potential and where $\varepsilon > 0$ is a parameter that measures the thickness of an approximate diffuse interface. One may also use an anisotropic extension of the following algorithm due to Bence, Merriman and Osher [2] : given a closed set $E \subset \mathbb{R}^n$, define $T_h(E) = \{x \in \mathbb{R}^n, u_h(x, h) \geq 1/2\}$, where u_h is the solution to the parabolic problem

$$\begin{cases} \partial_t u_h(x,t) & = & \Delta_\phi u_h(x,t), & t > 0, x \in \mathbb{R}^n, \\ u_h(x,0) & = & 1_E(x). \end{cases}$$

and approximate $E(t)$ by $E_h(t) = T_h^{[t/h]}E$.

The above approximation methods have to cope with the discretization of the strongly nonlinear operator Δ_ϕ [4, 8]. We propose a different approach, based on a linearization of the anisotropic Laplacian in the Fourier space, which leads

to a linear approximating algorithm. Let $\mathcal{F}(u)$ denote the Fourier transform of a function $u \in L^2(\mathbb{R}^n)$ and consider the operator defined by

$$\tilde{\Delta}_\phi u \quad = \quad \mathcal{F}^{-1}\left(-4\pi^2\phi^o(\xi)^2\mathcal{F}(u)(\xi)\right).$$

Given $E \subset \mathbb{R}^n$, we define $\tilde{T}_h(E) = \{x \in \mathbb{R}^n, u_h(x,h) \geq 1/2\}$, where $\tilde{u}_h$ is the solution to the parabolic problem

$$\left\{ \begin{array}{lcll} \partial_t \tilde{u}_h(x,t) & = & \tilde{\Delta}_\phi \tilde{u}_h(x,t), & t>0, x \in \mathbb{R}^n, \\ \tilde{u}_h(x,0) & = & 1_E(x). & \end{array} \right.$$

and approximate $E(t)$ by $\tilde{E}_h(t) = \tilde{T}_h^{[t/h]}E$. In [3], we connect this scheme to the work of Ishii, Pires and Souganidis [7] on anisotropic mean-curvature flow where the heat kernel in (1) is replaced by a smooth positive kernel with bounded second-order moments. In our case, however, the corresponding kernel does not have quite the same positivity and boundedness properties.

In [3], we also report numerical results obtained via the numerical approximation of a modified anisotropic Allen-Cahn equation, using the approximate anisotropic Laplacian $\tilde{\Delta}_\phi$. More precisely, we consider the PDE

$$\left\{ \begin{array}{lcll} \partial_t \tilde{u} & = & \tilde{\Delta}_\phi \tilde{u} - \varepsilon^{-2}W'(u), & x \in Q, t>0, \\ \tilde{u}(x,0) & = & q\left(\frac{\mathrm{dist}(x,\partial E)}{\varepsilon}\right), & \end{array} \right. \tag{2}$$

where $Q = [-1/2, 1/2]^n$ is a fixed computational box, that strictly contains the initial set E, and where the profile $q(s) = \tanh(\frac{s}{\sqrt{2}})$. We use a splitting scheme to treat first the diffusion term, then the reaction term at each time step. Assuming periodic boundary conditions on ∂Q, the linearity of $\tilde{\Delta}_\phi$ reduces the first step to a simple multiplication in Fourier space, which makes the scheme simple to implement and computationally very fast. Numerical experiments on evolutions from a Wulff set, show very good agreement between the theoretical and calculated solutions [3]. Figure 1 shows a 3D example of evolution with conserved volume from a torus, for the anisotropic density $\phi(\xi) = |\xi_1|+|\xi_2|+|\xi_3|$ using this algorithm.

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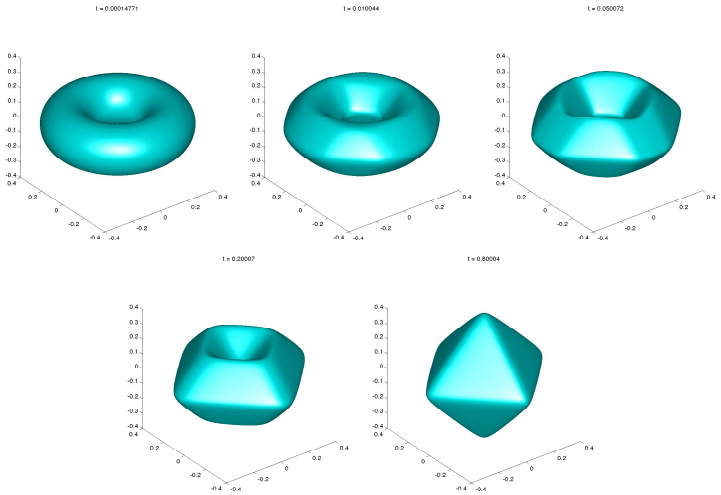

FIGURE 1. Evolution by anisotropic mean curvature from a torus.

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Some estimates for non self-adjoint operators. Intersection of cone and disk

MICHEL CROUZEIX

Holomorphic functions of one operator frequently occur in pure as applied mathematics. For instance, the exponential function is related to the semi-group, or the group theory, i.e. in particular to parabolic equations, linear Schroedinger, ... The cosine function plays a similar role for second order evolution equations. Time discretizations of these problems (for instance by Runge-Kutta methods) lead to use rational approximations of the exponential or of the cosine function. Explicit methods and Krylov type methods correspond with polynomial approximations. For the theory, as well as for the numerical analysis of these problems, estimates of these holomorphic functions are needed. In the case of a self-adjoint operator, spectral theory provides a very efficient tool, but the situation is much more complicated if we consider non normal operators. For them, spectral sets were introduced and studied by J. von Neumann in 1951

Let us consider a linear operator $A \in \mathcal{L}(D(A), H)$ on a Hilbert space H. A set $X \subset \mathbb{C}$ with spectrum $\sigma(A) \subset X$ is called a K-spectral set for the operator A if the

following inequality

$$\|r(A)\| \leq K \sup_{z\in X} |r(z)|$$

holds for all rational functions bounded on $X \subset \mathbb{C}$. *If the inequality holds with* $K = 1$, X *is called a spectral set.*

A work of B.&F. Delyon (1999) has pointed out the role that the numerical ranges can play in the theory of spectral sets. Recall that the numerical range $W(A)$ of the linear operator A is defined by

$$W(A) = \{\langle Av, v\rangle \,; v \in D(A),\ \|v\|_H = 1\},$$

$\langle\cdot,\cdot\rangle$ denoting the inner product in the complex (or complexified) Hilbert space H. Inspired by their paper, I have obtained the following result

The numerical range $W(A)$ *is a* $\mathcal{Q}$*-spectral set for* A, *with* $\mathcal{Q} \leq 11.08$.

(I conjecture that this result holds with $\mathcal{Q} = 2$). I believe that this result may be useful in many situations, in particular for evolution problems, for getting error estimates or stability results from the numerical analysis point of view, as well as for theoretical purposes, see [2] for different applications. The numerical range is a natural object for a numerical analyst who more often works with variational formulations based on the sesquilinear forms $\langle Av, v\rangle$ than by using directly the differential operator A. Note that assumptions of the often used Lax-Milgram lemma may be translated in term of numerical range. An important feature of the numerical range is its stability with respect to discretisation. Generally, the numerical range $W(A_h)$ of an approximation A_h of an operator A is closed to the numerical range $W(A)$. It is well-known that this is not the case for the spectra $\sigma(A_h)$ and $\sigma(A)$ in non self-adjoint situations... But a drawback of the numerical range is that it can be too large, recall that the numerical range is convex (the Toeplitz-Hausdorff theorem).

The aim of this lecture is to show the possibility to make a hole in the numerical range (more general results may be found in [1])

Theorem 1. *We assume that*

$$\|(A-\omega)^{-1}\| \leq \frac{1}{\rho} \quad \textit{and} \quad |\mathrm{Im}\langle Av, v\rangle| \leq \tan\theta\, \mathrm{Re}\langle (A-\gamma)v, v\rangle, \quad \forall v \in D(A),$$

with $\rho, \omega, \gamma, \theta \in \mathbf{R}$, $0 < \rho$, $\omega < \gamma$, $0 < \theta \leq \frac{\pi}{2}$. *Then,* X *is a* K*-spectral set for* A, *where*

$$X = \{z \,; |z-\omega| \geq \rho \textit{ and } |\mathrm{Im}\, z| \leq \tan\theta\, \mathrm{Re}(z-\gamma)\},$$

and $K = 3 + 2\sqrt{3}$.

Note that the set X is the intersection of the cone of vertex γ, angle 2θ, with the exterior of the disk centered in ω, radius ρ. The assumption $|\mathrm{Im}\langle Av, v\rangle| \leq \tan\theta\, \mathrm{Re}\langle (A-\gamma)v, v\rangle$ means that the numerical range of A is contained in the cone.

We want to prove that

$$\|r(A)\| \leq 3 + 2\sqrt{3}, \quad \text{for all rational functions } r \text{ bounded by 1 in } X.$$

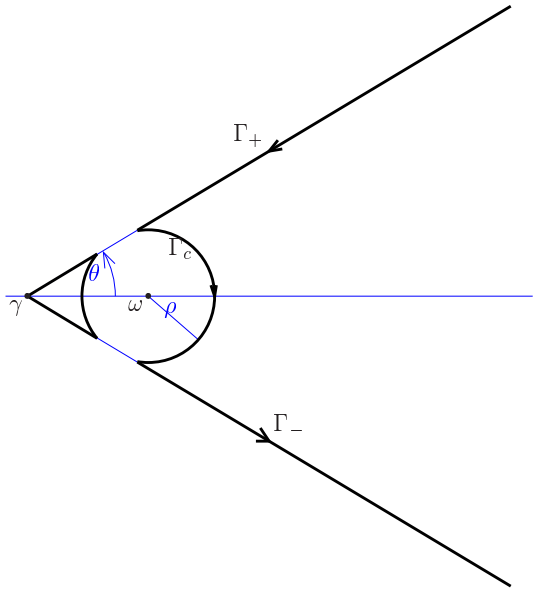

FIGURE 1. The K-spectral set X.

The ingredients of the proof of this result is first an integral representation of $r(A)$ using in particular the Poisson Kernel on the boundary of the disk, the double layer potential on the boundary of the cone, and a residual part. Then, for bounding such integrals, we have used the two following lemmata.

We let E be a closed subset of $\mathbb{C}$, m be a complex-valued measure on E, and $M(.) \in C(E; \mathcal{L}(H)$ be a bounded and continuous function with bounded operator values on a Hilbert space H. To a rational function bounded on E, $r \in \mathcal{R}(E)$, we associate the operators

$$g_1(r) = \int_E r(\sigma)\, M(\sigma)\, dm(\sigma) \quad \text{and} \quad g_2(r) = \int_E r(\sigma)\, (M(\sigma))^{-1} dm(\sigma).$$

Then we have

Lemma 1. *We assume that the operator values $M(.)$ are self-adjoint positive semi-definite, $M(\sigma) = M(\sigma)^* \geq 0$, for all $\sigma \in E$. Then, for all $r \in \mathcal{R}(E)$, we have*

$$\|g_1(r)\|_{H\to H} \leq \left\| \int_E M(\sigma)\, |dm(\sigma)| \right\|_{H\to H} \sup_{\sigma\in E} |r(\sigma)|.$$

Lemma 2. *We assume that there exist a function $N(.) \in C(E; \mathcal{L}(H)$ with bounded self-adjoint operator values and a positive number α such that*

$\frac{1}{2}(M(\sigma) + M(\sigma)^*) \geq N(\sigma) \geq \alpha > 0$, *for all $\sigma \in E$.*

Then, for all $r \in \mathcal{R}(E)$, we have

$$\|g_2(r)\|_{H\to H} \le \left\| \int_E (N(\sigma))^{-1} \, |dm(\sigma)| \, \right\|_{H\to H} \sup_{\sigma\in E} |r(\sigma)|.$$

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Exponential integrators for (a very few) hyperbolic problems

Petra Csomós

(joint work with Alexander Ostermann)

Exponential integrators are efficient methods for solving semilinear problems numerically. Their convergence was widely analysed, for example, in [1] for parabolic problems. In the present short talk we show our results when the exponential Euler method is applied to two hyperbolic problems where the exact solutions are known: the linear transport equation and Burgers' equation.

We suppose that the operator A is the generator of the strongly continuous semigroup $\left(\mathrm{e}^{tA}\right)_{t\ge 0}$ on the Banach space X and the function g maps from X to X. For a fixed time step $\tau > 0$, the solution to the semilinear problem $\dot{u} = Au + g(u)$ with a given initial value $u(0) = u_0$ is obtained from the variation-of-constants formula. The numerical solution $u_n \approx u(t_n)$ at time $t_n = n\tau$ by using the exponential Euler method reads as

$$u_{n+1} = \mathrm{e}^{\tau A} u_n + \int_0^\tau \mathrm{e}^{(\tau-s)A} g(u_n) \mathrm{d}s, \qquad n \in \mathbb{N}.$$

We applied the exponential Euler method for the linear transport equation $\partial_t w(t,x) = \alpha\partial_x w(t,x) + \beta(t)\partial_x w(t,x)$ with a given initial function $w(0,x) = w_0(x)$, parameters $\alpha, \beta(\cdot) \in \mathbb{R}$, and $u(t) = w(t,\cdot)$. The spatial derivative was discretised by the upwind scheme with grid size $h > 0$. The resulting stability conditions are the following for all $n \in \mathbb{N}$:

(1)	exponential Euler without upwind:	$-1 \le \frac{\beta(t_n)}{\alpha} \le 0$,
(2)	exponential Euler with upwind:	$\tau \le \frac{h}{\lvert\beta(t_n)\rvert}$,
(3)	upwind without exponential Euler:	$\tau \le \frac{h}{\lvert\alpha+\beta(t_n)\rvert}$.

Hence, for a fixed grid size h and for $\frac{\beta(t_n)}{\alpha} \ge -\frac{1}{2}$, one can choose larger time step τ in case (2) as in case (3), that is, it is worth applying an exponential integrator.

We have also shown that the exponential Euler method is first-order consistent for this equation in both cases (1) and (2).

We were interested in the stability of the exponential Euler method also for the inviscid Burgers equation $\partial_t w(t,x) = \alpha\partial_x w(t,x) + w(t,x)\partial_x w(t,x)$. Our numerical results met our expectations, that is, the numerical approximation to $w(t_n,\cdot)$ plays the role of $\beta(t_n)$. This means that instability occurs for too "big" or too "small" values of α (in comparison to the maximum of $w(t_n,\cdot)$).

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Global existence of equivariant wave maps on a curved background

Piero D'Ancona
(joint work with Qidi Zhang)

Given a product $\mathbb{R}\times M^n$, with M^n a Riemannian manifold, the wave map system defined on $\mathbb{R}\times M^n$ with values in a manifold N^ℓ (which we may assume isometrically embedded in some Euclidean $\mathbb{R}^m$), in local coordinates, takes the form

$$u^a_{tt} - \Delta_M u^a + \Gamma^a_{bc}(u)\partial_\alpha u^b \partial^\alpha u^c = 0, \tag{1}$$

where Δ_M is the negative Laplace-Beltrami operator on M^n and $\Gamma^a_{bc}(u)$ are the Christoffel symbols on N^ℓ. This equation has received a great deal of attention in recent years, however most results concern the case $M^n=\mathbb{R}^n$ with flat metric. To our knowledge, there are very few results on the nonflat case. In [12] the stability of equivariant, stationary wave maps on $\mathbb{S}^2$ with values in $\mathbb{S}^2$ is proved, while [4] considers the local existence on Robertson-Walker spacetimes. More recently, in [5] global existence of small wave maps is proved in the case when $M^n=M^4$ is a four dimensional small perturbation of flat $\mathbb{R}^4$, and the stability of equivariant wave maps defined on $\mathbb{H}^2$ is studied in [6].

Our goal is to initiate the study of equivariant solutions of (1) on more general base manifolds M^n, $n\geq 3$. Our main result is the global existence of equivariant wave maps for small data in the critical norm, provided the base manifold belongs to a class of manifolds which we call *admissible*. The class of admissible manifolds is rather large, and includes in particular asymptotically flat manifolds and perturbations of real hyperbolic spaces; see some examples in Remark 3 below. The definition is the following:

Definition 1 (Admissible manifolds)**.** Let $n\geq 3$. We say that a smooth manifold M^n is *admissible* if its metric has the form $dr^2 + h(r)^2 d\omega^2_{\mathbb{S}^{n-1}}$ and $h(r)$ satisfies:

(i) $\exists h_\infty \geq 0$ such that $H(r) := h^{\frac{1-n}{2}}(h^{\frac{n-1}{2}})'' = h_\infty + O(r^{-2})$ for $r >> 1$.

(ii) $H^{(j)}(r) = O(r^{-1})$ and $(h^{-\frac{1}{2}})^{(j)} = O(r^{-\frac{1}{2}-j})$ for $r >> 1$ and $1\leq j\leq [\frac{n-1}{2}]$.

(iii) There exist $c, \delta_0 > 0$ such that for $r > 0$ we have $h(r) \geq cr$ while the function $P(r) = rH(r) - rh_\infty + \frac{1-\delta_0}{4r}$ satisfies the condition $P(r) \geq 0 \geq P'(r)$.

Now assume M^n and N^ℓ are rotationally symmetric manifolds, with global metrics

$$M^n : \quad dr^2 + h(r)^2 d\omega^2_{\mathbb{S}^{n-1}}, \qquad N^\ell : \quad d\phi^2 + g(\phi)^2 d\chi^2_{\mathbb{S}^{\ell-1}} \tag{2}$$

where $d\omega^2_{\mathbb{S}^{n-1}}$ and $d\chi^2_{\mathbb{S}^{\ell-1}}$ are the standard metrics on the unit sphere. We recall the *equivariant ansatz* (see [10]): writing the map $u = (\phi, \chi)$ in coordinates on N^ℓ, the radial component $\phi = \phi(t, r)$ depends only on time and r, the radial coordinate on M^n, while the angular component $\chi = \chi(\omega)$ depends only on the angular coordinate ω on M^n. It follows that $\chi : \mathbb{S}^{n-1} \to \mathbb{S}^{\ell-1}$ must be a harmonic map, which imposes on n, ℓ the relation $\ell = k(k + n - 2)$ for some integer $k \geq 1$. On the other hand $\phi(t, r)$ must satisfy the *ℓ-equivariant wave map equation*

$$\phi_{tt} - \phi_{rr} - (n-1)\frac{h'(r)}{h(r)}\phi_r + \frac{\ell}{h(r)^2} g(\phi)g'(\phi) = 0 \tag{3}$$

for which one considers the Cauchy problem with initial data

$$\phi(0, r) = \phi_0(r), \qquad \phi_t(0, r) = \phi_1(r). \tag{4}$$

When $h(r) = r$ the base space is the flat $\mathbb{R}^n$ and (3) reduces to the equation originally studied in [11].

In the following statement we use the notation $|D_M| = (-\Delta_M)^{\frac{1}{2}}$, where Δ_M is the Laplace-Beltrami operator on M^n. If $v : M^n \to N^\ell$ is an equivariant map of the form $v = (\phi(r), \chi(\omega))$ with $\chi : \mathbb{S}^{n-1} \to \mathbb{S}^{\ell-1}$ a fixed harmonic map, its Sobolev $H^s(M^n; N^\ell)$ norm can be equivalently expressed as

$$\|v\|_{H^s(M^n;N^\ell)} \simeq \|\phi\|_{H^s} := \|(1 - \Delta_M)^{\frac{s}{2}}\phi\|_{L^2(M^n)}.$$

We define also the weighted Sobolev space $H^s_q(w)$ of radial functions on M^n with norm

$$\|\phi\|_{H^s_q(w)} := \|w^{-1}(|x|)\phi(|x|)\|_{H^s_q(\mathbb{R}^{n+2k})}, \qquad w(r) := r^k \frac{r^{\frac{n-1}{2}}}{h(r)^{\frac{n-1}{2}}}.$$

and we choose the indices (p, q) as

$$p = \frac{4(m+1)}{m+3}, \qquad q = \frac{4m(m+1)}{2m^2 - m - 5}, \qquad m = n + 2k. \tag{5}$$

The notation $L^\infty H^s \cap CH^s$ denotes the space of continuous bounded functions from $\mathbb{R}$ to H^s, while $L^p H^s_q(w)$ is the space of functions $\phi(t, r)$ which are L^p in time with values in $H^s_q(w)$. Our main result is the following:

Theorem 1 (Global existence in the critical norm)**.** *Let $n \geq 3$, $k \geq 1$, $\ell = k(k+n-2)$ and p, q as in* (5)*. Assume M^n and N^ℓ are two rotationally invariant manifolds with metrics given by* (2)*, with M^n admissible, and let h_∞ be the limit of $h^{\frac{1-n}{2}}(h^{\frac{n-1}{2}})''$ as $r \to \infty$. Consider the Cauchy problem* (3)*,* (4)*.*

If $h_\infty > 0$ *and* $\|\phi_0\|_{H^{\frac{n}{2}}} + \|\phi_1\|_{H^{\frac{n}{2}-1}}$ *is sufficiently small, the problem has a unique global solution* $\phi(t,r) \in L^\infty H^{\frac{n}{2}} \cap CH^{\frac{n}{2}} \cap L^p H_q^{\frac{n-1}{2}}(w)$.

If $h_\infty = 0$ *and* $\||D_M|^{\frac{1}{2}}\phi_0\|_{H^{\frac{n-1}{2}}} + \||D_M|^{-\frac{1}{2}}\phi_1\|_{H^{\frac{n-1}{2}}}$ *is sufficiently small, the problem has a unique global solution* $\phi(t,r)$ *with* $|D_M|^{\frac{1}{2}}\phi \in L^\infty H^{\frac{n-1}{2}} \cap CH^{\frac{n-1}{2}}$ *and* $\phi \in L^p H_q^{\frac{n-1}{2}}(w)$.

Remark 1. The solution is global in the critical space $H^{\frac{n}{2}} \times H^{\frac{n}{2}-1}$, and it enjoys additional L^pL^q integrability properties, determined by the Strichartz estimates used in the proof. This has the usual drawback that uniqueness holds only in a restricted space. *Unconditional uniqueness* in the critical space without additional restrictions was proved recently for general wave maps on Minkowski space in [7]. We conjecture that a similar result holds also in our situation; as a partial workaround, we prove that if the regularity of the initial data is increased by $\delta = \frac{1}{m+1}$ then uniqueness hods in the space $CH^{\frac{n}{2}+\frac{1}{m+1}}$.

Remark 2. The proofs of the previous results are contained in a paper under review.

Remark 3 (Examples of admissible manifolds). We can prove that suitable perturbations of admissible manifolds are also admissible; this allows to substantially enlarge the list of explicit examples. The following manifolds are included in the class:

- The Euclidean $\mathbb{R}^n$ and, more generally, rotationally invariant, asymptotically flat spaces of dimension $n \geq 3$, with a radial component of the metric of the form $h_\epsilon(r) = r + \mu(r)$, with $\mu : \mathbb{R}^+ \to \mathbb{R}$ such that for small $\epsilon > 0$
$$|\mu(r)| + r|\mu'(r)| + r^2|\mu''(r)| + r^3|\mu'''(r)| \leq \epsilon r \quad \text{for all} \quad r > 0$$
and
$$|\mu^{(j)}(r)| \lesssim r^{1-j} \quad \text{for} \quad r >> 1, \quad j \leq [\tfrac{n-1}{2}] + 2.$$
- Real hyperbolic spaces $\mathbb{H}^n$ with $n \geq 3$, for which $h(r) = \sinh r$; more generally, rotationally invariant perturbations of $\mathbb{H}^n$ with a metric $h_\epsilon(r) = \sinh r + \mu(r)$, with $\mu : \mathbb{R}^+ \to \mathbb{R}$ such that for small $\epsilon > 0$
$$|\mu(r)| + |\mu'(r)| + |\mu''(r)| + |\mu'''(r)| \leq \epsilon \langle r \rangle^{-3} \sinh r \quad \text{for all} \quad r > 0$$
and
$$|\mu^{(j)}(r)| \lesssim r^{-1} e^r \quad \text{for} \quad r >> 1, \quad j \leq [\frac{n-1}{2}] + 2.$$
- Some classes of rotationally invariant manifolds with a metric $h(r)$ of polynomial growth $h(r) \sim r^M$, where M can be any $M \geq 1$.

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Invariance of convex sets for non-autonomous evolution equations governed by forms

Dominik Dier
(joint work with Wolfgang Arendt and El Maati Ouhabaz)

We consider a non-autonomous form $\mathfrak{a}\colon [0,T]\times V\times V\to\mathbb{C}$ where V is a Hilbert space which is densely and continuously embedded in another Hilbert space H. Denote by $\mathcal{A}(t)\in\mathcal{L}(V,V')$ the associated operator. Given $f\in L^2(0,T,V')$, one knows that for each $u_0\in H$ there is a unique solution $u\in H^1(0,T;V')\cap L^2(0,T;V)$ of

$$\dot{u}(t)+\mathcal{A}(t)u(t)=f(t),\ u(0)=u_0.$$

This result by J. L. Lions is well-known. The aim of this talk is to present a criterion for the invariance of a closed convex subset $\mathcal{C}$ of H. More precisely, suppose $PV\subset V$ and

$$\operatorname{Re}\mathfrak{a}(t,Pv,v-Pv)\geq\operatorname{Re}\langle f(t),v-Pv\rangle$$

for all $v\in V$. Then we show that $u(t)\in\mathcal{C}$ for all $t\in[0,T]$ provided that $u_0\in\mathcal{C}$.

In the autonomous case for $f=0$, the criterion is known and even equivalent to invariance by a result proved in [2]. We give applications to positivity and comparison of solutions to heat equations with non-autonomous Robin boundary conditions.

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A comparison of triple jump and Suzuki fractals for obtaining high order from an almost symmetric Strang splitting scheme

Lukas Einkemmer, Alexander Ostermann

We consider the time discretization of ordinary and partial differential equations. More specifically, we assume that the considered problem can be written as the following abstract Cauchy problem

$$u' = Au + B(u), \qquad u(0) = u_0. \tag{1}$$

In this context splitting methods can be applied if the partial flows generated by A and B have an analytical representation, or if an efficient algorithm for finding their exact solution is known. However, the assumption that the partial flow generated by the nonlinear operator B can be computed exactly is usually a very strong requirement.

To remedy this deficiency of classic splitting methods, we propose and analyze splitting schemes which approximate the partial flow generated by B by that of an inhomogeneous linear differential equation. That is, we consider the linearized problem given by

$$v' = b(u_\star)v + d, \tag{2}$$

where we assume that, once a value $u_\star$ is substituted, the flow corresponding to (2) can be computed efficiently.

In this context, we can formulate the (classic) Strang splitting scheme as follows

$$M_\tau(u_0) = \mathrm{e}^{\frac{\tau}{2}A}\varphi_\tau^{b(u_{1/2})}(\mathrm{e}^{\frac{\tau}{2}A}u_0), \qquad u_{1/2} = \varphi_{\frac{\tau}{2}}^{b(u_0)}(\mathrm{e}^{\frac{\tau}{2}A}u_0), \tag{3}$$

where the flow corresponding to equation (2) is denoted by $\varphi_t^{b(u_\star)}$; τ is the step size. The Strang splitting above is no longer symmetric. The lacking symmetry of the method does not severely affect performance; however, if composition is used as a means to construct higher order methods, symmetry is a desirable property.

To remedy the lack of symmetry we consider the symmetric scheme $u_1 = S_\tau(u_0)$ given by the solution of the following implicit equation

$$u_1 = \mathrm{e}^{\frac{\tau}{2}A} \circ \varphi_{\frac{\tau}{2}}^{b(u_1)}(u_{1/2}).$$

Note that solving this equation is computationally not attractive in practice. Therefore, we proposed to employ fixed-point iterations to compute an approximation to u_1 (see [1] and [2]). The resulting one-step methods are not symmetric but they are symmetric up to a given order (a precise definition is given in [1]). Thus, it is possible to construct composition methods of arbitrary (even) order.

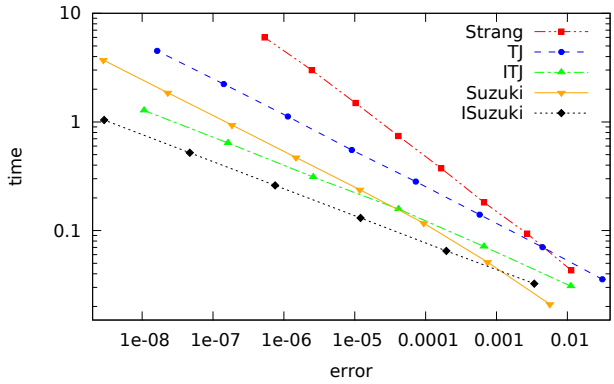

FIGURE 1. Work-precision plot (using the discrete infinity norm) for a charged particle in an inhomogeneous magnetic field integrated up to $T = 100$.

In [1] and [2] we have performed numerical simulations for a variety of ordinary and partial differential equations. To obtain methods of order four we have used the well-known triple jump composition. However, at least for some ODEs, the so called Suzuki fractals give better performance (see, e.g., [3]). In this case, five Strang splitting schemes have to be combined to obtain a scheme of order four (as compared to the three used in the triple jump approach). However, the error constant is expected to be smaller in the case of Suzuki fractals. Furthermore, the size of the negative time steps is significantly reduced compared to the triple jump scheme.

First, let us consider the ODEs describing a charged particle in an inhomogeneous magnetic field (more detailed information is provided in [1]). From the results shown in Figure 1, we observe that the iterated method (as described above) using the Suzuki fractals is clearly the method of choice for that particular problem. In [1] we have already shown that the splitting approach using the iterated triple jump scheme yields superior performance as compared to a fourth order Runge–Kutta method.

Second, we consider the parabolic Brusselator system (see [2]). In this case it is not possible to take negative time-steps. However, similar to the triple jump scheme, complex coefficients (with positive real part) can be determined that, using five compositions, yield a symmetric method of fourth order. While we observe from Figure 2 that the iterated triple jump schemes gives superior performance for medium to high precision (or equivalently long integration times), no additional benefit is observed by using the Suzuki fractals (in fact the run time for a given error is almost identical for the triple jump and the Suzuki fractal based methods).

Our last example is the hyperbolic KdV equation (see [2]). In this case, the Suzuki fractal method (that uses the classic Strang splitting and not its iterated version) already shows order four over some range in step size and thus, due to

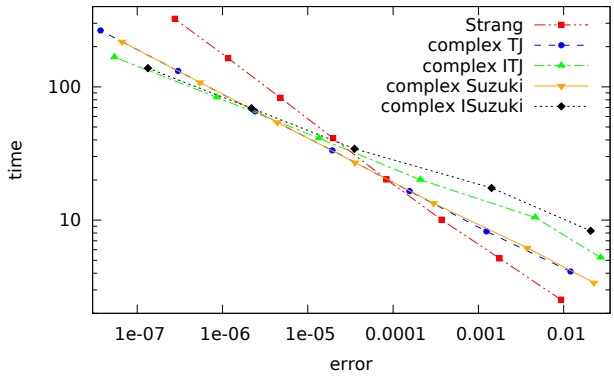

FIGURE 2. Work-precision plot (using the discrete infinity norm) for the Brusselator integrated up to $T = 0.25$ with 2^{10} grid points per dimension (in total we have 2^{21} grid points).

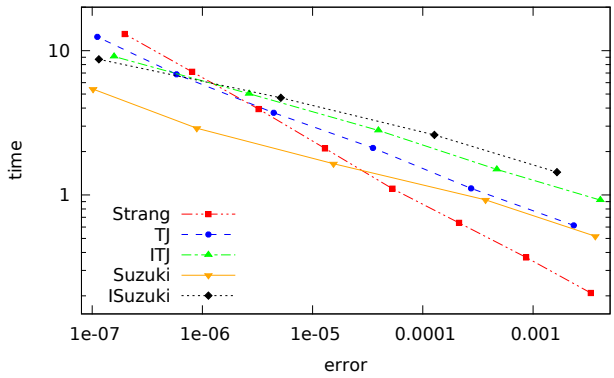

FIGURE 3. Work-precision plot (using the discrete infinity norm) for the KdV equation with the sech soliton initial value integrated up to $T = 0.05$ with 2^{10} grid points.

the low computational cost, proves to be the most efficient method (see Figure 3). We further note that the iterated schemes (both based on the Suzuki fractal as well as on the triple jump scheme) show similar performance.

In addition, work-precision plots for a number of problems (including the Van der Pol oscillator) are given in [4].

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Geodesic completeness for Sobolev H^s-metrics on the diffeomorphism group of the circle

Joachim Escher
(joint work with Boris Kolev)

The interest in right-invariant metrics on the diffeomorphism group of the circle started when it was discovered by Kouranbaeva [6] that the Camassa–Holm equation can be recast as the Euler equation of the right-invariant metric on $\mathrm{Diff}^\infty(\mathbb{S}^1)$ induced by the H^1-Sobolev inner product on the corresponding Lie algebra $\mathrm{C}^\infty(\mathbb{S}^1)$. The well-posedness of the geodesics flow for the right-invariant metric induced by the H^k inner product was obtained by Constantin and Kolev [2], for $k \in \mathbb{N}$, $k \geq 1$, following the pioneering work of Ebin and Marsden [3]. These investigations have been extended to the case of fractional order Sobolev spaces H^s with $s \in \mathbb{R}$, $s \geq 1/2$ by Escher and Kolev [4]. The method used to establish local existence of geodesics is to extend the metric and its spray to the Hilbert approximation $\mathcal{D}^q$ (the Hilbert manifold of diffeomorphisms of class H^q) and then to show that the (extended) spray is smooth. This was proved to work in [4] for $s \geq 1/2$ provided we choose $q > 3/2$ and $q \geq 2s$. The well-posedness on $\mathrm{Diff}^\infty(\mathbb{S}^1)$ follows as $q \to \infty$ from a regularity preserving result of the geodesic flow.

A Riemannian metric is *strong* if at each point it induces a topological isomorphism between the tangent space and the cotangent space. It is *weak* if it defines merely an injective linear mapping between the tangent space and the cotangent space. Note that on $\mathrm{Diff}^\infty(\mathbb{S}^1)$ only weak metrics exist. Furthermore we also mention that the extended metric on $\mathcal{D}^q$ is not strong but only weak as soon as $q > 2s$.

It is known that on a Banach manifold equipped with a strong metric, the geodesic semi-distance induced by the metric is in fact a distance. This is no longer true for weak metrics. It was shown by Bauer, Bruveris, Harms, and Michor [1] that this semi-distance identically vanishes for the H^s metric if $0 \leq s \leq 1/2$, whereas it is a distance for $s > 1/2$. This distance is nevertheless probably not complete on $T\mathcal{D}^q$. Indeed, although for a strong metric topological completeness implies geodesic completeness, this is generally not true for a weak metric. Finally, we recall that the metric induced by the H^1-norm (or equivalently by $A := I - D^2$) is not geodesically complete. The main result of this talk is the following, cf. [5]

Let $s > 3/2$ be given. Then the geodesic flow on $\mathcal{D}^q$ for $q \geq 2s + 1$ and on $\mathrm{Diff}^\infty(\mathbb{S}^1)$*, respectively, is complete for the weak Riemannian metric induced by the H^s-inner product.*

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Landau damping in Sobolev spaces for the Vlasov-HMF model

Erwan Faou

(joint work with Frédéric Rousset)

In this talk we consider the Vlasov-HMF model. This model has received much interest in the physics literature for many reasons: It is a simple ideal toy model that keeps several features of the long range interactions, it is a simplification of physical systems like charged or gravitational sheet models and it is rather easy to make numerical simulations on it. We study the long time behavior of solutions to this model for initial data that are small perturbations in a weighted Sobolev space to a spatially homogeneous stationary state satisfying a Penrose type stability condition. In [3], we prove that the solution scatters when times goes to infinity towards a modified state close to the initial data in a Sobolev space of lower order. This result implies a nonlinear Landau damping effect with polynomial rate for the solution which converges weakly towards a modified spatially homogeneous state. In the case of analytic or Gevrey regularity, this result has been shown to hold for a large class of Vlasov equations that contains the Vlasov-Poisson system by Mouhot and Villani [4] (see also the recent simplified proof [2]). The related problem of the stability of the Couette flow in the two-dimensional Euler equation has been also studied recently [1].

The Vlasov-HMF model reads

$$\partial_t f(t,x,v) + v\partial_x f(t,x,v) = \partial_x\Big(\int_{\mathbb{R}\times\mathbb{T}} P(x-y) f(t,y,u)\mathrm{d}u\mathrm{d}y\Big)\partial_v f(t,x,v), \tag{1}$$

where $(x,v) \in \mathbb{T}\times\mathbb{R}$ and the kernel $P(x)$ is given by $P(x) = \cos(x)$. Note that the main difference with the Vlasov-Poisson equation is the regularity of the kernel: in this latter case, $P(x) = \sum_{k\geq 0} k^{-2}\cos(kx)$ is the kernel associated with the inverse of the Laplace operator. The HMF model is thus the simplest nonlinear model with the structure (1). We consider initial data under the form $f_0(x,v) = \eta(v) + \varepsilon r_0(x,v)$ where ε is a small parameter and r_0 is of size one (in

a suitable functional space). This means that we study small perturbations of a stationary solution $\eta(v)$. We shall thus write the solution at time t under the form

$$f(t,x,v) = \eta(v) + \varepsilon r(t,x,v).$$

We are interested in the study of the behavior of f when time goes to infinity. To filter the effect of the free transport, it is convenient to introduce (as in [4], [2]) the unknown $g(t,x,v) = r(t,x+tv,v)$ that is solution of the equation

$$\partial_t g = \{\phi(t,g),\eta\} + \varepsilon\{\phi(t,g),g\}. \tag{2}$$

where

$$\phi(t,g)(x,v) = \int_{\mathbb{R}\times\mathbb{T}} (\cos(x-y+t(v-u)))g(t,y,u)\mathrm{d}u\mathrm{d}y \tag{3}$$

and $\{f,g\} = \partial_x f \partial_v g - \partial_v f \partial_x g$ is the usual microcanonical Poisson bracket. We shall usually write $\phi(t)$ when the dependence in g is clear.

We shall work in the following weighted Sobolev spaces, for $m_0 > 1/2$ be given, we set

$$\|f\|^2_{\mathcal{H}^n} = \sum_{|p|+|q|\le n} \int_{\mathbb{T}\times\mathbb{R}} (1+|v|^2)^{m_0} |\partial_x^p \partial_v^q f|^2 \mathrm{d}x\mathrm{d}v, \tag{4}$$

and we shall denote by $\mathcal{H}^n$ the corresponding function space. We shall denote by $\hat{\cdot}$ or $\mathcal{F}$ the Fourier transform on $\mathbb{T}\times\mathbb{R}$ given by

$$\hat{f}_k(\xi) = \frac{1}{2\pi}\int_{\mathbb{T}\times\mathbb{R}} f(x,v) e^{-ikx-i\xi v}\mathrm{d}x\mathrm{d}v.$$

Note that due to the regularity of the interaction kernel and the conservation of the L^p norms, it is very easy to prove the global well-posedness of the Vlasov-HMF model in $\mathcal{H}^s$ for every $s \ge 0$. Nevertheless, in order to study the asymptotic behaviour of g, the regularity of the kernel is not of obvious help. Indeed, when performing energy estimate on (2), it costs one positive power of t each time one puts a v derivative on the kernel.

We shall need a stability property of the reference state η in order to control the linear part of the Vlasov equation (2). Let us denote by η, the spatially homogeneous stationary state and let us define the function

$$K(n,t) = -np_n\, nt\, \hat{\eta}_0(nt)\mathbb{1}_{t\ge 0}, \quad t\in\mathbb{R}, \quad n\in\mathbb{Z},$$

where $(p_k)_{k\in\mathbb{Z}}$ are the Fourier coefficients of the kernel $P(x)$. We shall denote by $\hat{K}(n,\tau) = \int_{\mathbb{R}} e^{-i\tau t}K(n,t)\,dt$ the Fourier transform of $K(n,\cdot)$. We shall assume that η satisfies the following condition

$$(\mathbf{H}) \quad (1+v^2)\eta(v) \in \mathcal{H}^5 \quad \text{and} \quad \exists\,\kappa>0, \quad \inf_{\mathrm{Im}\,\tau\le 0} |1-\hat{K}(n,\tau)| \ge \kappa, \quad n=\pm 1.$$

This assumption is very similar to the one used in [4], [2] and can be related to the standard statement of the Penrose criterion.

In the evolution of the solution $g(t,x,v)$ of (2), an important role is played by the quantity

$$\zeta_k(t) = \hat{g}_k(t,kt), \quad k\in\{\pm 1\}, \tag{5}$$

such that

$$\phi(t,g) = \frac{1}{2}\sum_{k\in\{\pm 1\}} e^{ikx}e^{iktv}\zeta_k(t).$$

Note that for $k \neq 0$, $\zeta_k(t)$ is the Fourier coefficient in x of the density $\rho(t,x) = \int_{\mathbb{R}} f(t,x,v)\,dv$. This quantity also plays a key part in the analysis of [4], [2]. Note that here we need only to control two Fourier modes due to our simple interaction kernel.

Let us define for every $s \geq 4$ and $T \geq 0$ the weighted norm

$$Q_{T,s}(g) = \sup_{t\in[0,T]} \frac{\|g(t)\|_{\mathcal{H}^s}}{\langle t\rangle^3} + \sup_{t\in[0,T]}\sup_{k\in\{\pm 1\}} \langle t\rangle^{s-1}|\zeta_k(t)| + \sup_{t\in[0,T]} \|g(t)\|_{\mathcal{H}^{s-4}}.$$

Our main result is:

Theorem 1. *Let us fix $s \geq 7$ and $R_0 > 0$ such that $Q_{0,s}(g) \leq R_0$ and assume that $\eta \in \mathcal{H}^{s+4}$ satisfies the assumption (**H**). Then there exists $R > 0$ and $\varepsilon_0 > 0$ such that for every $\varepsilon \in (0,\varepsilon_0]$ and for every $T \geq 0$, we have the estimate*

$$Q_{T,s}(g) \leq R.$$

As a consequence, we obtain the following scattering result:

Corollary 1. *Under the assumption of Theorem 1, there exists a constant C and $g^\infty(x,v) \in \mathcal{H}^{s-4}$ such that for all $r \leq s-4$ and $r \geq 1$,*

$$\forall\, t \geq 0, \quad \|g(t,x,v) - g^\infty(x,v)\|_{\mathcal{H}^r} \leq \frac{C}{\langle t\rangle^{s-r-3}}. \tag{6}$$

The consequence of such results is the following nonlinear Landau damping effect: as $g(t,x,v)$ is bounded in $\mathcal{H}^{s-4}$, the solution $f(t,x,v) = \eta(v) + \varepsilon r(t,x,v) = \eta(v) + \varepsilon g(t,x-tv,v)$ satisfies

$$\forall\, n \in \mathbb{Z}^*, \quad \forall \xi \in \mathbb{R}, \quad \forall\, \alpha+\beta = s-4, \quad |\hat{f}_n(t,\xi)| = \varepsilon|\hat{g}_n(t,\xi+nt)| \leq \frac{C\varepsilon}{\langle \xi+nt\rangle^\alpha \langle n\rangle^\beta}.$$

This yields that for every $n \neq 0$, $\hat{f}_n(t,\xi)$ tends to zero with a polynomial rate.

Moreover, by setting

$$\eta^\infty(v) := \eta(v) + \frac{\varepsilon}{2\pi}\int_{\mathbb{T}} g^\infty(x,v)\mathrm{d}x,$$

we have by the previous corollary that for $r \leq s-4$,

$$\forall\, \xi \in \mathbb{R}, \quad |\hat{f}_0(\xi) - \hat{\eta}_0^\infty(\xi)| \leq \frac{C}{\langle \xi\rangle^r \langle t\rangle^{s-r-3}}$$

In other words, $f(t,x,v)$ converges weakly towards $\eta^\infty(v)$.

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Stability of plane waves in the nonlinear Schrödinger equation: analysis and numerics

Ludwig Gauckler
(joint work with Erwan Faou, Christian Lubich)

We consider the cubic nonlinear Schrödinger equation

$$\mathrm{i}\frac{\partial}{\partial t}\psi = -\Delta\psi + \lambda|\psi|^2\psi, \qquad \psi = \psi(x,t), \tag{1}$$

with 2π-periodic boundary conditions in d space dimensions, $x \in \mathbb{T}^d = \mathbb{R}^d/(2\pi\mathbb{Z}^d)$. We consider the defocusing case ($\lambda = +1$) as well as the focusing case ($\lambda = -1$).

This nonlinear Schrödinger equation (1) has some special solutions, the *plane wave solutions*

$$\psi(x,t) = \rho\, \mathrm{e}^{\mathrm{i}m\cdot x}\, \mathrm{e}^{-\mathrm{i}\omega t} \tag{2}$$

with $\rho \geq 0$, $m \in \mathbb{Z}^d$ and $\omega = |m|^2 + \lambda\rho^2$. In the talk, the stability of the these plane wave solutions under small perturbations of the initial value has been discussed, both for the exact solution and for a numerical solution.

For the exact solution, the main result of [2] has been presented. This result states that plane wave solutions (2) behave orbitally stable over long times under generic perturbations of the initial value in a high order Sobolev norm. The considered time intervals extend to arbitrary negative powers of the smallness parameter. More precisely, for fixed but arbitrary $N > 1$ and $\rho_0 > 0$ with $1 + 2\lambda\rho_0^2 > 0$, there exist $s_0 > 0$, $C \geq 1$ and a set $\mathcal{P}$ of full Lebesgue measure in the interval $(0, \rho_0]$ such that for every $s \geq s_0$ there exists $\varepsilon_0 > 0$ such that the following holds for every $m \in \mathbb{Z}^d$: if $\|\psi(\bullet, 0)\|_{L^2} \in \mathcal{P}$ and

$$\big\|\mathrm{e}^{-\mathrm{i}m\cdot\bullet}\psi(\bullet,0) - \psi_m(0)\big\|_{H^s} = \varepsilon \leq \varepsilon_0,$$

then the solution ψ of the nonlinear Schrödinger equation (1) with this initial value satisfies

$$\big\|\mathrm{e}^{-\mathrm{i}m\cdot\bullet}\psi(\bullet,t) - \psi_m(t)\big\|_{H^s} \leq C\varepsilon$$

over long times

$$0 \leq t \leq \varepsilon^{-N}.$$

Here, $\psi_m(t)$ denotes the mth Fourier coefficient of $\psi(\bullet, t)$, and $\|\cdot\|_{H^s}$ denotes the Sobolev H^s-norm. In the talk, this result has been illustrated by numerical simulations and the required conditions have been discussed.

For the proof of long-time orbital stability as given in [2], one first applies a sequence of reductions and transformations to the nonlinear Schrödinger equation (1). These reductions and transformations eliminate the m-th Fourier mode (which is not small) using the conservation of mass and the gauge invariance of the equation. This leads to a new system of equations that has small initial values and is, surprisingly, still of Hamiltonian form and has, again surprisingly, non-resonant frequencies. It fits perfectly into the abstract frameworks considered in [1, 4]. Applying the long-time stability results of these works to the special situation and transforming back to the original variables then leads to the long-time orbital stability of plane wave solutions.

In the second part of the talk, the focus turned from analysis to numerics. The main question was: Does a standard numerical method have the same stability behaviour near plane waves as the exact solution? The considered numerical method was the *split-step Fourier method*, one of the most popular methods for the nonlinear Schrödinger equation (1). This method combines a Fourier collocation method for the discretization in space with a Strang splitting for the discretization in time, see, for instance, [5].

The split-step Fourier method can indeed reproduce the long-time orbital stability of plane waves under small perturbations of the initial value. This main result of [3] has been presented in the talk. For this long-time result for the numerical solution one has to control numerical frequencies and in particular *numerical resonances* in addition to the assumptions that were needed for the stability of the exact solution. For small m in (2), these numerical resonances can be excluded—at least generically—under a Courant–Friedrichs–Lewy (CFL) type condition on the discretization parameters. The condition requires essentially that the product of the time step-size h and the largest frequency of the semi-discretization in space is bounded by $\pi/(2N+1)$ for orbital stability of the numerical solution on a long time interval

$$0 \le t_n = nh \le \min\bigl(\varepsilon^{-N}, h^{-2N}\bigr).$$

In the regime outside the CFL condition, one can still verify the required conditions for suitable choices of the time step-size and apply the results of [3] to obtain long-time orbital stability of the numerical solution. This was illustrated in the talk by numerical experiments.

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Splitting of dissipative evolution equations

Eskil Hansen

(joint work with Tony Stillfjord)

The aim of this note is to give an overview of some recent progress when analyzing splitting schemes applied to nonlinear evolution equations. Such equations are frequently encountered in biology, chemistry and physics, as they describe reaction-diffusion systems, as well as the damped wave equation. More precisely, we will consider splitting based discretizations for evolution equations of the form

$$\dot{u} = (F + G)u, \quad u(0) = u_0, \tag{1}$$

where F is typically the vector field of a nonlinear diffusion process and G is a nonlinear source term. Due to the nonlinearity of $F + G$, we can not expect the solution to exhibit any higher-order temporal regularity. For example, if

$$(F + G)u = \Delta(|u|^m u) + 0,$$

then the solution of (1) is given by the classical Barenblatt solution which is not continuously differentiable in time nor space. Because of the lack of time-regularity it is not, in general, possible to prove that a time discretization of the problem converges with an order greater than $p = 1$. Furthermore, due to the presence of diffusion, a spatial discretization of the equation will result in a stiff ODE system and therefore implicit schemes are required. Of the few remaining numerical methods the implicit Euler scheme is then the natural choice, but it is often computationally costly. An alternative is given by splitting methods, where the flows related to F and G are are approximated separately. This can dramatically reduce the computational cost. We consider several (formally) first-order splitting schemes given by the time stepping operators

$$S_h = (I - hF)^{-1} P_{hG},$$

where $S_h^n u_0$ is an approximation of the solution u at time $t = nh$ and the operator P_{hG} is chosen depending on the structure of G. Three standard choices are $P_{hG} = I + hG$, $(I - hG)^{-1}$ or e^{hG}.

The foundation of our numerical analysis is to assume that (1) is given on a real Banach space X and the nonlinear operator $F + G$ is m-dissipative, i.e., the resolvent

$$R_h = (I - h(F + G))^{-1}$$

is defined on X and it has a Lipschitz constant of the form $L[R_h] \leq 1 + Ch$ for sufficiently small values of h. The m-dissipativity of the operator $F+G$ guarantees

the existence of a unique mild solution to (1), which is one of the core results of the nonlinear semigroup theory [1, 4]. The main idea of the existence proof is to establish that $\{R_{t/n}^n u_0\}_{n\geq 1}$ is a Cauchy sequence in X, and then define the mild solution as

$$u(t) = \lim_{n\to\infty} R_{t/n}^n u_0.$$

In other words, the solution is given by the limit of the implicit Euler discretization. Furthermore, as part of the original proof [4, Theorem I] a convergence order of $p = 1/2$ was derived for the implicit Euler approximation $R_h^n u_0$, i.e.,

$$\|R_h^n u_0 - u(nh)\| \leq C_T h^{1/2}, \quad 0 \leq nh \leq T,$$

when $u_0 \in \mathcal{D}(F+G)$. Note that a convergence order of $p = 1$ is typically observed when X is finite dimensional, but the obtained order of one half is in fact optimal for general m-dissipative vector fields, as exemplified by [8].

The convergence of splitting schemes can be proven in this m-dissipative framework, see e.g. [3] where the case $P_{hG} = (I - hG)^{-1}$ is treated. Also [2] and [9] prove convergence of several similar splitting schemes in different contexts. However, the aim of this work is to prove *orders of convergence*. Our basic idea, which unifies the theory for different P_{hG}, is to prove that the splitting approximation $S_h^n u_0$ is within an $\mathcal{O}(h^q)$-vicinity of the implicit Euler approximation $R_h^n u_0$ when the operator G has some further structure:

Theorem 1. *Let F, G and $F + G$ all be m-dissipative operators on the real Banach space X, $u_0 \in \mathcal{D}(F + G)$ be given and $h \leq h_0$. If P_{hG} is stable, i.e., $L[P_{hG}] \leq 1 + Ch$, and satisfies the consistency bound*

$$\|\big(hGR_h + I - P_{hG}\big)R_h^j u_0\| \leq Ch^{1+q}, \tag{2}$$

for all $j = 0, \ldots, n$, then

$$\|S_h^n u_0 - u(nh)\| \leq C_T(h^p + h^q), \quad 0 \leq nh \leq T,$$

where u is the mild solution of (1) and $p \in [1/2, 1]$ is the convergence order of the implicit Euler scheme.

The proof follows by the telescopic sum

$$\begin{aligned}
\|R_h^n u_0 - S_h^n u_0\| &\leq \sum_{j=1}^{n} \|S_h^{n-j} R_h^j u_0 - S_h^{n-j+1} R_h^{j-1} u_0\| \\
&\leq \sum_{j=1}^{n} L[S_h]^{n-j} L[(I - hF)^{-1}] \|\big((I - hF)R_h - P_{hG}\big)R_h^{j-1} u_0\| \\
&\leq C \sum_{j=1}^{n} \|(hGR_h + I - P_{hG})R_h^{j-1} u_0\|.
\end{aligned}$$

While the stability assumption $L[P_{hG}] \leq 1 + Ch$ is natural for a well-defined scheme, the question of when the consistency (2) is true remains. We give three different examples of this, which illustrate the wide applicability of the theory.

Example 1. Let G be Lipschitz continuous and choose $P_{hG} = I + hG$. This gives the implicit-explicit Euler method where the diffusive term F is approximated by the implicit scheme and the non-stiff perturbation by the explicit scheme. In this case it is seen that Equation (2) holds with $q = 1$. For example, the evolution of s competing species can be modeled by the system

$$\dot{u}_\ell = \Delta(u_\ell)^{m+1} + G_\ell(u_1, u_2, \dots, u_s), \quad \ell = 1, \dots, s,$$

which can be cast into the setting of an m-dissipative evolution equation on $X = [L^1(\Omega)]^s$. Splitting the system means that the diffusion terms decouple, and hence their approximations can be parallelized. See [5] for details.

Example 2. A nonlinear parabolic equation with delay such as

$$\dot{u}(t) = \nabla\cdot\big(|\nabla u(t)|^m \nabla u(t)\big) + G_1 u(t-1) + G_2 \int_{-1}^{0} u(t+s)\mathrm{d}s$$

can be formulated on an appropriate Banach space so that the operators are again m-dissipative. With $P_{hG} = (I - hG)^{-1}$, we get that the consistency assumption (2) holds with $q > 1/2$. Such equations e.g. model electrical circuits and more realistic population dynamics which takes gestation periods into account. We refer to [6].

Example 3. The abstract Riccati equation

$$\dot{u} = A^* \circ u + u \circ A + v - u \circ u$$

can be treated in the setting of Hilbert-Schmidt operators on $L^2(\Omega)$ when $-A$ is a linear elliptic differential operator. If $Fu = A^* \circ u + u \circ A + v$ and $Gu = -u \circ u$, respectively, then $P_{hG} = \mathrm{e}^{hG}$ has a closed-form expression and it can be proven that Equation (2) holds with $q = 1$. Riccati equations arise in e.g. linear quadratic regulator problems, where their solutions provide the link between the system states and the optimal feedback. For more details, see [7].

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Overcoming the order reduction of dimension splitting methods due to corner singularities

Tobias Hell
(joint work with Alexander Ostermann, Michael Sandbichler)

A *dimension splitting method* may suffer from a severe order reduction when applied to an inhomogeneous evolution equation of the form

$$u'(t) = \mathcal{L}u(t) + g(t)\,, \quad u(0) = u_0 \tag{1}$$

on $L^2(\Omega)$, where $\Omega = (0,1)^2$ and $\mathcal{L} = \partial_x(a\partial_x) + \partial_y(b\partial_y)$ is an uniformly strongly elliptic operator with $a, b \in \mathcal{C}^2(\overline{\Omega})$ and $D(\mathcal{L}) = H^2(\Omega) \cap H_0^1(\Omega)$. For instance, the *Lie resolvent splitting* involving the split operators $\mathcal{A} = \partial_x(a\partial_x)$ and $\mathcal{B} = \partial_y(b\partial_y)$ converges with order of $1/4 - \varepsilon$ in time for arbitrarily small $\varepsilon > 0$ if the condition $\mathcal{L}^{-1}g(t) \in D(\mathcal{AB})$ is not fulfilled for all $t \geq 0$, cf. [1]. By applying the modification described in [1], the full convergence order of 1 is achieved.

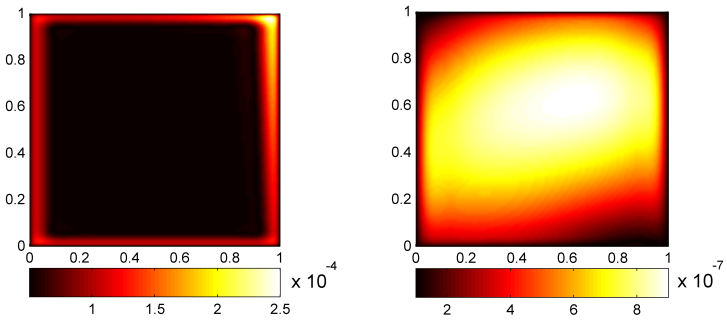

Figure 1. Pointwise error of the standard (left) and the modified Strang splitting (right) applied to (1) with an incompatible g.

By applying this modification to the *exponential Strang splitting* investigated in [2], the pointwise error is shifted into the interior of the domain, see Figure 1 and [3]. However, a convergence analysis of the numerically observed second-order convergence will be subject to further research.

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Endpoint Strichartz estimates and the cubic Dirac equation

Sebastian Herr
(joint work with Ioan Bejenaru)

In the first part of the talk we review the Strichartz estimates for the wave equation $\Box u = 0$ in dimension $d = 3$, see [5] for more information on this topic and further references. We focus on the estimate

$$\|e^{it|\nabla|}\phi\|_{L_t^p(\mathbb{R},L_x^q(\mathbb{R}^3))} \lesssim \|\phi\|_{\dot{H}_x^{\frac{2}{p}}(\mathbb{R}^3)} \tag{1}$$

and explain the proof in the case $\frac{1}{p}+\frac{1}{q} = \frac{1}{2}$, $p > 2$, which is reduced by Littlewood-Paley theory, the TT^*-argument and the Hardy-Littlewood-Sobolev-Lemma to the decay bound

$$|K_j(t,x)| \lesssim 2^{3j}(1 + 2^j|t|)^{-1}$$

for the kernel

$$K_j(t,x) := \int_{\mathbb{R}^3} e^{i(t,x)\cdot(|\xi|,\xi)}\chi_0^2(\xi)d\xi = \mathcal{F}^{-1}(\chi_j^2\sigma_{\text{cone}})(t,x)$$

where $(\chi_j)_{j\in\mathbb{Z}}$ is s smooth partition of unity satisfying $\text{supp}(\chi_j) \in \{\xi \in \mathbb{R}^3 \colon 2^{j-2} < |\xi| \leq 2^{j+2}\}$. The decay for $|t| \to \infty$ follows from a stationary phase argument.

In the second part of the talk, we turn to the endpoint Strichartz estimate for the Klein-Gordon equation $(\Box + 1)u = 0$. This reduces to the estimate

$$\|e^{it\langle\nabla\rangle}\phi\|_{L_t^2(\mathbb{R},L_x^\infty(\mathbb{R}^3))} \lesssim \|\phi\|_{\dot{H}_x^1(\mathbb{R}^3)}\,, \tag{2}$$

which is known to *fail* similar to the case of the wave equation (i.e. (1) with $p = 2$ and $q = \infty$), see [9] for the latter case. As above, the problem comes from the slow decay of the corresponding kernel, which is determined by curvature properties of the characteristic sets. However, it can be improved in certain directions: Let $\mathcal{C}_j$ be a covering of $\mathbb{S}^2$ by $\sim 2^{2j}$ maximally separated caps κ of radius 2^{-j} and center $\omega(\kappa)$ and let $(\eta_\kappa)_{\kappa\in\mathcal{C}_j}$ be a subordinate smooth angular partition of unity on $\mathbb{R}^3$ with $\text{supp}(\eta_\kappa) \subset \{\text{dist}(\omega(\kappa), \frac{\xi}{|\xi|}) \lesssim 2^{-j})\}$. For the angularly localized kernel

$$K_{j,\kappa}(t,x) = \int_{\mathbb{R}^3} e^{i(t,x)\cdot(\langle\xi\rangle,\xi)}\chi_j^2(\xi)\eta_\kappa(\xi)d\xi = \mathcal{F}^{-1}(\chi_j^2\eta_\kappa\sigma_{\text{hyperboloid}})(t,x)$$

we observe $|K_{j,\kappa}(t,x)| \lesssim 2^j(1 + 2^{-j}|t|)^{-\frac{3}{2}}$, which implies $\|K_{j,\kappa}\|_{L_t^1L_x^\infty} \lesssim 2^{2j}$. This can be improved substantially by introducing new coordinates, similar to the ideas from the wave maps problem [12, 11, 6, 7]: Let $\lambda = \lambda_j = \sqrt{1 + 2^{-2j}}$, $\omega = \omega(\kappa)$ and $\Theta_{\lambda,\omega} = (1 + \lambda^2)^{-\frac{1}{2}}(\lambda, \omega)$, $\Theta_{\lambda,\omega}^\perp = (1 + \lambda^2)^{-\frac{1}{2}}(-1, \lambda\omega)$, and $t_\kappa = \Theta_{\lambda,\kappa} \cdot (t,x)$, $x_\kappa^1 = \Theta_{\lambda,\kappa}^\perp \cdot (t,x)$ and (t_κ, x_κ) the associated orthogonal coordinates. We obtain $\|K_{j,\kappa}\|_{L_{t_\kappa}^1 L_{x_\kappa}^\infty} \lesssim 1$, and as a consequence:

Theorem 1 (see Theorem 2.1 in [2]). *For all $j \geq 0$, $\kappa \in \mathcal{C}_j$ and $\phi \in L^2(\mathbb{R}^3)$ such that* $\mathrm{supp}(\widehat{\phi}) \subset \{|\xi| \sim 2^j\}$ *we have*

$$2^{-j}\|e^{it\langle\nabla\rangle}P_\kappa\phi\|_{L^2_tL^\infty_x} + \|e^{it\langle\nabla\rangle}P_\kappa\phi\|_{L^2_{t_\kappa}L^\infty_{x_\kappa}} \lesssim \|\phi\|_{L^2},$$
$$\sum_{\kappa\in\mathcal{C}_j}\|e^{it\langle\nabla\rangle}P_\kappa\phi\|_{L^2_{t_\kappa}L^\infty_{x_\kappa}} \lesssim \|\phi\|_{\dot{H}^1},$$

where $\widehat{P_\kappa\phi} = \eta_\kappa\widehat{\phi}$.

In addition, there are corresponding energy-estimates in these coordinates (see Theorem 2.4 in [2]): If $\widehat{\phi}$ has angular support in another cap κ' and $\mathrm{dist}(\kappa,\kappa') = \alpha \gg 2^{-j}$, then

$$\|e^{it\langle\nabla\rangle}\phi\|_{L^\infty_{t_\kappa}L^2_{x_\kappa}} \lesssim \alpha^{-1}\|\phi\|_{L^2}.$$

In the third part of the talk, we briefly discuss an application of the above ideas to the cubic Dirac equation

$$(-i\gamma^\mu\partial_\mu + M)\psi = (\gamma^0\psi,\psi)\psi,$$

for the spinor field $\psi : \mathbb{R}^4 \to \mathbb{C}^4$, where $M > 0$, using the summation convention with respect to $\mu = 0,\dots,3$. Here, $\gamma^\mu \in \mathbb{C}^{4\times4}$ are the Dirac matrices and $(\cdot,\cdot)$ is the standard inner product on $\mathbb{C}^4$. This is a model in quantum field theory for self-interacting Dirac fermions [10, 4]. By scaling considerations, it turns out that the equation is $H^1(\mathbb{R}^3)-$critical. We prove the conjectured critical well-posedness result for small data, extending the results for radial data [8] and for data of subcritical regularity [3]:

Theorem 2 (cp. Theorem 1.1 in [2]). *The cubic Dirac equation is globally well-posed for small initial data in $H^1(\mathbb{R}^3)$ and these solutions scatter to free solutions as $t \to \pm\infty$.*

Here, the notion of well-posedness includes existence of mild solutions which are continuous curves in $H^1(\mathbb{R}^3)$, uniqueness in a certain subspace, and smooth dependence on initial data. The strategy of proof is as follows:

(1) reduction to a nonlinear system of evolution equations with null structure
(2) nonlinear estimates and contraction mapping principle in adapted function spaces
(3) Key ingredient: endpoint Strichartz and energy estimates imply bilinear L^2-estimates

For the first step, we adapt the ideas which have been developed for the Dirac-Klein-Gordon system in [1]: Multiplying the equation from the left by the matrix $\gamma^0 =: \beta$ leads to

$$-i(\partial_t + \alpha\cdot\nabla + i\beta)\psi = (\psi,\beta\psi)\beta\psi$$

where $\alpha^j = \gamma^0\gamma^j$ and $\alpha\cdot\nabla = \alpha^j\partial_j$. The operator $\alpha\cdot\nabla + i\beta$ has the symbol $\alpha\cdot\xi+\beta$. Because of $(\alpha\cdot\xi+\beta)^2 = (|\xi|^2+1)I_4$ the eigenvalues are $\pm\langle\xi\rangle$. Let $\Pi_\pm(\xi)$ denote

the projections on the eigenspaces. Then, we can write down an equivalent system for $\psi_\pm = \Pi_\pm \psi$, $\psi = \Pi_+\psi + \Pi_-\psi$, which is

$$\begin{aligned}(i\partial_t + \langle\nabla\rangle)\psi_+ &= -\Pi_+((\psi, \beta\psi)\beta\psi),\\ (i\partial_t - \langle\nabla\rangle)\psi_- &= -\Pi_-((\psi, \beta\psi)\beta\psi).\end{aligned}$$

This system has a null-structure (cp. [1]) which damps down products of waves with parallel frequencies:

$$\Pi_\pm(\xi)\Pi_\mp(\eta) = \mathcal{O}(\angle(\xi,\eta)) + \mathcal{O}(\langle\xi\rangle^{-1} + \langle\eta\rangle^{-1}).$$

Then, we construct function spaces $N^\pm$, $S^\pm$ such that

$$\begin{aligned}\|\Pi_\pm\psi\|_{S^\pm} &\lesssim \|\Pi_\pm\psi(0)\|_{H^1(\mathbb{R}^3)} + \|(i\partial_t \pm \langle\nabla\rangle)\Pi_\pm\psi\|_{N^\pm},\\ \|\Pi_\pm((\Pi_\pm\psi_1, \beta\Pi_\pm\psi_2)\beta\Pi_\pm\psi_3)\|_{N^\pm} &\lesssim \prod_{j=1}^{3}\|\Pi_\pm\psi_j\|_{S^\pm},\end{aligned}$$

which allows us to apply the contraction mapping principle in a small ball in $S^+ \times S^-$. The key ingredient for the above nonlinear estimate is a bilinear L^2L^2-estimate, which we prove using versions of $L^\infty L^2$ (energy) and L^2L^∞ (endpoint Strichartz) in frames, and the null-structure to perform the summation with respect to cap decompositions. We refer the reader to [2] for more details.

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Analysis of nematic liquid crystals: an approach via evolution equations

Matthias Hieber

(joint work with Manuel Nesensohn, Jan Prüss and Katharina Schade)

We consider the following system of equations modeling the flow of nematic liquid crystal materials in a bounded domain $\Omega \subset \mathbb{R}^3$ with smooth boundary $\partial\Omega$

$$\begin{cases} \partial_t u + (u\cdot\nabla)u &= \operatorname{div}\sigma & \text{in } (0,T)\times\Omega,\\ \operatorname{div} u &= 0 & \text{in } (0,T)\times\Omega,\\ d\times\left(g+\operatorname{div}\left(\frac{\partial W}{\partial(\nabla d)}\right)-\frac{\partial W}{\partial d}\right) &= 0 & \text{in } (0,T)\times\Omega,\\ |d| &= 1 & \text{in } (0,T)\times\Omega,\\ (u,\partial_\nu d) &= (0,0) & \text{on } (0,T)\times\partial\Omega,\\ (u,d)_{|t=0} &= (u_0,d_0) & \text{in } \Omega. \end{cases}$$

This system was developed by Ericksen and Leslie and is nowadays called the *Ericksen-Leslie* model for nematic liquid crystals. Here, the function $u : (0,\infty)\times\Omega\to\mathbb{R}^3$ denotes the velocity field and the function $d : (0,\infty)\times\Omega\to\mathbb{R}^3$ represents the macroscopic orientation of the molecules. The stress tensor σ is given by

$$\sigma := -pI - \frac{\partial W}{\partial([\nabla d]^T)}\nabla d + \sigma^L,$$

where p denotes the pressure of the fluid, σ^L the Leslie stress tensor and W the Oseen-Frank energy density functional.

Based on the the maximal regularity approach of evolution equations we prove local wellposedness of this system in the L^p-setting under various assumptions on the Leslie and elasticity coefficients, see [2]. For recent results concerning this system and in particular global weak solutions for this system in two space dimensions, we refer to [3] and the references therein.

The dynamics of this system is best understood for the case of the so called simplified model. Here $W(d,\nabla d)=\frac{1}{2}|\nabla d|^2$ and the set of equations then reads as

$$\begin{cases} \partial_t u + (u\cdot\nabla)u - \nu\Delta u + \nabla\pi &= -\lambda\operatorname{div}([\nabla d]^{\mathsf{T}}\nabla d) & \text{in } (0,T)\times\Omega,\\ \partial_t d + (u\cdot\nabla)d &= \gamma(\Delta d + |\nabla d|^2 d) & \text{in } (0,T)\times\Omega,\\ \operatorname{div} u &= 0 & \text{in } (0,T)\times\Omega,\\ (u,\partial_\nu d) &= (0,0) & \text{on } (0,T)\times\partial\Omega,\\ (u,d)_{|t=0} &= (u_0,d_0) & \text{in } \Omega. \end{cases}$$

Again we impose the condition

$$|d| = 1 \qquad \text{in } (0,T)\times\Omega.$$

Pioneering work concerning this system goes back to Lin and Liu [4].

Analyzing the above equation as a quasilinear parabolic evolution equation in the L^p-setting we show first the existence of a unique local strong solution. This solutions extends to a global strong solution provided the initial data are close to an equilibrium point. In this case the solution converges exponentially to an

equilibrium. We further show that the equilibria of this system are given by the set

$$\mathcal{E} = \{(u_*, d_*) : u_* = 0, d_* \in \mathbb{R}^n, |d = 1|\}$$

Finally, the solution is shown to be real analytic, jointly in time and space, see [1].

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A preconditioned Krylov method for an exponential integrator for non-autonomous parabolic problems

David Hipp

(joint work with Marlis Hochbruck)

In this note we briefly discuss some aspects of the implementation of the explicit exponential integrator for the non-autonomous parabolic Cauchy problems we proposed in [2] and analyzed in [1]. For problems arising from finite element discretizations of partial differential equations on time-dependent domains this integrator requires the approximation of $\mathbf{x} = f(-\tau \mathbf{S}\mathbf{M}^{-1})\mathbf{y} = \mathbf{M}f(-\tau \mathbf{M}^{-1}\mathbf{S})\mathbf{u}$ with $\mathbf{u} = \mathbf{M}^{-1}\mathbf{y}$. Here f is an analytic function consisting of exponentials or φ-functions defined as

$$\varphi_j(z) = \int_0^1 e^{(1-\theta)z} \frac{\theta^{j-1}}{(j-1)!} \mathrm{d}\theta,$$

τ is the time step size and $\mathbf{M}$ and $\mathbf{S}$ denote the mass matrix and the stiffness matrix, respectively. For large scale problems, we suggest to approximate $\mathbf{x}$ by polynomial or rational Krylov subspace methods, see [3] for a comprehensive discussion and [4] for detailed algorithms.

We consider the stiff ordinary differential equation

$$\frac{d}{dt}\Big(\mathbf{M}(t)\mathbf{u}(t)\Big) = -\mathbf{S}(t)\mathbf{u}(t), \qquad \mathbf{u}(0) = \mathbf{u}_0.$$

Formally, the exponential integrator of [1] is applied to a transformed problem given in the coordinates $\mathbf{y}(t) = \mathbf{M}(t)\mathbf{u}(t)$. Let $\mathbf{M}_n = \mathbf{M}(t_n)$, $\mathbf{S}_n = \mathbf{S}(t_n)$, where $t_n = n\tau$ and $\Phi_1(z) = \varphi_2(z) - 2\varphi_3(z)$, $\Phi_2(z) = 4\varphi_3(z) - \varphi_2(z)$. The integrator of

[2] applied to the above problem can then be recast as:

$$\begin{aligned}
\widetilde{\mathbf{y}}_{n+1} &= \exp\left(-\tau\mathbf{M}_n^{-1}\mathbf{S}_n\right)\mathbf{u}_n, \qquad \widetilde{\mathbf{y}}_{n+1/2} = \exp\left(-\tfrac{\tau}{2}\mathbf{M}_n^{-1}\mathbf{S}_n\right)\mathbf{u}_n,\\
\widetilde{\mathbf{u}}_{n+1} &= \mathbf{M}_{n+1}^{-1}\mathbf{M}_n\widetilde{\mathbf{y}}_{n+1},\\
\widetilde{\mathbf{z}}_{n+1/2} &= \mathbf{M}_{n+1}^{-1}\Big(\mathbf{S}_n\widetilde{\mathbf{y}}_{n+1/2} - \mathbf{S}_{n+1/2}\mathbf{M}_{n+1/2}^{-1}\mathbf{M}_n\widetilde{\mathbf{y}}_{n+1/2}\Big),\\
\widetilde{\mathbf{z}}_{n+1} &= \mathbf{M}_{n+1}^{-1}\Big(\mathbf{S}_n\widetilde{\mathbf{y}}_{n+1} - \mathbf{S}_{n+1}\widetilde{\mathbf{u}}_{n+1}\Big),\\
\mathbf{u}_{n+1} &= \widetilde{\mathbf{u}}_{n+1} + \Phi_1\left(-\tau\mathbf{M}_{n+1}^{-1}\mathbf{S}_{n+1}\right)\left(4\tau\widetilde{\mathbf{z}}_{n+1/2}\right) + \Phi_2\left(-\tau\mathbf{M}_{n+1}^{-1}\mathbf{S}_{n+1}\right)\left(\tau\widetilde{\mathbf{z}}_{n+1}\right).
\end{aligned}$$

Observe that besides the computational costs for the necessary "autonomous" parts $\widetilde{\mathbf{y}}_{n+1}$ and $\widetilde{\mathbf{y}}_{n+1/2}$, each intermediate step is cheap since

- matrix-functions are only applied to vectors of norm $\mathcal{O}(\tau)$ and
- linear systems of equations $\mathbf{Ax} = \mathbf{b}$ are only solved for small $\mathbf{b}$ or a good initial guess $\mathbf{x}_0$ is available, as in $\mathbf{M}_{n+1}^{-1}\mathbf{M}_n\mathbf{x} = \mathbf{b}$ with $\mathbf{x}_0 = \mathbf{b}$, for instance. For these linear systems, the conjugate gradient method converged so fast that preconditioning did not pay off.

To approximate $f(-\tau\mathbf{M}_n^{-1}\mathbf{S}_n)\mathbf{u}$ with polynomial or rational Krylov methods (KM), each iteration requires the following operations involving $\mathbf{M}_n$ and $\mathbf{S}_n$:

operations	$\mathbf{M}_n\mathbf{u}$	$\mathbf{S}_n\mathbf{u}$	$\mathbf{M}_n^{-1}\mathbf{u}$	$\left(\gamma\mathbf{M}_n + \tau\mathbf{S}_n\right)^{-1}\mathbf{u}$
polynomial KM (per iteration)	1	1	1	–
rational KM (per iteration)	2	–	–	1

For the solution of linear systems with $\gamma\mathbf{M}_n + \tau\mathbf{S}_n$ arising in rational Krylov methods, we used multigrid preconditioning based on the initial mesh. This simplification (we are actually using time-dependent meshes) is justified since our assumptions on the evolving domain imply that the difference between a mesh and its finer version stays of the same scale as the spatial error for the whole time interval.

We now compare the performance of the two Krylov methods used to approximate matrix-functions up to a tolerance of 10^{-6}. The numerical tests were done with a Matlab implementation of the example considered in [1, Sect. 4]. The table below shows the CPU-times in milliseconds of one time step with the above implementation of the integrator (given $\mathbf{M}_n$, $\mathbf{S}_n$, and $\mathbf{u}_n$):

d.o.f. of spatial discretization	144	534	2052	8040	31824
polynomial KM ($\tau = 0.01$)	0.742	1.296	6.546	76.05	836.6
rational KM ($\tau = 0.01$)	2.398	3.415	9.983	35.73	302.5
polynomial KM ($\tau = 0.1$)	5.257	5.31	39.34	465.9	5123.0
rational KM ($\tau = 0.1$)	8.598	15.45	20.17	116.8	363.4

Clearly, the computational cost for a single iteration of the rational Krylov method is higher than for a polynomial method, but since the convergence is independent of the spatial discretization (cf. [3]), it requires the same number of iterations on all grids. In contrast, polynomial Krylov methods require larger and larger Krylov subspaces if the discretization is refined and thus the norms of

the matrices grow. Accordingly, the CPU-times show that the integrator using the rational Krylov method is faster for fine spatial discretizations. The difference becomes more evident for the larger time step size.

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Mathematical challenges in non-linear glass-fiber optics

Dirk Hundertmark

(joint work with M. Burak Ergogan, Young-Ran Lee, William Green)

The one-dimensional non-linear Schrödinger equation (NLS) with periodically varying dispersion coefficient

$$iu_t + d(t)u_{xx} + |u|^2 u = 0. \tag{1}$$

describes the amplitude of a signal transmitted via amplitude modulation of a carrier wave through a fiber-optical cable where the dispersion is varied periodically along the fiber, see, e.g., [13, 15, 17]. In (1) t corresponds to the distance along the fiber, x denotes the (retarded) time, and $d(t)$ is the dispersion along the waveguide, which, for practical purposes, one can assume to be piecewise constant.

The dispersion management technique uses alternating sections of (nearly) opposite dispersion, introducing a rapidly varying dispersion $d(t)$ with small average dispersion which leads to well-localized stable soliton-like pulses changing periodically along the fiber. It turned out to be very fruitful (see, *e.g.*, [1, 3, 4, 9, 16, 17]).

Writing $d(t) = d_0(t) + d_{av}$, the envelope equation takes the form

$$iu_t + d_0(t)u_{xx} + d_{\mathrm{av}}u_{xx} + |u|^2 u = 0. \tag{2}$$

Here $d_0(t)$ is the mean zero part which we assume to be piecewise constant, and d_{av} the average dispersion over one period. Since the full equation (2) is very hard to study, Gabitov and Turitsyn suggested to separate the free motion given by the solution of $iu_t + d_0(t)u_{xx} = 0$ in (2), and to average over one period, [3, 4]. This yields the following dispersion management equation for the "averaged" solution v

$$iv_t + d_{\mathrm{av}}v_{xx} + Q(v, v, v) = 0, \quad \text{where} \tag{3}$$

$$Q(v_1, v_2, v_3) := \int_0^1 T_r^{-1}\Big(T_r v_1 \overline{T_r v_2} T_r v_3\Big) dr, \tag{4}$$

and $T_r = e^{ir\partial_x^2}$. In some sense, v is the slowly varying part of the solution u of (1) and the varying dispersion is interpreted as a fast background oscillation, justifying formally the above averaging procedure. The averaging procedure yielding (3) is similar to Kapitza's treatment of the unstable pendulum, see [10], and is well-supported by numerical and theoretical studies, see, for example, [1, 16, 17]. It was rigorously justified in [18] in the limit of large local dispersion $d_0(t)$.

One can find stationary solutions of (3) by making the ansatz $v(t,x) = e^{i\omega t} f(x)$, yielding the time independent equation

$$-\omega f = -d_{\rm av} f_{xx} - Q(f,f,f) \tag{5}$$

describing stationary soliton-like solutions, the so-called dispersion managed solitons. Despite the enormous interest in dispersion managed solitons, there are few rigorous results available. One reason for this is that it is a nonlinear and nonlocal equation. Existence and smoothness of solutions of (5) had first been rigorously established in [18] for positive average dispersion $d_{\rm av} > 0$. In the case $d_{\rm av} = 0$, the existence was obtained in [8], for the model case where $d_0 = \mathbf{1}_{[0,1)} - \mathbf{1}_{[1,2)}$, i.e., the simplest piecewise constant dispersion profile. See [7] for a simplified proof, which extend to large class of dispersion profiles, covering all physically relevant cases. Smoothness in the model case with case $d_{av} = 0$ was established in [14].

The decay of solutions was first addressed by Lushnikov in [11]. He gave convincing but non-rigorous arguments that any (weak) solution f of (5) for $d_{av} = 0$ satisfies

$$f(x) \sim |x| \cos(a_0 x^2 + a_1 x + a_2) e^{-b|x|} \quad \text{as } x \to \infty \tag{6}$$

for some suitable choice of real constants a_j and $b > 0$, see also [12]. In particular, he predicted that f and $\widehat{f}$ decay exponentially at infinity. For $d_{av} = 0$, the first rigorous x-space decay bounds were established in [6], where it was shown that both f and $\widehat{f}$ decay faster than any polynomial in the case $d_{\rm av} = 0$. In particular, any weak solution is a Schwartz function.

Our main result confirms Lusnikov's exponential decay prediction, [2]:

Theorem 1. *Let $f \in L^2$ be a (weak) solution of (5) with $d_{av} = 0$ in the model case $d_0 = \mathbf{1}_{[0,1)} - \mathbf{1}_{[1,2)}$. Then there exists $\mu > 0$ such that*

$$|f(x)| \lesssim e^{-\mu|x|}, \quad |\widehat{f}(\xi)| \lesssim e^{-\mu|\xi|},$$

where $\widehat{f}$ is the Fourier transform of f.

The difficulty in obtaining any decay for weak solutions comes from the fact that the kernel of the free Schrödinger evolution has no point-wise decay. Instead one has to use the multi-linear structure of $\mathcal{Q}$ and the oscillation of the kernel. In particular, our result follows from exponentially weighted multi-linear estimates for $\mathcal{Q}$. These are refinements of the x-space Strichartz estimates which were developed in [6] to prove, in conjunction with well-known Fourier space Strichartz estimates, that any weak solution $f \in L^2(\mathbb{R})$ of (5) for $d_{\rm av} = 0$ is already a Schwartz function.

We would also like to stress that our proof of Theorem 1 uses only the new exponentially weighted multi-linear estimates for $\mathcal{Q}$ together with $f \in L^2$ for any

weak solution and as such does not require any of the previous smoothness or decay results.

This decay result for solutions has been extended in [5] from the model case to a large class of dispersion profiles d_0, covering all physically relevant cases.

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Dispersive equations with quadratic nonlinearities

Herbert Koch
(joint work with Junfeng Li)

1. The Kadomtsev-Petviashvili II equation in 2d and 3d

The Kadomtsev-Petviashvili equation

$$\begin{cases} \partial_x \left(\partial_t u + \partial_x^3 u + \partial_x (u^2) \right) \pm \triangle_{\vec{y}} u = 0 \\ u(0, x, \vec{y}) = u_0(x, \vec{y}) \quad (x, \vec{y}) \in \mathbb{R} \times \mathbb{R}^2 \end{cases} \tag{1}$$

with $x, t \in \mathbb{R}$ and $y \in \mathbb{R}^d$, $d = 1, 2$ is a higher dimensional generalization of the Korteweg-de-Vries equation which takes into account weakly transversal effects. It has many interesting solutions. The soliton for the Korteweg-de-Vries equation is a solution to the Kadomtsev-Petviashvili equation with a trivial dependence on the transversal variable, which however is not square integrable since it does not decay in the transversal directions.

The KP equation comes in two flavors, depending on the relative size of the surface tension. KP-I is the equation with the minus sign, for which the line soliton is supposed to be unstable, and KP-II is the one with the + sign for which the line soliton is expected to be stable. The KP-II equation has better analytic properties, and it is the one we study here.

There are a number of symmetries:

- Translation in x, y and t
- Scaling: $\lambda^2 u(\lambda^3 t, \lambda x, \lambda^2 y)$ with $\lambda > 0$ is a solution if and only if u satisfies the KP equation.
- Galilean symmetry: $u(t, x - v \cdot y - |v|^2 t, y + 2tv)$ is a solution whenever u satisfies the KP equation and the velocity c is in $\mathbb{R}^d$

We will study well-posedness for initial data in a function space which is invariant under those symmetries. The scaling symmetry becomes an anisotropic scaling of the Fourier transform. Let $(\tau, \xi, \eta) \in \mathbb{R} \times \mathbb{R} \times \mathbb{R}^d$ be the independent variables of the space-time or spatial Fourier transform. The Galilean symmetry corresponds to a shear in the Fourier variables, leaving ξ invariant, or, as a consequence, it is a translation of the η variable for fixed ξ.

We call function spaces with bounded action of the symmetries *critical*. This simplest critical function spaces (for initial data) are defined by $\|u_0\|_{\dot{H}^{-1/2,0}} = \||\xi|^{-1/2} \hat{u}_0\|_{L^2}$ for $d = 1$ and $\|u_0\|_{\dot{H}^{1/2,0}} = \||\xi|^{1/2} \hat{u}_0\|_{L^2}$ for $d = 2$. There is the strong conviction that understanding well-posedness in critical spaces considerably improves the understanding of the nonlinear interaction. More down to earth it implies scattering, and global asymptotics for small initial data.

In [3] Hadac, Herr and the author have shown that the two dimensional ($d = 1$) KP-II equation is globally well-posed for small data. The three dimensional setting

is more challenging. Together with J. Li [4] we could prove that KP-II is well-posed in a function space invariant under Galilean symmetry and scaling, but this requires a nontrivial modification both, of the proofs and the function spaces.

Previous results include the seminal work by Bourgain in L^2 [1], Takaoka& Tzvetkov , Isaza & Mejia , Takaoka and Hadac [2] in the 2-dimensional subcritical setting, and Tzvetkov, Isaza, Lopez & Mejia, Hadac in the three-dimensional subcritical setting.

It is fairly recent that many critical problems of that type became accessible. It is the purpose of this notes to describe the crucial techniques in a broader context. The structure of the argument is best described in Schottdorf's thesis on Systems of Klein-Gordon equations with quadratic nonlinearities.

2. Function spaces

We will describe the function spaces in the context of the KP-II equation. The construction however is relevant in other situations as well. The linear KPII equation

$$u_t + u_{xxx} - \partial_x^{-1}\Delta_y u = 0$$

defines a unitary group $S(t)$ (Fourier transform). Bourgain has defined function spaces adapted to $S(t)$ by

$$\|u\|_{X^{s,b}} = \|S(-t)u(t)\|_{H^b(\mathbb{R};H^{s,0}(\mathbb{R}^{1+d}))} = \left\||\xi|^s|\tau - \xi^3 + |\eta|^2/\xi|^b\hat{u}\right\|_{L^2}.$$

For critical spaces one would want to use $X^{0,1/2}$ and $X^{0,-1/2}$ - which causes problems due to the failure of imbeddings like $H^{1/2} \not\subset C(\mathbb{R})$, $L^1 \not\subset H^{-1/2}$.

Tataru (unpublished) , Koch and Tataru [5] and Hadac & Herr & Koch [3] introduced a replacement by functions of bounded p variation V^p and their predual U^q which have several advantages compared to the $X^{s,b}$ spaces: Functions in V^2 are bounded, they satisfy better Strichartz and bilinear estimates, and they have good duality properties. They have been used in probability and harmonic analysis among others by Wiener, Lepingle, Bourgain and Lyons.

We will try to estimate quadratic nonlinearities by duality arguments. This leads to trilinear integrals

$$\int fghdx = (2\pi)^d \hat{f} * \hat{g} * \hat{h}(0).$$

Contributions come only from points in the support adding up to zero. The characteristic surface

$$\Sigma = \left\{(\tau,\xi,\eta) : \phi(\tau,\xi,\eta) = \tau - \xi^3 + \eta^2/\xi = 0\right\}$$

is of particular importance. The Fourier transform of a solution to the linear equation with initial data u_0 is

$$2\pi u_0(\xi,\eta)\delta_{\phi=0}$$

and the distance to Σ measures the deviation from being a solution.

Now, if $(\tau_i, \xi_i, \eta_i) \in \Sigma$, we define ω as

$$\begin{aligned}\omega =&\tau_1 + \tau_2 - (\xi_1 + \xi_2)^3 + (\eta_1 + \eta_2)^2/(\xi_1 + \xi_2)\\ =&-\xi_1\xi_2(\xi_1 + \xi_2)\left(3 + \left|\frac{\frac{\eta_1}{\xi_1} - \frac{\eta_2}{\xi_2}}{\xi_1 + \xi_2}\right|^2\right).\end{aligned}$$

The sum of two points on the characteristic surface has a vertical distance $|\omega|$ to the characteristic surface. This situation is called *nonresonant*.

Let

$$\hat{u}_\lambda = \begin{cases} \hat{u} & \lambda \le |\xi| < 2\lambda \\ 0 & \text{otherwise.}\end{cases}$$

and expand the product to

$$\int u_{\lambda_1} u_{\lambda_2} u_{\lambda_3} dxdydt$$

where $\lambda_1 \le \lambda_2 \le \lambda_3$. It suffices to consider $\lambda_1 = \mu \le \lambda = \lambda_2 = \lambda_3$. We expand $u_\lambda = u^h_\lambda + u^l_\lambda$ where the low modulation part u^l denotes the Fourier projection to the set where $|\omega| \le \mu\lambda^2/100$. Then

$$\int u^l v^l w^l dx\, dt\, dy = 0$$

and at least one term has to have high modulation, otherwise the integral vanishes. The high modulation part satisfies good L^2 estimates:

$$\|u^h\|_{L^2} \le c|\omega|^{-1/2}\|S(-t)u\|_{\dot{H}^{\frac{1}{2}}(\mathbb{R});L^2)} \le c\|u\|_{V^2}.$$

The remaining product has to be estimated in L^2 via a bilinear estimate

$$\|u_\mu v_\lambda\|_{L^2} \le C(\mu/\lambda)^{1/2}\|u_\mu\|_{U^2}\|v_\lambda\|_{U^2}$$

if $d = 1$ and

$$\|u_\mu v_\lambda\|_{L^2} \le C\mu\|u_\mu\|_{U^2}\|v_\lambda\|_{U^2}$$

if $d = 2$. This almost suffices if $d = 1$. A more sophisticated modification is needed if $d = 2$.

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On the backward Euler approximation of the stochastic Allen-Cahn equation

Mihály Kovács
(joint work with Stig Larsson and Fredrik Lindgren)

Let $\mathcal{D} \subset \mathbb{R}^d$, $d \leq 3$, be a spatial domain with smooth boundary $\partial\mathcal{D}$ and consider the stochastic partial differential equation written in the abstract Itô form

$$\mathrm{d}u + Au\,\mathrm{d}t + f(u)\,\mathrm{d}t = \mathrm{d}W,\ t \in (0,T]; \quad u(0) = u_0, \tag{1}$$

where $\{W(t)\}_{t\geq 0}$ is an $L^2(\mathcal{D})$-valued Q-Wiener process on a filtered probability space $(\Omega, \mathcal{F}, \mathbb{P}, \{\mathcal{F}_t\}_{t\geq 0})$ with respect to the normal filtration $\{\mathcal{F}_t\}_{t\geq 0}$. We use the notation $H = L^2(\mathcal{D})$ with inner product $\langle \cdot, \cdot \rangle$ and induced norm $\|\cdot\|$. Moreover, $A\colon D(A) \subset H \to H$ with $D(A) = H^2(\mathcal{D}) \cap H_0^1(\mathcal{D})$ denotes the linear elliptic operator $Au = -\nabla\cdot(\kappa\nabla u)$, where $\kappa(x) > \kappa_0 > 0$ is smooth. Finally, $f\colon D_f \subset H \to H$ is given by $(f(u))(x) = F'(u(x))$, where $F(s) = c(s^2-\beta^2)^2$ $(c > 0)$ is a double well potential. Note that f is not Lipschitz continuous and it does not satisfy a linear growth condition. It does, however, satisfy a global one-sided Lipschitz condition, which is a key property when proving uniform moment bounds.

We consider a fully implicit Backward Euler discretization of (1) via the iteration

$$u^j - u^{j-1} + \Delta t\, Au^j + \Delta t\, f(u^j) = \Delta W^j,\ j = 1, 2, \ldots, N; \quad u^0 = u_0,$$

where $\Delta t > 0$. Note that this scheme is implicit also in the drift term f but, in return, it preserves key qualitative aspects of the solution of (1) such as moment bounds. More precisely, let $I_n = \{1, 2, \ldots, n\}$, $1 \leq n \leq N$, $t_n = n\Delta t$ and $T = N\Delta t$. If $\|A^{\frac{1}{2}}Q^{\frac{1}{2}}\|_{\mathrm{HS}} < \infty$, where $\|\cdot\|_{\mathrm{HS}}$ denotes the Hilbert-Schmidt norm, $\mathbb{E}\|A^{\frac{1}{2}}u_0\|^p < \infty$, and $T^{p-1}\Delta t \leq \frac{1}{2}$ for some $p \geq 2$, then

$$\mathbb{E}\sup_{l\in I_n}\|u^l\|^p + \mathbb{E}\sup_{l\in I_n}\|A^{\frac{1}{2}}u^l\|^p \leq C(T, p, u_0).$$

Regarding convergence we show that if $\|A^{\frac{1}{2}+\varepsilon}Q^{\frac{1}{2}}\|_{\mathrm{HS}} < \infty$ for some small $\varepsilon > 0$, $\|A^{\frac{1}{2}}u_0\| < \infty$ almost surely, and $0 \leq \gamma < \frac{1}{2}$, then there are finite random variables $K \geq 0$ and $\Delta t_0 > 0$ such that, almost surely,

$$\sup_{t_n\in[0,T]}\|u(t_n) - u^n\| \leq K\Delta t^\gamma, \quad \Delta t \leq \Delta t_0.$$

Furthermore, we prove that if, in addition to $\|A^{\frac{1}{2}+\varepsilon}Q^{\frac{1}{2}}\|_{\mathrm{HS}} < \infty$ for some small $\varepsilon > 0$, we have $p \geq 1$ and $\mathbb{E}\|u_0\|_1^q < \infty$ for $q > p$, $q \geq 2$, then

$$\lim_{\Delta t\to 0}\mathbb{E}\sup_{t_n\in[0,T]}\|u(t_n) - u^n\|^p = 0.$$

The results are due to appear in [1].

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Duality in refined Sobolev-Malliavin spaces and weak error analysis for SPDE

Stig Larsson
(joint work with Adam Andersson, Raphael Kruse)

We develop a new method for the analysis of the weak error of numerical approximations of semilinear parabolic stochastic partial differential equations of the form

$$\mathrm{d}X(t) + AX(t)\,\mathrm{d}t = F(X(t))\,\mathrm{d}t + \mathrm{d}W(t),\ t \in (0,T]; \quad X(0) = X_0. \tag{1}$$

Here $(H, \|\cdot\|, \langle\cdot,\cdot\rangle)$ is a separable Hilbert and $-A$ the generator of an analytic semigroup of bounded linear operators on H. Both space-time white noise and trace class noise are considered and the nonlinearity F is allowed to be a Nemytskii operator.

Most works concerning weak error analysis for such equations are based on Ito's formula and Kolmogorov's equation in Hilbert space, which leads to technical difficulties and restrictive assumptions. Our new approach avoids these to a large extent, but it is still restricted to equations with additive noise. Another advantage is that it applies to non-Markovian equations, where the Kolmogorov equation is not available, for example, equations with memory.

Let $(H, \|\cdot\|, \langle\cdot,\cdot\rangle)$ be a separable Hilbert space and $Q \in \mathcal{L}(H)$ be a selfadjoint positive semidefinite linear operator on H. We define the space $H_0 = Q^{\frac{1}{2}}(H)$ and let $\mathcal{L}_2^0 = \mathcal{L}_2(H_0, H)$ be the space of Hilbert-Schmidt operators from H_0 to H. We consider a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in[0,T]}, \mathbf{P})$ on which an $L^2([0,T], H_0)$-isonormal process is defined. For a differentiable random variable X the Malliavin derivative $DX = (D_tX)_{t\in[0,T]}$ with respect to the isonormal process, is an $\mathcal{L}_2^0$-valued stochastic process. We introduce, for $p, q \geq 2$, the refined Sobolev-Malliavin spaces $\mathbf{M}^{1,p,q}(H)$ of random variables $X \in L^2(\Omega, H)$ such that

$$\|X\|_{\mathbf{M}^{1,p,q}(H)} = \Big(\|X\|^p_{L^p(\Omega,H)} + \|DX\|^p_{L^p(\Omega,L^q([0,T],\mathcal{L}_2^0))}\Big)^{\frac{1}{p}} < \infty.$$

The classical Sobolev-Malliavin spaces are obtained for $q = 2$ and the new spaces are "refined" by the introduction of the additional Hölder exponent q.

We use the refined spaces in a duality argument based on the Gelfand triple

$$\mathbf{M}^{1,p,q}(H) \subset L^2(\Omega, H) \subset \mathbf{M}^{1,p,q}(H)^*.$$

A key ingredient is the following inequality for the H-valued stochastic Itō-integral $\int_0^T \Phi\,\mathrm{d}W$ in the dual norm of $\mathbf{M}^{1,p,q}(H)$, where W is a cylindrical Q-Wiener

process and $\Phi \in L^p(\Omega, L^2([0,T], \mathcal{L}_2^0))$ is a predictable stochastic process. We show

$$\Big\| \int_0^T \Phi(t)\,\mathrm{d}W(t) \Big\|_{\mathbf{M}^{1,p,q}(H)^*} \leq \big\|\Phi\big\|_{L^{p'}(\Omega, L^{q'}([0,T],\mathcal{L}_2^0))}, \tag{2}$$

where p', q' are the conjugate exponents to $p, q \geq 2$. We apply this inequality in situations, where one usually relies on the Burkholder-Davis-Gundy inequality. In that case the $L^2(\Omega, H)$-norm of the stochastic integral is bounded in terms of the $L^p(\Omega, L^2([0,T], \mathcal{L}_2^0))$-norm of Φ. Here, the dual norm of the integral is bounded by the $L^{p'}(\Omega, L^{q'}([0,T], \mathcal{L}_2^0))$-norm of Φ and, as $p', q' \leq 2$, this admits stronger singularities.

Let $X, Y \in L^2(\Omega, H)$ and $\varphi\colon H \to \mathbf{R}$ have two continuous Fréchet derivatives of polynomial growth. Our technique relies on the following linearization of the weak error

$$\mathbf{E}\big[\varphi(X) - \varphi(Y)\big] = \mathbf{E}\big[\big\langle \tilde{\varphi}, X - Y\big\rangle\big], \quad \text{where} \quad \tilde{\varphi} = \int_0^1 \varphi'(\varrho X + (1-\varrho)Y)\,\mathrm{d}\varrho.$$

Our method is the following: If $V \subset L^2(\Omega, H) \subset V^*$ is a Gelfand triple such that $\tilde{\varphi} \in V$, then we obtain by duality

$$\big|\mathbf{E}\big[\varphi(X) - \varphi(Y)\big]\big| \leq \big\|\tilde{\varphi}\big\|_V \big\|X - Y\big\|_{V^*}.$$

With a good choice of V, the error converges in the V^*-norm with twice the rate of convergence in the $L^2(\Omega, H)$-norm which is the expected rate of weak convergence. For linear equations we prove that $V = \mathbf{M}^{1,p,p}(H)$ is a good choice for some $p > 2$. The main part of the error $X - Y$ is then a stochastic convolution $\int_0^T E(T-t)\,\mathrm{d}W(t)$. Bounding the error operator $E(T-t)$ in the appropriate norm yields convergence at the price of a singularity at $t = T$. By using the inequality (2) on this integral with sufficiently large $p = q > 2$, we may integrate a stronger singularity and obtain a higher rate of convergence. For semilinear equations the main difference is that a term involving $F(X) - F(Y)$ appears. We then use $V = \mathbf{G}^{1,p}(H) = \mathbf{M}^{1,p,p}(H) \cap L^{2p}(\Omega, H)$. We show that $F\colon V^* \to V^*$ is locally Lipschitz with a constant depending on $\|X\|_{\mathbf{M}^{1,2p,p}(H)}$, $\|Y\|_{\mathbf{M}^{1,2p,p}(H)}$. The choice of a stronger V-norm is necessary in order to control the nonlinearity in this way. After bounding these norms, we may use a standard Gronwall argument to bound $\|X - Y\|_{V^*}$.

We exploit this to $X_{h,k}^n$, the approximation of $X(t_n)$ by means of a standard finite element method in space with mesh parameter h and the semi-implicit Euler method with time step k and $t_n = nk$. For example, with trace class noise we prove for $\gamma \in [0,1)$ and $h, k \in (0,1]$ the weak convergence

$$\max_{n \in \{1,\dots,N\}} \big|\mathbf{E}\big[\varphi(X(t_n)) - \varphi(X_{h,k}^n)\big]\big| \leq C\big(h^{2\gamma} + k^{\gamma}\big).$$

This work is published in [1].

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Global existence of regular solutions arising in a 3-D fluid structure interaction with a moving interface

Irena Lasiecka
(joint work with M. Ignatova (Princeton), I. Kukavica (USC), A. Tuffaha)

Introduction. We consider a system of fluid-structure interaction with interior structural damping evolving in a domain which consists of two parts, the exterior part $\Omega_f(t)$ for the fluid and the interior part $\Omega_e(t)$ for the elastic structure. The dynamics of the fluid are described by the incompressible Navier-Stokes equations

$$\partial_t u - \Delta u + (u \cdot \nabla)u + \nabla p = 0 \text{ in } \Omega_f(t) \tag{1}$$

$$\nabla \cdot u = 0 \text{ in } \Omega_f(t) \tag{2}$$

while the elastic structural dynamics are governed by a damped linear wave equation

$$w_{tt} - \Delta w + \alpha w_t + \beta w = 0 \text{ in } \Omega_e \tag{3}$$

The interaction takes place at a common interface denoted by $\Gamma_c(t)$ and is described by the transmission boundary conditions matching the velocities and the stress forces at the interface. Our main result is global existence of solutions and their exponential decay for small initial data, in the presence of the interior structural damping but without the boundary feedback used in [IKLT2].

Fluid-structure interaction models have attracted considerable attention in both engineering and mathematical literature. This is due to a broad range of applications in various areas of applied sciences, from fluid mechanics to biochemistry and medicine. These models were originally considered within a finite element framework which is natural for numerical computations. More recently the issues related to the mathematical theory of existence, uniqueness, and stability of solutions for such systems have become topics of major interest in the mathematical literature. The fluid-structure interaction model considered here is an example of a quasilinear system exhibiting parabolic-hyperbolic coupling with an interface. The mismatch of regularity between hyperbolic and parabolic dynamics constitutes, as noted in [CS1], presents a defining feature and an obstacle in the analysis of the problem.

Earlier results in the area were mostly obtained for models where "hyperbolicity" of an elastic body was disguised by either (i) a strong damping added to the equation (thus making it parabolic) [B, DEGL] or (ii) replacing the wave component with a rigid body modeled by an ODE [F, SS]. In both cases, the analysis is more amenable due to the parabolic smoothing effects.

In the remainder of the introduction we describe the ideas behind the proof of the main global existence theorem. When proving well-posedness of the quasilinear system one is faced with a twofold task: obtaining suitable a-priori estimates which close at some desired level of regularity and construction of actual solutions to which a priori estimates can be applied. Somewhat unexpectedly, the presence of a boundary damping helps with the construction of solutions by providing a nice variational formulation with a cancellation of boundary terms. In the absence of

boundary damping, the construction is very laborious and not always in accordance with the a priori estimates needed for global decay of the energy as seen in [CS1]. In view of the above, our approach to the construction is based on introducing an approximating boundary term $\gamma\partial w/\partial\nu$, which provides a regularizing effect at the level of hyperbolic theory via "hidden" regularity. However, in order to obtain solutions to the original problem, after passing through the limit $\gamma \to 0$, one needs to obtain a priori estimates independent of $\gamma \to 0$. This leads to the second challenging aspect of this paper, whereby one needs to provide a priori bounds for the integrals of potential energy, without relying on the bound on normal derivatives for the solution of the wave equation (secured by $\gamma > 0$). This is accomplished by applying suitable multipliers which cancel the effects of boundary over-spill along with developing a series of estimates on tangential derivatives by methods reminiscent to ADN elliptic theory [GS]. We note that a related trick was employed in [LL2]; however, in [LL2] the Dirichlet-Stokes map served as the necessary multiplier.

The Main Result. We consider the Navier-Stokes equation

$$\partial_t u - \Delta u + (u \cdot \nabla)u + \nabla p = 0 \text{ in } \Omega_f(t) \tag{4}$$

$$\nabla \cdot u = 0 \text{ in } \Omega_f(t) \tag{5}$$

and a damped wave equation with $\alpha, \beta > 0$.

$$w_{tt} - \Delta w + \alpha w_t + \beta w = 0 \text{ in } \Omega_e \tag{6}$$

The system we consider is in the Lagrangian variables with a precise set-up as follows. Let $\eta(\cdot, t)\colon \Omega \to \Omega$ be the flow map under which the initial domain configurations Ω_f and Ω_e evolve with time, such that $\Omega_f(t) = \eta(\Omega_f, t)$ and $\Omega_e(t) = \eta(\Omega_e, t)$. For simplicity, we assume that the domains are flat, i.e.,

$$\Omega_f = \big\{x = (x_1, x_2, x_3) : h_1 \le x \le h_2 \text{ or } h_3 \le x \le h_4\big\}$$

$$\Omega_e = \big\{x = (x_1, x_2, x_3) : h_2 \le x \le h_3\big\}$$

with periodic boundary conditions with period 1 in the lateral directions.

Then, the incompressible Navier-Stokes equation has the form

$$v_t^i - \partial_j(a_l^j a_l^k \partial_k v^i) + \partial_k(a_i^k q) = 0 \text{ in } \Omega_f \times (0,T), \quad i = 1,2,3 \tag{7}$$

$$a_i^k \partial_k v^i = 0 \text{ in } \Omega_f \times (0,T) \tag{8}$$

where $v(x,t) = \eta_t(x,t) = u(\eta(x,t),t)$ and $q(x,t) = p(\eta(x,t),t)$ denote the Lagrangian velocity and the pressure of the fluid over the initial domain Ω_f. For the displacement function $w(x,t) = \eta(x,t) - x$, the linear damped elasticity equation reads

$$w_{tt}^i - \Delta w^i + \alpha w_t^i + \beta w^i = 0 \quad \text{ in } \Omega_e \times (0,T), \quad i = 1,2,3. \tag{9}$$

We thus seek a solution (v, w, q, a, η) to the damped fluid-structure system (7)–(9), where the dynamics of the Lagrangian matrix a and the flow map η are described

by the ODEs

$$a_t = -a : \nabla v : a \text{ in } \Omega_f \times (0,T) \tag{10}$$

$$\eta_t = v \text{ in } \Omega_f \times (0,T) \tag{11}$$

with the initial conditions $a(x,0) = I$ and $\eta(x,0) = x$ in Ω_f.

On the common boundary Γ_c, we assume continuity of the velocities

$$w_t^i = v^i \text{ on } \Gamma_c \times (0,T), \tag{12}$$

and the stresses

$$\partial_j w^i N_j = a_l^j a_l^k \partial_k v^i N_j - a_i^k q N_k \text{ on } \Gamma_c \times (0,T), \tag{13}$$

while on the outside fluid boundary Γ_f, we assume the non-slip boundary condition

$$v^i = 0 \text{ on } \Gamma_f \times (0,T) \tag{14}$$

for $i = 1,2,3$, where $N = (N_1, N_2, N_3)$ is the unit outward normal with respect to Ω_e.

We now state the main result of the paper. Let $\alpha, \beta > 0$. Assume that $v_0 \in V \cap H^4(\Omega_f)$, $w_0 \in H^3(\Omega_e)$, and $w_1 \in H^2(\Omega_e)$ are sufficiently small with the appropriate compatibility conditions on Γ_c

$$\begin{gathered} w_1 = v_0 \\ \Delta w_0 - \alpha w_1 - \beta w_0 = \Delta v_0 - \nabla q_0, \end{gathered} \tag{15}$$

$$\begin{gathered} \tfrac{\partial w_0}{\partial N} \cdot \tau = \tfrac{\partial v_0}{\partial N} \cdot \tau, \\ \tfrac{\partial w_1}{\partial N} \cdot \tau = \tfrac{\partial}{\partial N}\big(\Delta v_0 - \nabla q_0\big) \cdot \tau, \end{gathered} \tag{16}$$

also on Γ_c, for tangential vectors τ, and

$$v_0 = 0, \Delta v_0 - \nabla q_0 = 0, \ on \ \Gamma_f \tag{17}$$

Then, there exists a unique global-in-time smooth solution (v, w, q, a, η) to (7)–(11) which satisfies

$$v \in L^\infty([0,\infty); H^3(\Omega_f)) \tag{18}$$

$$v_t \in L^\infty([0,\infty); H^2(\Omega_f)) \tag{19}$$

$$v_{tt} \in L^\infty([0,\infty); L^2(\Omega_f)) \tag{20}$$

$$\nabla v_{tt} \in L^2([0,\infty); L^2(\Omega_f)) \tag{21}$$

$$\partial_t^j w \in C([0,\infty); H^{3-j}(\Omega_e)), \quad j = 0,1,2,3 \tag{22}$$

with $q \in L^\infty([0,\infty); H^2(\Omega_f))$, $q_t \in L^\infty([0,\infty); H^1(\Omega_f))$, $a, a_t \in L^\infty([0,\infty); H^2(\Omega_f))$, $a_{tt} \in L^\infty([0,\infty); H^1(\Omega_f))$, $a_{ttt} \in L^2([0,\infty); L^2(\Omega_f))$, and $\eta|_{\Omega_f} \in C([0,\infty); H^3(\Omega_f))$.

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Numerical reconstruction of small perturbations in the electromagnetic coefficients from dynamic measurements

Stephanie Lohrengel

(joint work with Marion Darbas)

The reliable reconstruction of defects from measurements on a given test piece is a challenging task that has important applications in medical imaging or non-destructive testing. If no further information about the defects is available, the resulting non-linear inverse problem is in general severely ill-posed. In the present communication, we aim to identify small perturbations in the magnetic permeability of a dielectric material from total or partial dynamic boundary measurements. Hence, the defects can be viewed as a perturbation of a smooth background, and the asymptotic expansion in terms of a parameter representing the (known) smallness of the defects allows their identification with help of a non-iterative numerical method. Former studies (see for instance Ammari, Vogelius, Volkov among others) have been devoted to the elaboration of asymptotic expansions as well as the numerical inversion procedure in the case of stationary or time-harmonic problems. Here, we are interested in the localization of defects from time-dependent measurements and the direct model problem is deduced from the transient Maxwell equations. For practical reasons, the method will be presented for a two-dimensional configuration. We refer the reader to [3] for all details of this communication.

In order to describe the defects mathematically, let Ω be a smooth bounded domain of $\mathbb{R}^2$ with boundary Γ. We assume that the homogeneous background medium is characterized by the electromagnetic parameters $\varepsilon = 1$ and $\mu = 1$. Now,

let $\alpha > 0$ be a constant parameter representing the common order of magnitude of all the defects and consider a smooth function p such that $\mathrm{supp}(p) \subset \Omega' \subset \Omega$ where Ω' is a smooth subdomain of Ω. This means that the defects are localized in the interior of the test piece at a fixed distance from the boundary. We further assume that p is bounded by a positive constant M: $|p(x)| \leq M$ for any $x \in \Omega'$. Then, the magnetic permeability of the damaged test piece is given in Ω by $\mu_\alpha(x) = 1 + \alpha p(x)$.

In order to formulate the inverse problem, we consider the electric field $E_\alpha(x,t)$ at time $t \in [0,T]$ on a damaged test piece. We assume that the impressed sources are acting on the boundary. The field E_α thus satisfies the second-order time-dependent Maxwell equations with given initial states E_0 and E_1,

$$
(1) \qquad \begin{cases} \partial_t^2 E_\alpha + \mathbf{curl}(\mu_\alpha^{-1}\mathrm{curl} E_\alpha) = 0, \quad \text{in } \Omega \times (0,T), \\ \mathrm{div} E_\alpha = 0, \quad \text{in } \Omega \times (0,T), \\ E_\alpha \times n = F, \quad \text{on } \Gamma \times (0,T), \\ E_\alpha(0) = E_0, \partial_t E_\alpha(0) = E_1, \quad \text{in } \Omega. \end{cases}
$$

Now, we consider the same problem on a healthy reference piece ($\alpha = 0$) and denote by $E(x,t)$ its solution. We refer to E as the background solution.

The inverse problem then consists in the reconstruction of the function p from partial boundary measurements $\mathrm{curl}\, E_\alpha$ and $\mathrm{curl}\, E$ on $\Gamma_0 \subset \Gamma$ for a given final time $T > 0$ and given data $\{F, E_0, E_1\}$. The key point of the reconstruction method is an asymptotic formula derived by Ammari [1]. In order to express this formula, we will consider from now on special boundary and initial data such that the background solution E coincides with a plane wave $E(x,t) = \eta^\perp e^{i\eta\cdot x - i|\eta|t}$ for a given wave vector $\eta \in \mathbb{R}^2$ and a unit vector $\eta^\perp$ perpendicular to η. Then, Ammari's formula states the following relation between the function p and the variation in the measurements due to the perturbation: for any $\eta \in \mathbb{R}^2$,

$$
(2) \qquad \int_0^T \int_{\Gamma_0} e^{i|\eta|t}\partial_t(e^{-i|\eta|t}G_\eta)(\mathrm{curl}\, E_\alpha - \mathrm{curl}\, E) d\sigma dt = -\alpha|\eta|^2 \int_{\Omega'} p(x) e^{2i\eta\cdot x} dx + \mathcal{O}(\alpha^2).
$$

On the right hand side of formula (2), one recognizes the Fourier transformation of the function p up to a scaling factor. Recall that the parameter α is the known order of magnitude of the perturbation. The term on the left hand side depends on the wave vector η through the measurements $\mathrm{curl}\, E$ and $\mathrm{cur}\, E_\alpha$ on a part Γ_0 of the boundary and a boundary control G_η. More precisely, for a given wave vector η, G_η is defined on Γ_0 such that the solution U_η of an auxiliary Maxwell problem comes to rest at the final time T: $U_\eta(T) = \partial_t U_\eta(T) = 0$ on Ω. Roughly speaking, U_η satisfies the second-order time-dependent Maxwell equations in the homogeneous background with boundary condition $U_\eta \times n = G_\eta$ on $\Gamma_0 \times (0,T)$, $U_\eta \times n = 0$ elsewhere on Γ, and truncated initial conditions $\{\beta E_0, 0\}$ at $t = 0$. The existence of such a boundary control is guaranteed for a class of initial conditions whenever the final time T and the part of the boundary Γ_0 control Ω geometrically in the sense of Bardos, Lebeau and Rauch. We refer to [5] for the observability estimate in the case of the first-order Maxwell equations and to [2] for the second-order system.

The main issue of our work is the effective computation of the quantity

$$M_\alpha(\eta) \stackrel{\text{def}}{=} -\frac{1}{\alpha|\eta|^2}\int_0^T \int_{\Gamma_0} e^{i|\eta|t}\partial_t(e^{-i|\eta|t}G_\eta)(\operatorname{curl} E_\alpha - \operatorname{curl} E)d\sigma dt$$

for wave vectors η belonging to a discrete sampling domain. Then, the perturbation p can be reconstructed up to an error $\mathcal{O}(\alpha)$ through inverse Fourier transformation:

$$p(-\pi x) = \frac{1}{\pi^2}(\mathcal{F}^{-1}M_\alpha)(x).$$

In order to get $M_\alpha(\eta)$, we need to compute the control term G_η. This will be achieved by the Hilbert Uniqueness Method (HUM) of J.-L. Lions which yields a constructive way to compute a boundary control with minimal L^2-norm. We refer to [2] for a detailed description of this method in the context of the second-order Maxwell equations and its numerical implementation. HUM is based on the inversion of the HUM-operator Λ which can be defined by the following four steps:

(1) solve a forward adjoint problem with homogeneous boundary condition and regular initial data $\{\psi_0, \psi_1\}$,
(2) compute a boundary term depending on the solution ψ of the first step,
(3) solve a backward problem with the boundary term derived in the second step and zero final data,
(4) evaluate the solution U of the third step at initial time 0 and let $\Lambda(\{\psi_0, \psi_1\}) = \{\partial_t U(0), -U(0)\}$.

The Maxwell solver that computes the respective solutions of the forward and backward problem, is based on a regularized formulation of the Maxwell equations which allows to use Lagrange Finite Elements (of type Q1 in the applications) for discretization in space. Discretization in time is performed through an implicit second-order Newmark scheme. An interesting alternative for efficient time integration could be rational Krylov subspace methods which yield very precise approximations of the matrix exponential function of the semi-discrete problem [4]. This issue has been discussed during the Oberwolfach Workshop and the authors aim to implement this novel approach which could lead to a significative gain of computational time.

The numerical reconstruction method has been tested successfully for various configurations such as multiple or near-by defects (see [3] for the numerical results). The method can handle noisy data up to 3% noise. The influence of the parameter α that models the common order of magnitude of the perturbations has not yet been fully analyzed.

Summing up, we presented a direct method for multiple defect localization of common order of magnitude. The method does not need any *a priori* knowledge of the number of defects and is able to reconstruct the shape of the defects. It allows limited-view data which are of interest in practical applications. The method is rather time-consuming since it is based on the numerical computation of a boundary control term for any wave vector in the discrete sampling domain.

Notice however that these computations are independent of the defects and can be achieved once for all before starting inverse Fourier transformation.

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Fast convolution quadrature for impedance boundary conditions

María Lopéz-Fernández

(joint work with Ralf Hiptmair, Alberto Paganini)

We consider a linear transient eddy current problem with the conductors occupying the bounded and connected polyhedron $\Omega_C \subset \mathbb{R}^3$. With a finite element discretization in mind we artificially truncate the fields to a simple bounded computational domain $\Omega \subset \mathbb{R}^3$ with $\overline{\Omega}_C \subset \Omega$.

For rapidly changing fields and high conductivities the skin effect prevents the fields from penetrating deep into the conductors. This permits us to model the effect of the conductor on the fields by means of an *impedance boundary condition* on the surface $\Gamma := \partial\Omega_C$ of the conductor. On *time-domain* this leads to a time-convolution on the boundary condition on Γ. More precisely, we adopt the following notation for convolutions

$$K(\partial_t)g := \int_0^t k(t-s)g(s)\,ds, \tag{1}$$

with K denoting the Laplace transform of k and fix a final time T. Then the evolution equations for the electromagnetic fields in $D := \Omega \setminus \Omega_C$ are given by

$$\mathbf{curl}\,\mathbf{curl}\,\mathbf{E} = \mathbf{f}(\boldsymbol{x},t) \quad , \quad \operatorname{div}\mathbf{E} = 0 \qquad \text{in } D\times(0,T)\,, \tag{2a}$$

$$\mathbf{curl}\,\mathbf{E}\times\boldsymbol{n} = \eta(\boldsymbol{x})K(\partial_t)\mathbf{E}_T \qquad \text{on } \Gamma\times(0,T)\,, \tag{2b}$$

$$\mathbf{E}_T = 0 \qquad \text{on } \partial\Omega\times(0,T)\,, \tag{2c}$$

$$\mathbf{E}(\cdot,0) = 0 \qquad \text{on } D\,. \tag{2d}$$

where η is a strictly positive function depending on the material and $K(s) = \sqrt{s}$. This is the so-called **E**-based formulation of an eddy current problem [1, Sect. 2.1]. The zero divergence condition on **E** in (2a) should be regarded as a *gauging*, which ensures uniqueness of the electric field outside Ω_C. The right hand side **f** stands

for a source current producing an exciting magnetic field. We assume that it is compatible in the sense that $\mathbf{f}(0) = 0$ and

$$\operatorname{supp} \mathbf{f}(\cdot, t) \subset D \quad \forall 0 \le t \le T \quad , \quad \mathbf{f} \in H^1(0, T; \boldsymbol{H}(\operatorname{div} 0, D)) \, , \tag{3}$$

where $\boldsymbol{H}(\operatorname{div} 0, D)$ is the space of solenoidal (divergence-free) vector fields on D.

We propose a numerical method for the full discretization of (2) in space and time that also allows a highly efficient implementation. Further details can be found in [5]. We will follow the *method of lines* and consider in the first place the spatial variational formulation of (2). Impedance boundary conditions require the electric fields to belong to the "energy space"

$$U := \{\mathbf{u} \in \boldsymbol{H}(\mathbf{curl}, \Omega) : \ \mathbf{u}_{T\,|\Gamma} \in \boldsymbol{L}^2_T(\Gamma), \ \mathbf{u}_T = 0 \text{ on } \partial\Omega\} \, , \tag{4}$$

which is a Hilbert space, when endowed with the usual graph norm $\|\cdot\|_U$. Here, $\boldsymbol{L}^2_T(\Gamma)$ stands for the space of square integrable tangential vector fields on Γ. In order to take into account the gauge condition $\operatorname{div} \mathbf{E} = 0$ the spatial variational formulation of (2) is posed on the function space V defined through the U-orthogonal *Helmholtz decomposition* [3, Thm. 3.3]

$$U = V \oplus \mathbf{grad}\, H^1_0(D) \, . \tag{5}$$

Obviously, V is a closed subspace of U and it will be equipped with the same norm. Then the spatial variational formulation of (2) reads: seek $\mathbf{E} \in H^1(0, T; V)$, $\mathbf{E}(0) = 0$, such that

$$\underbrace{(\mathbf{curl}\, \mathbf{E}, \mathbf{curl}\, \mathbf{v})_0}_{=:\, a(\mathbf{E},\mathbf{v})} + K(\partial_t) \underbrace{\int_\Gamma \eta(\boldsymbol{x}) \mathbf{E}_T \cdot \mathbf{v}_T \, \mathrm{d}S}_{=:\, b(\mathbf{E},\mathbf{v})} = (\mathbf{f}, \mathbf{v})_0 \tag{6}$$

for all $\mathbf{v} \in V$ and on $[0, T]$. Here $(\cdot, \cdot)_0$ designates the $L^2(D)$ inner product.

Theorem 1. *Under the assumptions* (3) *on the source term,* (6) *has a unique solution* $\mathbf{E} \in H^1(0, T; V)$.

For the spatial discretization we equip D with a tetrahedral mesh $\mathcal{M}$ with mesh-width h and write $\mathcal{E}(\mathcal{M})$ for the finite-dimensional space of lowest order $\boldsymbol{H}(\mathbf{curl}, D)$-conforming edge element functions on $\mathcal{M}$ [4, Sect. 3] and set

$$U_h := U \cap \mathcal{E}(\mathcal{M}) = \{\mathbf{v}_h \in \mathcal{E}(\mathcal{M}) : \ (\mathbf{v}_h)_T = 0 \text{ on } \partial\Omega\} \, . \tag{7}$$

In analogy to (5) the appropriate discrete variational space V_h will be a component of the U-orthogonal *discrete Helmholtz decomposition*

$$U_h = V_h \oplus \mathbf{grad}\, S_h \, , \tag{8}$$

where $S_h \subset H^1_*(D)$ denotes the space of piecewise linear continuous finite element functions on $\mathcal{M}$ that vanish on $\Gamma \cup \partial\Omega$. Then the spatially discrete variational problem reads: seek $\mathbf{E}_h \in H^1(0, T; V_h)$, $\mathbf{E}_h(0) = 0$, such that on $[0, T]$

$$a(\mathbf{E}_h, \mathbf{v}_h) + K(\partial_t) b(\mathbf{E}_h, \mathbf{v}_h) = (\mathbf{f}, \mathbf{v}_h)_0 \quad \forall \mathbf{v}_h \in V_h \, . \tag{9}$$

For the time discretization of the semi-discrete problem (9) we apply the Convolution Quadrature (*CQ*) method [6, 7, 2]. For CQ based on a Runge–Kutta method with abscissae $c_1, \ldots, c_m$ the approximation of (1) takes the form

$$(K(\partial_{\Delta t})g)(t) := \sum_{j=0}^{\infty} \boldsymbol{\omega}_j (g(t - t_j + c_l \Delta t))_{l=1}^m , \tag{10}$$

with (vector-)weights $\boldsymbol{\omega}_j$ depending on K and the particular Runge–Kutta method. The approximation of (9) by CQ can then be formulated as follows

$$a(\mathbf{E}_{h,\Delta t}, \mathbf{v}_h) + K(\partial_{\Delta t}) b(\mathbf{E}_{h,\Delta t}, \mathbf{v}_h) = (\mathbf{f}, \mathbf{v}_h)_0 \quad \forall \mathbf{v}_h \in V_h , \tag{11}$$

where the solution $\mathbf{E}_{h,\Delta t} \in L^2(0,T;U)$.

Theorem 2. *Let $\mathbf{E}_{h,\Delta t}$ be the fully-discrete approximation in* (11)*. Assume that the underlying Runge–Kutta method is stiffly accurate, has classical order p and stage order q. Then*

$$\|\mathbf{E} - \mathbf{E}_{h,\Delta t}\|_{L^2(0,T;U)} \leq C_1 e^{\sigma T} h + C_2 \Delta t^{\min(p,q+1)} \|\mathbf{f}\|_{H^{r+1}(0,T;U)} .$$

Fast algorithms for the implementation of CQ have also been developed in the last decade. A fast "oblivious" algorithm for *approximate* CQ (FOCQ) with considerably reduced memory requirements is presented in [8] and we follow these ideas. The FOCQ is based on a special quadrature to compute the convolution weights in (10) and a sophisticated organization of the computations during the time-stepping procedure. A necessary condition for FOCQ is that K in (1) is *sectorial*, this is

$$\begin{array}{l} K \text{ is analytic in a sector } |\arg(s-\sigma)| < \pi - \varphi, \quad \text{with } \varphi < \frac{1}{2}\pi\} \\ \text{and in this sector } |K(s)| \leq C|s|^\nu, \text{ for some } C > 0 \text{ and } \nu \in \mathbb{R}. \end{array} \tag{12}$$

Then for N the total number of time steps and ε the accuracy in the approximation of the weights, the algorithm in [8] reduces the number of multiplications of a naive implementation of CQ from $\mathcal{O}(N^2)$ to $\mathcal{O}(N \log N \log(\frac{1}{\varepsilon}))$ and the memory requirements from $O(N)$ to $\mathcal{O}(\log N \log(\frac{1}{\varepsilon}))$. The logarithmic dependance on ε of these estimates allows to take ε as small as the machine accuracy if required.

Theorem 3. *The impact of the fast and oblivious implementation of our CQ methods can be bounded by*

$$\|\mathbf{E}_{h,\Delta t} - \widetilde{\mathbf{E}}_{h,\Delta t}\|_{L^2(0,T;U)} \leq M \frac{C^2 N^3 e^{2\sigma T} \varepsilon}{1 - C N^2 e^{\sigma T} \varepsilon} \|\mathbf{f}\|_{L^2(0,T;U')} , \tag{13}$$

for $\widetilde{\mathbf{E}}_{h,\Delta t}$ the final result of our numerical method, M the number of degrees of freedom in space and N the total number of time steps. The constant $C > 0$ depends only on the underlying Runge-Kutta method.

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Error analysis of operator splitting for the Korteweg – de Vries equation

Christian Lubich
(joint work with Helge Holden and Nils Henrik Risebro)

We provide an error analysis of operator splitting for the equations of the type $u_t = Au + uu_x$ where A is a linear differential operator such that the equation is well-posed. Particular examples include the viscous Burgers' equation, the Korteweg–de Vries (KdV) equation, the Benney–Lin equation, and the Kawahara equation. We show that the Strang splitting method converges with the expected rate if the initial data are sufficiently regular. In particular, for the KdV equation we obtain second-order convergence in H^r for initial data in H^{r+5} with arbitrary $r \geq 1$. This result of [1] improves that of [2], where second-order convergence is shown in H^r for initial data in H^{r+9} with $r \geq 8$.

The proof derives local error estimates, for the error after one step with starting value in H^{r+5}, by interpreting the principal error terms as quadrature errors as in [3, 4]. Using the local error estimates and regularity results for the KdV and Burgers equations, boundedness of the numerical solution in H^{r+5} is shown over arbitrary finite time intervals. These estimates are combined, via Lady Windermere's fan with error propagation by the exact solutions of the KdV equation, to finally obtain the result.

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Kolmogorov equations in infinite dimensions

Alessandra Lunardi

My talk is about a family of Ornstein–Uhlenbeck operators in infinite dimension,

$$\mathcal{L}_\gamma\varphi(x) = \frac{1}{2}\,\mathrm{Tr}\,[Q^{1-\gamma}D^2\varphi(x)] - \frac{1}{2}\langle x, Q^{-\gamma}D\varphi(x)\rangle$$

where the space variable x runs in an infinite dimensional separable Hilbert space X, with norm $\|\cdot\|$, $Q \in \mathcal{L}(X)$ is a self-adjoint positive operator with finite trace, and $0 \le \gamma \le 1$. We fix once and for all an orthonormal basis in X, $\{e_k :\ k \in \mathbb{N}\} =$ such that $Qe_k = \lambda_k e_k$. Denoting by D_k the derivative in the direction of e_k, and setting $x_k = \langle x, e_k\rangle$, the operator $\mathcal{L}_\gamma$ may be written (at least, formally) as

$$\mathcal{L}_\gamma\varphi(x) = \frac{1}{2}\sum_{k=1}^{\infty}\lambda_k^{1-\gamma}D_{kk}\varphi(x) - \frac{1}{2}\sum_{k=1}^{\infty}\lambda_k^{-\gamma}x_kD_k\varphi(x).$$

$\mathcal{L}_\gamma$ is the Kolmogorov operator of the stochastic problem

$$dX_\gamma(t,x) = -\frac{1}{2}Q^{-\gamma}X_\gamma(t,x)dt + Q^{(1-\gamma)/2}dW(t), \quad X_\gamma(0,x) = x,$$

where $W(t)$ is a standard cylindrical Wiener process in X.

The operators $\mathcal{L}_\gamma$ share an important feature, namely for each $\gamma \in [0,1]$ the Gaussian measure μ with mean 0 and covariance Q is symmetrizing for $\mathcal{L}_\gamma$. That is,

$$\int_X v\mathcal{L}_\gamma u\,d\mu = \int_X u\mathcal{L}_\gamma v\,d\mu = -\frac{1}{2}\int_X \langle Q^{(1-\gamma)/2}Du, Q^{(1-\gamma)/2}Dv\rangle\,d\mu,$$

for all good u, v, for instance for u, $v \in C_b^2(X)$.

The most important among the operators $\mathcal{L}_\gamma$ are $\mathcal{L}_0$ and $\mathcal{L}_1$:

$$\mathcal{L}_0\varphi(x) = \frac{1}{2}\,\mathrm{Tr}\,[QD^2\varphi(x)] - \frac{1}{2}\langle x, D\varphi(x)\rangle,$$

is the operator that arises in the Malliavin calculus, while

$$\mathcal{L}_1\varphi(x) = \frac{1}{2}\,\mathrm{Tr}\,[D^2\varphi(x)] - \frac{1}{2}\langle x, AD\varphi(x)\rangle,$$

(with $A = Q^{-1}$) is the generator of the Ornstein-Uhlenbeck semigroup $T_1(t)$ with the best smoothing properties; if A is the realization of a partial differential operator in X the associated stochastic equation is a SPDE.

Let $\mathcal{O} \subset X$ be an open nonempty set, with regular boundary. The Sobolev spaces $W^{1,2}_\gamma(\mathcal{O},\mu)$ and $W^{2,2}_\gamma(\mathcal{O},\mu)$ are defined for instance in [4].

We study weak solutions to Dirichlet and Neumann problems for the equations (with data $\lambda > 0$ and $f \in L^2(\mathcal{O},\mu)$)

$$\lambda\varphi(x) - \mathcal{L}_\gamma\varphi(x) = f(x), \quad \text{in } \mathcal{O}.$$

For the Dirichlet problem, the setting is the Sobolev space $\mathring{W}^{1,2}_\gamma(\mathcal{O},\mu) \subset W^{1,2}_\gamma(\mathcal{O},\mu)$ of the functions $\varphi : \mathcal{O} \mapsto \mathbb{R}$ whose null extension to the whole X belongs to

$W^{1,2}_\gamma(X,\mu)$. A weak solution is a function $\varphi \in \mathring{W}^{1,2}_\gamma(\mathcal{O},\mu)$ such that

$$\lambda \int_{\mathcal{O}} \varphi\, v\, d\mu + \frac{1}{2}\int_{\mathcal{O}} \langle Q^{(1-\gamma)/2} D\varphi, Q^{(1-\gamma)/2} Dv\rangle\, d\mu = \int_{\mathcal{O}} f\, v\, d\mu, \quad \forall v \in \mathring{W}^{1,2}_\gamma(\mathcal{O},\mu).$$

Representation formulae for the weak solutions, and for the associated semigroups, were studied in [4]. The results of [4] were extended to Banach space settings and more general operators in [1].

For the Neumann problem, the setting is just $W^{1,2}_\gamma(\mathcal{O},\mu)$. A weak solution is a function $u \in W^{1,2}_\gamma(\mathcal{O},\mu)$ such that

$$\lambda \int_{\mathcal{O}} u\, v\, d\mu + \frac{1}{2}\int_{\mathcal{O}} \langle Q^{(1-\gamma)/2} Du, Q^{(1-\gamma)/2} Dv\rangle\, d\mu = \int_{\mathcal{O}} f\, v\, d\mu, \quad \forall v \in W^{1,2}_\gamma(\mathcal{O},\mu).$$

In both cases, the existence of a unique weak solution follows easily from the Lax-Milgram Lemma. It coincides with $R(\lambda, L_\gamma)f$, where L_γ is the self–adjoint operator associated to the quadratic form. If $\mathcal{O} = X$, maximal Sobolev regularity results were established long ago. This talk is devoted to

- $W^{2,2}$ regularity of the weak solutions for $\mathcal{O} \neq X$,
- meaning of the boundary condition,
- extension to a class of gradient systems.

1. $W^{2,2}$ regularity. Concerning the Dirichlet problem, we were able to prove $W^{2,2}$-regularity only for $\gamma = 0$, under geometric conditions on the boundary of $\mathcal{O}$ ([5]).

Concerning the Neumann problem, $W^{2,2}$-regularity for $0 \le \gamma \le 1$ in convex domains was established in [2]. $W^{2,2}$-regularity for a class of gradient systems, still in convex domains, $\gamma = 1$ is in [6].

2. Meaning of the boundary condition. It is possible to define the traces at $\partial\mathcal{O}$ of functions in the Sobolev spaces $W^{1,p}_\gamma(\mathcal{O},\mu)$, $p > 1$, as elements of $L^q(\partial\mathcal{O},\sigma)$, for each $q \in [1,p)$, see [3]. Here, σ is the Hausdorff–Gauss surface measure of Feyel and de La Pradelle ([7]).

The solutions to Dirichlet problems have null trace at $\partial\mathcal{O}$. The solutions φ to the Neumann problems are such that $\langle Q^{(1-\gamma)/2} D\varphi, Q^{(1-\gamma)/2} DG\rangle$ has null trace at $\partial\mathcal{O}$, if $\mathcal{O} = \{x \in X :\ G(x) < 0\}$ for a good function $G : X \mapsto \mathbb{R}$.

3. Maximal regularity for gradient systems. We consider the operator

$$\mathcal{K}u(x) = \frac{1}{2}\,\mathrm{Tr}\,[D^2u(x)] - \frac{1}{2}\langle x, ADu(x)\rangle - \langle DU(x), Du(x)\rangle$$

where $A = Q^{-1}$ and $U : X \mapsto [0,+\infty]$ is a convex, lower semicontinuous function, such that $U < \infty$, μ–a.e., the subdifferential $\partial U(x)$ is not empty for μ–a.e. x, and denoting by $D_0U(x)$ the element with minimal norm in $\partial U(x)$, $\|D_0U\| \in L^p(X,\mu)$, for some $p > 2$.

The measure that symmetrizes $\mathcal{K}$ is $\nu := e^{-2U}\mu$. A weak solution of the Neumann problem for the equation $\lambda u - \mathcal{K}u = f$ is a function $u \in W^{1,2}(\mathcal{O},\nu)$ such

that

$$\lambda \int_{\mathcal{O}} u\, v\, d\nu + \frac{1}{2} \int_{\mathcal{O}} \langle Du, Dv \rangle\, d\nu = \int_{\mathcal{O}} f\, v\, d\nu, \quad \forall v \in W^{1,2}(\mathcal{O}, \nu).$$

The following result was proved in [6].

Theorem. Let $\mathcal{O} = \{x \in X : G(x) < 0\}$, G convex and

$$G \in C^1(X) \cap_{p>1} W_0^{2,p}(X, \mu), \quad DG \neq 0 \text{ at } \partial\mathcal{O}, \quad \frac{1}{\|Q^{1/2} DG\|} \in \cap_{p>1} L^p(X, \mu),$$

Then, for $\lambda > 0$ and $f \in L^2(\mathcal{O}, \nu)$ the weak solution u of the Neumann problem for the equation $\lambda u - \mathcal{K}u = f$ belongs to $W^{2,2}(\mathcal{O}, \nu)$, $\|A^{1/2} Du\| \in L^2(\mathcal{O}, \nu)$ and

$$\|u\|^2_{W^{2,2}(\mathcal{O},\nu)} + \int_{\mathcal{O}} \|A^{1/2} Du\|^2 d\nu \leq C \int_{\mathcal{O}} \|f\|^2 d\nu.$$

Moreover $\langle Du, DG \rangle = 0$ in the sense of traces, at $\partial\mathcal{O}$.

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Consistent methods for Navier–Stokes–Korteweg systems

Charalambos Makridakis

In this work we construct reliable a posteriori estimates for some discontinuous Galerkin schemes applied to nonlinear systems of hyperbolic conservation laws, and in particular to Euler's equations. We make use of appropriate reconstructions of the discrete solution together with the relative entropy stability framework.

The methodology we use is quite general and allows for a posteriori control of discontinuous Galerkin schemes with standard flux choices which appear in the approximation of conservation laws.

In addition to the analysis, we present some numerical benchmarking to test the robustness of the resultant estimator.

Well-posedness of nonlinear Maxwell equations

Dominik Müller

In this talk we investigate the nonlinear homogeneous Maxwell equations

$$\begin{aligned}
\partial_t D(t,x) &= \operatorname{rot} H(t,x) && \big(t \in [0,T],\, x \in \Omega\big),\\
\partial_t B(t,x) &= -\operatorname{rot} E(t,x) && \big(t \in [0,T],\, x \in \Omega\big),\\
\operatorname{div} D(t,x) &= 0 && \big(t \in [0,T],\, x \in \Omega\big),\\
\operatorname{div} B(t,x) &= 0 && \big(t \in [0,T],\, x \in \Omega\big),
\end{aligned}$$

on a domain $\Omega \subseteq \mathbb{R}^3$, where $E(t,x), D(t,x) \in \mathbb{R}^3$ and $H(t,x), B(t,x) \in \mathbb{R}^3$ are the electric and magnetic fields respectively. These equations are complemented with material laws between the fields, suitable boundary conditions (unless $\Omega = \mathbb{R}^3$) and initial conditions. The material laws which we consider are given by $D = E + P(E)$, $B = H + M(H)$, for at least differentiable vector fields P and M, which are referred to as the polarisation and magnetisation. Nonlinear couplings occur in many applications. A well-known example is the Kerr nonlinearity in nonlinear optics, where $P(E) = \chi|E|^2E$ for some scalar function χ. The well-posedness theory for the resulting quasilinear Maxwell equations is far from being complete. The relevant literature is restricted to special cases or it requires regular data, see e.g. [2, 3, 4, 5]. Using an approach introduced by T. Kato, [1], we were able to establish a general abstract framework which applies to these Maxwell problems and improves the above results in various aspects. This approach relies on detailed linear estimates of a related problem where a solution has been frozen inside the nonlinearity.

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Fast and oblivious convolution quadrature

Achim Schädle

(joint work with María López-Fernández, Christian Lubich)

In the numerical solution of Volterra integral equations of convolution type one has to approximate and evaluate for $0 = t_0 < t_1 <, \ldots, < t_N = T$ expressions like

$$\int_0^{t_n} f(t_n - \tau)u(\tau)d\tau \approx \sum_{j=0}^{n} \omega_{n-j} u(t_j) \quad n = 1, \ldots, N \tag{1}$$

for a given kernel f, which may be operator valued, and an unknown, possibly vector valued, u, which is calculated step by step. In case of an equidistant time-grid, i.e. $t_n = nh$, with step-size $h = T/N$ the convolution weights could be defined using convolution quadrature, cf. [4]. We will assume that it is the Laplace-transform $F(s) = \int_0^\infty f(t)e^{-ts}dt$ of the kernel rather than the kernel itself, which can easily be evaluated. Applications we have in mind are the discretization of fractional time derivatives [5], where $K(s) = s^{-\alpha}$ or a fractional constitutive equation from viscoelasticity [1], where $K(s) = (1 + s^\alpha)^{-1}$ for some $0 < \alpha < 1$.

We present an algorithm to evaluate (1) for $n = 0, \ldots, N$ that is

(1) *fast* as it requires only $\mathcal{O}(N \log(N)$ operations and
(2) *oblivious* as it requires only $\mathcal{O}(\log(N))$ active memory, hence not all $u(t_j)$ for $j = 0, \ldots, N$ have to be kept in memory.

For an extension to adaptive time stepping and for a more detailed description we refer to [2]. Here we restrict the presentation to the equidistant case.

We assume that F is sectorial, i.e. analytic in a sector of the complex plane, that in this sector $|F(s)| \leq M|s|^{-\nu}$ for some $\nu > 0$, such that the Bromwich contour Γ in the Laplace inversion formula

$$f(t) = \frac{1}{2\pi i} \int_\Gamma F(s)\mathrm{e}^{st} ds$$

can be transformed to the left part of a hyperbola with foci on the real line

The three ingredients of the algorithms are a numerical inverse Laplace transform based on hyperbolas as developed in [3], a reduction to logarithmically few ordinary differential equations and a splitting of the integral in (1) in such a way that the error from the numerical inverse Laplace transform is uniformly bounded in each part and that only logarithmically few solutions of the ordinary differential equations have to be kept in memory.

There is a convergence result in [3] which states that for a suitable parametrization, by the complex sine function, and discretization with the trapezoidal rule, the quadrature error becomes exponentially small in K, i.e.

$$\left\| \frac{1}{2\pi i} \int_\Gamma F(s)\mathrm{e}^{st} ds - \sum_{k=-K}^{K} w_k \mathrm{e}^{t\lambda_k} F(\lambda_k) \right\| \leq C\mathrm{e}^{-cK}$$

uniformly for $t \in [a, \Lambda a]$, with $\Lambda > 1$, $2K$ being the number of quadrature points and constants $C, c > 0$, where C and c depend on Λ but not on a. Note that the length of the approximation interval $[a, \Lambda a]$ can be large, whereas the ratio of the end points is fixed. This result is illustrated for $F(s) = s^{-1/2}$ in Fig. 1.

We split the convolution integral according to

$$(2)\quad \int_0^{nh} f(nh-\tau)u(\tau)d\tau = \int_{(n-1)h}^{nh} f(nh-\tau)u(\tau)d\tau + \sum_{\ell=1}^{L} \int_{b_\ell h}^{b_{\ell-1}h} f(nh-\tau)u(\tau)d\tau$$

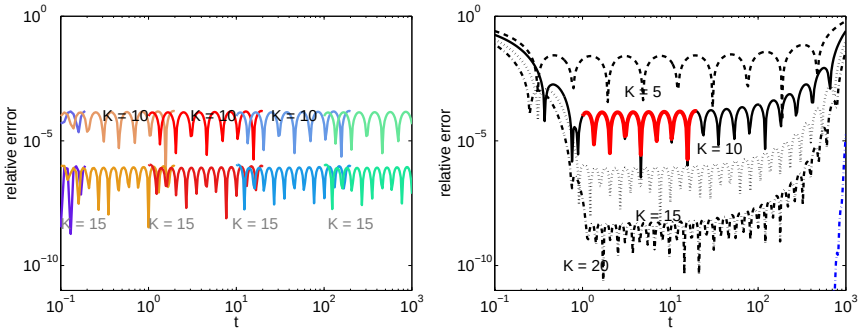

FIGURE 1. Approximation error of the inverse Laplace-transform versus t. Left: Each color in the figure corresponds to one contour. The "upper" row is for $K = 10$ the "lower" row for $K = 15$; $\Lambda = 20$. Right: Choosing only one contour for the approximation interval $[10^{-1}, 10^3]$, would give large errors towards the end points.

where L is the smallest integer such that $n < 2B^L$ for $B \in \mathbb{N}$. The choice of he b_ℓs will be discussed below. First we consider one summand

$$(3)\quad \int_{b_\ell h}^{b_{\ell-1}h} f(nh-\tau)u(\tau)d\tau = \int_\Gamma F(\lambda)\mathrm{e}^{(nh-b_{\ell-1}h)\lambda} \underbrace{\int_{b_\ell h}^{b_{\ell-1}h} \mathrm{e}^{(b_{\ell-1}h-\tau)\lambda}u(\tau)d\tau}_{y(\lambda,b_\ell,b_{\ell-1})}\, d\lambda$$

and recognize $y(\lambda, b_\ell, b_{\ell-1})$ as the solution at $t = b_{\ell-1}h$ of linear inhomogeneous ordinary differential equation

$$(4)\qquad \dot{y}(t) = \lambda y(t) + u(t)\ ; \quad y(b_\ell h) = 0$$

which may be solved by the exponential Euler method or any other integrator.

The splitting of the integral, i.e. the choice of the b_ℓs, is best explained with the aid of Fig. 2. To keep the memory requirement low as few as possible copies of the ODE solutions, $y(\lambda, b_\ell, b_{\ell-1})$, should be kept in memory. On the other hand in view of the discretization error in the numerical Laplace inversion the exponent in the exponential $nh - b_{\ell-1}h$ as well as $nh - b_\ell$ should be in an interval like $[B^{\ell-1}h 2B^\ell h]$, in which case $\Lambda = 2B$ and which requires the b_ℓs to change with n. In order to reuse an ODE solution the b_ℓs should change as rarely as possible. A compromise is depicted in Fig. 2, for the case $B = 2$.

Every color corresponds to an approximation interval, red, $\ell = 1$, to $[h, 3h] \subset [B^0h, 2B^1h]$, blue, $\ell = 2$ to $[2h, 7h] \subset [B^1h, 2B^2h]$, yellow, $\ell = 3$ to $[4h, 15h]$ and green, $\ell = 4$ to $[8h, 31h]$. If one draws a vertical line the b_ℓ are given by the intersection with the horizontal lines of the L-shapes. For example for $t_n = 16h$ we have $b_0 = n - 1 = 15$, $b_1 = 14$, $b_2 = 12$, $b_3 = 8$ and $b_4 = 0$, such that

$$\int_0^{16h} \cdots = \int_{15h}^{16h} \cdots + \int_{14h}^{15h} \cdots + \int_{12h}^{14h} \cdots + \int_{8h}^{12h} \cdots + \int_0^{8h} \cdots$$

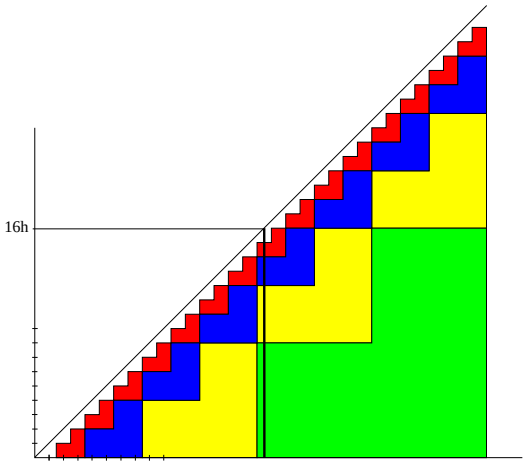

FIGURE 2. Splitting of the $t - \tau$ plane.

The integral closest to the diagonal has to be treated separately in a direct fashion [2]. The number of copies of ODE solutions that have to be kept in memory is small as L-shapes of one color do not overlap if one views Fig. 2 along the vertical or horizontal axis.

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Numerical study of the travelling waves for the nonlinear Schrödinger equation in dimension 2.

CLAIRE SCHEID
(joint work with David Chiron)

We are interested in the numerical study of the two dimensional travelling waves of the Nonlinear Schrödinger Equation for a general nonlinearity and with nonzero condition at infinity. The equation is given by

$$\text{(NLS)} \qquad i\frac{\partial \Psi}{\partial t} + \Delta\Psi + \Psi f(|\Psi|^2) = 0,\ |\Psi(t,x)| \to 1 \text{ as } |x| \to \infty$$

where Ψ is the wave function, and f is the nonlinearity. Several types of non linearities can be encountered depending on the physical phenomenon one has to describe. The most common one is given by $f(\rho) = 1 - \rho$, giving rise to the classical Gross-Pitaevskii (GP) equation. If the wave function Ψ does not vanish, the Madelung transform recasts (NLS) into a hydrodynamical type system. After linearization around $|\Psi| = 1$, one obtain an acoustic wave system with speed of sound $\mathfrak{c}_s = \sqrt{-2f'(1)}$. For (NLS) with nonzero condition at infinity, the travelling waves play a fundamental role in the dynamics. These are particular solutions of the form $\Psi(t, x) = u(x_1 - ct, x_2)$ where c is the speed of propagation, and u is a solution to the equation

$$\Delta u + uf(|u|^2) = ic\partial_{x_1} u \qquad (\mathrm{TW}_c)$$

with the condition at infinity $|u(x)| \to 1$ as $|x| \to \infty$. The qualitative properties of the travelling waves of the (GP) equation have been studied by C. Jones and P. Roberts in [1] in dimensions two and three. The study relies on numerical simulations with a Newton method and formal asymptotic expansions. For a general nonlinearity, the 1D case has been studied in details in [2] and uncommon qualitative behaviors have been put forward. therefore the question whether we can find similar behaviors in 2D is natural. In this work, we are interested in extending these studies: we use a numerical approach to investigate the qualitative behavior of (NLS) solutions in 2D and for a general type of nonlinearity. In this prospect, we would like to develop a numerical method based on the variational characterization of the equation (TW_c). Indeed, let us define the energy $E(u) = \int_{\mathbb{R}^2} |\nabla u|^2 + \int_{\mathbb{R}^2} F(|u|^2)$, with $F(\rho) = -\int_1^\rho f$, and the momentum $P(u) = \int_{\mathbb{R}^2} \langle i(u-1), \partial_{x_1} u\rangle$, with $\langle \cdot, \cdot \rangle$ the real scalar product on $\mathbb{C}$. Then the solution are critical points of the action $\mathcal{F}_c(u) \overset{\text{def}}{=} E(u) - cP(u)$ on a suitable energy space $\mathcal{X}$.

Numerical strategy. We expect the solutions to be saddle points of the action functional $\mathcal{F}_c$ that is not so easy to compute numerically. Therefore, we use relaxed functionals inspired from [3] to transform the saddle point problem into a local minimizers problem. More precisely, we mainly used the following functional $\mathcal{L}(u, \mu) \overset{\text{def}}{=} E(u) + \frac{M}{2}\Big(\mu - P(u)\Big)^2$, where M is a large constant and $\mu \in \mathbb{R}$. Then we consider the minimization problem $\mathfrak{L}_{\min}(\mu) \overset{\text{def}}{=} \inf\Big\{\mathcal{L}(u, \mu), u \in \mathcal{X}\Big\}$, for which we can use heat flow techniques to compute the local minimum. However, this strategy will only allow us to capture dynamically stable waves. To capture the unstable ones, we used a continuation with respect to the speed c. Following [1], [3], we look for solutions that respect two symmetries of the problem: u is assumed to satisfy $u(x) = u(x_1, x_2) = u(x_1, -x_2) = \bar{u}(-x_1, x_2)$, thus reducing the computational domain to $\mathbb{R}^2_+$. The latter is mapped (via a coordinate stretching) onto the square $[0, \pi/2]^2$. This strategy avoids dealing with artificial type of boundary conditions. However this comes at the price of two arbitrary constants (defining the length scale of the stretching) that have to be fixed along the computation. We discretize the equations in the framework of Finite Differences using second

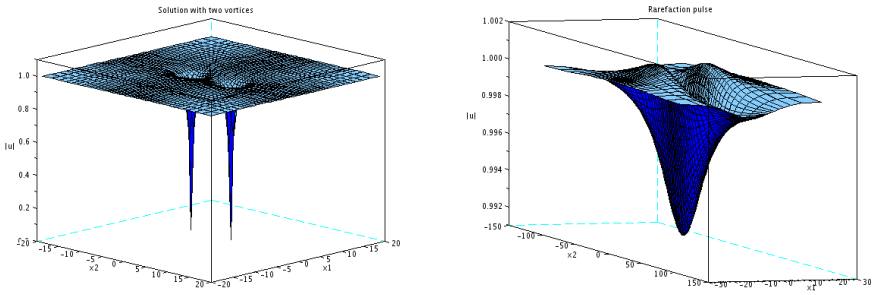

FIGURE 1. Plot of the modulus of u; Left: Vortices, Right: Rarefaction pulse.

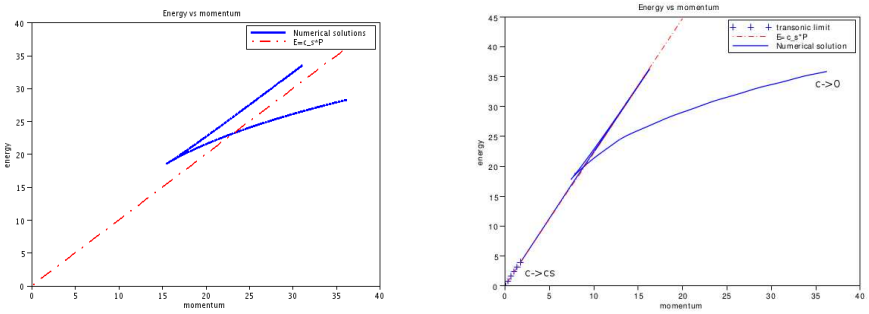

FIGURE 2. E-P diagram for : Left, a cubic-quintic-septic nonlinearity; Right: a saturated exponential nonlinearity.

order approximations. To start the computations either from $c \approx 0$ or $c \approx \mathfrak{c}_s$, we use the knowledge of a priori asymptotic behaviors, such as vortices as $c \approx 0$ (see Figure 1, Left) and rarefaction pulses as $c \approx \mathfrak{c}_s$ (see Figure 1, Right).

Results. We plot the energy-momentum diagrams for different type of nonlinearities. In the following we only present some of them. Through various examples, we show that even though the nonlinearity has the same behaviour as the well-known Gross-Pitaevskii (GP) nonlinearity, the qualitative properties of the travelling waves may be extremely different. For instance, we observe cusps (see Figures 2 and 3), a modified Kadomtsev-Petviashvili I (KP-I) asymptotic as $c \to \mathfrak{c}_s$, multiplicity results (see Figure 3, Right) and self intersection (see Figure 2, Right). The interested reader is referred to [4] for further details.

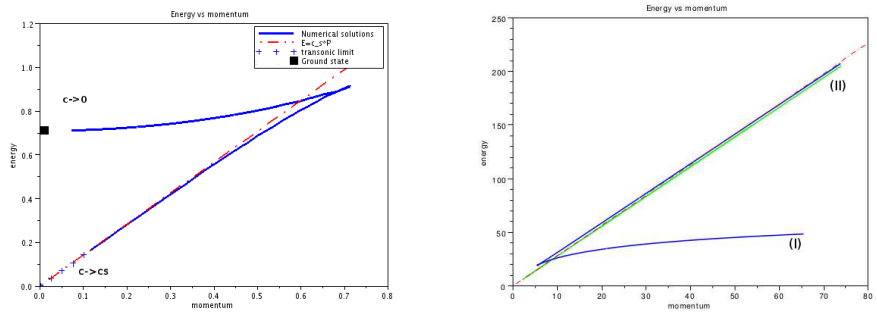

FIGURE 3. E-P diagram for : Left, a cubic-quintic nonlinearity; Right, a cubic-quintic-septic nonlinearity.

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Efficient time integration of Klein-Gordon-type equations in high-frequency limit regimes

KATHARINA SCHRATZ

(joint work with Erwan Faou)

In this talk we firstly explain how efficient time integrators for Klein-Gordon equations in the non-relativistic limit regime with gauge invariant nonlinearities can be constructed. Numerically the simulation with standard integrators is very delicate due to the high oscillations in the solution. The presented idea is based on the asymptotic expansion of the exact solution with respect to the inverse of the large parameter producing the high oscillations in time. This allows us to filter out the high oscillations in the exact solution explicitly and the numerical task can be reduced to the simulation of the non-oscillatory nonlinear Schrödinger limit system. Hence, the computational costs can be drastically reduced, see [1] for more details.

As this ansatz turns out to be very promising in the second part of the talk we will give some ideas how to extend the results to the Klein-Gordon-Zakharov system, which is a model in Plasma physics describing the interaction between Langmuir waves (oscillations of the electron density) and ion sound waves in a

plasma. In the high-plasma frequency and subsonic regime the solution becomes highly oscillatory in time, which leads to huge computational costs. Therefore we try to adapt the approach from above: The idea is again to filter out the high oscillations in the solution explicitly and numerically only solve the non-oscillatory limit systems which are of Zakharov and Klein-Gordon type, respectively. This limit analysis is well studied on $\mathbb{R}^d$ for $d \geq 3$, see for instance [2] and references therein. However, for practical implementation issues we need to consider the equation on a bounded domain, such as for instance the torus, which yields new open questions: Due to the so-called "loss of derivative" local wellposedness of the problem on the torus on a time interval independent of the plasma-frequency does not hold trivially. Furthermore the limit analysis has to be established for bounded domains, and a well-suited time integrator for the Zakharov limit system has to be constructed. Within this talk we give a formal overview on the analysis and discuss open problems.

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Space-time compressive, adaptive Galerkin discretizations for parabolic evolution equations in infinite dimensions

Christoph Schwab

Space-time variational formulations of parabolic evolution PDEs are reviewed; function spaces on finite or infinite-dimensional domains of definition are admitted. Among them are saddle point formulations [1], as well as a fractional derivative formulation which accommodates monotone nonlinearities (see [6] and the references there).

Particular instances are parabolic (pseudo)differential equations considered in [1], (Navier-)Stokes flow [2], infinite-dimensional Fokker-Planck (FP) and Ornstein-Uhlenbeck (OU) equations for functions on an infinite-dimensional, separable Hilbert space H which arise as Kolmogoroff equations for SPDEs [3]. The well-posedness of several space-time variational formulations of these equations in the Hilbert space $L^2(H, \gamma)$ of functions on H, which are square-integrable with respect to a Gaussian measure γ on H, is proved.

Adaptive space-time Galerkin discretizations, based on tensorized Riesz bases for the respective Bochner spaces, are introduced. These bases are built from biorthogonal, continuous piecewise polynomial wavelet bases in time and from suitable Riesz bases for the "spatial" Gel'fand triplet.

The (Navier)-Stokes equations for viscous, incompressible flow, are included as special cases *provided* Riesz bases for spaces of divergence-free functions are

employed; suitable Riesz bases of continuous, piecewise polynomials of divergence-free spaces have recently become available, at least in axiparallel domains, in [7].

Alternatively, novel space-time saddle-point formulations which enforce the divergence-free constraint weakly, by means of a Lagrange multiplier, are available. In [2, Theorem 4.6], an isomorphism result has been shown for the instationary Stokes operator. This isomorphism result for the Stokes operator shown there answers, in particular, a question posed by R. Temam regarding the precise function space setting for the pressure variable in the nonstationary (Navier)-Stokes case, under the provision of $H^2(D)$-regularity of the stationary Stokes operator (known to hold, eg., for convex polyhedral domains and no-slip boundary conditions). For Navier-Stokes equations, corresponding well-posed space-time-pressure saddle point formulations is available in [2, Theorem 5.3]. In this result, which is proved by linearization and the 'usual' fixed point argument (and which is, therefore, restricted to the small data setting), *a smallness assumption on the exact solution in* $W^{1,\infty}(0,T;W^{1,\infty}(D))$ is required.

For parabolic evolution equations on infinite-dimensional "spatial domains", the Wiener-Hermite polynomial chaos in the Wiener-Ito decomposition of $L^2(H,\gamma)$, is a Riesz-basis and shown in [3] to satisfy all requirements for the adaptive space-time Galerkin algorithms from [1, 4]. For (linear) Kolmogoroff equations for SPDEs, we prove in [3] that these algorithms produce sequences of finitely supported Galerkin approximations which converge at rates which are optimal with respect to the best N-term benchmark rate to the exact solution, with work and memory bounded only by an absolute multiple of the support length of the coefficient vectors of the approximation, thereby constituting the first such optimality result for a parabolic evolution equation on infinite-dimensional "domain of definition".

Adaptive space-time Galerkin discretizations, using Riesz bases in the respective Bochner spaces where the (linearized) evolution operator is an isomorphism, built from biorthogonal, piecewise polynomial wavelet bases in time and suitable Riesz bases in the evolution triplet (such as wavelet bases [1, 7] or the Wiener-Hermite polynomial chaos in the Wiener-Ito decomposition of $L^2(H,\gamma)$ [3] are introduced.

The resulting space-time adaptive Wiener-Hermite polynomial Galerkin discretizations of the infinite-dimensional evolution PDE are proved to converge quasioptimally: they produce sequences of finite-dimensional approximations which attain the best possible algebraic rates afforded by the tensor-products of multiresolution (wavelet) time-discretizations and, for Kolmogoroff equations for SPDEs [3], of systems of tensorized Wiener-Hermite polynomial chaos expansions in $L^2(H,\gamma)$. Here, quasioptimality is understood with respect to the nonlinear, best N-term approximation benchmark of the solution in the tensorized (wavelet-Wiener Polynomial Chaos) basis.

We prove, in particular, that the space-time adaptive Galerkin solution algorithms exhibit dimension-independent performance, which is optimal with respect to the algebraic best N-term approximation rate afforded by the solution and

the regularity of the multiresolution (wavelet) time-discretizations in the finite-dimensional case. All constants in our error and complexity bounds are independent of the number of "active" coordinates identified by the proposed adaptive Galerkin approximation algorithms.

The constants are also independent of the time-horizon. The proposed adaptive approaches therefore constitute first classes of provably optimal algorithms for the space-time adaptive, compressive approaches for the problem of long-term evolution of parabolic equations. For nonlinear evolution equations, such as the Navier-Stokes equations, these results require optimal, adaptive evaluations of the nonlinearities in [8].

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Four classes of exponential EPIRK integrators

Mayya Tokman
(joint work with Greg Rainwater)

Introduction. Exponential time integrators have recently gained much interest among numerical analysts and practitioners [2, 8, 1, 4, 5, 6, 7, 9, 12, 17, 10]. The key to constructing an efficient exponential integrator is designing a method that has low computational cost per time step. The evaluation of the matrix functions-vector products $\varphi(A)v$ is the most computationally expensive part of any exponential integrator. The framework of the Exponential Propagation Iterative Methods of Runge-Kutta type (EPIRK) was designed to enable construction of schemes with fewer and less costly evaluations of $\varphi(A)v$. We proposed four classes of EPIRK methods for different types of problems. Below we describe these

categories of schemes including the newest class of implicit-exponential (IMEXP) methods constructed to overcome certain limitations of the standard exponential integrators.

Classes of EPIRK methods. The general formulation of an s-stage EPIRK method for initial value problems $y' = f(y), y(t_0) = y_0$ ($y \in \mathbb{R}^n$) is given by

$$\begin{array}{l} Y_i = y_0 + a_{i1}\psi_{i1}(g_{i1}hA_{i1})hf(y_0) + \sum_{j=2}^{i} a_{ij}\psi_{ij}(g_{ij}hA_{ij})h\Delta^{(j-1)}r(y_0) \\ y_1 = y_0 + b_1\psi_{s1}(g_{s1}hA_{s1})hf(y_0) + \sum_{j=2}^{s} b_j\psi_{sj}(g_{sj}hA_{ij})h\Delta^{(j-1)}r(y_0), \end{array} \tag{1}$$

with Δ denoting the forward difference operator and $i = 1, .., (s-1)$. For a general function $f(y)$ a non-split EPIRK method can be constructed by choosing $A_{ij} = \frac{Df}{Dy}(y_0)$, $r(y) = f(y) - f(y_0) - \frac{Df}{Dy}(y_0)(y - y_0)$ and $\psi_{ij}(z)$ functions to be linear combinations of exponential-like functions $\varphi_k(z) = \int_0^1 e^{z(1-s)}\frac{s^(k-1)}{(k-1)!}ds$ [18]. If $f(y)$ can be split into a stiff linear and a nonlinear parts as $f(y) = Ly+N(y)$ then we can define a class of split EPIRK methods by choosing $A_{ij} = L$, $r(y) = N(y)$ and same $\psi_{ij}(z)$ as before [13]. Since an s-stage split EPIRK method cannot have an order higher than s, hybrid EPIRK methods have been proposed for the split problems where $A_{i1} = L$ and $A_{ij} = L + \frac{DN}{dy}$ for $j > 1$ [13].

While these classes of EPIRK methods have been shown to offer computational advantages for different types of problems, neither of these categories address the following limitations of all exponential integrators compared to implicit methods. For many problems the main computational advantage of an implicit method comes from the availability of a very efficient preconditioner for solving the linear systems inside the integrator. No efficient preconditioning strategies currently exist for exponential integrators. Additionally, for some split problems the stiffness is not limited to the linear part Ly and the nonlinearity $N(y)$ can become stiff at least for a portion of time integration interval. Often implicit-explicit (IMEX) methods are used to solve the split problems [14, 15]. These techniques treat the linear part implicitly and the nonlinearity explicitly. This approach will obviously encounter stability limitations for the problems where the nonlinear part can become stiff. Below we introduce the fourth category of the EPIRK methods, the implicit-exponential (IMEXP) integrators, designed to address these two challenges.

Implicit-exponential (IMEXP) methods and numerical example. We use the flexibility in the choice of $\psi_{ij}(z)$ and A_{ij} to construct an IMEXP method by setting some of the $\psi_{ij}(z)$ functions to be rational. For example, for a two-stage method we can choose $\psi_{i,1}(z) = 1/(1-z)$ and the rest of the $\psi_{ij}(z)$ functions to be exponential to yield

$$\begin{array}{l} Y_1 = y_0 + (I - g_{11}hL)^{-1}\, hf(y_0) \\ y_1 = y_0 + (I - g_{21}hL)^{-1}\, hf(y_0) + \frac{1}{2}\psi_{22}\,(g_{22}hJ)\,\Delta N(y_0), \end{array} \tag{2}$$

where $\Delta N(y_0) = N(Y_1) - N(y_0)$. The function $\psi_{22}(z)$ can be chosen to be any linear combinations of φ_k's exponential functions. For instance, if $\psi_{22}(z) = (e^z - 1)/z$ and $g_{11} = g_{21} = g_{22} = \frac{1}{2}$ we obtain a method of order 2 which we label

IMEXP2. All of the second terms on the right-hand-side of each of the stages in (2) can be computed using an efficient preconditioner if such is available for a given problem. In general, an IMEXP method can be constructed by choosing some $\psi_{ij}(z)$ functions to be rational while others to be exponential and setting A_{ij} to be either the linear part L or a full Jacobian. These techniques are most closely related to IMEX schemes but carry improved stability properties since the nonlinearity is handled via exponential method rather than explicitly. The following numerical example illustrates the performance of a two-stage second order IMEXP2 scheme.

We consider the 2D Schnakenberg system [16, 14] on $\Omega = [0,1] \times [0,1]$ with periodic boundary conditions and initial conditions given in Example 5 from [14]:

$$\begin{aligned} u_t &= 1000(0.126779 - u + u^2 v) + \Delta u \\ v_t &= 1000(0.792366 - u^2 v) + 10\Delta v. \end{aligned} \tag{3}$$

In [14] the second order semi-implicit BDF method (sBDF2) was identified as the most efficient scheme for the test problem (3) out of several IMEX methods analyzed. Thus we compare the performance of the IMEX sBDF2 with our new IMEXP2 scheme. Similarly to [14] we use $N = 256$ nodes and second order finite differences in spatial discretization, and integrate to time $t_{\text{final}} = 0.1$. For IMEXP2 the largest time step size used is 10^{-3} which is then halved five times. The adaptive Krylov subspace projection algorithm [11, 19] coupled with the Padé approximation algorithm of Higham [3] is used for evaluating $\varphi_1(A)v$ in IMEXP2. The infinity norm of the error is defined with respect to a reference solution obtained with MATLAB's *ode15s* integrator with tolerances set to 10^{-14}. The precision diagrams for the two methods is presented in Figure 1.

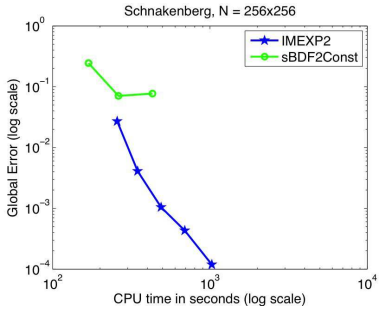

FIGURE 1. Precision diagrams

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Analysis and control of some fluid-structure interaction problems

Marius Tucsnak
(joint work with Takéo Takahashi, George Weiss)

We consider a coupled system described by nonlinear partial and ordinary differential equations modelling the motion of a rigid body inside a viscous incompressible fluid in a bounded domain Ω. The fluid flow is described by the classical *Navier-Stokes equations* (see (1)–(2) below), whereas the motion of the ball-shaped rigid body is governed by the *Newton laws* (see (6)-(7) below), including an external control force denoted by u acting on the ball.

The domain occupied by the fluid and the rigid ball is $\Omega \subset \mathbb{R}^3$, a connected open bounded set with C^2 boundary. The rigid ball has radius 1 and its center is located at the (variable) point h which is at a distance > 1 from the boundary $\partial\Omega$. We denote by $\mathcal{B}$ the closed set occupied by the ball. The fluid is homogeneous with *density* $\rho > 0$ and *viscosity* $\nu > 0$ and it occupies the domain

$$\mathcal{F}(h) = \Omega \setminus \mathcal{B}(h).$$

The full system of equations modelling the system, for $t \geqslant 0$, is

$$\rho\dot{v} - \nu\Delta v + \rho(v\cdot\nabla)v + \nabla p = 0, \qquad x \in \mathcal{F}(h(t)), \tag{1}$$

$$\operatorname{div} v = 0, \qquad x \in \mathcal{F}(h(t)), \tag{2}$$

$$v = 0, \qquad x \in \partial\Omega, \tag{3}$$

$$\dot{h} = g, \tag{4}$$

$$v = g(t) + \omega(t)\times(x-h), \qquad x \in \partial\mathcal{B}(h(t)), \tag{5}$$

$$m\dot{g} = -\int_{\partial\mathcal{B}(h)} \sigma(v,p)n \,\mathrm{d}\Gamma + u, \tag{6}$$

$$J\dot{\omega} = -\int_{\partial\mathcal{B}(h)} (x-h)\times\sigma(v,p)n \,\mathrm{d}\Gamma, \tag{7}$$

$$h(0) = h_0, \quad \dot{h}(0) = g_0, \quad \omega(0) = \omega_0, \tag{8}$$

$$v(x,0) = v_0(x), \qquad x \in \mathcal{F}(h_0). \tag{9}$$

In the above system the *state variables* are $v(x,t)$ (the Eulerian velocity field of the fluid), $h(t)$ (the position of the center of the rigid ball), its time derivative $g(t)$, and $\omega(t)$ (the angular velocity of the ball). The function $p(x,t)$ is the pressure of the fluid, which is not a state variable, because at any time instant it can be computed from v at the same instant, up to an additive constant. We have denoted by $n(x,t)$ the unit normal to $\partial\mathcal{B}(h(t))$ at the point $x \in \partial\mathcal{B}(h(t))$, directed to the interior of the ball, and by m and J the mass and the moment of inertia of the rigid ball. (If we would take $\nu = 0$, then (1)–(2) would be called *Euler's equations*, but then the other equations and the nature of the system would change.) We have denoted by $\sigma(v,p)$ the tensor defined by

$$\sigma_{ij}(v,p) = -p\delta_{ij} + \nu\left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i}\right) \qquad (i,\, j \in \{1,2,3\}). \tag{10}$$

This is the *stress tensor* in the fluid, and $\mathrm{d}\Gamma$ is the surface measure on $\partial\mathcal{B}(h)$. We also need a notation for the set of points where the center of the ball can be:

$$\Omega^\circ = \{x \in \Omega \mid d(x,\partial\Omega) > 1\}, \tag{11}$$

and we assume this set to be connected.

The main difficulty in the analysis of (1)–(9) is that the Navier-Stokes equations are valid in a non-cylindrical space-time domain. This domain depends on the solution, so that we have here a free boundary problem. Early references addressing these difficulties are Conca, San Martín and Tucsnak [1], Desjardins and Esteban [3] and Hoffman and Starovoitov [7]. The case $\Omega = \mathbb{R}^3$ has been considered in Galdi and Silvestre [5] and by Cumsille and Takahashi [2]. The global existence and uniqueness of strong solutions has been proved for sufficiently small v_0 in Takahashi [9]. Local existence of strong solutions in the L^p context has been proved in Geissert, Götze and Hieber [6]. The existence of global weak solutions for $u = 0$ (with possible contacts between the rigid body and $\partial\Omega$) has been proved in San Martín, Starovoitov and Tucsnak [8] and in Feireisl [4]. Most the above references considered the case in which u in (6) is a given function of time (often

identically equal to zero). The literature on this subject is large and, inevitably, we have left out some relevant references. For instance, there are several works studying a fluid with a rigid body moving according to a prescribed trajectory (that is independent of the fluid).

One of the contributions of our work is that we prove the existence and uniqueness of global (in time) strong solutions of (1)–(9) when u is given by a feedback law of the form

$$u(t) = k_p[h_1 - h(t)] - k_d\dot{h}(t), \tag{12}$$

with a given $h_1 \in \Omega^\circ$ and $k_p > 0$, $k_d \geqslant 0$. This feedback may be regarded as a proportional-derivative (PD) controller, as is often used in control engineering. Another interpretation is that the force u from (12) is generated by a spring (with constant k_p) and a mechanical damper (with constant k_d) connected between $h(t)$ and a fixed anchor point h_1. This result assumes that the initial velocity field v_0 as well as the initial data g_0, ω_0 and $h_1 - h_0$ are sufficiently small in a suitable sense. In addition to existence and uniqueness of global strong solutions of (1)–(9) with (12), we show that these solutions satisfy

$$\lim_{t\to\infty} h(t) = h_1, \qquad \lim_{t\to\infty} g(t) = 0, \qquad \lim_{t\to\infty} \omega(t) = 0, \tag{13}$$

$$\lim_{t\to\infty} \|v(\cdot,t)\|_{\mathcal{H}^1(\mathcal{F}(h(t)))} = 0. \tag{14}$$

The last formula of course implies that $\lim_{t\to\infty} \|v(\cdot,t)\|_{L^2(\mathcal{F}(h(t)))} = 0$, a fact that is proved before proving (14), using energy estimates.

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A stochastical maximal function and maximal regularity estimates for parabolic SPDE

Lutz Weis

Let us first discuss the maximal regularity problem for parabolic SPDE for the model equation

$$dY = -A\,dt \;+\; g d\beta(t), \quad Y(0) = 0, \quad t \in [0,T], \tag{1}$$

where $(-A)$ is the generator of an analytic semigroup on a Banach space X and $g : [0,T] \times \Omega \longrightarrow X$ is a process adapted to the Brownian notation $\beta(t)$. It is well known -due to the nonsmooth pathes of $\beta(\cdot)$- the best regularity estimate one can hope for is

$$|||A^{\frac{1}{2}} Y||| \;\leq |||g||| \tag{2}$$

for some appropriate norm for X-valued processes.
For Hilbert spaces X (see [2], [1]) and, more recently, for subspaces of L^p-spaces with $p > 2$ (see [8]), inequality (2) was established with

$$|||g||| = \Big(\mathbb{E}\|g\|^q_{L^q([0,T],X)}\Big)^{\frac{1}{q}},$$

where $q \in (0,\infty)$, if A has a bounded H^∞-functional calculus. The latter is e.g. the case if e^{-zA} is a contraction on L^p, but also for many systems of elliptic operators and Stokes Operators (see e.g. [3], [6]). In our talk we presented a new maximal regularity result for the same class of operators A and all UMD Banach spaces X. The class of UMD spaces X is the natural context for a satisfying theory of stochastic Ito-integrals of X-valued processes (see [7]) and includes all subspaces and quotient spaces of L^p-spaces with $1 < p < \infty$, as well as products and mixed norm spaces, in short "all" reflexive Banach spaces of interest in applications. However, to prove (2) it seems necessary to replace the Bochner norm considered so far by a more sophisticated norm. We propose a new maximal function for adapted processes $g : [0,T] \times \Omega \longrightarrow X$ given by

$$g^\diamond(t) = \sup_{\delta>0} \Big\| \delta^{-\frac{1}{2}} \int_{t-\delta}^{t} g(s) d\beta(s) \Big\|_{L^q(\Omega,X)}, \qquad t \in [0,T] \tag{3}$$

(the factor $\delta^{-\frac{1}{2}}$ assures that $g^\diamond = g$ for a constant g). Then one introduces for $q \in (2,\infty)$ the norm

$$|||g||| = \Big(\int_0^T g^\diamond(t)^q dt \Big)^{\frac{1}{q}}, \qquad r \in (2,\infty). \tag{4}$$

For adapted processes it satisfies

$$\left(\mathbb{E}\,\|g\|^q_{L^r([0,T],X)}\right)^{\frac{1}{q}} \lesssim |||g|||, \tag{5}$$

so that the completion of elementary processes with respect to the norm (4) truly consists of X-valued processes in a reasonable Bochner space of function. Furthermore, if X is a subspace of an L^p-space with $p > 2$ (or more generally a UMD space of type 2), then the two norms in (5) are equivalent. Therefore the following theorem extends the result from [8] quoted earlier.

Theorem. *Let X be a UMD space. If A has a H^∞-functional calculus of angle $\sigma < \frac{\pi}{2}$ on X, then the solution g of equation (1) satisfies the maximal regularity estimate (2) with respect to the norm (4).*
Conversely, if A on X and A^* on X^* satisfy the maximal regularity estimate for all g with finite norm (5) then A has a bounded H^∞-calculus.

One can also give a characterization of the stochastic maximal regularity property for A alone in terms of square function estimates which read in the L^p-case as

$$\left\| \left(\int_0^\infty \left| A^{\frac{1}{2}} e^{-tA} x(\cdot) \right|^2 \right)^{\frac{1}{2}} \right\|_{L^p} \lesssim \|x\|_{L^p} \tag{6}$$

for all $x \in L^p$. However, the H^∞-calculus seems to be the appropriate assumption for stochastic maximal regularity.

As in the deterministic case the estimate (2) is the basic ingredient to find strong local solutions for a large class of nonlinear equations

$$dY(t) = [AY(t) + f(t, Y(t))]dt + \sum_{j=1}^{\infty} b_i(t, Y(t))d\beta_j(t) \tag{7}$$

with local Lipschitz nonlinearities

$$\begin{aligned} f(\cdot,\cdot) &: [0,T] \times D(A) \longrightarrow X \\ b(\cdot,\cdot) &: [0,T] \times D(A^{\frac{1}{2}}) \longrightarrow X \end{aligned}$$

subject to conditions that assure e.g. the convergence of the stochastic term involving a sequence $\beta_j(t)$ of independent Brownian motions (see [8] for details and a formulation in terms of cylindrical Wiener processes).
A typical example for (7) would be the Zakai equation

$$dY(t) = \Delta Y(t) + \nabla Y(t) d\beta(t)$$

with $X = L^p(\mathbb{R}^d)$, $D(A) = W^{2,p}(\mathbb{R}^d)$ and $D(A^{\frac{1}{2}}) = W^{1,p}(\mathbb{R}^d)$. Here the gradient $\nabla Y(t)$ in front of the stochastic perturbation $d\beta(t)$ is essential and requires a stochastic maximal regularity estimate. Furthermore, the parapolic SPDE's of Krylov's L^p-theory (see [5]) belong to our class (7).
Similarly as in Krylov's theory the proof of the theorems depend on a combination of methods from harmonic analysis (here coached in terms of the theory of the

H^∞-calculus), stochastic analysis (in particular Burkholder-Gundy inequalities for X-valued processes, see [7]) and evolution equations. They are linked together by square function estimates such as (6) and extensions of (6) to UMD spaces (see [9], [4]).

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Participants

Prof. Dr. Helmut Abels
Fakultät für Mathematik
Universität Regensburg
93040 Regensburg
GERMANY

Prof. Dr. Georgios Akrivis
Department of Computer Science
University of Ioannina
Ioannina 45100
GREECE

Prof. Dr. Wolfgang Arendt
Abteilung Angewandte Analysis
Universität Ulm
89069 Ulm
GERMANY

Prof. Dr. Wolf-Jürgen Beyn
Fakultät für Mathematik
Universität Bielefeld
Postfach 100131
33501 Bielefeld
GERMANY

Prof. Dr. Eric Bonnetier
Lab. de Modelisation et Calcul
L J K
Univ. Joseph Fourier Grenoble I
B.P. 53
38041 Grenoble Cedex 9
FRANCE

Simone Buchholz
Fakultät für Mathematik
Institut für Analysis
Karlsruher Institut f. Technologie (KIT)
76128 Karlsruhe
GERMANY

Dr. Maria Paz Calvo
Departamento de Matematica Aplicada
Facultad de Ciencias
Universidad de Valladolid
Prado de la Magdalena s/n
47011 Valladolid
SPAIN

Dr. Michel Crouzeix
I. R. M. A. R.
Équipe d'Analyse Numérique
Université de Rennes I
Campus de Beaulieu
35042 Rennes Cédex
FRANCE

Dr. Petra Csomos
MTA-ELTE Research Group
"Numerical Analysis and Large Networks"
Hungarian Academy of Sciences
Pazmany Peter Setany 1/C
Budapest 1117
HUNGARY

Prof. Dr. Piero D'Ancona
Dipartimento di Matematica
Universita di Roma "La Sapienza"
Istituto "Guido Castelnuovo"
Piazzale Aldo Moro, 2
00185 Roma
ITALY

Dominik Dier
Abteilung für Mathematik
Universität Ulm
Helmholtzstr. 18
89081 Ulm
GERMANY

Johannes Eilinghoff
Karlsruher Institut f. Technologie (KIT)
Inst. f. Angew. & Numerische Mathematik
Englerstr. 2
76131 Karlsruhe
GERMANY

Lukas Einkemmer
Institut für Mathematik
Universität Innsbruck
Technikerstr. 19a
6020 Innsbruck
AUSTRIA

Prof. Dr. Joachim Escher
Institut für Angewandte Mathematik
Leibniz Universität Hannover
Welfengarten 1
30167 Hannover
GERMANY

Prof. Dr. Erwan Faou
Dept. de Mathématiques
ENS Cachan Bretagne
Campus Ker Lann
Avenue Robert Schumann
35170 Bruz
FRANCE

Dr. Ludwig Gauckler
Institut für Mathematik
Sekr. MA 3-3
Technische Universität Berlin
Straße des 17. Juni 136
10623 Berlin
GERMANY

Dr. Eskil Hansen
Centre for Mathematical Sciences
Lund University
221 00 Lund 7
SWEDEN

Dr. Tobias Hell
Institut für Mathematik
Numerical Analysis
Universität Innsbruck
Technikerstr. 19a
6020 Innsbruck
AUSTRIA

Prof. Dr. Sebastian Herr
Fakultät für Mathematik
Universität Bielefeld
Postfach 100131
33501 Bielefeld
GERMANY

Prof. Dr. Matthias Hieber
Fachbereich Mathematik
TU Darmstadt
Schloßgartenstr. 7
64289 Darmstadt
GERMANY

David Hipp
Karlsruher Institut f. Technologie (KIT)
Institut f. Angew. & Numerische Mathematik
Englerstr. 2
76131 Karlsruhe
GERMANY

Prof. Dr. Ralf Hiptmair
Seminar für Angewandte Mathematik
ETH-Zentrum
Rämistr. 101
8092 Zürich
SWITZERLAND

Prof. Dr. Marlis Hochbruck
Karlsruher Institut f. Technologie (KIT)
Institut f. Angew. & Numerische Mathematik
Englerstr. 2
76131 Karlsruhe
GERMANY

Prof. Dr. Dirk Hundertmark
Fakultät für Mathematik
Institut für Analysis
Karlsruher Institut f. Technologie (KIT)
76128 Karlsruhe
GERMANY

Dr. Tobias Jahnke
Karlsruher Institut f. Technologie (KIT)
Inst. f. Angew. & Numerische Mathematik
76131 Karlsruhe
GERMANY

Prof. Dr. Herbert Koch
Mathematisches Institut
Universität Bonn
53115 Bonn
GERMANY

Dr. Mihály Kovács
Dept. of Mathematics & Statistics
University of Otago
P.O. Box 56
Dunedin
NEW ZEALAND

Prof. Dr. Stig Larsson
Department of Mathematics
Chalmers University of Technology
412 96 Göteborg
SWEDEN

Prof. Dr. Irena Lasiecka
Department of Mathematics
University of Virginia
Kerchof Hall
P.O.Box 400137
Charlottesville, VA 22904-4137
UNITED STATES

Dr. Yuning Liu
Fakultät für Mathematik
Universität Regensburg
93040 Regensburg
GERMANY

Dr. Stephanie Lohrengel
Dept. de Mathématiques
Université de Reims
B.P. 347
51 062 Reims Cedex
FRANCE

Dr. Maria Lopez-Fernandez
Institut für Mathematik
Universität Zürich
Winterthurerstrasse 190
8057 Zürich
SWITZERLAND

Prof. Dr. Christian Lubich
Mathematisches Institut
Universität Tübingen
Auf der Morgenstelle 10
72076 Tübingen
GERMANY

Prof. Dr. Alessandra Lunardi
Dip. di Matematica e Informatica
Universita di Parma
Parco Area delle Scienze 53/A
43124 Parma
ITALY

Prof. Dr. Charalambos Makridakis
Department of Mathematics
University of Sussex
Falmer
Brighton BN1 9QH
UNITED KINGDOM

Dhia Eddine Mansour
Mathematisches Institut
Universität Tübingen
Auf der Morgenstelle 10
72076 Tübingen
GERMANY

Marcel Mikl
Karlsruher Institut f. Technologie (KIT)
Inst. f. Angew. & Numerische Mathematik
Englerstr. 2
76131 Karlsruhe
GERMANY

Prof. Dr. Peter Monk
Department of Mathematical Sciences
University of Delaware
501 Ewing Hall
Newark, DE 19716-2553
UNITED STATES

Dominik Müller
Fakultät für Mathematik
Karlsruher Institut f. Technologie (KIT)
76128 Karlsruhe
GERMANY

Prof. Dr. Thieu-Huy Nguyen
Fachbereich Mathematik
TU Darmstadt
64289 Darmstadt
GERMANY

Prof. Dr. Serge Nicaise
ISTV, MACS
Université de Valenciennes
Le Mont Houy
59313 Valenciennes Cedex 9
FRANCE

Prof. Dr. Alexander Ostermann
Institut für Mathematik
Universität Innsbruck
Technikerstr. 13
6020 Innsbruck
AUSTRIA

Dr. César Palencia de Lara
Departamento de Matemática Aplicada
etsi Telecomunicación
Universidad de Valladolid
47011 Valladolid
SPAIN

Prof. Dr. Jan Prüß
Institut für Mathematik
Fakultät NW II
Universität Halle-Wittenberg
06099 Halle
GERMANY

Prof. Giuseppe Savare
Dipartimento di Matematica
"F. Casorati"
Universita di Pavia
Via Ferrata, 1
27100 Pavia
ITALY

Dr. Achim Schädle
Mathematisches Institut
Heinrich-Heine-Universität
Universitätsstr. 1
40225 Düsseldorf
GERMANY

Prof. Dr. Claire Scheid
Laboratoire J.-A. Dieudonné
Université de Nice Sophia Antipolis
Parc Valrose
06108 Nice Cedex 02
FRANCE

Dr. Roland Schnaubelt
Fakultät für Mathematik
Karlsruher Institut f. Technologie (KIT)
76128 Karlsruhe
GERMANY

Katharina Schratz
Karlsruher Institut f. Technologie (KIT)
Inst. f. Angew. & Numerische Mathematik
76131 Karlsruhe
GERMANY

Dr. Mayya Tokman
School of Natural Sciences
University of California
5200 North Lake Road
Merced CA 95343
UNITED STATES

Prof. Dr. Christoph Schwab
Seminar f. Angew. Mathematik
ETH Zentrum
HG G 57.1
Rämistr. 101
8092 Zürich
SWITZERLAND

Prof. Dr. Marius Tucsnak
Institut Elie Cartan
- Mathématiques -
Université Henri Poincare, Nancy I
Boite Postale 239
54506 Vandoeuvre-les-Nancy Cedex
FRANCE

Roberta Tittarelli
L2EP-USTL
Bat. P2
59655 Villeneuve d'Ascq Cedex
FRANCE

Prof. Dr. Lutz Weis
Karlsruher Institut für Technologie
Universität Karlsruhe
Institut für Analysis
76128 Karlsruhe
GERMANY