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Mini-Workshop: Infinite Dimensional Hopf Algebras

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Abstract. This is a report of the above mini-workshop. It contains brief accounts of all 17 talks given at the meeting, with commentary on their interconnections. A selection of the numerous open questions discussed at and generated by the meeting is provided in a separate section. The cumulative references listed for each of the talks together provide an up-to-date guide to the fast-growing literature on the topics covered.

Mathematics Subject Classification (2010): 16T05.

Introduction by the Organisers

The workshop *Infinite Dimensional Hopf Algebras*, organised by Ken Brown (Glasgow), Ken Goodearl (Santa Barbara), Tom Lenagan (Edinburgh), and James Zhang (Seattle), was well attended with 17 participants with broad geographic representation from four continents. This workshop was a nice blend of researchers with various backgrounds. In particular, three related but somewhat independent programmes of work on the algebraic structure of infinite dimensional Hopf algebras satisfying some finiteness conditions were well represented. These programmes are all experiencing vigourous activity in the present century. One (initiated by K. A. Brown) is aimed at developing general structure theory for noetherian Hopf algebras; a second (initiated by N. Andruskiewitsch and H.-J. Schneider) is concerned with the classification of pointed Hopf algebras; while the third (initiated by K. A. Brown, K. R. Goodearl, D.-M. Lu, Q.-S. Wu and J. J. Zhang) looks to describe Hopf algebras of small growth and/or small coradical. The workshop brought together leading figures from all these programmes. Progress in the different directions was reviewed, connections and areas of overlap were identified, and joint plans for future research were developed.

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Mini-Workshop: Infinite Dimensional Hopf Algebras

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Abstracts

On infinite dimensional Hopf algebras

Nicolás Andruskiewitsch

Here is an approach to the classification of Hopf algebras in suitable classes [2, 4]. Let H be a Hopf algebra with bijective antipode $\mathcal{S}$. The *Hopf coradical* of H is the subalgebra $H_{[0]}$ generated by H_0 and the *standard filtration* is defined by $H_{[n]} = \wedge^{n+1} H_{[0]}$, $n \in \mathbb{N}$. Then $H = \bigcup_{n\geq 0} H_{[n]}$, $H_{[0]}$ is a Hopf subalgebra of H and $\operatorname{gr} H = \oplus_{n\geq 0} H_{[n]}/H_{[n-1]}$ is a graded Hopf algebra. If $\pi : \operatorname{gr} H \to H_{[0]}$ is the homogeneous projection, then $R := (\operatorname{gr} H)^{\operatorname{co}\pi} = \oplus_{n\geq 0} R^n$ is a connected graded Hopf algebra in ${}^{H_{[0]}}_{H_{[0]}}\mathcal{YD}$ and $\operatorname{gr} H \cong R\# H_{[0]}$. A class $\mathcal{C}$ of Hopf algebras (defined by a property that is applicable to braided Hopf algebras) is *suitable* when

$$H \in \mathcal{C} \overset{\text{(i)}}{\Longleftrightarrow} \operatorname{gr} H \in \mathcal{C} \overset{\text{(ii)}}{\Longleftrightarrow} R, H_{[0]} \in \mathcal{C}. \tag{1}$$

For instance, the class of finite-dimensional Hopf algebras is suitable. Let $\mathcal{C}$ be an interesting class of Hopf algebras. We propose to consider the following questions:

(a). Let C be a cosemisimple coalgebra and $S : C \to C$ a bijective anti-coalgebra map. Classify all Hopf algebras $L \in \mathcal{C}$ generated by C, such that $\mathcal{S}|_C = S$.

(b). Given L as in (a), classify all connected graded Hopf algebras $R \in {}^L_L\mathcal{YD}$ such that $R \in \mathcal{C}$, or alternatively $R\#L \in \mathcal{C}$.

(c). For L and R as in (a), (b), classify all Hopf algebras H such that $\operatorname{gr} H \cong R\#L$.

When $\mathcal{C}$ is suitable, complete answers to questions (a), (b), (c) amount to the classification of Hopf algebras in $\mathcal{C}$. But for various interesting classes, some implications in (1) are open problems; e.g., if $\mathcal{C}$ is the class of noetherian Hopf algebras, then the implication $\Leftarrow$ in (i) is a standard fact, while $\Rightarrow$ is open. Even when $\mathcal{C}$ is not suitable, the questions would help to advance in the classification and to furnish families of examples. If $V = R^1$ (the infinitesimal braiding of H), then the Nichols algebra $\mathfrak{B}(V)$ is a subquotient of R. As an example, we mention the classification of pointed Hopf algebras with finitely generated abelian group $G(H)$ and generic infinitesimal braiding that are domains and have finite Gelfand-Kirillov dimension [1, 5]; and among them, those that are reductive [3].

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Distinguished pre-Nichols algebras

Iván Angiono

Let $(q_{ij}) \in \mathbf{k}^{\theta\times\theta}$ be a matrix such that the corresponding Nichols algebra $\mathcal{B}(V)$ of diagonal type is finite-dimensional. Let $\chi : \mathbb{Z}^\theta \times \mathbb{Z}^\theta \to \mathbf{k}^\times$ be the bicharacter such that $\chi(\alpha_i, \alpha_j) = q_{ij}$. Among all the pre-Nichols algebras (i.e., braided graded Hopf algebras $R = \oplus_{n\geq 0} R_n$ generated as an algebra by $R_1 = V$) there exists one, denoted by $\widetilde{\mathcal{B}}(V)$ and called the *distinguished pre-Nichols algebra* of χ, which admits all the Lusztig isomorphisms as $\mathcal{B}(V)$. For example if (q_{ij}) is a braiding corresponding to a finite-dimensional quantized enveloping (super)algebra $U_q(\mathfrak{g})$ at a root of unity q, then $\widetilde{\mathcal{B}}(V)$ is precisely $U_q^+(\mathfrak{g})$ while $\mathcal{B}(V)$ is the positive part of small quantum group $\mathfrak{u}_q(\mathfrak{g})$, obtained as a quotient of $U_q(\mathfrak{g})$.

The distinguished pre-Nichols algebra $\widetilde{\mathcal{B}}(V)$ has a PBW basis with the same generators E_α of degree $\alpha \in \Delta_+^\chi$ as for $\mathcal{B}(V)$, but some of them in a set $\mathcal{O}(V)$ have infinite height. Moreover $E_\alpha^{N_\alpha} = 0$ if and only if $\alpha \notin \mathcal{O}(V)$, $N_\alpha = \operatorname{ord}\chi(\alpha,\alpha)$. We prove that $\widetilde{\mathcal{B}}(V)$ is a Noetherian braided Hopf algebra such that $\operatorname{GKdim}\widetilde{\mathcal{B}}(V) = |\mathcal{O}(V)|$ since it admits a filtration (coming from the PBW basis) such that the associated graded Hopf algebra is a quotient of a quantum affine space.

We also prove that the subalgebra $Z^+(V)$ generated by $E_\alpha^{N_\alpha}$, $\alpha \in \mathcal{O}(V)$, is a braided Hopf subalgebra whose elements q-commute with the whole $\widetilde{\mathcal{B}}(V)$: in particular, it is a quantum affine space in generators $E_\alpha^{N_\alpha}$. Finally $\widetilde{\mathcal{B}}(V)$ is a free finite $Z^+(V)$-module.

Under some mild conditions of (q_{ij}) the algebra $Z^+(V)$ is commutative and then there exists a surjective map from the (isomorphism classes of) irreducible modules of $\widetilde{\mathcal{B}}(V)$ to $\operatorname{Spec} Z^+(V)$, which will help to study the representation theory of $\widetilde{\mathcal{B}}(V)$. Moreover the graded dual of $\widetilde{\mathcal{B}}(V)$ is the corresponding Lusztig's divided power algebra containing the Nichols algebra $\mathcal{B}(V)^* = \mathcal{B}(V^*)$.

By bosonization of $\widetilde{\mathcal{B}}(V)$ with suitable abelian group algebras and taking Drinfeld doubles of these we obtain new examples of Noetherian pointed Hopf algebras of finite Gelfand-Kirillov dimension. Moreover each of them contains a q-commutative Hopf subalgebra such that the quotient is a Hopf algebra obtained by the same process over the Nichols algebra.

These results generalize those for quantum groups at roots of unity in [2, 3].

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Hopf algebras under finiteness conditions

Ken Brown

This talk was a survey of recent progress in the study of infinite dimensional Hopf algebras, usually over an algebraically closed field k of characteristic 0, with H satisfying one or both of two finiteness conditions, namely the finiteness of Gelfand-Kirillov dimension, or the noetherian condition, that is the ascending chain condition on one-sided ideals. It was in some sense a continuation and an updating of two earlier survey articles, [3] and [6]. The third survey article in this informal sequence, [4], gives a much fuller account of the ground covered in the talk, as well as other aspects which we did not have time to cover.

The topics addressed were as follows. First, the noetherian property and the finiteness of the Gelfand-Kirillov dimension were reviewed: the description of all Hopf algebras of certain types which satisfy one or other of these finiteness conditions was considered. Relations of these conditions with each other, and with finite generation of the algebra, were considered, as well as some discussion on the prime and primitive spectra of Hopf algebras satisfying finiteness conditions. In particular we discussed recent work of Bell, Leung, Walton and Sierra, [2], [9]. A recurring theme here is that, for affine Hopf k-algebras, finiteness of the Gelfand-Kirillov dimension may be a more stringent condition than the noetherian property.

We then specialised to the classes of pointed and connected Hopf algebras. For such a Hopf algebra H, the *coalgebra filtration* $\{H_n\}$ [7, Chapter 5] is both a coalgebra *and* an algebra filtration, so that the associated graded algebra gr$H$ is also a Hopf algebra, and is (respectively) pointed or connected, providing an obvious tool for the study of H. In the *pointed* case, that is when H_0 is the group algebra of group-like elements of H, when H is furthermore generated as an algebra in degree one (with respect to the coradical filtration), the resulting class of Hopf algebras has been the focus of the research programme initiated by Andruskiewitsch and Schneider at the end of the last millenium, (see e.g. [1]). Aspects of this programme were discussed in detail in other talks in the miniworkshop. We therefore focussed in the latter part of this talk on the case of *connected* Hopf algebras of finite GK-dimension - that is, where $H_0 = k$. In this case, by a result which in part goes back to Sweedler [8, Theorem 11.2.5a], made more precise by Zhuang [11], gr$H$ is a commutative polynomial k-algebra in GKdim$H$ variables. We reviewed developments from this result, from the papers [11], [10], [5].

The talk listed a number of open questions; these, together with many others, can be found in [4].

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Hopf algebras of low GK-dimension

K. R. Goodearl

The talk surveyed classification efforts directed at Hopf algebras of Gelfand-Kirillov dimension four or less. Taking account of principles from quantum groups and non-commutative algebraic geometry, and recalling that coordinate rings of connected algebraic groups are domains, one is prompted to focus on affine or noetherian Hopf algebras which are domains or prime rings. (Under these assumptions, the only Hopf algebra of GK-dimension 0 is the base field.) All the reported classification results require an algebraically closed base field of characteristic zero.

Brown and Zhang [1] investigated the class $\mathcal{H}_1$ of prime affine Hopf algebras of GK-dimension 1 with finite global dimension. They presented a class of infinite-dimensional Taft algebras and a class generalizing the Hopf algebras introduced by Liu [3], and proved that any Hopf algebra in $\mathcal{H}_1$ with prime PI-degree is one of the following four types: an enveloping algebra $U(\mathfrak{g})$ with $\dim \mathfrak{g} = 1$; a group algebra $k\Gamma$ with Γ infinite cyclic or infinite dihedral; an infinite dimensional Taft algebra; or a generalized Liu algebra. No new Hopf algebras in the class $\mathcal{H}_1$ have been discovered, and it is conjectured that the mentioned classes cover $\mathcal{H}_1$.

Goodearl and Zhang [2] investigated the class $\mathcal{H}_2$ of affine or noetherian Hopf algebras of GK-dimension 2 which are domains. Several canonical examples are known; they proved that any Hopf algebra H in $\mathcal{H}_2$ with $\mathrm{Ext}^1_H(k,k) \neq 0$ is either in one of the canonical example classes or in one of three new constructed families. Wang, Zhang, and Zhuang [4] then found a family of Hopf algebras H in $\mathcal{H}_2$ satisfying $\mathrm{Ext}^1_H(k,k) = 0$. It is conjectured that this family, together with the previous ones, covers $\mathcal{H}_2$. Wang, Zhang, and Zhuang verified this conjecture for the subclass of $\mathcal{H}_2$ in which the Hopf algebras are generated by grouplikes and skew primitives, and do not contain sub-Hopf-algebras from either of two specific one-parameter families [4].

In GK-dimensions 3 and 4, the connected Hopf algebras have been completely classified (again assuming an algebraically closed base field of characteristic zero). This was done by Zhuang [6] for GK-dimension 3 and by Wang, Zhang, and Zhuang [5] for GK-dimension 4.

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Nichols algebras

Istvan Heckenberger

Most of the deep results in this survey use in some way the Weyl groupoid of a Nichols algebra of a semi-simple Yetter-Drinfeld module.

Let V be a vector space over a field $\mathbf{k}$ and let $c \in \mathrm{Aut}_{\mathbf{k}}(V \otimes V)$ which satisfies the braid equation. Then (V, c) is called a *braided vector space*. The *Nichols algebra of* (V, c) is (by one of the many definitions) the braided $\mathbb{N}_0$-graded Hopf algebra $T(V)/\mathcal{I}(V)$, where $\mathcal{I}(V)$ is the unique maximal coideal of $T(V)$ contained in $\oplus_{n\geq 2} T^n(V)$. If I is any Hopf ideal of $T(V)$, then $T(V)/I$ is a pre-Nichols algebra. This definition implies that whenever (V, c) is graded by a group Γ, then $\mathcal{B}(V)$ is a Γ-graded braided Hopf algebra. In particular, assume that $V = \oplus_{i=1}^{\theta} V_i$ is the direct sum of subspaces such that $c(V_i \otimes V_j) \subseteq V_j \otimes V_i$ for all i, j. Then $\mathcal{B}(V)$ is $\mathbb{Z}^\theta$-graded via $\deg V_i = \alpha_i$ for all i, where $(\alpha_i)_{1\leq i\leq\theta}$ is the standard basis of $\mathbb{Z}^\theta$. This grading is a crucial clue to understand $\mathcal{B}(V)$.

Our leading questions for the moment are the following.

(1) What is the general structure of $\mathcal{B}(V)$?
(2) When is $\mathcal{B}(V)$ finite dimensional/noetherian/of finite GK-dimension?

Let (V, c) be a braided vector space of diagonal type and let $\theta = \dim V$. Kharchenko proved that any pre-Nichols algebra generated by V has a restricted PBW basis, where the PBW generators are labeled by Lyndon words. (No Weyl groupoid here!) Viewing the $\mathbb{Z}^\theta$-degrees of the PBW generators of $\mathcal{B}(V)$ as the positive half of a generalized root system leads one to a very fruitful combinatorics of Nichols algebras. Moreover, $\mathcal{B}(V)$ is noetherian if and only if this set is finite.

Let now V be the direct sum of θ simple objects in a category ${}^H_H\mathcal{YD}$, where H is a Hopf algebra with invertible antipode. Then in interesting cases (e. g. if dim $\mathcal{B}(V) < \infty$) one can show that $\mathcal{B}(V) \simeq \otimes_{j\in J}\mathcal{B}(\overline{V}_j)$ as $\mathbb{Z}^\theta$-graded objects in ${}^H_H\mathcal{YD}$, where each $\overline{V}_j$ is a subobject of $\mathcal{B}(V)$. Then $\mathcal{B}(V)$ is called *decomposable.* If moreover J is a finite set, then $\mathcal{B}(V)$ has only finitely many graded right coideal subalgebras, and all proper maximal chains of right coideal subalgebras have the same length. Moreover, if $\mathcal{B}(V)$ is noetherian then $\mathcal{B}(V)$ is decomposable, J is finite, and all $\mathcal{B}(\overline{V}_j)$ are noetherian.

Assume that V is a finite-dimensional semi-simple Yetter-Drinfeld module over a group algebra. In the last years, the classification of those V with finite-dimensional/noetherian Nichols algebra was intensively advanced by various authors, and additional information on $\mathcal{B}(V)$ (e.g. defining relations) was obtained.

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Actions of finite dimensional Hopf algebras on AS regular algebras

Ellen Kirkman

This talk is a survey of a project to extend classic invariant theory (the study of the subring of invariants $k[x_1, \cdots, x_d]^G$ under the action of a finite group G) to a noncommutative setting. Let k be an algebraically closed field of characteristic zero. We replace $k[x_1, \cdots, x_d]$ by an Artin-Schelter regular algebra A of dimension d (global and GK-dimensions) that is generated in degree 1; hence, when commutative, A is a commutative polynomial ring. Let G be a group of graded automorphisms of A or, more generally, a semisimple Hopf algebra H, so that A is an H-module algebra and the H action on A is inner faithful and preserves the grading of A. Noncommutative algebras are more rigid than $k[x_1, \cdots, x_d]$ in that they generally have fewer graded automorphisms (for example, the only graded automorphisms of $k_{q_{i,j}}[x_1, \cdots, x_d]$ are diagonal automorphisms (torus actions), unless $q_{i,j} = \pm 1$), so considering Hopf algebra actions on A can sometimes produce new actions (e.g. the 8 dimensional Hopf algebra of Kac-Paljutkin K_8 acts on $k_i[u, v]$ for $i = \sqrt{-1}$), and for some algebras A, we obtain fixed subalgebras $A^H \neq A^G$, for G a finite group (see [8]). However, there are other situations (see [5] and [3]) that force $H = kG$. In this talk I discuss what is known about the cases where A^H is AS Gorenstein, and where A^H is AS regular.

Extending results of F. Klein (1884) and Watanabe (1974) on $k[x_1, \cdots, x_d]^G$ for any finite subgroup G of $SL_d(\mathbb{C})$, we consider when A^H is AS Gorenstein. Jørgensen and Zhang extended the notion of the determinant of a group action on $k[x_1, \cdots, x_d]$ to the *homological determinant* of a group action on A, and this notion was extended further to a Hopf action of *trivial homological determinant* in [8]. Extending Klein's results to this context, all finite dimensional Hopf actions with trivial homological determinant on AS regular algebras of dimension 2 are classified in [4] using results of [2].

The Shephard-Todd-Chevalley Theorem (1954) states that $k[x_1, \cdots, x_d]^G$ is a polynomial ring (i.e. a commutative AS regular algebra) if and only if G is a reflection group (a group generated by elements with invariant subspaces of codimension 1), and the groups with these properties were completely classified. Extending this result to our noncommutative setting, a group element g is called a *reflection* of an AS regular algebra A of dimension d if its trace function $Tr_A(g,t)$, when written as a rational function, has a pole of order $d-1$ at $t = 1$. There are new reflections (that we call *mystic reflections*) and new groups, such as the dicyclic groups, that become reflection groups for $k_{-1}[x_1, \cdots, x_n]$ (see [7] and [9]). The case of $k_{q_{i,j}}[x_1, \cdots, x_d]$ has been studied in detail [9], and it has been shown recently [1] that if $A = k_{q_{i,j}}[x_1, \cdots, x_n]^G$ is an AS regular algebra then the group algebra kG is isomorphic as an algebra to kG', where G' is a reflection group. A project to classify all semisimple noncocommutative Hopf actions on small dimensional AS regular algebras is underway. There are actions of K_8 on $k_{-1}[u,v]$ and on $k_{\pm i}[u,v]$ that produce AS regular fixed rings; for Hopf algebras of dimension 12 there are even non-PI algebras A of dimension 3 with AS regular fixed rings. The problem of completely characterizing semisimple Hopf algebras H that act on AS regular algebras A with AS regular fixed rings A^H (Hopf algebras we might call *reflection Hopf algebras* for A) is largely open.

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Coideal subalgebras of $U_q(\mathfrak{g})$

Stefan Kolb

Let H be a Hopf algebra with invertible antipode. Let $C \subseteq H$ be a right coideal subalgebra (RCSA) of H. Building on work by M. Takeuchi, A. Masuoka, and H.-J. Schneider one says that C is a quantum homogeneous space for H if H is faithfully flat as a right C-module, see [4]. Quantum homogeneous spaces for pointed Hopf algebras have a particularly straightforward description.

Proposition. ([6, Proposition 1.4]) *Let H be a pointed Hopf algebra and $C \subseteq H$ a RCSA. The following are equivalent.*

(1) $G(C) = \{c \in C \mid \Delta(c) = c \otimes c\}$ *is a group.*
(2) H *is a faithfully flat right C-module.*
(3) H *is a free right C-module.*

The quantum enveloping algebra $U_q(\mathfrak{g})$ of a simple complex Lie algebra $\mathfrak{g}$ is an important example of an infinite dimensional pointed Hopf algebra. Large classes of quantum homogeneous spaces for $U_q(\mathfrak{g})$ are known. These include in particular the quantum symmetric pair coideal subalgebras introduced by G. Letzter [5] which have recently attracted renewed interest in representation theory. However, the general classification of quantum homogeneous spaces for $U_q(\mathfrak{g})$ remains open.

Let $U^{\geq}$ denote the positive Borel part of $U_q(\mathfrak{g})$, and let U^0 denote the coradical. V. Kharchenko and collaborators classified all RCSAs of $U_q(\mathfrak{g})$ which contain U^0 for $\mathfrak{g}$ of type A_n, B_n, and G_2. They observed that the Hopf subalgebra $U^{\geq}$ of $U_q(\mathfrak{g})$ contains $|W|$ such RCSAs. Here $|W|$ denotes the order of the Weyl group W of $\mathfrak{g}$. The situation was clarified by I. Heckenberger and H.-J. Schneider in the much wider context of bozonisations of Nichols algebras [3]. This is the starting point for the classification of larger classes of RCSAs of $U_q(\mathfrak{g})$ in terms of Weyl group combinatorics in [1], [2] as summarized by the following table. Progress towards the bottom right corner of the table is desirable.

C RCSA	$C \subseteq U^{\geq}$	$C \subseteq U_q(\mathfrak{g})$
$U^0 \subseteq C$	[3]: $U^+[w]U^0$ for $w \in W$	[2]
$G(C)$ group	[1]	**?**

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On the representation theory of right coideal subalgebras of quantized enveloping algebras

Stéphane Launois
(joint work with Jason P. Bell and Karel L. Casteels)

Let $\mathfrak{g}$ be a complex simple Lie algebra of rank n, and let $\pi := \{\alpha_1, \ldots, \alpha_n\}$ be the set of simple roots associated to a triangular decomposition $\mathfrak{g} = \mathfrak{n}^- \oplus \mathfrak{h} \oplus \mathfrak{n}^+$. The Weyl group W of $\mathfrak{g}$ is endowed with the Bruhat order. To any $w \in W$ corresponds a nilpotent Lie algebra $\mathfrak{n}_w := \mathfrak{n}^+ \cap \mathrm{Ad}_w(\mathfrak{n}^-)$, where Ad stands for the adjoint action. A quantum analogue of the enveloping algebra of this nilpotent Lie algebra was defined in [2] by using the braid group action of W on the quantized enveloping algebra $U_q(\mathfrak{g})$ induced by Lusztig automorphisms. The resulting (quantum) algebra is denoted by $U_q[w]$. These algebras are strongly related to homogeneous right coideal subalgebras of the positive Borel part of $U_q(\mathfrak{g})$ as explained in [3].

The aim of this talk was to study the representation theory of these algebras $U_q[w]$. As usual for infinite-dimensional algebras, it is a very difficult problem, and so we follow Dixmier's approach and study the annihilators of simple modules, the so-called primitive ideals, of these algebras. These primitive ideals are somehow well understood (up to localization), as it follows from the Stratification Theorem of Goodearl and Letzter, and works of Mériaux-Cauchon and Yakimov, that the set of primitive ideals $\mathrm{Prim}(U_q[w])$ of $U_q[w]$ admits a stratification of the form:

$$\mathrm{Prim}(U_q[w]) = \bigsqcup_{v \leq w} \mathrm{Prim}_v(U_q[w]), \tag{1}$$

with $\mathrm{Prim}_v(U_q[w]) \simeq (\mathbb{C}^*)^{d(v)}$. The aim of this talk was to present the main result of [1] which gives a formula for the dimension $d(v)$ of $\mathrm{Prim}_v(U_q[w])$.

Theorem. For all $v \leq w$, we have $d(v) = \ker(v + w)$.

This result was obtained independently, and by completely different methods, by Yakimov [4].

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The totally nonnegative grassmannian (and totally nonnegative matrices)

Tom Lenagan

This talk outlined the work of Postnikov on the totally nonnegative grassmannian, [3], and then surveyed work of Goodearl, Launois and Lenagan, [1, 2] on cell recognition in totally nonnegative matrices.

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Folding of Nichols algebras and quantum groups

Simon D. Lentner

The Nichols algebra $\mathcal{B}(M)$ of a Yetter-Drinfel'd module M over a group Γ is a quotient of the tensor algebra $T(M)$. It has a natural structure of a Hopf algebra in a braided category satisfying a certain universal property. Finite-dimensional Nichols algebras arise naturally in the classification of finite-dimensional pointed Hopf algebras [1]. For example, they form the quantum Borel part in the small quantum groups $u_q(\mathfrak{g})$. Heckenberger classified all finite-dimensional Nichols algebras for Γ abelian [3] and there has been much development concerning the case Γ nonabelian by Andruskiewitsch, Heckenberger, Schneider, Vendramin and others.

In a previous Oberwolfach talk [4] I have presented a new method to construct Nichols algebras over nonabelian central extensions $G \to \Gamma$ as direct sum of certain Bigalois objects of a known Nichols algebra over Γ with outer automorphisms.

In the present talk I explain this construction from a Lie-theoretic view, give several families of examples [5] and discuss other recent developments.

Diagram folding is a phenomenon known from semisimple Lie algebras: Let $\mathfrak{g}$ be a Lie algebra and σ a suitable Dynkin diagram automorphism. Then the σ-orbits of roots form the root system of the sub-Lie algebra $\mathfrak{g}^\sigma$. For example $(E_6)^\sigma = F_4$.

As it turns out, this is precisely the impact of the previously discussed construction on the root system of the Nichols algebra in the sense of [2]: The Dynkin diagram of the newly constructed Nichols algebra over the nonabelian group G is a folding of the Dynkin diagram of the known diagonal Nichols algebra. Also, the center of G depends on the diagram and is related to certain symplectic forms in $\mathbb{F}_2^n$, which I classified in [6] and which unify my previous case-by-case arguments.

For example, folding of $u_q(\mathfrak{g})^+$ for $\mathfrak{g} = E_6$ and $q = i$ leads to a new Nichols algebra of dimension 2^{36} of type $\mathfrak{g}^\sigma = F_4$ over conjugacy classes of length $1, 1, 2, 2$.

Recent results of Heckenberger and Vendramin in the classification of finite-dimensional Nichols algebras of rank > 1 over nonabelian groups suggest that the complex Nichols algebras in [5] are in fact the only examples in large rank.

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Maximal Pointed Hopf Subalgebras

D.-M. Lu

(joint work with Z.-P. Fan, X.-L. Yu)

For each Hopf algebra H (over an algebraically closed field), we introduce its maximal pointed Hopf subalgebra H', which is based on a generalized direct sum decomposition of coalgebras and BDK's filtration [1]. As an application, we realized a transfer of a Nichols algebra over a cosemisimple Hopf algebra to one over a group algebra.

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Cleftness for universal quantum groups

Akira Masuoka

(joint work with Yuji Tsuno)

We work over an arbitrary field k. Given a Hopf algebra H, a right H-comodule algebra $A \neq 0$ is said to be an H*-Galois extension* over the subalgebra $B = A^{coH}$ of H-coinvariants, if the left A-linearization $A \otimes_B A \to A \otimes H$ of the structure map $A \to A \otimes H$ on A is a bijection. Those extensions include a tractable special class, called *cleft comodule algebras*. We say that A is *cleft*, if there exists a convolution-invertible H-colinear map $H \to A$. Such an A is characterized as an H-Galois extension with the *normal basis property*, i.e. $A \simeq B \otimes H$ as left B-module and right H-comodules ([3]). This is also characterized as an H-crossed product, $B\#_\sigma H$, which is constructed by a weak H-action on B and a two-cocycle

$\sigma : H \otimes H \to B$ (the wrong credit at my talk should be corrected to [1]). Note that if A is a Hopf algebra given a quotient Hopf algebra $\pi : A \to H$, then A is regarded as a right H-comodule algebra with respect to $(\mathrm{id}_A \otimes \pi) \circ \Delta_A$. In this situation, A is known to be cleft if it is pointed or finite-dimensional.

Closed embeddings of quantum groups correspond to, or are even the same as, Hopf algebra quotients. The closed embeddings of universal quantum groups here mean the Hopf algebra quotients $H(n) \to H_d(F)$ of Takeuchi's free Hopf algebra $H(n)$ that were constructed by Bichon (for $d = 1$) and by Chirvăsitu [2] (for $d > 1$). Let $n > 1$, $d \geq 1$ be integers, and let $F \in \mathrm{GL}_n(k)$. We let $H(n)$ denote the free Hopf algebra on the dual coalgebra $C = M_n(k)^*$ of the $n \times n$ matrix algebra $M_n(k)$. This is universal among the Hopf algebras given a coalgebra map from C. The inner-automorphism $X \mapsto {}^tF^{-1}X\,{}^tF$ on $M_n(k)$ given by the transpose ${}^tF$ of F is dualized to a coalgebra automorphism on C, which uniquely extends to a Hopf-algebra automorphism, say $(\ )^F$, on $H(n)$. By definition, $H_d(F)$ is the quotient Hopf algebra of $H(n)$ divided by the relation $S^{2d}(a) = a^F$, $a \in H(n)$.

Theorem Assume $d > 1$. Then $H(n)$ is cleft as a right (as well as left) $H_d(F)$-comodule algebra.

As for the case when $d = 1$, we only have partial results so far. The excellent paper [2] by Chirvăsitu gives especially the complete description of the Grothendieck ring $K(H_d(F))$ of the tensor category of finite-dimensional $H_d(F)$-comodules, assuming that k is algebraically closed. Our theorem refines some of his proofs and explains some phenomena of his results in $d > 1$.

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On the values of Frobenius-Schur indicators for Hopf algebras

Susan Montgomery

Let H be a semisimple Hopf algebra over $\mathbb{C}$, and let V be an irreducible representation of H with character χ. Let Λ be the integral of H with $\epsilon(\Lambda) = 1$. The nth Frobenius-Schur (FS) indicator of V is defined by

$$\nu_n(V) := \chi(\Lambda^{[n]}),$$

where for any $x \in H$, $x^{[n]} = \sum x_1 x_2 \cdots x_n$ is the nth Hopf power of x. This agrees with the classical FS-indicator for finite groups.

A more conceptual definition of the indicator has been given in [2]. Consider the nth tensor power $V^{\otimes n}$ of V. H acts diagonally on V, and in fact commutes with the action of the cyclic permutation $\alpha = (1, 2, \ldots, n)$ applied to $V^{\otimes n}$. Then $\nu_n(V) = trace(\alpha|_{(V^{\otimes n})^H})$.

Indicators have become an important tool in studying Hopf algebras; for example they are used in [2] to prove a version of Cauchy's theorem: if a prime p divides the dimension of H, then it also divides the exponent of H. Indicators are also gauge invariants, that is, invariants of the tensor category $\mathcal{C}$ of representations of H [4],[5]. It is an open question as to whether the set of all indicator values for all representations of H determines $\mathcal{C}$ up to tensor equivalence.

Thus it is important to know what indicator values can occur. For groups, all values of $\nu_n \in \mathbb{Z}$, but this can fail for Hopf algebras [2], although they must always be nth cyclotomic integers. One important Hopf algebra is $D(G)$, the Drinfel'd double of a finite group G, and it was hoped that in this case, all indicator values were integers. In current joint work with Iovanov and Mason [1], we find a necessary and sufficient condition for all indicators of $D(G)$ to be integers, as well as some sufficient conditions. We then prove that many groups do have this property: alternating and symmetric groups, $PSL_2(q)$, the Mathieu groups M_{11} and M_{12}, and all regular nilpotent groups (this includes all p-groups of order $\leq p^p$). We do not know if it is true for all simple groups.

However we show that there exists an irregular nilpotent group of order 5^6 with non-integer indicators.

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The orbit relation for an action of a Hopf algebra

Serge Skryabin

Let H be a Hopf algebra over a field, and let A be an H-module algebra. With each prime ideal P of A we associate a collection of ideals $(P_C)_{C\in\mathcal{F}}$ indexed by the directed set $\mathcal{F}$ of finite dimensional subcoalgebras of H. When H is the group algebra of a group G, those ideals are precisely finite intersections of G-conjugates of P, and the prime ideals of A minimal over ideals in the family associated with P recover the G-orbit of P.

This observation suggests the following definition for an arbitrary H. Let us say that a subset $O \subset \operatorname{Spec} A$ is an H-orbit if for each $P \in O$ and each $P' \in \operatorname{Spec} A$ one has $P' \in O$ if and only if P' is a prime minimal over P_C for some $C \in \mathcal{F}$. It is not clear, however, that the H-orbit containing P exists. For example, if P' is a prime minimal over some P_C, one has to know that P has a similar relationship with respect to P'.

Denote by $\operatorname{Spec}_f A$ the set of those prime ideals P of A for which there exists no infinite strictly ascending chain $P_0 \subset P_1 \subset \cdots$ in $\operatorname{Spec} A$ starting at $P_0 = P$. In other words, $P \in \operatorname{Spec} A$ is in $\operatorname{Spec}_f A$ if and only if the factor ring A/P satisfies ACC on prime ideals.

Theorem. *If A is an H-module algebra module-finite over its center, then the set $\operatorname{Spec}_f A$ is a disjoint union of H-orbits. Thus there is an equivalence relation $\sim_H$ on $\operatorname{Spec}_f A$ such that for $P, P' \in \operatorname{Spec}_f A$ one has $P \sim_H P'$ if and only if P' is a prime minimal over P_C for some $C \in \mathcal{F}$.*

Under the hypotheses stated, for each $P \in \operatorname{Spec}_f A$ and each $C \in \mathcal{F}$ the factor algebra A/P_C is shown to have an artinian classical quotient ring $Q(A/P_C)$. The rings $Q(A/P_C)$ with a fixed P and with C running over $\mathcal{F}$ form an inverse system. An essential step in the proof of the above theorem consists in checking that the linearly compact H-module algebra $L_P(A) = \varprojlim Q(A/P_C)$ is topologically H-simple, i.e., it has no nontrivial H-stable closed ideals. This extends an earlier result on H-simplicity of H-prime artinian algebras which appeared in [1].

When A is also assumed to be noetherian and H-semiprime, it is shown as an application that A itself has an artinian (actually quasi-Frobenius) classical quotient ring $Q(A)$. This result does not require any restrictions on H, while the same question remains open in general for H-module algebras which are not module-finite over centers (see [1]). Complete proofs of all results discussed in my talk are presented in [2].

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Hopf actions on commutative domains

Chelsea Walton

(joint work with Pavel Etingof)

Let k be an algebraically closed field of characteristic zero. The purpose of this work is understand quantum symmetry in the sense of studying quantum analogues of group actions on k-algebras. We restrict our attention to the actions of finite quantum groups, i.e. of finite dimensional Hopf algebras H. Two important subclasses of such H are those that are *semisimple* (i.e., semisimple as an algebra; such Hopf algebras are always finite dimensional) and those that are *pointed* (i.e., all simple H-comodules are 1-dimensional). Moreover, we are also motivated from the viewpoint of classic invariant theory and algebraic geometry, where the examination of Hopf actions on commutative domains over k is naturally of interest.

The classification of semisimple Hopf actions on commutative domains over k is completely understood. Namely, Etingof and I produced the following result.

Theorem. [2, Theorem 1.3] If a semisimple Hopf algebra H over k acts inner faithfully on a commutative domain over k, then H is a finite group algebra.

For the proof, we use methods from the study of fusion categories to show that there are only finitely many right coideal subalgebras for a semisimple Hopf algebra. However, it was brought to the attention of the speaker that the aforementioned result was proved by H. Schneider during a series at lectures at National University of Córdoba in 1996. In any case, the theorem above fails if any of the hypotheses are omitted: see [2, Remark 4.3], [1], and the material below for the omission of 'commutative', 'domain', and 'semisimple', respectively.

On the other hand, the non-semisimple case is much more complicated. In [3] and [4], Etingof and I study actions of finite dimensional (not necessarily semisimple) Hopf algebras H on commutative domains, particularly when H is pointed of finite Cartan type. The work in [3] begins by reducing to the case where H acts *inner faithfully* on a field (so that the action does not factor through a smaller quotient Hopf algebra). Such a Hopf algebra is referred to as *Galois-theoretical*. We present examples of these Hopf algebras, which include the Taft algebras, $u_q(\mathfrak{sl}_2)$, and some Drinfeld twists of other small quantum groups. We also give many examples of finite dimensional Hopf algebras which are not Galois-theoretical. Classification results on finite dimensional pointed Galois-theoretical Hopf algebras of finite Cartan type will be provided in [4].

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Modular derivations and twisted Poincaré dualities

Quanshui Wu

(joint work with Juan Luo, Shengqiang Wang)

Poisson structures naturally appear in classical/quantum mechanics, in the deformation theory of commutative algebras and in mathematical physics. They play an important role in Poisson geometry, in algebraic geometry and non-commutative geometry. Poisson cohomologies are important invariants of Poisson structures. They are closely related to Lie algebra cohomology, but Poisson cohomology is finer in general. The set of Casimir elements of the Poisson structure is the 0-th cohomology; Poisson derivations modulo Hamiltonian derivations is the 1-st cohomology. Poisson cohomology appears as one considers deformations of Poisson algebras.

Poisson homology has close relations with the Hochschild homology of its deformation algebra. In some cases, Poisson homology is computable. On the other hand, one can compute the Poisson (co)homology via the Hochschild (co)homology of the deformation algebra [1]. By using the semiclassical limit of the dualizing bimodule between the Hochschild homology and Hochschild cohomology of the corresponding quantum affine space, Launois-Richard [1] obtained a new Poisson module which provides a twisted Poincaré duality between Poisson homology and cohomology for the polynomial algebra with quadratic Poisson structure. Following this idea, Zhu [3] constructed a new Poisson module structure, which comes from the dualizing bimodule resulting in the Poincaré duality between Hochschild homology and cohomology of the universal enveloping algebra, and obtained a twisted Poincaré duality for linear Poisson algebras. Motivated by their work, A version of the twisted Poincaré duality is proved between the Poisson homology and cohomology of a polynomial Poisson algebra R (or more generally, a smooth Poisson algebra R with trivial canonical bundle) with values in an arbitrary Poisson module. The duality is induced from an explicit isomorphism between the Poisson cochain complex of R with values in M and the Poisson chain complex of R with values in the twisted Poisson module M_t. The Poisson module structure M_t is twisted by the modular derivation of R, which is a Poisson derivation. In the case of the Poisson structure is unimodular, the twisted Poincaré duality reduces to the Poincaré duality in usual sense. A general duality theorem is proved by Huebschmann [2] in the setting of Lie-Rinehart algebra.

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Homological techniques pertaining to Hopf algebras

James Zhang

This talk is to review some homological techniques that are used in the study of infinite dimensional noetherian Hopf algebras. Part of the talk is based on methods introduced by Brown, Goodearl, Lu, Wu and Zhang and results in [1, 2, 3, 4, 5]. The following is a list specific topics.

(1) Homological properties of finite dimensional Hopf algebras.
(2) Artin-Schelter Gorenstein properties of noetherian PI Hopf algebras.
(3) Homological integral and applications in infinite dimensional case.

(4) Rigid dualizing complexes and Calabi-Yau properties of noetherian Artin-Schelter regular Hopf algebras.
(5) Nakayama automorphism and homological identities involving Hopf algebras.

Similar techniques are also used in noncommutative invariant theory, noncommutative algebraic geometry and mathematical physics. The aim of the talk is to search for hidden homological invariants and to find common features for objects from different subjects. Several open questions are presented. One immediate question for us is how to apply these homological methods in the study of possibly infinite dimensional braided Hopf algebras (e.g., Nichols algebras).

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Open Questions

All participants

Please see the articles above for notation and terminology.

Question 1 (N. Andruskiewitsch)**.** Let H be a Hopf algebra with bijective antipode. As in my talk, let $H_{[0]}$ be the *Hopf coradical* generated as algebra by the coradical H_0. Define the *standard filtration* of H by $H_{[n]} = \wedge^{n+1} H_{[0]}$. Then $\operatorname{gr} H \cong R\#H_{[0]}$ where R is a Yetter-Drinfel'd Hopf algebra over $H_{[0]}$.

Consider the statements:

a) H is noetherian (respectively has finite GK-dimension).
b) $\operatorname{gr} H \cong R\#H_{[0]}$ is noetherian (respectively has finite GK-dimension).
c) R and $H_{[0]}$ are noetherian (respectively have finite GK-dimension).

Are these statements equivalent? Some of the implications are well-known and others follow easily, while some of them are difficult. Several of these and other questions were discussed by the participants in a joint problem session:

By freeness $b) \Rightarrow c)$ seems to hold easily in both cases. For noetherian, $b) \Rightarrow a)$ is a standard results [20] Thm. 1.6.3, while $a) \Rightarrow b)$ has an easy counterexample in finite characteristic and $c) \Rightarrow b)$ seems open. For GK-dimension, $a) \Rightarrow b)$ follows from the dimension inequality [20] Lm. 8.3.20; provided $\operatorname{gr} H$ is affine the converse $b) \Rightarrow a)$ holds due to dimension equality [20] Prop. 8.6.5. In general, $c) \Rightarrow b)$ has a group algebra counterexample (non-nilpotent-by-finite-polycyclic), but may hold in the present case if the filtration is nice enough.

Question 2 (K.A. Brown and K.R. Goodearl)**.** Suppose H is a noetherian Hopf algebra over an algebraically closed field. Are the quotients H/P, for minimal prime ideals P of H, Morita equivalent to each other? (This is clearly true in the finite dimensional case.) If not, what properties do these quotients share? In particular,

a) If one of the quotients H/P is PI, are all of them PI (in which case H itself is PI)?
b) Do the quotients H/P all have the same GK-dimension? Positive answers are known in two cases: (1) H is an affine PI algebra [21, Theorem 5.6]. (In fact, the affine hypothesis can be replaced by the assumption that H remains noetherian under any extension of the base field.) (2) H has an exhaustive nonnegative filtration such that $\operatorname{gr} H$ is connected graded noetherian with enough normal elements [19, Theorem 0.4].

Question 3 (K.A. Brown and K.R. Goodearl)**.** Are all noetherian Hopf algebras strongly noetherian? Here "strongly noetherian" is meant in the sense of Artin and Zhang [9]: an algebra A over a field k is strongly noetherian provided $A \otimes_k B$ is noetherian for all commutative noetherian k-algebras B.

Question 4 (K.A. Brown and K.R. Goodearl)**.** Many open questions concerning noetherian Hopf algebras are given in the surveys [10, 11, 13, 12].

Question 5 (S. Lentner)**.** The Lusztig quantum groups of divided powers are infinite dimensional Hopf algebras that contain a finite-dimensional pointed Hopf algebra (the small quantum groups) and a Frobenius-homomorphism to a universal enveloping of a Lie algebra. Especially they are not generated in degree 1.

a) Give other examples of Hopf algebra extensions of an enveloping algebra $U(\mathfrak{g}')$ by a finite-dimensional pointed Hopf algebra H, especially when the coradical is a nonabelian group algebra, say $\mathbb{D}_4, \mathbb{S}_3$.
b) Classify these extensions for a fixed finite-dimensional pointed Hopf-algebra H: Say, for $H = U_q(\mathfrak{g})^+$, for $H = U_q(\mathfrak{g})$, or for a bosonized Nichols algebra over a nonabelian group. How are the root systems of $\mathfrak{g}, \mathfrak{g}'$ related?
c) Classify these extensions for a fixed GK-dimension, which pins down $U(\mathfrak{g}')$.

We expect examples with diagram folding $\mathfrak{g}' = \mathfrak{g}^\sigma$ already in the diagonal case, e.g. (partially dual to) the example in Goodearl's talk for $\mathfrak{g} = A_1 \times \ldots A_1, \mathfrak{g}' = A_1$. Angiono's talk provide (again dually) a surprising and rather different type of examples where H is diagonal and non-Cartan and $\mathfrak{g}'$ are the Cartan roots.

Question 6 (S. Lentner)**.** My talk was about the construction of large-rank finite-dimensional complex Nichols algebras from known diagonal Nichols algebras by diagram folding [18].

a) Is there a closed construction of the known finite-dimensional Nichols algebras (arbitrary rank, arbitrary characteristic) from diagonal Nichols algebras?
b) For each case in a), construct pointed Hopf algebras (liftings) by folding the known liftings of the diagonal Nichols algebras. Are these all liftings?

c) Prominent examples of liftings in the diagonal case are the small quantum groups $u_q(\mathfrak{g})$, and their representation categories have ties to the representation theory of the Lie group $G(\mathfrak{g})$ over the finite field $\mathbb{F}_q$. What is the interpretation of the diagram folding construction under this correspondence?

Question 7 (D.-M. Lu)**.** For each Hopf algebra H (over an algebraically closed field), we introduce its maximal pointed Hopf subalgebra H', which is based on a generalized direct sum decomposition of coalgebras and BDK's filtration. As an application, we realized a transfer of a Nichols algebra over a cosemisimple Hopf algebra to one over a group algebra.

a) Is there a Hopf-ideal I such that $H = H' \oplus I$?
b) What algebraic and homological relations between H and H'?

Question 8 (A. Masuoka)**.** Alexandr Zubkov and I aim to *classify all reductive affine algebraic supergroups over an algebraically field k of characteristic $\neq 2$.* The category of affine supergroups G is anti-isomorphic to the category of super-commutative Hopf superalgebras A; G is the group-valued functor defined on the category of super-commutative superalgebras which is represented by the corresponding A. Our aim is translated into Hopf-algebra language as follows:

Question: Classify all super-commutative Hopf superalgebras A over an algebraically field k of characteristic $\neq 2$ such that

a) A is finitely generated as an algebra,
b) A is generated by the coradical $\operatorname{Corad} A$ of the coalgebra A, and
c) the commutative Hopf algebra $\overline{A}$ contains neither non-trivial idempotent nor non-zero nilpotent element, where $\overline{A} = A/(A_1)$ denotes the (largest) quotient Hopf algebra of A divided by (A_1), the ideal generated the odd component A_1 of A.

Here are two remarks. (1) Those super-commutative Hopf superalgebra A which are cosemisimple, i.e. $A = \operatorname{Corad} A$, and which are not Hopf algebras, i.e. $A_1 \neq 0$, are rather restricted even in characteristic zero. (2) The question may be regarded as a variant of the first step of the classification program proposed by N. Andruskiewitsch and J. Cuadra, which requests to *classify all Hopf algebras A that are generated by* $\operatorname{Corad} A$.

Question 9 (S. Montgomery)**.** Let $\mathcal{C}$ be the category of modules of a semisimple Hopf algebra H, so that $\mathcal{C}$ is generated by the irreducible modules for H, call them $V_1, V_2, \dots V_n$. Let m be the exponent of H and let $\nu_j(V_i)$ be the value of the jth Frobenius-Schur indicator on V_i. Consider the set

$$S = \{\nu_j(V_i) \mid 1 \leq i \leq n, 1 \leq j \leq m\}.$$

By work of Mason-Ng-Schauenburg, the set S consists of gauge invariants; that is, the values are invariants of the category $\mathcal{C}$. The question is the converse:
If Hopf algebra H and H' have the same set S of indicator values, are their module categories $\mathcal{C}$ and $\mathcal{C}'$ gauge equivalent?

Note that $\nu_m(V_i) = dim(V_i)$, so if the sets are the same, then $H' \cong H$ as algebras. The question is true trivially if H is the group algebra of an abelian group. This question was raised by the proposer in 2006, though it still seems to be open for group algebras. Richard Ng has suggested that if it is false as stated, one could also require that the two categories have the same fusion rules.

Question 10 (S. Skryabin)**.** Let H be any Hopf algebra over a field with a bijective antipode. Does every H-semiprime right noetherian H-module algebra have an artinian classical right quotient ring?

Question 11 (S. Skryabin)**.** Can the existence of H-orbits of prime ideals be proved for some classes of H-module algebras other than the algebras module-finite over their centers, e.g., for PI algebras?

Question 12 (S. Skryabin)**.** Suppose that an H-module algebra A is H-simple, semiprime, noetherian and module-finite over its center. Does then A always have a finite global dimension?

Question 13 (S. Skryabin)**.** Suppose that A is H-simple, noetherian and module-finite over its center, but not necessarily semiprime. Is it possible to conclude that A is Gorenstein?

Question 14 (C. Walton)**.** As an extension of the theorem in the abstract above, Pavel Etingof and I posed the following question [14, Question 5.9] on Hopf actions on PI algebras (algebras satisfying a polynomial identity).

Question: If a cosemisimple Hopf algebra H over k acts inner faithfully on a PI domain of PI degree d, must then $\text{PIdeg}(H^*) \leq d^2$?

If the answer is affirmative, then we have that the bound d^2 is sharp due to [14, Example 5.10].

Question 15 (C. Walton)**.** In [15, 16], the authors examine the Galois-theoretical property of the most extensively studied finite dimensional, pointed Hopf algebras: those of finite Cartan type. Given Heckenberger's classification of finite dimensional, pointed Hopf algebras over abelian groups [17], we also propose the following tasks: Study the Galois-theoretical property of finite dimensional, pointed Hopf algebras H with $G(H)$ abelian of *standard type* (which properly includes finite Cartan type) [7], *super type* [2], and *unidentified type* [8]. Moreover, it would be interesting to achieve this task for the known finite dimensional, pointed Hopf algebras H with $G(H)$ non-abelian.

Question 16 (J.J. Zhang)**.** Let A be a noetherian Artin-Schelter regular algebra. Is A a braided Hopf algebra? It was verified by Heckenberger and others during the workshop that every noetherian Artin-Schelter regular algebra of global dimension two that is generated in degree 1 is a braided Hopf algebra. Question 1 was also verified for some other examples of Artin-Schelter regular algebras of higher global dimension. The same question can be asked for Artin-Schelter Gorenstein algebras, but the answer could be "no" in this general setting.

Question 17 (J.J. Zhang)**.** Given an Artin-Schelter regular algebra A, is there a natural way of constructing finite dimensional Hopf algebras H that act on A inner faithfully? This question is motivated by recent work of Chan, Kirkman, Walton and Zhang on the classification of finite dimensional Hopf algebra actions on Artin-Schelter regular algebras of dimension two.

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