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Hilbert Modules and Complex Geometry

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Abstract. The major topics discussed in the workshop were Hilbert modules of analytic functions on domains in $\mathbb{C}^n$, Toeplitz and Hankel operators, reproducing kernel Hilbert spaces and multiplier algebras, the interplay of complex geometry and operator theory, non-commutative function theory and operator theory, Hilbert bundles on symmetric spaces.

Mathematics Subject Classification (2010): 32A35, 32A36, 32M15, 46E22, 46J15, 46L52, 47A13, 47B32, 47B35, 47A53.

Introduction by the Organisers

Multivariable operator theory is a comparatively young branch of functional analysis dealing with the structure, classification and applications of systems of linear operators on Hilbert or Banach spaces. The language of analytic Hilbert modules was developed in the framework of multivariable operator theory as a synthesis of topological homology, commutative algebra, complex analytic and algebraic geometry. Domains of holomorphy, or more general Kähler manifolds, give rise to interesting Hilbert spaces of holomorphic functions and associated operators or operator algebras. The Cowen-Douglas theory provides a tool to study operators or systems of operators in terms of complex differential geometry via holomorphic vector bundles related to the eigenvalues or the Taylor spectrum of the given operators. A very dynamic new direction of research is represented by the theory of functions of non-commuting variables and their interactions with non-commutative operator theory. The aim of the meeting was to bring together leading researchers and talented young mathematicians from these areas to report about exciting new developments and to identify crucial problems which are of central importance for future progress.

The main topics included Hilbert modules of analytic functions on different types of domains in $\mathbb{C}^n$, spectral properties of Toeplitz and Hankel operators, reproducing kernel Hilbert spaces and the structure of their multiplier algebras, applications of complex geometry to operator theory, Hilbert bundles of holomorphic and non-holomorphic type and operator theory on symmetric spaces, moment and interpolation problems, non-commutative function and operator theory. The workshop was attended by 47 participants from well over ten different countries. On average there were three talks in the morning session and three talks in the afternoon session leaving ample room for mathematical discussions in the breaks. The varied backgrounds of the participants and the unique atmosphere of the research institute has lead to a number of new joint research projects started in Oberwolfach. The following section of abstracts contains summaries of all the lectures given during the workshop.

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Workshop: Hilbert Modules and Complex Geometry

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Hilbert modules of holomorphic and non-holomorphic type over symmetric domains

Harald Upmeier

Bounded symmetric domains are natural generalizations of the unit disk and the unit ball. They can be realized as quotient spaces

$$D = G/K$$

where G is a semi-simple Lie group and K is a maximal compact subgroup. The fundamental example is the unit ball in the matrix space $\mathbf{C}^{r\times s}$ of arbitrary rank. Here $G = U(r,s)$ is the pseudo-unitary group of signature (r,s) and $K = U(r) \times U(s)$. The group action generalizes the classical Moebius transformations

$$z \mapsto (az+b)(cz+d)^{-1}.$$

In order to define the well known Hilbert modules of scalar valued holomorphic functions, such as the weighted Bergman spaces and the Hardy space, one introduces the so-called quasi-determinant $\Delta(z,w)$ which for matrices equals

$$\Delta(z,w) = \det(1 - zw^*).$$

Using suitable negative powers

$$K_\nu(z,w) = \Delta(z,w)^{-\nu}$$

one arrives at the reproducing kernel functions for these spaces. For example, the Hardy space corresponds to the parameter $\nu = d/r$ and the standard Bergman space corresponds to the so-called genus $\nu = p$ of the underlying domain. For matrices, $p = r + s$. There also exist natural analogues for the Drury-Arveson space in this setting.

These scalar-valued function spaces correspond to suitable holomorphic line bundles $\mathcal{L}$ over D and their powers $\mathcal{L}^\nu$. In order to construct holomorphic vector bundles of higher rank, and the associated Hilbert modules of vector-valued holomorphic functions on D, we consider instead positive integer powers

$$K_n(z,w) = \Delta(z,w)^n$$

of the quasi-determinant. In this case one obtains polynomials of bounded degree, more precisely these kernel functions have a canonical decomposition (under the natural action of K) into irreducible submodules labelled by integer partitions $m_1 \geq m_2 \geq \ldots \geq m_r \geq 0$ of length r such that $m_1 \leq n$. There are $\binom{n+r}{r}$ such partitions. One can define a "big" Hilbert space of holomorphic mappings from D into the polynomials $\mathcal{P}^n$ spanned by the kernel functions $K_n(z,w)$. On the other hand, each partition gives rise to a "little" Hilbert space of holomorphic mappings from D into the polynomials $\mathcal{P}_{m_1,\ldots,m_r}$ belonging to the irreducible K-submodule. The main result, obtained for the unit ball ($r = 1$) in collaboration with Gadadhar Misra, is an explicit construction of G-equivariant intertwining operators from the little Hilbert spaces into the big Hilbert space. This leads to a classification of the invariant hermitian metrics on holomorphic vector bundles which are multiplicity-free.

More ambitious is the project to introduce and study Hilbert modules of non-holomorphic type. Each bounded symmetric domain D has a compact symmetric dual space M, which is called the conformal compactification. For matrices M coincides with the Grassmann manifold of r-dimensional subspaces in $\mathbf{C}^{r+s}$. The group G does not act transitively on M but instead it has $r+1$ open orbits $\mathcal{O}$ which in general are not domains of holomorphy (pseudo-convex). Instead of holomorphic functions one considers Dolbeault cohomology classes $H^{0,q}(\mathcal{O})$ of $(0,q)$-differential forms, with respect to the $\overline{\partial}$-operator. Following ideas from integral geometry introduced into complex analysis by S. Gindikin we describe the restriction of these cohomology classes to the Shilov boundary S via a generalized Radon transform. This leads to the complementary G-invariant subspaces of $L^2(S)$ besides the Hardy space $H^2(S)$.

Generalizations of the radial parts of invariant operators

Jonathan Arazy

Introduction

Let D be a Cartan domain of rank r in $\mathbb{C}^n$, i.e. an irreducible bounded symmetric domain in its Harish Chandra realization (namely - as the open unit ball of a Jordan-triple $Z := (\mathbb{C}^n, \{\cdot,\cdot,\cdot\})$). Let $G = \mathrm{Aut}(D)$ be the group of all holomorphic automorphisms of D and let $\mathcal{D}$ be the algebra of all G-invariant differential operators on D. It is well known that $\mathcal{D}$ is a commutative algebra, whose minimal number of generators is r. Moreover, the spherical functions $\{\psi_\lambda\}_{\lambda\in\mathbb{C}^r}$ form a complete set of joint eigenfunctions for the members of $\mathcal{D}$, and the eigenvalue map $\mathcal{D} \ni T \mapsto \tilde{T}(\lambda) = T\psi_\lambda(0)$ (called also the " Harish Chandra transform") yields a canonical isomorphism between $\mathcal{D}$ and the polynomial algebra $\mathbb{C}[x_1, x_2, \cdots, x_r]$. Let μ_0 be the unique (up to a constant multiple) G-invariant measure on D. Then the members of $\mathcal{D}$ are (unbounded) commuting normal operators on $L^2(D, \mu_0)$, and this gives rise to a spectral measure E on $\mathbb{C}^r$. Via the functional calculus $\varphi \mapsto \int_{\mathbb{C}^r} \varphi(\lambda)\, dE(\lambda)$ one can consider other classes of interesting G-invariant operators (for instance - the Berezin transforms), and they can be studied also as operators on suitable G-invariant L^p-spaces.

Let $K := \{g \in G; g(0) = 0\}$ be the maximal compact subgroup of G. It is well known that many interesting problems concerning the G-invariant operators depend only on the behavior of their K-radial parts (identified with their restrictions to the K-invariant functions). The Cartan decomposition $G = KAK$ and the fact that $D \equiv G/K$ imply that the radial parts of the G-invarianl operators are identified with operators on functions spaces on (subsets of) $A \equiv \mathbb{R}^r$. It is important to notice that in this realization the characteristic dimensions a, b and n of D, coming from the Jordan-triple structure of $Z := (\mathbb{C}^n, \{\cdot,\cdot,\cdot\})$, become *parameters.*

I am interested in the study of the class (denoted by $\mathcal{R}$) of the generalization of the radial parts of the G-invariant operators when the characteristic dimensions a, b and n are replaced by continuous parameters. The study of special aspects and of individual members of $\mathcal{R}$ is at least 30 years old, and in addition to its importance in analysis and geometry it is of great importance also in combinatorics. I make no attempt to survey here this study, but I would like to mention only the fundamental theory of Heckman and Optam [1] on general root systems.

In addition to my interest in the study of individual members of $\mathcal{R}$, I am interested in the *structural properties* of $\mathcal{R}$. Here are some natural (and not completely independent) questions on this theme: Does there exist a natural geometric realization for the members of $\mathcal{R}$? Is $\mathcal{R}$ commutative? Are the members of $\mathcal{R}$ normal operators with respect to the generalization of the radial part of μ_0? Does there exist large families of joint eigenfunctions for the members of $\mathcal{R}$ which allow an analogue of the Harish Chandrd transform? Are the members of $\mathcal{R}$ functions (in a canonical way) of a special set of generators of $\mathcal{R}$?

The affirmative answers to the analogous questions in the context of the G-invariant operators on D (and thus - for their radial parts) does not imply the affirmative answers for the members of $\mathcal{R}$. Thus the study of these interesting and difficult questions in the setup of $\mathcal{R}$ requires more efforts, and most likely - the development of new techniques. Of course - the individual results mentioned above (using different terminology and notation) will be of great help.

In what follows I will report (without details or proofs) on a work in progress (jointly with Leonid Zelenko) in the simplest case of the rank-1 domains, namely the open (Euclidean) unit ball B_n of $\mathbb{C}^n$, $n \in \mathbb{N}$. We study the spectral properties of the generalizations of the radial parts of the invariant Laplacian and Berezin transforms on B_n, and obtain interesting results. This suggests affirmative answers to some of the above questions in the rank - 1 case.

Generalization of the radial part of the invariant Laplacian

Let $\gamma > 0$ be fixed and consider on $(0,1)$ the operator

$$L_\gamma := x(1-x)^2 D^2 + (1-x)(\gamma - x)D, \qquad (D := \frac{d}{dx})$$

and the measure

$$d\mu_\gamma(x) := x^{\gamma-1}(1-x)^{-\gamma-1}\, dx.$$

If $\gamma = n \in \mathbb{N}$ then (up to normalizing constants) L_γ is the radial part of the G-invariant Laplacian on B_n, and μ_γ is the radial part of the G-invariant measure on B_n. Therefore the continuous parameter γ can be considered as a "generalized dimension", and L_γ and μ_γ - as the "generalized Laplacian" and the" generalized invariant measure", respectively.

Let us write every $\lambda \in \mathbb{C}$ as $\lambda = \beta(\beta - \gamma)$, where $\Re(\beta) \geq \gamma/2$, and define

$$e_\beta(x) := (1-x)^\beta\, {}_2F_1(\beta, \beta; \gamma; x), \quad x \in (0,1)$$

$$f_\beta(x) := (1-x)^\beta\, {}_2F_1(\beta, \beta; 1 + 2\beta - \gamma, 1 - x), \quad x \in (0,1),$$

where ${}_2F_1(\cdot, \cdot; \cdot; \cdot)$ is Gauss' hypergeometric function

Theorem: *Let $\lambda \in \mathbb{C}$. Then $ker(L_\gamma - \lambda I) = span\{e_\beta, f_\beta\}$. The resolvent $R_{L_\gamma}(\lambda) := (\lambda I - L_\gamma)^{-1}$ of L_γ is the integral operator*

$$R_{L_\gamma}(\lambda)f(x) := \int_0^1 G_\lambda(x,y)f(y)\,d\mu_\gamma(y),$$

whose kernel (the Green function) G_λ is given by

$$G_\lambda(x,y) := c(\lambda)\, f_\beta(max\{x,y\})\, e_\beta(min\{x,y\}), \quad c(\lambda) := \frac{\Gamma(\beta)^2}{\Gamma(\gamma)\Gamma(1+2\beta-\gamma)}.$$

The knowledge of the asymptotics of the hypergeometric functions yield easily the characterization of $e_\beta, f_\beta \in L^p(\mu_\gamma)$. This knowledge and the symmetry of $G_\lambda(x,y)$ allow the study of the boundedness of $R_{L_\gamma}(\lambda)$ in the spaces $L^p(\mu_\gamma)$ via Schur's lemma on integral operators. This leads to the following result.

Theorem: *Let $1 \le p,q \le \infty$ be so that $p^{-1}+q^{-1} = 1$, and let $\lambda \in \mathbb{C}$. The following are equivalent:*

a: *$R_{L_\gamma}(\lambda)$ is bounded as an operator on $L^p(\mu_\gamma)$,*
b: *$R_{L_\gamma}(\lambda)$ is bounded as an operator on $L^q(\mu_\gamma)$,*
c: *$\Re(\beta) > \gamma/min(p,q)$.*

Consequently, the spectrum of L_γ as an operator on $L^p(\mu_\gamma)$, $p \neq 2$, is the closed "parabola domain"

$$\begin{aligned} \sigma_{L^p(\mu_\gamma)}(L_\gamma) &= \{\lambda \in \mathbb{C}; \lambda = \beta(\beta-\gamma), \gamma/2 \le \Re(\beta) \le \gamma/min(p,q)\} \\ &= \{x+iy;\ x,y \in \mathbb{R}, \quad x \le -\frac{\gamma^2}{pq} - \frac{y^2}{\gamma^2(\frac{1}{p}-\frac{1}{q})^2}\}, \end{aligned}$$

and the spectrum of L_γ as an operator on $L^2(\mu_\gamma)$ is the interval $(-\infty, -\gamma^2/4]$.

Theorem: *L_γ is symmetric with respect to μ_γ. Moreover, it admits a self-adjoint (unbounded) extension to the Sobolev space $W^{2,2}(0,1)$.*

Generalization of the radial part of Berezin transforms

For $\nu > \gamma$ define an operator $\mathcal{B}_\nu$ (depending also on γ) by

$$\mathcal{B}_\nu f(x) := \int_0^1 K_\nu(x,y)\, f(y)\, d\mu_\gamma(y),$$

where

$$K_\nu(x,y) := B(\gamma, \nu-\gamma)^{-1}\,(1-x)^\nu\,(1-y)^\nu\,{}_2F_1(\nu,\nu;\gamma;xy).$$

Notice that if $\gamma = n \in \mathbb{N}$ then $\mathcal{B}_\nu$ is the radial part of the Berezin transform with parameter ν on B_n. It is easy to see that $\mathcal{B}_\nu$ is doubly stochastic, hence $\|\mathcal{B}_\nu\|_{B(L^1(\mu_\gamma))} = \|\mathcal{B}_\nu\|_{B(L^\infty(\mu_\gamma))} = 1$. Thus, by interpolation, $\mathcal{B}_\nu$ is bounded also on $L^p(\mu_\gamma)$, $1 < p < \infty$, and admits the a-priori norm estimate $\|\mathcal{B}_\nu\|_{B(L^p(\mu_\gamma))} \le 1$. It is also clear that $\mathcal{B}_\nu$ is positive definite as an operator on $L^2(\mu_\nu)$.

Lemma: *Consider an integral operator* $T_K f(x) = \int_0^1 K(x,y)\, f(y)\, d\mu_\gamma(y)$ *with a* C^2*-kernel* K*. Then*

$$T_K\, L_\gamma = L_\gamma\, T_K \iff L_\gamma(x)K(x,y) = L_\gamma(y)K(x,y), \quad \forall x,y \in (0,1),$$

where $L_\gamma(x)$ *is the operator* L_γ *acting on the variable* x*, and similarly for* $L_\gamma(y)$*.*

As a corollary of the lemma we obtain the following important result.

Theorem: *Let* $0 < \gamma < \nu$*. Then* $L_\gamma \mathcal{B}_\nu = \mathcal{B}_\nu L_\gamma$*.*

Proposition: $e_\beta \in dom(\mathcal{B}_\nu)$ *if and only if* $\Re(\beta) < \nu$*, and in this case*

$$\mathcal{B}_\nu e_\beta = \frac{\Gamma(\nu-\gamma+\beta)\,\Gamma(\nu-\beta)}{\Gamma(\nu-\gamma)\,\Gamma(\nu)}\; e_\beta.$$

Remark: For $\gamma > 1$ and any $\lambda \in \mathbb{C}$, f_β is not an eigenfunction of $\mathcal{B}_\nu$, because $\mathcal{B}_\nu f_\beta$ is bounded near 0 and $f_\beta(x) \equiv c x^{1-\gamma}$ as $x \downarrow 0$.

Theorem: *Let* $0 < \gamma < \nu$*, and let* $1 \le p, q \le \infty$ *be so that* $\frac{1}{p} + \frac{1}{q} = 1$*. Then*

$$\|\mathcal{B}_\nu\|_{B(L^p(\mu_\gamma))} = \rho_{L^p(\mu_\gamma)}(\mathcal{B}_\nu) = \rho^{(point)}_{L^p(\mu_\gamma)}(\mathcal{B}_\nu) = \frac{\Gamma(\nu-\frac{\gamma}{p})\Gamma(\nu-\frac{\gamma}{q})}{\Gamma(\nu)\Gamma(\nu-\gamma)},$$

where

$$\rho_{L^p(\mu_\gamma)}(\mathcal{B}_\nu) \quad \text{and} \quad \rho^{(point)}_{L^p(\mu_\gamma)}(\mathcal{B}_\nu)$$

denote the spectral radius and the supremum of the eigenvalues, respectively.

Theorem: *Let* $\gamma \ge 2$*. Then, as an operator on* $L^p(\mu_\gamma)$*,* $\mathcal{B}_\nu$ *is the following function of* L_γ*:*

$$\mathcal{B}_\nu = \prod_{k=0}^{\infty} \left(I - \frac{L_\gamma}{(k+\nu)(k+\nu-\gamma)} \right)^{-1} = \frac{\Gamma(\nu-\gamma/2+A)\,\Gamma(\nu-\gamma/2-A)}{\Gamma(\nu)\,\Gamma(\nu-\gamma},$$

where $A = (L_\gamma - \gamma^2/4)^{1/2}$*.*

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Toeplitz operators on bounded symmetric domains and Dixmier trace

Genkai Zhang
(joint work with Harald Upmeier)

The trace of commutators of Toeplitz operators on strictly pseudo-convex domains has been studied for quite some time and for different motivations. Among many results there is the well-known Helton-Howe formula [2] expressing the anticommutators of $2n$-Toeplitz operators $T(\phi_1), \cdots, T(\phi_{2n})$ in terms of the integration of the form $d\phi_1 \wedge \cdots d\phi_{2n}$. In our earlier work [1] we found a Dixmier trace formula for the product $[T(\phi_1), T(\psi_1)] \cdots [T(\phi_n), T(\psi_n)]$ of commutators of pairs of Toeplitz operators. For bounded symmetric domains of higher rank the commutators of Toeplitz operators are however not compact [3]. We shall construct operator calculus by taking proper orthogonal projections and compute their trace and Dixmier trace. More precisely let D be an irreducible bounded symmetric domain of rank $r \geq 2$, viewed a realization of a Hermitian symmetric space K. The Hardy space $H^2(D)$ on D has an irreducible decomposition under K into subspaces indexed by tuples of integers, $m_1 \geq \cdots \geq m_r \geq 0$. We consider the Toeplitz operators of the form $T_1(\phi) = P_1 T(\phi) P_1$ where P_1 is the orthogonal projection onto the sum of spaces with $m_1 \geq m_2 = \cdots = m_r = 0$. It turns out that these operators, after conjuagating with certain K-invariant differential operators, can be realized as Toeplitz operators on certain circle bundle over a compact Hermitian symmetric space with symbols being differential operators in the sense of Boutet de Monvel and Guillemin [4]. In the case of Type I domain of complex $n \times m$-matrices the compact Hermitian symmetric space is the product $\mathbb{P}^{n-1} \times \mathbb{P}^{m-1}$ of the projective spaces, and in the case of Lie ball it is the nil cone in the projective space, i.e. the lines $[z]$ with $(z, z) = 0$. We find certain determinant type operators and compute explicitly their Dixmier trace in terms of integrations on the circle bundle. (This is a joint work in progress with H. Upmeier.)

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Cauchy–Riemann operators on generalized flag manifolds

Benjamin Schwarz

Let (X, h) be a Kähler manifold, and let $E \to X$ be a holomorphic vector bundle. Let $T^{1,0}$ and $T^{0,1}$ denote the holomorphic and the anti-holomorphic tangent bundle on X. The composition of the $\overline{\partial}$-operator and the isomorphism h_* induced by the Riesz-isomorphism $(T^{0,1})^* \cong T^{1,0}$ given by h defines the *Cauchy–Riemann operator* $\bar{D}$,

$$\bar{D} := h_* \circ \overline{\partial} \colon C^\infty(X, E) \xrightarrow{\overline{\partial}} C^\infty(X, E \otimes (T^{0,1})^*) \xrightarrow{h_*} C^\infty(X, E \otimes T^{1,0}).$$

Since $E \otimes T^{1,0}$ is again a holomorphic vector bundle, iterates of the Cauchy–Riemann operator are defined in the obvious way. By abuse of notation, we simply write

$$\bar{D}^m := \bar{D} \circ \cdots \circ \bar{D} : C^\infty(X, E) \to C^\infty(X, E \otimes (T^{1,0})^{\otimes m}).$$

Due to symmetry properties of the Kähler metric, it turns out that the image of $\bar{D}^m$ is actually contained in $C^\infty(X, E \otimes (T^{1,0})^{\otimes m})$, where Sym^m denotes the m'th symmetric power of the tangent bundle $T^{1,0}$. The kernels $\mathcal{N}^m(X, E) := \ker \bar{D}^{m+1}$ of these higher order Cauchy–Riemann operators define the filtered vector space of the so called *nearly holomorphic sections*,

$$\mathcal{N}(X, E) := \bigcup_{m \geq 0} \mathcal{N}^m(X, E) \quad \text{with} \quad \mathcal{N}^0(X, E) \subseteq \mathcal{N}^1(X, E) \subseteq \mathcal{N}^2(X, E) \subseteq \cdots .$$

We note that $\mathcal{N}^0(X, E) = \mathcal{O}(X, E)$ is the space of holomorphic sections. The concept of nearly holomorphic functions (associated to the trivial line bundle) was introduced by Shimura [9]. The corresponding Cauchy–Riemann operators have been generalized to the present form by Engliš and Peetre [1].

One of the first natural questions concerns the existence of non-trivial nearly holomorphic sections. On domains that admit a Kähler potential, it is well-known how to describe the space of nearly holomorphic sections, see e.g. [7]. On compact Kähler manifolds, this is a highly non-trivial question. The transition between local and global existence of nearly holomorphic sections is encoded in cohomology theory. We thus study the *sheaf of nearly holomorphic sections* (*of degree* $\leq m$), denoted by $\mathcal{N}^m(E)$, and defined by the assignment

$$U \subseteq X \text{ open} \mapsto \mathcal{N}^m(E)(U) := \mathcal{N}^m(U, E).$$

Let $\iota \colon \mathcal{N}^{m-1}(E) \hookrightarrow \mathcal{N}^m(E)$ denote the natural inclusion. One of the main results presented here is the following fact.

Theorem 1. *The sequence*

$$0 \to \mathcal{N}^{m-1}(E) \xrightarrow{\iota} \mathcal{N}^m(E) \xrightarrow{\bar{D}^m} \mathcal{O}(E \otimes \mathrm{Sym}^m) \to 0$$

is an exact sequence of coherent sheaves.

As an immediate consequence of the coherence, we obtain that $\mathcal{N}^m(X, E)$ is finite dimensional in case of a connected compact Kähler manifold. Applying the cohomology functor to the exact sequence, we obtain

Corollary 1. *If $H^1(X, E \otimes \mathrm{Sym}^\ell) = 0$ for all $0 < \ell < m$, then*
$$\mathcal{N}^m(X,E)/\mathcal{N}^{m-1}(X,E) \cong \mathcal{O}(X, E \otimes \mathrm{Sym}^m).$$

In the case of generalized flag manifolds, representation theory provides another approach to nearly holomorphic sections. Let $X = G/P$ be the generalized flag manifold with G a complex simple simply-connected Lie group, and $P \subseteq G$ a parabolic subgroup, and let $E = G \times^P E_o$ be the G-homogeneous holomorphic vector bundle associated a simple P-module E_o. We fix a maximal compact subgroup U in G and a U-invariant Kähler metric h on X. Then it is immediate from the definition that the Cauchy–Riemann operators are U-equivariant, so for each m, $\mathcal{N}^m(X,E)$ is U-invariant. Moreover, since X is compact, it follows that each nearly holomorphic section is U-finite. We are able to show that nearly holomorphic sections exhaust all U-finite smooth sections.

Theorem 2. *Let $X = G/P$, and $E = G \times^P E_o$ be the G-homogeneous vector bundle associated to a simple P-module E_o. Then,*
$$\mathcal{N}(X,E) = C^\infty(X,E)_{U-\mathit{finite}}.$$
In particular, $\mathcal{N}(X,E)$ is a dense subspace of $C^\infty(X,E)$ (with respect to uniform convergence).

We next consider the decomposition of $\mathcal{N}(X,E)$ into simple U-modules. According to the representation theory of compact Lie groups, we obtain a decomposition
$$\mathcal{N}(X,E) = \bigoplus_{\lambda \in \Lambda} m_\lambda^E \, V_\lambda^*,$$
where Λ parametrizes simple U-modules by their highest weights, V_λ^* denotes the U-module dual to V_λ, and m_λ^E denotes its multiplicity in $\mathcal{N}(X,E)$ which is known to be finite. For simplicity, we restrict our attention in the following to the case where the parabolic subgroup P is a Borel subgroup, and denote it by $P = B$. Then, simple B-modules are one-dimensional and parametrized by elements of the weight lattice, so $E = G \times^P \mathbb{C}_{-\mu}$ is a line bundle. Then, applying Theorem 2 it follows that the multiplicity $m_\lambda^\mu := m_\lambda^E$ is given by Kostant's multiplicity formula,
$$m_\lambda^\mu = \sum_{w \in W} (-1)^{|w|} \wp(w.\lambda - \mu),$$
where W is the Weyl group and $\wp(\nu)$ denotes the Kostant partition function, which counts the number of ways to write a weight ν as a sum of positive roots. So far, these are classical results from representation theory. The new ingredient here is the U-invariant filtration of the space of nearly holomorphic sections. We thus may ask for the U-decomposition with respect to this filtration, i.e. we introduce the polynomials
$$m_\lambda^\mu(q) := \sum_{k \geq 0} m_{\lambda,k}^\mu \, q^k,$$
where $m_{\lambda,k}^\mu$ denotes the multiplicity of V_λ^* in $\mathcal{N}^m(X,E)/\mathcal{N}^{m-1}(X,E)$. By definition, $m_\lambda^\mu(1) = m_\lambda^\mu$. Our main theorem states that these polynomials are precisely

given by Luszig's q-analog of Kostant's multiplicity formula [5]. To state the result, let $\wp(\nu;q)$ be the polynomial in q, whose coefficient of q^k counts the number of ways to write ν as a sum of precisely k (not necessarily distinct) positive roots.

Theorem 3. *Let $X = G/B$ and $E = G\times^B \mathbb{C}_{-\mu}$ be the (dual) G-homogeneous line bundle associated to a dominant weight $\mu \in \Lambda$. Then, the q-multiplicity $m_\lambda^\mu(q)$ is given by*

$$m_\lambda^\mu(q) = \sum_{w\in W} (-1)^{|w|}\wp(w.\lambda - \mu; q),$$

which is Lusztig's q-analog of Kostant's weight multiplicity formula.

Lusztig's polynomials occur in various branches of representation theory. In the special case $\mu = 0$ (corresponding to the trivial line bundle), the polynomials $m_\lambda^0(q)$ were first constructed, independently, by Hesselink [3] and Peterson [6]. They discovered that the polynomials $m_\lambda^0(q)$ are the coefficients of the Hilbert series corresponding to the (graded) coordinate ring of the nilpotent cone in the Lie algebra $\mathfrak{g}$ of G. Prior to this, Kostant determined this Hilbert series in terms of generalized exponents by an investigation of G-harmonic polynomials on $\mathfrak{g}$, see [4]. For general μ, Lusztig and Kato proved that $m_\lambda^\mu(q)$ are closely related to certain Kaszdan–Lusztig polynomials, see [2], and hence encode deep combinatorial and geometric information.

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Sharp estimates for Hankel operators and related invariants

Magnus Goffeng
(joint work with Heiko Gimperlein)

This report is concerned with recent work on spectral estimates for the commutator of the Szegö projection with functions of low regularity. Two methods will be discussed. The first method uses direct kernel estimates and a Theorem of Russo [6] (the method is described in [4]). The second method is joint work with Heiko Gimperlein, it gives sharper estimates at the cost of using heavier machinery (details can be found in [3]). The original motivation for this work was to obtain explicit analytic formulas for differential geometric and topological invariants under low regularity assumptions, in an approach much inspired by [2, Chapter 2.α, Proposition 3]. Applications in this direction can be found in [3, 4].

1. Geometric setup and main results

The geometric setup goes as follows. We assume $\Omega \subseteq M$ to be a relatively compact domain in a complex manifold of n complex dimensions with strictly pseudo-convex C^∞-boundary. We pick an associated contact form θ on $\partial\Omega$. The Hilbert space that we are interested in is the closed subspace

$$H^2(\partial\Omega) := \{f \in L^2(\partial\Omega) : f \text{ admits a holomorphic extension to } \Omega\} \subseteq L^2(\partial\Omega).$$

The choice of volume form on $\partial\Omega$ is irrelevant, but $\theta\wedge(\mathrm{d}\theta)^{n-1}$ would be a canonical choice. The orthogonal projection onto $H^2(\partial\Omega)$ will be denoted by $P_{\partial\Omega}$, this operator is called the Szegö projection. A standard way of studying the structure of the Szegö projection is through the Heisenberg structure on the boundary. With the contact form θ we associate the Heisenberg structure $H := \ker\theta \subseteq T\partial\Omega$. Since θ is a contact form, $\mathrm{d}\theta|_H$is non-degenerate and H is bracket generating:

$$H + [H, H] = T\partial\Omega, \quad \text{for } n > 1. \tag{1}$$

The Heisenberg structure H provides $T\partial\Omega$ with the structure of a Heisenberg group over each point on $\partial\Omega$. We choose a Riemannian metric on H. It follows from (1) and Chow's theorem that for $n > 1$, the Carnot-Caratheodory metric d_{CC}, which measures distances along piecewice smooth paths parallel to H, is well defined. For $n > 1$, we let $C^\alpha_{CC}(\partial\Omega)$ denote the space of functions Hölder continuous of exponent $\alpha \in (0, 1]$ with respect to d_{CC}. This is a Banach algebra in the norm

$$\|a\|_{C^\alpha_{CC}} := \|a\|_C + |a|_{C^\alpha_{CC}} \quad \text{where} \quad |a|_{C^\alpha_{CC}} := \sup_{x\neq y} \frac{|a(x) - a(y)|}{d_{cc}(x, y)^\alpha}.$$

There are *strict* inclusions $C^\alpha(\partial\Omega) \subsetneq C^\alpha_{CC}(\partial\Omega) \subsetneq C^{\alpha/2}(\partial\Omega)$. For $n = 1$, the natural analogy of $C^\alpha_{CC}(\partial\Omega)$ is $C^{\alpha/2}(\partial\Omega)$ for $\alpha \in (0, 2)$ and $Lip(\partial\Omega)$ for $C^2_{CC}(\partial\Omega)$.

We use the notation $(\mu_k(T))_{k\in\mathbb{N}}$ for the singular values of a compact operator T. For $p \in [1, \infty)$, we let $\mathcal{L}^p(\mathcal{H})$ denote the p^{th} Schatten ideal consisting of operators T with $\|T\|_{\mathcal{L}^p(\mathcal{H})} := \left(\sum_k \mu_k(T)^p\right)^{1/p} < \infty$. We denote the p^{th} weak Schatten ideal by $\mathcal{L}^{p,\infty}(\mathcal{H})$, it consists of operators T with $\|T\|_{\mathcal{L}^{p,\infty}(\mathcal{H})} := \sup_k k^{1/p}\mu_k(T) < \infty$.

The main result discussed in this report is the following estimate. There is a constant $C_{\partial\Omega}$ such that

$$\|[P_{\partial\Omega}, a]\|_{\mathcal{L}^{\frac{2n}{\alpha},\infty}(L^2(\partial\Omega))} \leq C_{\partial\Omega}\|a\|_{C^\alpha_{CC}}. \tag{2}$$

The exponent $2n/\alpha$ should be thought of as a dimension in the sense of noncommutative geometry. It appears since $2n$, and not the Euclidean dimension $2n-1$, governs the Weyl law in the Heisenberg calculus. In more geometric terms, $2n$ is the Hausdorff dimension of $(\partial\Omega, d_{CC})$ by Mitchell's theorem.

2. Proving a non-sharp version of the estimate (2)

It is possible to prove a non-sharp version of (2) using fairly coarse knowledge about the singularity of the kernel of the Szegö projection at the diagonal. To describe the singularity of the Szegö projection, we focus on the case of spheres. The difficulties in the general case is captured by this case as the Szegö projection microlocally looks like that on the sphere.

For the sphere $S^{2n-1} \subseteq \mathbb{C}^n$, a contact form is given by $\theta = 2\mathrm{Im}(\bar{z}\cdot \mathrm{d}z)$. The Szegö projection for the usual surface density $\mathrm{d}S$ on S^{2n-1} can be computed as

$$P_{S^{2n-1}}f(z) := \lim_{r\uparrow 1}\int_{S^{2n-1}} \frac{f(w)\mathrm{d}S(w)}{(1-rz\cdot\bar{w})^n},$$

after a suitable normalization. The integral kernel of $P_{S^{2n-1}}$ is equivariant for the transitive $SU(n)$-action, hence the structure of the kernel is characterized by its behaviour as $rw \to z = (1,0,\ldots,0)$. We take polar coordinates $w = (\sqrt{1-|w'|^2}\mathrm{e}^{i\varphi}, w')$ around $(1,0,\ldots,0)$, here $w' \in \mathbb{C}^{n-1}$. In these coordinates, as $rw \to z$

$$(1-rz\cdot\bar{w})^{-n} = (1-r\sqrt{1-|w'|^2}\mathrm{e}^{-i\varphi})^{-n} \sim (i\varphi + |w'|^2)^{-n}. \tag{3}$$

We note that $\theta|_{(1,0)} = \mathrm{d}\varphi$. Equation (3) implies that the integral kernel of the Szegö projection behaves like $d_{CC}(z,w)^{-2n}$, explaining why we use the metric d_{CC} that distinguishes the direction of the contact form. Russo's theorem [6, Theorem 1] implies that

$$\|[P_{S^{2n-1}}, a]\|_{\mathcal{L}^p(L^2(S^{2n-1}))} \leq \left(\int_{S^{2n-1}}\left(\int_{S^{2n-1}} \frac{|a(w)-a(z)|^{p'}}{d_{CC}(z,w)^{2np'}}\mathrm{d}S(w)\right)^{p/p'}\mathrm{d}S(z)\right)^{1/p},$$

where $p' = p/(p-1)$. After a direct integral estimate, we conclude that for any $\epsilon > 0$ there is a C_ϵ with $\|[P_{S^{2n-1}}, a]\|_{\mathcal{L}^{2n/\alpha+\epsilon}(L^2(S^{2n-1}))} \leq C_\epsilon |a|_{C^\alpha_{CC}}$.

3. Proving the sharp estimate (2)

We prove (2) using a trick from noncommutative geometry first introduced in [7]. It states that if D is an invertible and self-adjoint, but possibly unbounded operator, on a Hilbert space $\mathcal{H}$ and a is an antiself-adjoint operator preserving the domain of D, then as self-adjoint operators on $\mathcal{H}$:

$$-\|[D,a]\|_{\mathcal{B}(\mathcal{H})}|D|^{-1} \leq [D|D|^{-1}, a] \leq \|[D,a]\|_{\mathcal{B}(\mathcal{H})}|D|^{-1}. \tag{4}$$

If $[D, a]$ has no bounded extension, we interpret these inequalities as empty statements. It reduces spectral estimates of commutators $[F, a]$, where $F = 2P - 1$, to finding a D with $F = D|D|^{-1}$ and a suitable spectral behavior.

In the one-dimensional case $\partial\Omega = S^1 \cong \mathbb{R}/2\pi\mathbb{Z}$, the usage of (4) collapses to a slick argument of a result of Aleksandrov-Peller [1, Lemma 8.3]. The Heisenberg structure $H = 0$ is not bracket generating but we simply turn to the differential operator $D = -i\frac{\mathrm{d}}{\mathrm{d}\varphi} + \frac{1}{2}$. This choice of D has the ON-eigenbasis $(e_k)_{k\in\mathbb{Z}}$ where $e_k = (2\pi)^{-1/2}\mathrm{e}^{ik\varphi}$ because $De_k = (k+1/2)e_k$. Since $H^2(S^1)$ is spanned by $(e_k)_{k\in\mathbb{N}}$ it holds that $D|D|^{-1} = 2P_{S^1} - 1$. The identity $\|[D, a]\|_{\mathcal{B}(L^2(S^1))} = |a|_{Lip(S^1)}$ holds by definition. Combining (4) with interpolation gives us the desired estimate

$$\|[P_{S^1}, a]\|_{\mathcal{L}^{1/\alpha,\infty}(L^2(S^1))} \leq 2^{1-2\alpha}|a|_{C^\alpha(S^1)}.$$

For $n > 1$, the role of D is played by an elliptic operator in the Heisenberg calculus of order 1. The right spectral behaviour $|D|^{-1} \in \mathcal{L}^{2n,\infty}$ follows from the Weyl law for Heisenberg operators. Finally, a longer proof using a $T1$-theorem from [5] provides us with a constant $C_D > 0$ such that $\|[D, a]\|_{\mathcal{B}} \leq C_D\|a\|_{C^1_{CC}}$. From this construction, and studying the real interpolation of the scale $C^\alpha_{CC}(\partial\Omega)$, the estimate (2) follows from (4).

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Gelfand theory of a class of Toeplitz algebras on Bergman spaces over the unit ball

Wolfram Bauer

(joint work with Nikolai Vasilevski)

Let $\mathbb{B}^n$ denote the open Euclidean unit ball in $\mathbb{C}^n$ and consider the family $\{v_\lambda\}_{\lambda>-1}$ of standard weighted measures on $\mathbb{B}^n$ given by

$$dv_\lambda = c_\lambda(1 - |z|^2)^\lambda dv(z).$$

Here we write dv for the usual Lebesgue volume form and $c_\lambda > 0$ denotes a normalizing constant such that $v_\lambda(\mathbb{B}^n) = 1$. Let $\mathcal{A}^2_\lambda(\mathbb{B}^n)$ be the weighted Bergman space of v_λ-square integrable holomorphic functions on $\mathbb{B}^n$. As is well-known

$\mathcal{A}^2_\lambda(\mathbb{B}^n)$ forms a closed subspace of $L^2(\mathbb{B}^n, dv_\lambda)$ and hence the orthogonal projection $P : L^2(\mathbb{B}^n, dv_\lambda) \to \mathcal{A}^2_\lambda(\mathbb{B}^n)$ is well-defined. Given as symbol $f \in L^\infty(\mathbb{B}^n)$ the Toeplitz operator T_f acts on $\mathcal{A}^2_\lambda(\mathbb{B}^n)$ by the rule

$$T_f(g) = P(fg), \qquad \text{where} \qquad g \in \mathcal{A}^2_\lambda(\mathbb{B}^n).$$

Take a subset $\mathcal{S} \subset L^\infty(\mathbb{B}^n)$ and consider the normed closed subalgebra $\mathcal{T}_\lambda(\mathcal{S})$ of the algebra of all bounded operators on $\mathcal{A}^2_\lambda(\mathbb{B}^n)$ which is generated by Toeplitz operators with symbols in $\mathcal{S}$. In general, $\mathcal{T}_\lambda(\mathcal{S})$ is non-commutative, however, the question arises for which classes $\mathcal{S}$ of symbols $\mathcal{T}_\lambda(\mathcal{S})$ is a commutative Banach or even C^*-algebra (uniformly for all weights $\lambda > -1$). If $n = 1$ and in the C^*-algebra setting this question has been treated in [6, 10]. Roughly speaking the result is as follows: *A C^*-algebra generated by Toeplitz operators is commutative on each weighted Bergman space if and only if the corresponding symbols $f \in \mathcal{S}$ of the operators are constant on orbits of a maximal commutative subgroup of the Möbius transforms of the unit disc.*

The paper [7] extends this observation to the higher dimensional setting $n > 1$. In this case for $\mathcal{T}_\lambda(\mathcal{S})$ to be a commutative C^*-algebra the operator symbols need to be constant on the orbits of a maximal abelian subgroup of the automorphism group $\mathrm{Aut}(\mathbb{B}^n)$ of $\mathbb{B}^n$. Up to equivalence there are $n+2$ of such subgroups which then lead to $n+2$ essentially different "model" commutative C^*-algebras. Moreover, the spectral representation of its elements can be calculated explicitly in each case (cf. [7]). As it turns out other interesting classes of commutative Banach algebras generated by Toeplitz operators exist in the case $n > 1$. These algebras are induced by the maximal commutative subgroups of $\mathrm{Aut}(\mathbb{B}^n)$ that include a torus action. More precisely, these are the quasi-elliptic, quasi-parabolic, quasi-hyperbolic and quasi-nilpotent group, the latter ones only exist in dimensions $n \geq 3$. In terms of their generators these algebras have been classified in [4, 5, 8, 9].

In a second step their Gelfand theory (description of the maximal ideals, the Gelfand transform, radical, $\cdots$) is of interest and leads to various applications to the spectral theory of Toeplitz operators. In case of the Toeplitz Banach algebras subordinate to the quasi-elliptic group this projects has been carried out in a series of papers [1, 2, 3].

As a common feature it turns our that the algebras $\mathcal{T}_\lambda(\mathcal{S})$ studied in [1, 2, 3] are generated by two of its sub-algebras $\mathcal{A}$ and $\mathcal{B}$ whose elements mutually commute. Moreover, $\mathcal{A}$ is an infinitely generated commutative C^*-algebra of diagonal operators with respect to the standard basis of $\mathcal{A}^2_\lambda(\mathbb{B}^n)$ and $\mathcal{B}$ is a finitely generated commutative Banach algebra of non-diagonal operators and not $*$-invariant. We describe the maximal ideals $M_{\mathcal{A}}$ of $\mathcal{A}$ and $M_{\mathcal{B}}$ of $\mathcal{B}$ separately and determine the maximal ideals of $\mathcal{T}_\lambda(\mathcal{S})$ as a certain subset of the Cartesian product $M_{\mathcal{A}} \times M_{\mathcal{B}}$. As an important ingredient we have to define certain fibrations of $M_{\mathcal{A}}$ and $M_{\mathcal{B}}$. It turns out that $\mathcal{T}_\lambda(\mathcal{S})$ is not semi-simple and in various interesting cases we can calculate its radical and the Gelfand transform restricted to a dense sub-algebra. As an application, we remark various consequences to the spectral theory of the elements in $\mathcal{T}_\lambda(\mathcal{S})$. In some of the cases (i.e. whenever the maximal ideal space of $\mathcal{T}_\lambda(\mathcal{S})$ has a "simple structure") we conclude that these algebras are spectral invariant in all bounded operators on $\mathcal{A}^2_\lambda(\mathbb{B}^n)$. In particular, spectral invariance holds for all algebras that appear in low dimensions $n \leq 3$.

It is an interesting problem to find an explicit description of the C^*-algebras generated by the above diagonal operators as sub-algebras of $\ell^\infty(\mathbb{Z}^m_+)$ where $m \leq n$. The case $m = 1$ is understood and the description is given by an oscillation condition on bounded sequences. However, if $m > 1$ new effects arise and the proofs do not generalize. The structural analysis of the commutative Banach algebras induced by the quasi-parabolic, quasi-hyperbolic and quasi-nilpotent group as well remains an open problem and shall be studied in a future project.

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Some Reccent Applications of Hilbert Modules

Ronald Douglas

In my book with Pasulsen, we pursued the study of general contractive Hilbeert modules via "isometric" resolutions by the Hardy module. We proposed this approach for other classes of Hilbert modules. In this talk we suggested a different approach, one paralelling what is done in commutative algebra.

In particular, fix a reproducing kernel Hilbert module R on a domain G in $\mathbb{C}^m$ over the polynomial in m variables. Define the "free modules" for this study as the tensor product of R by a coefficient Hilbert space E, perhaps limiting E to be of finite dimension. Consider the class of Hilbert modules obtained as a quotient of a free module and a projective module M as one for which there exists another module N such that the direct sum of M and N is free. A basic question is whether the analogue of the Quillen-Suslin Theorem holds for G the unit ball and E finite dimensional. The answer for a RKHM R can be shown to depend only on the multiplier algebra for R. The result holds for the Hardy module and the weighted Bergman modules since all have the same multiplier algebra and the result for the Hardy module follows from the Beurling-Lax-Halmos Theorem.

In addition, we showed how to derive an analytic operator-valued function on the unit disk which is bounded below but has no analytic left-inverse based on the fact that the unilateral and Bergman shifts are not similar. Such an example was first given by Treil.

Finally, we showed that multiplication by a finite Blaschke product on the Bergman space can be represented as a generalized bundle shift.

Flag structure for operators in the Cowen-Douglas class

Gadadhar Misra

(joint work with Kui Ji, C. Jiang and D. Keshari)

The Cowen-Douglas class $B_n(\Omega)$ consists of those bounded linear operators T on a complex separable Hilbert space $\mathcal{H}$ which possess an open set $\Omega \subset \mathbb{C}$ of eigenvalues of constant multiplicity n and admit a holomorphic choice of eigenvectors: $s_1(w), \ldots, s_n(w)$, $w \in \Omega$, in other words, there exists holomorphic functions $s_1, \ldots, s_n : \Omega \to \mathcal{H}$ which span the eigenspace of T at $w \in \Omega$.

The holomorphic choice of eigenvectors $s_1, \ldots, s_n$ defines a holomorphic Hermitian vector bundle E_T via the map

$$s : \Omega \to \mathrm{Gr}(n, \mathcal{H}),\ s(w) = \ker(T - w) \subseteq \mathcal{H}.$$

In the paper [3], Cowen and Douglas show that there is a one to one correspondence between the unitary equivalence class of the operators T in $B_n(\Omega)$ and the equivelence classes of the holomorphic Hermitian vector bundles E_T determined by them.

They also find a set of complete invariants for this equivalence consisting of the curvature $\mathcal{K}$ of E_T and a certain number of its covariant derivatives.

Unfortunately, these invariants are not easy to compute unless n is 1.

Finding similarity invariants for operators in the class $B_n(\Omega)$ has been somewhat difficult from the beginning. The conjecture made by Cowen and Douglas in [3] was shown to be false [1, 2]. However, significant progress on the question of similarity has been made recently (cf. [6, 9]).

We isolate a subset of irreducible operators in the Cowen-Douglas class $B_n(\Omega)$ for which a complete set of tractable unitary invariants is relatively easy to identify. We also determine when two operators in this class are similar.

We introduce below this smaller class $\mathcal{F}B_2(\Omega)$ of operators in $B_2(\Omega)$ leaving out the more general definition for now.

Definition 1. *We let $\mathcal{F}B_2(\Omega)$ denote the set of bounded linear operators T for which we can find operators T_0, T_1 in $\mathcal{B}_1(\Omega)$ and an intertwiner S between T_0 and T_1, that is, $T_0S = ST_1$ so that*

$$T = \begin{pmatrix} T_0 & S \\ 0 & T_1 \end{pmatrix}.$$

An operator T in $B_2(\Omega)$ admits a decomposition of the form (cf. [9, Theorem 1.49, pp. 48]) $\begin{pmatrix} T_0 & S \\ 0 & T_1 \end{pmatrix}$ for some pair of operators T_0 and T_1 in $B_1(\Omega)$. Conversely, an operator T, which admits a decomposition of this form for some choice of T_0, T_1 in $B_1(\Omega)$ can be shown to be in $B_2(\Omega)$. In defining the new class $\mathcal{F}B_2(\Omega)$, we are merely imposing one additional condition, namely that $T_0S = ST_1$.

We show that T is in the class $\mathcal{F}B_2(\Omega)$ if and only if there exist a frame $\{\gamma_0, \gamma_1\}$ of the vector bundle E_T such that $\gamma_0(w)$ and $t_1(w) := \frac{\partial}{\partial w}\gamma_0(w) - \gamma_1(w)$ are orthogonal for all w in Ω. This is also equivalent to the existence of a frame $\{\gamma_0, \gamma_1\}$ of the vector bundle E_T such that $\frac{\partial}{\partial w}\|\gamma_0(w)\|^2 = \langle \gamma_1(w), \gamma_0(w)\rangle$, $w \in \Omega$. Our first main theorem on unitary classification is given below.

Theorem 1. *Let $T = \begin{pmatrix} T_0 & S \\ 0 & T_1 \end{pmatrix}$ and $\tilde{T} = \begin{pmatrix} \tilde{T}_0 & \tilde{S} \\ 0 & \tilde{T}_1 \end{pmatrix}$ be two operators in $\mathcal{F}B_2(\Omega)$. Also let t_1 and $\tilde{t}_1$ be non-zero sections of the holomorphic Hermitian vector bundles E_{T_1} and $E_{\tilde{T}_1}$ respectively. The operators T and $\tilde{T}$ are equivalent if and only if $\mathcal{K}_{T_0} = \mathcal{K}_{\tilde{T}_0}$ (or $\mathcal{K}_{T_1} = \mathcal{K}_{\tilde{T}_1}$) and $\frac{\|S(t_1)\|^2}{\|t_1\|^2} = \frac{\|\tilde{S}(\tilde{t}_1)\|^2}{\|\tilde{t}_1\|^2}$.*

Cowen and Douglas point out in [3] that an operator in $B_1(\Omega)$ must be irreducible. However, determining which operators in $B_n(\Omega)$ are irreducible is a formidable task. It turns out that the operators in $\mathcal{F}B_2(\Omega)$ are always irreducible. Indeed, if we assume S is invertible, then T is strongly irreducible.

Recall that an operator T in the Cowen-Douglas class $B_n(\Omega)$, up to unitary equivalence, is the adjoint of the multiplication operator M on a Hilbert space $\mathcal{H}$ consisting of holomorphic functions on $\Omega^* := \{\bar{w} : w \in \Omega\}$ possessing a reproducing kernel K. What about operators in $\mathcal{F}B_n(\Omega)$? For $n = 2$, a model for these operators is described below.

Let $\gamma = (\gamma_0, \gamma_1)$ be a holomorphic frame for the vector bundle E_T, $T \in \mathcal{F}B_2(\Omega)$. Then the operator T is unitarily equivalent to the adjoint of the multiplication operator M on a reproducing kernel Hilbert space $\mathcal{H}_\Gamma \subseteq \mathrm{Hol}(\Omega^*, \mathbb{C}^2)$ possessing a reproducing kernel $K_\Gamma : \Omega^* \times \Omega^* \to \mathbb{C}^{2\times 2}$. It is easy to write down the kernel K_Γ explictly: For $z, w \in \Omega^*$, we have

$$\begin{aligned} K_\Gamma(z,w) &= \begin{pmatrix} \langle \gamma_0(\bar{w}), \gamma_0(\bar{z})\rangle & \langle \gamma_1(\bar{w}), \gamma_0(\bar{z})\rangle \\ \langle \gamma_0(\bar{w}), \gamma_1(\bar{z})\rangle & \langle \gamma_1(\bar{w}), \gamma_1(\bar{z})\rangle \end{pmatrix} \\ &= \begin{pmatrix} \langle \gamma_0(\bar{w}), \gamma_0(\bar{z})\rangle & \frac{\partial}{\partial \bar{w}}\langle \gamma_0(\bar{w}), \gamma_0(\bar{z})\rangle \\ \frac{\partial}{\partial z}\langle \gamma_0(\bar{w}), \gamma_0(\bar{z})\rangle & \frac{\partial^2}{\partial z \partial \bar{w}}\langle \gamma_0(\bar{w}), \gamma_0(\bar{z})\rangle + \langle t_1(\bar{w}), t_1(\bar{z})\rangle \end{pmatrix}, \end{aligned}$$

where t_1 and $\gamma_0 := S(t_1)$ are frames of the line bundles E_{T_1} and E_{T_0} respectively. It follows that $\gamma_1(w) := \frac{\partial}{\partial w}\gamma_0(w) - t_1(w)$ and that $t_1(w)$ is orthogonal to $\gamma_0(w)$, $w \in \Omega$.

Setting $K_0(z,w) = \langle \gamma_0(\bar w), \gamma_0(\bar z)\rangle$ and $K_1(z,w) = \langle t_1(\bar w), t_1(\bar z)\rangle$, we see that the reproducing kernel K_Γ has the form:

$$K_\Gamma(z,w) = \begin{pmatrix} K_0(z,w) & \frac{\partial}{\partial \bar w}K_0(z,w) \\ \frac{\partial}{\partial z}K_0(z,w) & \frac{\partial^2}{\partial z \partial \bar w}K_0(z,w) + K_1(z,w) \end{pmatrix}. \tag{1}$$

We now give examples of natural classes of operators that belong to $\mathcal{F}B_2(\Omega)$. Indeed, we were led to the definition of this new class $\mathcal{F}B_2(\Omega)$ of operators by trying to understand these examples better.

An operator T is called *homogeneous* if $\phi(T)$ is unitarily equivalent to T for all ϕ in Möb which are analytic on the spectrum of T.

If an operator T is in $\mathcal{B}_1(\mathbb{D})$, then T is homogeneous if and only if $\mathcal{K}_T(w) = -\lambda(1-|w|^2)^{-2}$, for some $\lambda > 0$. The paper [10] provides a model for all homogeneous operators in $B_n(\mathbb{D})$. We describe them for $n = 2$. For $\lambda > 1$ and $\mu > 0$, set $K_0(z,w) = (1 - z\bar w)^{-\lambda}$ and $K_1(z,w) = \mu(1 - z\bar w)^{-\lambda-2}$. An irreducible operator T in $B_2(\mathbb{D})$ is homogeneous if and only if it is unitarily equivalent to the adjoint of the multiplication operator on the Hilbert space $\mathcal{H} \subseteq \mathrm{Hol}(\mathbb{D}, \mathbb{C}^2)$ determined by the positive definite kernel given in equation (1). The similarity as well as a unitary classification of homogeneous operators in $B_n(\mathbb{D})$ were obtained in [10] using non-trivial results from representation theory of semi-simple Lie group. For $n = 2$, this classification is a consequence of Theorem 1.

An operator T in $B_1(\Omega)$ acting on a Hilbert space $\mathcal{H}$ makes it a module over the polynomial ring via the usual point-wise multiplication. An important tool in the study of these modules is the localization. This is the Hilbert module $J\mathcal{H}^{(k)}_{\mathrm{loc}}$ corresponding to the spectral sheaf $J\mathcal{H} \otimes_{\mathcal{P}} \mathbb{C}^k_w$, where

(1) $\mathcal{P}$ is the polynomial ring,
(2) $\mathbb{C}^k_w$ is a k - dimensional module over the polynomial ring,
(3) the module action on $\mathbb{C}^k_w$ is via the map $\mathcal{J}(w)$, see [7, (2.8) pp. 376];
(4) $J : \mathcal{H} \to \mathrm{Hol}(\Omega, \mathbb{C}^k)$ is the jet map, namely, $Jf = \sum_{\ell=0}^{k-1} \partial^\ell f \otimes \varepsilon_{\ell+1}$, $\varepsilon_1, \ldots, \varepsilon_k$ are the standard unit vectors in $\mathbb{C}^k$.

We now consider the localization with $k = 2$. If we assume that the operator T has been realized as the adjoint of the multiplication operator on a Hilbert space of holomorphc function possessing a kernel function, say K, then the kernel $JK^{(2)}_{\mathrm{loc}}$ for the localization (of rank 2) given in [7, (4.2) pp. 393] coincides with K_Γ of equation (1). In this case, we have $K_1 = K = K_0$.

As is to be expected, using the complete set of unitary invariants given in Theorem 1, we see that the unitary equivalence class of the Hilbert module $\mathcal{H}$ is in one to one correspondence with that of $J\mathcal{H}^{(2)}_{\mathrm{loc}}$.

Thus the class $\mathcal{F}B_2(\Omega)$ contains two very interesting classes of operators. For $n > 2$, we find that there are competing definitions. One of these contains the homogeneous operators and the other contains the Hilbert modules obtained from the localization.

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Reducing subspace for multiplication operators of the Bergman space

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Let $L^2_a(\mathbb{D})$ be the Bergman space on the unit disk $\mathbb{D}$. For a bounded holomorphic function ϕ on the unit disk, define the multiplication operator M_ϕ on $L^2_a(\mathbb{D})$ by

$$M_\phi(h) = \phi h,\ h \in L^2_a(\mathbb{D}).$$

An invariant subspace $\mathcal{M}$ for M_ϕ is a closed subspace of $L^2_a(\mathbb{D})$ satisfying $\phi\mathcal{M} \subseteq \mathcal{M}$. If, in addition, $M^*_\phi\mathcal{M} \subseteq \mathcal{M}$, we call $\mathcal{M}$ a reducing subspace of M_ϕ. We say $\mathcal{M}$ is a minimal reducing subspace if there is no nontrivial reducing subspace for M_ϕ contained in $\mathcal{M}$. Obviously the problem of classifying the reducing subspaces of M_ϕ is equivalent to finding the projections in the commutant algebra $\mathcal{A}_\phi = \{\mathcal{M}_\phi, \mathcal{M}^*_\phi\}'$. Therefore, in the Bergman space $L^2_a(\mathbb{D})$ framework, one can use essentially the same proof (see [1, 9, 10]) to show that for a "nice" analytic function f, there exists a finite Blaschke product ϕ such that $\{M_f\}' = \{M_\phi\}'$. Therefore, the structure of the reducing subspaces of the multiplier M_f is the same as that for M_ϕ.

Much progress in understanding the lattice of reducing subspaces of M_ϕ has been made in recent years [2, 4, 5, 6, 7, 8, 11]. A major breakthrough was made be Douglas, Sun and Zheng [2] using a systematic analysis of the local inverses of the ramified finite fibration $\phi^{-1} \circ \phi$ over the disk. They proved that the linear dimension of the commutant $\mathcal{A}_\phi = \{\mathcal{M}_\phi, \mathcal{M}^*_\phi\}'$ is finite and equal to the number of connected components of the Riemann surface of $\phi^{-1} \circ \phi$. The authors raised the following question, whose validity they have established in degree $n \leq 8$.

Conjecture. For a Blaschke product ϕ of finite order, the double commutant algebra $\mathcal{A}_\phi$ is abelian.

Several notable corollaries would follow once one proves the conjecture. For instance, the commutativity of the algebra A_ϕ implies that, for every finite Blaschke product ϕ, the minimal reducing subspaces of M_ϕ are mutually orthogonal; in addition, their number is equal to the number q of connected components of the Riemann surface of $\phi^{-1} \circ \phi$.

In the recent paper [3] we offer an affirmative answer to the above conjecture.

Theorem. Let ϕ be a finite Blaschke product of order n. Then the von Neumann algebra $\mathcal{A}_\phi = \{M_\phi, M_\phi^*\}'$ is commutative of dimension q, and hence $\mathcal{A}_\phi \cong \underbrace{\mathbb{C} \oplus \cdots \oplus \mathbb{C}}_{q}$, where q is the number of connected components of the Riemann surface of $\phi^{-1} \circ \phi$.

It also allows us to provide an indirect description of the reducing subspaces. Following [2], let $\{G_1, \cdots, G_q\}$ be the partition of the local inverses coming from the Riemann surface $\phi^{-1} \circ \phi$. For two integers $0 \leq j_1, j_2 \leq n-1$, write $j_1 \sim j_2$ if

$$\sum_{\rho_k \in G_i} \zeta^{k\, j_1} = \sum_{\rho_k \in G_i} \zeta^{k\, j_2} \text{ for any } 1 \leq i \leq q.$$

This equivalence relation divided the set $\{0, 1, \cdots, n-1\}$ into a dual partition $\{G'_1, \cdots, G'_p\}$. It was proved in [3] that there is exact one minimal reducing subspace for each G'_j. From this operator theory technique, we deduced the following intrinsic geometry property for any finite Blaschke product.

Corollary. The number of components in the dual partition is also equal to q, the number of connected components of the Riemann surface for $\phi^{-1} \circ \phi$.

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Spectra of Multiplier Algebras of Dirichlet Type Spaces

Richard Rochberg

1. Introduction

The Hardy space, H^2, is a fundamental example of a Hilbert space with reproducing kernel which has a complete Nevanlinna Pick kernel (HSRK with a CNPK). We now understand that some properties that were once seen as special to the Hardy space are true (or have versions that are true) for this more general class of spaces.

Classically, it is known that the spectrum of H^∞, the multiplier algebra of the Hardy space, has rich structure. We ask about the extent to which the spectra of multiplier algebras of other Hilbert spaces in the class also have a rich structure. We outline some preliminary results, a detailed presentation is in [10].

2. General Definitions

We refer to the books [5] and [6] for basic facts about the Hardy space, H^2, and its multiplier algebra, H^∞, the algebra of bounded holomorphic functions on the unit disk, $\mathbb{D}$. For the basic facts about HSRK with a CNPK see [1] and [11].

We consider the generalized Dirichlet spaces. For $0 \leq \alpha \leq 1$ let $\mathcal{D}_\alpha$ be the Hilbert space of functions which are holomorphic on the unit disk, and for which, with $f = \sum a_n z^n$,

$$\|f\|_\alpha^2 = \sum (n+1)^{1-\alpha} |a_n|^2 \approx \|f\|_{H^2}^2 + \int\int_{\mathbb{D}} |f'|^2 (1-|z|^2)^\alpha dA < \infty.$$

This is a nested family of Hilbert spaces, The largest, $\mathcal{D}_1$, is the Hardy space, the smallest, $\mathcal{D}_0$, is the Dirichlet space. Each is a HSRK with a CNPK.

We denote the kernel function for $\mathcal{D}_\alpha$ by k and the normalized kernel functions by $\hat{k}$. We will use a particular type of domination for kernel functions; we say that k satisfies DOM if

$$\text{(DOM)} \qquad \exists C > 0, \forall x, y \in \mathbb{D},\ |k(x,y)| \leq C \operatorname{Re} k(x,y).$$

All of the $\mathcal{D}_\alpha$ EXCEPT the Hardy space $\mathcal{D}_1$ satisfy DOM.

We denote the multiplier algebra of $\mathcal{D}_\alpha$ by $M(\mathcal{D}_\alpha)$. This is a commutative Banach algebra (using the operator norm) and has maximal ideal space $\mathfrak{M}(M(\mathcal{D}_\alpha))$. For $m \in M(\mathcal{D}_\alpha)$ the Gelfand transform, $\hat{m}$ is the function on $\mathfrak{M}(M(\mathcal{D}_\alpha))$ whose value at ξ is $\hat{m}(\xi) = \xi(m)$.

Associated with a RKHS H of functions on $\mathbb{D}$ is a metric δ_H defined by

$$\delta_H(x,y) = \sqrt{1 - \left|\left\langle \hat{k}_x, \hat{k}_y \right\rangle\right|^2},$$

[2]. The metric δ_1 for the Hardy space is the psuedohyperbolic metric. For $0 < \alpha < 1$ the metric δ_α is not very different from δ_1. The metric δ_0 for the Dirichlet space is quite different, perhaps explaining some of the differences in the results which follow.

The metric δ is useful in studying the maximal ideal space. They are connected through the following.

Proposition 1. *For H a HSRK of functions on $\mathbb{D}$, and having a CNPK, the metric δ_H can alternatively be defined by, for $\zeta_1, \zeta_2 \in \mathbb{D}$:*

$$\begin{aligned} \delta_H(\zeta_1\zeta_2) &= \sup\left\{\rho(\hat{m}(\zeta_1), \hat{m}(\zeta_2)) : \|m\|_{\mathfrak{M}(M(H))} = 1\right\} \\ &= \sup\left\{|\hat{m}(\zeta_2)| : \hat{m}(\zeta_1) = 0 : \ \|m\|_{\mathfrak{M}(M(H))} = 1\right\}, \end{aligned}$$

Using these formulas as definitions, δ_H is a metric on $\mathfrak{M}(M(H))$ where it continues to satisfy satisfy $0 \le \delta \le 1$. The condition $\delta(\alpha, \beta) < 1$ is an equivalence relation on $\mathfrak{M}(M(H))$.

For the Hardy space the equivalence of the two suprema follows easily from the fact that H^∞ is a uniform algebra. In general we don't have that, and that equivalence requires von Neumann's inequality. The equivalence of this definition with the earlier one using the kernel functions uses the fact that we have a CNPK.

The function δ on $\mathfrak{M}$ is sometimes called the Gleason distance and the equivalence classes under the relation $\delta < 1$ are called parts or Gleason parts.

We are interested in analytic structure in $\mathfrak{M}(M(\mathcal{D}_\alpha))$, in particular analytic disks. An analytic disk is a nonconstant map $\Phi : \mathbb{D} \to \mathfrak{M}(M(\mathcal{D}_\alpha))$ such that, for all $m \in M(\mathcal{D}_\alpha)$, $\Phi(\hat{m}(z)) \in \mathrm{Hol}(\mathbb{D})$. For each α, the natural inclusion of $\mathbb{D}$ into $\mathfrak{M}(M(\mathcal{D}_\alpha))$ gives an example of an analytic disk, the only obvious example. It is easy to see that any analytic disk is contained in a single part; hence one approach in the search for analytic structure is to first look for parts that are not singletons.

In the classical work on H^∞, and in our work, nontrivial parts are related to interpolating sequences. A sequence $Z \subset \mathbb{D}$ is an *interpolating sequence*, $Z \in \mathcal{IS}(M(\mathcal{D}_\alpha))$, for $M(\mathcal{D}_\alpha)$ if $M(\mathcal{D}_\alpha)|_Z = \ell^\infty(Z)$. We denote the closure of the sequence Z in the compact space $\mathfrak{M}(M(\mathcal{D}_\alpha))$ by $\bar{Z}$. The interpolating sequences, $Z \in \mathcal{IS}(M(\mathcal{D}_\alpha))$, have been described. A necessary and sufficient condition is that the points of Z be separated in the metric δ_α and that the collection of points is not too thick in any region (a measure built from Z must be a Carleson measure for $\mathcal{D}_\alpha$). The rather simple proof of this, using DOM, and valid for all the $\mathcal{D}_\alpha$ EXCEPT the Hardy space is due to Bøe [3]. The case of the Hardy space is the classical theorem of Carleson [4]

3. Classical Results for the Hardy Space

The first results about analytic structure in $M(\mathcal{D}_\alpha) = \mathfrak{M}(H^\infty)$ was due to I. J. Schark in 1961 [11].

Theorem 1. *Suppose $Z \in IS(M(\mathcal{D}_1))$. Any $\mathfrak{m} \in \bar{Z}$ is the center of an analytic disk in $\mathfrak{X}_0$.*

This was substantially refined by Hoffman [7]

Theorem 2. *Suppose $\mathfrak{s} \in \mathfrak{M}(M(\mathcal{D}_1))$ and let $\mathcal{P}(\mathfrak{s})$ be the Gleason part containing $\mathfrak{s}$. Then*

1. *Either $\mathcal{P}(\mathfrak{s}) = \mathfrak{s}$, or $\mathcal{P}(\mathfrak{s})$ is an analytic disk.*
2. *The second case occurs exactly when there is a $Z \in IS(M(\mathcal{D}_1))$ with $\mathfrak{s} \in \bar{Z}$.*

4. Results for Generalized Dirichlet Spaces

The first of those results extends to the $M(\mathcal{D}_\alpha)$ for $0 < \alpha < 1$ but the result is not known for $\alpha = 0$.

Theorem 3. *For $0 < \alpha < 1$, if $\{z_n\} = Z \in IS(M(\mathcal{D}_\alpha))$ and $\mathfrak{m} \in \bar{Z}$ then $\mathfrak{m}$ is the center of an analytic disk in $\mathfrak{M}(M(\mathcal{D}_\alpha))$.*

The proof follows the pattern in [11]. A map Φ of the disk into $\mathfrak{M}$ is constructed as a limit of disk automorphisms. If one can find a function $B_Z \in M(\mathcal{D}_\alpha)$ which vanishes on Z and with control on $\{B_Z'(z_n)\}$ one can use that function as a test function to show Φ is nonconstant by showing that $(\Phi \circ B_Z)'(0) \neq 0$. The Blaschke product with zero set Z is in the multiplier algebra [13] and can be used as the required test function.

5. A No-Go Theorem for the Dirichlet Space

The previous proof does not extend to the Dirichlet space; neither the Dirichlet space nor its multiplier algebra contain any infinite Blaschke products. However there is a more fundamental obstacle which shows that a proof of that general kind cannot work. Any map of the disk into $\mathfrak{M}(M(\mathcal{D}_0))$ constructed as a limit of maps of the disk into itself must be constant.

Proposition 2. *Suppose $\{\alpha_n\}, \{\beta_n\} \subset \mathbb{D}$ and $\exists C, \forall n,\ \delta_1(\alpha_n, \beta_n) < C < 1$. If $|\alpha_n| \to 1$; then $\delta_0(\alpha_n, \beta_n) \to 0$.*

The proof of this is by a detailed elementary comparison of δ_0 and δ_1

Corollary 1. *Suppose $\{L_\alpha\}$ are analytic maps of $\mathbb{D}$ to $\mathbb{D}$ with $|L_n(0)| \to 1$. Then any limit map, $L = \lim L_\alpha$ mapping $\mathbb{D}$ into $\mathfrak{M}(M(\mathcal{D}_0))$ is a constant map..*

6. A Result for the Dirichlet Space

We do not know if there are nontrivial analytic disks in $\mathfrak{M}(M(\mathcal{D}_0))$. However there is some structure in $\mathfrak{M}(M(\mathcal{D}_0))$. An indication is given by the following:

Theorem 4. *If $Z \in IS(M(\mathcal{D}_0))$ and $\mathfrak{m} \in \bar{Z}$ the Gleason part containing $\mathfrak{m}$ contains infinitely many points.*

The heart of the proof of this is finding a second point in the part. Here is the idea. Suppose $\tilde{Z}$ is a slight perturbation of Z in the δ_0 metric. Then $\tilde{Z}$ is also an interpolating sequence and furthermore, its closure contains a point $\tilde{\mathfrak{m}}$ is the same part as $\mathfrak{m}$. To show that $\tilde{\mathfrak{m}} \neq \mathfrak{m}$ one needs to find a function in $M(\mathcal{D}_0)$ which separates them. Such a function is easy to find as soon as one shows that $\tilde{Z} \cup Z$ is an interpolating sequence.

7. Final Comments

The proof of the previous theorem requires that we be working with a HSRK with a CNPK whose kernel functions satisfy DOM (which is used in characterizing the interpolating sequences and also in showing their stability under perturbation and forming unions). Hence the result holds, with the same proof, for a large variety of spaces, including for instance, the $\mathcal{D}_\alpha$, $0 < \alpha < 1$ and various Besov spaces of functions in one or several variables. (However, the full ubiquity of DOM is unclear.) The proof, however, does not apply directly to H^∞ in one or several variables. For more about H^∞ in this context, see [9]. For recent general discussion of analytic structure in spectra see [8].

More broadly, much of what is known about the maximal ideal space of H^∞, beginning with the early results described in [6], suggest natural questions about the multiplier algebras of Hilbert spaces, and also Banach spaces, of analytic functions. Almost nothing is know.

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Multiplier algebras of embedded discs

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(joint work with Kenneth R. Davidson and Orr Shalit)

The results mentioned are contained in [3].

We are concerned with multiplier algebras of certain reproducing kernel Hilbert spaces with the complete Nevanlinna-Pick property. By a result of Agler and McCarthy [1], every such space can be regarded as a space of functions on a variety V in a complex ball $\mathbb{B}_d$. More precisely, for $1 \leq d \leq \infty$, the Drury-Arveson space H^2_d is the reproducing kernel Hilbert space on the open unit ball $\mathbb{B}_d$ in $\mathbb{C}^d$ (which is understood as ℓ^2 if $d = \infty$) with reproducing kernel

$$K(z,w) = \frac{1}{1 - \langle z, w \rangle} \quad \text{for } z, w \in \mathbb{B}_d.$$

Given a separable irreducible complete Nevanlinna-Pick space $\mathcal{H}$, there is a subset $V \subset \mathbb{B}_d$ such that the multiplier algebra $\mathrm{Mult}(\mathcal{H})$ of $\mathcal{H}$ can be identified with $\mathcal{M}_V := \mathrm{Mult}(H^2_d)\big|_V$. Moreover, V can be chosen to be a *variety*, that is, there is a family $S \subset \mathrm{Mult}(H^2_d)$ such that

$$V = \{z \in \mathbb{B}_d : f(z) = 0 \text{ for all } f \in S\}.$$

In [5], Davidson, Ramsey and Shalit studied the connection between the geometry of the variety V and the structure of the operator algebra $\mathcal{M}_V$. In particular, they investigated the isomorphism problem for multiplier algebras of complete Nevanlinna-Pick spaces by studying the following question: Let $V, W \subset \mathbb{B}_d$ be two varieties. When are $\mathcal{M}_V$ and $\mathcal{M}_W$ (algebraically, isometrically) isomorphic?

The authors of [5] show (in the case of $d < \infty$) that $\mathcal{M}_V$ and $\mathcal{M}_W$ are isometrically isomorphic if and only if there is a conformal automorphism of $\mathbb{B}_d$ which maps V onto W. The question of algebraic isomorphism turns out to be harder. The results of [4] and [6] combine to show that in the case of homogeneous varieties, $\mathcal{M}_V$ and $\mathcal{M}_W$ are algebraically isomorphic if and only if V and W are biholomorphically equivalent.

For general varieties, the following result is shown in [5]. A biholomorphism $F : V \to W$ is called a *multiplier biholomorphism* if the coordinates of F are contained in $\mathcal{M}_V$, and the coordinates of F^{-1} are contained in $\mathcal{M}_W$.

Theorem (Davidson-Ramsey-Shalit). *Let $V, W \subset \mathbb{B}_d$ be varieties with $d < \infty$ which are the union of finitely many irreducible varieties and a discrete variety. If $\mathcal{M}_V$ and $\mathcal{M}_W$ are algebraically isomorphic, then V and W are multiplier biholomorphic.*

Davidson, Ramsey and Shalit also show that the converse of this is false by exhibiting two multiplier biholomorphic Blaschke sequences whose multiplier algebras are not isomorphic.

Here, we ask for a possible converse in the case of "nice" irreducible varieties. More precisely, we are concerned with varieties which are biholomorphic to the unit disc $\mathbb{D}$. The prototypical result in this setting is the following theorem due to Alpay, Putinar and Vinnikov [2], which, roughly speaking, says that for nicely embedded discs V, the algebra $\mathcal{M}_V$ is isomorphic to H^∞.

Theorem (Alpay-Putinar-Vinnikov). *Let $f : \mathbb{D} \to V \subset \mathbb{B}_d$ be a biholomorphism with $d < \infty$. If*

(1) f extends to an injective C^2 function on $\overline{\mathbb{D}}$,
(2) $f'(z) \neq 0$ for $z \in \overline{\mathbb{D}}$,
(3) $||f(z)|| = 1$ if and only if $|z| = 1$, and
(4) $\langle f(z), f'(z)\rangle \neq 0$ when $|z| = 1$.

Then $\mathcal{M}_V$ is isomorphic to H^∞.

We show that the transversality condition (4) is automatically satisfied. We also prove that for a slight weakening of the above conditions, the conclusion is no longer valid.

Theorem. *There exists a rational function $f : \overline{\mathbb{D}} \to V \subset \mathbb{B}_2$ with poles off $\overline{\mathbb{D}}$ as in the Alpay-Putinar-Vinnikov theorem, except for $f(1) = f(-1)$. In this case, $\mathcal{M}_V$ is not isomorphic to H^∞, and $f^{-1} \notin \mathcal{M}_V$.*

Note that the embedding f above is not a multiplier biholomorphism.

We also study a special class of embeddings $f : \mathbb{D} \to \mathbb{B}_\infty$ into the infinite dimensional ball. These embeddings are of the form

$$f(z) = (b_1 z, b_2 z^2, b_3 z^3, \ldots),$$

where $(b_n) \in \ell^2$ with $||(b_n)||_2 = 1$ and $b_1 \neq 0$. For every such choice of (b_n), the image $f(\mathbb{D})$ is a variety, and any two varieties of this type are multiplier biholomorphic. However, we exhibit an uncountable family of such embeddings whose multiplier algebras $\mathcal{M}_{f(\mathbb{D})}$ are not isomorphic. Concretely, for $s \leq 0$, let $\mathcal{H}_s$ be the reproducing kernel Hilbert space on $\mathbb{D}$ with reproducing kernel

$$K(z, w) = \sum_{n=0}^{\infty} (n+1)^s (z\overline{w})^n.$$

Note that $\mathcal{H}_0 = H^2$, and $\mathcal{H}_{-1}$ is the Dirichlet space.

Theorem. *For $s \in [-1, 0]$, there are embeddings $f_s : \mathbb{D} \to V_s \subset \mathbb{B}_\infty$ as above such that $\mathcal{M}_{V_s} \cong \mathrm{Mult}(\mathcal{H}_s)$. Any two of these algebras are not isomorphic.*

In the infinite dimensional ball, we encounter some surprises, such as when we extend the above scale of multiplier algebras beyond $s = -1$. For $s < -1$, there are still embeddings of the form $f_s(z) = (b_{s,1} z, b_{s,2} z^2, \ldots)$ as above such that $\mathcal{M}_{f_s(\mathbb{D})}$ can be identified with $\mathrm{Mult}(\mathcal{H}_s)$, but now $||(b_{s,n})||_2 = r_s < 1$.

Theorem. *For $s < -1$, the set $\overline{f_s(\mathbb{D})} \subset r_s\mathbb{B}_\infty$ is a* compact variety*. Any two of the algebras $M_{\overline{f_s(\mathbb{D})}}$ are not isomorphic.*

This is in stark contrast to the finite dimensional case, where every compact variety consists of only finitely many points.

The study of varieties in $\mathbb{B}_\infty$ is further complicated by the nature of the maximal ideal space of $\mathrm{Mult}(H^2_\infty)$. If $d < \infty$, then the only character ρ on $\mathrm{Mult}(H^2_d)$ with $\rho(Z_i) = 0$ for $1 \le i \le d$ is the character given by evaluation at the origin. This is no longer true for $d = \infty$. Indeed, there are interpolating sequences (z_n) for $\mathrm{Mult}(H^2_\infty)$ which converge weakly to zero. In this case, the adjoint of the surjective homomorphism

$$\Phi : \mathrm{Mult}(H^2_\infty) \to \ell^\infty, \quad \Phi(f) = (f(z_n)),$$

can be regarded as an embedding of $\beta\mathbb{N}$ into the maximal ideal space of $\mathrm{Mult}(H^2_\infty)$. Every character ρ in the set $\Phi^*(\beta\mathbb{N} \setminus \mathbb{N})$ satisfies $\rho(Z_i) = 0$ for all i.

The following is a list of questions that arises from our work:

(1) Is there a local definition of a variety in our sense? Is every (classical) irreducible component again a variety?
(2) Let $V, W \in \mathbb{B}_d$ with $d < \infty$ be *irreducible* varieties which are multiplier biholomorphic. Are $\mathcal{M}_V$ and $\mathcal{M}_W$ isomorphic?
(3) Is multiplier biholomorphism an equivalence relation on irreducible varieties in $\mathbb{B}_d$, $d < \infty$? It is not an equivalence relation on Blaschke sequences in $\mathbb{D}$.
(4) Let $f : \mathbb{D} \to V \subset \mathbb{B}_\infty$ be a proper embedding which extends continuously to $\overline{\mathbb{D}}$. Assume that $f(0) = 0$. Is the character of point evaluation at 0 the only character ρ on $\mathcal{M}_V$ with $\rho(Z_i) = 0$ for all i?

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Applications of two variable de Branges-Rovnyak spaces to toral curves

Greg Knese
(joint work with Kelly Bickel)

The purpose of this talk is to explain how understanding Hilbert spaces associated to bounded analytic functions on the bidisk $\mathbb{D}^2$ can lead to information about important classes of polynomials, such as stable polynomials and polynomials which define toral curves.

Let ϕ denote a holomorphic function $\phi : \mathbb{D}^2 \to \overline{\mathbb{D}}$ or more generally an operator valued holomorphic $\phi : \mathbb{D}^2 \to \mathcal{B}_1(E, E_*)$ where $\mathcal{B}_1(E, E_*)$ denotes the operator norm unit ball of the bounded linear operators from a Hilbert space E to a Hilbert space E_*. Because ϕ is a multiplier of $H^2(\mathbb{T}^2)$ (or an appropriate vector-valued space), the following kernel function is positive semi-definite

$$\frac{1 - \phi(z)\phi(w)^*}{(1 - z_1\bar{w}_1)(1 - z_2\bar{w}_2)}. \tag{1}$$

By general theory of reproducing kernel Hilbert spaces, there is an associated Hilbert space for which this kernel is its reproducing kernel. In analogy with one variable theory, we call this space the de Branges-Rovnyak space associated to ϕ. De Branges-Rovnyak spaces in one variable are important in operator models for contractions and can be used to address function theoretic problems such as the existence of angular derivatives at the boundary. Two variable de Branges-Rovnyak spaces are equally interesting and useful but the full space with kernel (1) is in some sense too large to be of use. Instead, it helps to notice, as J. Agler did, that the kernel (1) can be broken into two kernels; one corresponding to a space invariant under multiplication by z_1 and the other invariant under z_2:

$$\frac{1 - \phi(z)\phi(w)^*}{(1 - z_1\bar{w}_1)(1 - z_2\bar{w}_2)} = \frac{K_1(z,w)}{1 - z_1\bar{w}_1} + \frac{K_2(z,w)}{1 - z_2\bar{w}_2}.$$

Here K_1, K_2 are positive semi-definite kernels.

This *Agler decomposition* has numerous applications to function theory on the bidisk (see [1, 4, 3]) but there is an added mystery to it due to the fact that Agler decompositions are not unique and Agler's original proof of their existence used Andô's inequality and a Hahn-Banach cone separation argument and hence was non-constructive; see [1]. Other, more constructive, proofs have been found. Ball, Sadosky, and Vinnikov found a multidimensional scattering theory approach [5]. Kummert has an approach that produces an Agler decomposition at least for rational inner functions using a one variable matrix Fejér-Riesz factorization [11], while Geronimo and Woerdeman have a related approach that also applies only to rational inner functions and amounts to a certain truncated trigonometric moment problem on the two-torus $\mathbb{T}^2$ [8]. In the work [7], we have in some sense tried to unify these different approaches because the latter two yield exceptionally detailed information about the possible kernels K_1, K_2 while the approach of Ball-Sadosky-Vinnikov is the most general. We present here some applications of this unified approach to understanding stable polynomials and determinantal representations of certain varieties.

A scalar valued rational inner function on $\mathbb{D}^2$ is a function of the form $\phi(z) = \frac{\tilde{p}(z)}{p(z)}$ where $p \in \mathbb{C}[z_1, z_2]$ is a polynomial with no zeros on $\mathbb{D}^2$, $\tilde{p}$ is the "reverse" of p:

$$\tilde{p}(z) = z_1^n z_2^m \overline{p(1/\bar{z}_1, 1/\bar{z}_2)}$$

where p has bidegree (n, m), and we assume $\gcd(p, \tilde{p}) = 1$. A result of applying the more detailed Agler decomposition to ϕ is the formula

$$|p(z)|^2 - |\tilde{p}(z)|^2 = (1 - |z_1|^2) \sum_{j=1}^{n} |A_j(z)|^2 + (1 - |z_2|^2) \sum_{j=1}^{m} |B_j(z)|^2 \tag{2}$$

where $A_j, B_j \in \mathbb{C}[z_1, z_2]$. The number of squares in the above decomposition is minimal. It turns out that the detailed matrix valued analogue of this formula yields a similar decomposition for some three variable polynomials using a argument of Kummert [12]. In [6] we show that if $p \in \mathbb{C}[z_1, z_2, z_3]$ has no zeros in $\mathbb{D}^3$, no factors in common with $\tilde{p}$ and multidegree $(1, 1, n)$, then

$$|p(z)|^2 - |\tilde{p}(z)|^2 = \sum_{j=1}^{3} (1 - |z_j|^2) SOS_j(z) \tag{3}$$

where SOS_1, SOS_2, SOS_3 are sums of squared moduli of polynomials with SOS_1, SOS_2 at most two squares and SOS_3 at most $2n$ squares. Such decompositions do not exist in general for three variable polynomials with no zeros on $\mathbb{D}^3$. The number of squares is minimal at least when p has degree $(1, 1, 1)$, and we suspect it is minimal in general.

The above sums of squares formulas yield determinantal representations for certain classes of *stable* polynomials. The term *stable* is used in a variety of contexts and is perhaps not consistently used. Nevertheless, let us say $p \in \mathbb{C}[z_1, \dots, z_d]$ is *scattering Schur stable* if p has no zeros in $\mathbb{D}^d$ and no factors in common with $\tilde{p}$ and *self-reverse stable* if $p = \tilde{p}$ and p has no zeros in $\mathbb{D}^d$. The zero sets of self-reverse stable polynomials are *toral* in the sense that they intersect $\mathbb{T}^d$ in a $(d-1)$-dimensional set. Self-reverse stable polynomials can be converted to *real stable* polynomials through a Cayley transform. A polynomial $p \in \mathbb{R}[z_1, \dots, z_d]$ is *real-stable* if p has no zeros in $\mathbb{C}_+^d$. Here $\mathbb{C}_+$ is the upper half-plane. The connection between scattering Schur stable polynomials and self-reverse stable polynomials is that any irreducible self-reverse stable polynomial q can be written as $q = p + \tilde{p}$ where p is scattering Schur stable. This key observation along with a "lurking isometry argument" is enough to prove the following determinantal formulas.

If $q \in \mathbb{C}[z_1, z_2]$ self-reverse stable of degree (n, m), then there exists a $(n+m) \times (n+m)$ unitary U such that

$$q(z) = c \det(I - U\Delta(z))$$

where c is a constant and $\Delta(z) = \begin{pmatrix} z_1 I_n & 0 \\ 0 & z_2 I_m \end{pmatrix}$. Similarly, if $q \in \mathbb{C}[z_1, z_2, z_3]$ is self-reverse stable of degree $(1, 1, n)$, then there exists a $(4+2n) \times (4+2n)$ unitary

U such that $q(z)$ divides $\det(I - U\Delta(z))$

$$\Delta(z) = \begin{pmatrix} z_1 I_2 & 0 & 0 \\ 0 & z_2 I_2 & 0 \\ 0 & 0 & z_3 I_{2n} \end{pmatrix}.$$

We can only show q divides such a representation because of the extra squares required in (3). This issue is related to the Generalized Lax Conjecture; see [13] for more information. It is possible to convert these formulas to determinantal representations for real stable polynomials by applying an appropriate Cayley transform [10]. See [9] for other proofs of some determinantal representations.

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NC holomorphic functions: Three bears, but no Goldilocks

JOHN E. MCCARTHY
(joint work with Jim Agler)

A free polynomial, or nc polynomial (nc stands for non-commutative), is a polynomial in non-commuting variables. Let $\mathbb{P}^d$ denote the algebra of free polynomials in d variables. If $p \in \mathbb{P}^d$, it makes sense to think of p as a function that can be evaluated on matrices. Let $\mathbb{M}_n$ be the set of n-by-n complex matrices, and $\mathbb{M}^{[d]} = \cup_{n=1}^{\infty} {\mathbb{M}_n}^d$.

To make this precise, define a *graded function* to be a function f, with domain some subset of $\mathbb{M}^{[d]}$, and with the property that if $x \in \mathbb{M}_n^d$, then $f(x) \in \mathbb{M}_n$. An *nc-function* is a graded function f defined on a set $\Omega \subseteq \mathbb{M}^{[d]}$ such that

i) If $x, y, x \oplus y \in \Omega$, then $f(x \oplus y) = f(x) \oplus f(y)$.

ii) If $s \in \mathbb{M}_n$ is invertible and $x, s^{-1}xs \in \Omega \cap \mathbb{M}_n^d$, then $f(s^{-1}xs) = s^{-1}f(x)s$.

Free polynomials are examples of nc-functions. Nc-functions have been studied for a variety of reasons: by Anderson [2] as a generalization of the Weyl calculus; by Taylor [14], in the context of the functional calculus for non-commuting operators; Popescu [10, 11, 12, 13], in the context of extending classical function theory to d-tuples of bounded operators; Ball, Groenewald and Malakorn [3], in the context of extending realization formulas from functions of commuting operators to functions of non-commuting operators; Alpay and Kalyuzhnyi-Verbovetzkii [1] in the context of realization formulas for rational functions that are J-unitary on the boundary of the domain; Helton [4] in proving positive matrix-valued functions are sums of squares; and Helton, Klep and McCullough [5, 6] and Helton and McCullough [7] in the context of developing a descriptive theory of the domains on which LMI and semi-definite programming apply. Recently, Kaliuzhnyi-Verbovetskyi and Vinnikov have written a monograph on the subject [8].

We need to introduce topologies on $\mathbb{M}^{[d]}$.

We shall say that a set $\Omega \subseteq \mathbb{M}^{[d]}$ is an *nc domain* if it is closed under direct sums and unitary conjugations, and $\Omega \cap {\mathbb{M}_n}^d$ is open for every n. We shall say that a topology is an *admissible topology* if it has a basis of bounded nc domains.

Let τ be an admissible topology on $\mathbb{M}^{[d]}$, and let Ω be a τ-open set. A τ-holomorphic function is an nc-function $f : \Omega \to \mathbb{M}^{[1]}$ that is τ locally bounded.

We discussed three different admissible topologies, the fine, fat and free topologies. In the fine and the fat topologies, there is an inverse function theorem, due to J. Pascoe [9], and an implicit function theorem. However, holmorphic functions are not approximable by nc polynomials even pointwise. A consequence of the implicit function theorem is that if $p(X, Y)$ is an nc polynomial in 2 variables, and one looks at solutions to the matrix equation $p(X, Y) = 0$, then, for generic p and X, the only Y's that sole the equation must commute with X.

In the free topology, there is no implicit function theorem, but there is an Oka-Weil theorem: on compact, polynomially convex sets, one can approximate free holomorphic functions uniformly by nc polynomials. (In particular, one has pointwise approximation).

However, there is no Goldilocks topology: you cannot have both an implicit function theorem, and pointwise polynomial approximation.

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Multivariable operator theory and representations of the Cuntz algebras

Palle E. T. Jorgensen

The Cuntz algebra $\mathcal{O}_N$ is indexed by an integer $N > 1$, where N is the number of generators. As a C^*-algebra (denoted $\mathcal{O}_N$), it is defined by its generators and relations (the Cuntz-relations), and $\mathcal{O}_N$ is known to be a simple, purely infinite C^*-algebra. Further its K-groups are known. But its irreducible representations are highly subtle. To appreciate the importance of the study of representations of $\mathcal{O}_N$, recall that to specify a representation of $\mathcal{O}_N$ amounts to identifying a system of isometries in a Hilbert space $\mathcal{H}$, with mutually orthogonal ranges, and adding up to $\mathcal{H}$. But such orthogonal splitting in Hilbert space may be continued iteratively, and as a result, one gets links between the study of $\mathcal{O}_N$-representation on the one hand, to such neighboring areas as symbolic dynamics; and to filters used in signal processing, corresponding to a system of N uncorrelated frequency bands.

Returning to the subtleties of the representations of $\mathcal{O}_N$, and their equivalence classes, it is known that, for fixed N, that the set of equivalence classes of irreducible representations of $\mathcal{O}_N$, does not admit a Borel cross section; i.e., the equivalence classes, under unitary equivalence, does not admit a parameterization in the measurable Borel category. (Intuitively, they defy classification.) Nonetheless, special families of inequivalent representations have been found, and they have a multitude of applications, both to mathematical physics, to the study of wavelets, to harmonic analysis, to the study of fractals as iterated function systems and to the study of $\mathrm{End}(\mathcal{B}(\mathcal{H}))$ (= endomorphisms) where $\mathcal{H}$ is a fixed Hilbert space. Hence it is of interest to identify both discrete and continuous series of representations of $\mathcal{O}_N$; as they arise in such applications. In addition to unitary equivalence for pairs of representations of $\mathcal{O}_N$, we study quasi-equivalence (i.e., isomorphism of the associated von Neumann algebras.)

We show that $\mathcal{O}_N$ has a certain representation $\pi_{(universal)}$ which is universal in the following sense: $\pi_{(universal)}$ is multiplicity free (i.e., its commutant is abelian.) Every multiplicity free representation of $\mathcal{O}_N$ is unitarily equivalent to a subrepresentation of $\pi_{(universal)}$. Moreover, every representation of $\mathcal{O}_N$ is quasi-equivalent to a subrepresentation of $\pi_{(universal)}$. Some of this report includes joint research papers between Dorin Dutkay and the presenter.

We begin with a systematic study of $\mathrm{Rep}(\mathcal{O}_N, \mathcal{H})$ where $\mathcal{H}$ is a fixed Hilbert space. We compute, starting with a fixed representation of $\mathcal{O}_N$ (the Cuntz algebra with N generators), an associated spectral resolution of a maximal abelian algebra computed from the symbolic presentation of $\mathcal{O}_N$. This takes the form of a projection valued measure P on the Borel subsets of the Cantor group $\mathcal{K}_N$, an infinite Cartesian product of $\mathbb{Z}_N$, or equivalently, the set of all infinite words in a fixed alphabet of N letters. The relevance of these projection valued measures includes wavelet analysis, e.g., the study of decompositions of $L^2(\mathbb{R})$ with respect to wavelet packets; as well as to general and canonical decomposition of representations of $\mathcal{O}_N$. More generally we show (among our applications) that representations of $\mathcal{O}_N$ include both discrete series (such as permutative representations), as well as continuous series (e.g., wavelet representations) from low pass filters, and more generally monic representations, i.e., representations realized in $L^2(\mu)$ measure-spaces.

The material presented includes joint research with Ola Bratteli, and of more recently, with Dorin Dutkay; especially three joint papers (Dutkay-J) to appear, and in the arXiv. We refer to this regarding the representations, and the Hilbert modules realized this way; see also [1, 6, 7, 5, 4, 3, 2].

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Curvature invariant on noncommutative polyballs

Gelu Popescu

In [3] and [4], Arveson introduced and studied a notion of curvature for finite rank contractive Hilbert modules over $\mathbb{C}[z_1, \dots, z_n]$, which is basically a numerical invariant for commuting n-tuples $T := (T_1, \dots, T_n)$ in the unit ball

$$[B(\mathcal{H})^n]_1^- := \{(X_1, \dots, X_n) \in B(\mathcal{H})^n : \ I - X_1X_1^* - \cdots - X_nX_n^* \geq 0\},$$

with rank $\Delta_T < \infty$, where $\Delta_T := I - T_1T_1^* - \cdots - T_nT_n^*$ and $B(\mathcal{H})$ is the algebra of bounded linear operators on a Hilbert space $\mathcal{H}$. Subsequently, the author [21], [22] and, independently, Kribs [15] defined and studied a notion of curvature for arbitrary elements in $[B(\mathcal{H})^n]_1^-$ and, in particular, for the full Fock space $F^2(H_n)$ with n generators. Some of these results were extended by Muhly and Solel [16] to a class of completely positive maps on semifinite factors. The theory of Arveson's curvature on the symmetric Fock space $F_s^2(H_n)$ with n generators was significantly expanded due to the work by Greene, Richter, and Sundberg [12], Fang [9], and Gleason, Richter, and Sundberg [13]. English remarked in [8] that using Arveson's ideas one can extend the notion of curvature to complete Nevanlinna-Pick kernels. The extension of Arveson's theory to holomorphic spaces with non Nevanlinna-Pick kernels was first considered by Fang [11] who was able to show that the main results about the curvature invariant on the symmetric Fock space carry over, with different proofs and using commutative algebra techniques, to the Hardy space $H^2(\mathbb{D}^k)$ over the polydisc, in spite of its extremely complicate lattice of invariant subspaces (see [29]). Inspired by some results on the invariant subspaces of the Dirichlet shift obtained by Richter [28], the theory of curvature invariant was extended to the Dirichlet space by Fang [10]. In the noncommutative setting, a notion of curvature invariant for noncommutative domains generated by positive regular free polynomials was considered in [23].

Our goal is to develop a theory of curvature invariant for regular noncommutative polyballs. In particular, our results allow one to formulate a theory of curvature invariant and multiplicity invariant for the tensor product of full Fock spaces $F^2(H_{n_1}) \otimes \cdots \otimes F^2(H_{n_k})$ and also for the tensor product of symmetric

Fock spaces $F_s^2(H_{n_1}) \otimes \cdots \otimes F_s^2(H_{n_k})$. To prove the existence of the curvature and its basic properties in these settings requires a new approach based on non-commutative Berezin transforms and multivariable operator theory on polyballs and varieties (see [20], [24], [25], and [26]), and also certain summability results for completely positive maps which are trace contractive. In particular, we obtain new proofs for the existence of the curvature on the full Fock space $F^2(H_n)$, the Hardy space $H^2(\mathbb{D}^k)$ (which corresponds to $n_1 = \cdots = n_k = 1$), and the symmetric Fock space $F_s^2(H_n)$.

We remark that one can re-formulate the results of the paper in terms of Hilbert modules [7] over the complex semigroup algebra $\mathbb{C}[\mathbb{F}_{n_1}^+ \times \cdots \times \mathbb{F}_{n_k}^+]$ generated by the direct product of the free semigroups $\mathbb{F}_{n_1}^+, \ldots, \mathbb{F}_{n_k}^+$. In this setting, the Hilbert module associated with the universal model $\mathbf{S}$ acting on the tensor product $F^2(H_{n_1}) \otimes \cdots \otimes F^2(H_{n_k})$ plays the role of rank-one free module in the algebraic theory [14]. The commutative case can be re-formulated in a similar manner.

In a forthcoming paper, we introduce and study the Euler characteristic associated with the elements of polyballs, and obtain a version of Gauss-Bonnet-Chern theorem from Riemannian geometry, which connects the curvature to the Euler characteristic of some associated algebraic modules.

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Tensor algebras and weighted shifts

Baruch Solel
(joint work with Paul S. Muhly)

1. Introduction

In recent years, together with Paul Muhly, we have been studying noncommutative Hardy algebras (associated with W^*-correspondences) and viewed this study as a study of noncommutative function theory on domains which are certain unit balls.

Currently, inspired by a work of Popescu ([3]), we considered more general domains and that seems to suggest the study of “weighted tensor algebras”. In my talk I introduce some of our new results. These results show that domains of a certain type do correspond to certain weighted Hardy algebras in the sense that elements in these algebras can be viewed as (operator valued) functions on these domains.

However, this study also suggest that the theory of noncommutative function theory on more general domains (on one hand) and the theory of tensor algebras associated with more general weights (on the other hand) is worth studying and there is much more to do.

In the next section, I will describe the representations of the (unweighted) tensor algebras. In the succeeding sections I will briefly describe some of our results concerning more general domains.

2. Preliminaries: noncommutative Hardy (and tensor) algebras

Given a von Neumann algebra M and a W^*-correspondence E over M (assumed to be self dual), we can form the Fock correspondence.

$$\mathcal{F}(E) := M \oplus E \oplus E^{\otimes 2} \oplus E^{\otimes 3} \oplus \dots .$$

It is a W^*-correspondence over M and $\mathcal{L}(\mathcal{F}(E))$ is a von Neumann algebra. We consider two families of operators on $\mathcal{F}(E)$. For every $a \in M$, $\varphi_\infty(a)$ is defined on $\mathcal{F}(E)$ by $\varphi_\infty(a)(\xi_1 \otimes \cdots \xi_k) = \varphi_E(a)\xi_1 \otimes \cdots \otimes \xi_k$ where φ_E is the left action of M on E. For every $\xi \in E$, T_ξ is defined on $\mathcal{F}(E)$ by $T_\xi(\xi_1 \otimes \cdots \xi_k) = \xi \otimes \xi_1 \otimes \cdots \xi_k$.

The norm closed algebra generated by $\{\varphi_\infty(a), T_\xi : \ a \in M, \ \xi \in E\}$ is the *tensor algebra* $\mathcal{T}_+(E)$ and its w^*-closure is the *Hardy algebra* $H^\infty(E)$.

Examples 1.
1. *If $M = E = \mathbb{C}$, $\mathcal{F}(E) = \ell^2$, $\mathcal{T}_+(E) = A(\mathbb{D})$ and $H^\infty(E) = H^\infty(\mathbb{D})$.*
2. *If $M = \mathbb{C}$ and $E = \mathbb{C}^d$ then $\mathcal{F}(E) = \ell^2(\mathbb{F}_d^+)$, $\mathcal{T}_+(E)$ is Popescu's $\mathcal{A}_d$ and $H^\infty(E)$ is F_d^∞ (Popescu) or $\mathcal{L}_d$ (Davidson-Pitts). These algebras are generated by d shifts.*

Theorem 1. [1] *Every completely contractive representation of $\mathcal{T}_+(E)$ on a Hilbert space H is given by a pair $(\sigma, \mathfrak{z})$ where*

(1) σ is a normal representation of M on $H = H_\sigma$. ($\sigma \in NRep(M)$)
(2) $\mathfrak{z} : E \otimes_\sigma H \to H$ is a contraction that satisfies

$$\mathfrak{z}(\varphi(\cdot) \otimes I_H) = \sigma(\cdot)\mathfrak{z}.$$

We write $\sigma \times \mathfrak{z}$ for the representation and we have $(\sigma \times \mathfrak{z})(\varphi_\infty(a)) = \sigma(a)$ and $(\sigma \times \mathfrak{z})(T_\xi)h = \mathfrak{z}(\xi \otimes h)$ for $a \in M$, $\xi \in E$ and $h \in H$.

Write $\mathcal{I}(\varphi \otimes I, \sigma)$ for the intertwining space and D_σ for the open unit ball there. Thus the c.c. representations of the tensor algebra are parameterized by the family $\{\overline{D_\sigma}\}_{\sigma \in NRep(M)}$.

Examples 2.
(1) *$M = E = \mathbb{C}$. So $\mathcal{T}_+(E) = A(\mathbb{D})$, σ is the trivial representation on H, $E \otimes H = H$ and D_σ is the (open) unit ball in $B(H_\sigma)$.*
(2) *$M = \mathbb{C}$, $E = \mathbb{C}^d$. $\mathcal{T}_+(E) = \mathcal{A}_d$ (Popescu's algebra) and D_σ is the (open) unit ball in $B(\mathbb{C}^d \otimes H, H)$. Thus the c.c. representations are parameterized by row contractions $(T_1, \dots, T_d)$.*
(3) *M general, $E =_\alpha M$ for an automorphism α.*
$\mathcal{T}_+(E) =$ the analytic crossed product.
The intertwining space $\mathcal{I}(\varphi \otimes I, \sigma)$ can be identified with $\{\mathfrak{z} \in B(H) : \sigma(\alpha(T))\mathfrak{z} = \mathfrak{z}\sigma(T), T \in B(H)\}$ and the c.c. representations are $\sigma \times \mathfrak{z}$ where $\mathfrak{z}$ is a contraction there.

3. The weighted Tensor and Hardy Algebras

In the analysis above, the c.c. representations were parameterized by points in a closed unit ball (of $\mathcal{I}(\varphi \otimes I, \sigma)$). Now we turn to discuss more general domains.

The domains

To define the domains, we consider now a sequence $X = \{X_k\}_{k=1}^{\infty}$ of operators satisfying

- $X_k \in \mathcal{L}(E^{\otimes k}) \cap \varphi_k(M)'$ (i.e. a bimodule map).
- $X_k \geq 0$ for all $k \geq 1$ and X_1 is invertible.
- $\overline{lim}||X_k||^{1/k} < \infty$.

Definition 1. *A sequence $X = \{X_k\}_{k=1}^{\infty}$ satisfying (1)-(3) above is said to be admissible.*

Associated to an admissible sequence X, we now set

$$\overline{D}_{X,\sigma} := \{\mathfrak{z} \in \mathcal{I}(\varphi \otimes I, \sigma) : \quad ||\sum_{k=1}^{\infty} \mathfrak{z}^{(k)}(X_k \otimes I_{H_\sigma})\mathfrak{z}^{(k)*}|| \leq 1\}$$

where $\mathfrak{z}^{(k)} = \mathfrak{z}(I_E \otimes \mathfrak{z}) \cdots (I_{E^{\otimes k}} \otimes \mathfrak{z}) : E^{\otimes k} \otimes H \to H$.

Examples 3.

- *If $X_1 = I_E$ and $X_k = 0$ for $k > 1$, $\overline{D}_{X,\sigma} = \overline{D}_\sigma$.*
- *If $E = M = \mathbb{C}$, σ is on H and $X_k = x_k \in \mathbb{C}$, $\overline{D}_{X,\sigma} = \{T \in B(H) : \sum_k x_k T^k T^{*k} \leq I\}$.*
- *If $M = \mathbb{C}$, $E = \mathbb{C}^d$, σ is on H and X_k is the $d^k \times d^k$ matrix $(x_{\alpha,\beta})$ (where α, β are words of length k in $\{1, \dots, d\}$),*

$$\overline{D}_{X,\sigma} = \{T = (T_1, \dots, T_d) : \sum_{|\alpha|=|\beta|} x_{\alpha,\beta} T_\alpha T_\beta^* \leq I\}$$

where $T_\alpha = T_{\alpha_1} \cdots T_{\alpha_k}$.
- *If $E =_\alpha M$, $x_k \in Z(M)$ and*

$$\overline{D}_{x,\sigma} = \{T \in B(H_\sigma) : T\sigma(\alpha(\cdot)) = \sigma(\cdot)T, \sum_k T^k \sigma(x_k) T^{k*} \leq I\}.$$

Theorem 2. *Given an admissible sequence X, one can construct a sequence of operators $Z = \{Z_k\}$ such that*

- *$Z_k \in \mathcal{L}(E^{\otimes k}) \cap \varphi_k(M)'$.*
- *$Z_k \geq 0$ and invertible for all $k \geq 1$.*
- $\sup_k ||Z_k|| < \infty$.
- *The sequence Z can be computed from X via the equation $Z^{(k)*}Z^{(k)} = (\sum_{k=\sum i_j, i_j \in \mathbb{N}} X_{i_1} \otimes X_{i_2} \otimes \cdots \otimes X_{i_l})^{-1}$ where*

$$Z^{(m)} = Z_m(I_E \otimes Z_{m-1}) \cdots (I_{E^{\otimes(m-1)}} \otimes Z_1).$$

The sequence Z will be the **sequence of weights**.

Fix X and (associated) Z as above.

For $\xi \in E$, define the "Z-weighted shift" operator $W_\xi \in \mathcal{L}(\mathcal{F}(E))$ by

$$W_\xi(\xi_1 \otimes \xi_2 \otimes \cdots \otimes \xi_n) = Z_{n+1}(\xi \otimes \xi_1 \otimes \xi_2 \otimes \cdots \otimes \xi_n).$$

and $W_\xi b = Z_1(\xi b)$.

Definition 2. (1) *The norm-closed algebra generated by $\varphi_\infty(M)$ and $\{W_\xi : \xi \in E\}$ will be called the* ***Z-tensor algebra*** *of E and denoted $\mathcal{T}_+(E,Z)$.*

(2) *The ultra-weak closure of $\mathcal{T}_+(E,Z)$ will be called the* ***Z-Hardy algebra*** *of E and denoted $H^\infty(E,Z)$.*

Theorem 3. *Every completely contractive representation π of $\mathcal{T}_+(E,Z)$ on H is given by a pair $(\sigma, \mathfrak{z})$ where*

(1) σ is a normal representation of M on $H = H_\sigma$. ($\sigma \in NRep(M)$)

(2) $\mathfrak{z} \in \overline{D}_{X,\sigma}$.

In fact, $\pi(\varphi_\infty(a)) = \sigma(a)$ and $\pi(W_\xi)h = \mathfrak{z}(\xi \otimes h)$.

Conversely, every such pair gives rise to a c.c. representation.

Thus the c.c. representations of the Z-tensor algebra are parameterized by the family $\{\overline{D}_{X,\sigma}\}_{\sigma \in NRep(M)}$.

We write $\sigma \times \mathfrak{z}$ for the representation π above.

Lemma 1. *Given $\mathfrak{z} \in D_{X,\sigma}$, the map defined by $\Phi_{\mathfrak{z}}(T) = \sum_{k=1}^\infty \mathfrak{z}^{(k)}(X_k \otimes T)\mathfrak{z}^{(k)*}$ (where the convergence is in ultraweak operator topology) is a completely positive map on $\sigma(M)'$ and the sequence $\{\Phi_{\mathfrak{z}}^m(I)\}$ is decreasing. (Write $Q_{\mathfrak{z}}$ for its limit)*

An important role is played by the induced representations (of the tensor algebra). They can be viewed as a generalization of representing a (classical) weighted shift (on ℓ_2) by a weighted shift with multiplicity (on $\ell_2 \otimes K$).

If π is a normal representation of M on K then $X \in \mathcal{T}_+(E,Z) \mapsto X \otimes I_K \in B(\mathcal{F}(E) \otimes_\pi K)$ is a contractive representation (called *an induced representation*). The associated pair will be denoted $(Ind(\pi), \mathfrak{w}_K)$.

If σ is a normal representation of M on H and $\mathfrak{z} \in D_{X,\sigma}$, we write $\Delta_*(\mathfrak{z})$ for $(I_H - \sum_{k=1}^\infty \mathfrak{z}^{(k)}(X_k \otimes I_H)\mathfrak{z}^{(k)*})^{1/2}$. We also write $\mathcal{D}(\mathfrak{z})$ (or simply $\mathcal{D}$) for the subspace $\overline{\Delta_*(\mathfrak{z})H}$ and note that it is reducing for $\sigma(M)$. Also we write $K_{\mathcal{D}}$ for $\mathcal{F}(E) \otimes_\sigma \mathcal{D}$.

Theorem 4. *Let σ be a normal representation of M on H and $\mathfrak{z} \in D_{X,\sigma}$. Then there is a Hilbert space $\mathcal{U}$ with a normal representation τ of M on $\mathcal{U}$ and an element $\mathfrak{v} \in D_{X,\tau}$ such that*

(1) $\mathfrak{w}_{\mathcal{D}} \oplus \mathfrak{v}$ is a co-extension of $\mathfrak{z}$; that is, H can be identified with a subspace of $K_{\mathcal{D}} \oplus \mathcal{U}$ that is co-invariant for the representation of $\mathcal{T}_+(E,Z)$ associated with $\mathfrak{w}_{\mathcal{D}} \oplus \mathfrak{v}$ and $(\mathfrak{w}_{\mathcal{D}} \oplus \mathfrak{v})^*|H = \mathfrak{z}^*$.

(2) $\sum_{k=1}^\infty \mathfrak{v}^{(k)}(X_k \otimes I_{\mathcal{U}})\mathfrak{v}^{(k)*} = I_{\mathcal{U}}$.

If $Q_{\mathfrak{z}} = 0$, we get $\mathcal{U} = \{0\}$ and the representation associated with $\mathfrak{z}$ is a compression of an induced representation.

Remark 1. *By imposing extra conditions in the theorem above, we can require that $\mathfrak{v}$ (and therefore also $\mathfrak{w}_{\mathcal{D}} \oplus \mathfrak{v}$) will be extendable to a C^*-representation of the C^*-algebra generated by $\mathcal{T}_+(E,Z)$. It can then be viewed as an "isometric dilation" result.*

4. Elements of the Hardy algebra as families of functions

Given $F \in \mathcal{T}_+(E,Z)$, we define a family $\{\widehat{F}_\sigma\}_{\sigma \in NRep(M)}$ of (operator valued) functions.

Each function $\widehat{F}_\sigma$ is defined on $\overline{D}_{X,\sigma}$ and takes values in $B(H_\sigma)$:

$$\widehat{F}_\sigma(\mathfrak{z}) = (\sigma \times \mathfrak{z})(F).$$

Here $NRep(M)$ is the set of all normal representations of M.

Note that the family of domains is a matricial family in the following sense.

Definition 3. *A family of sets $\{\mathcal{U}(\sigma)\}_{\sigma \in NRep(M)}$, with $\mathcal{U}(\sigma) \subseteq \mathcal{I}(\varphi \otimes I, \sigma)$, satisfying $\mathcal{U}(\sigma) \oplus \mathcal{U}(\tau) \subseteq \mathcal{U}(\sigma \oplus \tau)$ is called a* matricial family *of sets (or an nc set).*

Definition 4. *Suppose $\{\mathcal{U}(\sigma)\}_{\sigma \in NRep(M)}$ is a matricial family of sets and suppose that for each $\sigma \in NRep(M)$, $f_\sigma : \mathcal{U}(\sigma) \to B(H_\sigma)$ is a function. We say that $f := \{f_\sigma\}_{\sigma \in NRep(M)}$ is a* matricial family of functions *(or an nc function) in case*

$$C f_\sigma(\mathfrak{z}) = f_\tau(\mathfrak{w}) C \tag{1}$$

for every $\mathfrak{z} \in \mathcal{U}(\sigma)$, every $\mathfrak{w} \in \mathcal{U}(\tau)$ and every $C \in \mathcal{I}(\sigma \times \mathfrak{z}, \tau \times \mathfrak{w})$ (equivalently, $C \in \mathcal{I}(\sigma, \tau)$ and $C\mathfrak{z} = \mathfrak{w}(I_E \otimes C)$).

Theorem 5. *For every $F \in \mathcal{T}_+(E,Z)$, the family $\{\widehat{F}_\sigma\}$ is a matricial family of functions on $\{\overline{D}_{X,\sigma}\}_\sigma$.*

Does the converse hold?

In the unweighted case we proved the converse in [2].

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Arveson-Douglas conjecture and Toeplitz operators

Miroslav Engliš

(joint work with Jörg Eschmeier)

Let $\mathbf{B}^d$ be the unit ball in $\mathbf{C}^d$, $d \geq 1$. The Drury-Arveson space H^2_d consists of all holomorphic functions $f(z) = \sum_\nu f_\nu z^\nu$ on $\mathbf{B}^d$ such that

$$\|f\|^2_{DA} := \sum_\nu |f_\nu|^2 \frac{\nu!}{|\nu|!} < \infty,$$

equipped with the corresponding norm and inner product. The operators

$$M_{z_j} : f(z) \mapsto z_j f(z)$$

of multiplication by the coordinate functions are bounded on H^2_d, and commute with each other. This endows H^2_d with the structure of a module over the polynomial ring $\mathbf{C}[z_1, \dots, z_d]$, a polynomial p corresponding to the operator $M_p = p(M_{z_1}, \dots, M_{z_d})$ of multiplication by p on H^2_d. If $\mathcal{M} \subset H^2_d$ is a (closed) subspace invariant under all M_{z_j}, $j = 1, \dots, d$, we can therefore consider the restrictions $M_{z_j}|_{\mathcal{M}}$, which are commuting bounded linear operators on $\mathcal{M}$, as well as the compressions

$$S_j := P_{\mathcal{M}^\perp} M_{z_j}|_{\mathcal{M}^\perp}, \qquad j = 1, \dots, d,$$

of the M_{z_j} to the orthogonal complement $\mathcal{M}^\perp = H^2_d \ominus \mathcal{M}$, which are commuting bounded linear operators on $\mathcal{M}^\perp$.

The following conjecture was originally made by Arveson with d in the place of $\dim Z(p)$, and refined to the current form by Douglas. (In both cases, it was also formulated for the more general case of modules $\mathcal{M}$ in $H^2_d \otimes \mathbf{C}^N$ generated by $\mathbf{C}^N$-valued homogeneous polynomials, with some finite $N \geq 1$.)

Arveson-Douglas Conjecture. *Assume $\mathcal{M}$ is generated, as a module, by finitely many homogeneous polynomials $p_1, \dots, p_m \in \mathbf{C}[z_1, \dots, z_d]$. Then the commutators $[S_j, S_k^*]$, $j, k = 1, \dots, d$, belong to the Schatten class $\mathcal{S}^q$ for all $q > \dim Z(p)$, where $\dim Z(p)$ is the complex dimension of the zero-set $Z(p) \equiv Z(p_1, \dots, p_m)$ of the polynomials $p_1, \dots, p_m$.*

The Arveson conjecture, and in some cases also its refined version due to Douglas, have so far been proved in various special settings: by Arveson himself when $p_1, \dots, p_m$ are monomials; by Guo and Wang for $m = 1$ or $d \leq 3$; by Douglas and Wang when $m = 1$ and $\mathcal{M}$ is a submodule of the Bergman space $L^2_{\text{hol}}(\mathbf{B}^d)$ on $\mathbf{B}^d$ (instead of H^2_d) generated by an arbitrary, not necessarily homogeneous polynomial p; by Fang and Xia for submodules of the same type in certain weighted (Sobolev-)Bergman spaces on $\mathbf{B}^d$, which included $L^2_{\text{hol}}(\mathbf{B}^d)$ as well as the Hardy space $H^2(\partial\mathbf{B}^d)$ on $\mathbf{B}^d$, but not H^2_d (unless $d = 1$); by Kennedy and Shalit when $p_1, \dots, p_m$ are homogeneous polynomials such that the linear spans of $Z(p_1), \dots, Z(p_m)$ in $\mathbf{C}^d$ have mutually trivial intersections; etc. See the original paper by Douglas for more on the motivation and applications to K-homology and index theory.

There is also a reformulation of (a weaker version of) the Arveson-Douglas conjecture in terms of varieties. Namely, denote by $I(p)$ the ideal in $\mathbf{C}[z_1, \dots, z_d]$ generated by $p_1, \dots, p_m$; then $\mathcal{M}$ is the closure of $I(p)$ in H^2_d, and $I(p)$ is a *homogeneous* (or *graded*) ideal. Denoting for any ideal J in $\mathbf{C}[z_1, \dots, z_d]$ by

$$Z(J) := \{z \in \mathbf{C}^d : q(z) = 0 \ \forall q \in J\}$$

the zero set of J, we then have $Z(p) = Z(I(p))$, which is a homogeneous variety in $\mathbf{C}^d$, i.e. $z \in Z(p)$, $t \in \mathbf{C}$ implies $tz \in Z(p)$. Conversely, for any subset $X \subset \mathbf{C}^d$,

$$I(X) := \{q \in \mathbf{C}[z_1, \dots, z_d] : q(z) = 0 \ \forall z \in X\}$$

is an ideal in $\mathbf{C}[z_1, \dots, z_d]$, which is homogeneous if X is. The correspondences $J \mapsto Z(J)$, $X \mapsto I(X)$ are not one-to-one: one always has $I(Z(J)) \supset J$, with equality if and only if J is a radical ideal; also, $Z(J_1) = Z(J_2)$ if and only if $\sqrt{J_1} = \sqrt{J_2}$ (this is Hilbert's Nullstellensatz). Specializing to modules generated by radical ideals, we thus get the following "geometric version" of the Arveson-Douglas conjecture, due to Kennedy and Shalit.

Geometric Arveson-Douglas conjecture. *Let V be a homogeneous variety in $\mathbf{C}^d$ and $\mathcal{M} = \{f \in H^2_d : f(z) = 0$ for all $z \in V \cap \mathbf{B}^d\}$. Then $[S_j, S_k^*] \in \mathcal{S}^q$ for all $q > \dim_{\mathbf{C}} V$.*

As already mentioned in passing, one can consider the above conjectures not only for H^2_d, but also for other spaces of holomorphic functions on $\mathbf{B}^d$ on which the multiplication operators M_{z_j}, $j = 1, \dots, d$, act boundedly. These include the (weighted Bergman) spaces

$$A^2_\alpha(\mathbf{B}^d) \equiv A^2_\alpha := L^2_{\text{hol}}(\mathbf{B}^d, d\mu_\alpha)$$

of holomorphic functions on $\mathbf{B}^d$ square-integrable with respect to the probability measure

$$d\mu_\alpha(z) := \frac{\Gamma(\alpha+d+1)}{\Gamma(\alpha+1)\pi^d}(1-|z|^2)^\alpha \, dz, \qquad \alpha > -1,$$

where dz denotes the Lebesgue volume on $\mathbf{C}^d$ and the restriction on α ensures that these spaces are nontrivial (and contain all polynomials). In terms of the Taylor coefficients $f(z) = \sum_\nu f_\nu z^\nu$, the norm in A^2_α is given by

$$\|f\|^2_\alpha = \sum_\nu |f_\nu|^2 \frac{\nu!\,\Gamma(d+\alpha+1)}{\Gamma(|\nu|+d+\alpha+1)}.$$

The right-hand side makes actually sense and is positive-definite for all $\alpha > -d-1$, and we can thus extend the definition of A^2_α also to α in this range; in particular, this will give, in addition to the weighted Bergman spaces for $\alpha > -1$ (including the ordinary — i.e. unweighted — Bergman space $L^2_{\text{hol}}(\mathbf{B}^d)$ for $\alpha = 0$), also the Hardy space

$$A^2_{-1} = H^2(\partial\mathbf{B}^d, d\sigma)$$

with respect to the normalized surface measure $d\sigma$ on $\partial\mathbf{B}^d$ for $\alpha = -1$, as well as the Drury-Arveson space

$$A^2_{-d} = H^2_d$$

for $\alpha = -d$. Furthermore, passing to the equivalent norm

$$\|f\|^2_{\alpha\circ} := \sum_\nu \frac{|f_\nu|^2}{(|\nu|+1)^{d+\alpha}} \frac{\nu!}{|\nu|!},$$

one can even define the corresponding spaces $A^2_{\alpha\circ}$ for any real α, with $A^2_{\alpha\circ} = A^2_\alpha$ (as sets, with equivalent norms) for $\alpha > -d-1$ (hence, in particular, $A^2_{-d,\circ} = H^2_d$ for $\alpha = -d$, $A^2_{-1,\circ} = H^2(\partial\mathbf{B}^d)$ for $\alpha = -1$, and $A^2_{\alpha\circ} = A^2_\alpha$ for $\alpha > -1$). Actually, $A^2_{\alpha\circ}$ are precisely the subspaces of holomorphic functions

$$A^2_{\alpha\circ} = W^{-\alpha/2}_{\text{hol}}(\mathbf{B}^d) := \{f \in W^{-\alpha/2}(\mathbf{B}^d) : \ f \text{ is holomorphic on } \mathbf{B}^d\}$$

in the Sobolev spaces $W^{-\alpha/2}(\mathbf{B}^d)$ on $\mathbf{B}^d$ of order $-\frac{\alpha}{2}$, for any real α. The coordinate multiplications M_{z_j}, $j = 1, \dots, d$, are continuous on $A^2_{\alpha\circ}$ for any $\alpha \in \mathbf{R}$, and one can consider the Arveson-Douglas conjecture in this setting.

Our main result is the proof of the geometric variant of the Arveson-Douglas conjecture — that is, proof of the Arveson-Douglas conjecture for subspaces $\mathcal{M}$ generated by a radical homogeneous ideal — in all these settings for smooth submanifolds.

Main Theorem. *Let V be a homogeneous variety in $\mathbf{C}^d$ such that $V \setminus \{0\}$ is a complex submanifold of $\mathbf{C}^d \setminus \{0\}$ of dimension n, $\alpha \in \mathbf{R}$, and $\mathcal{M}$ the subspace in $A^2_{\alpha\circ}$, or in A^2_α if $\alpha > -d-1$, of functions vanishing on $V \cap \mathbf{B}^d$. Then $[S_j, S_k^*] \in \mathcal{S}^q$, $j, k = 1, \dots, d$, for all $q > n$.*

Our method of proof relies on two ingredients: the results of Beatrous about restrictions of functions in $A^2_{\alpha\circ}$ to submanifolds, and the theory of Boutet de Monvel and Guillemin of Toeplitz operators on the Hardy space with pseudodifferential symbols (so-called "generalized Toeplitz operators"). It actually turns out that the Boutet de Monvel and Guillemin theory can also be used to replace the results of Beatrous mentioned above, at least those that we need here.

Details are available in the authors' paper [1].

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The noncommutative Choquet boundary and a connection to essential normality

Matthew Kennedy

(joint work with Kenneth R. Davidson, Orr Shalit)

1. The noncommutative Choquet boundary

In 1969, Arveson [1] conjectured the existence of noncommutative analogues of the Shilov boundary and the Choquet boundary. Let $\mathcal{S}$ be an operator system, i.e. a unital self-adjoint subspace of a (potentially noncommutative) C*-algebra, and let $\mathcal{A} = \mathrm{C}^*(\mathcal{S})$. A closed two-sided ideal $\mathcal{I}$ in $\mathcal{A}$ is said to be a *boundary ideal* if the quotient map $\mathcal{A} \to \mathcal{A}/\mathcal{I}$ is completely isometric on $\mathcal{S}$. An irreducible representation $\pi : \mathcal{A} \to \mathcal{B}(H)$ is said to be a *boundary representation* of $\mathcal{S}$ if the restriction $\pi|_{\mathcal{S}}$ has a unique extension to a completely positive map on $\mathcal{A}$. The set of all boundary representations of $\mathcal{S}$ is called the *noncommutative Choquet boundary* of $\mathcal{S}$.

Conjecture 1 (Arveson 1969)**.** *Let $\mathcal{S}$ be an operator system, and let $\mathcal{A} = \mathrm{C}^*(\mathcal{S})$.*

(1) There exists a unique minimal boundary ideal $\mathcal{I}$ of $\mathcal{A}$ called the Shilov boundary ideal *of $\mathcal{S}$.*

(2) *The Shilov boundary ideal $\mathcal{I}$ satisfies $\mathcal{I} = \cap_\pi \ker \pi$, where the intersection is taken over the noncommutative Choquet boundary of $\mathcal{S}$.*

Remark 1. *Conjecture 1 is equivalent to the statement that the Choquet boundary of $\mathcal{S}$ completely norms $\mathcal{S}$.*

Definition 1. *Let $\mathcal{S}$ be an operator system, and let $\mathcal{A} = \mathrm{C}^*(\mathcal{S})$. If the Shilov boundary ideal $\mathcal{I}$ of $\mathcal{S}$ exists, then the C*-algebra $\mathrm{C}^*_\mathrm{e}(\mathcal{S}) = \mathcal{A}/\mathcal{I}$ is called the* C*-envelope *of $\mathcal{S}$.*

Although Arveson was able to prove Conjecture 1 in certain special cases, he could not prove it in general. This was the situation until 1979, when Hamana [10] gave a proof of the existence of the C*-envelope of an operator system using ideas from the theory of injective Banach spaces.

Theorem 1 (Hamana 1979)**.** *The C*-envelope of an operator system always exists.*

The existence proof of Hamana did not readily lead to an identification of the C*-envelope. Moreover, it did not say anything at all about the noncommutative Choquet boundary. However, Hamana's proof was the only one available for over 25 years, until 2005, when Dritschel-McCullough [7] gave a new proof of the existence of the C*-envelope using a notion of maximality for completely positive maps. Building on these ideas, in 2007, nearly 40 years after stating it, Arveson [4] proved Conjecture 1 for separable operator systems.

Theorem 2 (Arveson 2007)**.** *The Choquet boundary of a separable operator system $\mathcal{S}$ completely norms $\mathcal{S}$. In other words, Conjecture 1 holds for separable operator systems.*

Arveson's proof of Theorem 2 uses the theory of direct integral decompositions of C*-algebras, which is a highly technical and very nonconstructive method of decomposing a C*-algebra as a "measurable direct sum." Moreover, for measure-theoretic reasons, this theory does not work in the non-separable setting. Since many interesting operator systems are non-separable, for example many dual operator algebras, this limited the applicability of Arveson's result.

In the beginning of 2013, Ken Davidson and I [6] proved Conjecture 1 in complete generality. Our proof avoids the use of direct integral decompositions, which means it can be applied in the non-separable setting. The techniques used to prove this result result provides a new, more constructive proof of the existence of the C*-envelope.

Theorem 3 (Davidson-Kennedy 2013)**.** *The Choquet boundary of any operator system $\mathcal{S}$ completely norms $\mathcal{S}$. In other words, Conjecture 1 holds for every operator system.*

2. Connections to essential normality

For fixed $d \geq 1$, let $\mathbb{C}[z] = \mathbb{C}[z_1, \ldots, z_d]$ denote the algebra of complex polynomials in d variables. The *Drury-Arveson space* H^2_d is the completion of $\mathbb{C}[z]$ with respect to the inner product $\langle \cdot, \cdot \rangle$, defined on monomials by

$$\langle z^\alpha, z^\beta \rangle = \delta_{\alpha\beta} \frac{\alpha_1! \cdots \alpha_d!}{(\alpha_1 + \cdots + \alpha_d)!}, \quad \alpha, \beta \in \mathbb{N}_0^d.$$

Elements in H^2_d can be viewed as analytic functions on the complex unit ball in $\mathbb{C}^d$. The coordinate multiplication operators $M_{z_1}, \ldots, M_{z_d}$, defined on $\mathbb{C}[z]$ by

$$(M_{z_i} p)(z_1, \ldots, z_d) = z_i p(z_1, \ldots, z_d), \quad p \in \mathbb{C}[z],\ 1 \leq i \leq d,$$

extend to bounded linear operators on H^2_d, and the d-tuple $M_z = (M_{z_1}, \ldots, M_{z_d})$ forms a contractive d-tuple of operators called the *d-shift*.

Let $I \lhd \mathbb{C}[z]$ be an ideal, and let N denote the closure of I in H^2_d. Then N is an invariant subspace for the tuple M_z, and with the respect to the orthogonal decomposition $H^2_d = N \oplus N^\perp$, we can write

$$M_{z_i} = \begin{bmatrix} A_i & 0 \\ * & * \end{bmatrix}, \quad 1 \leq i \leq d.$$

The d-tuple $A = (A_1, \ldots, A_d)$ is commuting, and a result of Arveson [2] implies that, in fact, every commuting d-tuple of operators on a Hilbert space arises in precisely this way (although it may be necessary to consider vector-valued polynomials).

A far-reaching conjecture of Arveson and Douglas (see e.g. [3]) suggests a connection between the operator-algebraic behavior of the d-tuple A, and the geometric structure of the variety $V = \{\lambda \in \mathbb{C}^d \mid p(\lambda) = 0\ \forall p \in I\}$ corresponding to I. Specifically, the Arveson-Douglas conjecture states that if I is homogeneous, i.e. is generated by homogeneous polynomials, then it is *essentially normal*, meaning that the self-commutators $A_i^* A_j - A_j A_i^*$, $1 \leq i, j \leq d$ belong to the ideal $\mathcal{K}$ of compact operators.

A positive solution to the Arveson-Douglas conjecture would imply that the sequence

$$0 \longrightarrow \mathcal{K} \longrightarrow \mathrm{C}^*(A_1, \ldots, A_d) + \mathcal{K} \longrightarrow C(V \cap \partial\mathbb{B}_d) \longrightarrow 0$$

is exact, where $C(V \cap \partial\mathbb{B}_d)$ denotes the C*-algebra of continuous functions on $V \cap \partial\mathbb{B}_d$. In this case, the C*-algebra $\mathrm{C}^*(A_1, \ldots, A_d)$ can be considered as a noncommutative invariant of the variety V, containing information such as the fundamental class of V.

The Arveson-Douglas conjecture is known to hold in certain cases. Arveson himself [3] showed that it holds if I is generated by monomials, and Guo-Wang [9] showed that it holds if I is generated by a single element, or if $d \leq 3$. The conjecture was also shown to hold for some other classes of ideals in [11] and [12]. We also mention that a major advance in the problem has recently been made by Englis and Eschmeier [8].

Orr Shalit [13] and I were able to find a surprising connection between the essential normality of a commuting tuple of operators and the structure of the noncommutative Choquet boundary. In order to state our result we require a notion of rigidity for the generators of a C*-algebra due to Arveson [5].

Definition 2. *Let $\mathcal{S}$ be an operator system, and let $\mathcal{A} = \mathrm{C}^*(\mathcal{S})$. Then $\mathcal{S}$ is said to be* hyperrigid *if for every non-degenerate *-representation $\pi : \mathcal{A} \to \mathcal{B}(H)$, the restriction $\pi|_{\mathcal{S}}$ has the unique extension property.*

Roughly speaking, a C*-algebra "inherits" many of the properties of a hyperrigid generating set, which makes it desirable to find hyperrigid generating sets which are as small as possible.

Theorem 4 (Kennedy-Shalit 2013)**.** *Let $I \lhd \mathbb{C}[z]$ be a (potentially non-homogeneous) ideal. Then I is essentially normal if and only if the corresponding tuple of operators is hyperrigid.*

Theorem 4 shows that the essential normality of an arbitrary commuting tuple of operators on a Hilbert space is closely related to the structure of the Choquet boundary of the tuple. More generally, it is an important problem to be able to compute the Choquet boundary, (and hence the C*-envelope), of such a tuple. As a consequence of our work, we obtain the following result.

Theorem 5 (Kennedy-Shalit 2013)**.** *Let $A = (A_1, \ldots, A_d)$ be a commuting d-tuple of operators on a Hilbert space such that $\sum A_k A_k^* = I$. Then the C*-envelope of A is commutative.*

Examples of tuples of operators have been constructed for which a slightly stronger variant of the Arveson-Douglas conjecture fails. Since Theorem 5 applies to these examples, it would be interesting to study the boundary representations of these examples, with an eye towards identifying the specific obstruction to essential normality.

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Moments, Idempotents, and Interpolation

Florian-Horia Vasilescu

The aim of this talk is to present a new approach to truncated moment problems, based on the use of spaces of characters of certain associated finite dimensional commutative Banach algebras. The existence of representing measures for such functionals is characterized via some intrinsic conditions. More details can be found in [10].

Numerous relatively recent contributions in this area are due to R. Curto and L. Fialkow (see [2]-[4]). Other contributors are M. Putinar, M. Laurent, H. M. Möller, S. Burgdorf and I. Klep, etc.

Solving a truncated moment problems means, roughly speaking, that giving a finite multi-sequence of real numbers $\gamma = (\gamma_\alpha)_{|\alpha|\le 2m}$ with $\gamma_0 > 0$, where α's are multi-indices of a fixed length $n \ge 1$, and $m \ge 0$ is an integer, one looks for a positive measure μ on $\mathbf{R}^n$ (usually called a *representing measure* for γ) such that $\gamma_\alpha = \int t^\alpha d\mu$ for all monomials t^α with $|\alpha| \le 2m$.

If such a measure exists, we may always assume it to be atomic, via Tchakaloff's theorem [9].

Speaking about Tchakaloff's theorem, we obtain, with our methods, the following version of it (for other versions, see also [5], [7], [1] etc).

Theorem. [10] *Let μ be a positive Borel measure on $\mathbf{R}^n$ such that*

$$\int_{\mathbf{R}^n} (t_1^2 + \cdots + t_n^2) d\mu(t) < +\infty.$$

Then there exist a subset $\Xi = \{\xi^{(1)}, \dots, \xi^{(d)}\} \subset \mathbf{R}^n$ and positive numbers $\lambda_1, \dots, \lambda_d$ ($d \le n+1$) such that

$$\int_{\mathbf{R}^n} p(t) d\mu(t) = \sum_{j=1}^{d} \lambda_j p(\xi^{(j)}),\ \ p \in \mathcal{P}_2.$$

Moreover, the weights $\lambda_1, \dots, \lambda_d$, and the nodes $\xi^{(1)}, \dots, \xi^{(d)}$ as well, are given by explicit formulas.

We fix an integer $n \ge 1$ associated with the euclidean space $\mathbf{R}^n$, and for every integer $m \ge 0$ we denote by $\mathcal{P}_m$ (resp. $\mathcal{RP}_m$) the vector space of all polynomials in n real variables, with complex (resp. real) coefficients, of total degree less or equal to m. The vector space of all polynomials in n real variables, with complex (resp. real) coefficients, will be denoted by $\mathcal{P}$ (resp. $\mathcal{RP}$).

Let us fix an integer $m \geq 0$, and let us consider a linear map $\Lambda : \mathcal{P}_{2m} \mapsto \mathbf{C}$ with the properties

(1) $\Lambda(\bar{p}) = \overline{\Lambda(p)},\ p \in \mathcal{P}_{2m}$;
(2) $\Lambda(|p|^2) \geq 0,\ p \in \mathcal{P}_m$;
(3) $\Lambda(1) = 1$.

A map $\Lambda : \mathcal{P}_{2m} \mapsto \mathbf{C}$ with the properties (1)-(3) is called a *unital square positive functional* [6] (briefly, a *uspf*).

Every *uspf* $\Lambda : \mathcal{P}_{2m} \mapsto \mathbf{C}$ satisfies the *Cauchy-Schwarz inequality.* We set $\mathcal{I}_\Lambda = \{p \in \mathcal{P}_m; \Lambda(|p|^2) = 0\}$, which is a vector space, via the Cauchy-Schwarz inequality.

The quotient space $\mathcal{H}_\Lambda = \mathcal{P}_m/\mathcal{I}_\Lambda$ is a Hilbert space, whose scalar product is given by $\langle p + \mathcal{I}_\Lambda, q + \mathcal{I}_\Lambda\rangle = \Lambda(p\bar{q}),\ p, q \in \mathcal{P}_m$.

The symbol $\mathcal{RH}_\Lambda$ designate the space $\{\hat{p} \in \mathcal{H}_\Lambda; p \in \mathcal{RP}_m\}$, which is a real Hilbert space.

Definition. [10] An element $\hat{p} \in \mathcal{RH}_\Lambda$ is called Λ-*idempotent* (or simply *idempotent*) if it is a solution of the equation $\|\hat{p}\|^2 = \langle\hat{p}, \hat{1}\rangle$.

Note that $\hat{p} \in \mathcal{RH}_\Lambda$ is an idempotent if and only if $\Lambda(p^2) = \Lambda(p)$.

Theorem. [10] *For every uspf* $\Lambda : \mathcal{P}_{2m} \mapsto \mathbf{C}$*, the space* $\mathcal{H}_\Lambda$ *has othogonal bases consisting of idempotents.*

Our Main Theorem characterizes the existence of representing measures for a *uspf* $\Lambda : \mathcal{P}_{2m} \mapsto \mathbf{C}$, having $d = \dim \mathcal{H}_\Lambda$ atoms, in terms of orthogonal bases of $\mathcal{H}_\Lambda$ consisting of idempotent elements. In other words, we use only intrinsic conditions.

Definition. [10] Let $\Lambda : \mathcal{P}_{2m} \mapsto \mathbf{C}$ be a uspf and let $\mathcal{B} = \{\hat{b}_1, \ldots, \hat{b}_d\}$ be an orthogonal basis of $\mathcal{H}_\Lambda$ consisting of idempotent elements. We say that the basis $\mathcal{B}$ is Λ-*multiplicative* if $\Lambda(t^\alpha b_j)\Lambda(t^\beta b_j) = \Lambda(b_j)\Lambda(t^{\alpha+\beta} b_j)$ whenever $|\alpha| + |\beta| \leq m$, $j = 1, \ldots, d$.

Main Theorem. [10] *The uspf* $\Lambda : \mathcal{P}_{2m} \mapsto \mathbf{C}$ *has a representing measure in* $\mathbf{R}^n$ *possessing* $d := \dim \mathcal{H}_\Lambda$ *atoms if and only if there exists a* Λ*-multiplicative basis of the space* $\mathcal{H}_\Lambda$.

Our main result implies that all *uspf* $\Lambda : \mathcal{P}_2 \mapsto \mathbf{C}$ have representing measures in $\mathbf{R}^n$ with $d = \dim \mathcal{H}_\Lambda$ atoms. Indeed, if $\mathcal{B} = \{\hat{b}_1, \ldots, \hat{b}_d\}$ is an arbitrary orthogonal basis of $\mathcal{H}_\Lambda$ consisting of idempotent elements, then the condition$\Lambda(t^\alpha b_j)\Lambda(t^\beta b_j) = \Lambda(b_j)\Lambda(t^{\alpha+\beta} b_j)$ is automatically fulfilled when $|\alpha| + |\beta| \leq 1$, $j = 1, \ldots, d$.

In this case, we may write explicitly all representing measures of Λ. This remark provides a proof of our version of Tchakaloff's theorem, stared above.

The main result also leads to a system of quadratic equations, which can be used, at least in principle, to get a solution of the moment problem having a number of atoms equal to dim$\mathcal{H}_\Lambda$. Looking for a Λ-multiplicative basis $\{\hat{b}_1, \ldots, \hat{b}_d\}$, and setting $b_j = \sum_\alpha x_{j\alpha} t^\alpha$, where $x_{j\alpha} = 0$ if $|\alpha| > m$, we should solve the following equations:

$$\sum_{\alpha,\beta} \gamma_{\alpha+\beta} x_{j\alpha} x_{j\beta} = \sum_\alpha \gamma_\alpha x_{j\alpha},\ j = 1, \ldots, d,$$

which is an **idempotent equation**,

$$\sum_{\alpha,\beta} \gamma_{\alpha+\beta} x_{j\alpha} x_{k\beta} = 0,\ j,k = 1,\dots,d,\ j < k,$$

which is an **orthogonality equation**, and

$$\sum_{\xi,\eta} \gamma_{\alpha+\xi}\gamma_{\beta+\eta} x_{j\xi} x_{j\eta} = \sum_{\xi,\eta} \gamma_\xi\ \gamma_{\alpha+\beta+\eta} x_{j\xi} x_{j\eta},$$
$$0 \neq |\alpha| \leq |\beta|,\ |\alpha| + |\beta| \leq m,\ j = 1,\dots d,$$

which is a **Λ-multiplicativity equation** (see [10] for more details).

Finding a solution $\{x_{j\alpha},\ j = 1,\dots,d,\ |\alpha| \leq m\}$ of the equations from above, with $b_1,\dots,b_d$ nonnull, provided it exists, means to solve the corresponding moment problem.

To find a general solution of these equations seems to be a difficult problem, but in some particular cases this is possible.

The main result also leads to some special connection, including a version of the K-moment problem (i.e., looking for solutions supported by a closed subset $K \subset \mathbf{R}^n$, a version of the full moment problem, as well as some connctions with the classical inerpolation problem.

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Joint Spectrum and Commutativity of Operator Tuples

Kehe Zhu

(joint work with Isaak Chagouel, Michael Stessin)

For an n-tuple $\mathbb{A} = (A_1,\cdots,A_n)$ of compact operators on a Hilbert space H we define the joint point spectrum of $\mathbb{A}$ as the set $\sigma_p(\mathbb{A})$ consisting $z = (z_1,\cdots,z_n) \in \mathbb{C}^n$ such that

$$\ker(I + z_1 A_1 + \cdots + z_n A_n) \neq (0).$$

Theorem 1. *Suppose $\mathbb{A}$ is an n-tuple of compact self-adjoint operators on a separable Hilbert space H. Then the operators in $\mathbb{A}$ pairwise commute if and only if $\sigma_p(\mathbb{A})$ consists of countably many, locally finite, hyperplanes in $\mathbb{C}^n$.*

Note that a hyperplane in $\mathbb{C}^n$ is simply the zero set of a linear polynomial: $\lambda_1 z_1 + \cdots + \lambda_n z_n + \lambda_0 = 0$. When $n = 2$, a hyperplane will also be called a complex line.

Theorem 2. *Suppose $\mathbb{A}$ is an n-tuple of $N \times N$ normal matrices. Then the following three conditions are equivalent:*

(a) The operators in $\mathbb{A}$ pairwise commute.

(b) The polynomial

$$p_{\mathbb{A}}(z) = \det(I + z_1 A_1 + \cdots + z_n A_n)$$

can be factored into a product of linear polynomials.

(c) The joint spectrum $\sigma_p(\mathbb{A})$ consists of finitely many hyperplanes.

The following are easy consequences of Theorem 1.

Corollary 3. *A compact operator A is normal if and only if $\sigma_p(A, A^*)$ consists of countably many, locally finite, complex lines in $\mathbb{C}^2$.*

Corollary 4. *Two compact operators A and B on H are normal and commute if and only if $\sigma_p(A, A^*, B, B^*)$ consists of countably many, locally finite, hyperplanes in $\mathbb{C}^4$.*

Corollary 5. *Two compact operators A and B commute completely (that is, A commutes with both B and B^*) if and only if each of the four sets $\sigma_p(A \pm A^*, B \pm B^*)$ consists of countably many, locally finite, complex lines in $\mathbb{C}^2$.*

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Generalized cycles and local intersection numbers; extended abstract

Mats Andersson
(joint work with D. Eriksson, H. Samuelsson, E. Wulcan and A. Yger)

Let $\mathcal{Z}(\mathbb{P}^N)$ denote the group of analytic cycles on $\mathbb{P}^N$, i.e., formal finite sums

$$Z = \sum_j \alpha_j Z_j,$$

where Z_j are irreducible subvarieties of $\mathbb{P}^N$. If Z is irreducible itself, then at each point $x \in Z$ there is a well-defined positive integer $\text{mult}_x Z$, the multiplicity of Z at x. Roughly speaking one takes a generic plane through x of complimentary dimension, moves it slightly and counts the number of intersection points close to x. There is also a positive integer $\deg Z$ which is the total number of intersection points with a generic such plane. These two numbers extend to arbitrary cycles by linearity.

Let Z be the cusp $\{x_1^3 - x_2^2 x_0 = 0\}$ in $\mathbb{P}^2$. Then $p = [1, 0, 0]$ is the only non-smooth point. We have that

$$\text{mult}_x Z = 1, x \in Z \setminus p, \ \text{mult}_p Z = 2, \quad \deg Z = 3.$$

Recall that a current is a continuous linear functional on the space of smooth forms $\mathcal{E}(\mathbb{P}^N)$. For instance, given a cycle Z we have the associated Lelong current $[Z]$ defined by

$$[Z].\xi = \int_{\mathbb{P}^N} [Z] \wedge \xi, \quad \xi \in \mathcal{E}(\mathbb{P}^N).$$

Clearly Z is determined by its Lelong current, and this representation makes it possible to give analytic definitions of multiplicity and degree: We have that

$$\text{mult}_x Z = \ell_x[Z],$$

where the right hand side is the Lelong number of the current $[Z]$ at x, this is a measure of the mass concentration at x. Furthermore,

$$\deg Z = \int_{\mathbb{P}^N} [Z] \wedge \omega^{\dim Z}$$

(provided that Z has pure dimension), where $\omega = dd^c \log |x|^2$ is the Fubini-Study metric form.

If $Z, W \in \mathcal{Z}(\mathbb{P}^N)$ have pure dimensions and $\dim(Z \cap W) = \dim Z + \dim W - N$, then there is a well-defined cycle

$$Z \cdot W = \sum \alpha_\ell V_\ell,$$

called the proper intersection, where V_ℓ are the irreducible components of the set-theoretical intersection V of Z and W, and α_ℓ are integers. The classical definition is geometric and/or algebraic, but by means of the Lelong current representation we have

$$[Z \cdot W] = [Z] \wedge [W],$$

where the product on right hand side is defined by choosing suitable regularizations of the currents and go the the limit. For instance, the proper intersection of the cusp and a generic line through p is equal to $2\{p\}$, whereas the intersection with the line $x_2 = 0$ is $3\{p\}$.

In the classical non-proper case, see [4], the intersection product $Z \cdot W$ is a certain Chow class on V of dimension $\dim Z + \dim W - N$; this means that it is represented by a cycle on V that is determined only up to rational equivalence. In particular, it has a well-defined degree and the Bezout equality

$$\deg(Z \cdot W) = \deg Z \cdot \deg W$$

holds, provided that $\dim Z + \dim W \geq n$; otherwise $Z \cdot W$ is zero.

For instance, the self-intersection of the cusp Z above is represented by the set of 9 points obtained by taking one of the Z and move it slightly so that one gets a proper intersection. (More precisely any divisor of a generic section of the line bundle $\mathcal{O}(3)$ restricted to Z is a representative.)

In the 90's Tworzewski, [6], Gaffney-Gassler, [5], and Achilles-Manaresi, [1], independently introduced integers

$$\epsilon_k(Z, W, x), \; k = 0, 1, \ldots, \dim V,$$

called the *local intersection numbers* at x, where k describes the complexity of the local intersection at x on dimension k. The definition in [6] and [5] is geometric and relies on a local variant of the so-called Stückrad-Vogel procedure, [7], whereas the definition in [1] is algebraic. In [3] we found an analytic definition as the Lelong numbers of certain currents.

If the intersection is proper, then $\epsilon_k(Z, W, x) = \operatorname{mult}_x(Z \cdot W)$ for $k = \dim V$ and 0 otherwise. If $Z = W$ is the cusp, then

$$\epsilon(Z, Z, x) = (0, 1), \; x \in Z \setminus \{p\}, \quad \epsilon(Z, Z, p) = (3, 2),$$

that is, at the point p we have the local intersection number 3 on dimension 0 and 2 on dimension 1.

It is clear that no representative of the self-intersection $Z \cdot Z$ of the cusp can represent the local intersection numbers. Tworzewski, [6], proved however that there is a unique analytic cycle $Z \circ W$ such that (lower index denotes component of dimension k)

$$\sum_k \operatorname{mult}_x (Z \circ W)_k = \sum_\ell \epsilon_\ell(Z, W, x).$$

For instance, if Z is the cusp, then

$$Z \circ Z = Z + 3\{p\}.$$

Notice however that $\deg(Z \circ Z) = 6 \neq 9 = 3 \cdot 3 = (\deg Z)^2$ so the Bezout equality is not fufilled (and clearly there is no cycle at all with the right multiplicities that also satisfies the Bezout equality in this case).

We introduce, for any subvariety X of $\mathbb{P}^N$, a group $\mathcal{B}(X)$ of currents that we call generalized cycles on X. If we identify classical cycles with the associated Lelong currents we get an inclusion $\mathcal{Z}(X) \subset \mathcal{B}(X)$. We also have a natural inclusion $\mathcal{B}(X) \subset \mathcal{B}(X')$ if $X \subset X'$, and $\mathcal{B}(X)$ is precisely the subgroup of the μ in $\mathcal{B}(\mathbb{P}^N)$ whose support $|\mu|$ is contained in X. Each generalized cycle μ has a natural decomposition $\mu = \mu_0 + \mu_1 + \cdots$, where μ_k has dimension k. It turns out that $\operatorname{mult}_x \mu := \ell_x \mu$ and

$$\deg \mu := \int \mu \wedge \omega^{\deg \mu}$$

are integers. Intuitively generalized cycles are obtained as certain mean values of classical cycles. For instance,

$$\omega_p := dd^c \log(|x_1|^2 + |x_2|^2)$$

is a generalized cycle that is singular only at the point p. It has degree 1 and the multiplicity at p is 1; at each other point the multiplicity is zero. In fact, ω_p is a mean value of all lines through p.

Our main result is the following:

Theorem 1. *There is a bilinear pairing $\mathcal{B}(X) \times \mathcal{B}(X') \to \mathcal{B}(X \cap X')$, $(Z, W) \mapsto Z \bullet W$ with the following properties:*

(i) $\mathit{mult}_x (Z \bullet W)_k = \epsilon(Z, W, x)$ for all x and k

(ii) $\deg(Z \bullet W) = \deg Z \cdot \deg W$ provided that $\dim(|Z| \cap |W|) \geq \dim Z + \dim W - N$,

(iii) $Z \bullet W$ coincides with $Z \cdot W$ on "cohomology level".

It follows that $Z \bullet W = Z \cdot W$ if the intersection is proper. If ℓ is a line, then $\ell \bullet \ell = \ell$. If Z is the cusp above in $\mathbb{P}^2$, then

$$Z \bullet Z = Z + 3\{p\} + \mu,$$

where μ is a generalized cycle on Z of dimension 0 and total mass 3, intuitively meaning 3 points that move around on Z. Notice that the total degree of $Z \bullet Z$ is 9 as expected.

The formal definition of $\mathcal{B}(X)$ is the following: Let $f\colon Y \to X$ be any proper holomorphic mapping and let α be a product of Chern forms of Hermitian line bundles over Y. Then $\mu = f_*\alpha$ defines a generalized cycle with support on X and $\mathcal{B}(X)$ is defined so that the element is independent of the choice of Chern forms (Hermitian metrics). For instance, if $i\colon Z \to X$ is an inclusion, then $[Z] = i_*1$. Let $\pi\colon \mathbb{P}^2$ be the blow-up of $\mathbb{P}^2$ at p and let α be minus the Chern form of the exceptional divisor. Then $\omega_p = \pi_*\alpha$.

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Carleson Measures in Hilbert Spaces of Analytic Functions

Brett D. Wick

(joint work with Michael T. Lacey, Eric T. Sawyer, Chun-Yen Shen, Igancio Uriarte-Tuero, Alexander Volberg)

Let $\Omega \subset \mathbb{C}^n$ be a domain and let $\mathcal{H}$ be a reproducing kernel Hilbert space of analytic functions over Ω. Associated to each point $\lambda \in \Omega$ we have a function $K_\lambda \in \mathcal{H}$, the reproducing kernel at point $\lambda \in \Omega$, such that

$$f(\lambda) = \langle f, K_\lambda \rangle_{\mathcal{H}} .$$

Given a non-negative Borel measure μ on Ω one would like a 'geometric' or 'testing' condition on the measure μ so that the following inequality holds:

$$\int_\Omega |f(z)|^2 \, d\mu(z) \leq C(\mu)^2 \, \|f\|^2_{\mathcal{H}} \quad \forall f \in \mathcal{H} \tag{1}$$

with $C(\mu)$ denoting the norm of the measure μ under this embedding. This is an embedding question about when $\mathcal{H} \subset L^2(\Omega;\mu)$ and measures for which (1) holds are called *Carleson measures* after important work by L. Carleson, [5] in the context when $\Omega = \mathbb{D}$ and $\mathcal{H} = H^2(\mathbb{D})$. One can encounter many problems in analysis when having a characterization of (1) is extremely useful.

It is clear that a necessary condition for (1) to hold is the following:

$$\sup_{\lambda\in\Omega} \int_\Omega |k_\lambda(z)|^2 \, d\mu(z) < \infty$$

where k_λ is the normalized reproducing kernel for $\mathcal{H}$. In many situations this obvious necessary condition is also sufficient; but in several situations this is no longer the case and one must instead rely upon a more complicated characterization of the measures for which (1) holds. We now highlight two recent successes of harmonic analysis in studying Carleson measures of analytic functions where the simple necessary condition is no longer sufficient, yet nevertheless we are able to obtain a characterization of the Carleson measures.

1. Carleson Measures for the Space K_ϑ

Let $H^2(\mathbb{D})$ denote the Hardy space of analytic functions on the unit disk $\mathbb{D}$. Let ϑ be an inner function on $\mathbb{D}$, namely an analytic function such that $|\vartheta(\xi)| = 1$ for almost every $\xi \in \mathbb{T}$. The space $K_\vartheta \equiv H^2(\mathbb{D}) \ominus \vartheta H^2(\mathbb{D})$ is called the *model space associated to* ϑ. It is easy to see that this is a reproducing kernel Hilbert space with kernel

$$K_\lambda(z) \equiv \frac{1 - \overline{\vartheta(\lambda)}\vartheta(z)}{1 - \overline{\lambda} z}.$$

Function theoretic properties of the space K_ϑ are of significant interest, and we concentrate here on Carleson measures for the space. Recall that a measure μ is a K_ϑ-Carleson measure if we have the following estimate holding:

$$\int_{\overline{\mathbb{D}}} |f(z)|^2 \, d\mu(z) \leq C(\mu)^2 \, \|f\|^2_{K_\vartheta} \quad \forall f \in K_\vartheta.$$

This problem has been intensely studied by numerous authors, with the question of characterization posed by Cohn [6] in 1982. An attractive special case when ϑ satisfies the 'one-component', or 'connected level set' condition, namely that the enlargement of the spectrum, given by

$$\Omega(\epsilon) \equiv \{z \in \mathbb{D} \,:\, |\vartheta(z)| < \epsilon\}, \qquad \epsilon > 0$$

is connected for some $\epsilon > 0$. In this case, Cohn *op. cite* and Treil and Volberg [12], showed that μ is K_ϑ-Carleson if and only if the Carleson condition $\mu(B_I) \lesssim |I|$ holds for all intervals I such that the Carleson box B_I intersects $\Omega(\epsilon)$. See also the alternate proof obtained by Aleksandrov in [2]. For more general ϑ, see however the counterexample of Nazarov-Volberg [9], based on the famous counterexample of Nazarov [8] to the Sarason conjecture. Apparently, there are very few results known for general ϑ, with one of these being the remarkable results of Aleksandrov [1] characterizing those μ for which K_ϑ isometrically embeds into $L^2(\overline{\mathbb{D}}; \mu)$, under the natural embedding map.

It is possible to recast the problem about Carleson measures for K_ϑ in terms of the boundedness of the Cauchy transform between two weighted Hilbert spaces, which then can be studied via recent techniques in harmonic analysis. Our characterization of the Carleson measures for K_ϑ is given by the following theorem.

Theorem 1 (Lacey, Sawyer, Shen, Uriarte-Tuero, Wick, [7]). *Let μ be a non-negative Borel measure supported on $\overline{\mathbb{D}}$ and let ϑ be an inner function on $\mathbb{D}$ with Clark measure σ. Set $\nu_{\mu,\vartheta} = |1 - \vartheta|^2 \mu$. The following are equivalent:*

(i) *μ is a Carleson measure for K_ϑ, namely,*

$$\int_{\mathbb{D}} |f(z)|^2 \, d\mu(z) \le C(\mu)^2 \, \|f\|^2_{K_\vartheta} \quad \forall f \in K_\vartheta;$$

(ii) *The Cauchy transform C is a bounded map between $L^2(\mathbb{T};\sigma)$ and $L^2(\overline{\mathbb{D}};\nu_{\mu,\vartheta})$, i.e., $\mathsf{C} : L^2(\mathbb{T};\sigma) \to L^2(\overline{\mathbb{D}};\nu_{\vartheta,\mu})$ is bounded. Here σ is the Clark measure associated to ϑ;*

(iii) *The three conditions below hold for the pair of measures σ and $\nu_{\mu,\vartheta}$:*

$$\sigma(\mathbb{T}) \cdot \nu_{\mu,\vartheta}(\overline{\mathbb{D}}) + \sup_{z\in\mathbb{D}} \big\{ \mathsf{P}(\sigma \mathbf{1}_{\mathbb{T}\setminus I})(z) \mathsf{P}\nu_{\mu,\vartheta}(z) + \mathsf{P}\sigma(z)\mathsf{P}(\nu_{\mu,\vartheta}\mathbf{1}_{\overline{\mathbb{D}}\setminus B_I})(z) \big\} \equiv \mathscr{A}_2,$$

$$\sup_I \sigma(I)^{-1} \int_{B_I} |\mathsf{C}_\sigma \mathbf{1}_I(z)|^2 \nu_{\mu,\vartheta}(dA(z)) \equiv \mathscr{T}^2,$$

$$\sup_I \nu_{\mu,\vartheta}(B_I)^{-1} \int_I |\mathsf{C}^*_{\nu_{\mu,\vartheta}} \mathbf{1}_{B_I}(w)|^2 \sigma(dw) \equiv \mathscr{T}^2,$$

where these conventions hold. The last two inequalities are uniform over all intervals $I \subset \mathbb{T}$, with $|I| \le \frac{1}{2}$, $B_I \equiv \{z \in \overline{\mathbb{D}} : z = r\,\mathrm{e}^{i\theta}, |1-r| \le |I|, \mathrm{e}^{i\theta} \in I\}$ is the Carleson box over I.

Moreover,

$$C(\mu) \simeq \|\mathsf{C}\|_{L^2(\mathbb{T};\sigma)\to L^2(\overline{\mathbb{D}};\nu_{\vartheta,\mu})} \simeq \mathscr{A}_2^{1/2} + \mathscr{T}.$$

2. Carleson Measures for Besov-Sobolev Spaces

The space $B^2_\sigma(\mathbb{B}_{2d})$ is the collection of analytic functions on the unit ball $\mathbb{B}_{2d}$ in $\mathbb{C}^d$ and such that for any integer $m \ge 0$ and any $0 \le \sigma < \infty$ such that $m + \sigma > \frac{d}{2}$ we have the following norm being finite:

$$\|f\|^2_{B^\sigma_2} \equiv \sum_{j=0}^{m-1} |f^{(j)}(0)|^2 + \int_{\mathbb{B}_{2d}} |(1-|z|^2)^{m+\sigma} f^{(m)}(z)|^2 \frac{dV(z)}{(1-|z|^2)^{d+1}}.$$

One can show that these spaces are independent of m, the choice of derivative (within reason) is unimportant, and are reproducing kernel Hilbert spaces with obvious inner products. The reproducing kernels are given by $K^\sigma_\lambda(z) \equiv \frac{1}{(1-\overline{\lambda}\cdot z)^{2\sigma}}$.

An important question in the study of these spaces is a characterization of the measures μ for which

$$\int_{\mathbb{B}_{2d}} |f(z)|^2 d\mu(z) \le C(\mu)\|f\|^2_{B^\sigma_2(\mathbb{B}_{2d})}.$$

For the range $\frac{d}{2} \le \sigma$, the characterization of the Carleson measures is given by the simple necessary condition: testing on the reproducing kernels. For the range $0 \le \sigma \le \frac{1}{2}$ these Carleson measures in the complex ball were initially characterized by Arcozzi, Rochberg and Sawyer. In [3] Arcozzi, Rochberg and Sawyer developed the theory of "trees" on the unit ball and then demonstrated that the inequality they wished to prove was related to a certain two-weight inequality on these trees. Once they have the characterization of the trees, they can then deduce the corresponding characterization for the space of analytic functions. Again, in the range $0 < \sigma \le \frac{1}{2}$

a proof more in the spirit of what appears in this paper was obtained by E. Tchoundja, [10, 11].

An important open question in the theory of Besov–Sobolev spaces was a characterization of the Carleson measures in the difficult range $\frac{1}{2} < \sigma < \frac{d}{2}$, see for example [4]. The characterization of Carleson measure for $B_2^\sigma(\mathbb{B}_{2d})$ is contained in the following result.

Theorem 2 (Volberg, Wick [13]). *Suppose that $0 < \sigma$. Let μ be a positive Borel measure in $\mathbb{B}_{2d}$. Then the following conditions are equivalent:*

(a) *μ is a $B_2^\sigma(\mathbb{B}_{2d})$-Carleson measure;*
(b) *$T_{\mu,2\sigma} : L^2(\mathbb{B}_{2d};\mu) \to L^2(\mathbb{B}_{2d};\mu)$ is bounded;*
(c) *There is a constant C such that*
  (i) *$\|T_{\mu,2\sigma}\chi_Q\|^2_{L^2(\mathbb{B}_{2d};\mu)} \le C\,\mu(Q)$ for all Δ-cubes Q;*
  (ii) *$\mu(B_\Delta(x,r)) \le C\,r^{2\sigma}$ for all balls $B_\Delta(x,r)$ that intersect $\mathbb{C}^d \setminus \mathbb{B}_{2d}$.*

Above, the sets B_Δ are balls measured with respect to a naturally occurring metric in the problem. The operator $T_{\mu,2\sigma}$ is a Bergman-type Calderón–Zygmund operator with respect to this metric Δ for which we can apply the methods of harmonic analysis to study. And, the set Q is a "cube" defined with respect to the metric Δ.

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On the problem of characterizing multipliers for the Drury-Arveson space

Jingbo Xia

(joint work with Quanlei Fang)

Let $\mathbf{B}$ be the open unit ball in $\mathbf{C}^n$. We assume that the complex dimension n is greater than or equal to 2. Recall that the Drury-Arveson space H_n^2 is the Hilbert space of analytic functions on $\mathbf{B}$ that has the function

$$\frac{1}{1 - \langle \zeta, z \rangle}$$

as its reproducing kernel. A newcomer in the family of reproducing-kernel Hilbert spaces, the Drury-Arveson space has been the subject of intense study in recent years. Perhaps this intense interest in H_n^2 is mainly due to its close connection with a number of important topics, such as the von Neumann inequality for commuting row contractions, the corona theorem, and the Arveson conjecture. But this interest in H_n^2 is also attributable to the fascinating (some might say mysterious) properties of the space itself.

One source of fascination with the Drury-Arveson space is its collection of *multipliers*. Recall that a function $f \in H_n^2$ is said to be a multiplier of the Drury-Arveson space if $fh \in H_n^2$ for every $h \in H_n^2$ [2]. Let $\mathcal{M}$ denote the collection of the multipliers of H_n^2. Also recall from [2] that if $f \in \mathcal{M}$, then the multiplication operator M_f is bounded on H_n^2. The operator norm $\|M_f\|$ on H_n^2 is also called the *multiplier norm* of f. It is well known that the H^∞-norm $\|f\|_\infty$ does not dominate the multiplier norm of f [2]. What is more, for $f \in \mathcal{M}$, $\|f\|_\infty$ fails to dominate even the essential norm of M_f on H_n^2 [3].

An enduring challenge in the theory of the Drury-Arveson space, since its very inception, has been the quest for a good characterization of the membership in $\mathcal{M}$. Let $k \in \mathbf{N}$ be such that $2k \geq n$. Then given any $f \in H_n^2$, one can define the measure $d\mu_f$ on $\mathbf{B}$ by the formula

$$(***) \qquad d\mu_f(z) = |(R^k f)(z)|^2 (1 - |z|^2)^{2k-n} dv(z),$$

where dv is the normalized volume measure on $\mathbf{B}$ and R denotes the radial derivative $z_1\partial_1 + \cdots + z_n\partial_n$. In [4], Ortega and Fàbrega showed that $f \in \mathcal{M}$ if and only if $d\mu_f$ is an H^2_n-Carleson measure. That is, $f \in \mathcal{M}$ if and only if there is a C such that

$$\int |h(z)|^2 d\mu_f(z) \le C\|h\|^2$$

for every $h \in H^2_n$. In [1], Arcozzi, Rochberg and Sawyer gave a characterization for all the H^2_n-Carleson measures on $\mathbf{B}$.

So the combination of the result of Arcozzi, Rochberg and Sawyer and the result of Ortega and Fàbrega is *a* characterization of the membership $f \in \mathcal{M}$. But this characterization is quite complicated, because the condition for a general $d\mu$ to be an H^2_n-Carleson measure is quite complicated.

But if we are only interested in multipliers, then we are only interested in the $d\mu_f$ given by $(***)$, not the general $d\mu$ on $\mathbf{B}$. So the question is, for the subclass of measures $d\mu_f$ given by $(***)$, is there a simpler, or more direct, condition that determines when it is an H^2_n-Carleson measure? Equivalently, is there a simpler, or more direct, characterization of the membership $f \in \mathcal{M}$?

Since the Drury-Arveson space is a reproducing-kernel Hilbert space, it is natural to turn to the reproducing kernel for possible answers. Recall that the normalized reproducing kernel for H^2_n is given by the formula

$$k_z(\zeta) = \frac{(1-|z|^2)^{1/2}}{1-\langle \zeta, z\rangle},$$

$z, \zeta \in \mathbf{B}$. One of the frequent tools in the study of reproducing-kernel Hilbert spaces is the *Berezin transform*. But for any $f \in H^2_n$, the Berezin transform

$$\langle f k_z, k_z \rangle$$

is none other than $f(z)$ itself. Given what we know about H^2_n, the boundedness of Berezin transform on $\mathbf{B}$ is not expected to guarantee the membership $f \in \mathcal{M}$. Here we use the phrase "not expected", because this is not an issue that has been settled in the literature. Note that Arveson's example in [2] only shows that for an analytic function f on $\mathbf{B}$, the finiteness of $\|f\|_\infty$ does not guarantee $f \in H^2_n$. But if one starts with an $f \in H^2_n$, and then one assumes $\|f\|_\infty < \infty$, does it follow that $f \in \mathcal{M}$? In the literature one cannot find answer to this very simple question, although the answer is not expected to be affirmative.

Even if one accepts that for $f \in H^2_n$, the boundedness of the Berezin transform $\langle f k_z, k_z\rangle$ is not enough to guarantee the membership $f \in \mathcal{M}$, what about something stronger than the Berezin transform? For example, anyone who gives any thought about multipliers is likely to come up with the following natural and basic

Question 1.1. For $f \in H^2_n$, does the condition

$$\sup_{|z|<1} \|f k_z\| < \infty$$

imply the membership $f \in \mathcal{M}$?

Prima facie, one would think that there is at least a fair chance that the answer to Question 1.1 might be affirmative. And that was what we thought for quite a while. What makes this question particularly tempting is that an affirmative answer would give a very simple characterization of the membership $f \in \mathcal{M}$. But that would be too simple a characterization, as it turns out. After a long struggle, we have finally arrived at the conclusion that, tempting though the question may be, its answer is actually negative. The following is our main result:

Theorem 1.2. *There exists an $f \in H_n^2$ satisfying the conditions $f \notin \mathcal{M}$ and*

$$\sup_{|z|<1} \|fk_z\| < \infty.$$

The proof of this theorem involves a construction that is quite technical. Indeed it involves numerous estimates and requires everything that we know about the Drury-Arveson space. As it turns out, the same construction also shows that the function-theoretic operator theory on the Drury-Arveson space is quite different from that on the more familiar reproducing-kernel Hilbert spaces, such as the Hardy space and the Bergman space.

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On interpolating sequences for Hardy Sobolev spaces of the ball in $\mathbb{C}^n$

Eric Amar

We shall work with the Hardy-Sobolev spaces H_s^p. For $1 \leq p < \infty$ and $s \in \mathbb{R}$, H_s^p is the space of holomorphic functions in the unit ball $\mathbb{B}$ in $\mathbb{C}^n$ such that the following expression is finite

$$\|f\|_{s,p}^p := \sup_{r<1} \int_{\partial\mathbb{B}} |(I+R)^s f(rz)|^p \, d\sigma(z),$$

where I is the identity, $d\sigma$ is the Lebesgue measure on $\partial\mathbb{B}$ and R is the radial derivative

$$Rf(z) = \sum_{j=1}^n z_j \frac{\partial f}{\partial z_j}(z).$$

For $s \in \mathbb{N}$, this norm is equivalent to

$$\|f\|_{s,p}^p = \max_{0\leq j\leq s} \int_{\partial\mathbb{B}} \left|R^j f(z)\right|^p d\sigma(z).$$

This means that $R^j f \in H^p(\mathbb{B}),\ j = 0, \dots,\ s.$

For $s = 0$ these spaces are the classical Hardy spaces $H^p(\mathbb{B})$ of the unit ball $\mathbb{B}$.

Let p' the conjugate exponent for p ; the Hilbert space H_s^2 is equipped with reproducing kernels :

$$\forall a \in \mathbb{B},\ k_a(z) = \frac{1}{(1-\bar{a}\cdot z)^{n-2s}},\ \|k_a\|_{s,p} := \|k_a\|_{H_s^p} \simeq (1-|a|^2)^{s-n/p'}$$

i.e. $\forall a \in \mathbb{B},\ \forall f \in H_s^p,\ f(a) = \langle f, k_a \rangle$, where $\langle \cdot, \cdot \rangle$ is the scalar product of the Hilbert space H_s^2. In the case $s = n/2$ there is a log for k_a.

Definition 1. *The* **measure** μ *in* $\mathbb{B}$ *is* **Carleson** *for* $H_s^p,\ \mu \in C_{s,p}$*, if we have the embedding*

$$\forall f \in H_s^p,\ \int_{\mathbb{B}} |f|^p \, d\mu \leq C\|f\|_{s,p}^p.$$

Definition 2. *The* **sequence** S **is Carleson, CS**,*in* $H_s^p(\mathbb{B})$*, if the associated measure*

$$\nu_S := \sum_{a\in S} \|k_{s,a}\|_{s,p'}^{-p}\delta_a$$

is Carleson for $H_s^p(\mathbb{B})$.

Definition 3. *The* **multipliers algebra** $\mathcal{M}_s^p$ *of* H_s^p *is the algebra of functions* m *on* $\mathbb{B}$ *such that*

$$\forall h \in H_s^p,\ mh \in H_s^p.$$

The norm of a multiplier is its norm as an operator from H_s^p *into* H_s^p.

We already know that $\forall p,\ \mathcal{M}_0^p(\mathbb{B}) = H^\infty(\mathbb{B})$ and $\mathcal{M}_s^p = H^\infty \cap Carleson\ Condition$ so they are characterized by Volberg and Wick [12] for $p = 2$, and for any p in the range $n - 1 \leq ps \leq n$, see [9].

Definition 4. *The* **sequence** S *of points in* $\mathbb{B}$ *is* **interpolating** *in the multipliers algebra* $\mathcal{M}_s^p$ *of* $H_s^p(\mathbb{B})$ *if there is a* $C > 0$ *such that*

$$\forall \lambda \in \ell^\infty(S),\ \exists m \in \mathcal{M}_s^p :: \forall a \in S,\ m(a) = \lambda_a \text{ and } \|m\|_{\mathcal{M}_s^p} \leq C\|\lambda\|_\infty.$$

Definition 5. *The* **sequence** S *of points in* $\mathbb{B}$ *is* **interpolating** *in* $H_s^p(\mathbb{B})$ *if there is a* $C > 0$ *such that*

$$\forall \lambda \in \ell^p(S),\ \exists f \in H_s^p(\mathbb{B}) :: \forall a \in S,\ f(a) = \lambda_a\|k_a\|_{s,p'},\ \|f\|_{H_s^p} \leq C\|\lambda\|_p.$$

Definition 6. *Let* S *be an interpolating sequence in* H_s^p *we say that* S *has a* **bounded linear extension operator, BLEO**, *if there is a a bounded linear operator* $E : \ell^p(S) \to H_s^p$ *and a* $C > 0$ *such that*

$$\forall \lambda \in \ell^p(S),\ E(\lambda) \in H_s^p,\ \|E(\lambda)\|_{H_s^p} \leq C\|\lambda\|_p \ :\ \forall a \in S,\ E(\lambda)(a) = \lambda_a\|k_a\|_{s,p'}.$$

We have the table concerning the interpolating sequences :

$H^\infty(\mathbb{D})$	$H^\infty(\mathbb{B})$	$\mathcal{M}_s^p(\mathbb{B})$
Characterized by L. Carleson [8]	No characterization	characterised for $p = 2,\ n - 1 < 2s \leq n$ by A.R.S. [5]
IS $\Rightarrow$ BLEO by P. Beurling [7]	IS $\Rightarrow$ BLEO by A. Bernard [6]	IS $\Rightarrow$ BLEO by E. A. Th 1

Theorem 1. *If* S *is interpolating for* $\mathcal{M}_s^p$ *and* $p \geq 2$*, then* S *has a bounded linear extension operator.*

Definition 7. *The* **sequence** S *of points in* $\mathbb{B}$ *is* **dual bounded** *(or minimal, or weakly interpolating) in the multipliers algebra* $\mathcal{M}_s^p$ *of* $H_s^p(\mathbb{B})$ *if there is a bounded sequence* $\{\rho_a\}_{a\in S} \subset \mathcal{M}_s^p$ *such that*

$$\forall a, b \in S,\ \rho_a(b) = \delta_{ab} \text{ and } \exists C > O\ :\ \forall a \in S,\ \|\rho_a\| \leq C.$$

Definition 8. *The* **sequence** S *of points in* $\mathbb{B}$ *is* δ **separated** *in* $\mathcal{M}_s^p$ *if*

$$\forall a, b \in S,\ a \neq b,\ \exists m \in \mathcal{M}_s^p :: m(a) = 0,\ m(b) = 1,\ \|m\|_{\mathcal{M}_s^p} \leq \delta^{-1}.$$

Now we have

$H^\infty(\mathbb{D})$	$H^\infty(\mathbb{B})$
DB $H^\infty \Rightarrow$ IS H^p, $\forall p \leq \infty$ with BLEO	DB $H^\infty \Rightarrow$ IS H^p, $\forall p < \infty$ with BLEO, E.A. [2]
IS $H^\infty \Rightarrow$ CS L. Carleson [8]	IS $H^\infty \Rightarrow$ CS, N. Varopoulos [11] DB $H^\infty \Rightarrow$ CS, E. A. [4]
Union IS separated $\Rightarrow$ IS	Union IS separated $\Rightarrow$ IS N. Varopoulos [10]

$\mathcal{M}_s^p(\mathbb{B})$
IS $\mathcal{M}_s^p \Rightarrow$ IS H_s^p, $\forall p \geq 2$ with BLEO, E.A. Th 2
IS $\mathcal{M}_s^p \Rightarrow$ CS H_s^p, E. A. Th 3
Union IS separated $\Rightarrow$ IS for $s = 1$, $\forall p$ and $p = 2$, $\forall s$ E. A. Th 4 and Cor 1

Theorem 2. *Let S be an interpolating sequence for the multipliers algebra $\mathcal{M}_s^p$ of $H_s^p(\mathbb{B})$ then S is also an interpolating sequence for H_s^p provided that $p \geq 2$.*

Theorem 3. *Let S be an interpolating sequence for $\mathcal{M}_s^p$ then S is Carleson $H_s^p(\mathbb{B})$.*

Theorem 4. *Let S_1 and S_2 be two interpolating sequences in $\mathcal{M}_s^p$ such that $S := S_1 \cup S_2$ is separated, then S is still an interpolating sequence in $\mathcal{M}_s^p$, provided that $s = 1$ or $s = n/2$.*

Theorem 5. [1]*Let σ_1 and σ_2 be two interpolating sequences in the spectrum of the commutative algebra of operators A, such that $\sigma := \sigma_1 \cup \sigma_2$ is separated, then σ is an interpolating sequence for A.*

Corollary 1. *Let S_1 and S_2 be two interpolating sequences in $\mathcal{M}_s^2$ such that $S := S_1 \cup S_2$ is separated, then S is still an interpolating sequence in $\mathcal{M}_s^2$.*

If $n - 1 \leq 2s \leq n$ then the Hilbert space $H_s^2(\mathbb{B})$ has the Pick property, i.e. its interpolating sequences are the same than those of $\mathcal{M}_s^2$ hence the results for $\mathcal{M}_s^2$ are valid for H_s^2. We also have some specific results to H_s^p as the following theorem, which is a weak version of [3] we have in $H^p(\mathbb{B})$.

Theorem 6. *Let S be a sequence of points in $\mathbb{B}$ such that*

- *there is a sequence $\{\rho_a\}_{a \in S}$ in H_s^p such that* $\forall a, b \in S,\ \rho_a(b) \simeq \delta_{ab} \|\rho_a\|_{s,p} \|k_a\|_{s,p'}$.
- *If* $0 < s < \frac{n}{2} \min(\frac{1}{p'}, \frac{1}{q'})$ *with* $\frac{1}{r} = \frac{1}{p} + \frac{1}{q}$, *i.e.* $s < \frac{n}{2p'}$ *and* $\frac{p}{2} < r < p$, *we have*

$$\forall j \leq s,\ \left\|R^j(\rho_a)\right\|_p \lesssim \left\|R^j(k_a)\right\|_p \Rightarrow \|\rho_a\|_{s,p} \lesssim \|k_a\|_{s,p}.$$

- *S is Carleson in $H_s^q(\mathbb{B})$.*

Then S is H_s^r interpolating with the bounded linear extension property, provided that $p \leq 2$.

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Separating structures of finite rank and vessels of commuting selfadjoint operators

Victor Vinnikov

(joint work with Daniel Alpay, and Daniel Estévez & Dmitry Yakubovich)

Separating structures of finite rank. A separating structure of finite rank consists of

- $\mathcal{K}$, a Hilbert space,
- $A_1, A_2 \colon \mathcal{K} \to \mathcal{K}$, (bounded) commuting selfadjoint operators,
- $\mathcal{K} = \mathcal{K}_- \oplus \mathcal{K}_+$, an orthogonal decomposition,

such that the hermitian form $\langle(\xi_1 A_1 + \xi_2 A_2)k_-, k_+\rangle$ on $\mathcal{K}_- \times \mathcal{K}_+$ has finite rank for all $\xi \in \mathbb{R}^2$ (the same is then true for the corresponding hermitian form on $\mathcal{K}_+ \times \mathcal{K}_-$).

We set

$$\mathcal{K}_-^0 = \left\{k_- \in \mathcal{K}_- \colon \langle(\xi_1 A_1 + \xi_2 A_2)k_-, k_+\rangle = 0 \ \forall k_+ \in \mathcal{K}_+ \forall \xi \in \mathbb{R}^2\right\},$$
$$\mathcal{K}_+^0 = \left\{k_+ \in \mathcal{K}_+ \colon \langle(\xi_1 A_1 + \xi_2 A_2)k_-, k_+\rangle = 0 \ \forall k_- \in \mathcal{K}_- \forall \xi \in \mathbb{R}^2\right\},$$

and $\mathcal{M}_- = \mathcal{K}_- \ominus \mathcal{K}_-^0$, $\mathcal{M}_+ = \mathcal{K}_+ \ominus \mathcal{K}_+^0$, so that we have a decomposition

$$\mathcal{K} = \underbrace{\mathcal{K}_-^0 \oplus \mathcal{M}_-}_{\mathcal{K}_-} \oplus \underbrace{\mathcal{M}_+ \oplus \mathcal{K}_+^0}_{\mathcal{K}_+}.$$

With respect to this decomposition A_j has a block structure

$$A_j = \begin{bmatrix} \star & \star & 0 & 0 \\ \star & \Lambda_{j,-1} & R_{j,-1} & 0 \\ 0 & \Lambda_{j,-1} & R_{j,0} & 0 \\ 0 & 0 & \star & \star \end{bmatrix}, j = 1, 2.$$

Notice that $\dim \mathcal{M}_- = \dim \mathcal{M}_+ < \infty$ (the "defect spaces"), we set $\mathcal{M} = \mathcal{M}_- \oplus \mathcal{M}_+$.

To avoid pathologies, we introduce a *nondegeneracy assumption*:

$$\nexists \xi = (\xi_1, \xi_2) \in \mathbb{C}^2 \setminus 0 \text{ s.t. } P_{\mathcal{M}_\pm}(\xi_1 A_1 + \xi_2 A_2)|_{\mathcal{M}_\mp} \text{ both non-invertible,}$$

and a *minimality assumption*: $\mathcal{M}_\pm$ are cyclic subspaces for $\mathbb{C}[A_1, A_2]$.

Why are these interesting objects?

- They are a natural generalization of two evolution continuous time scattering systems, where we have a pair of orthogonal subspaces which are invariant under a pair of (usually unbounded) commuting selfadjoint operators (the infinitesimal generators).
- Special case: subnormal operators of finite type (Xia [9, 10, 11], Yakubovich [12, 13]),

  $$S\colon \mathcal{H} \to \mathcal{H},\ S = N|_{\mathcal{H}},\ N\colon \mathcal{K} \to \mathcal{K} \text{ normal}, \mathcal{H} \subseteq \mathcal{K},$$
  $$\dim \operatorname{im}[S^*, S] < \infty.$$

  correspond to a separating structure of finite rank

  $$N = A_1 + iA_2,\ cK_- = \mathcal{K} \ominus \mathcal{H},\ cK_+ = \mathcal{H};$$
  $$\mathcal{M}_+ = \operatorname{im}[S^*, S],$$

  with a special property: $P_{\mathcal{M}_-}(A_1 + iA_2)|_{\mathcal{M}_+} = 0$ (a complex linear combination of A_1 and A_2 leaves one of the two subspaces invariant).
- They provide a natural framework for dilation theory for pairs of commuting nonselfadjoint operators in the nondissipative case.
- The subspaces $\mathcal{M}_-$ and $\mathcal{M}_+$ "propagate" under the action of A_1 and A_2 yielding a representation of A_1 and A_2 as doubly infinite block Jacobi matrices; hence the study of separating structures is closely related to the study of commuting block Jacobi matrices.

An example, and a functional model. Let

- $X = X_- \cup X_{\mathbb{R}} \cup X_+$ be a compact real Riemann surface of dividing type;
- $\mathbf{V}_\chi \otimes \Delta$ a positive parahermitian bundle of multiplicative half–order differentials ($\Delta \otimes \Delta = K_X$, V_χ is obtained by reflection from a unitary flat vector bundle on X_+);
- λ_1 and λ_2 a pair of real meromorphic functions on X with no real poles birationally imbedding X in the plane.

Then the multiplication operators by λ_1 and λ_2 on

$$L^2(X_{\mathbb{R}}, \mathbf{V}_\chi \otimes \Delta) = H^2(X_-, \mathbf{V}_\chi \otimes \Delta) \oplus H^2(X_+, \mathbf{V}_\chi \otimes \Delta)$$

form a separating structure of finite rank. Here $\mathcal{M}_{\pm}$ is spanned by the Cauchy kernels of χ at the poles of λ_1, λ_2 on $X_{\pm}$. (For the detailed definitions of all the notions involved, see [1, 3, 4].)

Main result, in the baking: this example yields the general functional model for (minimal nondegenerate) separating structures of finite type, *up to two modifications*,

- We have to consider a collection of Riemann surfaces with a vector bundle on each one and with some "gluing data", corresponding to a torsion free sheaf on a possibly singular and reducible algebraic curve.
- We have to allow for this curve having certain degenerate components (which are not real but come in pairs of complex conjugates) corresponding to the point spectrum of the pair of operators A_1 and A_2.

Selfadjoint commutative vessels. An operator vessel, as originally introduced by M.S. Livsic in the 1980s, is a collection of spaces and operators that reflect an interplay between a tuple of operators that commute, or more generally satisfy some commutation relations; it correspond to an overdetermined multidimensional linear input/state/output system together with compatibility conditions for its input and outpout signals. See [6, 8, 4], as well as the recent preprint [7] for the noncommutative setting.

A selfadjoint commutative vessel is an impedance conservative overdetermined 2D input/state/output linear system:

$$i\frac{\partial x}{\partial t_1} = -A_1 x + \widetilde{B}\sigma_1 u,\ i\frac{\partial x}{\partial t_2} = -A_2 x + \widetilde{B}\sigma_2 u,\ y = iB^* x + Du,$$

with compatibility conditions

$$\sigma_2\frac{\partial u}{\partial t_2} - \sigma_1\frac{\partial u}{\partial t_1} + i\gamma u = 0,\ \sigma_2^*\frac{\partial y}{\partial t_2} - \sigma_1^*\frac{\partial y}{\partial t_1} + i\gamma^* y = 0$$

for the input and the output signals; we will assume $\dim \mathcal{E} < \infty$. Here $A_1, A_2 \colon \mathcal{H} \to \mathcal{H}$ selfadjoint, commutative, $\widetilde{B} \colon \mathcal{E} \to \mathcal{H}$, $D \colon \mathcal{E} \to \mathcal{H}$, $\sigma_1, \sigma_2, \gamma \colon \mathcal{E} \to \mathcal{E}$ selfadjoint, and the following relations are satisfied:

$$\begin{aligned}
&A_1\widetilde{B}\sigma_2 - A_2\widetilde{B}\sigma_1 + \widetilde{B}\gamma = 0 \text{ (the vessel condition)},\\
&\sigma_1^* D = -D^*\sigma_1,\ \sigma_2^* D = -D^*\sigma_2,\\
&\gamma^* D = -D^*\gamma - i\sigma_1^*\widetilde{B}^*\widetilde{B}\sigma_2 + i\sigma_2^*\widetilde{B}^*\widetilde{B}\sigma_1 \text{ (the linkage conditions)}.
\end{aligned}$$

The real algebraic curve $\mathbf{C}$ in $\mathbb{P}^2$ with a selfadjoint determinantal representation $\det(\lambda_1\sigma_2 - \lambda_2\sigma_1 + \gamma) = 0$ (here we use affine coordinates) is called the discriminant curve of the vessel. Assuming that $\mathbf{C}$ is irreducible with the desingularizing compact real Riemann surface $X \to \mathbf{C}$, and under certain additional maximality assumption on the determinantal representation [2, 5], we can then construct, similarly to [8, 4], a functional model for the vessel. The operators A_1 and A_2 become in the model multiplication operators by the coordinate functions λ_1 and λ_2 on the L^2 space of sections of a certain vector bundle on X over $X_{\mathbb{R}}$ with respect to an appropriate measure $d\mu$. The vector bundle is obtained by pulling back the input/output family of vector spaces along $\mathbf{C}$: $\boldsymbol{\mathcal{E}}(\lambda) = \ker\,(\lambda_1\sigma_2 - \lambda_2\sigma_1 + \gamma)$ to X, and the measure $d\mu$ can be obtained either by compressing the joint spectral measure of A_1 and A_2 to the space $\mathcal{E}$ via the mapping $\widetilde{B}$ or from the joint characteristic function of the vessel.

It is a nontrivial fact of life that we can always embed a separating structure of finite rank into a selfadjoint commutative vessel with $\mathcal{H} = \mathcal{K}$, $\mathcal{E} = \mathcal{M}$, $\widetilde{B} = P_{\mathcal{M}} : \mathcal{K} \to \mathcal{M}$. The main result in the baking is then obtained using the functional model for this vessel, beefed up to deal with reducible curves and generically maximal (rather than maximal) determinantal representations. The main steps in the proof are to show that the real irreducible components of the discriminant curve $\mathbf{C}$ (there may be also pairs of complex conjugate components with finitely many real points) are of dividing type, that the determinantal representation there is positive and generically maximal, and that $d\mu$ there is the Lebesgue measure.

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The Fundamental operator(s)

Tirthankar Bhattacharyya

The purpose of this research report is to give an idea about two things:

(1) details about my talk,
(2) report of recent research on two inhomogeneous domains - the symmetrized bidisc in $\mathbb{C}^2$ and the tetrablock in $\mathbb{C}^3$.

The important object of the talk and of the research is the fundamental operator of a Γ-contraction. The closed symmetrized bidisc Γ is the set

$$\{(z_1 + z_2, z_1 z_2) : |z_1|, |z_2| \leq 1\}.$$

A pair of commuting bounded operators (S, P) on a Hilbert space $\mathcal{H}$ is called a Γ-contraction if Γ is a spectral set for (S, P). The closed symmetrized bidisc Γ is polynomially convex. Thus, a pair of commuting bounded operators (S, P) is a Γ-contraction if and only if

$$\|f(S, P)\| \leq \sup_{(s,p)\in\Gamma} |f(s, p)|$$

for all polynomials f in two variables.

For a contraction P and a bounded commutant S of P, we seek a solution X of the operator equation

$$S - S^*P = (I - P^*P)^{\frac{1}{2}} X (I - P^*P)^{\frac{1}{2}},$$

where X is a bounded operator on $\overline{\mathrm{Ran}}(I - P^*P)^{\frac{1}{2}}$ with numerical radius of X being not greater than 1. We show the existence and uniqueness of solution to the operator equation above for a Γ-contraction (S, P). This allows us to construct an explicit Γ unitary dilation of a Γ-contraction (S, P). A Γ-unitary dilation means a pair of commuting normal operators (R, U) on a Hilbert space $\mathcal{K}$ containing $\mathcal{H}$ which satisfies

$$P_{\mathcal{H}} f(R, U)|_{\mathcal{H}} = f(S, P)$$

for any rational function with poles off Γ and

$$\sigma_T(R, U) \subseteq \partial\Gamma = \{(z_1 + z_2, z_1 z_2) : |z_1|, |z_2| = 1\}.$$

The other way holds too, i.e, for a commuting pair (S, P) with $\|P\| \leq 1$ and the spectral radius of S being not greater than 2, the existence of a solution to the above equation implies that (S, P) is a Γ-contraction.

This was the first role of the fundamental operator. The second role that we describe in the talk is a recent work of Pal and Shalit [6]. They showed that for every pair of matrices (S, P), having the closed symmetrized bidisc Γ as a spectral set, there is a one dimensional complex algebraic variety Λ in Γ such that for every matrix valued polynomial $f(z_1, z_2)$,

$$\|f(S, P)\| \max_{(z_1,z_2)\in\Lambda} \|f(z_1, z_2)\|.$$

The variety Λ is shown to have the determinantal representation

$$\Lambda = \{(s, p) \in \Gamma : \det(F + pF^* - sI) = 0\},$$

where F is the fundamental operator of the Γ-contraction (S, P). When (S, P) is a strict Γ-contraction, then Λ is a distinguished variety in the symmetrized bidisc, i.e., a one dimensional algebraic variety that exits the symmetrized bidisc through its distinguished boundary. All distinguished varieties of the symmetrized bidisc are characterized by a determinantal representation as above.

Recent work on the fundamental operator described in [3] characterizes operators pairs F and G which can arise as fundamental operators of a Γ-contraction (S, P) and its adjoint (S^*, P^*). This issue is important because an explicit construction of the Γ-unitary dilation involves both the fundamental operators. This shows that it is of interest to know which pair of operators F and G, defined on different Hilbert spaces in general, satisfying $w(F) \leq 1$ and $w(G) \leq 1$, qualify as fundamental operators. In other words, does there always exist a Γ-contraction (S, P) such that F is the fundamental operator of (S, P) and G is the fundamental operator of (S^*, P^*)? If there is such an (S, P), then it forces a relation between F, G and P.

For a contraction P on a Hilbert space $\mathcal{H}$, define

$$\Theta_P(z) = [-P + zD_{P^*}(I_{\mathcal{H}} - zP^*)^{-1}D_P]|_{\mathcal{D}_P} \text{ for all } z \in \mathbb{D}.$$

The function Θ_P is called the *characteristic function* of the contraction P. By virtue of the relation $PD_P = D_{P^*}P$(see ch.1, sec.3 of [8]), it follows that each $\Theta_P(z)$ is an operator from $\mathcal{D}_P$ into $\mathcal{D}_{P^*}$. The characteristic function induces an operator M_{Θ_P} in $\mathcal{B}(H^2_{\mathcal{D}_P}(\mathbb{D}), H^2_{\mathcal{D}_{P^*}}(\mathbb{D}))$ defined by

$$M_{\Theta_P} f(z) = \Theta_P(z) f(z) \text{ for all } z \in \mathbb{D}.$$

The way the characteristic function relates to the fundamental operator is as follows. Let (S, P) on a Hilbert space $\mathcal{H}$ be a Γ-contraction and F, G be the fundamental operators of (S, P) and (S^*, P^*) respectively. Then

$$\Theta_P(z)(F + F^*z) = (G^* + Gz)\Theta_P(z) \tag{1}$$

holds, where Θ_P is characteristic function of P.

Since the above gives a necessary condition, it is natural to ask about sufficiency. A contraction P is called *pure* if P^{*n} strongly converges to 0 as n goes to infinity. Sz.-Nagy and Foias called it a $C_{\cdot 0}$ contraction. The unilateral shift is a pure contraction. So are its compressions to all co-invariant subspaces. A Γ-contraction (S, P) is called pure if the contraction P is pure. Let P be a pure contraction on a Hilbert space $\mathcal{H}$. Let $F \in \mathcal{B}(\mathcal{D}_P)$ and $G \in \mathcal{B}(\mathcal{D}_{P^*})$ be two operators with numerical radius not greater than one. If (1) holds, then there exists an operator S on $\mathcal{H}$ such that (S, P) is a Γ-contraction and F,G are fundamental operators of (S, P) and (S^*, P^*) respectively.

Another inhomogeneous domain having close connection to the symmetrized bidisc has also been our object of research in recent times. This is called the tetrablock. We did not talk about it, but it is worth mentioning it here because of its relevane. The tetrablock, roughly speaking, is the set of all linear fractional maps that map the open unit disc to itself. It can be identified with a polynomially convex subset of $\mathbb{C}^3$. A tuple of commuting bounded operators (A, B, P) which has the tetrablock as a spectral set is called a tetrablock contraction. The motivation comes from the success of model theory in the symmetrized bidisc Γ. The two domains are related intricately. Given a triple (A, B, P) as above, we associate with it a pair (F_1, F_2), called its fundamental operators. We show that (A, B, P) dilates if the fundamental operators F_1 and F_2 satisfy certain commutativity conditions. Moreover, the dilation space is no bigger than the minimal isometric dilation space of the contraction P. Whether these commutativity conditions are necessary too is not known. What we have shown is that if there is a tetrablock isometric dilation on the minimal isometric dilation space of P, then those commutativity conditions necessarily get imposed on the fundamental operators.

The structure of a tetrablock unitary (this is the candidate as the dilation triple) and a tertrablock isometry (the restriction of a tetrablock unitary to a joint invariant subspace) have been deciphered completely.

The methods applied in the study of the tetrablock are motivated by [4]. Although the calculations are lengthy and more complicated, they go through beautifully to reveal that the dilation depends on the mutual relationship of the two fundamental operators so that certain conditions need to be satisfied. The question of whether all tetrablock contractions dilate or not is open.

The relevant articles are as follows.

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Projective spectrum in Banach algebras

Rongwei Yang

For a tuple $A = (A_1,\ A_2,\ ...,\ A_n)$ of elements in a unital algebra $\mathcal{B}$ over $\mathbb{C}$, its *projective spectrum* $P(A)$ (or $p(A)$) is the collection of $z \in \mathbb{C}^n$ (or respectively $z \in \mathbb{P}^{n-1}$) such that $A(z) = z_1A_1 + z_2A_2 + \cdots + z_nA_n$ is not invertible in $\mathcal{B}$. In finite dimensional case, projective spectrum is a projective hypersurface. When A is commuting, $P(A)$ is a union of hyperplanes that looks like a bundle over the Taylor spectrum of A. The *projective resolvent set* $P^c(A) := \mathbb{C}^n \setminus P(A)$ can be identified with $\mathcal{B}^{-1} \cap span\{A_1,\ A_2,\ ...,\ A_n\}$. For every Banach algebra $\mathcal{B}$, $P^c(A)$

is a domain of holomorphy. $\mathcal{B}$-valued Maurer-Cartan type 1-form $A^{-1}(z)dA(z)$ reveals the topology of $P^c(A)$. There is a map from multilinear functionals on $\mathcal{B}$ to the de Rham cohomology $H_d^*(P^c(A),\, \mathbb{C})$. In finite dimensional commutative case, this map is a surjective homomorphism by a theorem of Brieskon and Arnold. In noncummutative case, this map links the cyclic cohomology of $\mathcal{B}$ to $H_d^*(P^c(A),\, \mathbb{C})$. Further, there exists a higher order form of the classical Jacobi's formula.

Spectral data for several matrices and multivariable analogs of Livshits characteristic functions

Yuri Neretin

$\mathrm{U}(\infty)$ is the group of finitary unitary matrices of infinite size,

$$\mathrm{U}(\infty) = \varinjlim \mathrm{U}(n).$$

We write them as block matrices of size $\alpha + \infty$

$$g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

Let $K \subset \mathrm{U}(\infty)$ be the group of unitary matrices of the form

$$s = \begin{pmatrix} 1 & 0 \\ 0 & u \end{pmatrix}.$$

Definition. A multicolligation is a conjugacy class of $\mathrm{U}(\infty) \times \cdots \times \mathrm{U}(\infty)$ with respect to conjugations by elements of K. This means that we consider tuples $(g_1, \dots, g_m) \in \mathrm{U}(\infty)^m$ determined up to the equivalence

$$(g_1, \dots, g_m) \sim (sg_1s^{-1}, \dots, sg_ms^{-1}), \qquad s \in K.$$

Definition. Multiplication of tuples $\{g_j\} \circ \{h_j\}$

$$\mathfrak{g} = \{g_j\} = \left\{ \begin{pmatrix} a_j & b_j \\ c_j & d_j \end{pmatrix} \right\} \qquad \mathfrak{h} = \{h_j\} = \left\{ \begin{pmatrix} p_j & q_j \\ r_j & t_j \end{pmatrix} \right\}$$

is

$$\begin{pmatrix} a_j & b_j \\ c_j & d_j \end{pmatrix} \circ \begin{pmatrix} p_j & q_j \\ r_j & t_j \end{pmatrix} := \begin{pmatrix} a_j & b_j & 0 \\ c_j & d_j & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} p_j & 0 & q_j \\ 0 & 1 & 0 \\ r_j & 0 & t_j \end{pmatrix} = \begin{pmatrix} a_jp_j & b & a_jq_j \\ c_jp_j & d_j & c_jq_j \\ r_j & 0 & t_j \end{pmatrix}.$$

The size of the new matrix is $\alpha + (\infty + \infty) = \alpha + \infty$. This is a well-defined assoaciative operation on conjugacy classes.

Perverse equation for eigenvectors ($m = 2$)

$$\begin{pmatrix} q_1 \\ x_1 \\ q_2 \\ x_2 \end{pmatrix} = \begin{pmatrix} a_1 & b_1 & 0 & 0 \\ c_1 & d_1 & 0 & 0 \\ 0 & 0 & a_2 & b_2 \\ 0 & 0 & c_2 & d_2 \end{pmatrix} \begin{pmatrix} p_1 \\ s_{11}x_1 + s_{12}x_2 \\ p_2 \\ s_{21}x_1 + s_{22}x_2 \end{pmatrix}.$$

Here

$$S = \begin{pmatrix} s_{11} & s_{12} \\ s_{21} & s_{22} \end{pmatrix}$$

is a $m \times m$ matrix.
Eliminating x, we get a dependence

$$q = \chi_{\mathfrak{g}}(S)p,$$

where $\chi_{\mathfrak{g}}$ is a rational function

$$\chi_{\mathfrak{g}} : \mathrm{Mat}(m) \to \mathrm{Mat}(m\alpha)$$

Theorem 1. *(a) $\chi_{\mathfrak{g}}(S)$ depends only on a conjugacy class.*
(b) $\chi_{\mathfrak{g}\circ\mathfrak{h}}(S) = \chi_{\mathfrak{g}}(S) \cdot \chi_{\mathfrak{h}}(S)$.
(c) $\chi_{\mathfrak{g}}$ is an inner function, i.e

$$\begin{aligned} \|S\| \le 1 \quad &\Rightarrow \quad \|\chi_{\mathfrak{g}}(S)\| \le 1 \\ S \textit{ is unitary} \quad &\Rightarrow \quad \chi_{\mathfrak{g}}(S) \textit{ is unitary.} \end{aligned}$$

(d) Let Λ be a diagonal matrix, then

$$\chi_{\mathfrak{g}}(\Lambda S \Lambda^{-1}) = \Lambda \chi_{\mathfrak{g}}(S) \Lambda^{-1}.$$

Remark. $\chi_{\mathfrak{g}}$ can be regarded as a rational map from a Grassmannian to a Grassmannian.

Perverse equation for eigenvalues $N = S^{-1}$, $m = 3$

$$\det \begin{pmatrix} \nu_{11} - d_1 & \nu_{12} & \nu_{13} \\ \nu_{21} & \nu_{22} - d_2 & \nu_{23} \\ \nu_{31} & \nu_{32} & \mu_{33} - d_3 \end{pmatrix} = 0.$$

This is a divisor in a Grassmannian.
Conjecture. To obtain all rational inner functions of matrix arguments

$$\mathrm{Mat}(m) \to \mathrm{Mat}(m\alpha),$$

it is sufficient to eliminate x from the equation

$$\begin{pmatrix} q \\ x_1 \\ \vdots \\ x_m \end{pmatrix} = \begin{pmatrix} a & b_1 & \dots & b_m \\ c_1 & d_{11} & \dots & d_{1m} \\ \vdots & \vdots & \ddots & \vdots \\ c_m & d_{m1} & \dots & d_{mm} \end{pmatrix} \begin{pmatrix} p \\ s_{11}x_1 + \dots + s_{1m}x_m \\ \vdots \\ s_{m1}x_1 + \dots + s_{mm}x_m \end{pmatrix},$$

where the matrix is a unitary block matrix of size

$$\alpha + \infty + \dots + \infty.$$

Industry
1. Lie groups.
Let G be a group, K a compact subgroup. Let

$$K \setminus G/K$$

be double cosets (i.e., space of sets $KgK \subset G$). Let $\mathcal{M}$ be the space of compactly supported measures on G, invariant with respect to left and right shifts by elements

of K. Then $\mathcal{M}$ is an algebra with respect to convolutions (Hecke-Iwahori algebra). Let ρ be a unitary representation of G. Let H^K be the space of K-invariant vectors. Then $\mathcal{M}$ acts in H^K (a standard fact).
2. Infinite-dimensional groups.
Example: $G = \mathrm{U}(\alpha + N)$, $K = \mathrm{U}(N)$. Pass to limit as $N \to \infty$.
Fact. In limit, we get a multiplication on $K \setminus G/K$.
Fact. The semigroup $K \setminus G/K$ acts in H^K.
Remark. Conjugacy classes are special case of double cosets. Indeed, conjugacy classes of G by K are double cosets $K \setminus (G \times K)/K$.

Complete positivity and representations of ball semigroups

Karl-Hermann Neeb
(joint work with Daniel Beltiţă)

This project aims at a more systematic understanding of unitary representations of unitary (resp. orthogonal) groups $\mathrm{U}(\mathcal{A}) = \{a \in \mathcal{A} : a^*a = aa^* = \mathbf{1}\}$ of real unital seminormed involutive algebras $(\mathcal{A}, p)$. This means that p is a submultiplicative seminorm on $\mathcal{A}$ satisfying $p(a^*) = p(a)$. Typical examples we have in mind are unital C^*-algebras, $\mathcal{A} = C^\infty(X, M_n(\mathbb{K}))$, where X is a smooth manifold and $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}, \mathbb{H}\}$, or algebras of smooth vectors for Lie automorphism groups of unital C^*-algebras.

If $G = \mathrm{U}(\mathcal{A})$ is the unitary group of a C^*-algebra, then every irreducible representation $(\pi, \mathcal{H})$ of $\mathcal{A}$ provides for every partition $\lambda = (\lambda_1, \ldots, \lambda_k)$ of N an irreducible representation $(\pi_\lambda, \mathbb{S}_\lambda(\mathcal{H}))$ (and their dual representations) by a straightforward generalizations of the classical Schur–Weyl theory to Hilbert spaces (see [3] and [5] for an extension to type II_1-factor representations).

For the special case where $\mathcal{A} = K(\mathcal{H})$ (compact operators on an infinite dimensional Hilbert space) old results from A. Kirillov and G. Olshanski [9, 12] assert that continuous unitary representations of $\mathrm{U}(\mathcal{A})$ are direct sums of irreducible ones which are of the form $\mathbb{S}_\lambda(\mathcal{H}) \otimes \mathbb{S}_\mu(\mathcal{H})^*$ and there are natural generalizations to $G = \mathrm{U}(K(\mathcal{H}))$, where $\mathcal{H}$ is a real or a quaternionic Hilbert space. Therefore the Schur–Weyl construction is exhaustive in these cases. The key method to obtain these results is to show that continuous unitary representations are generated by dilation, resp., the GNS-construction, from operator-valued positive definite functions ϕ of the form $\phi(g) = \rho(pgp)$, where p is a hermitian projection of finite rank and ρ is a continuous representation of the semigroup $S = \overline{\mathrm{ball}}(p\mathcal{A}p)$ of contractions in $p\mathcal{A}p$, where $p\mathcal{A}p \cong M(n, \mathbb{K})$ for $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}, \mathbb{H}\}$. It turns out that the representations ρ of S for which ϕ is positive definite are precisely the completely positive representations of S ([10]).

This observation was our motivation to take a closer look at completely positive representations of the semigroup

$$S := \mathrm{ball}(\mathcal{A}, p) := \{a \in \mathcal{A} : p(a) < 1\}$$

for a seminormed real involutive algebra $(\mathcal{A}, p)$. An application that we have in mind is that any completely positively representation ρ of $\mathrm{ball}(p\mathcal{A}p)$ for a hermitian projection p, leads to a unitary representation of $\mathrm{U}(\mathcal{A})$. Our main result generalizes work of Arveson on non-linear states on balls of C^*-algebras ([2]), of Ando/Choi on series expansion of nonlinear completely positive functions on C^*-algebras ([1]), and of Hiai/Nakamura on non-linear completely positive C^*-valued functions on balls of C^*-algebras ([7]).

Here is our main **Theorem** ([4]): *Let $(\mathcal{A}, p)$ be a real seminormed involutive algebra, V a complex Hilbert space and $\phi : \mathrm{ball}(\mathcal{A}, p) \to B(V)$ be a bounded function. Then the following are equivalent:*

(i) *ϕ is completely positive.*
(ii) *ϕ is positive definite and analytic with respect to any vector topology for which p is continuous.*
(iii) *There exists a linear completely positive map $\Phi : e^{C^*(\mathcal{A},p)} \to B(V)$ with $\Phi \circ \Gamma = \phi$. Here $e^{C^*(\mathcal{A})}$ is the c_0-direct sum of the C^*-algebras $S^n(C^*(\mathcal{A}))$ ([2]).*

An important consequence of this theorem is that it leads to a one-to-one correspondence between bounded analytic representations of $\mathrm{ball}(\mathcal{A}, p)$ with the representations of the C^*-algebra $e^{C^*(\mathcal{A},p)}$, which thus plays the role of a host algebra in the sense of [6] for the bounded analytic representations of $\mathrm{ball}(\mathcal{A}, p)$.

Our goal is to fit the classification results on norm-continuous representation for groups of the form $C^\infty(X, \mathrm{SU}_N(\mathbb{C})) \cong \mathrm{SU}_N(C^\infty(X))$ and the Kirillov–Olshanskii classification into a natural framework that applies to Lie groups associated to more general classes of $*$-algebras, such as AF-algebras (see also [5, 11] for related recent results).

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