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Modular Forms

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Abstract. The theory of Modular Forms has been central in mathematics with a rich history and connections to many other areas of mathematics. The workshop explored recent developments and future directions with a particular focus on connections to the theory of periods.

Mathematics Subject Classification (2010): 11xx.

Introduction by the Organisers

The workshop *Modular Forms*, organized by Jan Hendrik Bruinier (Darmstadt), Atsushi Ichino (Kyoto), Tamotsu Ikeda (Kyoto) and Özlem Imamoglu (Zürich) consisted of 19 one-hour long lectures and covered various recent developments in the theory of modular and automorphic forms and related fields.

A particular focus was on the connection of modular forms to periods, since there have been important developments in that direction in recent years. In this context, the topics that the workshop addressed include the global Gross-Prasad conjecture and its analogs, which predict a relationship between periods of automorphic forms and central values of L-functions, the theory of liftings and their applications to period relations, as well as explicit aspects of these formulas and relations with a view towards the arithmetic properties of periods.

There are two fundamental ways in which automorphic forms are related to periods. First, according to the conjectures of Deligne, Beilinson and Scholl, special values of motivic automorphic L-functions at integral arguments should be given by periods and encode important arithmetic information, such as ranks of

Chow groups and Selmer groups. Second, the Fourier coefficients of automorphic forms are often given by periods. For instance, by the work of Waldspurger, the coefficients of half integral weight eigenforms are given by period integrals of their Shimura lifts. The majority of the lectures (in particular talks by Wee-Teck Gan, Erez Lapid, Kazuki Morimoto, Anantharam Raghuram, Abhishek Saha and Shunsuke Yamana) discussed (or were motivated by) relations of periods and special values of automorphic L-functions. Periods related to classes in cohomology and Chow groups of Shimura varieties and their connections to automorphic forms were addressed in the talks by Kathrin Bringmann, Yingkun Li, Yifeng Liu, and Tonghai Yang.

Other talks discussed the role of automorphic forms in geometry, for instance in context of the Kudla program (Stephan Ehlen, Valery Gritsenko, Jürg Kramer, Stephen Kudla and Martin Raum). Aspects of the analytic theory of automorphic forms played an important role in the talks by Valentin Blomer, Gautam Chinta, Tomoyoshi Ibukiyama and Ren He Su.

In total, 53 researchers participated in the workshop. Out of these, 37 came from 12 countries different from Germany. Beyond the talks, the participants enjoyed ample time for discussions and collaborative research activities. The traditional hike on Wednesday afternoon led us to the Ochsenwirtshof in Schapbach. A further highlight was a piano recital on Thursday evening by Valentin Blomer.

The organizers and participants of the workshop thank the *Mathematisches Forschungsinstitut Oberfwolfach* for hosting the workshop and providing such an ideal working environment.

Acknowledgement: The MFO and the workshop organizers would like to thank the National Science Foundation for supporting the participation of junior researchers in the workshop by the grant DMS-1049268, "US Junior Oberwolfach Fellows". Moreover, the MFO and the workshop organizers would like to thank the Simons Foundation for supporting Stephen S. Kudla in the "Simons Visiting Professors" program at the MFO.

Workshop: Modular Forms

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Abstracts

Product formulas for Borcherds forms

Stephen Kudla

In a now classic pair of Inventiones papers in 1995 and 1998, Borcherds constructed meromorphic modular forms on the arithmetic quotient of a bounded domain D associated to a rational quadratic space V, $(\ ,\ )$ of signature $(n, 2)$. These forms have various remarkable properties, for example, their divisor is explicitly given. But perhaps most striking is that, in a suitable neighborhood of each 0-dimensional boundary component, they are given by a product formula reminiscent of that for the Dedekind η function. In this talk, I will describe analogous product formulas for Borcherds forms, now valid in a suitable neighborhood of each 1-dimensional boundary component, assuming that V admits 2-dimensional isotropic subspaces.

Let

$$D = \{w \in V(\mathbb{C}) \mid (w, w) = 0,\ (w, \bar{w}) < 0\ \}/\mathbb{C}^\times \subset \mathbb{P}(V(\mathbb{C}))$$

be the 'quadric' model of the symmetric space associated to V. Fix an even integral lattice $M \subset M^\vee$ in V, let

$$\Gamma \subset \Gamma_M = \{\gamma \in \mathrm{SO}(V) \mid \gamma M = M, \quad \gamma|_{M^\vee/M} = 1\}$$

be a subgroup of finite index, and let $X_\Gamma = \Gamma\backslash D$ be the corresponding arithmetic quotient. Let $S_M = \mathbb{C}[M^\vee/M]$ be the group algebra of the discriminant group of M, which we view as a subspace of $S(V(\mathbb{A}_f))$, the space of locally constant functions of compact support on the finite adèle points of V. The group $\mathrm{SL}_2(\mathbb{Z})$, for n even, or a central extension of it, for n odd, acts on the space S_M via the Weil representation ρ_M. Recall that in [2], Borcherds takes as input a weakly holomorphic modular form $F : \mathfrak{H} \to S_M$ of weight $1-\frac{n}{2}$ and type ρ_M. In particular F has a Fourier expansion

$$F(\tau) = \sum_m c(m)\, q^m, \qquad c(m) \in S_M$$

with only a finite number of nonvanishing coefficients $c(m)$ for $m < 0$. Assuming that for $m \le 0$, $c(m) \in \mathbb{Z}[M^\vee/M]$, Borcherds associates to F a meromorphic modular form $\Psi(F)$ on D of weight $c(0)(0)/2$ with respect to a finite index subgroup of Γ_M.

Now suppose that

$$V = U + V_0 + U'$$

is a Witt decomposition of V, where U is an isotropic 2-plane, U' is an isotropic complement and $V_0 = (U + U')^\perp$. The complex curve

$$\mathcal{C}(U) = \{w \in U(\mathbb{C}) \mid \mathrm{span}\{w, \bar{w}\} = U(\mathbb{C})\}/\mathbb{C}^\times \simeq \mathbb{P}(U(\mathbb{C})) - \mathbb{P}(U(\mathbb{R}))$$

lies in the closure of D in $\mathbb{P}(V(\mathbb{C}))$ and defines a 1-dimensional rational boundary component in the Bailey-Borel compactification X_Γ^{BB} of X_Γ.

Choose a $\mathbb{Z}$-basis e_1, e_2 for the lattice $M \cap U$ and let e_1' and e_2' be a dual basis for U'. The Witt decomposition then determines an isomorphism

$$D \xrightarrow{\sim} \{(\tau_1, w_0, \tau_2') \in \mathbb{C} \times V_0(\mathbb{C}) \times \mathbb{C} \mid 4v_1v_2' - Q(w_0 - \bar{w}_0) > 0\},$$

where $v_1 = \Im(\tau_1)$, $v_2' = \Im(\tau_2')$, $Q(x) = \frac{1}{2}(x,x)$, and the inverse map is obtained by taking

$$w(\tau_1, w_0, \tau_2') = -\tau_2' e_1 + (\tau_1\tau_2' - Q(w_0))e_2 + w_0 + \tau_1 e_1' + e_2'.$$

Note that, as $v_2' \to \infty$ for τ_1 and w_0 in bounded sets, the isotropic line $\mathbb{C}w$ in D goes to the isotropic line $\mathbb{C}(-e_1 + \tau_1 e_2)$ in $\mathcal{C}(U)$.

Theorem. *In a region of the form*

$$\{w(\tau_1, w_0, \tau_2') \mid v_2' > Av_1 + (Q(\Im(w_0)) + B)v_1^{-1}\ \},$$

for suitable positive constants A and B, the Borcherds form $\Psi(F)(w)$ is given as the product of the following factors:

(a)

$$\prod_{\substack{x \in M^\vee \\ (x,e_2)=0 \\ (x,e_1)>0 \\ \operatorname{mod} M\cap\mathbb{Q}e_2}} \big(1 - e(-(x,w))\big)^{c(-Q(x))(x)},$$

(b)

$$\prod_{\substack{x \in M^\vee \cap U^\perp \\ \operatorname{mod} M\cap U \\ Q(x)\neq 0}} \left(\frac{\vartheta(-(x,w),\tau_1)}{\eta(\tau_1)} e((x,w) - \frac{1}{2}(x_U, w))^{(x,e_1')} \right)^{c(-Q(x))(x)/2},$$

(c)

$$\prod_{\substack{x \in M^\vee \cap U/M\cap U \\ x \neq 0}} \left(\frac{\vartheta(-(x,w),\tau_1)}{\eta(\tau_1)} e(\frac{1}{2}(x,w))^{(x,e_1')} \right)^{c(0)(x)/2},$$

(d)

$$\kappa\, \eta(\tau_1)^{c(0)(0)}\, q_2^{I_0},$$

where $\vartheta(z,\tau)$ is the Jacobi theta function and

$$I_0 = -\sum_m \sum_{\substack{x \in M^\vee \cap U^\perp \\ \operatorname{mod} M\cap U}} c(-m)(x)\, \sigma_1(m - Q(x)).$$

Here $q_2 = e(\tau_2')$ and $\sigma_1(n)$ is the sum of the positive divisors of n if $n > 0$, $\sigma_1(0) = -1/24$, and $\sigma_1(n) = 0$ if $n < 0$. Finally, κ is a scalar of absolute value 1.

The quantity q_2 only appears in factors (a) and (d), and the infinite product in (a) converges absolutely in the given region and goes to 1 as v_2' goes to infinity, i.e., as q_2 goes to zero. In fact, in a smooth toroidal desingularization of a neighborhood of the boundary component of X_Γ^{BB} defined by $\mathcal{C}(U)$, the compactifying divisor $\mathcal{B}(U)$ is a Kuga-Sato variety cut out locally by the equation $q_2 = 0$. Thus, $\Psi(F)$ extends to this desingularization and I_0 is its order of vanishing along $\mathcal{B}(U)$. The value of $q_2^{-I_0}\Psi(F)$ on $\mathcal{B}(U)$ is the product of (b), (c), and (d) without the q_2 factor. It is a theta function on the Kuga-Sato variety of a type considered by Looijenga [8] and gives the first Fourier-Jacobi coefficient of $\Psi(F)$. Other Fourier-Jacobi coefficients can be computed by expanding (a).

Examples of product formulas of this type occur in Borcherds [1], and in many papers of Gritsenko [4], Gritenko-Nikulin [5],[6],[7], and others [3]. Our result shows that they arise for all Borcherds forms and a uniform proof is given.

The proof is analogous to that of [1] and is based on a computation of the Fourier expansion of the regularized theta lift of F along the unipotent radical of the parabolic subgroup G_U of G stabilizing U. The classical modular forms $\vartheta(z,\tau)$ and $\eta(\tau)$ arise via the first and second Kronecker limit formulas, [10], which are encountered along the way.

The product formula of the Theorem is essentially simpler than that of Borcherds; for example, no choice of Weyl chamber or determination of Weyl vector is involved. This is due, on the one hand, to the fact that the singularities of $\Psi(F)$ near the boundary component $\mathcal{C}(U)$ are accounted for by the finite product of theta functions in (b) and hence do not otherwise disturb convergence. On the other hand, the geometry of the desingularization is quite simple in a neighborhood of $\mathcal{B}(U)$, whereas the desingularization of a 0-dimensional boundary component involves a choice of rational polyhedral cones, etc., [9].

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Distribution of mass of holomorphic cusp forms

Valentin Blomer

(joint work with Rizwanur Khan and Matthew Young)

Let $f \in S_k$ be an L^2-normalized cusp form of even weight k for the modular group $\Gamma = SL_2(\mathbb{Z})$. A basic question is to understand the size of $F(z) = y^{k/2}f(z)$ and the distribution of its mass on $\Gamma\backslash\mathbb{H}$ as k becomes large. This can be made quantitative in various ways, e.g. by bounding the L^p-norm of F for $2 < p \leq \infty$. A first guess might be that the mass of F should be nicely distributed on $\Gamma\backslash\mathbb{H}$ such that F has no essential peaks, but one sees quickly some limitations to equidistribution:

As the dimension $\dim S_k \sim \mathrm{vol}(\Gamma\backslash\mathbb{H})k/(4\pi)$ is large, it is reasonable to restrict to *Hecke eigenforms* which enjoy a multiplicity one property. Next, the exceptional behaviour of Whittaker functions produces bumps of F high in the cusp. Writing the Fourier expansion of the Hecke eigenform F as

$$\sum_n \frac{\lambda(n)}{n^{1/2}} e(nx)W_k(4\pi ny), \quad W_k(y) = y^{k/2}e^{-y/2}\Gamma(k)^{-1/2}$$

(so that with the convention $\lambda(1) = 1$ the function is roughly L^2-normalized), we see that

$$\|F\|_\infty \geq \Big|\int_0^1 F(z)e(-x)dx\Big| = |W_k(4\pi y)| \asymp k^{1/4}, \quad y = k/(4\pi).$$

This argument works in great generality (for instance, one can similarly show for certain Siegel cusp forms in $S_k(\mathrm{Sp}_{2n}(\mathbb{Z}))$ that $\|\det(\cdot)^{k/2}f\|_\infty \gg k^{(n^2+n)/8}$).

On the other hand, the Fourier expansion implies $\|F\|_\infty \ll k^{1/4+\varepsilon}$, so that by interpolation

$$\|F\|_p \ll k^{1/4-1/(2p)+\varepsilon}. \tag{1}$$

This can be viewed as the trivial bound.

In this talk the main focus is on the 4-norm which features an interesting interplay with L-functions. Let B_k denote a Hecke eigenbasis of S_k. By a triple product period formula ([8, 4]) we have

$$\|F\|_4^4 = \sum_{g\in B_{2k}} |\langle F^2, G\rangle|^2 = \frac{\pi^3}{2(2k-1)L(1,\mathrm{sym}^2 f)^2} \sum_{g\in B_{2k}} \frac{L(1/2,g)L(1/2,\mathrm{sym}^2 f\times g)}{L(1,\mathrm{sym}^2 g)}.$$

It is important to note that all L-values here are non-negative [5, 6], and the L-values at 1 can be bounded conveniently from above and below by $k^{o(1)}$ [2]. The first result is the following mean value estimate for the degree 6 L-function [1]:

Theorem 1. *For a Hecke eigenform $f \in S_k$ we have*

$$\sum_{g\in B_{2k}} L(1/2,g)L(1/2,\mathrm{sym}^2 f\times g) \ll k^{1+\varepsilon}.$$

Using bounds for $L(1/2,g)$ [7], we obtain the following improvement of (1) in the case $p = 4$.

Corollary 1. *For a Hecke eigenform $f \in S_k$ we have $\|F\|_4^4 \ll k^{1/3+\varepsilon}$.*

This should be seen as a Weyl-type bound for the 4-norm, comparable in strength to Weyl's subconvexity estimate for the Riemann zeta-function. One can also obtain bounds for the following geodesic restriction problem:

Corollary 2. *For a Hecke eigenform $f \in S_k$ we have $\int_0^\infty |F(iy)|^2 \frac{dy}{y} \ll k^{1/4+\varepsilon}$.*

This is the first nontrivial geodesic restriction result for holomorphic forms of large weight; the trivial bound here (obtainable in a variety of ways) is $k^{1/2+\varepsilon}$.

Finally let $g \in S_{2k}$ with k odd, and let $F_g \in S_{k+1}(\mathrm{Sp}_4(\mathbb{Z}))$ be the (L^2-normalized) associated Saito-Kurokawa lift. In the following we consider its restriction $F_g|_\Delta$ to the diagonal $(\Gamma\backslash\mathbb{H}) \times (\Gamma\backslash\mathbb{H})$. If all spaces are equipped with probability measures, then a formula of Ichino [3] implies

$$\|F_g|_\Delta\|_2^2 = \frac{\pi^2}{15\, L(3/2, g) L(1, \mathrm{sym}^2 g)} \cdot \frac{12}{k} \sum_{f \in B_{k+1}} L\big(1/2, \mathrm{sym}^2 f \times g\big).$$

The method of proof of Theorem 1 gives

Corollary 3. *We have*

$$\frac{12}{2k-1} \sum_{g \in B_{2k}} \|F_g|_\Delta\|_2^2 = 2 + O(k^\eta)$$

for some $\eta > 0$.

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The Shimura-Waldspurger correspondence for Mp_{2n}

Wee-Teck Gan

In this talk, we revisit the Shimura-Waldspurger (SW) correspondence which gives a precise description of the automorphic discrete spectrum of the metaplectic double cover Mp_2 of $\mathrm{SL}_2 = \mathrm{Sp}_2$, and formulate a conjectural extension to general Mp_{2n}. Since the treatment is adelic, one first has a local analog.

1. **Local SW correspondence**

Let F be a nonarchimedean local field. Let W be the $2n$-dimensional symplectic F-vector space, and let V^+ and V^- be the two $2n+1$-dimensional quadratic spaces with trivial discriminant, with V^+ split. The following was shown in [3].

Fix a nontrivial additive character ψ of F. The theta correspondence with respect to ψ gives a bijection

$$\mathrm{Irr}_\epsilon(\mathrm{Mp}W)) \longleftrightarrow \mathrm{Irr}(\mathrm{SO}(V^+)) \sqcup \mathrm{Irr}(\mathrm{SO}(V^-)),$$

where we consider genuine representations of $\mathrm{Mp}(W)$ on the LHS.

When F is archimedean, the analogous theorem was obtained by Adams-Barbasch [1]. Further, the above result was obtained in [3] under the hypothesis that the residual characteristic of F is $p \neq 2$, as the Howe duality conjecture was used. During the duration of the Oberwolfach workshop, Takeda and I have been able to show the Howe duality conjecture for (almost) equal rank dual pairs (see [4]) so that the $p \neq 2$ hypothesis is no longer necessary.

2. **Global SW correspondence**

Now assume that we are working over a number field k. It is natural to attempt to use the global theta correspondence to obtain a precise description of the automorphic discrete spectrum of $\mathrm{Mp}(W_{\mathbb{A}})$. For readers familiar with Waldspurger's work [5, 6] in the case when $\dim W = 2$, it will be apparent that there is an obstruction to this approach: the global theta lift $\Theta(\pi)$ of a cuspidal representation π of $\mathrm{Mp}(W_{\mathbb{A}})$ or $\mathrm{SO}(V_{\mathbb{A}})$ may be 0 and it is nonzero precisely when $L(1/2, \pi) \neq 0$. This obstruction already occurs when $\dim W = 2$, and was not easy to overcome. Waldspurger had initially alluded to results of Flicker proved by the trace formula. Nowadays, one could appeal to a result of Friedberg-Hoffstein, stating that if $\epsilon(1/2, \pi) = 1$, then there exists a quadratic Hecke character χ such that $L(1/2, \pi \times \chi) \neq 0$. When $\dim W > 2$, however, the analogous analytic result does not seem to be forthcoming and may be very hard. We are going to suggest a new approach in the higher rank case, but before that, we would like to describe the analog of Arthur's conjecture for Mp_{2n}.

3. **Arthur's conjecture for** Mp_{2n}

For a fixed additive automorphic character ψ, one expects that

$$L^2_{disc} = \bigoplus_{\Psi} L^2_{\Psi,\psi} \quad \text{where } \Psi = \bigoplus_i \Psi_i = \bigoplus_i \Pi_i \otimes S_{r_i}$$

is a global discrete A-parameter for Mp_{2n}; it is also an A-parameter for SO_{2n+1}. Here, S_{r_i} is the r_i-dimensional representation of $\mathrm{SL}_2(\mathbb{C})$ and Π_i is a cuspidal representation of GL_{n_i} such that

$$\begin{cases} L(s, \Pi_i, \wedge^2) \text{ has a pole at } s = 1, \text{ if } r_i \text{ is odd;} \\ L(s, \Pi_i, \mathrm{Sym}^2) \text{ has a pole at } s = 1, \text{ if } r_i \text{ is even.} \end{cases}$$

Moreover, we have $\sum_i n_i r_i = 2n$ and the summands Ψ_i are mutually distinct.

For a given Ψ, one inherits the following additional data:

- for each v, one inherits a local A-parameter
$$\Psi_v = \bigoplus_i \Psi_{i,v} = \bigoplus_i \Pi_{i,v} \otimes S_{r_i}.$$
By the LLC for GL_N, we may regard each $\Pi_{i,v}$ as an n_i-dimensional representation of the Weil-Deligne group WD_{k_v}.
- one has a "global component group" $A_\Psi = \bigoplus_i \mathbb{Z}/2\mathbb{Z} \cdot a_i$, which is a $\mathbb{Z}/2\mathbb{Z}$-vector space equipped with a distinguished basis indexed by the Ψ_i's. Similarly, for each v, we have the local component group A_{Ψ_v} which is defined as the component group of the centralizer of the image of Ψ_v, thought of as a representation of $WD_{k_v} \times \mathrm{SL}_2(\mathbb{C})$. There is a natural diagonal map $\Delta : A_\Psi \longrightarrow \prod_v A_{\Psi_v}$.
- For each v, one has a local A-packet associated to Ψ_v and ψ_v:
$$\Pi_{\Psi_v,\psi_v} = \{\sigma_{\eta_v} : \eta_v \in \mathrm{Irr}(A_{\Psi_v})\},$$
consisting of unitary representations (possibly zero, possibly reducible) of $\mathrm{Mp}_{2n}(k_v)$ indexed by the set of irreducible characters of A_{Ψ_v}. On taking tensor products of these local A-packets, we obtain a global A-packet
$$A_{\Psi,\psi} = \{\sigma_\eta : \eta = \otimes_v \eta_v \in \mathrm{Irr}(\prod_v A_{\Psi_v})\}$$
consisting of abstract unitary representations $\sigma_\eta = \otimes_v \sigma_{\eta_v}$ of $\mathrm{Mp}_{2n}(\mathbb{A})$ indexed by the irreducible characters $\eta = \otimes_v \eta_v$ of $\prod_v A_{\Psi_v}$.
- Arthur has attached to Ψ a quadratic character (possibly trivial) ϵ_Ψ of A_Ψ, This character plays an important role in the multiplicity formula for the automorphic discrete spectrum of SO_{2n+1}. For Mp_{2n}, we need to define a modification of ϵ_Ψ. Set
$$\eta_\Psi(a_i) = \begin{cases} \epsilon(1/2, \Pi_i), \text{ if } L(s, \Pi_i, \wedge^2) \text{ has a pole at } s = 1; \\ 1, \text{ if } L(s, \Pi_i, \mathrm{Sym}^2) \text{ has a pole at } s = 1. \end{cases}$$
The modified quadratic character of A_Ψ in the metaplectic case is $\tilde{\epsilon}_\Psi = \epsilon_\Psi \cdot \eta_\Psi$.

We can now state the conjecture.

Arthur Conjecture for Mp_{2n}: For each such Ψ,
$$L^2_{\Psi,\psi} \cong \bigoplus_{\eta \in \mathrm{Irr}(\prod_v A_{\Psi_v}) : \Delta^*(\eta) = \tilde{\epsilon}_\Psi} \sigma_\eta$$

4. A new approach

In an ongoing work, we are developing a new approach for the Arthur conjecture described above. Namely, by results of Arthur [2], one now has a classification of the automorphic discrete spectrum of SO_{2r+1} for all r. Instead of trying to construct the automorphic discrete spectrum of Mp_{2n} by theta lifting from SO_{2n+1}, one could attempt to use theta liftings from SO_{2r+1} for $r \geq n$. Let us illustrate this in the case when $\dim W = 2$.

Let π be a cuspidal representation of $\mathrm{PGL}_2(\mathbb{A}) = \mathrm{SO}(V_{\mathbb{A}}^+)$. Then π gives rise to a near equivalence class in the automorphic discrete spectrum of Mp_2. If $L(1/2, \pi) \neq 0$, this near equivalence class can be exhausted by the global theta lifts of π and its Jacquet-Langlands transfer to inner forms of PGL_2. When $L(1/2, \pi) = 0$, we consider the A-parameter $\psi = \pi \otimes S_1 \oplus 1 \otimes S_2$ for SO_5. This is a so-called Saito-Kurokawa A-parameter. By Arthur, ψ indexes a near equivalence class in the automorphic discrete spectrum of SO_5. Piatetski-Shapiro gave a construction of the Saito-Kurokawa representations by theta lifting from Mp_2, using Waldspurger's results as initial data. However, *one can turn the table around.*

Namely, taking the Saito-Kurokawa near equivalence classes as given by Arthur, one can consider their theta lift back to Mp_2. By the Rallis inner product formula, such a theta lift is nonzero if the partial L-function

$$L^S(s, \Phi_\psi) = L^S(s, \pi) \cdot \zeta(s + \frac{1}{2}) \cdot \zeta(s - \frac{1}{2})$$

has a pole at $s = 3/2$, or equivalently if $L^S(3/2, \pi) \neq 0$. Now this is certainly much easier to ensure than the nonvanishing at $s = 1/2$! In this way, one can construct the desired near equivalence class for Mp_2 associated to π and by studying the local theta correspondence in detail, one can recover Waldspurger's results from 30 years ago.

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Special values of *L*-functions and congruences for automorphic forms

Anantharam Raghuram
(joint work with Baskar Balasubramanyam)

Hida proved the following beautiful theorem in [4]: suppose f is a primitive weight k, level N, holomorphic cusp form on the upper half-plane then the value at $s = 1$ of the degree 3 adjoint L-function $L(1, \mathrm{Ad}^0, f)$ is essentially the Petersson norm (f, f) of f up to an algebraic number; let's denote this algebraic number as $L^{\mathrm{alg}}(1, \mathrm{Ad}^0, f)$. Furthermore, if p is a large enough rational prime that divides

$L^{\text{alg}}(1, \text{Ad}^0, f)$, then p is a congruence prime for f, i.e., there is another primitive weight k, level N, cusp form g such $f \equiv g \pmod{p}$, i.e., $a_n(f) \equiv a_n(g) \pmod{p}$ for all $n \geq 1$.

Such a result has since been generalized to various GL(2)-contexts:

(1) Eknath Ghate [3] proved a version of this theorem for Hilbert modular forms of parallel weight.
(2) Mladen Dimitrov [2] generalized it further for Hilbert modular forms of any algebraic weight.
(3) Eric Urban [6] had separately generalized Hida's theorem to the context of GL_2 over an imaginary quadratic field; in this situation he observes that $L(1, \text{Ad}^0, f)$ is a non-critical value.
(4) Namikawa [5] has very recently generalized this result to GL_2 over any number field.

In [1] we generalize Hida's theorem above to the context of cohomological cuspidal automorphic representation of GL_n over any number field. This also generalizes all the above mentioned works. For first main result is:

Theorem 1. *Let π be a cohomological cuspidal automorphic representation of GL_n over a number field F. Let ε be a permissible signature for π. Define:*

$$L^{\text{alg}}(1, \text{Ad}^0, \pi, \varepsilon) \ := \ \frac{L(1, \text{Ad}^0, \pi)}{\Omega_F \cdot \Omega_{\text{ram}}(\pi) \cdot p_\infty(\pi) \cdot p^\varepsilon(\pi) \cdot q^{\tilde{\varepsilon}}(\tilde{\pi})}.$$

(Here Ω_F is a nonzero constant that depends only on F; $\Omega_{\text{ram}}(\pi)$ is a nonzero constant that depends only on the ramified local representations of π; $p_\infty(\pi)$ is a nonzero constant that depends only on the representation at infinity; $p^\varepsilon(\pi)$ (resp., $q^{\tilde{\varepsilon}}(\tilde{\pi})$) is a period defined by comparing a rational structure on Whittaker model and a rational structure on a cohomological model in bottom (resp., top) degree cuspidal cohomology.) For all $\sigma \in \text{Aut}(\mathbb{C})$ we have

$$\sigma(L^{\text{alg}}(1, \text{Ad}^0, \pi, \varepsilon)) \ = \ L^{\text{alg}}(1, \text{Ad}^0, {}^\sigma\pi, {}^\sigma\varepsilon).$$

In particular, $L^{\text{alg}}(1, \text{Ad}^0, \pi, \varepsilon) \in \mathbb{Q}(\pi)$ which is a number field.

Our second main result is technical, but roughly speaking it says that:

Theorem 2. *If p is a prime such that $p|L^{\text{alg}}(1, \text{Ad}^0, \pi, \varepsilon)$, and suppose p is outside a finite set of exceptions, then p is a congruence prime for π.*

The meaning of this theorem is that there is another cohomological automorphic representation π', which contributes to inner cohomology, such that

$$\pi \equiv \pi' \pmod{p}.$$

If the highest weight on GL_n, with respect to which we take cohomology, happens to be a regular weight, then we are assured that π' is also cuspidal. Note that the congruence of two automorphic representations is defined in terms of their Satake parameters: suppose $\alpha_1, \dots, \alpha_n$ (resp, $\alpha'_1, \dots, \alpha'_n$) are the Satake parameters of

π and π' at some unramified prime l, then to say that π and π' are congruent modulo p, we require:

$$\sum_{i_1<i_2\cdots<i_j} \alpha_{i_1}\cdots\alpha_{i_j} \equiv \sum_{i_1<i_2\cdots<i_j} \alpha'_{i_1}\cdots\alpha'_{i_j} \pmod p$$

for all unramified l, and for all $1 \le j \le n$.

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Construction of liftings to vector valued Siegel modular forms

Tomoyoshi Ibukiyama

Partly motivated by conjectures on Shimura type correspondence between Siegel modular forms of integral weight and half-integral weight, we construct two kinds of liftings from pairs of elliptic modular forms, one is to vector valued Siegel modular forms of integral weight of odd degree, and the other to vector valued Siegel modular forms of half-integral weight of even degree, as well as the description of L functions. We explain the motivation part first and then report on the liftings. We denote by H_n the Siegel upper half space of degree n, by Γ_n the Siegel modular group $Sp_n(\mathbb{Z}) \subset M_{2n}(\mathbb{Z})$ of degree n. We define the automorphy factor of weight 1/2 for the group

$$\Gamma_0^{(n)}(4) = \left\{ g = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \Gamma_n; C \equiv 0 \bmod 4 \right\}$$

by $\theta(\gamma Z)/\gamma(Z)$ for $\gamma \in \Gamma_0^{(n)}(4)$ and $Z \in H_n$, where $\theta(Z) = \sum_{p\in\mathbb{Z}^n} e^{2\pi i\, {}^t pZp}$. We define a character ψ of $\Gamma_0^{(n)}(4)$ by $\psi(\gamma) = \left(\frac{-4}{\det(D)}\right)$ where $(-4/*)$ is the Kronecker character modulo 4. Let (Sym_j, V_j) be the j-th symmetric tensor representation of $GL_n(\mathbb{C})$ and χ a character of $\Gamma_0^{(n)}(4)$. For $k \in \mathbb{Z}_{>0}$, a holomorphic function $F : H_n \to V_j$ is a vector valued Siegel modular form of weight $\det^k Sym(j)$ if it satisfies $F(\gamma Z) = \det(CZ+D)^k Sym_j(CZ+D)F(Z)$ for any $\gamma \in \Gamma_n$, and of weight $\det^{k-1/2} Sym(j)$ of $\Gamma_0^{(n)}(4)$ with character χ if $F(\gamma Z) = \chi(\gamma)(\theta(\gamma Z)/\theta(Z))^{2k-1} Sym_j(CZ+D)F(Z)$ for any $\gamma \in \Gamma_0^{(n)}(4)$, and is a Siegel cusp form if it vanishes on the boundary. We denote by $S_{k,j}(\Gamma_n)$ and $S_{k-1/2,j}(\Gamma_0^{(n)}(4),\chi)$

the spaces of such cusp forms, omitting χ when χ is trivial. To extract the level one part of $S_{k-1/2,j}(\Gamma_0^{(n)}(4),\psi^l)$, the *plus subspace* $S^+_{k-1/2,j}(\Gamma_0^{(n)}(4),\psi^l)$ is defined. For $F=\sum_T a(T)exp(2\pi i\mathrm{Tr}(TZ))\in S_{k-1/2,j}(\Gamma_0^{(n)}(4),\psi^l)$ ($l=0$ or 1), F belongs to the plus subspace if $a(T)=0$ unless $T-(-1)^{k+l-1}(\mu_i\mu_j)_{1\le i,j\le n}$ is 4 times a half integral matrix for some integers μ_i with $1\le i\le n$. By virtue of Tsushima's conjectural dimension formulas (which we have proved in half of the cases by some structure theorem of vector valued Jacobi forms [11]), we have

Theorem 1. *For integers k, j with $k\ge 3$ and j even, assuming some standard vanishing theorem of cohomology, we have*

$\dim S^+_{k-1/2,j}(\Gamma_0^{(2)}(4),\psi)=\dim S_{j+3,2k-6}(\Gamma_2)$.

$\dim S^+_{k-1/2,j}(\Gamma_0^{(2)}(4))=\dim S^+_{k-1/2,j}(\Gamma_0^{(2)}(4),\psi)+\dim S_{2k-4}(\Gamma_1)\times S_{2k+2j-2}(\Gamma_1)$.

Based on these dimensional relations and a lot of numerical evidences, we propose the following conjectures. Here we note that (1) below has been already given in [8] and (2) for $j=0$ in [6], but (2) for $j>0$ and (3) are new.

Conjecture ([6], [8], [9])**.** *(1) We have a Hecke equivariant isomorphism*

$$S^+_{k-1/2,j}(\Gamma_0^{(2)}(4),\psi)\cong S_{j+3,2k-6}(\Gamma_2).$$

(2) There is an injective lifting $L: S_{2k-4}(\Gamma_1)\times S_{2k+2j-2}(\Gamma_1)\to S^+_{k-1/2,j}(\Gamma_0^{(2)}(4))$.

(3) Denoting by $S^{+,0}_{k-1/2,j}(\Gamma_0^{(2)}(4))$ the orthogonal complement of the image of the above conjectural L in $S^+_{k-1/2,j}(\Gamma_0^{(2)}(4))$, we have a Hecke equivariant isomorphism

$$S^{+,0}_{k-1/2,j}(\Gamma_0^{(2)}(4))\cong S_{j+3,2k-6}(\Gamma_2).$$

These conjectures have a good application to Harder's conjecture on congruences, in particular the last one (See [8], [9]).

Now, we construct two kinds of general liftings, including the above L. First we explain the differential operator which is crucial for the construction for general j. We denote by $W(F)$ the restriction of functions $F(Z)$ of $Z=\begin{pmatrix}\tau & z\\ {}^tz & \omega\end{pmatrix}\in H_m$ to $(\tau,\omega)\in H_{m-1}\times H_1$ (i.e. to $z=0$). For $g_1=\begin{pmatrix}A_1 & B_1\\ C_1 & D_1\end{pmatrix}\in Sp_{m-1}(\mathbb{R})$ and $g_2=\begin{pmatrix}a & b\\ c & d\end{pmatrix}\in SL_2(\mathbb{R})=Sp_1(\mathbb{R})$, we write $\iota(g_1,g_2)=\begin{pmatrix}A & B\\ C & D\end{pmatrix}\in Sp_m(\mathbb{R})$ for the natural diagonal embedding ι. For any integer $j\ge 0$ and any $\kappa\in(1/2)\mathbb{Z}$, there exists a holomorphic linear V_j-valued differential operator $\mathbb{D}_{\kappa,j}$ of constant coefficients (unique up to constants) which satisfies the following condition ([7]).

Condition. *Notations being as above, for any holomorphic functions $F: H_m\to\mathbb{C}$, any $g_1\in Sp_{m-1}(\mathbb{R})$, and any $g_2\in SL_2(\mathbb{R})$, we have*

$$\begin{aligned}&W\left[\mathbb{D}_{\kappa,j}\left(\det(CZ+D)^{-\kappa}F(\iota(g_1,g_2)Z)\right)\right]\\&\quad=\det(C_1\tau+D_1)^{-\kappa}Sym_j(C_1\tau+D_1)^{-1}(c\omega+d)^{-\kappa-j}W(\mathbb{D}_{\kappa,j}F)(g_1\tau,g_2\omega).\end{aligned}$$

Here the branch of the κ-th power is fixed consistently if $\kappa \notin \mathbb{Z}$.

(1) **The case when the target is of integral weight.** Assume that k is even. Let $f \in S_{2k-2n}(\Gamma_1)$ be a Hecke eigenform. T. Ikeda constructed a lifting from f to $I(f) \in S_k(\Gamma_{2n})$. For any Hecke eigenform $g \in S_{k+j}(\Gamma_1)$, we define

$$\mathcal{F}_{f,g}(\tau) = \int_{\Gamma_1 \backslash H_1} W(\mathbb{D}_{k,j} I(f))(\tau,\omega) g(\omega) d\omega \quad \text{for the Petersson measure } d\omega.$$

Theorem 2. *We have $\mathcal{F}_{f,g} \in S_{k,j}(\Gamma_{2n-1})$. If $\mathcal{F}_{f,g} \neq 0$, then this is a Hecke eigenform and its L functions are explicitly given (though details are omitted here). In partirular when $n = 2$ (i.e. a lift to degree 3), the spinor L function is given by*

$$L(s, \mathcal{F}_{f,g}, Sp) = L(s-k+2, g) l(s-k+3, g) L(s, f \otimes g).$$

When $j = 0$, this is nothing but the Ikeda-Miyawaki lift by Ikeda, the results for the spinor L being supplied by Heim ($n = 3$) and Hayashida (general n). We also note that the case $n = 2$ is a realization of a part of the conjectures given in [1].

(2) **The case when the target is of half-integral weight.** Again we assume that k is even, $f \in S_{2k-2n}(\Gamma_1)$ a Hecke eigenform, and take the Ikeda lift $I(f) \in S_k(\Gamma_{2n})$. Let Φ_1 be the first Fourier Jacobi coefficient of $I(f)$ w.r.t. the last component of H_{2n}. Then Φ_1 corresponds with an element $H \in S^+_{k-1/2}(\Gamma_0^{(2n-1)}(4))$. We define $\mathbb{D}_{\kappa,j}$ and W to the partition $2n-1 = (2n-2)+1$ and $\kappa = k - 1/2$ (so $\tau \in H_{2n-2}$, $\omega \in H_1$). For any Hecke eigenform $h \in S^+_{k+j-1/2}(\Gamma_0^{(1)}(4))$, we define

$$\mathcal{H}_{f,h}(\tau) = \int_{\Gamma_0^{(1)}(4) \backslash H_1} W(\mathbb{D}_{k-1/2,j} H)(\tau,\omega) h(\omega) d\omega.$$

We denote by g the Hecke eigenform in $S_{2k+2j-2}(\Gamma_1)$ corresponding to h by the usual Shimura correspondence.

Theorem 3. *We have $\mathcal{H}_{f,h} \in S^+_{k-1/2,j}(\Gamma_0^{(2n-2)}(4))$. If $\mathcal{H}_{f,h} \neq 0$, this is a Hecke eigenform and its L function in the sense of Zhuravlev is given explicitly in general (though omitted here). In particular when $n = 2$, we have*

$$L(s, \mathcal{H}_{f,h}) = L(s, g) L(s-j-1, f).$$

When $j = 0$, the proofs of Theorem 2 for the spinor L function and Theorem 3 were given by S. Hayashida, using his characterization of Fourier-Jacobi coefficients of $I(f)$ and H, which is a natural generalization of the Maass relation for Saito-Kurokawa liftings (See [2], [3], [4]. [5].) The case $j > 0$ can be similarly proved by using the properties of $\mathbb{D}_{\kappa,j}$.

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Meromorphic cycle integrals

Kathrin Bringmann
(joint work with Ben Kane)

This talk generalizes classical sums of quadratic forms, which are cusp forms and which played a key role in connection with the Shimura/Shintani lift, to the meromorphic setting. This is work in progress.

Let me first recall the classical situation for cusp forms. Let $\mathcal{Q}_D$ denote the set of integral/binary quadratic forms with discriminant D. For $D > 0$, we then define for $k > 1$ the following quadratic form Poincaré series ($\tau \in \mathbb{H}$)

$$f_{k,D}(\tau) := \frac{D^{k-\frac{1}{2}}}{\binom{2k-2}{k-1}\pi} \sum_{Q \in \mathcal{Q}_D} Q(\tau, 1)^{-k}.$$

This function was introduced by Zagier in connection with the Doi-Naganuma lift (between modular forms and Hilbert modular forms) and is a cusp forms of weight $2k$ for $\mathrm{SL}_2(\mathbb{Z})$. It arises from a Hilbert modular form by restricting to the diagonal. Kohnen and Zagier showed that the $f_{k,D}$ are the Fourier coefficients of holomorphic kernel functions for the Shimura resp. Shintani lifts between half-integral and integral weight cusp forms. More precisely, for $\tau, z \in \mathbb{H}$, define

$$\Omega(\tau, z) := \sum_{0 < D \equiv 0,1 \pmod 4} f_{k,D}(\tau)\, e^{2\pi i D z}.$$

Then Ω is a modular form of weight $2k$ in the variable τ and weight $k + \frac{1}{2}$ in the variable z. Furthermore, integrating Ω against a cusp form f of weight $2k$ (resp. $k + \frac{1}{2}$) with respect to the first (resp. second) variable yields the Shintani (resp. Shimura) lift.

The functions $f_{k,D}$ also give important examples of modular forms with rational periods. These were studied by Kohnen and Zagier and have appeared more recently in work of Duke, Imamoglu, and Toth where they were related to the error to modularity of certain holomorphic functions which are defined via cycle integrals.

The quadratic form Poincaré series can also be decomposed into restricted sums where one only sums over equivalence classes of quadratic forms. To be more precise, for $\mathcal{A}$ an equivalence class of quadratic forms with discriminant D define

$$f_{k,D,\mathcal{A}}(\tau) := \frac{D^{k-\frac{1}{2}}}{\binom{2k-2}{k-1}\pi} \sum_{Q\in\mathcal{A}} Q(\tau,1)^{-k}.$$

Kramer showed that the functions $f_{k,D,\mathcal{A}}$ generate S_{2k} as D runs through all discriminants and $\mathcal{A}$ over all classes of forms with discriminant D.

The $f_{k,D,\mathcal{A}}$ are of big importance as integrating against them yields cycle integrals. To be more precise, for $f \in S_{2k}$, define

$$r_Q(f) := \int_{\Gamma_Q\backslash C_Q} f(z)Q(z,1)^{k-1}dz,$$

where Γ_Q is the subgroup of $\mathrm{SL}_2(\mathbb{Z})$ fixing Q. Moreover C_Q is given by

$$a|\tau|^2 + b\mathrm{Re}(\tau) + c = 0.$$

Then

$$\left\langle f, f_{k,D,[Q]}\right\rangle \doteq r_Q(f).$$

The functions $f_{k,D}$ also occur as images of a certain theta lift. To describe this, we write $\tau = u + iv \in \mathbb{H}$, $z = x + iy \in \mathbb{H}$, and denote, for $Q = [a,b,c] \in \mathcal{Q}_D$,

$$Q_\tau := \frac{1}{v}\left(a|\tau|^2 + bu + c\right).$$

Shintani's theta function projected into Kohnen's plus space is defined as

$$\Theta(\tau,z) := v^{-2k} y^{\frac{1}{2}} \sum_{D\in\mathbb{Z}} \sum_{Q\in\mathcal{Q}_D} Q(\tau,1)^k e^{-4\pi Q_\tau^2 y} e^{2\pi i D z}.$$

The function $\Theta(-\overline{\tau},z)$ transforms like a modular form of weight $k+\frac{1}{2}$ in z and weight $2k$ in τ. Integrating the Dth weight $k+\frac{1}{2}$ (cuspidal) Poincaré series in Kohnen's plus space, $P_{k+1/2,D}$, against Θ yields $f_{k,D}$. To be more precise, we define the theta lift

$$\Phi(H)(\tau) := \langle H, \Theta(\tau,\cdot)\rangle$$

for functions H that are modular of weight $k+1/2$ and satisfy an appropriate growth condition so that the integral converges absolutely. Then we have

$$\Phi\left(P_{k+\frac{1}{2},D}\right) \doteq f_{k,D}.$$

Let me now come to the functions of interest for this talk, meromorphic quadratic form Poincaré series. Define for $-D<0$ a discriminant

$$f_{k,-D}(\tau) := D^{\frac{k}{2}} \sum_{Q\in\mathcal{Q}_{-D}} Q(\tau,1)^{-k}.$$

This function now has poles at the roots of Q. Towards ∞ it grows like a cusp form. Following Petersson, we call such functions *meromorphic cusp forms.* It would be interesting to see whether this function comes from restricting Bianci modular forms. Also it would be interesting to investigate whether one can build some kind of generating function out of the $f_{k,-D}$.

Theorem 1 (B. - Kane)**.** *We have*

$$\Phi\left(P_{k+\frac{1}{2},-D}\right) = f_{k,-D}$$

where $P_{k+\frac{1}{2},-D}$ is the $-D$th Poincaré series in Kohnen's plus space which basically has principal part q^{-D}.

Note that the Petersson scalar product has to be regularized.

Let me now come to the question of integrating against the $f_{k,-D}$s. Again I define the associated form restricted to quadratic form classes. For $D>0$, write

$$f_{k,-D,\mathcal{A}}(\tau) := D^{\frac{k}{2}} \sum_{Q\in\mathcal{A}} Q(\tau,1)^{-k},$$

where $\mathcal{A}$ is a class of quadratic forms with discriminant $-D$. This function is again a meromorphic cusp form.

The question is what happens if you integral meromorphic cusp forms against $f_{k,-D,[Q]}$. Since the naive inner product diverges, we must regularize these integrals and denote the associated inner products by $\langle\cdot,\cdot\rangle_{\mathrm{mer}}$.

Theorem 2. *If f is a weight $2k$ meromorphic cusp form and $k>3$, then*

$$\left\langle f, f_{k,-D,[Q]}\right\rangle_{\mathrm{mer}} \doteq \sum_{\substack{z\in\mathbb{H}\\ z\neq z_Q}} \mathrm{Res}_{\tau=z}\left(f(\tau)Q(\tau,1)^{k-1}\right)\int_0^{\operatorname{arctanh}\left(\frac{\sqrt{D}}{Q_z}\right)} \sinh^{2k-2}(\theta)d\theta.$$

In particular $f_{k,-D,[Q]}$ is orthogonal to cusp forms.

In the special case that the poles of f are all simple, $\mathrm{Res}_{\tau=z}\left(f(\tau)Q(\tau,1)^{k-1}\right)$ has a particularly nice shape, leading to the following corollary.

Corollary 3. *If the poles of f modulo $\mathrm{SL}_2(\mathbb{Z})$ are at $z_1,\ldots,z_r$ and they are all simple, then*

$$\left\langle f, f_{k,-D,[\mathcal{A}]}\right\rangle_{\mathrm{mer}} \doteq \sum_{\ell=1}^{r} \mathrm{Res}_{\tau=z_\ell} f(\tau) \sum_{Q\in\mathcal{A}} Q\left(z_\ell,1\right)^{k-1} \int_0^{\operatorname{arctanh}\left(\frac{\sqrt{D}}{Q_{z_\ell}}\right)} \sinh^{2k-2}(\theta)d\theta.$$

These cycle integrals yield to new automorphic functions. Define

$$\mathcal{G}(z) := \sum_{Q\in\mathcal{A}} Q(z,1)^{k-1} \int_0^{\operatorname{arctanh}\left(\frac{\sqrt{D}}{Q_{z_\ell}}\right)} \sinh^{2k-2}(\theta)d\theta.$$

Theorem 4 (B. - Kane)**.** *The function $\mathcal{G}$ is a meromorphic harmonic Maass form of weight $2-2k$. To be more precise*

$$\left(\frac{1}{2\pi i}\frac{\partial}{\partial z}\right)^{2k-1}(\mathcal{G}) \doteq \xi_{2-2k}(\mathcal{G}) \doteq f_{k,-D,\mathcal{A}},$$

where $\xi_k := 2iy^k\overline{\frac{\partial}{\partial \overline{z}}}$.

It still remains to investigate modularity properties in the case of higher order poles.

Central critical L-values and Selmer groups for triple product motives

Yifeng Liu

In this talk, we provide new examples of the Bloch–Kato conjecture in the rank-0 case.

Let K be a number field. Consider a Chow motive (with rational coefficients) M over K equipped with a polarization $M \times M^\vee \to \mathbb{Q}(1)$ and of pure weight -1. Associated to M, there is an L-function $L(s, M)$ defined for s with $\Re s$ sufficiently large. For each prime p, we have the p-adic realization M_p, which is a finite-dimensional p-adic Galois representation of K. Denote by $\mathrm{H}^1_f(K, M_p)$ the Bloch–Kato Selmer group [1], which is a $\mathbb{Q}_p$-subspace of $\mathrm{H}^1(K, M_p)$.

Conjecture 1 (Bloch–Kato)**.** *Let the notation be as above. We have*

(1) *the L-function $L(s, M)$ has a meromorphic continuation to the entire complex plane and satisfies the functional equation*

$$L(s, M) = \epsilon(M)c(M)^{-s}L(-s, M)$$

for some root number $\epsilon(M) \in \{\pm 1\}$ and conductor $c(M) \in \mathbb{Z}_{>0}$;

(2) *for all primes p,*

$$\mathrm{ord}_{s=0}L(s, M) = \dim_{\mathbb{Q}_p}\mathrm{H}^1_f(K, M_p).$$

Now let F be a real quadratic field with the Galois involution θ. Consider a rational elliptic curve E of conductor N and another elliptic curve A over F. The F-motive $\mathrm{h}^1(A) \otimes \mathrm{h}^1(A^\theta)$ has a natural descent to a $\mathbb{Q}$-motive $\mathrm{As}\,\mathrm{h}^1(A)$, called the *Asai motive*. Put $M_{E,A} = \mathrm{h}^1(E) \otimes \mathrm{As}\,\mathrm{h}^1(A)(2)$. Then $M_{E,A}$ is canonically polarized of symplectic type, and has pure weight -1.

Theorem 1. *Let the notation be as above.*

(1) *Part (1) of the previous conjecture holds for $M_{E,A}$.*

(2) *Suppose that N is prime to both the conductor of A and the discriminant of F; neither E nor A has geometric complex multiplication; and if a prime $v \mid N$ is inert in F, then $v \| N$. If $L(0, M_{E,A})$ is non-vanishing, then*

$$\dim_{\mathbb{Q}_p}\mathrm{H}^1_f(\mathbb{Q}, (M_{E,A})_p) = 0$$

for all but finitely many p.

In the above theorem, part (1) is a consequence of the theory of triple product L-functions and the recent result of [3]; and part (2) is one the main theorems of [4]. Combining with the main theorem of [2], we have the following corollary to the previous theorem.

Corollary 2. *Let E_1 and E_2 be two rational elliptic curves of conductors N_1 and N_2, respectively. Suppose that neither E_1 nor E_2 has geometric complex multiplication; N_1 and N_2 are coprime; and E_1 has multiplicative reduction at least one finite place. Consider the motive $M = \mathrm{h}^1(E_1) \otimes \mathrm{Sym}^2\mathrm{h}^1(E_2)(2)$. If $L(0, M)$ is non-vanishing, then for all but finitely many primes p,*

$$\dim_{\mathbb{Q}_p} \mathrm{H}^1_f(\mathbb{Q}, M_p) = 0.$$

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Borcherds Products Everywhere Theorem

Valery Gritsenko

(joint work with Cris Poor and David Yuen)

This is a report on my joint results (see [10]) with Cris Poor and David Yuen about Borcherds Products on groups that are simultaneously orthogonal and symplectic, the paramodular groups Γ_t of degree two and the elementary divisors $(1, t)$. This work began as an attempt to make Siegel paramodular cusp forms that are simultaneously Borcherds Products and additive Jacobi lifts (or Gritsenko lifts for Γ_t constructed in [3]–[4]). On the face of it, this phenomenon may seem the least interesting type of a Borcherds product but it is the only known way to control the weight of constructed series of Borcherds product. Additionally, for computational purposes, a paramodular form that is both a Borcherds product and a Gritsenko lift is very useful; such a form has simple Fourier coefficients because it is a lift (this fact is important in the theory of Lorentzian Kac–Moody Lie algebras) and a known divisor because it is a Borcherds product. In the case of weight 3, a Borcherds product gives the canonical divisor class of the moduli space of $(1, t)$-polarized abelian surfaces. Therefore the construction of infinite families of such Siegel paramodular forms is interesting for applications to algebraic geometry. We give nine infinite families of modular forms, including a family of weight 3, which are simultaneously Borcherds Products and Gritsenko lifts. This is the first appearance of such infinite families in the literature.

All these Borcherds products are made by starting from certain special Jacobi forms that are **theta blocks** without theta denominator. Main Theorem gives a rather unexpected and surprising way to construct holomorphic Borcherds products starting from theta blocks of positive weight. As it is rather easy to search for theta blocks, we call this the Borcherds Products Everywhere Theorem. The proof uses the theory of Borcherds products for paramodular forms as given by Gritsenko and Nikulin [7]–[9], the recent theory of theta blocks due to Gritsenko, Skoruppa and Zagier [11], and a theory of generalized valuations on rings of formal series presented in section 4 of [10].

Let η be the Dedekind Eta function and ϑ be the odd Jacobi theta function and write $\vartheta_\ell(\tau, z) = \vartheta(\tau, \ell z)$. The most general theta block [11] can be written $\eta^{f(0)} \prod_{\ell \in \mathbb{N}} (\vartheta_\ell/\eta)^{f(\ell)}$ for a sequence $f : \mathbb{N} \cup \{0\} \to \mathbb{Z}$ of finite support. Here we consider only theta blocks *without theta denominator,* meaning that f is nonnegative on $\mathbb{N}$.

Main Theorem. *Let $v, k, t \in \mathbb{N}$. Let ϕ be a weak Jacobi form of weight k and index t that is a theta block without theta denominator and that has vanishing order v in $q = e^{2\pi i \tau}$. If v is odd assume that ϕ is a holomorphic (at infinity) Jacobi form Then $\psi = (-1)^v \phi|V_2/\phi$, where V_2 is the Hecke operator $J_{k,t} \to J_{k,2t}$, is a weakly holomorphic Jacobi form of weight 0 and index t and the Borcherds lift of ψ is a* **holomorphic** *paramodular form of level t and some weight $k' \in \mathbb{N}$. Moreover the Borcherds product is* **antisymmetric** *when v is an odd power of two and otherwise symmetric. If $v = 1$ then $k = k'$ and the first two Fourier Jacobi coefficients of the Borcherds lift of ψ and the Gritsenko lift of ϕ agree.*

In order to complete the line of thought that began this research and to completely characterize the paramodular forms that are both Gritsenko lifts of theta blocks without theta denominator and Borcherds Products, it would suffice to prove the following conjecture.

Conjecture. *Let $\phi \in J_{k,t}$ be a theta block without theta denominator and with vanishing order one in $q = e(\tau)$. Then $\mathrm{Grit}(\phi) = \mathrm{Borch}(\psi)$ for $\psi = -\frac{\phi|V_2}{\phi}$.*

We know in the above conjecture that $\mathrm{Borch}(\psi)$ and $\mathrm{Grit}(\phi)$ are both symmetric forms in $M_k(\Gamma_t)$ and that they have identical first and second Fourier Jacobi coefficients. The following theorem proves Conjecture for weights k satisfying $4 \le k \le 11$. The proof based on the results of [5] proceeds by demonstrating an exhaustive list of examples.

Theorem (Theta-products of order one). *Let $\ell \in \mathbb{N}$ be in the range $1 \le \ell \le 8$, and let $d_1, \dots, d_\ell \in \mathbb{N}$ with $(d_1 + \dots + d_\ell) \in 2\mathbb{N}$. Then Conjecture above is true for the Jacobi form*

$$\eta^{3(8-\ell)} \vartheta_{d_1} \cdot \ldots \cdot \vartheta_{d_\ell} \in J_{k,t}, \text{ where } k = 12 - \ell \text{ and } t = (d_1^2 + \dots + d_\ell^2)/2.$$

Additionally, this product is a Jacobi cusp form if $\ell < 8$ or if $\ell = 8$ and $\frac{(d_1 \cdot \ldots \cdot d_8)}{d^8}$ is even where $d = (d_1, \dots, d_8)$ is the greatest common divisor of the d_i.

We can also construct a *ninth* infinite series of such modular forms of weight 3. Let us take the simplest non-trivial theta blocks, i.e., with a single η factor in the denominator. These are the so-called **theta-quarks** (see [11] and [2, Corollary

3.9]); for $a, b \in \mathbb{N}$, set

$$\theta_{a,b} = \frac{\theta_a \theta_b \theta_{a+b}}{\eta} \in J_{1,a^2+ab+b^2}(\chi_3), \qquad \chi_3 = \epsilon_\eta^8, \qquad \chi_3^3 = 1.$$

The theta-quark $\theta_{a,b}$ is a Jacobi cusp form if $a \not\equiv b \mod 3$. The following theorem is a direct corollary of [5, Theorem 4.2] about the strongly reflective modular form of weight 3 with respect to $\mathrm{O}^+(2U \oplus 3A_2(-1))$.

Theorem (On theta-quarks.) *For $a_1, b_1, a_2, b_2, a_3, b_3 \in \mathbb{N}$, we have*

$$\mathrm{Grit}(\theta_{a_1,b_1}\theta_{a_2,b_2}\theta_{a_3,b_3}) = \mathrm{Borch}(\psi) \in M_3(\Gamma_t)$$

where $t = \sum_{i=1}^3 (a_i^2 + a_i b_i + b_i^2)$ and $\psi = -\dfrac{(\theta_{a_1,b_1}\theta_{a_2,b_2}\theta_{a_3,b_3})|V_2}{\theta_{a_1,b_1}\theta_{a_2,b_2}\theta_{a_3,b_3}}$.

This example is very interesting because a paramodular cusp form of weight 3 with respect to Γ_t induces a canonical differential form on the moduli space of $(1,t)$-polarized abelian surfaces, see [4]. Therefore the divisor of the modular form in this example gives the class of the canonical divisor of the corresponding Siegel modular 3-fold.

In a subsequent publication, we hope to show that the identity proven as the last example of section 2, $\mathrm{Grit}(\phi_{2,37}) = \mathrm{Borch}(\psi_{2,37})$, is also a member of an infinite family of identities for Siegel paramodular forms of weight 2.

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Multiple Dirichlet series and prehomogeneous vector spaces

Gautam Chinta

I would like to describe some examples of *Multiple Dirichlet series* i.e. Dirichlet series in several complex variables, and different ways they arise in

- the theory of automorphic forms
- zeta functions of prehomogeneous vector space

In recent years a general theory of Whittaker functions of *metaplectic Eisenstein series* (i.e. Whittaker functions of Eisenstein series on metaplectic covers of reductive groups) has started to be developed. There is some overlap in the kinds of series that arise in this manner with those which arise in the theory of Shintani zeta functions, but neither subsumes the other. I hope to indicate how the two theories can inform one another to further progress in both fields.

The first example below is originally due to Siegel [8], who used the theory of Eisenstein series of half-integer weight. An alternate approach to this same series, via the theory of prehomogeneous vector spaces, was given by Shintani [7]. This is described in Section 2. This connection between Eisenstein series and Shintani zeta functions of quadratic forms is more fully explored in the work of Ibukiyama and Saito [4]. In Section 3 I describe the work of my student J. Wen [9] who studies a three variable Shintani zeta function associated to the space of integer cubes. This turns out also to be related to Eisenstein series, this time on the metaplectic double cover of $GL(4)$. In the final section, I report on my ongoing joint work with T. Taniguchi on zeta functions of cubic orders.

1. Siegel and half-integer weight Eisenstein series

The first example of the kind of multiple Dirichlet series I would like to describe arises in the work of Siegel. See also the paper of Goldfeld and Hoffstein [3] for an elaboration and applications of Siegel's work. Start with the 1/2-integral weight Eisenstein series $\tilde{E}(z,s)$ on $\Gamma = \Gamma_0(4)$. Maass [5] computed its Fourier expansion and showed that the coefficients could be expresssed in terms quadratic Dirichlet L-functions. Siegel takes the Mellin transform of the Eisenstein series to produce a double Dirichlet series $Z(s,w)$, which is roughly of the form

$$\sum_d \frac{L(s,\chi_d)}{d^w}. \tag{1}$$

This series has

- two commuting functional equations — one coming from the functional equation of the Eisenstein series and one from the Mellin transform
- a meromorphic continuation to $\mathbb{C}^2$.

In fact, it turns out that $Z(s,w)$ actually satisfies a group of 12 functional equations! There are various ways to realize these extra "hidden" functional equations. On the one hand, we can see them by simply interchanging the order of summation and using quadratic reciprocity. On the other hand, this double Dirichlet series which we constructed as a Mellin transform of a half-integral weight Eisenstein series on the double cover of $SL(2)$ happens to coincide with a Whittaker function of a minimal parabolic Eisenstein series on the metaplectic double cover of $GL(3)$.

2. Shintani zeta function of binary quadratic forms

Next I would like to describe another manifestation of this same series, this time via the theory of *zeta functions of prehomogeneous vector spaces* initiated by Sato and Shintani.

Let $B_2(\mathbb{Z})$ be the subgroup of upper triangular matrices in $SL_2(\mathbb{Z})$. This subgroup acts on the space of integral binary quadratic forms. Conceptually, it will be more illuminating to consider the equivalent action of $B_2(\mathbb{Z})$ on the space of integral binary cubic forms $ax^2y + by^2 + cy^3$ with a root at infinity. We have two invariants for this action: $b^2 - 4ac$ and b.

The associated *Shintani zeta function* is

$$Z_{\text{Shintani}}(s_1, s_2) = \sum_{a>0} \frac{1}{|a|^{s_1}} \sideset{}{'}\sum_{\substack{b\in\mathbb{Z}\\ 0\le b\le 2a-1}} \frac{1}{|b^2-4ac|^{s_2}} \tag{2}$$

where the prime on the summation indicates that we omit terms for which $b^2 - 4ac = 0$. Playing around with this a little, we see that this series is essentially the same as the series (1) of Siegel introduced in the previous section.

3. Work of Jun Wen

Another example of a Shintani zeta function in several variables has recently been studied by my student Jun Wen. Let $V_{\mathbb{Z}}$ be the space $\mathbb{Z}^2 \otimes \mathbb{Z}^2 \otimes \mathbb{Z}^2$ and $G = SL_2(\mathbb{Z}) \times SL_2(\mathbb{Z}) \times SL_2(\mathbb{Z})$. Bhargava [1] carefully studies the G orbits on $V_{\mathbb{Z}}$ and derives numerous arithmetic applications. Wen considers instead the action of the parabolic subgroup $P = B_2(\mathbb{Z}) \times B_2(\mathbb{Z}) \times SL_2(\mathbb{Z})$ on $V_{\mathbb{Z}}$. This action has three relative invariants. Wen shows that the the associated Shintani zeta function is equal to a Whittaker function of a metaplectic Eisenstein series on a double cover of GL_4. This series is roughly of the form

$$\sideset{}{'}\sum_{\text{rank}(O)=2} \frac{\zeta_O(s_1)\zeta_O(s_3)}{|\mathtt{disc}(O)|^{s_2}}$$

where the sum is over all quadratic rings of nonzero discriminant.

4. Zeta functions of cubic rings

In this section I describe ongoing joint work with T. Taniguchi.

In the previous sections we've seen two examples involving sums of zeta functions of quadratic rings. One might wonder whether we can construct a natural series involving zeta functions of cubic (or higher rank) rings. Indeed, Shintani [6] studied a zeta function associated to the space of binary cubic forms. This example looks like it could be a special value of a multivariate series involving zeta functions of cubic rings.

How might we begin to construct such a series? In our first example, we saw that in order to parametrize zeta functions of quadratic rings we needed to look not at the space of binary quadratic forms, but at the space of *binary cubic forms with a degeneracy condition*, namely a rational root.

Inspired by this, we look at *quartic rings*. Bhargava [2], following Wright-Yukie [10], considers the action of $G = SL_2(\mathbb{Z}) \times SL_3(\mathbb{Z})$ on pairs of integral ternary quadratic forms $V_{\mathbb{Z}} = \mathbb{Z}^2 \otimes \mathtt{sym}^2\mathbb{Z}^3$. He shows (essentially) that orbits correspond to quartic rings.

In joint work with T. Taniguchi, we choose an appropriate parabolic subgroup P of G and show that the Shintani zeta function corresponding to the action of P on a suitable sublattice of $V_{\mathbb{Z}}$ involves a sum of zeta functions of cubic orders.

This result is probably not surprising to the experts — in any event it is not too hard to prove once everything is set up correctly. What *is* surprising is that this series affords an interchange of summation which lets us rewrite it in terms of (sums of) the Shintani zeta function of binary cubic forms. This is a remarkable fact! The existence of this meaningful interchange of summation plays a key role in the analytic continuation of the series, which is rather elaborate and requires techniques not previously used in this context.

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Symmetric Formal Fourier Jacobi Series and Kudla's Conjecture

Martin Raum
(joint work with Jan Hendrik Bruinier)

We can attach a Fourier Jacobi expansion to every (classical) Siegel modular for of genus ≥ 2:

$$f(\tau) = \sum_{0 \leq m \in \mathbb{Z}} \phi_m(\tau_1)\, \exp(2\pi i\, m\tau_2), \text{ where } \tau = \left(\begin{smallmatrix} \tau_1 & z \\ {}^{\mathrm{t}}z & \tau_2 \end{smallmatrix}\right)$$

lies in the Siegel upper half space of genus g, denoted by $\mathbb{H}_g$. We have decomposed τ into $\tau_1 \in \mathbb{H}_{g-1}$, $\tau_2 \in \mathbb{H}_1$, and $z \in \mathbb{C}^{g-1}$. Expansions of this kind are ubiquitous in the study of Siegel modular forms, as they allow to reduce considerations to Jacobi forms ϕ_m of genus $g-1$. To name some examples, confer work on the Saito-Kurokawa Conjecture [1, 9, 10, 11, 14], on the spinor L-series [7, 3], and on computations of Siegel modular forms [13, 12].

We formalize the notion of Fourier Jacobi expansions: A series of Jacobi forms whose Fourier coefficients satisfy a natural symmetry condition is called a *formal Fourier Jacobi expansion*. We obtain a map $\mathrm{M}_k^{(g)} \longrightarrow \mathrm{FM}_k^{(g)}$ from the space of Siegel modular forms to the space of formal Fourier Jacobi expansions. Our main theorem states that this map is an isomorphism.

Our main application is a proof of Kudla's conjecture. On orthogonal Shimura varieties X there is a natural family $Z(t)$ of cycles, index by positive definite, symmetric matrices $t \in \mathrm{Mat}^{\mathrm{T}}_{\mathbb{Q}}$ with rational entries (for matters of presentation, we restrict to the easiest case). Kudla and Millson [4, 5, 6] studied the attached generating series

$$f_X(\tau) = \sum_t Z(t)\, \exp\big(2\pi i\, \mathrm{trace}(t\tau)\big)$$

and proved that it is a Siegel modular form with coefficients in *cohomology*. Inspired by these finding, it was later conjectured that the generating series with coefficients in the Chow group was also a modular form [8]. Zhang [15] proved in his thesis that f_X is a formal Fourier Jacobi expansion. From our result, we hence infer modularity of f_X.

(Classical) Siegel modular forms of genus $g > 1$ are holomorphic functions on

$$\mathbb{H}_g = \big\{\tau \in \mathrm{Mat}^{\mathrm{T}}_g(\mathbb{C}) \,:\, \Im(\tau) \text{ positive definite}\big\},$$

where $\mathrm{Mat}^{\mathrm{T}}_g$ denotes the set of symmetric matrices of size g. In the simplest case, we have $k \in 2\mathbb{Z}$ and, by definition, a Siegel modular form of weight k is a holomorphic function $f : \mathbb{H}_g \to \mathbb{C}$ that satisfies

$$f\big((a\tau + b)(c\tau + d)^{-1}\big) = \det(c\tau + d)^k f(\tau)$$

for all $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{Sp}_g(\mathbb{Z})$. We denote the space of genus g, weight k Siegel modular forms by $\mathrm{M}_k^{(g)}$. The Fourier Jacobi expansion of $f \in \mathrm{M}_k^{(g)}$ is of the form

$$f(\tau) = \sum_{0 \le m \in \mathbb{Z}} \phi_m(\tau_1, z) \exp(2\pi i\, m\tau_2)$$

as above. To formalize this, we define genus $g-1$ Siegel Jacobi forms of weight k and index $m \in \mathbb{Z}$ as holomorphic functions $\phi : \mathbb{H}_{g-1} \times \mathbb{C}^{g-1}$ such that $\phi(\tau_1, z) \exp(2\pi i\, m\tau_2)$ transforms like a Siegel modular form under

$$\mathrm{Stab}_{\mathrm{Sp}_g(\mathbb{Z})}\Big(\mathrm{span}\left(e_1, \ldots, e_{g-1},\, e_{g+1} \ldots, e_{2g-1}\right)\Big),$$

where $e_1, \ldots, e_{2g}$ is a standard basis of $\mathbb{Z}^{2g}$. In the case $g = 2$ (i.e., $g - 1 = 1$), we impose an additional growth condition. The space of Siegel Jacobi forms is denoted by $\mathrm{J}_{k,m}^{(g-1)}$.

Definition: A formal series

$$\sum_{0 \le m \in \mathbb{Z}} \phi_m(\tau_1, z) \exp(2\pi i\, m\tau_2) \in \prod_{0 \le m \in \mathbb{Z}} \mathrm{J}_{k,m}^{(g-1)}$$

is called symmetric, if its (formal) Fourier coefficients $c(t)$, $t \in \mathrm{Mat}_g^{\mathrm{T}}(\mathbb{Q})$ satisfy $c({}^{\mathrm{t}}utu) = c(t)$ for all $u \in \mathrm{GL}_g(\mathbb{Z})$. We write $\mathrm{FM}_k^{(g)}$ for the space of such expansions. For geometric reasons, we call them *formal Fourier Jacobi expansions*.

Theorem (Bruinier, R.): For $g > 1$, we have $\mathrm{FM}_k^{(g)} = \mathrm{M}_k^{(g)}$.

Our work [2] will cover vector valued Siegel modular forms for the metaplectic cover of $\mathrm{Sp}_g(\mathbb{Z})$, half-integral weights, and Fourier Jacobi expansions with Jacobi forms of arbitrary positive genus. This is, in fact, necessary to prove Kudla's conjecture: Zhang has established that the generating series f_X mentioned above is a vector valued formal Fourier Jacobi expansion with Jacobi forms of genus 1.

Our theorem is reminiscent of rigidity theorems in formal geometry. It seems feasible but technically difficult to reprove our theorem by means of formal methods. This is ongoing work.

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Symmetric square L-functions of $\mathrm{GL}(n)$

Shunsuke Yamana
(joint work with Eyal Kaplan)

The symmetric square L-function of an irreducible cuspidal automorphic representation π of $\mathrm{GL}_n(\mathbb{A})$ is defined by the Euler product

$$L(s,\pi,\mathrm{sym}^2)=\prod_v L(s,\pi_v,\mathrm{sym}^2),$$

where $\mathbb{A}$ is the adele ring of a number field F. For almost all places v of F Hecke theory associates to the local component π_v of π a conjugacy class in $\mathrm{GL}_n(\mathbb{C})$, represented by a diagonal matrix $\mathrm{diag}[\alpha_{v,1},\dots,\alpha_{v,n}]$, and the local symmetric square L-factor is defined by

$$L(s,\pi_v,\mathrm{sym}^2)=\prod_{1\le i\le j\le n}(1-\alpha_{v,i}\alpha_{v,j}q_v^{-s})^{-1},$$

where q_v is the cardinality of the residue field of the completion F_v of F at v.

Assume that $n\ge 2$. It is interesting to ask when $L(s,\pi,\mathrm{sym}^2\otimes\chi)$ has a pole. If n is even, then its pole at $s=1$ is characterized in terms of functorial transfers from general spin groups, while if n is odd, then its pole at $s=1$ is characterized in terms of functorial transfers from symplectic groups. Following Bump-Ginzburg and Takeda, we develop a theory of symmetric square L-functions for $\mathrm{GL}(n)$ and give another characterization of its pole at $s=1$ in terms of nonvanishing of certain period integrals of trilinear type.

The construction of the symmetric square L-function involves certain small genuine automorphic representations of the double cover $\bar{G}_{n,\mathbb{A}}$ of $\mathrm{GL}_n(\mathbb{A})$, known as exceptional representations, constructed by Kazhdan and Patterson [2] for general

k-fold covers of $\mathrm{GL}_n(\mathbb{A})$. Let θ^ψ denote the exceptional representation of $\bar{G}_{n,\mathbb{A}}$ associated to a nontrivial character ψ of $F\backslash\mathbb{A}$. Let $|\cdot|$ denote the standard idele norm of $\mathbb{A}^\times$. Put

$$\mathrm{GL}_n(\mathbb{A})^1 = \{g \in \mathrm{GL}_n(\mathbb{A}) \mid |\det g| = 1\}.$$

Theorem. Let π be an irreducible cuspidal automorphic representation of $\mathrm{GL}_n(\mathbb{A})$ with central character ω_π. The function $L(s, \pi, \mathrm{sym}^2)$ has a pole at $s = 1$ if and only if $\omega_\pi^2 = 1$ and there are $\varphi \in \pi$ and $\Theta, \Theta' \in \theta^\psi$ such that

$$\int_{\mathrm{GL}_n(F)\backslash\mathrm{GL}_n(\mathbb{A})^1} \varphi(g)\Theta(g)\overline{\Theta'(g)}\,\mathrm{d}g \neq 0.$$

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Whittaker coefficients of cuspidal representations of the metaplectic group

Erez M. Lapid

(joint work with Zhengyu Mao)

Given a quasi-split reductive group G over a number field F (with ring of adeles $\mathbb{A}$) with maximal unipotent subgroup N and a non-degenerate character ψ_N of $N(\mathbb{A})$, trivial on $N(F)$, consider the Whittaker–Fourier coefficient

$$\mathcal{W}(\varphi) = \mathcal{W}^{\psi_N}(\varphi) := \int_{N(F)\backslash N(\mathbb{A})} \varphi(n)\psi_N(n)^{-1}\,dn$$

of an automorphic form φ on $G(F)\backslash G(\mathbb{A})$. The problem that we study is the relation between this coefficient and the Petersson inner product

$$(\varphi, \varphi^\vee) = \int_{G(F)\backslash G(\mathbb{A})} \varphi(g)\varphi^\vee(g)\,dg$$

for a cuspidal representation π of $G(\mathbb{A})$. (For simplicity of notation we assume that the center of G is anisotropic. We normalize the invariant measures so that $\mathrm{vol}(N(F)\backslash N(\mathbb{A})) = \mathrm{vol}(G(F)\backslash G(\mathbb{A})) = 1$.) For the general linear group, such a relation is given by the theory of Rankin–Selberg integrals, developed in higher rank by Jacquet, Piatetski-Shapiro and Shalika (cf. [11]). It involves the residue at $s = 1$ of $L(s, \pi \otimes \pi^\vee)$.

Let us try to make this more precise and at the same time formulate a question for other groups. (See [16] for more details.) By local multiplicity one, there exists a constant $c_\pi^{\psi_N}$, depending on π, such that

$$(1)\quad \mathcal{W}^{\psi_N}(\varphi)\mathcal{W}^{\psi_N^{-1}}(\varphi^\vee) = (c_\pi^{\psi_N})^{-1}\frac{\Delta_G^S(1)}{L^S(1,\pi,\mathrm{Ad})}\int_{N(F_S)}^{st}(\pi(n)\varphi,\varphi^\vee)\psi_N(n)^{-1}\,dn.$$

Here $\Delta_G^S(s)$ is a certain explicit abelian (partial) L-function (depending only on G, not on π), S is a sufficiently large finite set of places including all the archimedean and the ramified places, the measure on $N(F_S)$ is normalized so that $\mathrm{vol}(N(\mathcal{O}_S)\backslash N(F_S)) = 1$ where $\mathcal{O}_S$ is the ring of S-integers and $\int^{st}$ is a certain regularized integral which in the p-adic case is simply the stable limit of the integrals over compact open subgroups of $N(F_v)$. (The integral converges absolutely if π_v is square-integrable but not otherwise.) Implicit here is the holomorphy and non-vanishing of the adjoint L-function $L^S(s,\pi,\mathrm{Ad})$ at $s = 1$. The proportionality constant $c_\pi^{\psi_N}$, which exists by local uniqueness of Whittaker model, is independent of S by the Casselman–Shalika formula. (This is why the factor $\Delta_G^S(s)$ is introduced.)

The Rankin–Selberg theory for GL_n alluded to above shows that $c_\pi^{\psi_N} = 1$ for any irreducible cuspidal representation π of GL_n. For other quasi-split groups $c_\pi^{\psi_N}$ depends on the automorphic realization of π (unless of course there is multiplicity one, which is at least expected for classical groups. Note that $O(2n)$ is a classical group, but not $SO(2n)$.)

It turns out that a sensible expression for $c_\pi^{\psi_N}$ is feasible if we admit Arthur's conjectures (for the discrete spectrum) in a strong form, namely a canonical decomposition

$$L^2_{\mathrm{disc}}(G(F)\backslash G(\mathbb{A})) = \widehat{\bigoplus_{\phi}}\,\overline{\mathcal{H}_\phi}$$

according to elliptic Arthur's parameters. The latter are equivalence classes of (certain) homomorphisms from the direct product of the (hypothetical) Langlands group with $\mathrm{SL}_2(\mathbb{C})$ into the dual group of G, whose image has a finite centralizer modulo the center. (In passing we mention the recent work of V. Lafforgue who made dramatic progress towards establishing the above decomposition in the function field case [14]. One of the difficulties that he successfully confronts is how to uniquely characterize the spaces $\mathcal{H}_\phi$. It is unclear how to resolves this in the number field case.)

Except for GL_n, the spaces $\mathcal{H}_\phi$ are not irreducible (or even multiplicity free) in general. To a large extent the reducibility of $\mathcal{H}_\phi$ is measured by a certain finite group $\mathcal{S}_\phi$ (and its local counterparts) attached to ϕ [1] – a phenomenon which goes back to Labesse–Langlands ([15], cf. [13]). For instance, if G is split then the group $\mathcal{S}_\phi$ is the quotient of the centralizer of the image of ϕ in the complex dual $\widehat{G}$ of G by the center of $\widehat{G}$. (For GL_n, $\mathcal{S}_\phi$ is always trivial.) The relevant Arthur's parameters in our context are those of Ramanujan type, namely those which are trivial on SL_2. (Otherwise $\mathcal{W}^{\psi_N}$ vanishes on $\mathcal{H}_\phi$ [20].) For these ϕ, $\mathcal{H}_\phi$ is contained in the cuspidal spectrum and we can (conjecturally) single out a distinguished irreducible ψ_N-generic subspace $\pi^{\psi_N}(\phi)$ of $\mathcal{H}_\phi$.

Conjecture 1. *For any elliptic Arthur's parameter ϕ of Ramanujan type we have $c^{\psi_N}_{\pi^{\psi_N}(\phi)} = |\mathcal{S}_\phi|$.*

The conjecture is modeled after recent conjectures and results of Ichino–Ikeda [8] which sharpen the Gross–Prasad conjecture, which in turn go back to classical results of Waldspurger [23, 22]. (See [5] for a recent extension of these conjectures by Gan–Gross–Prasad.) More recently, Sakellaridis–Venkatesh formulated conjectures in the much broader scope of periods over spherical subgroups (at least in the split case) [21]. Conjecture 1 can be viewed as a strengthening of the conjectures of [21] in the case at hand.

For quasi-split classical groups one may formulate Conjecture 1 more concretely thanks to the work of Cogdell–Kim–Piatetski-Shapiro–Shahidi, Ginzburg–Rallis–Soudry and others [4, 6]. More precisely, if G is a quasi-split classical group and ψ_N is as before, there is a one-to-one correspondence $\{\pi_1, \dots, \pi_k\} \mapsto \sigma = \sigma^{\psi_N}(\{\pi_1, \dots, \pi_k\})$ between the sets of (distinct) cuspidal representations of general linear groups GL_{n_i} of certain self-duality type depending on G and with $n_1 + \dots + n_k = m$ where m is determined by G, and ψ_N-generic cuspidal representation of $G(\mathbb{A})$. (For convenience we exclude even orthogonal groups which require extra care.) The bijection is given explicitly by the descent method of Ginzburg–Rallis–Soudry and the functorial transfer of σ to GL_m is the isobaric sum $\pi_1 \boxplus \dots \boxplus \pi_k$. In particular, one can describe $L(1, \sigma, \mathrm{Ad})$ in terms of known L-functions of GL_n.

Conjecture 1 translates into the following:

Conjecture 2. *Let $\sigma = \sigma^{\psi_N}(\{\pi_1, \dots, \pi_k\})$. Then $c^{\psi_N}_\sigma = 2^{k-1}$.*

The descent method applies equally well to the metaplectic groups $\widetilde{\mathrm{Sp}}_n$ – the two-fold cover of the symplectic groups Sp_n – and we can also formulate an analogous (but modified) conjecture as follows.

Conjecture 3. *Assume that σ is the ψ_N-descent of $\{\pi_1, \dots, \pi_k\}$ to $\widetilde{\mathrm{Sp}}_n$. Let π be the isobaric sum $\pi_1 \boxplus \dots \boxplus \pi_k$. Then*

$$\mathcal{W}^{\psi_N}(\varphi)\mathcal{W}^{\psi_N^{-1}}(\varphi^\vee) = 2^{-k}\Delta^S_{\mathrm{Sp}_n}(1)\frac{L^S(\frac{1}{2}, \pi)}{L^S(1, \pi, \mathrm{sym}^2)}\int^{st}_{N(F_S)}(\sigma(n)\varphi, \varphi^\vee)\psi_N(n)^{-1}\,dn.$$

(The analogue of the Casselman–Shalika formula in this context is due to Bump–Friedberg–Hoffstein [2].) We note that in the case of $\widetilde{\mathrm{Sp}}_n$, the image of the ψ_N-descent consists of the cuspidal ψ_N-generic spectrum whose ψ-theta lift to $\mathrm{SO}(2n-1)$ vanishes where ψ is determined by ψ_N. (See [6, §11] for more details.) In the case $n = 1$, this excludes the so-called exceptional representations.

The case of the metaplectic two-fold cover of SL_2 (i.e., $n = 1$) goes back to the classical result of Waldspurger on the Shimura correspondence [22] which was followed up by many authors. A different approach, pursued by Jacquet [10] and completed by Baruch–Mao (for $n = 1$) [3], is via the relative trace formula. Recently, Wei Zhang [24, 25] proved the Gan–Gross–Prasad conjecture for unitary groups under certain local restrictions using the relative trace formula of Jacquet–Rallis [12].

In the series of papers [17, 18, 19] we prove Conjecture 3 under the assumption that F is totally real and the archimedean component σ_∞ is square-integrable. Our main tool is the descent method of Ginzburg–Rallis–Soudry and its local counterpart. (We do not use the relative trace formula.) As a bonus we derive in [9] applications to the formal degree conjecture of Hiraga–Ichino–Ikeda [7].

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Real-dihedral harmonic Maass forms and CM-values of Hilbert modular functions

Yingkun Li

In the theory of modular forms, those of weight $k = 1$ are important because of their connection to Galois representations. By the Theorem of Deligne-Serre [7], one can functorially attach to each weight one newform f a continuous, odd, irreducible representation

$$\rho_f : \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \longrightarrow \mathrm{GL}_2(\mathbb{C}).$$

Let $\tilde{\rho}_f : \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \longrightarrow \mathrm{GL}_2(\mathbb{C})$ be the associated projective representation. If the image of $\tilde{\rho}_f$ is isomorphic to a dihedral group, then ρ_f is induced from a character of $\mathrm{Gal}(\overline{F}/F)$ for some quadratic field F in M. We say that f or ρ_f is real-dihedral if F is a real quadratic field.

A *harmonic Maass form* of weight $k \in \mathbb{Z}$ is a real-analytic function $\mathcal{F} : \mathbb{H} \longrightarrow \mathbb{C}$ such that it is modular and annihilated by the hyperbolic Laplacian Δ_k of weight k

$$\begin{aligned}\Delta_k &:= y^2\left(\tfrac{\partial^2}{\partial x^2} + \tfrac{\partial^2}{\partial y^2}\right) - iky\left(\tfrac{\partial}{\partial x} + i\tfrac{\partial}{\partial y}\right) = \xi_{2-k} \circ \xi_k,\\ \xi_k &:= 2iy^k\overline{\partial_{\overline{z}}},\end{aligned} \tag{1}$$

where we write $z = x + iy$. Furthermore, it is only allowed to have polar-type singularities in the cusps. They were introduced in [2] to study theta-liftings. Every harmonic Maass form $\mathcal{F}$ can be written canonically as the sum of a holomorphic part $\tilde{f}$ and a non-holomorphic part f^*. The holomorphic part $\tilde{f}$ is also known as a *mock-modular form*, which has been extensively studied by many people [1, 3, 8] after Zwegers' groundbreaking thesis [18] (see [17] for a good exposition) and has connections to many different areas of mathematics (see [13] for a comprehensive overview). When $k = 1$, we call $\mathcal{F}$ real-dihedral if $\xi_1(\mathcal{F})$ is a real-dihedral newform.

We are interested in studying a family of real-dihedral harmonic Maass forms and relate their Fourier coefficients to CM-values of Hilbert modular functions.

Suppose $D \equiv 1 \pmod 4), p \equiv 5 \pmod 8$ are primes satisfying conditions

$$\begin{aligned} &F = \mathbb{Q}(\sqrt{D}) \text{ has class number one,}\\ &p\mathcal{O}_F = \mathfrak{p}\mathfrak{p}',\\ &\mathrm{ord}_{\mathfrak{p}}(u_F^{(p-1)/4} - 1) > 0, \end{aligned}$$

where $u_F > 1$ is the fundamental unit of F. Let $\chi_D(\cdot) = \left(\frac{\cdot}{D}\right)$ be the quadratic character of conductor D and ϕ_p the character of conductor p and order 4. The space of cusp forms $S_1(Dp, \chi_D\phi_p)$ is one-dimensional and spanned by a newform

$$f_\varphi(z) := \sum_{\mathfrak{a}\subset\mathcal{O}_F} \varphi(\mathfrak{a})q^{\mathrm{Nm}(\mathfrak{a})} = \sum_{n\geq 1} c_\varphi(n)q^n, \tag{2}$$

where $q = e^{2\pi iz}$ and φ is a ray class group character of F. When $D = 5, p = 29$, the form f_φ was studied by Stark in the context of producing explicit generators of class fields of real-quadratic fields from special values of L-functions [15, 16].

Since $S_1(Dp, \chi_D\phi_p)$ is one-dimensional, there exists a harmonic Maass form $\mathcal{F}_\varphi(z)$ such that $\xi_1(\mathcal{F}_\varphi) = f_\varphi$ and its holomorphic part $\tilde{f}_\varphi$ has the following Fourier expansion at infinity

$$\tilde{f}_\varphi(z) = c_\varphi^+(-1)q^{-1} + c_\varphi^+(0) + \sum_{\substack{n\geq 2\\ \chi_D(n)\neq -1}} c_\varphi^+(n)q^n.$$

Furthermore, with a mild condition on the growths of $\mathcal{F}_\varphi$ at other cusps of $\Gamma_0(Dp)$, the form $\mathcal{F}_\varphi$ is *unique* and the coefficients $c_\varphi^+(-1), c_\varphi^+(0)$ can be written explicitly as algebraic multiples of $\log u_F$.

Let $F_2 = \mathbb{Q}(\sqrt{p})$, $\mathcal{O}_{F_2}$ its ring of integers and X_{F_2} the open Hilbert modular surface whose complex points are $\mathrm{SL}_2(\mathcal{O}_{F_2})\backslash\mathbb{H}^2$. It is a connected component of the moduli space parametrizing isomorphisms of abelian surfaces with real multiplication. Let M_8 denote the field fixed by $\ker\tilde{\rho}_\varphi$. It contains two pairs of CM extensions K_4/F_2 and $\tilde{K}_4/\tilde{F}_2$, which are reflex fields of each other under the appropriate CM types $\Sigma = \{1, \sigma\}$ and $\tilde{\Sigma} = \sigma^3\Sigma = \{1, \sigma^{-1}\}$. Here, σ is an element of order 4 in the dihedral group $\mathrm{Gal}(M_8/\mathbb{Q}) \cong D_8$ of order 8.

Let $\mathrm{Cl}_0(K_4)$ be the kernel of the norm map $\mathrm{Nm} : \mathrm{Cl}(K_4) \longrightarrow \mathrm{Cl}(F_2)$ on class groups. Each class in $\mathrm{Cl}_0(K_4)$ gives rise to an isomorphism class of abelian surfaces on X_{F_2} with complex multiplication by (K_4, Σ), which is a "big" CM point in the sense of [4]. For $\mathcal{A} \in \mathrm{Cl}_0(K_4)$, let $Z_{\mathcal{A},\Sigma} \in X_{F_2}(\mathbb{C})$ denote the corresponding CM point. Since the 2-rank of $\mathrm{Cl}(K_4)$ is 1, it has a unique quadratic character ψ_2. Then we could define the twisted CM 0-cycle $\mathcal{CM}(K_4, \psi_2)$ by

$$\mathcal{CM}(K_4, \Sigma, \psi_2) := \sum_{\mathcal{A}\in\mathrm{Cl}_0(K_4)} \psi_2(\mathcal{A})Z_{\mathcal{A},\Sigma}, \tag{3}$$

$$\mathcal{CM}(K_4, \psi_2) := \sum_{j=0}^{3} \mathcal{CM}(K_4, \sigma^j\Sigma, \psi_2). \tag{4}$$

It is algebraic and defined over the real quadratic field F. For $m \in \mathbb{N}$, let T_m be the m^{th} Hirzebruch-Zagier divisor on X_{F_2}. Given any normalized integral Hilbert modular function $\Psi(z_1, z_2)$ on X_{F_2} in the sense of Theorem 1.1 in [5] with divisor

$$\sum_{\substack{m\geq 1\\ \gcd(pD,m)=1}} c(-m)T_m,$$

where $c(-m) \in \mathbb{Z}$, we will show that the value of Ψ at $\mathcal{CM}(K_4, \psi_2)$ are related to the coefficients $c_\varphi^+(n)$ by

$$\log|\Psi(\mathcal{CM}(K_4, \psi_2))| = -\frac{c_\varphi(p)h^+_{\tilde{F}_2}}{h_{\tilde{F}_2}} \sum_{m\geq 1} c(-m)b_\varphi(m), \tag{5}$$

where $h_{\tilde{F}_2}$ and $h^+_{\tilde{F}_2}$ are the class number and narrow class number of $\tilde{F}_2 = \mathbb{Q}(\sqrt{Dp})$ respectively, and

$$b_\varphi(m) := \sum_{d|m} a_\varphi\left(\frac{m^2}{d^2}\right)\phi_p(d), \tag{6}$$

$$a_\varphi(n) := \sum_{k\in\mathbb{Z}} c_\varphi^+\left(\frac{Dn-pk^2}{4}\right)\delta_D(k), \tag{7}$$

$$\delta_D(k) := \begin{cases} 1 & D \nmid k, \\ 2 & D \mid k. \end{cases} \tag{8}$$

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Structure and arithmeticity for nearly holomorphic Siegel cusp forms of degree 2

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(joint work with Ameya Pitale, Ralf Schmidt)

This joint project with Ameya Pitale and Ralf Schmidt is a detailed study of the representations generated by nearly holomorphic Siegel cusp forms of degree 2. In particular, we prove a close link between such forms and holomorphic vector valued Siegel cusp forms, and this allows us to deduce many arithmetic results.

Introduction. Let $\mathbb{H}_2$ denote the Siegel upper half space of degree 2, consisting of two-by-two complex matrices that are symmetric and whose imaginary part is positive definite. Let p be a non-negative integer. We let $N^p(\mathbb{H}_2)$ denote the space of all polynomials of degree $\leq p$ in the entries of Y^{-1} (writing $Z \in \mathbb{H}_2$ as $Z = X + iY$) with holomorphic functions on $\mathbb{H}_2$ as coefficients. The space

$$N(\mathbb{H}_2) = \bigcup_{p \geq 0} N^p(\mathbb{H}_2)$$

is the space of *nearly holomorphic functions* on $\mathbb{H}_2$. Evidently, $N(\mathbb{H}_2)$ is a ring, and

$$N^p(\mathbb{H}_2)N^q(\mathbb{H}_2) \subset N^{p+q}(\mathbb{H}_2).$$

Given any congruence subgroup Γ of $\mathrm{Sp}_4(\mathbb{Z})$ and any integer k, we let $N_k^p(\Gamma)$ denote the space of functions $F : \mathbb{H}_2 \to \mathbb{C}$ such that

(1) $F \in N^p(\mathbb{H}_2)$
(2) $F|_k\gamma = F$ for all $\gamma \in \Gamma$.

The space $N_k^p(\Gamma)$ is the space of nearly holomorphic modular forms of degree 2, weight k for Γ. We let $R_k^p(\Gamma) \subset N_k^p(\Gamma)$ denote the subspace of cusp forms.

Nearly holomorphic modular forms come up naturally as special values of Eisenstein series, and so are important in proving algebraicity of special L-values via the method of integral representations. However, despite a lot of work, especially by Shimura, they have not really been properly understood in the framework of adelic automorphic representations.

Results. In our project, we completely explicate the $(\mathfrak{g}, K)$-modules generated by nearly holomorphic modular forms of degree 2. We explain how these forms arise as vectors in representations that also contains vectors corresponding to holomorphic vector valued Siegel cusp forms. This allow us to deduce a structure theorem for the space of nearly holomorphic Siegel modular forms of degree 2 with respect to an arbitrary congruence subgroup.

More precisely, let $V_m \simeq \mathrm{sym}^m(\mathbb{C}^2)$ be the space of all homogeneous polynomials of total degree m in the two indeterminates X and Y with complex coefficients and let $\hat{\rho}_{l,m}$ be the representation of $\mathrm{GL}_2(\mathbb{C})$ on the vector space V_m. Let $M_{l,m}(\Gamma)$ denote the space of holomorphic functions $F : \mathbb{H}_2 \to V_m$ such that

(1) F is holomorphic everywhere, including the cusps.
(2) $F(\gamma Z) = \hat{\rho}_{l,m}((CZ + D))F(Z)$.

The space $M_{l,m}(\Gamma)$ is the space of holomorhic vector modular forms of degree 2, weight-type (l, m) for Γ. We let $S_{l,m}(\Gamma) \subset M_{l,m}(\Gamma)$ denote the subspace of cusp forms.

Theorem 1. *For any pair of integers l, m with $m \geq 0$ and m even, and any non-negative integer v, there exists a linear map $\Delta^v_{l,m}$ from $S_{l,m}(\Gamma)$ to $R^{m/2+2v}_{l+m+2v}(\Gamma)$. This map has the following properties:*

- *It preserves rationality of Fourier coefficients, is Hecke-equivariant and has an explicit formula in terms of differential operators.*
- *The ratio of Peterson inner products $\langle \Delta^v_{l,m} F, \Delta^v_{l,m} F\rangle / \langle F, F\rangle$ does not depend on F.*

Furthermore, the images of spaces of vector-valued cusp forms under the above map gives a direct sum decomposition of the space of nearly holomorphic cusp forms. In other words, have

$$R^p_k(\Gamma) = \bigoplus_{\substack{l \geq 2, m \geq 0 \\ l \equiv k \bmod 2,\ m \equiv 0 \bmod 2 \\ k-p \leq l+m/2 \leq l+m \leq k}} \Delta^{(k-l-m)/2}_{l,m} \left(S_{l,m}(\Gamma)\right).$$

The proof of the above theorem relies on an extensive study of the $(\mathfrak{g}, K)$-modules generated by nearly holomorphic modular forms as well as various calculations involving moving between the vectors in various K-types.

An important application of the structure theorem above is to arithmeticity of Petersson norms for nearly holomorphic cusp forms.

Theorem 2. *Let F, G be elements of $R^p_k(\Gamma)$ with F a Hecke eigenform. Then, for any $\sigma \in \mathrm{Aut}(\mathbb{C})$,*

$$\sigma\left(\frac{\langle F, G\rangle}{\langle F, F\rangle}\right) = \frac{\langle \sigma(F), \sigma(G)\rangle}{\langle \sigma(F), \sigma(F)\rangle}.$$

The above result is a significant generalization of results of Shimura and Garrett.

Applications. We have various applications in mind for the above results. Perhaps the most notable one involves algebraicity of special values of L-functions. A well-known problem in the arithmetic theory of automorphic forms is Deligne's conjecture on algebraicity of critical values of L-functions. The simplest example of this conjecture is the classical fact that for all positive integers n, one has

$$\frac{\zeta(2n)}{\pi^{2n}} := \frac{\sum_{k=1}^{\infty} k^{-2n}}{\pi^{2n}} \in \mathbb{Q}.$$

Deligne conjectured that this is a special case of a general fact, i.e., similar results ought to hold for certain special values (the so-called critical values) of any L-function that is "motivic" (roughly speaking, this means it is related to algebraic geometry via cohomology). This conjecture is one of the deep unsolved problems in mathematics. Partial progress has been made using various methods, such as the method of integral representations, methods involving cuspidal and Eisenstein cohomology, and Iwasawa theory.

As early as 1981, M. Harris proved a special case of Deligne's conjecture for the standard L-function of a Siegel modular form of full level. This result has since been extended by Shimura, Mizumoto and various others. Despite this, important cases remain open, even for degree 2 forms. For example, the case of vector valued forms of degree 2 has been solved only in the case of full level (due to Kozima). This project will extend Kozima's result to vector valued Siegel modular forms for arbitrary congruence subgroups of $\mathrm{Sp}_4(\mathbb{Z})$. This is still work in progress.

On special values of L-functions for quaternion unitary groups of degree 2 and GL(2)

Kazuki Morimoto

1. Deligne's conjecture on special values of L-functions.

Let $\mathcal{M}$ be a motive over $\mathbb{Q}$ with coefficients in an algebraic number field E. Put $R = E \otimes_{\mathbb{Q}} \mathbb{C}$. We have $E \subset R$ canonically. Then the motive $\mathcal{M}$ has the L-function $L(\mathcal{M}, s)$ taking values in R. Deligne defined the motivic periods $c^{\pm}(\mathcal{M}) \in R^{\times}/E^{\times}$ and conjectured that if $n \in \mathbb{Z}$ is a critical point of $\mathcal{M}$,

$$\frac{L(\mathcal{M}, n)}{(2\pi i)^{d^{\pm} n} c^{\pm}(\mathcal{M})} \in E$$

where $\pm$ is the same sign as $(-1)^n$ and $d^{\pm}$ is the dimension of the $\pm$-eigen space of the Betti realization of $\mathcal{M}$ (see Deligne [2, Conjecture 2.8]). We are interested in the special case of this conjecture when $\mathcal{M} = M \otimes N$, where M (resp. N) is the motive corresponding to a Siegel cuspform of degree 2 (resp. elliptic cuspform). In [12], Yoshida computed the Deligne's periods $c^{\pm}(M \otimes N)$, and he gave an explication of them by modular forms under the assumption that the above Deligne's conjecture holds for M. Using this computation, he gives a conjecture on an algberaicity of special values of degree 8 L-functions for GSp(4) and GL(2) (cf. [12, Theorem 13]).

2. L-functions for quaternion unitary groups of degree 2 and GL(2).

Let D be a quaternion algebra over $\mathbb{Q}$ such that $D\otimes_{\mathbb{Q}}\mathbb{R}\simeq \mathrm{Mat}_{2\times 2}(\mathbb{R})$, which is possibly split. Define

$$G_D=\left\{g\in \mathrm{GL}_2(D)\mid {}^t\bar{g}\begin{pmatrix}0&1\\1&0\end{pmatrix}g=\lambda(g)\cdot\begin{pmatrix}0&1\\1&0\end{pmatrix}\right\}$$

where $g\mapsto\bar{g}$ is the canonical involution of D. Then G_D is an inner form of $\mathrm{GSp}(4)$, and we have

$$G_D(\mathbb{R})\simeq \mathrm{GSp}(4,\mathbb{R})$$

by the assumption on D. In particular, we have

$$G_D\simeq \mathrm{GSp}(4)\quad\text{when}\quad D\simeq \mathrm{Mat}_{2\times 2}(\mathbb{Q}).$$

Let (Π,V_Π) be an irreducible cuspidal automorphic representation of $G_D(\mathbb{A}_{\mathbb{Q}})$ such that Π_∞ is the holomorphic discrete series representation with Harish-Chandra parameter (k_1+2k_2-1,k_1-2). Remark that when $D\simeq\mathrm{Mat}_{2\times 2}(\mathbb{Q})$, we can attach this automorphic representation to Siegel cuspforms of degree 2 and of weight $\rho_{(k_1,k_2)}:=\det^{k_1}\otimes\mathrm{Sym}^{2k_2}$ (cf. Saha [11]). We realize V_Π in the space of $V_{(k_1,k_2)}$-valued automorphic forms where $V_{(k_1,k_2)}$ is the representation space of $\rho_{(k_1,k_2)}$. Since $(\rho_{(k_1,k_2)},V_{(k_1,k_2)})$ is defined over $\mathbb{Q}$, it has a $\mathbb{Q}$-rational structure $V_{(k_1,k_2)}(\mathbb{Q})$. Then we fix a $\rho_{(k_1,k_2)}$-invariant hermitian form $\langle -,-\rangle_{(k_1,k_2)}$ on $V_{(k_1,k_2)}$ such that it takes values in $\mathbb{Q}$ on $V_{(k_1,k_2)}(\mathbb{Q})$.

Let (π,V_π) be an irreducible cuspidal automorphic representation of $\mathrm{GL}(2,\mathbb{A}_{\mathbb{Q}})$ such that π_∞ is the holomorphic discrete series representation of of weight l. For simplicity, we assume that the central characters of Π and π are trivial.

In [5, Main Thereom], we showed an algebraicity of special values of degree 8 L-functions $L(s,\Pi\times\pi)$ at various critical points when $l=k_1$ and $k_2=0$, which conforms with Yoshida's conjecture. When $D\simeq\mathrm{Mat}_{2\times 2}(\mathbb{Q})$, the algebraicity for this L-function was studied by various people; Furusawa [3], Böcherer–Heim[1], Pitale–Schmidt [7], Saha [9] [10] and Pitale–Saha–Schmidt [8]. In [6], we generalize [5, Main Thereom] to mixed weight cases including vector valued cases extending the method in [5].

Theorem 1 ([6])**.** *Let Π and π be as above. Assume that*

$$2k_2+6<l<2k_1+2k_2-6.$$

Put

$$c(k_1,k_2,l)=\max\left\{l-2k_2,2k_1+2k_2-l\right\}.$$

Let m be an integer such that

$$2<m\leq\frac{c(k_1,k_2,l)}{2}-1.$$

Then we have

$$\frac{L\left(m,\Pi\times\pi\right)}{\pi^{4m}\langle\Phi,\Phi\rangle\langle\Psi,\Psi\rangle}\in\overline{\mathbb{Q}}$$

where $\Phi \in V_\Pi$ and $\Psi \in V_\pi$ are arithmetic automorphic forms over $\overline{\mathbb{Q}}$ in the sense of Harris [4]*. Here, we define inner products by*

$$\langle \Phi, \Phi \rangle = \int_{G_D(\mathbb{Q})\mathbb{A}_{\mathbb{Q}}^\times \backslash G_D(\mathbb{A}_{\mathbb{Q}})} \langle \Phi(h), \Phi(h) \rangle_{(k_1,k_2)}\, dh$$

and

$$\langle \Psi, \Psi \rangle = \int_{\mathrm{GL}(2,\mathbb{Q})\mathbb{A}_{\mathbb{Q}}^\times \backslash \mathrm{GL}(2,\mathbb{A}_{\mathbb{Q}})} \Psi(g)\overline{\Psi(g)}\, dg$$

with the Tamagawa measures dh and dg.

From this algebraicity, we can show the following period relation.

Corollary 1. *Let (Π, V_Π) be as above. Assume that Π_v is tempered for almost all finite places v and that $k_1 \geq 8$. Further, suppose that there exists an irreducible cuspidal automorphic representation $(\Pi', V_{\Pi'})$ of $\mathrm{GSp}(4, \mathbb{A}_{\mathbb{Q}})$ such that*

$\Pi_\infty \simeq \Pi'_\infty$ and $\Pi_v \simeq \Pi'_v$ at almost all finite places v where $G_D(\mathbb{Q}_v) \simeq \mathrm{GSp}(4, \mathbb{Q}_v)$. Then for arithmetic automorphic forms $\Phi \in V_\Pi$ and $\Phi' \in V'_\Pi$, we have

$$\frac{\langle \Phi, \Phi \rangle}{\langle \Phi', \Phi' \rangle} \in \overline{\mathbb{Q}}.$$

Remark 1. *In* [6]*, we prove a similar algberaicity result over any totally real field without an assumption on central characters. Further, we prove the Galois equivariance of special values.*

Remark 2. *Saha* [11] *proved a period relation for Yoshida lifts using* [5, Main Theorem]*. In a similar argument as in* [11]*, we can generalize his result to a vector valued case using Theorem 1.*

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CM values of automorphic Green functions and L-functions

Tonghai Yang

1. Introduction

In 1980s, Gross and Zagier discovered a deep and direct relation between the height of a CM point in $J_0(N)$ and the central derivative of some Rankin-Selberg L-function [9] and its little cousin—a beautiful factorization formula for singular moduli [8]. In this talk, we will explain a new approach to these results and possible generalization to high dimensional Shimura varieties of orthogonal type $(n, 2)$ and unitary type $(n, 1)$, although our main focus in this talk is on the orthogonal type. The main ideas are regularized theta liftings started by Borcherds [2], Siegel-Weil formula, and a nice relation between incoherent Eisenstein series and coherent Eisenstein series.

2. Shimura Varieties, special divisors, and automorphic green functions

Let (V, Q) be a rational quadratic space over $\mathbb{Q}$ of signature $(n, 2)$. Let $H = \mathrm{GSpin}(V)$ and let $\mathbb{D}$ be the Hermitian domain of oriented negative 2-planes in $V_{\mathbb{R}}$. To a compact open subgroup K of $H(\mathbb{A}_f)$, one associates a Shimura variety X_K over $\mathbb{Q}$ with

$$X_K(\mathbb{C}) = H(\mathbb{Q})\backslash\mathbb{D} \times H(\mathbb{A}_f)/K.$$

For an element $x \in V$ with $Q(x) > 0$ and an element $h \in H(\mathbb{A}_f)$, one defines a natural divisor $Z(x, h)$ of X_K over $\mathbb{Q}$ as follows. Let

$$\mathbb{D}_x = \{z \in \mathbb{D} : z \perp x\}, \quad H_x = \{h \in H : h(x) = x, \text{ and } h(x^\perp) \subset x^\perp\}.$$

Then

$$Z(x, h)(\mathbb{C}) = (H_x(\mathbb{Q})\backslash\mathbb{D}_x \times H(\mathbb{A}_f)/(H(\mathbb{A}_f) \cap hKh^{-1}) \to X_K(\mathbb{C}), \ [z, h_1] \mapsto [z, h_1 h].$$

For every Schwartz function $\phi(\in S(V_f)^K)$, and $m \in \mathbb{Q}_{>0}$, one has Kudla's weighted special divisor ([10])

$$Z(m, \phi) = \sum_{h \in H_{x_0}\backslash H(\mathbb{A}_f)/K} Z(x_0^\perp, h)\phi(h^{-1}x_0) \in Z^1(X_K)$$

if there is some $x_0 \in V$ with $Q(x_0) = m$. Otherwise, we take $Z(m, \phi) = 0$. The weighted special divisors behave well under pullback, and does not depends on the choice of K.

Now let L be an even integral lattice of V. Let $S_L = \mathbb{C}[L'/L] \subset S(V_f)$. We assume for simplicity that K preserves L and acts trivially on L'/L. There is a Weil representation ω_L of $\mathrm{SL}_2(\mathbb{Z})$ on S_L, induced from its action on $S(V_f)$. Let $H_{1-\frac{n}{2}}(\omega_L)$ be the space of harmonic Maass forms $f : \mathbb{H} \to S_L$ of weight $1 - \frac{n}{2}$ and Weil representation ω_L ([4], [5], or [7]), one has Fourier expansion

$$f(\tau) = f^+(\tau) + f^-(\tau) = \sum_{m \gg \infty} c_f^+(m) q^m + \sum_{m<0} c_f^-(m) \Gamma(\frac{n}{2}, 4\pi|m|v) q^m.$$

Here $c_f^\pm(m) \in S_L$ and $\Gamma(s,x)$ is the partial Gamma function. The following theorem is due to Borcherds [2], Bruinier and Funke [3], and Schofer [11]:

Theorem 1. *Let*

$$\Phi(z,h,f) = \int_{\mathrm{SL}_2(\mathbb{Z})\backslash\mathbb{H}}^{reg} f(\tau)\theta_L(\tau,z,h) d\mu(\tau)$$

be the regularized theta lifting. Here θ_L is a usual Siegel theta function, viewed as a $(S_L^\vee, \omega_L^\vee)$-valued modular form of weight $\frac{n}{2} - 1$. Assume $c_f^+(m)$ is integral for $m < 0$. Then

1. *$\Phi(z,h,f)$ is a Green function for $Z(f) = \sum_{m>0} Z(m, c_f^+(-m))$. Moreover, it is harmonic if $c_f^+(0)(0) = 0$.*
2. *$\Phi(z,h,f)$ is well-defined everywhere on X_K.*
3. *When f is weakly holomoprhic, there is a memomorphic automorphic form $\Psi(f)$ with $Div(\Psi) = Z(f)$ and*

$$-\log|\Psi(f)|^2_{Pet} = \Phi(f).$$

Moreover, when V is isotropic, $\Psi(f)$ has Borcherds product expansion near a cusp.

3. Small CM values and Rankin-Selberg L-function

Let $U \subset V$ be a rational negative 2-plane. Then $U_\mathbb{R}$ gives two points (with two orientations) $z_U^\pm$ in $\mathbb{D}$. Let $k = \mathbb{Q}(\sqrt{-\det U})$ be an imaginary quadratic field. Then $\mathrm{GSpin}(U) = k^\times$, and we have a special small CM 0-cycle

$$Z(U) = \{z_U^\pm\} \times k^\times \backslash k_f^\times / U_K \to X_K, \quad U_K = k_f^\times \cap K,$$

in X_K, defined over $\mathbb{Q}$. The subspace U also gives orthogonal decomposition

$$V = V^+ \oplus U, \quad L \supset \mathcal{P} \oplus \mathcal{N}, \quad \mathcal{P} = L \cap V^+, \mathcal{N} = L \cap U.$$

Associated to $\mathcal{P}$ is a holomorphic modular form $\theta_\mathcal{P}$ valued in $S_\mathfrak{P}^\vee$ of weight $\frac{n}{2}$ and representation $\omega_\mathcal{P}^\vee$. Associated to $\mathcal{N}$ are a typical coherent Eisenstein series $E_\mathcal{N}(\tau, s, -1)$ and an incohrent Eisenstein series $E_\mathcal{M}(\tau, s, 1)$, both valued in $S_\mathcal{N}^\vee$ but with weight -1 and 1 respectively. They are related by

$$-2\bar{\partial}\left(E'_\mathcal{N}(\tau, 0; 1)\, d\tau\right) = E_\mathcal{N}(\tau, 0; -1)\, d\mu(\tau).$$

Let $\mathcal{E}_\mathcal{L}(\tau)$ be the 'holomorphic' part of $E'(\tau, 0, 1)$. Then Bruinier and I proved in 2009 [7] the following theorem, which is a simple generalization of Schofer's work on weakly holomorphic forms [11]. In Schofer's case $\xi(f) = 0$, so no L-function shows up.

Theorem 2. *Let $f \in H_{1-\frac{n}{2}}(\omega_L)$, and let $U \subset V$ be as above. Then*

$$\Phi(Z(U), f) = \deg Z(U) \left[CT(f^+ \theta_{\mathcal{P}} \mathcal{E}_{\mathcal{N}}) - L(\xi(f), \theta_{\mathcal{P}}, 0) \right].$$

Here

$$L(\xi(f), \theta_{\mathcal{P}}, s) = \langle \theta_{\mathcal{P}}(\tau) E_{\mathcal{N}}(\tau, s, 1), \xi(f) \rangle_{Pet}$$

is the Rankin-Selberg L-function of $\xi(f)$ and $\theta_{\mathcal{P}}$, which is automatically zero at $s = 0$.

When $n = 1$, we used it to give a totally different proof of a variant of the Gross-Zagier formula in the same article. When $n = 2$, Bruinier and I are working on to give a new proof of the Gross-Zagier formula. This formula also indicates some simple conjectural relation between Faltings height of a CM cycle and the central derivative of the Rankin-Selberg L-function. The conjectural formula was verified in special cases for $n \le 2$ in [7] and for general n in a upcoming joint work of Andreatta, Goren, Howard, and Mafapusi [1]. Its analogue in unitary case was proved by Bruinier, Howard, and myself [4].

4. Big CM values and L-series

In this section we assume that $n = 2d$ is even. Let F be a totally really number field of degree $d+1$ with real embeddings σ_i, $i = 0, 1, \cdots, d$. Let $W = (W, Q_F)$ be a quadratic space over F of signature $(0, 2)$ at σ_0 and $(2, 0)$ at other infinite primes. Let $\mathrm{Res}_{F/\mathbb{Q}} W$ be the $\mathbb{Q}$-vector space W with $\mathbb{Q}$-quadratic form $Q(x) = \mathrm{tr}_{F/\mathbb{Q}} Q_F(x)$. It is of signature $(2d, 2) = (n, 2)$. We assume $\mathrm{Res}_{F/\mathbb{Q}} W \cong V$. Then $W_{\sigma_0} = W \otimes_{F, \sigma_0} \mathbb{R}$ is a negative 2-plane of $V_{\mathbb{R}}$, and gives two big CM points $z_0^{\pm} \in X_K$. Clearly $\mathrm{Res}_{F/\mathbb{Q}} \mathrm{SO}(W) \subset \mathrm{SO}(V)$. Let T be the preimage of $\mathrm{Res}_{F/\mathbb{Q}} \mathrm{SO}(W)$ in $H = \mathrm{GSpin}(V)$. Then T is a maximal torus of H (thus the name big CM points). The associated CM cycle

$$Z(W, \sigma_0) = \{z_0^{\pm}\} \times T(\mathbb{Q}) \backslash T(\mathbb{A}_f)/K_T, \quad K_T = T(\mathbb{A}_f) \cap K$$

is defined over F. Let $Z(W)$ is the formal sum of its Galois conjugates (see [5] for more detailed description), which is a big CM cycle defined over $\mathbb{Q}$. Associated to L is an incohrent Hilbert Eisenstein series $E_L(\vec{\tau}, s)$ valued in $S_L^{\vee}$ of F of weight $(1, \cdots, 1)$, which is automatically zero at $s = 0$. Let $\mathcal{E}(\tau)$ be the 'holomorphic' part of $E_L'(\tau, 0)$ (with $\tau \in \mathbb{H}$ diagonally embedded into $\mathbb{H}^{d+1}$). Define

$$L(\xi(f), W, s) = \int_{\mathrm{SL}_2(\mathbb{Z}) \backslash \mathbb{H}} E_L(\tau, s) \overline{\xi(f)} v^{n+2} d\mu(\tau).$$

In [5], Bruinier, Kudla, and I proved the following theorem, which is a generalization of [6] and [8].

Theorem 3. *Let the notation be as above. Then*

$$\Phi(Z(W), f) = \deg Z(W) \left[CT(f^+ \mathcal{E}) - L'(\xi(f), W, 0) \right]$$

In the case $n = 2$, it is application to the Colmez conjecture [12].

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Eisenstein series in Kohnen plus space for Hilbert modular forms

Ren He Su

Let $r \geq 2$. In 1975, Cohen [1] introduced the so-called Cohen Eisenstein series $\mathcal{H}_r$ which is a modular form of weight $r + 1/2$ defined by

$$\mathcal{H}_r(z) = \zeta(1-2r)$$
$$+ \sum_{\substack{N\geq 0 \\ (-1)^r N \equiv 0,1 (\mathrm{mod} 4)}} \left(L(1-r, \chi_{D_{(-1)^r N}}) \sum_{d | f_{(-1)^r N}} \mu(d)\chi_{D_{(-1)^r N}d}(d) d^{r-1}\sigma_{2r-1}(f/d) \right) q^N$$

where for any integer n, D_n is the discriminant of $\mathbb{Q}(\sqrt{n})\mathbb{Q}$ and f_n is the positive integer such that $n = f_n^2 D_n$. Inspired by this, Kohnen [4] in 1980 introduced the plus spaces as

$$M^+_{r+1/2}(\Gamma_0(4)) = \left\{ f(z) = \sum_{(-1)^r N \equiv 0,1 (\mathrm{mod} 4)} a(N) q^N \in M_{r+1/2}(\Gamma_0(4)) \right\},$$
$$S^+_{r+1/2}(\Gamma_0(4)) = M^+_{r+1/2}(\Gamma_0(4)) \cap S_{r+1/2}(\Gamma_0(4)).$$

So we easily get that $\mathcal{H}_r \in M^+_{r+1/2}(\Gamma_0(4))$.

Recently, Hiraga and Ikeda [3] generalized the concept of Kohnen plus space to the case for general Hilbert modular forms of parallel weight. Let F be a totally real number field of degree n over $\mathbb{Q}$ with its ring of integers $\mathfrak{o}_F$ and different $\mathfrak{d}_F$ over $\mathbb{Q}$. We define the congruence subgroup $\Gamma \subset SL_2(F)$ by

$$\Gamma = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(F) \,\middle|\, a, d \in \mathfrak{o}_F, b \in \mathfrak{d}_F^{-1}, c \in 4\mathfrak{d}_F \right\}.$$

For any $\xi \in F$, we denote $\xi \equiv \square \pmod 4$ if there is an integer $x \in \mathfrak{o}_F$ such that $\xi - x^2 \in 4\mathfrak{o}_F$. Now let κ be an integer. The generalized Kohnen plus spaces are defined as

$$M^+_{\kappa+1/2}(\Gamma) = \left\{ f(z) = \sum_{(-1)^\kappa \xi \equiv \square (\mathrm{mod} 4)} a(\xi) q^\xi \in M_{\kappa+1/2}(\Gamma) \right\},$$
$$S^+_{\kappa+1/2}(\Gamma) = M^+_{\kappa+1/2}(\Gamma) \cap S_{\kappa+1/2}(\Gamma).$$

Here for any $z \in \mathfrak{h}^n$ and $\xi \in F$, $q^\xi = \exp(2\pi\sqrt{-1}\mathrm{Tr}(z\xi))$. So the definition coincides with the plus space given by Kohnen for the case $F = \mathbb{Q}$. Some analogues of the results of Kohnen are also showed by Hiraga and Ikeda. Now what we want to do is to get a generalization of the Cohen Eisenstein series in the generalized plus spaces. Indeed, we have the following theorem.

Theorem. Let κ be a positive integer which is not 1 if $F \neq \mathbb{Q}$ and χ' be a character of the ideal class group of F. Then we have $G(z) = G_{\kappa+1/2}(z, \chi') \in M^+_{\kappa+1/2}(\Gamma)$ which is defined by

$$G(z) = L_F(1-2\kappa, \overline{\chi'}^2) + \sum_{\substack{(-1)^\kappa \xi \equiv \square \bmod 4 \\ \xi \succ 0}} \chi'(\mathfrak{D}_{(-1)^\kappa \xi}) L_F(1-\kappa, \overline{\chi_{(-1)^\kappa \xi} \chi'}) \mathfrak{C}_\kappa((-1)^\kappa \xi) q^\xi.$$

where

$$\mathfrak{C}_\kappa(\xi) = \sum_{\mathfrak{a} | \mathfrak{F}_\xi} \mu(\mathfrak{a}) \chi_\xi(\mathfrak{a}) \chi'(\mathfrak{a}) N_{F/\mathbb{Q}}(\mathfrak{a})^{\kappa-1} \sigma_{2\kappa-1, \chi'^2}(\mathfrak{F}_\xi \mathfrak{a}^{-1}).$$

Here $\mathfrak{D}_\xi$ is the relative discriminant of $F(\sqrt{\xi})/F$, $\mathfrak{F}_\xi^2 \mathfrak{D}_\xi = (\xi)$, $\mathfrak{a}$ runs over all integral ideals dividing $\mathfrak{F}_\xi$, μ is the Möbius function for ideals and $\sigma_{k,\chi}$ is the sum of divisors function twisted by χ, that is,

$$\sigma_{k,\chi}(\mathfrak{A}) = \sum_{\mathfrak{b} | \mathfrak{A}} N_{F/\mathbb{Q}}(\mathfrak{b})^k \chi(\mathfrak{b})$$

for any integral ideal $\mathfrak{A}$ of F. Moreover, G is a Hecke eigenform.

Thus if h is the class number of F, then we got h such Eisenstein series. Also, we have that the Eisenstein series span the whole Kohnen plus space with the cusp forms. We write this in a theorem.

Theorem. The Kohnen plus space $M^+_{\kappa+1/2}(\Gamma)$ is a vector space over $\mathbb{C}$ spanned by cusp forms and the h Eisenstein series we got in the last theorem, that is,

$$M^+_{\kappa+1/2}(\Gamma) = S^+_{\kappa+1/2}(\Gamma) \oplus \bigoplus_{j=1}^{h} \mathbb{C} \cdot G_{\kappa+1/2}(z, \chi_j)$$

where $\chi_1, ..., \chi_h$ are the h distinct characters of the class group of F.

Together with the results of Ikeda and Hiraga [3], we get that $M^+_{\kappa+1/2}(\Gamma)$ is a direct sum of spaces spanned by Hecke eigenforms.

It is known that Cohen [1] used his Eisenstein series to give a generalization of Hurwitz's class number relation. Also Eichler and Zagier [2] showed that Cohen Eisenstein series have a deep relation with the Jacobi-Eisenstein series and Siegel modular forms of degree 2. One may expects that the generalized Cohen Eisenstein series can give some analogues of those results.

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Lattices with many Borcherds products

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(joint work with Jan Hendrik Bruinier, Eberhard Freitag)

In our joint [4] work we prove that there are only finitely many isomorphism classes of even lattices L of signature $(2, n)$ for which the space of cusp forms of weight $1 + n/2$ for the Weil representation of the discriminant group of L is trivial and compute the list of these lattices. They have the property that every Heegner divisor for the orthogonal group of L can be realized as the divisor of a Borcherds product. We obtain similar classification results in greater generality for finite quadratic modules.

Let L be an even lattice of signature $(2, n)$ and write $\mathrm{O}(L)$ for its orthogonal group. In his celebrated paper [1] R. Borcherds constructed a map from vector valued weakly holomorphic elliptic modular forms of weight $1-n/2$ to meromorphic modular forms for $\mathrm{O}(L)$ whose zeros and poles are supported on Heegner divisors. Since modular forms arising in this way have particular infinite product expansions, they are often called *Borcherds products*. They play important roles in different areas such as Algebraic and Arithmetic Geometry, Number Theory, Lie Theory, Combinatorics, and Mathematical Physics.

By Serre duality, the obstructions for the existence of weakly holomorphic modular forms with prescribed principal part at the cusp at ∞ are given by vector valued cusp forms of dual weight $1 + n/2$ transforming with the Weil representation associated with the discriminant group of L [2]. In particular, if there are no non-trivial cusp forms of this type, then there are no obstructions, and every Heegner divisor is the divisor of a Borcherds product. A lattice with this property is called *simple*.

It was conjectured by E. Freitag that there exist only finitely many isomorphism classes of such simple lattices. Under the assumptions that $n \geq 3$ and that the Witt rank of L is 2, it was proved by M. Bundschuh that there is an upper bound on the determinant of a simple lattice [5]. Unfortunately, this bound is very large and therefore not feasible to obtain any classification results. The argument of [5] is based on volume estimates for Heegner divisors and the singular weight bound for holomorphic modular forms for $\mathrm{O}(L)$.

We show that for any $n \geq 1$ (without any additional assumption on the Witt rank) there exist only finitely many isomorphism classes of even simple lattices of signature $(2, n)$. Second, we develop an efficient algorithm to determine all these lattices.

Along the way we obtain several results on modular forms associated with finite quadratic modules which are of independent interest and which we now briefly describe. A finite quadratic module is a pair consisting of a finite abelian group A together with a $\mathbb{Q}/\mathbb{Z}$-valued non-degenerate quadratic form Q on A, see [7], [9]. Important examples of finite quadratic modules are obtained from lattices. If L is an even lattice with dual lattice L', then the quadratic form on L induces a $\mathbb{Q}/\mathbb{Z}$-valued quadratic form on the discriminant group L'/L.

Recall that there is a Weil representation ρ_A of the the metaplectic extension $\mathrm{Mp}_2(\mathbb{Z})$ of $\mathrm{SL}_2(\mathbb{Z})$ on the group ring $\mathbb{C}[A]$ of a finite quadratic module A. If $k \in \frac{1}{2}\mathbb{Z}$, we write $S_{k,A}$ for the space of cusp forms of weight k and representation ρ_A for the group $\mathrm{Mp}_2(\mathbb{Z})$. For simplicity we assume throughout that $2k \equiv -\operatorname{sig}(A) \pmod 4$, since our application to simple lattices will only concern this case. We say that a finite quadratic module A is k-simple if $S_{k,A} = \{0\}$. With this terminology, an even lattice L is simple if and only if L'/L is $(1 + n/2)$-simple.

The dimension of the space $S_{k,A}$ can be computed by means of the Riemann-Roch theorem. Therefore a straightforward approach to showing that there are nontrivial cusp forms consists in finding lower bounds for the dimension of $S_{k,A}$. Unfortunately, the dimension formula involves rather complicated invariants of ρ_A at elliptic and parabolic elements, and it is a non-trivial task to obtain strong lower bounds. We show that the following asymptotic holds.

Theorem. *If $\varepsilon > 0$, then*

$$\dim(S_{k,A}) - \dim(M_{2-k,A(-1)}) = |A/\{\pm 1\}| \cdot \left(\frac{k-1}{12} + O_\varepsilon(N_A^{\varepsilon - 1/2}) \right)$$

for every finite quadratic module A and every weight $k \geq 3/2$ with $2k \equiv -\operatorname{sig}(A)$ $\pmod 4$. Here N_A is the level of A, and $A(-1)$ denotes the abelian group A equipped with the quadratic forms $-Q$. The constant implied in the Landau symbol is independent of A and k.

An a corollary we can give an affirmative answer to the conjecture by E. Freitag.

Corollary. *Let $r_0 \in \mathbb{Z}_{\geq 0}$. There exist only finitely many isomorphism classes of finite quadratic modules A with minimal number of generators $\leq r_0$ such that $S_{k,A} = \{0\}$ for some weight $k \geq 3/2$ with $2k \equiv -\operatorname{sig}(A) \pmod 4$.*

In particular, since $\dim S_{k,A} > 0$ for $k > 14$, there are only finitely many isomorphism classes of simple lattices. Note that there do exist infinitely many isomorphism classes of 1/2-simple finite quadratic modules, which has been shown by Skoruppa [8].

Moreover, we remark that bounding the minimal number of generators is essential.

Example. *If* $A = 3^{\varepsilon n}$ *with* $n \in \mathbb{Z}_{>0}$ *odd and* $\varepsilon = (-1)^{\frac{n-1}{2}}$*, then* $\operatorname{sig}(A) \equiv 2 \pmod 4$ *and* $S_{3,A} = \{0\}$.

This follows for instance from the dimension formula in [6], Chapter 5.2.1, p. 93.

Unfortunately, the implied constant in the Landau symbol in the above theorem is large. Therefore, it is a difficult task to compute the list of all k-simple finite quadratic modules for a bounded number of generators. We develop an efficient algorithm to address this problem. The idea is to first compute all *anisotropic* finite quadratic modules that are k-simple for some k. To this end we derive an explicit formula for $\dim(S_{k,A})$ in terms of class numbers of imaginary quadratic fields and dimension bounds that are strong enough to obtain a classification.

Next we employ the fact that an arbitrary finite quadratic module A has a unique anisotropic quotient A_0, and that there are intertwining operators for the corresponding Weil representations. For the difference $\dim S_{k,A} - \dim S_{k,A_0}$ very efficient bounds can be obtained. This can be used to classify all k-simple finite quadratic modules with a bounded number of generators.

Finally, all simple *lattices* of signature $(2, n)$ can be found by a applying a criterion of Nikulin [7] to determine which of these simple discriminant forms arise as discriminant groups L'/L of even lattices L of signature $(2, n)$.

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A geometrical approach to Jacobi forms, revisited

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(joint work with José Burgos Gil)

1. Introduction

Arakelov theory [3] was created to compute heights of rational points or, more generally, of cycles on varieties defined over number fields using arithmetic intersections. However, the original theory was limited to the use of vector bundles equipped with *smooth* hermitian metrics. By the work [1], Arakelov theory was extended to allow to incorporate vector bundles equipped with *logarithmically singular* hermitian metrics. This led to interesting applications for Shimura varieties of non-compact type and their automorphic vector bundles equipped with the natural invariant hermitian metric, e.g., a general foundation for the height used by Faltings in his proof of Mordell's conjecture; for further examples, see [5]. The key ingredient of our generalization was Mumford's observation [6] that Chern-Weil theory continues to apply in the case of logarithmically singular metrics.
Our next goal is to generalize arithmetic intersection theory to the case of *mixed* Shimura varieties of non-compact type. It turned out that new problems arise, namely that the natural invariant metrics of the natural vector bundles have singularities which are worse than logarithmically singular, at least in codimension 2. Therefore, we have begun in [2] by studying the simplest non-trivial example, on which we report here, namely the hermitian line bundle associated to the classical theta function $\theta_{1,1}$ on the universal elliptic curve over a modular curve.
The set-up is as follows: Let $\Gamma = \Gamma(N)$ $(N \geq 3)$ be the principal congruence subgroup of level N acting by fractional linear transformations on the upper half-plane $\mathbb{H}$. We let $Y(N) := \Gamma(N)\backslash\mathbb{H}$ and $E^0(N) := \Gamma(N) \ltimes \mathbb{Z}^2\backslash\mathbb{H} \times \mathbb{C}$. The modular curve $X(N)$ is obtained from $Y(N)$ by adding the cusps $P_1, \ldots, P_{p_N}$ and the universal elliptic curve $E(N)$ is obtained by compactifying $E^0(N)$ by N-gons $\bigcup_{\nu=0}^{N-1} \Theta_{j,\nu}$ ($\Theta_{j,\nu} \cong \mathbb{P}^1_{\mathbb{C}}$ with self-intersection -2) over the cusps P_j $(j = 1, \ldots, p_N)$.
We denote by $J_{k,m}(\Gamma(N))$ the $\mathbb{C}$-vector space of Jacobi forms of weight k, index m with respect to $\Gamma(N)$. We recall from [4] that the factor of automorphy in the definition of Jacobi forms gives rise to a 1-cocycle in $H^1(\Gamma(N) \ltimes \mathbb{Z}^2, \mathbb{C}^\times)$, and hence, to a line bundle $L^0_{k,m}$ on $E^0(N)$. Letting $j\colon E^0(N) \longrightarrow E(N)$ be the inclusion map, it has been shown in [4] that there is a distinguished subsheaf $\mathcal{F}_{k,m}$ of $j_*L^0_{k,m}$ such that $J_{k,m}(\Gamma(N)) \cong H^0(E(N), \mathcal{F}_{k,m})$, which enabled us to determine the dimension of $J_{k,m}(\Gamma(N))$ using the Riemann-Roch theorem on the surface $E(N)$.
Finally, we note that for $f \in J_{k,m}(\Gamma(N))$, the natural invariant metric is given by

$$\|f(\tau, z)\|^2_{\mathrm{Pet}} := |f(\tau, z)|^2\, e^{-4\pi m y^2/\eta}\, \eta^k \quad (\tau = \xi + i\eta \in \mathbb{H},\ z = x + iy \in \mathbb{C}).$$

It induces a hermitian metric $\|\cdot\|_{\mathrm{Pet}}$ on $L^0_{k,m}$; we put $\overline{L}^0_{k,m} := (L^0_{k,m}, \|\cdot\|_{\mathrm{Pet}})$.

2. Some definitions

Let X be a smooth, complex, projective variety of complex dimension d, $D \subset X$ a normal crossing divisor, and $U := X \setminus D$ with embedding $j\colon U \hookrightarrow X$. We call an open coordinate neighborhood V of X with coordinates $z_1, \ldots, z_d$ *adapted to* D, if D is locally given by the equation $z_1 \cdot \ldots \cdot z_k = 0$ for some $k \in \{1, \ldots, d\}$.

Definition. Let L be a line bundle on X and $\|\cdot\|$ a smooth hermitian metric on $L|_U$. We say that $\|\cdot\|$ has *logarithmic growth (along D)*, if for all $x \in X$, there is a coordinate neighborhood V of x adapted to D, a nowhere vanishing regular section s of L on V, and an integer $M > 0$ such that

$$\prod_{j=1}^{k} \log\left(\frac{1}{|z_j|}\right)^{-M} \ll \|s(z_1, \ldots, z_d)\| \ll \prod_{j=1}^{k} \log\left(\frac{1}{|z_j|}\right)^{M} \quad (|z_j| < e^{-e}).$$

Definition. We say that *a smooth hermitian line bundle* $\overline{L}^0 := (L^0, \|\cdot\|)$ *on* U *admits a Mumford-Lear extension to* X, if the following exist: A positive integer e, a line bundle L on X, an algebraic subset $S \subset D \subset X$ with $\operatorname{codim}_X(S) \geq 2$, a smooth hermitian metric $\|\cdot\|$ on $L|_U$ with logarithmic growth along $D \setminus S$, and an isometry $\alpha\colon (L^0, \|\cdot\|)^{\otimes e} \longrightarrow (L|_U, \|\cdot\|)$. The 5-tuple $(e, L, S, \|\cdot\|, \alpha)$ is called *a Mumford-Lear extension of* $\overline{L}^0$.

We introduce the directed set (with the obvious morphisms)

$$\begin{aligned}\operatorname{Bir}(X) :=\{&Y \text{ smooth, complex, projective variety} \mid \pi_Y\colon Y \longrightarrow X \text{ proper, bira-}\\ &\text{tional morphim such that } D_Y := \pi_Y^{-1}(D) \text{ normal crossing divisor}\}.\end{aligned}$$

Definition. We say that $\overline{L}^0$ *admits all Mumford-Lear extensions over* X, if $\pi_Y^*\overline{L}^0$ admits a Mumford-Lear extension from $U_Y := Y \setminus D_Y$ to Y for all $Y \in \operatorname{Bir}(X)$.

Remark. If $Y \in \operatorname{Bir}(X)$, s a rational section of L^0 (which can be viewed as a rational section of $\pi_Y^* L^0$), and $(e', L', S', \|\cdot\|', \alpha')$ is a Mumford-Lear extension of $\pi_Y^*\overline{L}^0$ to Y, we have the $\mathbb{Q}$-Cartier divisor $\operatorname{div}_Y(s) := e'^{-1}\operatorname{div}(\alpha'(s^{\otimes e'}))$.

Definition. Assume that $\overline{L}^0$ admits all Mumford-Lear extensions over X, and let s be a rational section of L^0. The b-*divisor associated to* s is defined as

$$\operatorname{div}(s) := \big(\operatorname{div}_Y(s)\big)_{Y \in \operatorname{Bir}(X)}.$$

Definition. A b-divisor $C = (C_Y)_{Y \in \operatorname{Bir}(X)}$ on a surface X is called *integrable*, if the limit $C \cdot C$ of intersection numbers $C_Y \cdot C_Y$ over $Y \in \operatorname{Bir}(X)$ exists.

3. First results and concluding remarks

Let $X := E(N)$, $D := E(N) \setminus E^0(N)$, let S denote the double points of D, and write H for the image of the zero section from $X(N)$ to $E(N)$. We then introduce

$$C := 8H + \sum_{j=1}^{p_N} \sum_{\nu=0}^{N-1} \left(N - 4\nu + \frac{4\nu^2}{N}\right)\Theta_{j,\nu} \quad \text{and} \quad L_{4\ell,4\ell} := \mathcal{O}_{E(N)}(\ell\, C).$$

Proposition. The 5-tuple $(1, L_{4\ell,4\ell}, S, \|\cdot\|_{\mathrm{Pet}}, \alpha)$ is a Mumford-Lear extension of the smooth hermitian line bundle $\overline{L}^0_{4\ell,4\ell}$ to $E(N)$ with $\alpha\colon \overline{L}^0_{4\ell,4\ell} \longrightarrow \overline{L}_{4\ell,4\ell}|_{E^0(N)}$

induced by the assignment $\theta_{1,1}^{8\ell} \mapsto s$, where s is chosen such that $\mathrm{div}(s) = \ell\, C$. The *proof* consists in determining the divisor of $\theta_{1,1}^{8\ell}$ on the surface $E(N)$ and in showing that the Petersson metric $\|\cdot\|_{\mathrm{Pet}}$ is of logarithmic growth on $D \setminus S$.

Theorem. The line bundle $\overline{L}^0_{4\ell,4\ell}$ admits all Mumford-Lear extensions over $E(N)$. The associated b-divisor $\mathrm{div}(\theta_{1,1}^{8\ell})$ is integrable, and we have the formula

$$\mathrm{div}\big(\theta_{1,1}^{8\ell}\big) \cdot \mathrm{div}\big(\theta_{1,1}^{8\ell}\big) = \frac{16\, p_N N\, \ell^2}{3}\,. \tag{1}$$

Concluding remarks. (i) We note that formula (1) can be rewritten as

$$\mathrm{div}\big(\theta_{1,1}^{8\ell}\big) \cdot \mathrm{div}\big(\theta_{1,1}^{8\ell}\big) = (4\ell)(4\ell)\,[\mathrm{PSL}_2(\mathbb{Z}) : \Gamma(N)]\,\frac{\zeta_{\mathrm{MT}}(2,2,2)}{\zeta(6)}\,,$$

where $\zeta(s)$ and $\zeta_{\mathrm{MT}}(s,s,s)$ are the Riemann and the Mordell-Tornheim ζ-function, respectively, and 4ℓ is the weight as well as the index of the Jacobi form in question.
(ii) By a suitable residue calculation, one can show that

$$\mathrm{div}\big(\theta_{1,1}^{8\ell}\big) \cdot \mathrm{div}\big(\theta_{1,1}^{8\ell}\big) = \int\limits_{E(N)} \mathrm{c}_1\big(\overline{L}_{4\ell,4\ell}\big)^{\wedge 2}.$$

(iii) Formula (1) has a nice toric interpretation as a limit of volumes of polytopes.
(iv) In compatibility with a Hilbert-Samuel formula for $\dim_{\mathbb{C}} J_{4\ell,4\ell}(\Gamma(N))$, one has

$$J_{4\ell,4\ell}\big(\Gamma(N)\big) = \varprojlim_{Y \in \mathrm{Bir}(E(N))} H^0\big(Y, \pi_Y^* L_{4\ell,4\ell}\big),$$

which allows to interpret Jacobi forms as (a limit of) global sections of a line bundle rather than as global sections of the subsheaf $\mathcal{F}_{4\ell,4\ell}$ of $j_* L^0_{4\ell,4\ell}$.
(v) By working on the Riemann-Zariski space

$$\mathfrak{X} := \varprojlim_{Y \in \mathrm{Bir}(E(N))} Y,$$

we can apply our generalization of Arakelov theory [1], there. Resulting (limit) calculations will be made explicit in our future research, e.g., by determining the arithmetic degree of the arithmetic b-divisor $\widehat{\mathrm{div}}(\theta_{1,1}^{8\ell}) := (\mathrm{div}(\theta_{1,1}^{8\ell}), \|\cdot\|_{\mathrm{Pet}})$.

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