Mathematisches Forschungsinstitut Oberwolfach

Report No. 26/2014

DOI: 10.4171/OWR/2014/26

Algebraic Structures in Low-Dimensional Topology

Organised by
Louis Hirsch Kauffman, Chicago
Vassily Olegovich Manturov, Moscow
Kent E. Orr, Bloomington
Robert Schneiderman, New York

25 May – 31 May 2014

Abstract. The workshop concentrated on important and interrelated invariants in low dimensional topology. This work involved virtual knot theory, knot theory, three and four dimensional manifolds and their properties.

Keywords. geometric topology, knot theory, virtual knot theory, invariants, parity, graph links, free knots, knot cobordism, virtual knot cobordism, groups, fundamental groups, braids, representations of groups, skein theory, knot polynomials, quandles, skein modules, quandle cohomology, distributive cohomology, manifolds, surgery.

Mathematics Subject Classification (2010): 57M25.

Introduction by the Organisers

The workshop "Algebraic Structures in Low-Dimensional Topology" organized by Louis Kauffman, Vassily Manturov, Kent Orr and Robert Schneidermann was well attended, with over 25 participants from an international commmunity of researchers. Talks were given on a wide variety of topics, including both three and four dimensional geometric topology, knot theory and virtual knot theory. The subject areas of this conference included specifically algebraic and combinatorial approaches to invariants sucn as parity in the theory of graph links, free knots and virtual knot theory, uses of surfaces and curves on surfaces to understand virtual knot cobordism and to understand relationships between classical and virtual knots, orderability in groups and fundamental groups, new approaches to the Alexander polynomial, braids and representations of braid groups, relationships of representation theory with the skein theory of knot polynomials, structure of quandles, structure of skein modules, and extensions of ideas in quandle cohomology to distributive cohomology. Along with these combinatorial and algebraic ideas there was much discussion of geometric/topological techniques such as branched coverings, structures on manifolds, cobordisms, surgery and dymnamics of surgery, and even relationships between Fourier series and representations of braids. There are many challenging problems in low dimensional topology, and a remarkable number of fertile ideas and methods. This conference was an excellent meeting place for the participants to work and share their ideas.

Acknowledgement: The MFO and the workshop organizers would like to thank the National Science Foundation for supporting the participation of junior researchers in the workshop by the grant DMS-1049268, “US Junior Oberwolfach Fellows”.

Workshop: Algebraic Structures in Low-Dimensional Topology

Table of Contents

Abstracts

Some very good formulas for the Alexander polynomial

Dror Bar-Natan

I will describe some very good formulas for a (matrix plus scalar)-valued extension of the Alexander polynnomial to tangles, then say that everything extends to virtual tangles, then roughly to simply knotted balloons and hoops in 4D, then the target space extends to (free Lie algebras plus cyclic words), and the result is a universal finite type of the knotted objects in its domain. Taking a cue from the BF topological quantum field theory, everything should extend (with some modifications) to arbitrary codimension-2 knots in arbitrary dimension and in particular, to arbitrary 2-knots in 4D. But what is really going on is still a mystery.

My talk's handout, video, and further links are at
`http://www.math.toronto.edu/~drorbn/Talks/Oberwolfach-1405/`.

Braids and the concordance group

Michael Brandenbursky
(joint work with Jarek Kedra)

The goal of this talk was to establish a non-trivial relation between a braid group $\mathbf{B}_n$ and the smooth concordance group $\mathrm{Conc}(\mathbf{S}^3)$ of knots in $\mathbf{S}^3$. Moreover, a discovery of this relation enabled us to answer the following 3 questions, which are apparently unrelated.

(1) This question is attributed to Livingston and Calegari [4]. Is there a connection between the stable commutator length scl in groups and the stable four ball genus in $\mathrm{Conc}(\mathbf{S}^3)$?

(2) This question was asked by Burago-Ivanov-Polterovich in their influential paper on diffeomorphism groups [3]. Roughly speaking they asked whether the existence of quasi-morphisms is equivalent to the existence of stably unbounded norms. More precisely, they asked the following: Does there exists a perfect group whose scl is zero, but which admits a stably unbounded conjugation-invariant norm?

(3) The following questions were asked by Paolo Lisca. Are there infinitely many non slice knots in the family $\{\widehat{(\sigma_1\sigma_2^{-1})^k}\}_{k=1}^{\infty}$? If yes, then what is the cardinality of this set? Here σ_i is the $i-th$ Artin generator of $\mathbf{B}_3$ and $\widehat{\alpha}$ denotes the closure of a braid α.

Let $n \in \mathbf{N}$. In [1] we defined a map $\Psi_n : \mathbf{B}_n \to \mathrm{Conc}(\mathbf{S}^3)$ which takes a braid $\alpha \in \mathbf{B}_n$ and associates to it a concordance class of the knot $\widehat{\alpha\sigma_\alpha}$, where σ_α is a suitably chosen braid in $\mathbf{B}_n$.

Our main theorem is the following:

Theorem 1 ([1]). *The map* $\Psi_n : \mathbf{B}_n \to \mathrm{Conc}(\mathbf{S}^3)$ *is a quasihomomorphism with respect to the four ball genus norm and with defect* $D_{\Psi_n} \leq 3n + 1$*. Its image contains all concordance classes represented by knots which are closures of braids on n strings. In addition, if* $n \in \mathbf{N} \cup \{\infty\}$ *this map is Lipschitz with respect to the biinvariant word norm on the braid group and the four ball genus norm* g_4 *on the concordance group. More precisely,*

$$\mathrm{g}_4(\Psi_n(\alpha)) \leq \frac{1}{2}\|\alpha\|$$

for all braids $\alpha \in \mathbf{B}_n$*.*

Let $[\mathbf{B}_n, \mathbf{B}_n]$ denote the commutator subgroup of $\mathbf{B}_n$. Theorem 1 yields 3 corollaries which answer positively the two first questions.

Corollary 2. *Let* $\alpha \in [\mathbf{B}_n, \mathbf{B}_n]$*. Then if the stable commutator length of* α *is trivial then the stable four ball genus of* $\Psi_n(\alpha)$ *is bounded above by the defect* D_{Ψ_n}*:*

$$\mathrm{scl}(\alpha) = 0 \qquad \Longrightarrow \qquad \mathrm{sg}_4(\Psi_n(\alpha)) \leq D_{\Psi_n}.$$

Corollary 3. *Let* $\alpha \in [\mathbf{B}_n, \mathbf{B}_n]$*. If* $\mathrm{scl}(\alpha) = 0$ *then the concordance classes* $\Psi_n(\alpha^k)$*, for* $k \in \mathbf{Z}$*, have uniformly bounded four ball genus.*

Corollary 4. *The commutator subgroup* $[\mathbf{B}_\infty, \mathbf{B}_\infty]$ *of the infinite braid group satisfies the conditions of the second question and hence gives a solution to the question of Burago-Ivanov-Polterovich.*

In a recent work in progress [2] the first author proved the following theorem, which in particular answers positively the first question of Paolo Lisca.

Theorem 5. *Let* $k \neq 0 \bmod 3$ *and* $\alpha_k = (\sigma_1\sigma_2^{-1})^k$*. Then the knot* $\widehat{\alpha_k}$ *is algebraically slice if and only if k is odd.*

As a corollary the first author obtained the following corollary in number theory. We would like to make a remark that Corollary 6 may be proved using elementary methods. Nevertheless, we think that it is interesting to give a purely topological proof of such statement.

Corollary 6. *Let* L_k *be the k-th Lucas number. Then for each* $k \in \mathbf{N}$

(1) $L_{12k\pm4} = 5 \bmod 3 \,\mathrm{or}\, 7 \bmod 3$
(2) $L_{12k\pm2} = 3 \bmod 3$
(3) $L_{12k\pm2} - 2$ *is a square number.*

References

[1] M. Brandenbursky and J. Kedra, *Concordance group and stable commutator length in braid groups,* Arxiv:1402.3191, 2014.
[2] M. Brandenbursky, *3-braids, concordance group and Lucas numbers,* in preparation.
[3] D. Burago, S. Ivanov and L. Polterovich *Conjugation-invariant norms on groups of geometric origin*, Groups of diffeomorphisms in Adv. Stud. Pure Math., **52** (2008), 221–250.
[4] C. Livingstone, *The stable 4-genus of knots,* Algebr. Geom. Topol., **10**, number 4 (2010), 2191–2202.

Towards a general construction for the unitary representation rings of Artin braid groups

John Bryden

1. Introduction

The braid groups lie at the nexus of algebra and topology, Understanding the structure of their representation rings could play an important role in many problems situated at this intersection. The construction of new quantum invariants is one such problem.

In 2000 G. Carlsson invented the notion of deformation K-theory (cf. [1]) to study complex representations of groups. Deformation K-theory, K_{def}, can be defined as the algebraic K-theory of the representation category, $R(G)$, of the group G. That is,

$$K^*_{\text{def}}(G) := K^*(R(G)) = \pi_{*+1}(BQ\mathcal{R}(G)) .$$

where $\mathcal{R}(G)$ is the category having one and only one object.

$$\text{Obj}(\mathcal{R}(G)) = \amalg_{n\geq 0} \text{ Hom}(G, U(n))$$

and the morphisms are the elements

$$\text{Mor}(\mathcal{R}(G)) = \amalg_{n\geq 0} U(n) \times \text{ Hom}(G, U(n))$$

Tyler Lawson (cf. [2]) proved that there is an E^∞-ring spectrum, $\mathcal{R}[G]$, associated to the representation category $\mathcal{R}(G)$ satisfying the condition:

Theorem 1.1.

$$K_{def}(G) \cong \pi_i(\mathcal{R}[G]) \cong H_i(Irr(G))$$

where $Irr(G)$ is the topological monoid of isomorphism classes of irreducible representations.

Remark 1.2. There is an associated differential graded complex $C_*(\text{Irr}(G))$ that can be used in conjunction with Theorem 1.1 to extract the irreducible representation of G (cf. [2]).

Dan Ramras then proved that the deformation K-theory of the free product of groups is excisive (cf. [3]).

Theorem 1.3. *For finitely generated discrete groups G and H the following diagram of spectra*

$$\begin{array}{ccc} K_{def}(G * H) & \longrightarrow & K_{def}(H) \\ \downarrow & & \downarrow \\ K_{def}(G) & \longrightarrow & K_{def}(\{1\}) \end{array}$$

is homotopy cartesian.

The expression $K_{\text{def}}(G)$ in this theorem refers to the representation ring spectrum $\mathcal{R}[G]$ defined in [2]. This notation is ambiguous. However, this diagram of spectra induces the same diagram at the level of groups.

There is a long exact Mayer-Vietrois type sequence associated to the homotopy cartesian square of Theorem 1.3. This exact sequence is constructed by splicing together the long exact sequences obtained from the fibre sequences of the vertical morphisms in the homotopy cartesian square,

$$\ldots \longrightarrow K_{\text{def}}(G * H) \longrightarrow K_{\text{def}}(G) \oplus K_{\text{def}}(H) \longrightarrow K_{\text{def}}(\{1\}) \longrightarrow \ldots .$$

Ramras (cf. [3]) further proved that for surface groups $\pi_1(\Sigma)$,

Theorem 1.4. *Let $K_{\mathbf{C}}(\Sigma)$ denote the complex topological K-theory of the surface Σ. Then,*

$$K_{def}(\pi_1(\Sigma)) \cong K_{\mathbf{C}}(\Sigma) .$$

Remark 1.5. Theorem 1.1 is not generally true for arbitrary groups. However, in this particular situation Theorem 1.1 can be applied to find the irreducible representations of $\pi_1(\Sigma)$.

Remark 1.6. Theorem 1.4 exhibits the feature of deformation K-theory that Carlsson was purposefully trying to build into deformation K-theory. That is, $K_{\text{def}} \cong K_{\mathbf{C}}$. This is known only for surface groups and finitely generated abelian groups.

Example 1.7. If G is a finite group with n irreducible representations, then $K_{\text{def}}(G) \sim \vee_n \mathbf{ku}$ where $\mathbf{ku}$ is the complex connective K-theory spectrum (cf. [2]).

When $G = \mathbf{Z/m}$, it follows that

$$K^i_{\text{def}}(G) \cong \left\{ \begin{array}{lll} 0 & \ldots & \text{if } i = \text{ odd,} \\ \mathbf{Z} & \ldots & \text{if } i = \text{ even.} \end{array} \right\}$$

Example 1.8. Suppose $G = PSL_2(\mathbf{Z}) \cong \mathbf{Z/2}*\mathbf{Z/3}$. Find $K_{\text{def}}(PSL_2)$. It follows from Example 1.7 that in odd dimensions the sequence

$$\ldots \longrightarrow K^{2i+1}_{\text{def}}(\mathbf{Z}/2 * \mathbf{Z}/3) \longrightarrow K^{2i+1}_{\text{def}}(\mathbf{Z}/2) \oplus K^{2i+1}_{\text{def}}(\mathbf{Z}/3) \longrightarrow K^{2i+1}_{\text{def}}(\{1\}) \longrightarrow \ldots$$

reduces to the sequence

$$\ldots \longrightarrow K^{2i+1}_{\text{def}}(\mathbf{Z}/2 * \mathbf{Z}/3) \longrightarrow 0 \oplus 0 \longrightarrow 0 \longrightarrow \ldots$$

Since the connecting morphisms turn out to be 0, it is clear that

$$K^{2i+1}_{\text{def}}(\mathbf{Z}/2 * \mathbf{Z}/3) \cong 0 .$$

Hence, in every even dimension there are short exact sequences:

$$0 \longrightarrow K^{2i}_{\text{def}}(\mathbf{Z}/2 * \mathbf{Z}/3) \longrightarrow K^{2i}_{\text{def}}(\mathbf{Z}/2) \oplus K^{2i}_{\text{def}}(\mathbf{Z}/3) \longrightarrow K^{2i}_{\text{def}}(\{1\}) \longrightarrow 0 .$$

Example 1.7 shows that these sequences reduce to the following short exact sequences,

$$0 \longrightarrow K^{2i}_{\text{def}}(\mathbf{Z}/2 * \mathbf{Z}/3) \longrightarrow \mathbf{Z}^2 \oplus \mathbf{Z}^3 \longrightarrow \mathbf{Z} \longrightarrow 0$$

Thus, $K^{2i}_{\text{def}}(PSL_2) \cong K^{2i}_{\text{def}}(\mathbf{Z}/2 * \mathbf{Z}/3) = \mathbf{Z}^4$.

2. Braid Groups

Let $C_n(\mathbf{R}^{2n})$ denote the configuration space of n points in the plane. The symmetric group S_n acts on configuration space in the natural way. Define the braid group, B_n, on n points to be

$$B_n = \pi_1(C_n(\mathbf{R}^{2n})/S_n)$$

and the pure braid group, PB_n, as:

$$PB_n = \pi_1(C_n(\mathbf{R}^{2n}))\,.$$

There is a well defined group homomorphism $\eta\colon B_n \longrightarrow S_n$, defined as follows: to each braid in B_n associate the permutation it induces on its strands in S_n. Notice that if σ_i is a standard generator of B_n, $\eta(\sigma_i) = (i, i+1)$ for $i = 1 \ldots, n-1$. The kernel of η is precisely the subgroup of B_n formed by the braids inducing the trivial permutation, that is, the pure braid group PB_n. Thus there is a short exact sequence

$$1 \longrightarrow PB_n \longrightarrow B_n \longrightarrow S_n \longrightarrow 1$$

Very little is known about the representation theory of the Artin braid groups or the pure braid groups. However, it may be possible to use the theoretical ideas described above to describe the representation rings of both the pure braid groups and Artin braid groups. Recently in this direction I proved that,

Theorem 2.1 (Bryden, J)**.** *The pure braid groups, PB_n, satisfy excision for deformation K-theory.*

Corollary 2.2 (Bryden, J)**.** *The Mayer - Vietoris type long exact sequence can be used to determine a formula for $K_{def}(PB_n)$ in terms of free groups.*

Remark 2.3. The construction of the formula for K_{def} uses the proof of Theorem 2.1, and so is omitted in this abstract.

Problem 2.4. Since the pure braid groups are residually nilpotent, it may be possible to extend Theorem 1.1 to this case.

Problem 2.5. A general construction for the unitary representation rings of the Artin braid groups can now be formulated by effecting the procedure described above. However,

(1) I do not yet know if Theorem 1.1 can be extended to this case,

(2) I have not yet proved that the deformation K-theory and complex topological K-theory are isomorphic in this case.

Remark 2.6. The multiplicative structure of the representation ring is determined by the structure of the derived category of modules over the E^{∞} ring spectrum $R[G]$.

References

[1] G. Carlsson, Structured stable homotopy theory and the descent problem for algebraic K-theory of fields, 2003, preprint.
[2] T. Lawson, *Completed Representation Ring Spectra of Nilpotent Groups*, Algebr. Geom. Topol. 6 (2006) 253-285.
[3] D. Ramras, *Excision for Deformation K-Theory of Free Products*, Algebr. Geom. Topol. 7 (2007) 2239-2270.

A possible topos-theoretic interpretation of Khovanov homology

J. Bryden, L. Kauffman

We note that the Khovanov homology of the link K, denoted $\mathrm{Kho}^*(K)$, is isomorphic to the cohomology of the nerve $\mathcal{N}$ of the cube category, $\mathrm{Cube}(K)$, associated with K, with coefficients in a category $\mathcal{C}$ of Frobenius algebras.:

$$\mathrm{Kho}^*(K) \cong H^*(\mathcal{N}\mathrm{Cube}(K); \mathcal{A}),$$

for $\mathcal{A} \in \mathrm{Obj}(\mathcal{C})$. This in turn is isomorphic with the cohomology of the classifying category of $\mathrm{Cube}(K)$ with the same coefficient pre-sheaf. That is,

$$\mathrm{Kho}^*(K) \cong H^*(\mathcal{B}\mathrm{Cube}(K); \mathcal{A})$$

The object $\mathcal{B}\mathrm{Cube}(K)$ is an object whose étale homotopy theory can be studied.

Conjecture 1. The Khovanov homology, $\mathrm{Kho}^*(K)$, of the link , K, can be represented by an appropriate étale homotopy type that is an invariant of the link K.

Cheeger-Gromov universal bounds of von Neumann L^2 rho-invariants

Jae Choon Cha

This is an extended abstract of my talk, which is based on part of the paper [2]. More related results, further discussions, and proofs can be found in [2].

In [4], Cheeger and Gromov defined and studied the von Neumann L^2 ρ-invariant $\rho^{(2)}(M,\phi) \in \mathbb{R}$ of a closed Riemannian $(4k-1)$-dimensional manifold M, which is associated to the regular cover determined by a homomorphism $\phi\colon \pi_1(M) \to G$. In their study of the topological invariance of $\rho^{(2)}(M,\phi)$, Cheeger and Gromov proved the following boundedness theorem, using deep analytic methods: *for any closed smooth $(4k-1)$-manifold M, there is a constant C_M such that $|\rho^{(2)}(M,\phi)| \le C_M$ for any homomorphism $\phi\colon \pi_1(M) \to G$ into any group G.*

A natural question arises from the work of Cheeger and Gromov: *can we understand the universal bound C_m topologically?* This is a question intriguing not only on its own but also for applications. Especially, since work of Cochran and Teichner [5], the Cheeger-Gromov universal bound C_M has been playing an important role in the study of low dimensional topology: concordance of knots and links in the 3-space, homology cobordism of 3-manifolds, and Whitney towers and gropes in 4-manifolds. While a topological understanding of the Cheeger-Gromov universal bound would improve many related results, almost nothing was known about the topological aspects of the universal bound, except its existence.

In this talk, we present a topological approach to the Cheeger-Gromov universal bound, with new applications to 3-manifold topology. Our first result is a topological proof of the existence of the universal bound, which applies to topological manifolds as well:

Theorem 1. *For any closed topological $(4k-1)$-manifold M, there is a constant C_M such that $|\rho^{(2)}(M,\phi)| \le C_M$ for any homomorphism $\phi\colon \pi_1(M) \to G$ into any group G.*

The proof employs the idea appeared in the work of Chang and Weinberger [3], and a functorial embedding of groups into acyclic groups due to Baumslag, Dyer, and Heller [1].

For 3-manifolds, we develop new methods to obtain explicit estimates of the universal bound, which relate it to the fundamental 3-manifold presentations: triangulations, Heegaard splittings, and surgery descriptions. In this talk a triangulation designates a simplicial complex structure. A 3-manifold is assumed to be closed. The *simplicial complexity* of a 3-manifold is defined to be the minimal number of 3-simplices in a triangulation.

Theorem 2. *If M is a 3-manifold with simplicial complexity n, then for any homomorphism ϕ of $\pi_1(M)$ into an arbitrary group G,*

$$|\rho^{(2)}(M,\phi)| \le 363090 \cdot n.$$

Recall that a mapping class h in the mapping class group $\mathrm{Mod}(\Sigma_g)$ on a genus g surface Σ_g gives a Heegaard splitting of a 3-manifold. Lickorish showed that the ± 1 Dehn twists along standard $3g-1$ curves on Σ_g generate $\mathrm{Mod}(\Sigma_g)$ [8]. For a given 3-manifold M, define the *Heegaard-Lickorish complexity* to be the minimal word length of a mapping class in $\mathrm{Mod}(\Sigma_g)$, with respect to the Lickorish twists, which gives a Heegaard splitting of M. Here g is arbitrary.

Theorem 3. *If a 3-manifold M has Heegaard-Lickorish complexity ℓ, then for any homomorphism ϕ of $\pi_1(M)$ into an arbitrary group G,*

$$|\rho^{(2)}(M,\phi)| \le 251258280 \cdot \ell.$$

Any 3-manifold is obtained by surgery on a (integral) framed link in S^3. For a framed link L with surgery coefficients $n_i \in \mathbb{Z}$, let $f(L) = \sum |n_i|$. We denote the crossing number of a link L by $c(L)$, that is, the minimal number of crossings in a planar diagram of L.

Theorem 4. *If a 3-manifold M is obtained by surgery on a framed link L in S^3, then for any homomorphism ϕ of $\pi_1(M)$ into an arbitrary group G,*

$$|\rho^{(2)}(M,\phi)| \le 69713280 \cdot c(L) + 34856640 \cdot f(L).$$

The above results establish relationships of the Cheeger-Gromov universal bound with combinatorial, geometric group theoretic, and knot theoretic aspects of 3-manifolds.

In addition, although the coefficients in Theorems 2, 3, and 4 are large, we have the following result:

Theorem 5. *The linear universal bounds in Theorems B, C, and D are asymptotically optimal.*

For a precise formulation of Theorem E, see [2]. We remark that finding an optimal or improved (smaller) coefficients in Theorems 2, 3, and 4 seems to be another intriguing problem.

Based on our results, we make a new application of the Cheeger-Gromov ρ-invariants, to the complexity theory of 3-manifolds. For 3-manifolds, a relaxed notion of a triangulation called a *pseudo-simplicial triangulation* is often considered. Briefly, a pseudo-simplicial triangulation is a union of tetrahedra with identifications of faces that gives the 3-manifold as a quotient space (see, for example, [6, Section 2]). The (pseudo-simplicial) *complexity* $c(M)$ of a 3-manifold M is defined to be the minimal number of tetrahedra in a pseudo-simplicial triangulation. We remark that Matveev developed basic theory of (an equivalent notion of) complexity in terms of spines [9].

Understanding the complexity of a 3-manifold is difficult. Finding an upper bound is easier, since any (pseudo-simplicial) triangulation gives an upper bound, but finding a lower bound has been recognized as a hard problem. There are some interesting results that give lower bounds, for instance due to Jaco, Matveev, Pervova, Petrionio, Rubinstein, Tillman, and Vesnin. However, for instance, even for the case of lens spaces, the complexity is not completely understood.

We obtain new lower bounds of the complexity of 3-manifolds, using the Cheeger-Gromov ρ-invariants:

Corollary 6. *Suppose M is a 3-manifold. Then for any homomorphism ϕ of $\pi_1(M)$,*

$$c(M) \geq \frac{1}{209139840} \cdot |\rho^{(2)}(M, \phi)|.$$

In spite of the large denominator, it turns out that the lower bound in Corollary 6 can be arbitrary larger than information that can be obtained by the previously known results. As explicit examples, consider the lens spaces $L(n, 1)$ [9, 6]. It was conjectured that $c(L(n, 1)) = n - 3$ for $n > 3$. It is known that $c(L(n, 1)) \leq n - 3$, due to Jaco and Rubinstein [6]. For even n, Jaco, Rubinstein, and Tillman proved that the conjecture holds [7]. The case of odd n remains open; in fact, the previous known lower bounds (do not apply or) are at most square root growth in n, while the conjectured complexity is linear. Using Corollary 6 and some explicit computation of the Cheeger-Gromov ρ-invariant, we prove the following:

Theorem 7. *For any knot in S^3, the complexity of its n-surgery manifold is precisely linear in n, i.e., $\Theta(n)$. As a special case, for the lens space $L(n, 1)$, we* *have*

$$\frac{n-3}{69713280} \leq c(L(n,1)) \leq n-3.$$

This proves that the conjecture for $c(L(n,1))$ is true asymptotically.

For the proofs of our main results, we develop two new ingredients, which seem interesting on their own. First, we introduce a construction of efficient 4-dimensional bordisms of 3-manifolds over a group, which may be viewed as a quantitative geometric incarnation of the Atiyah-Hirzebruch bordism spectral sequence; the "size" of the resulting bordism depends *linearly* on the simplicial complexity of a given 3-manifold and certain algebraic information. Second, we introduce the notion of *controlled chain homotopy*, which is a chain level algebraic analogue of controlled homotopy. We give uniformly controlled chain homotopy versions of known homological results, with explicit estimates of the diameter, including a controlled acyclic model theorem, controlled Eilenberg-Zilber theorem for products, and controlled approximations of Baumslag-Dyer-Heller's functorial embedding of groups into acyclic groups. Further details can be found in [2].

References

[1] G. Baumslag, E. Dyer, and A. Heller, *The topology of discrete groups*, J. Pure Appl. Algebra **16** (1980), no. 1, 1–47.

[2] Jae Choon Cha, *A topological approach to Cheeger-Gromov universal bounds for von Neumann rho-invariants*, arXiv:1405.1805.

[3] Stanley Chang and Shmuel Weinberger, *On invariants of Hirzebruch and Cheeger-Gromov*, Geom. Topol. **7** (2003), 311–319 (electronic).

[4] Jeff Cheeger and Mikhael Gromov, *Bounds on the von Neumann dimension of L^2-cohomology and the Gauss-Bonnet theorem for open manifolds*, J. Differential Geom. **21** (1985), no. 1, 1–34.

[5] Tim D. Cochran and Peter Teichner, *Knot concordance and von Neumann ρ-invariants*, Duke Math. J. **137** (2007), no. 2, 337–379.

[6] William Jaco and J. Hyam Rubinstein, *Layered-triangulations of 3-manifolds*, arXiv:math/0603601.

[7] William Jaco, Hyam Rubinstein, and Stephan Tillmann, *Minimal triangulations for an infinite family of lens spaces*, J. Topol. **2** (2009), no. 1, 157–180. MR 2499441 (2010b:57016)

[8] W. B. R. Lickorish, *A representation of orientable combinatorial* 3*-manifolds*, Ann. of Math. (2) **76** (1962), 531–540.

[9] S. V. Matveev, *Complexity theory of three-dimensional manifolds*, Acta Appl. Math. **19** (1990), no. 2, 101–130.

Knots in virtually fibered 3-manifolds and commensurability

Micah W. Chrisman

This talk discusses the results of the recent paper [8]. The objective is to study knots in compact orientable 3-manifolds using the methods of virtual knot theory.

Let M be a compact oriented 3-manifold and K an oriented knot in M. Suppose that M admits a covering space $\Pi : \Sigma \times (0,1) \to M$, where Σ is a compact oriented surface and Π is orientation preserving. Suppose that $\mathfrak{k}$ is an oriented knot in $\Sigma \times (0,1)$ such that $\Pi(\mathfrak{k}) = K$ with orientation preserved. Then $\mathfrak{k}$ may be considered as a knot in $\Sigma \times [0,1]$ via inclusion. The knot $\mathfrak{k}$ in $\Sigma \times I$ stabilizes to a virtual knot υ. This setup is called a *virtual cover* $(\mathfrak{k}^{\Sigma\times(0,1)}, \Pi, K^M)$ *with associated virtual knot* υ. If $\mathfrak{k}$ is contained in a fundamental region of Π, then the virtual cover is called a *fundamental virtual cover.*

In many situations, the associated virtual knot will be an ambient isotopy invariant of the knot K in M. For example, let M be the complement of a fibered link J in S^3. Let K_1 be a knot in M that is "close" to a fiber Σ of the given fiber bundle $M \to S^1$. The pullback provides the needed covering space $\Pi : \Sigma \times (0,1) \to M$. By "close", we mean that there is a knot $\mathfrak{k}_1$ in $\Sigma \times (0,1)$ such that $\Pi(\mathfrak{k}_1) = K$ and $\mathfrak{k}_1$ is contained in a fundamental region of Π. Let υ_1 be the associated virtual knot. Suppose that K_2 is another knot in M that is "close" to Σ and thereby providing an associated virtual knot υ_2. It follows from the theory of virtual covers that if K_1 and K_2 are ambient isotopic knots in M, then υ_1 and υ_2 are equivalent as virtual knots [9].

Several examples are provided. Knots in the complement of a trefoil, the complement of 6_2, the complement of the Hopf link, and the complement of the Borromean rings are considered in detail. Virtual covers are used to detect inequivalent knots in manifolds, demonstrate non-invertibility of a knot, and prove that a three component link is non-separable in S^3.

Virtual covers may also be used to study knots in manifolds M that are not fibered over S^1. In particular, they can be used to study knots in virtually fibered 3-manifolds. A compact orientable 3-manifold is said to be *virtually fibered* if it admits a finite index covering that is fibered over S^1. Thurston's *virtual fibering conjecture* [32] implies, if true, that all hyperbolic link complements are virtually fibered. Agol [1] has recently proved that if M is closed and hyperbolic, then M is virtually fibered.

The main theoretical tool needed for the extension to virtually fibered manifolds is *commensurability.* This approach is inspired the the work of Leininger [24] and Walsh [33]. We will say that knot A in the manifold M and the knot B in the manifold N are *elementary commensurable* if there is a knot C in a manifold P and finite index regular coverings $\Pi_M : P \to M$ and $\Pi_N : P \to N$ such that $\Pi_M(C) = A$, $\Pi_N(C) = B$, and C is contained in a fundamental region of Π_M and a fundamental region of Π_N. In this case we write $A^M \doteq B^N$. If there is a finite sequence of elementary commensurabilities taking A^M to B^N, we say that A^M and B^N are *commensurable.* This is written as $A^M \risingdotseq B^N$.

The main theorem of [8] can be stated briefly as follows. If the knot A^M has a fundamental virtual cover with associated virtual knot υ_A, the knot B^N has a fundamental virtual cover with associated virtual knot υ_B, and $A^M \risingdotseq B^N$, then

$\upsilon_A \leftrightharpoons \upsilon_B$ as virtual knots. A sketch of the proof is given during the talk.

The first example of a non-fibered virtually fibered manifold is due to Gabai [16]. It is a non-fibered hyperbolic link complement admitting a 2-fold regular covering by a fibered link complement. A detailed construction of the manifold is provided during the talk. Let M denote Gabai's example manifold.

The main theorem of [8] is applied to knots in Gabai's manifold M. Virtual covers are used to distinguish between inequivalent knots in M and prove the non-invertibility of a knot in M.

References

[1] I. Agol. The virtual Haken conjecture. *Doc. Math.*, 18:1045–1087, 2013. With an appendix by Agol, Daniel Groves, and Jason Manning.

[2] I. Agol, S. Boyer, and X. Zhang. Virtually fibered Montesinos links. *J. Topol.*, 1(4):993–1018, 2008.

[3] M. Boileau, S. Boyer, R. Cebanu, and G. S. Walsh. Knot commensurability and the Berge conjecture. *Geom. Topol.*, 16(2):625–664, 2012.

[4] M. O. Bourgoin. Twisted link theory. *Algebr. Geom. Topol.*, 8(3):1249–1279, 2008.

[5] G. Burde and H. Zieschang. *Knots*, volume 5 of *de Gruyter Studies in Mathematics*. Walter de Gruyter & Co., Berlin, second edition, 2003.

[6] J. S. Carter, S. Kamada, and M. Saito. Stable equivalence of knots on surfaces and virtual knot cobordisms. *J. Knot Theory Ramifications*, 11(3):311–322, 2002. Knots 2000 Korea, Vol. 1 (Yongpyong).

[7] J. S. Carter, D. S. Silver, and S. G. Williams. Invariants of Links in Thickened Surfaces. *arXiv:1304.4655v1[math.GT]*, April 2013.

[8] M. W. Chrisman. Knots in virtually fibered 3-manifolds *ArXiv e-prints*, May 2014

[9] M. W. Chrisman and V. O. Manturov. Fibered knots and virtual knots. *J. Knot Theory Ramifications*, 22(12):1341003, 23, 2013.

[10] H. A. Dye and L. H. Kauffman. Virtual crossing number and the arrow polynomial. *J. Knot Theory Ramifications*, 18(10):1335–1357, 2009.

[11] R. Fenn, C. Rourke, and B. Sanderson. The rack space. *Trans. Amer. Math. Soc.*, 359(2):701–740 (electronic), 2007.

[12] T. Fiedler. A small state sum for knots. *Topology*, 32(2):281–294, 1993.

[13] S. Friedl and T. Kitayama. The virtual fibering theorem for 3-manifolds. *ArXiv e-prints*, October 2012.

[14] D. Futer. Fiber detection for state surfaces. *Algebr. Geom. Topol.*, 13(5):2799–2807, 2013.

[15] D. Gabai. The Murasugi sum is a natural geometric operation. In *Low-dimensional topology (San Francisco, Calif., 1981)*, volume 20 of *Contemp. Math.*, pages 131–143. Amer. Math. Soc., Providence, RI, 1983.

[16] D. Gabai. On 3-manifolds finitely covered by surface bundles. In *Low-dimensional topology and Kleinian groups (Coventry/Durham, 1984)*, volume 112 of *London Math. Soc. Lecture Note Ser.*, pages 145–155. Cambridge Univ. Press, Cambridge, 1986.

[17] M. Goussarov, M. Polyak, and O. Viro. Finite-type invariants of classical and virtual knots. *Topology*, 39(5):1045–1068, 2000.

[18] D.M. Ilyutko and V. O. Manturov. *Virtul Knot Theory:State of the Art*, volume 51 of *Series on Knots and Everything*. World Scientific, 2013.

[19] D.P. Ilyutko, V. O. Manturov, and I. M. Nikonov. Virtual knot invariants arising from parities. *arXiv:1102.5081v1[math.GT]*, 2011.

[20] N. Kamada and S. Kamada. Abstract link diagrams and virtual knots. *Journal of Knot Theory and its Ramifications*, 9:93–106, 2000.

[21] L. H. Kauffman. Virtual knot theory. *European J. Combin.*, 20(7):663–690, 1999.
[22] V. A. Krasnov and V. O. Manturov. Graph-valued invariants of virtual and classical links and minimality problem. *J. Knot Theory Ramifications*, 22(12):1341006, 14, 2013.
[23] G. Kuperberg. What is a virtual link? *Algebraic and Geometric Topology*, 3:587–591, 2003.
[24] C. J. Leininger. Surgeries on one component of the Whitehead link are virtually fibered. *Topology*, 41(2):307–320, 2002.
[25] C. J. Leininger. Erratum to: "Surgeries on one component of the Whitehead link are virtually fibered" [Topology **41** (2002), no. 2, 307–320; MR1876892 (2002j:57037)]. *Topology*, 42(2):507–508, 2003.
[26] V. O. Manturov. Free knots and parity. In *Introductory lectures on knot theory*, volume 46 of *Ser. Knots Everything*, pages 321–345. World Sci. Publ., Hackensack, NJ, 2012.
[27] D. Rolfsen. *Knots and links*. Publish or Perish Inc., Berkeley, Calif., 1976. Mathematics Lecture Series, No. 7.
[28] J. Sawollek. On Alexander-Conway Polynomials for Virtual Knots and Links. *arXiv:9912173[math.GT]*, December 1999.
[29] E. H. Spanier. *Algebraic topology*. McGraw-Hill Book Co., New York-Toronto, Ont.-London, 1966.
[30] J. R. Stallings. Constructions of fibred knots and links. In *Algebraic and geometric topology (Proc. Sympos. Pure Math., Stanford Univ., Stanford, Calif., 1976), Part 2*, Proc. Sympos. Pure Math., XXXII, pages 55–60. Amer. Math. Soc., Providence, R.I., 1978.
[31] N. Steenrod. *The Topology of Fibre Bundles*. Princeton Mathematical Series, vol. 14. Princeton University Press, Princeton, N. J., 1951.
[32] W. P. Thurston. Three-dimensional manifolds, Kleinian groups and hyperbolic geometry. *Bull. Amer. Math. Soc. (N.S.)*, 6(3):357–381, 1982.
[33] G. S. Walsh. Great circle links and virtually fibered knots. *Topology*, 44(5):947–958, 2005.
[34] G. S. Walsh. Orbifolds and commensurability. *ArXiv e-prints*, March 2010.

A Revision of Levine's Stategy: towards a Wang-type sequence for classical knot concordance

Tim D. Cochran
(joint work with Christopher W. Davis)

In 1969 Jerome Levine successfully classified higher-odd-dimensional knot concordance in terms of simple invariants, namely linking numbers, of special links on an arbitrary Seifert surface. For knots in S^3, it was known that the situation is more complicated but this philosophy has nonetheless dominated the search for a characterization of slice knots. In this work we discuss unexpected failures of Levine's program and indicate refinements necessary to recover this strategy.

More specifically, a knot K is called *algebraically slice* if for any genus g Seifert surface Σ for K there is a half-rank summand of $H_1(\Sigma)$ on which the Seifert form vanishes. A basis for this summand can be realized by a g-component link, J, embedded on Σ. Levine showed that any slice knot is an algebraically slice knot, and in higher dimensions, that any algebraically slice knot is a slice knot. More precisely, given a smooth slice disk Δ for a knot K and any genus g Seifert surface for K, there exist g-component link, J, embedded on the Seifert surface that is *associated to* Δ. We call such a link a *derivative of* K. By definition such a link J has pairwise linking numbers zero. It was conjectured that J itself must be a slice link, or at least be algebraically slice. Indeed in higher dimensions

Levine proved that for any algebraically slice *simple* knot K any derivative is a slice link, from which it follows easily that K is a slice knot. For knots in S^3, in 1973 Casson and Gordon found additional obstructions to K being a slice knot. But, significantly, the Casson-Gordon invariants also can be expressed in terms of more elememtary invariants of the derivative link. Whereas Levine showed that the simplest invariants of a derivative link (linking numbers) obstruct K from being a slice knot, Casson-Gordon, Gilmer and Cooper showed that certain sums of signatures of the derivative link obstruct K from being a slice knot. In recent years many more obstructions have been found and in almost every case case they have been shown to be expressible in terms of lower order invariants of the derivative link. Thus hope has remained that Levine's philosophy/strategy was sound, namely that if K is a slice knot then, for any Seifert surface, there exists a derivative that is itself a slice link.

In the simplest situation, when a (classical) slice (or merely algebraically slice) knot bounds a genus one Seifert surface, Σ, it can be shown that (modulo orientation) there are precisely two derivatives for K, each of which is itself a knot. In this specific case, 1982 Kauffman conjectured that the converse is true (in support of Levine's philosophy):

Conjecture [Kauffman's Strong Conjecture] If K is a slice knot and Σ is a genus one Seifert surface for K then one of the two derivatives, J, is a slice knot.

It was specifically expected that the signatures of J would vanish. The authors recently showed this is false in some cases, contradicting Kauffman's conjecture [1]. Thus Levine's philosophy needs to be modified if it is to be used.

In this work we answer the question: given a smooth slice disk Δ for a knot K and a derivative link, J, associated to Δ, what CAN be said (geometrically) about J? The answer is not that J itself is slice but that, loosely speaking, J is equal to $(t_* - id)(L)$ for some link L. We also adress a *filtered* version of this question, with respect to the Cochran-Orr-Teichner n-solvable filtration of the knot concordance group. One application of our work is that the signature part of the conjecture DOES hold in certain cases. We also provide very strong answers to these questions for genus one knots.

Our main theorem is:

Main Theorem If K is a slice knot via a slice disk Δ to which J is associated then there exists some link L in a rational homology 3-sphere such that $J\#L\#-t_*(L)$ is *null-bordant with respect to the group G*, where G is the commutator subgroup of $\pi_1(B^4-\Delta)/<d_1,...,d_g>$ and d_i are the components of J.

Although it will be difficult for the reader to appreciate the implications of this theorem (and we will not here explain all the terminology), note that instead of the expected answer that J is trivial in the knot concordance group, the theorem is saying that the concordance class of J is of the form $(1-t_*)(L)$ for some link L. We also have a filtered version of the main theorem.

As one concrete application of our main theorem, we come close to resolving the situation completely for the signature conjecture for genus one knots:

Corollary Suppose that K admits a genus one Seifert surface, Σ, admitting a derivative J (consequently $\Delta_K(t) \doteq (mt-(m+1)((m+1)t-m)$. Suppose $m \notin \{0,-1\}$. Then:

1. If K is slice knot (or even a 2-solvable knot in the sense of Cochran-Orr-Teichner), (and J is associated to a slice disk), then the Levine-Tristram signature function of the $(c,1)$-cable of J is of the form
$$T_{(m,1)} \;\#\; -T_{(m+1,1)}$$
for some positive integer c and some knot T in a rational homology 3-sphere;
2. conversely, if the algebraic concordance type of J satisfies
$$[J_{(c,1)}] = [T_{(m,1)}] - [T_{(m+1,1)}]$$
for some c and some knot T in a homology 3-sphere, then K is a 1.5-solvable knot.

We also give very strong results about the injectivity of certain winding number zero satellite operators $R : \mathcal{C} \to \mathcal{C}$.

References

[1] Tim D. Cochran and Christopher W. Davis, *Counterexamples to Kauffman's Conjectures on Slice Knots*, preprint (2013) http://front.math.ucdavis.edu/1303.4418.

Ribbon tangles in the Kauffman bracket skein module

Michael Eisermann

The Fox–Milnor theorem of 1966 says that the Alexander polynomial of each slice knot $K \subset \mathbb{R}^3$ factors as $\Delta(K) = P(t)P(t^{-1})$ for some $P \in \mathbb{Z}[t]$. In particular the determinant $\det(K) = \Delta(K)_{t\mapsto -1}$ is a square, whence $\det(K) \equiv 1 \bmod 8$.

The Jones polynomial often provides more information but is difficult to interpret geometrically. In this talk I have presented what little is known about the Jones polynomial of ribbon links [1]: For every n–component ribbon link $L = L_1 \cup \cdots \cup L_n$ in $\mathbb{R}^3$, the Jones polynomial $V(L)$ is divisible by the polynomial $V(\bigcirc^n)$ of the n–component trivial link. This integrality property allows us to define a generalized determinant $\det V(L) := [V(L)/V(\bigcirc^n)]_{t\mapsto -1}$ and to prove $\det(L) \equiv \det(L_1)\cdots\det(L_n) \bmod 32$, whence $\det(L) \equiv 1 \bmod 8$. This property naturally extends to ribbon tangles in the Kauffman bracket skein module (work in progress joint with Emmanuel Wagner, Dijon) and generalizes to the Homflypt polynomial (at least partially; again this is still work in progress).

These algebraic obstructions are rather weak but suggest that there are more geometric information to be extracted from generalizations of the Jones polynomial such as the Homflypt polynomial or Khovanov homology.

References

[1] M. Eisermann, *The Jones polynomial of ribbon links*, Geometry & Topology **13** (2009) 623-660. Online avaiblabe under DOI 10.2140/gt.2009.13.623 and `arxiv.org/abs/0802.2287`.

[2] The slides of this talk are available online under `www.igt.uni-stuttgart.de/eiserm/conferences/2014-ribbontangles-Oberwolfach.print.pdf`

Invariants of homotopic classes of curves and graphs on 2-surfaces (Towards a bridge between knots and groups)

Denis Alexandrovich Fedoseev

(joint work with Vassily Olegovich Manturov)

Finitely presented groups often appear as invariants of some objects in low-dimension topology: knots as well as their generalizations and modifications — virtual knots, knots in 3-manifolds, etc. *Flat knot* is a simplification of the notion of virtual knot. Flat knot is an equivalence class of homotopic classes of curves up to *stabilization*. This object being an important source of virtual knots invariants can be considered from both algebraic and geometric points of view. Homotopic class of a curve on a surface can be given as a conjugation class of the fundamental group of the surface. Such objects have been long ago classified both algebraically and geometrically [1, 2]. Moreover, they admit the structure of Lie bialgebra (Goldman-Turaev, [3, 4]) of geometric nature. Topological study of such object is closely related to knot theory methods: isotopy classes of curves are equivalence classes of diagrams by Reidemeister moves, so it is possible to apply knot-theoretical methods to the objects, considered algebraically. A natural question arises: what other groups can be studied by the means of classical and virtual knot theories and which "geometric structures" arise on such groups.

The central idea is to consider different generalizations of the following construction originally done for a curve on a surface:

Theorem 1 (I.M. Nikonov)**.** *Let γ be a curve on a orientable closed 2-manifold S_g allowing a source-sink structure such that the complement to γ is a union of cells. Than*

$$G_\gamma \cong \pi_1(S_g)/\langle\gamma\rangle,$$

where $\langle\gamma\rangle$ denotes the normal closure of the homotopic class of the curve γ.

Here the group G_γ is defined via its presentation: the set of vertices is taken as the generator set and relations are obtained from polygons on the surface — every polygon formed by the curve's arcs gives a relation in the form $a_1 \dots a_n = 1$, where a_i are the vertices in a cyclic order induced by a compatible source-sink structure on the curve.

The explored generalisations are the following theorems about invariant groups for curves without a source-sink structure, for a set of curves on a surface and for $\Theta-$graphs.

First two theorems deal with the case of curves without a source-sink structure. In that case two approaches are valid. For the first one we construct a two-fold covering $\tilde{\Gamma}$ of the original curve, which allows a source-sink structure and present the invariant group G for the covering curve as before; the explicit description of the covering can be found in [6]. That approach leads us to the

Theorem 2. *Let there be a curve γ on a closed 2-surface M without a source-sink structure. Then the representation of the group $G_{\tilde{\Gamma}}$ is a homotopic invariant of the curve γ.*

Alternatively we can stick to the given curve, but improve the relations in the group G: give every generator a power plus or minus 1 depending on whether the vertex is compatible with a source-sink structure or not. To be more precise, we attribute every vertex an *arbitrary* source-sink structure and fix the direction we will walk along the polygons: clockwise or counterclockwise. Then for every vertex a_i in a polygon we write either a_i or a_i^{-1} in the corresponding relation depending on the compatibility of the chosen direction and the structure in the vertex. The resulting group is called $G^{\pm}$ and the following theorem holds:

Theorem 3. *Let γ be a curve on a closed 2-surface M without a source-sink structure and the complement to the curve is a union of cells. Then*

$$G_{\gamma}^{\pm} \cong \pi_1(M)/\langle\gamma\rangle,$$

where$\langle\gamma\rangle$ means the normal closure of homotopic class of γ.

The following theorem generalises the ideas above t the case of several curves on a surface:

Theorem 4. *Let $\gamma = \gamma_1 \cup \cdots \cup \gamma_k$ be a diagram of k curves which allows a source-sink structure and breaks the surface into a union of cells Δ_i. Then there is an isomorphism*

$$G_{\gamma} \cong (\pi_1(S_g)/\langle\gamma_1\rangle \dots \langle\gamma_k\rangle) * \mathbb{Z} * \cdots * \mathbb{Z}$$

where $\langle\gamma_i\rangle$ means a normal closure of homotopic class of the curve γ_i and the number of factors $\mathbb{Z}$ equals $k-1$.

If the surface has boundary, we simply ignore all the polygons with holes inside. Consequently we get

Theorem 5. *Let γ be a closed curve of an oriented 2-surface M with or without boundary. Let γ allow a source-sink structure. Then*

$$G_{\gamma} \cong \pi_1(M)/\langle\gamma\rangle,$$

where $\langle\gamma\rangle$ means normal closure of homotopic class of the curve and the group G_{γ} is described before.

Finally, if we consider a θ−graph and construct a group for it similar to the above-mentioned $G^{\pm}$ ignoring the 3-valent vertices in the process, we get an invariant group once again. This case is especially important, since it allows to establish a connection between geometric transformations and the most important Tietze transformation in a particular case of groups with two relations.

The structures considered above can be directly used in knot theory to construct knot invariants. Recall that a *knot* in 3-sphere is an embedding of a circle into S^3. *Knot diagram* is such a projection of a knot onto a plane, that all the intersection points (called crossings) are double and transversal and in every crossing there is an additional structure designating which arc is upper and which — lower. Two knot diagrams are called equivalent if they can be connected via a series of transformations of the following list: trivial isotopy and three Reidemeister moves $\Omega_1, \Omega_2, \Omega_3$. An equivalency class of diagrams is also called a knot.

A flat knot is an equivalency class of closed curves on a 2-surface M with all the intersection points being double and transversal with respect to *flat Reidemeister moves*. Unlike classical knots, upper and lower arcs are not distinguished in flat knot crossings.

A link (*flat link*) is a disjoint sum of several knots (flat knots) with double transversal intersection points.

The theorems proved in this work let us, on the one hand, use group methods to study knots and, on the other hand, use well-known in knot theory structures and theorems to study groups.

Among the structures which can be translated into group theory language there is, for example, a bracket, introduced by one of the authors in the work [7]. This object allows some minimality theorems to be proved. In particular, the bracket properties (see [7]) and the above-proved theorems lead to the following statement:

Theorem 6. *Let a diagram if some class of conjugacy of a curve on a surface be odd and irreducible. Than every other diagram of this class contains this diagram as a subdiagram.*

Here irreducibility means the absence of lunes and odd means the cycle, obtained from equivalency classes of opposite semiedges on the diagram, is odd. Since every class of a curve on a surface has a corresponding group this theorem at the same time is a statement about group structure.

References

[1] Hass, J. and Scott, P. (1994) *Shortening curves on surfaces, Topology 33, 1, pp. 25–43.*

[2] Reinhart, B. L. (1962) *Algorithms for Jordan curves on compact surfaces, Ann. of Math. 75, 2, pp. 209–222.*

[3] Goldman, W. (1986). *Invariant functions on Lie groups and Hamiltonian flows of surface group representations, Invent. Math. 85, pp. 263–302.*

[4] Turaev, V. G. (1991). *Skein quantization of Poisson algebras of loops on surfaces, Ann. Sci. École Norm. Sup. 4, 24, 635–704.*

[5] Fomenko, A.T. (1991), *The theory of invariants of multidimensional integrable hamiltonian systems (with arbitrary many degrees of freedom). Molecular table of all integrable systems with two degrees of freedom, Amer. Math. Soc., 6, P. 1–35.*

[6] Manturov V.O. (2005) *Khovanov complex for virtual knots, Fund. and apple. math., 11, 4, pp. 127–152.*

[7] Ilyutko D.P., Manturov V.O., Nikonov I.M. (2011) *Parity in knot and graph-link theories, Topology, 41, PFUR, M., 3–163*

How to Draw a Knot

Roger Fenn

Despite the title this was a talk about immersions of a circle in the plane. Other information such as over/under crossings etc can be added later. Other versions are given in [1]. It was shown how an immersed circle in an oriented surface can be coded by a permutation of the $2n$ numbers $\{1, 2, ..., 2n\}$ and a function from these numbers to a 2-element set $\{I, II\}$. From this information a permutation of the $4n$ numbers $\{1, 2, ..., 4n\}$ can be constructed and information from this tells us whether the immersion exists and what the genus of the surface is. The converse problem of actually drawing the immersion is provided by showing that any triangulation of a disk can be realised as a straight line embedding with convex boundary.[2].[3]. The metric information can be provided by a circle packing.[4] but as was pointed out in the talk: the interior tends to shrink exponentially. This aspect therefore needs further work.

References

[1] **Dowker, C. H.; Morwen B. Thistlethwaite** (1983).*"Classification of knot projections"*. Topology and its Applications 16 (1): (1983) 19–31.

[2] **Fáry, István** (1948), *"On straight-line representation of planar graphs"*, Acta Sci. Math. (Szeged) 11: 229233, MR 0026311

[3] **Stein, S. K.** (1951), *"Convex maps"*, Proceedings of the American Mathematical Society 2 (3): 464466, doi:10.2307/2031777, JSTOR 2031777, MR 0041425

[4] **Burt Rodin and Dennis Sullivan** (1987), *The convergence of circle packings to the Riemann mapping* J Differential Geom. Volume 26, Number 2 349-360.

Singularization of knots and closed braids and new volume conjectures

Thomas Fiedler

We construct a combinatorial counter part of a not yet existing "embedded Kontsevich-like integral for 1-parameter families of long knots". It is based on a method of constructing non symmetric solutions of a global tetrahedron equation. These solutions give rise to the first combinatorial 1-cocycle, called R_1, with values in the $\mathbb{Z}$-module of refined singular long knots and which represents a non trivial cohomology class in the topological moduli space of long knots. The evaluation of the 1-cocycle R_1 on Hatcher's loop $hat(K)$ for a long knot K and its expansion by using the Kauffman-Vogel HOMFLYPT skein relations for singular knots leads to the first quantum knot polynomial $PR_1(hat(K))$ which is *not multiplicative* for the connected sum of knots.

Each double point in $R_1(hat(K))$ comes with a sign. We replace each positive double point by a negative crossing and each negative double point by a positive crossing. Let us call this the *canonical resolution of* $R_1(hat(K))$. The result is an integer combination of ordinary long knots and each invariant derived from this combination of knots is an invariant of our original knot K. Moreover, we could iterate our construction and apply $R_1(hat)$ to each of the new knots. The result is the *canonical wave of* K in the space of all knots $\amalg_K M_K$ which starts from the given knot K. For example, $R_1(hat(5_2^+))$ consists of a single singular knot with a negative double point. If we replace the double point by a negative crossing then we get back of course our original knot 5_2^+. But if we replace it by a positive crossing then we obtain the knot 7_3^+. Replacing the positive double points by negative crossings for $R_1(hat(5_2^+))$ leads to a linear combination of the knots 0_1, 3_1^+ and 8_{20} (with writhe $w = +2$ for a minimal diagram). The wave is *contracting* (i.e. ends with the element 0) for all torus knots and for the knot 4_1 but it seems very likely that it is *expanding* (i.e. contains knots with arbitrary high crossing number) already starting from the knot 5_2^+.

It is well known that the couple of Vassiliev invariants $(v_2(K), v_3(K))$ is a complete invariant for torus knots (and it distinguishes also 4_1 from all torus knots). *A very optimistic conjecture would be that* (v_2, v_3) *evaluated on the canonical wave of K (i.e. each iteration gives an unordered linear combination of couples* (v_2, v_3)*) is already a complete knot invariant.*

It seems that the new invariant is closely related to the geometry of knots. For example, $R_1(hat(K))$ is identical 0 for all torus knots but it is not 0 in general. Let us specify the *quantum 1-cocycle invariant* $PR_1(hat(K))$ by $z = v^{1/2} - v^{-1/2}$ as for the specialization of the HOMFLYPT polynomial to the Jones polynomial. We forget about the signs of the double points. The embedded 1T-relation should not change the invariant. This leads to the substitution $A = v/(v^{1/2} + v^{-1/2})$ and $B = v^{-1}/(v^{1/2} + v^{-1/2})$ in Kauffman-Vogel's skein relations (compare Remark 2 in [1]).

Let us denote the resulting invariant for the *connected sum* of $m \in \mathbb{N}$ copies of K by $P_{K,m}(v)$. It was shown in [1] that in general $PR_1(hat(\sharp_m K))$ is *not* determined by $PR_1(hat(K))$ together with m.

Conjecture 1 (alternative volume conjecture for knots)**.** Let K be a knot. Then

$$\lim_{m\to\infty} log|1 + P_{K,m}(e^{2\pi i/m})|/m = Vol(S^3 \setminus K)/2\pi,$$

where $Vol(S^3 \setminus K)$ is the simplicial volume.

A variation of our combinatorial method produces universally defined combinatorial 1-cocycles for all closed n-braids which are knots. Let $\beta \in B_n$ be a pseudo-Anosov braid which closes to a knot. Hatcher's loop corresponds to the rotation of the closed braid $\hat{\beta}$ around the core of the solid torus V. We consider the quantum invariant $PR_1(hat(\hat{\beta}))$ in the HOMFLYPT skein module of the solid torus, which is obtained from $R_1(hat(\hat{\beta}))$ by applying the Kauffman-Vogel skein relations (here we do not need to normalize because Reidemeister I moves do not appear in an isotopy of closed braids and moreover we set for simplicity $A = A^+ = A^-$ and $B = B^+ = B^-$). Setting $v = 1$ and substituting $z = t^{1/2} - t^{-1/2}$ we obtain an element $\Delta R_1(hat(\hat{\beta}))$ in the Alexander skein module of the solid torus. Using skein relations we express $\Delta R_1(hat(\hat{\beta}))$ in the standard basis of the Alexander skein module, namely the isotopy classes in V of all closed positive permutation braids. Each closed permutation braid $\hat{\sigma}$ is now replaced by its 2-variable Alexander polynomial $\Delta_{\hat{\sigma}\cup L}(u, t) = det(uId - B^r_\sigma(t))$. Here $L = (0 \times \mathbb{R}) \cup \infty \subset S^3$ is the braid axes and $B^r_\sigma(t)$ is the reduced Burau matrix for σ. (The variable u corresponds to the meridian of L and the variable t corresponds to the meridian of $\hat{\sigma}$.) The result of the substitution $\Delta R_1(hat(\hat{\beta}))(A,B,t,u)$, with $A-B=t^{1/2}-t^{-1/2}$, is a Laurent polynomial in $t^{1/2}$ and a polynomial in u of degree $n-1$. We specialize the invariant by $t=-1$ and by the *magic relation* $A+B=1$. The result is a polynomial in u of degree $n-1$ which we denote shortly by $\Delta R_1(hat(\hat{\beta}))(u)$. *Let* u_β *denote the zero of* $\Delta R_1(hat(\hat{\beta}))(u)$ *with the greatest absolute value.*

The *entropy* $h(\beta)$ of a pseudo-Anosov braid is known to be equal to $log(\lambda_\beta)$, where $\lambda_\beta > 1$ is the *stretching factor* of one of the two transverse invariant measured foliations associated to β.

Conjecture 2 (entropy conjecture for pseudo-Anosov braids)**.** Let β be a pseudo-Anosov n-braid which closes to a knot. Then

$$\lim_{m\to\infty} log|u_{\beta^{mn+1}}|/(mn+1) = h(\beta).$$

Moreover, if the associated invariant measured foliation is transverse orientable then already $\log|u_\beta| = h(\beta)$.

The conjecture is true for the 3-braid $\beta=\sigma_1\sigma_2^{-1}$ and for the 4-braid $\beta=\sigma_1\sigma_2\sigma_3^{-1}$ we obtain already for $m=0$ that $|u_\beta|/\lambda_\beta = 1.00143....$

It would be extremely interesting to have computer programs in order to calculate $PR_1(hat(K))$ and $PR_1(hat(\hat{\beta}))$ for more examples.

References

[1] Fiedler T. : Singularization of knots and closed braids, arXiv: in preparation

Left-orderability and cyclic branched covers

Cameron Gordon
(joint work with Tye Lidman)

The left-orderability of the fundamental group of a 3-manifold M is related to the existence of taut foliations on M and also to the Heegaard Floer homology of M. In fact, for a prime rational homology 3-sphere M, it is conceivable that the following are equivalent:

(1) $\pi_1(M)$ is left-orderable
(2) M has a co-orientable taut foliation
(3) M is not an L-space.

It is known that (2) implies (3) [4], and that all three conditions are equivalent when M is a Seifert fibered space. The equivalence of (1) and (3) was explicitly conjectured in [1]. In the absence of a proof of the equivalence of (1), (2) and (3) in general, we say that a 3-manifold M is **excellent** if M has a co-orientable taut foliation and $\pi_1(M)$ is left-orderable, and is a **total L-space** if it is an L-space and $\pi_1(M)$ is not left-orderable.

The talk focused on the case where M is of the form $\Sigma_n(K)$, the n-fold cyclic branched covering of a knot K in the 3-sphere, $n > 1$.

For example, if K is a 2-bridge knot then $\Sigma_2(K)$ is a lens space, and hence a total L-space. More generally, if K is alternating then $\Sigma_2(K)$ is an L-space [5] and $\pi_1(\Sigma_2(K))$ is not left-orderable [1], so again $\Sigma_2(K)$ is a total L-space. We described various known results on $\Sigma_n(K)$ for K a 2-bridge knot and $n > 2$. In some cases, $\Sigma_n(K)$ is a total L-space for all n, while in others it is known that $\pi_1(\Sigma_n(K))$ is left-orderable for all sufficiently large n [2] but not left-orderable for some small values of n. There are also examples where we are able to show that $\Sigma_n(K)$ is excellent. To completely describe what happens for all 2-bridge knots is an interesting open problem.

Other classes of knots that we studied are torus knots, cable knots, Whitehead doubles, and pretzel knots.

For a torus knot K, $\Sigma_n(K)$ is a Seifert fibered space, and there we used the results of Eisenbud, Hirsch, Jankins, Naimi and Neumann characterizing the existence of horizontal foliations on Seifert fibered spaces to show that $\Sigma_n(K)$ is excellent if $\pi_1(\Sigma_n(K))$ is infinite (and a total L-space otherwise).

For the (p, q)-cable $C_{p,q}(K)$ of a knot K, we showed that $\Sigma_n(C_{p,q}(K))$ is excellent unless $n = q = 2$. When $n = q = 2$ there are examples where the manifold is excellent and also examples where it is a total L-space.

An important ingredient of the proof of the result for cables is a recent theorem of Li and Roberts [3] stating that for any non-trivial knot K there is an interval $(-a, b), a, b > 0$, such that for any slope $\alpha \in (-a, b)$, there is a co-orientable taut foliation on the exterior of K which meets the boundary in the foliation by circles of slope α. Moreover, they conjecture that this interval always contains $(-1, 1)$.

The untwisted Whitehead double of a knot K, $Wh(K)$, has trivial Alexander polynomial, and so it follows that $\Sigma_n(Wh(K))$ is an integral homology sphere for all n. Now it seems to be rare for an integral homology sphere to be an L-space: the only known examples are S^3 and connected sums of the Poincaré homology sphere. It is to be expected, therefore, that $\Sigma_n(Wh(K))$ should be excellent. We showed that this is true for $n = 2$. It follows that $\pi_1(\Sigma_n(Wh(K)))$ is left-orderable for all even n, and we showed further that this is also true for odd n provided the Li-Roberts Conjecture holds for K.

Teragaito recently showed [6] that the 3-fold cyclic branched cover of the (p, q, r)-pretzel knot, where p, q, r are odd and > 1, is an L-space. We showed that its fundamental group is not left-orderable, so it is a total L-space.

References

[1] S. Boyer, C. McA. Gordon and L. Watson, *On L-spaces and left-orderable fundamental groups*, Math. Ann. **356** (2013), no. 4, 1213–1245.

[2] Y. Hu, *The left-orderability and the cyclic branched coverings*, arXiv:1311.3291.

[3] T. Li and R. Roberts, *Taut foliations in knot complements*, arXiv:1211.3066.

[4] P. Ozsváth and Z. Szabó, *Holomorphic disks and genus bounds*, Geom. Topol. **8** (2004), 311–334.

[5] P. Ozsváth and Z. Szabó, *On the Heegaard Floer homology of branched double-covers*, Adv. Math. **194** (2005), no. 1, 1–33.
[6] M. Teragaito, *Cyclic branched covers of alternating knots and L-spaces*, arXiv:1404.6892.

The Geometry of the Knot Concordance Space

Shelly Harvey
(joint work with Tim D. Cochran)

Most of the 50-year history of the study of the set of (smooth) knot concordance classes, $\mathcal{C}$, has focused on its structure as an abelian group. Here we take a different approach, namely we study $\mathcal{C}$ as a metric space admitting many natural geometric operators, especially satellite operators.

We consider two different norms and their induced metrics on $\mathcal{C}$, which are defined as follows. For $K \in \mathcal{C}$, $\|K\|_s$, the slice genus, is the minimum g such that K is the boundary of a smoothly embedded compact oriented surface of genus g in B^4. The homology norm, $\|K\|_H$, is defined as the minimum of $\frac{1}{2}(\beta_2(V)+|\sigma(V)|)$ where V ranges over all smooth, oriented, compact, simply-connected 4-manifolds with $\partial V = S^3$ in which K is slice, that is, in which K bounds a smoothly embedded disk that represents 0 in $H_2(V, \partial V)$. The slice and homology metrics are defined as $d_s(K, J) = \|K-J\|_s$ and $d_H(K, J) = \|K-J\|_H$. We note that the homology norm (respectively metric) is only a pseudo norm (respectively pseudo metric). Moreover, d_H will be an honest metric if the smooth 4-dimensional Poincare conjecture is true.

We first consider the geometry of the spaces. We establish the existence of quasi-n-flats for every n, which implies that $\mathcal{C}$ (with either metric) admits no quasi-isometric embedding into a finite product of (Gromov) hyperbolic spaces. We then show that the geometries (even the coarse geometries) of $(\mathcal{C}, d_s)$ and $(\mathcal{C}, d_H)$ are very different. Specifically, we show that the identity map $i : (\mathcal{C}, d_s) \to (\mathcal{C}, d_H)$ is not a quasi-isometry.

We then fix a metric space $(\mathcal{C}, d_*)$ where $* = s$ or H and consider various natural satellite operators acting on this space. We show that every satellite operator $P : (\mathcal{C}, d_*) \to (\mathcal{C}, d_*)$ is within a bounded distance of a "simple" satellite operator, the $(n, 1)$-cable, $C_{n,1}$, where n is the (algebraic) winding number of P. This implies, for example, that winding number ± 1 operators induce quasi-isometries and winding number zero operators are quasi-contractions (with either metric). Moreover, when P is a strong winding number ± 1 operator, we can prove a much stronger result. We show that if P is a strong winding number ± 1 operator then $P : (\mathcal{C}, d_H) \to (\mathcal{C}, d_H)$ preserves the pseudo-norm d_H and is quasi-surjective; so that if the smooth 4-dimensional Poincaré conjecture is true then P is an isometric embedding of $(\mathcal{C}, d_H)$ that is quasi-surjective! These results contribute to the suggestion that $\mathcal{C}$ is a fractal space.

Graph-links: polynomial and quantum invariants

Denis P. Ilyutko
(joint work with Vassily Olegovich Manturov)

It is well known that classical knots can be represented by Gauss diagrams (chord diagrams with some framings), and the whole information about the knot and its invariants can be read out of any Gauss diagram encoding it. Whenever a chord diagram is not a Gauss diagram of any classical knot, one gets a virtual knot, where generic immersion points of intersections of edges of the knot are encircled.

It turns out that some information about the knot can be obtained from a more combinatorial data: the intersection graph of a Gauss diagram. The intersection graph is a graph without loops and multiple edges, whose vertices are in one-to-one correspondence with chords of the Gauss diagram. Two vertices of the intersection graph are *adjacent* whenever the corresponding chords of the Gauss diagram are *linked*, see Fig. 1. Each vertex of the intersection graph is endowed with the local writhe number of the corresponding crossing.

However, sometimes a chord diagram can be obtained from the intersection graph in a non-unique way, and some graphs (shown in Fig. 2) cannot be represented by chord diagrams at all.

Likewise virtual knots appear out of non-realizable chord diagram and thus generalize classical knots (which have realizable chord diagrams), graphs-links come out of intersection graphs: We may consider graphs which are realized by chord diagrams, and, in turn, by virtual links, and pass to arbitrary simple graphs which correspond to some mysterious objects generalizing links and virtual links.

Traldi and Zulli [10] constructed a self-contained theory of "non-realizable knots" (the theory of *looped interlacement graphs*) possessing lots of interesting knot theoretic properties by using Gauss diagrams. These objects are equivalence classes of (decorated) graphs modulo "Reidemeister moves".

The author and V. O. Manturov suggested another way of looking at knots and links and generalizing them (the theory of *graph-links*): whence a Gauss diagram corresponds to a transverse passage along a knot, one may consider a *rotating circuit* which never goes straight and always turns right or left at a classical crossing. One can also encode the type of smoothing (Kauffman's A-smoothing or Kauffman's B-smoothing) corresponding to the crossing where the circuit turns right or left and never goes straight, see Fig. 3. We note that chords of diagrams are naturally split into two sets: those corresponding to crossings where two opposite directions correspond to *emanating* edges with respect to the circuit and the other two correspond to *incoming* edges, and those where we have two consecutive (opposite) edges one of which is incoming and the other one is emanating.

After the two theories were constructed, some questions arose. The first question is whether or not every graph is Reidemeister equivalent (each theory has own Reidemeister moves) to an intersection graph of a virtual knot diagram. The second question is related to the existence of an equivalence between two theories. Other questions concern invariants and classifications of graph-links.

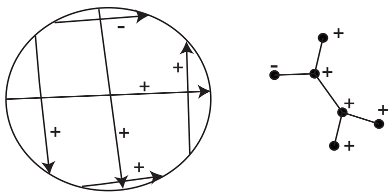


Figure 1. A Gauss diagram and its labeled intersection graph

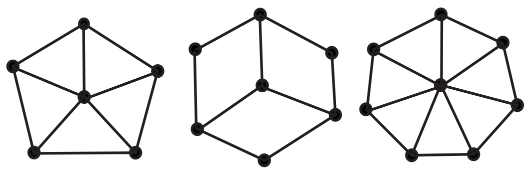


Figure 2. Non-realizable Bouchet graphs

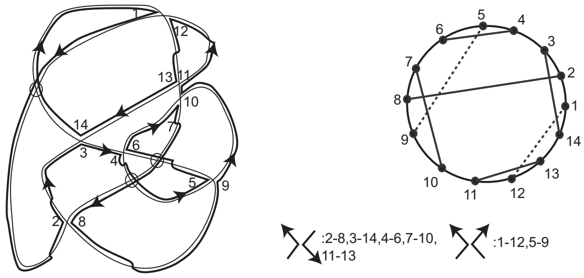


Figure 3. Rotating circuit shown by a thick line; chord diagram

The first question was resolved for graph-knots by using parity theory introduced by Manturov in [7] and for graph-links by the author in [5]. The equivalence of the two theories (the theory of looped interlacement graphs and the theory of graph-knots) was proved in [1]. Also, some invariants were constructed, see [2, 3, 4, 5, 9, 10].

It turns out that if we forget about the writhe number information for a link and only have the structure of opposite edges, we shall get non-trivial objects (modulo Reidemeister moves). Analogously we can construct the theory of free graph-links: At each vertex we have only one label.

In [8] V. O. Manturov constructed an invariant for free links, which is analogous to Kuperberg's quantum invariant [6]. By using this invariant V. O. Manturov gave a classification of free links without triangles. Applying that construction to the theory of free graph-links we can construct an invariant of free graph-links which will help us classify free graph-links without "triangles".

References

[1] D. P. Ilyutko, *An equivalence between the set of graph-knots and the set of homotopy classes of looped graphs*, Knot Theory and Its Ramifications J. **21**:1 (2012), DOI: 10.1142/S0218216512500010.
[2] D. P. Ilyutko, V. O. Manturov, *Introduction to graph–link theory*, Knot Theory and Its Ramifications J. **18**:6 (2009), 791–823.
[3] D. P. Ilyutko, V. O. Manturov, *Graph–links*, Doklady Mathematics **80**:2 (2009), 739–742 (Original Russian Text in Doklady Akademii Nauk **428**:5 (2009), 591–594).
[4] D. P. Ilyutko, V. O. Manturov, *Graph-links*, in: Proceedings of the Advanced Summer School on Knot Theory, Trieste, Series of Knots and Everything, World Scientific (2011), 135–161.
[5] D. P. Ilyutko, V. S. Safina, *Graph-links: non-realizability, orientation and the Jones polynomial*, Topology, CMFD **51** (2013), 33–63.
[6] G. Kuperberg, *The quantum G_2 link invariant*, Internat. J. Math. **5** (1994), 61–85.
[7] V. O. Manturov, *Parity in knot theory*, Math. sb. **201**:5 (2010), 693–733 (Original Russian Text in Mathematical sbornik **201**:5 (2010), 65–110).
[8] V. O. Manturov, *An almost classification of free knots*, Doklady Mathematics **88**:2 (2013), 556–558 (Original Russian Text in Doklady Akademii Nauk **452**:4 (2013), 371–374).
[9] I. Nikonov, *Khovanov homology of graph-links*, arXiv, math.GT/1005.2812.
[10] L. Traldi and L. Zulli, *A bracket polynomial for graphs*, Knot Theory and its Ramifications J. **18** (2009), 1681–1709.

Towards a Rasmussen Invariant for Virtual Knot Cobordisms

Aaron Kaestner
(joint work with Heather A. Dye, Louis H. Kauffman)

In this talk we gave an alternate formulation for the Manturov definition [8] of Khovanov Homology [4] [5] for virtual knots and links with arbitrary coefficients. This approached used cut loci on the knot diagram to induce a conjugation operator in the Frobenius algebra. We then discussed the implications of the maps induced in the aforementioned theory to the universal Frobenius algebra [6] and noted that for virtual knots the universal Frobenius algebra corresponds to the generalization of Lee's Algebra introduced by Bar-Natan [1]. Next we discussed how one can apply the Karoubi envelope approach of Bar-Natan and Morrison [2] on abstract link diagrams [3] with cross cuts to construct the canonical generators of the Khovanov-Lee Homology [7]. Using these generators we derived a generalization of the Rasmussen invariant [9] for virtual knot cobordisms.

References

[1] D. Bar-Natan, *On Khovanov's categorification of the Jones polynomial*, Algebr. Geom. Topol. **2** (2002) , 337–370.
[2] D. Bar-Natan and S. Morrison, *The Karoubi envelope and Lee's degeneration of Khovanov homology*, Algebr. Geom. Topol. **6** (2006) , 1459–1469.
[3] N. Kamada and S. Kamada, *Abstract link diagrams and virtual knots*, J. Knot Theory Ramifications **9**(1) (2000) , 93–106.
[4] M. Khovanov, *A categorification of the Jones polynomial*, Duke Math. J. **101**(3) (2000) , 359–426.
[5] M. Khovanov, *A functor-valued invariant of tangles*, Algebr. Geom. Topol. **2** (2002) , 665–741.

[6] M. Khovanov, *Link homology and Frobenius extensions*, Fundamenta Mathematicae **190** (2006) , 179–190.
[7] E. S. Lee, *An endomorphism of the Khovanov invariant*, Adv. Math. **197**(2) (2005) , 554–586.
[8] V. O. Manturov, *Khovanov homology for virtual knots with arbitrary coefficients*, J. Knot Theory Ramifications **16**(3) (2007) , 345–377.
[9] J. Rasmussen, *Khovanov homology and the slice genus*, Invent. Math. **182**(2) (2010) , 419–447.

Virtual Knot Theory and Virtual Knot Cobordism

Louis H. Kauffman

This talk reviews the definition of virtual knots, bracket polynomial, arrow polynomial and affine index polynomial [1, 2, 3, 4]. Then the talk defines *virtual knot cobordism* [5] as a combination of virtual isotopy, births of unknotted circles, deaths of unknotted circles and saddle transformations. See [6] for a theory of cobordism for free knots. Each cobordism from K to K' describes an abstract orientable surface S whose boundary is the union of K and K'. We call the *genus g of the cobordism* the genus of this surface S. We give a non-trivial example of a virtual slice knot (the virtual stevedore's knot) and show that it is non-classical by using the invariants reviewed at the beginning of the talk. We define two virtual knots to be *concordant* if there is a genus zero cobordism between them. A knot is slice if it is concordant to the unknot. This is equivalent to saying that it is slice if it is concordant to the empty knot (since an unknot bounds a disk). We say that a virtual knot is *ribbon* if it is concordant to the unknot via only deaths and saddles. The virtual stevedore is an example of a virtual ribbon knot. In general, any virtual knot is cobordant to the empty knot and so there is a least genus cobordism for that virtual knot. We define $g_4(K)$ to be the least such genus, and we call this the four-ball genus of K. We continue by raising the problem of band-pass equivalence for virtual knots (generalizing classical band passing for classical knots). Classical knots fall into two pass classes according to their Arf invariants. At this writing there is no known classification for the pass classes of virtual knots. Finally, we generalize the middle-level diagrams for classcial surfaces in four-space to virtual middle-level diagrams and we generalize the Yoshikawa moves [7] for such diagrams to the virtual case. At this writing it is not known if the isotopy relations generated by these new Yoshikawa moves are the same as generalized Roseman moves [8] for surface immersion diagrams for virtual embeddings into four-space. Slides corresponding to this talk are available at $< http://dl.dropbox.com/u/11067256/VirtualKnotCobordism.pdf >$.

References

[1] Kauffman, Louis H. Virtual Knot Theory , European J. Comb. (1999) Vol. 20, 663-690.
[2] Kauffman, Louis H. An extended bracket polynomial for virtual knots and links. (English summary) J. Knot Theory Ramifications 18 (2009), no. 10, 1369–1422.
[3] Kauffman, Louis H. Introduction to virtual knot theory. J. Knot Theory Ramifications 21 (2012), no. 13, 1240007, 37 pp.

[4] Kauffman, Louis H. An affine index polynomial invariant of virtual knots. J. Knot Theory Ramifications 22 (2013), no. 4, 1340007, 30 pp.
[5] Kauffman, Louis H. Virtual knot cobordism, to appear in "New Invariants in Low Dimensional Topology" edited by L.H. Kauffman and V.O. Manturov, World Scientific Pub. Co. (2014).
[6] *Manturov V. O.* Parity and Cobordisms of Free Knots// Mat. Sb., 203:2 (2012), 45–7 (Original Russian Text in Mathematical sbornik **203**:12 (2012), 45-76).
[7] Swenton, Frank J. On a calculus for 2-knots and surfaces in 4-space. J. Knot Theory Ramifications 10 (2001), no. 8, 1133–1141.
[8] Takeda, Yasushi. Introduction to virtual surface-knot theory. J. Knot Theory Ramifications 21 (2012), no. 14, 1250131, 6 pp.

On the structure of Takasaki quandles

Seongjeong Kim
(joint work with Y.Bae)

A Takasaki quandle $(T(G), *)$ is a quandle defined on an abelian group G with the binary operation $*$ defined by $a * b = 2b - a$. Since a Takasaki quandle is derived from an abelian group, its algebraic structure depends on the underlying abelian group.

Lemma 1. A Takasaki quandle $(T(G), *)$ is connected if and only if $2G = G$.

Lemma 2. Let $(T(G), *)$ be the Takasaki quandle of an abelian group $(G, +)$. If X is a subgroup of G, then X is a subquandle of $(T(G), *)$.

From the above lemma, we can expect that subgroups of G affect the quotient quandle of a Takasaki quandle. We can see the following statements;

Lemma 3. Let $(T(G), *)$ be the Takasaki quandle of an abelian group $(G, +)$ and Y a subquandle of $T(G)$. If there is a quandle automorphism f of $T(G)$ and a subgroup X of G such that $Y = f(X)$, then, for $a, b \in G$, either

$$(Y * a) \cap (Y * b) = \emptyset \text{ or } Y * a = Y * b.$$

Moreover, if $(T(G), *)$ is a connected quandle, then the converse is also true.

Theorem 1. Let $(T(G), *)$ be the Takasaki quandle of an abelian group $(G, +)$ and X a subgroup of G. Let $f : T(G) \to T(G)$ be a quandle automorphism and $Y = f(X)$. Define a binary operation $*'$ on $\{Y * a\}_{a \in G}$ by

$$(Y * a) *' (Y * b) = Y * (a * b)$$

Then $(\{Y * a\}_{a\in G}, *')$ is a quandle. In fact, $(\{Y * a\}_{a\in G}, *')$ is isomorphic to $T(2(G/X))$ as a quandle. Denote $\{Y * a\}_{a\in G}$ by $T(G)/Y$.

Moreover, if $T(G)$ is connected, then $(T(G)/Y, *')$ is isomorphic to $T(G/X)$. That is, for a connected Takasaki quandle $T(G)$, quotient quandle of $T(G)$ and quotient group of G are closely related.

References

[1] Y. Bae, S. Kim, *On qoutients of Takasaki quandles*, Journal of Knot Theory and Its Ramifications 22, No. 12(2013).

On algebraic, PL and Fourier degrees of knots and braids

Stephan Klaus

By a theorem of Alexander, an embedded torus $T \subset \mathbb{R}^3$ bounds a solid torus $S^1 \times D^2$ on the one side or the other. Thus, isotopy classes of embeddings of a torus T^2 in $\mathbb{R}^3$ correspond bijectively to isotopy classes of knots. The correspondence from knots to torus embeddings is given by the boundary of a tubular neighborhood around a knot.

Torus embeddings can also be constructed as affine real algebraic varieties, i.e. as the set of zeros of a real polynomial $p(x, y, z)$. For example, the standard embedding of a torus, which corresponds to the unknot, is defined by rotation of a circle around the z-axis, where the circle in the (x, z)-plane has radius b and center $(a, 0, 0)$. It is well-known that this embedding is given by the polynomial $(x^2 + y^2 + z^2 + a^2 - b^2)^2 - 4a^2(x^2 + y^2)$ of degree 4.

We give below a simple proof that every knot type can be represented by an affine real algebraic torus embedding, i.e. by a suitable polynomial $p(x, y, z)$. Hence we can define the **algebraic degree** a-deg(K) of a knot K as the minimal degree of a polynomial which represents K.

In [4] we have constructed the following polynomial of degree 14 which represents the trefoil knot $3_1 = T_{2,3}$:

$$\begin{gathered}p_{2,3}(x,y,z) := (-8(x^2+y^2)^2(x^2+y^2+1+z^2+a^2-b^2)\\ +4a^2(2(x^2+y^2)^2-(x^3-3xy^2)(x^2+y^2+1))+8a^2(3x^2y-y^3)z\\ +4a^2(x^3-3xy^2)z^2)^2-(x^2+y^2)(2(x^2+y^2)(x^2+y^2+1+z^2+a^2-b^2)^2\\ +8(x^2+y^2)^2+4a^2(2(x^3-3xy^2)-(x^2+y^2)(x^2+y^2+1))\\ -8a^2(3x^2y-y^3)z-4(x^2+y^2)a^2z^2)^2-0.0001\end{gathered}$$

This yields a-deg$(T_{2,3}) \leq 14$. The method of construction works by a suitable parametrization of two separate circles in the (x, z)-plane which rotate around their center of mass and at the same time rotate around the z-axes. Here the first rotation velocity has to be $3/2$ times faster than the second one. Algebraic variable elimination of the rotation parameters yields the above polynomial. Details can be found in [4].

In [5] we have generalized this method to all torus knots $T_{p,q}$ with p and q coprime by the construction of polynomials $P_{p,q}$ of degree $4p + 2q$. Hence this yields an upper bound for a-deg$(T_{p,q})$.

Now we recall the **minimal stick number** which we also call the **minimal PL-degree** of a knot K. PL-deg(K) is defind as a the minimal number of line segments necessary to represent the knot as a PL-knot.

Theorem: [6] *Any knot K can be represented by an affine real algebraic torus and it holds a-deg$(K) \leq 2$PL-deg(K).*

The proof is quite simple: Replace each line segment L_i in a PL-representation of K by a small ellipsoid E_i around L_i such that the ellipsoids of consecutive line segments are touching. Each E_i is given by a quadratic polynomial $p_i(x, y, z)$ such that p_i takes negative values inside E_i. Hence the product polynomial $p(x, y, z) := \prod_i p_i(x, y, z)$ gives the union of ellipsoids which is an affine algebraic variety V with singularities at the touching points of the ellipsoids. By construction, p is negative in the inner part of the elipsoids, i.e. inside of V, and positive outside of V. Hence the polynomial $q(x, y, z) := p(x, y, z) - a$ gives for small $a > 0$ a resolution of the singularities of V such that the new affine algebraic variety is a torus embedding. If the PL-representation has s line segments (sticks), then the degree of the polynomial q is $2s$, which proves the statement on a-deg and PL-deg.

The following facts on PL-deg are known:

- A knot with PL-deg$(K) < 6$ is the unknot O which has PL-deg$(O) = 3$.
- It holds PL-deg$(3_1) = 6$ (trefoil knot) and PL-deg$(4_1) = 7$ (figure eight knot).
- For the torus knots it holds PL-deg$(T_{p,q}) \leq 2p$ if $2 \leq p < q \leq 2p$ by a result of Jin [2].

It follows that a-deg$(3_1) \leq 12$, hence the algebraic representation of degree 14 given above is not optimal. The upper bound for torus knots in [5] is also not optimal in case of $2 \leq p < q \leq 2p$ by the result of Jin.

Now we recall the definition of the Fourier degree of a knot which was given by Trautwein [8] and Kauffman [3]. As the embedding $S^1 \to \mathbb{R}^3$ is a vector-valued periodic function, it has a Fourier representation. Up to isotopy, it can be given as a finite Fourier sum f. We call the highest frequency the degree of f. Then the **Fourier degree** F-deg(K) is defined as the minimal degree which is necessary to represent K up to isotopy by a finite Fourier sum.

The following facts on F-deg are known [8], [3]:

- A knot with F-deg$(K) < 3$ is the unknot O which has PL-deg$(O) = 1$.
- It holds F-deg$(3_1) =$ F-deg$(4_1) = 3$.
- There is a lower bound which is given by the super bridge number $sb(K)$ and it holds $b(K) < sb(K) \leq$F-deg(K) [8].
- For torus knots it holds F-deg$(T_{p,q}) \leq p + q$ by the standard parametrization.

In [6] we have proved the following relation between algebraic and Fourier degree:

Theorem: [6] *It holds a-deg$(K) \leq 12$F-deg(K) for any knot.*

The proof works by an algebraic construction of the tubular neighborhood around a knot K which is given as a finite Fourier series:

$$K(t) = p(c, s), \quad c := \cos(2\pi t), \quad s := \sin(2\pi t)$$

and $p(c, s)$ denotes three polynomials for the vector components of $k(t)$. Now the tube of radius r around K is given by the algebraic equations

$$|(x, y, z) - p(c, s)|^2 = r^2 \quad \text{and} \quad \langle (x, y, z), -\partial_1 p(c, s)s + \partial_2 p(c, s)c \rangle = 0$$

As a next step of the proof we eliminate the variables c and s using $c^2 + s^2 = 1$. The algebraic elimination of s can be carried out explicitely and for the remaining variable c we use the Sylvester determinant for the resolvent. Degree counting then gives the factor 12 which is probably not an optimal bound. Additionally, the theorem gives a second proof that each knot can be realized by an algebraic torus embedding. Details can be found in [6].

Open question: It would be very interesting to have also inequalities in the other direction between these degrees. In particular, it seems to be an open problem to compute the minimum/maximum of the three quotients which can be formed by a-deg(K), PL-deg(K) and F-deg(K). Moreover, do the limits of these quotients exist, e.g. if the crossing number of K tends to infinity?

Now we consider the Fourier degree also for braids. Here we interchange freely between periodic functions ($f(t+1) = f(t)$) and functions $g(z)$ defined on the unit circle $z \in S^1 \subset \mathbb{C}$ by $z = \exp(2\pi i t)$ and $g(z) = f(t)$. This gives the well-known translation between finite Fourier sums $f(t)$ and Laurent polynomials $g(z)$.

An n-braid β has a link closure $\hat{\beta}$ which is a knot if and only if the associated permutation σ_β of β is an n-cycle. We call such a braid a **cyclic braid**. Each strand of β can be considered as a mapping from $[0,1]$ to $\mathbb{C}$. If the braid is cyclic, we can concatenate its n strands in their consecutive order given by σ_β and then we obtain a periodic function (after increasing velocity by the factor n)

$$\beta^u : S^1 \to \mathbb{C}.$$

We call β^u the **unfolding** of the cyclic braid β. In this way we obtain a homeomorphism between the space CB_n of cyclic n-braids and the following space of periodic functions

$$UB_n := \{g : S^1 \to \mathbb{C} \mid \Delta_g^n(z) \neq 0\}$$

$$\Delta_g^n(z) := \prod_{i=1}^{\lfloor n/2 \rfloor} (g(\epsilon^i z) - g(z))$$

where $\epsilon := \exp(2\pi i/n)$ is the primitive n-th root of unity. The nonvanishing condition for the product is equivalent to the condition that different strands do not intersect. By multiplication of z with a suitable $z' \in S^1$, the non-intersection condition between the j-th and k-th strands can be shifted to the non-intersection condition of the $(j-k)$-th strand and the zeroth strand. This is the reason that the definition of the **n-th discriminant** $\Delta_g^n(z)$ of g only contains $\lfloor n/2 \rfloor$ factors (largest integer smaller than $n/2$).

Note that the set of path components $\pi_0(UB_n)$ is a set of free (non-based) homotopy classes and thus is in bijection to the subset of conjugacy classes in the braid group Br_n which are formed by cyclic braids.

Now we are able to define the **Fourier degree** F-deg(β) of a cyclic n-braid β as the minimal degree which is necessary to find a representative $g(z)$ in UB_n by a finite Fourier sum (Laurent polynomial).

Theorem: [6] *For a cyclic n-braid β and its knot closure $\hat{\beta}$ it holds F-deg($\hat{\beta}$) $\leq$ F-deg(β) $+ n$.*

This follows at once from the fact that the knot closure of β can be given in terms of its unfolded function $g(z)$ as

$$\hat{\beta}(t) = \left(\cos(2\pi nt)(1 + \frac{1}{2r}\mathrm{Re}(g(z))), \sin(2\pi nt)(1 + \frac{1}{2r}\mathrm{Re}(g(z))), \frac{1}{2r}\mathrm{Im}(g(z))\right)$$

where $z = \exp(2\pi it)$ and $r := \max\{|g(z)| \mid z \in S^1\}$ is the maximal elongation of $g(z)$.

Note that the bridge number of $\hat{\beta}$ is equal or smaller than the number n of strands which together with the result of Trautwein suggests that F-deg(β) represents the reduced part of the Fourier degree of a knot.

The unfoldings $g(z)$ allow algebraic manipulations which we summarize in a dictionary:

unfolded function $g(z)$	knot closure $\hat{\beta}$
$a + g(z)$	toroidal translation
$ag(z)$	toroidal rotation/dilatation
$\bar{g}(z$	mirror knot
$g(\bar{z})$	inverse knot (time reversal)
$g(uz), u \in S^1$	rotation around z-axis
$z^n g(z)$	Dehn twist

Unfortunately, concatenation of braids and Markov stabilization seem to have no simple algebraic counterpart for $g(z)$.

Note that each non-zero Laurent-polynomial $G(z)$ can be uniquely written as $G(z) = z^m p(z)$ with $m \in \mathbb{Z}$ and $p(z)$ a polynomial satisfying $p(0) \neq 0$. Then the condition $G(z) \neq 0$ on S^1 means that all zeros of $p(z)$ lie in the complement of the untit circle. This applies in particular to the discriminant. The **discriminant set**

$$V := \{g(z) \mid \Delta_g^n(z) \text{ has a zero on } S^1\}$$

has an interesting algebraic filtration $V_1 \subset V_2 \subset \ldots$ by Laurent/Fourier degree of g. This suggests that it should be possible to apply Vassiliv's method to construct a spectral sequence where the initial term is related to the Laurent/Fourier degree. It would be interesting to obtain in this way a better algebraic and topological understanding of the space of polynomials whose n-th discriminant does not vanish on S^1.

As a generalization, there is also a modification of unfolded cyclic braids which leads to an algebraic model for **knots in a full torus or in a thickened torus**. The condition that a braid β is in the complement of a fixed constant strand $z = a$ can be implemented algebraically by the condition $(g(z) - a)\Delta_g^n(z) \neq 0$ on S^1. More generally, braids avoiding m fixed constant strands in a_i can be described by the condition

$$(g(z) - a_1)(g(z) - a_2) \cdots (g(z) - a_m)\Delta_g^n(z) \neq 0$$

on S^1. For example, we can choose the a_m to be the m-th roots of unity and this gives $(g(z)^m - 1)\Delta_g^n(z) \neq 0$. This gives a space of unfolded functions which is related to the relative braid group $Br_{m,n}$, see Lambropoulou [7]. By the closure operation we obtain an algebraic model for knots in the complement of the trivial m-link. In the case $m = 1$, this complement is homeomorphic to the (open) full torus $S^1 \times D^2$. As the Hopf link can be represented by the Hopf 2-braid with the strands 0 and z, we furthermore obtain an algebraic description of cyclic n-braids avoiding the Hopf braid:

$$g(z)(g(z) - z^n)\Delta_g^n(z) \neq 0$$

on S^1. Here we have to replace z by z^n as each of the n strands has to be disjoint from the Hopf strand z. This condition is equivalent to $g(z) \neq z^n$ on S^1. In this case, the closure operation leads to knots in the complement of the Hopf link which is homeomorphic to the (open) thickened torus $S^1 \times S^1 \times D^1$.

Acknowledgements I like to thank Gert-Martin Greuel and Hanspeter Kraft for very helpful discussions on some algebraic problems connected with polynomial representations and variable elimination. Moreover, I like to thank Sofia Lambropoulou for many essential discussions on knots and braids in Oberwolfach and Athens. It was her suggestion to consider the Fourier degree also for braids and to consider the generalization for knots in a full torus or in a thickened torus. Last but not least I like to thank the organizers of this Oberwolfach Workshop for inviting me to give a talk on these results although I was not a regular workshop participant.

References

[1] Colin C. Adams, *The Knot Book*, W.H.Freeman (1994)
[2] Gyo Taek Jin, *Polygon indices and superbridge indices of torus knots and links*, Journal of Knot Theory and its Ramifications Vol.6 (1997), pp. 281–289
[3] Louis H. Kauffman, *Fourier knots*, pp. 364-373, in: *Series on Knots and Everything Vol.19: Ideal Knots*, World Scientific (1997)
[4] Stephan Klaus, *The solid trefoil knot as an algebraic surface*, pp. 2-4, featured article in CIM Bulletin (2010), Coimbra International Center for Mathematics
[5] Stephan Klaus, *The solid torus knots as algebraic surfaces*, preprint, Oberwolfach (2013)
[6] Stephan Klaus, *On algebraic, PL and Fourier degrees of knots and braids*, preprint, Oberwolfach (2013)
[7] Sofia Lambropoulou, *Braid structures in knot complements, handlebodies and 3-manifolds*, pp. 274-289, in: *Series on Knots and Everything Vol.24: Knots in Hellas '98, Proceedings of the International Conference on Knot Theory and Its Ramifications*, World Scientific (2000)
[8] Aaron K. Trautwein, *An introduction to harmonic knots*, pp. 353-363, in: *Series on Knots and Everything Vol.19: Ideal Knots*, World Scientific (1997)

Topological Surgery and Dynamics

Sofia Lambropoulou
(joint work with Stathis Antoniou, Nikola Samardzija, Ioannis Diamantis)

0.1. An *n-dimensional topological surgery* on an n-manifold M is, roughly, the topological procedure whereby an appropriate n-manifold with boundary is removed from M and is replaced by another n-manifold with the same boundary, using a 'gluing' homeomorphism, thus creating a new n-manifold $\chi(M)$ (not necessarily different from the starting one).

0.2. Surgery in Nature and its dynamics. (with Stathis Antoniou and Nikola Samardzija [1])

Apart from just being a formal topological procedure, topological surgery appears in nature in numerous and diverse processes of various scales for ensuring new results. Such processes are initiated by attracting or repelling forces between two points, or *'poles'*, which seem to be joined by some *'invisible thread'*.

In order to model topologically such phenomena, in [1] we introduce dynamics in 1-, 2- and 3-dimensional topological surgery, by means of attracting or repelling forces between two selected points in the manifold. We also introduce the notions of solid 1- and 2-dimensional topological surgery, and of truncated 1-, 2- and 3-dimensional topological surgery, which are more appropriate for modelling natural processes. As representative examples we will give the definitions of attracting solid 2-dimensional surgery and attracting truncated 3-dimensional surgery:

Definition 1. Start with the 3-ball of radius 1 with polar layering:

$$D^3 = \cup_{0<r\leq 1} S^2_r \cup \{C\},$$

where r the radius of a 2-sphere and C the limit point of the spheres, that is, the center of the ball. *Attracting solid 2-dimensional surgery* on D^3 is the topological procedure where: on all spheres S^2_r colinear pairs of antipodal points are specified, on which the same colinear attracting forces act, see Figure 1. Then attracting 2-dimensional surgeries are performed on the whole continuum of the concentric spheres using the same homeomorphism h. Attracting 2-dimensional surgery on the limit point C is defined to be the limit circle of the nested tori resulting from the continuum of 2-dimensional surgeries. The process is the same as first removing the center C from D^3, performing the 2-dimensional surgeries and then taking the closure of the resulting space. Namely we obtain:

$$\chi(D^3) := \cup_{0<r\leq 1} \chi(S^2_r) \cup \chi(C),$$

which is a solid torus.

Definition 2. Consider two points in 3-space, surrounded by spherical neighbourhoods, say B_1 and B_2 and assume that on these points strong attracting forces act. View Figure 2. As a result, a 'joining thread', say L, is created between the two points and 'drilling' along L is initiated. The joining arc L is seen as part of

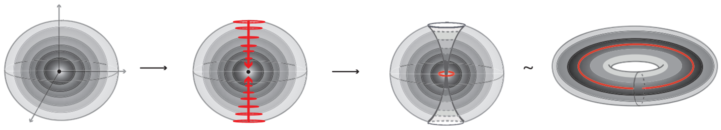


FIGURE 1. Attracting solid 2-dimensional surgery.

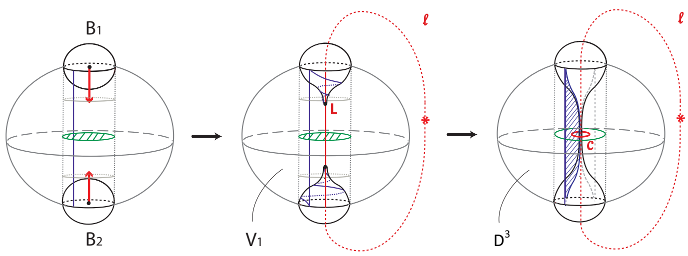


FIGURE 2. Attracting truncated 3-dimensional surgery.

a simple closed curve l passing by the point at infinity. This is the surgery curve. Further, the two 3-balls B_1 and B_2 together with the space in between make up a solid cylinder, the 'cork'. Let V_1 be a solid torus, which filled by the cork gives rise to a 3-ball D^3, such that the centers of the two balls B_1 and B_2 lie on its boundary. The process of 3-dimensional surgery restricted in D^3 shall be called *attracting truncated 3-dimensional surgery.*

To address some examples of natural phenomena exhibiting surgery: 1-dimensional surgery happens in DNA recombination and in the magnetic reconnection. Attracting 2-dimensional surgery is exhibited in the formation of whirls, in the Falaco solitons and in the gene transfer in bacteria. Repelling 2-dimensional surgery is exhibited in bubble blowing and in the cell mitosis. Finally, 3-dimensional surgery can be observed in the formation of tornadoes and in an electromagnetic field excited by a current loop.

On the theoretical level, these new notions allow one to visualize 3-dimensional surgery and to connect surgeries in different dimensions [1]. Our work is inspired by our connection of 3-dimensional topological surgery with a dynamical system (work in progress). Then, on one hand we will have a mathematical model for 3-dimensional surgery. On the other hand, through our connection many natural phenomena can be modelled through our dynamical system. For details see [1] and references therein.

0.3. The examples of 3-dimensional surgery above seem to exhibit *p/q-rational surgery along the unknot*, which, starting from S^3, results in the lens space $L(p, q)$

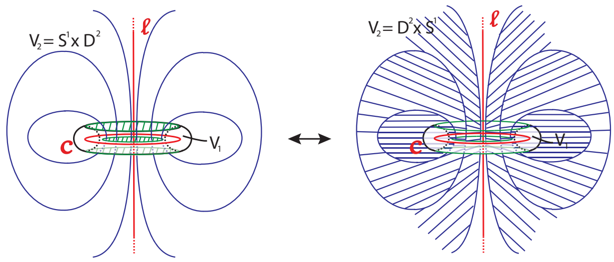


FIGURE 3. 3-dimensional surgery along l.

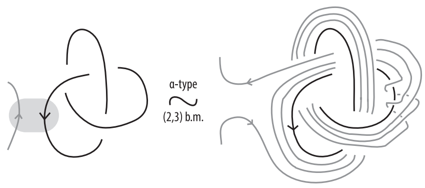


FIGURE 4. A $\mathbb{Q}$-band move.

(see Figure 3). Topologically speaking, a specified (p, q)-torus knot on a tubular neighbourhood of the surgery curve l before, now bounds a disc in $L(p, q)$. So, if a piece of arc approaches the surgery curve l, it will have to follow in parallel the (p, q)-torus knot.

0.4. Braid Equivalence for Rational Surgery. (with Ioannis Diamantis [2])

Studying the knot theory of a c.c.o. 3-manifold gives information about the 3-manifold. Further, there are c.c.o. 3-manifolds which have simpler description when obtained from S^3 by rational surgery instead of integral surgery. Even more so, there are whole (infinite) families of 3-manifolds described by rational surgery along the same link. Representative examples are the lens spaces $L(p, q)$, the homology spheres obtained by rational surgery $1/n$ along the trefoil knot, the Seifert manifolds, and manifolds obtained by surgery along torus knots.

In [2] we provide mixed braid equivalence, geometric as well as algebraic, for isotopic oriented links in a c.c.o. 3-manifold M obtained by rational surgery along a framed link in S^3. Our results are based on earlier results by Lambropoulou and Rourke for 3-manifolds with integral surgery description. More precisely, we first prove a sharpened version of the Reidemeister theorem for links in M, see Figure 4. We then give geometric formulations of the braid equivalence via mixed braids in S^3 using the L-moves and the $\mathbb{Q}$-braid band moves, see Figure 5.
We finally give algebraic formulations in terms of the mixed braid groups $B_{m,n}$ using *cabling*, and the techniques of *parting* and *combing* for mixed braids developed in earlier work of Lambropoulou and Rourke, see Figure 6.

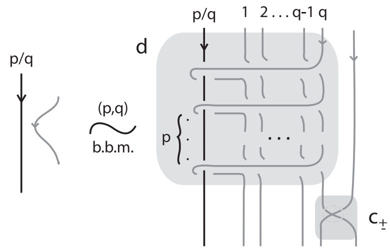


FIGURE 5. A $\mathbb{Q}$-braid band move locally.

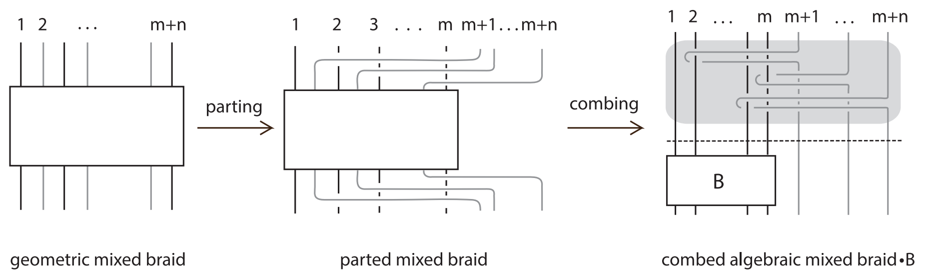


FIGURE 6. Parting and combing a geometric mixed braid.

Our result for $L(p,q)$ is then applied for computing the Homflypt skein module of $L(p,q)$ via the braid approach, after an appropriate change of basis of the Homflypt skein module of the solid torus (work in progress).

Our algebraic equivalences set a homogeneous ground for the algebraic braid equivalences for link isotopy in families of 3-manifolds. We provide concrete formuli of the braid equivalences in lens spaces, Seifert manifolds, homology spheres obtained from the trefoil and manifolds obtained from torus knots. The algebraic classification of links in a 3-manifold via mixed braids is a useful tool for studying skein modules of 3-manifolds and of families of 3-manifolds. For details see [2] and references therein.

Acknowledgments This research has been co-financed by the European Union (European Social Fund - ESF) and Greek national funds through the Operational Program "Education and Lifelong Learning" of the National Strategic Reference Framework (NSRF) - Research Funding Program: THALES: Reinforcement of the interdisciplinary and/or inter-institutional research and innovation.

REFERENCES

[1] S. Lambropoulou, S. Antoniou, N. Samardzija, *Topological surgery and its dynamics*, arXiv:1406.1106v1 [math.GT] 4 Jun 2014.

[2] I. Diamantis, S. Lambropoulou, *Braid equivalence in 3-manifolds with rational surgery description*, arXiv:1311.2465v1 [math.GT] 8 Nov 2013.

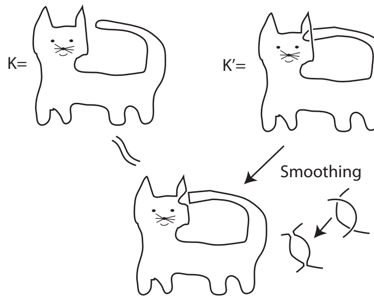


FIGURE 1. An odd irreducible diagram K is a subdiagram of a diagram K' equivalent to it

New Parities and Cobordisms in Low-Dimensional Topology

VASSILY OLEGOVICH MANTUROV

The talk deals with the notion of *parity*. Knots and virtual knots are encoded by *diagrams* modulo *moves*. Diagrams contain *crossings* which are connected by arcs. It turns out that if there is a natural way to distinguish between *even* and *odd crossings* that behave well under the moves, then there are easy ways to prove minimality theorems in a strong sense, construct various functorial mappings, enhance many new invariants etc.

What is a parity? There is a natural parity called *Gaussian:* If a knot diagram K is encoded by a Gauß diagram $G(K)$ then every crossing k corresponds to a chord $c(k)$; the crossing k and the chord c are called *even* if $c(k)$ is linked with evenly many chords and *odd* otherwise.

A Gauß diagram is *even* if all chords of it are even. It was noted by Gauß that Gauß diagrams of plane curves (and hence, classical knots) are all even.

However, classical crossings of virtual knots can be both even and odd (note that for virtual knot diagrams, chords on chord diagrams correspond to classical crossings, only). Moreover, the existence of odd crossings yields non-triviality of a diagram in a crucial sense.

Theorem 1. [10] If a (virtual) knot diagram K is *odd* and irreducible then every diagram K' equivalent to K contains K as a *smoothing*.

Here *oddness* of K means that all crossings of K are odd, and *irreducibility* means that there is no way to apply a (flattened version of the) second decreasing Reidemeister move to K.

View Fig. 1.

Here we illustrate how *Theorem 1* works in one turn: after performing a second Reidemeister move to K, we get K', and we can see K "inside" it.

Certainly, this Theorem has lots of consequences which allow one to make many important conclusions about *any* diagram of a virtual knot by looking at just *one* diagram of it, thus reducing properties of *knots* to properties of their *diagrams.*

In particular, this allowed the author to prove first the folloiwng

Theorem 2. [13] There exists a family of virtual knots K_n such that the minimal virtual crossing number of K_n grows quadratically with respect to the number of classical crossings of K_n.

Are there any other parities?

Theorem 1 uses simple axioms of the notion of parity. In fact, there is a general notion of parity such that all theorems can be formulated for **any** parity. Certainly, the notion of *odd* diagrams will depend on the type of parity one chooses.

Here we just mention two parites.

The component parity. For 2-component links, mixed crossings are odd and pure crossings (belonging to one component) are even.

The homology parity. When considering knots K in a thickened surface $S_g \times I$, take a cohomology class h such that $h(K) = 0 \in H^1(S_g \times I, \mathbb{Z}_2)$. Then for every crossing c of K, we have two "halves" of the knot which can be thought of as cycles in $H_1(S_g, \mathbb{Z}_2)$. Taking the parity for c to be the evaluation of h on any of these halves, we get a well-defined parity. This evaluation does not depend on the choice of the half.

Parities and Functorial mappings: Projections and Coverings

Key observation: If we remove all odd chords from a chord diagram of a knot K then the remaining chords will form a knot $f(K)$ which is an invariant of K; in other words, the deletion of all odd chords is a well-defined functorial operation.

One can reiterate this operation for many parities. Doing this accurately until the underlying genus becomes zero (and taking the stabilization into account), we get the following

Theorem 3. [12] There is a well-defined mapping pr on Gauß diagram such that for every K, $pr(K)$ is obtained from K by deleting some chords, $pr(K)$ is a classical Gauß diagram, and if K_1 is equivalent to K_2 then $pr(K_2)$ is equivalent to $pr(K_1)$.

There are in fact, many maps pr satisfying the condition above. Such a theorem proves that many minimal characteristics of classical knots are attained on their classical diagrams. For more detail, see [12, 1].

New Parities.

In Figure 2 we show a transformation (mutation or similar) of a link component which changes a pattern $P \to P'$; this transofrmation does not change algebraic invariants based on some bare count (polynomials, homological intersection, etc), so most of the invariants do not change under operations which do not change such numeric characteristics (say, mutations).

The new parities are based on the count of the intersection with some *pattern* P.

Thus, if this pattern changes slightly as shown in Fig. 2, it may turn out that parities of many crossings change seriously: in the right part of the Fig.2, there is no pattern P, thus, all crossings are *even*, whence some crossings in the left part are odd.

So, this leads to brand new parities for one component of a 2-component link (this will appear in [15]).

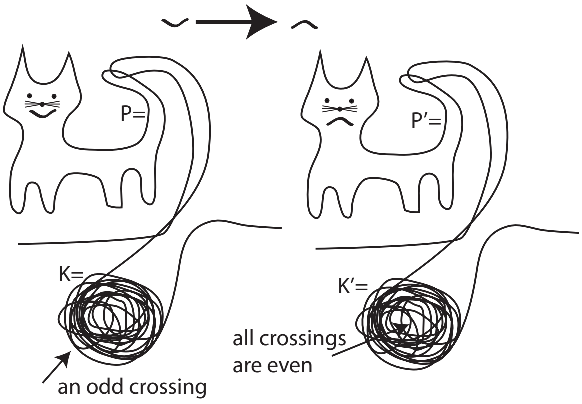


FIGURE 2. Parity coming from patterns

Parities and Cobordisms Another striking example is an invariant constructed from the Gaussian parity which provides an obstruction for free knots to be null-cobordant in some sense.

Let $G = \langle a, b, b' | a^2 = b^2 = b'^2 = e, ab = b'a \rangle$. Given a Gauss diagram D. We say that an odd chord is *of the first type* and *of the second type*, otherwise. With each even chord end we associate a, with each odd first type chord we associate b and with each odd second type chord we associate b'. Fix a point X on D and construct the word $\gamma_X(D)$ by walking along D.

One can check that the conjugacy class of $[\gamma_X(D)]$ in G is an invariant of the free knots.

The Cayley graph of G looks as follows: its vertices correspond to integer points on the plane Oxy with $y = 0$ or $y = 1$, vertical edges (connecting $y = 0$ to $y = 1$ with a fixed x) represent the element a, and horizontal edges (connecting two neighbouring vertices) represent b or b': an edge connecting (u, v) to $(u + 1, v)$ is marked by b whenever $u + v$ is even and by b' otherwise; the coordinate origin is the unit of the group. So, every element of G is a point with integer x and $y \in \{0, 1\} y = 1$. Those elements obtained from Gauss diagrams are represented by some points with $y = 0$, so, just by one integer number. The conjugation in G changes only the sign of this number, and the absolute value $l(D)$ is an invariant of free knots. Theorem gives a new series of counterexamples to Turaev's conjecture.

Carter,Turaev, Or et al. studied the following problem: given a generically immersed 2-curve γ in an oriented 2-surface S_g. Does there exist a 3-manifold $M, \partial M = S_g$ and a proper map $D \to M$ having only stable singularities such that $\partial D = D \cap S_g = g$? There are many topological obstructions coming from homology classes of S_g formed by loops of γ. We address a similar question with no surface and no manifolds, just with framed four-graphs.

Theorem 5.([11]) If for some framed 4-graph Γ we have $l(\Gamma) \neq 0$ then there is no way to span Γ by an image of the 2-disc (having Morse and Reidemeister singularities).

It turns out that the invariant l reduces to an integer number, which, in turn, is nothing but some count of odd crossings with sophisticated signs.

It would be interesting to find invariants of cobordisms and sliceness obstructions for free knots valued in pictures.

Parity can be extended to 2-knots

• Are there any parities coming from *patterns* which can be extended to 2-knots and lead to cobordism invariants?

Are there parities in classical knots?

There are no parities for diagrams of classical knots directly satisfying the parity axioms, but there are ways to use virtual knots for classical knot theory, see papers [2] and [9] and M.Chrisman's abstract in the present abstract book.

Acknowldegements: The author is partially supported by Laboratory of Quantum Topology of Chelyabinsk State University (Russian Federation government grant 14.Z50.31.0020) and by grants of the Russian foundation for Basic Resarch, 13-01-00830,14-01-91161, 14-01-31288.

References

[1] V.V.Chernov. Some corollaries of Manturov's projection theorem. *J. Knot Theory Rami cations* 22(12) (2013), no. 1, 1250139, 6 pp.

[2] M. W. Chrisman and V. O. Manturov. Fibered knots and virtual knots. *J. Knot Theory Ramifications*, 22(12) (2013) :1341003, 23.

[3] D. P. Ilyutko, *An equivalence between the set of graph-knots and the set of homotopy classes of looped graphs*, Knot Theory and Its Ramifications J. **21**:1 (2012), DOI: 10.1142/S0218216512500010.

[4] D. P. Ilyutko, V. O. Manturov, *Introduction to graph–link theory*, Knot Theory and Its Ramifications J. **18**:6 (2009), 791–823.

[5] D. P. Ilyutko, V. O. Manturov, *Graph–links*, Doklady Mathematics **80**:2 (2009), 739–742 (Original Russian Text in Doklady Akademii Nauk **428**:5 (2009), 591–594).

[6] D. P. Ilyutko, V. O. Manturov, *Graph-links*, in: Proceedings of the Advanced Summer School on Knot Theory, Trieste, Series of Knots and Everything, World Scientific (2011), 135–161.

[7] D. P. Ilyutko, V. S. Safina, *Graph-links: non-realizability, orientation and the Jones polynomial*, Topology, CMFD **51** (2013), 33–63.

[8] G. Kuperberg, *The quantum G_2 link invariant*, Internat. J. Math. **5** (1994), 61–85.

[9] V. A. Krasnov and V. O. Manturov. Graph-valued invariants of virtual and classical links and minimality problem. *J. Knot Theory Ramifications*, 22(12):1341006, 14, 2013.

[10] V. O. Manturov, *Parity in knot theory*, Math. Sb. **201**:5 (2010), 693–733 (Original Russian Text in Mathematical sbornik **201**:5 (2010), 65–110).

[11] *Manturov V. O.* Parity and Cobordisms of Free Knots// Mat. Sb., 203:2 (2012), 45–7 (Original Russian Text in Mathematical sbornik **203**:12 (2012), 45-76).

[12] *Manturov V.O.* Parity and projection from virtual knots to classical knots. J. Knot Theory Ramifications 22 (2013), no. 9, 1350044, 20 pp.

[13] *Manturov V.O.* Virtual crossing numbers for virtual knots. J. Knot Theory Ramifications 21 (2012), no. 13, 1240009, 13 pp.

[14] V. O. Manturov, *An almost classification of free knots*, Doklady Mathematics **88**:2 (2013), 556–558 (Original Russian Text in Doklady Akademii Nauk **452**:4 (2013), 371–374).

[15] V. O. Manturov, *New Parities and Coverings over Free Knots*, in preparation.

On Dijkgraaf-Witten invariants over $\mathbb{Z}_2$ of 3-manifolds

Sergei Matveev
(joint work with Vladimir Turaev)

Dijkgraaf and Witten [1] derived numerical topological invariants of closed manifolds from cohomology classes of finite groups. The DW-invariants have been extensively studied in the literature. A computation of the DW-invariant of a manifold requires a summation of several terms whose number depends exponentially on the first Betti number of the manifold. We show that if an integer $m \geq 1$ expands as a sum of two integral powers of 2, then the DW-invariant of m-dimensional manifolds associated with the non-trivial element of $H^m(\mathbb{Z}_2;\mathbb{Z}_2) = \mathbb{Z}_2$ can be computed from the Arf invariants of certain quadratic maps. This result applies, in particular, to m = 3.

Let M be a closed 3-manifold. We define the quadratic map $Q_M \colon H^1(M;\mathbb{Z}_2) \to \mathbb{Z}_2$ by the rule $Q_M(x) = \langle x^3, [M]\rangle$, where $x \in H^1(M;\mathbb{Z}_2)$, $x^3 \in H^3(M;\mathbb{Z}_2)$ and $[M]$ is the fundamental class of M.

Theorem. *Let M be a closed connected 3-manifold and $A \subset H^1(M;\mathbb{Z}_2)$ be the annihilator of the bilinear pairing ℓ_M corresponding to Q_M. If there is $x \in A$ such that $x^3 \neq 0$, then the DW-invariant $Z(M)$ of M is equal to 0. If for all $x \in A$ we have $x^3 = 0$, then $Z(M) = 2^{k+m-1}(-1)^{\mathrm{Arf}(Q_M)}$, where $m = \dim_{\mathbb{Z}_2}(A)$ and $k = \frac{1}{2}\dim_{\mathbb{Z}_2}(H^1(M;\mathbb{Z}_2)/A)$.*

The authors were partially supported by Laboratory of Quantum Topology of Chelyabinsk State University (Russian Federation government grant 14.Z50.31.0020).

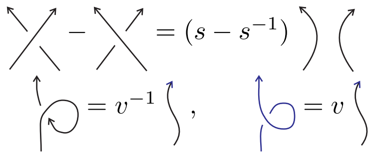
References

[DW] R. Dijkgraaf, E. Witten, *Topological gauge theories and group cohomology*, Comm. Math. Phys. **129** (1990) 393–429.

Global switching and smoothing in the Homflypt skein of T^2.

Hugh R. Morton
(joint work with Peter Samuelson)

The Homflypt skein $S(F)$ of an oriented (thickened) surface F consists of linear combinations of framed oriented diagrams on F up to Reidemeister moves R_{II}, R_{III} modulo the linear 'skein relations'



(Switch and smooth)

(Framing change)

using the ring $\Lambda = \mathbf{Z}[v^{\pm 1}, s^{\pm 1}]$ with denominators $s^r - s^{-r}, r > 0$ as coefficient ring. The framing of a diagram is understood to be the 'blackboard framing' from the surface F.

The skein $S(F)$ forms an algebra over Λ under the product induced by placing the diagram(s) representing an element D on top of the diagram(s) representing E to define DE.

In the case $S(F) = \mathcal{C}$ where F is the annulus this algebra is commutative and has been studied for some time. A recent account of some of its properties can be found in [4]. It has an interpretation as the algebra of symmetric functions in a large number of commuting variables $x_1, \dots, x_N$, and contains an element P_m for each m corresponding to the power sum $x_1^m + \cdots + x_N^m$. One representation of this element, due originally to Aiston [1], is a multiple of the sum of m closed m-braids.

In the case when $F = T^2$ the skein $H = S(T^2)$ is a non-commutative algebra, which can be generated by elements $P_{\mathbf{x}}$, one for each $\mathbf{x} \in \mathbf{Z}^2$, corresponding to free homotopy classes of curves in T^2.

For a primitive $\mathbf{x} = (m, n) \in \mathbf{Z}^2$ we represent $P_{\mathbf{x}}$ by the embedded (m, n) curve on T^2. It is an immediate consequence of the switch and smooth skein relation that the commutator $[P_{(1,0)}, P_{(0,1)}]$ satisfies

$$[P_{(1,0)}, P_{(0,1)}] = (s - s^{-1})P_{(1,1)}.$$

The same switching and smoothing relation shows that $[P_{\mathbf{x}}, P_{\mathbf{y}}] = (s - s^{-1})P_{\mathbf{x}+\mathbf{y}}$ when the primitive curves $\mathbf{x}$ and $\mathbf{y}$ cross once in the positive direction.

The main result presented in the talk is the following.

Theorem (Global switch and smooth).
The commutator $[P_{\mathbf{x}}, P_{\mathbf{y}}]$ in H satisfies

$$[P_{\mathbf{x}}, P_{\mathbf{y}}] = (s^k - s^{-k})P_{\mathbf{x}+\mathbf{y}}$$

where $k = \det(\mathbf{x}\ \mathbf{y})$ is the signed crossing number of $\mathbf{x}$ with $\mathbf{y}$.

Here the commutator can be regarded as a switch of curves $\mathbf{x}$ and $\mathbf{y}$, with $\mathbf{x}+\mathbf{y}$ as the simultaneous smoothing at the k crossings. This is an exact interpretation when $\mathbf{x}, \mathbf{y}$ and $\mathbf{x} + \mathbf{y}$ are all primitive.

In the statement of the theorem we must also specify $P_{m\mathbf{x}}$ for any multiple of a primitive $\mathbf{x}$. This is defined by decorating the embedded curve $\mathbf{x}$ by the element P_m from the skein $\mathcal{C}$ of the annulus.

The proof of the theorem relies on a result in [4] to establish that

$$[P_{(m,0)}, P_{(0,1)}] = (s^m - s^{-m})P_{(m,1)}.$$

Direct skein manipulation shows that

$$[P_{(0,-1)}, P_{(m,1)}] = (s^m - s^{-m})P_{(m,0)},$$

using Aiston's representation of P_m.

The full theorem follows from these two cases using induction on $\det(\mathbf{x}\ \mathbf{y})$.

Reasons for interest in the algebra H are its resemblance to a special case of the Hall algebra of an elliptic curve [2], and as an extension of the work of Frohman and Gelca [3] on the Kauffman bracket skein of T^2.

References

[1] A.K.Aiston, Skein theoretic idempotents of Hecke algebras and quantum group invariants. PhD thesis, University of Liverpool 1996.

[2] I. Burban and O. Schiffmann, On the Hall algebra of an elliptic curve, I, Duke Math. J. 161 (2012), 1171-1231.

[3] Charles Frohman and Răzvan Gelca, Skein modules and the noncommutative torus, Trans. Amer. Math. Soc. 352 (2000), no. 10, 4877–4888.

[4] H.R.Morton and P.M.G.Manchón, Geometrical relations and plethysms in the Homfly skein of the annulus. J. London Math. Soc. 78 (2008), 305-328.

Casson towers and slice links

Mark Powell

I gave results sharpening the minimal height of a Casson tower required to find a flat embedded disc inside a neighbourhood. In joint work with Jae Choon Cha, we showed that a Casson tower of height 4 contains an embedded flat disc. The previously best known result was for height 5 towers. The proof uses the disc embedding theorem of Freedman, and involves passing from Casson towers to gropes, and an improved understanding of the combinatorics of grope and tower height raising (see the book of Freedman and Quinn for an introduction). I also gave results on height 3 and 2 towers, which require some assumptions on the fundamental group, and I gave applications to showing that certain links are slice.

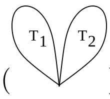
Progress in distributive homology: from q-polynomial of rooted trees to Yang-Baxter homology

Jozef H. Przytycki

We start with a long historical introduction beginning with Heinrich Kühn (1690-1769), C.L.G. Ehler (1685-1753), and Leonard Euler (1707-1783) and we argue that topology (geometria situs) started in Gdańsk (Danzig) about 1734. We mention the work of Celestyn Burstin (1888-1938) and Walter Mayer (1887–1948), (1929, distributivity) and Samuel Eilenberg (1913-1998) (homological algebra). We complete the historical summary by celebrating 30 years of the Jones polynomial (May 30, 1984, V.F.R.Jones wrote a letter to J.Birman announcing his construction of a new link polynomial). Thus it is appropriate to describe today a new simple invariant of rooted trees. Let T be a plane rooted tree then $Q(T) \in Z[q]$ is defined by the initial condition $Q(\bullet) = 1$ and the recursion relation

$$Q(T) = \sum_{v \in L(T)} q^{r(T,v)} Q(T - v), \text{ where } L(T) \text{ is the set of leaves of } T,$$

and $r(T,v)$ is the number of edges of T to the right of the path connecting v with the root v_0. For example $Q(\bigvee) = (1+q) = [2]_q$ or more generally $Q(T_n) = [n]_q!$, where T_n is a star with n rays and $[n]_q = 1 + q + ... + q^{n-1}$.

Theorem: Let $T_1 \vee T_2$ be the wedge (or root) product (

) . Then:

$$Q(T_1 \vee T_2) = \binom{E(T_1)+E(T_2)}{E(T_1)}_q Q(T_1)(Q(T_2)$$

Proof: We proceed by induction on $E(T)$, with obvious initial case of $E(T_1) = 0$ or $E(T_2) = 0$. Let T be a rooted plane tree with $E(T_1)E(T_2) > 0$, then we have:

$$Q(T) = \sum_{v \in L(T)} q^{r(T,v)} Q(T-v) =$$

$$\sum_{v \in L(T_1)} q^{r(T_1,v)+E(T_2)} Q((T_1 - v) \vee T_2) + \sum_{v \in L(T_2)} q^{r(T_2,v)} Q(T_1 \vee (T_2 - v)) \overset{inductive\ assumption}{=}$$

$$\sum_{v \in L(T_1)} q^{r(T_1,v)+E(T_2)} \binom{E(T_1)+E(T_2)-1}{E(T_1)-1, E(T_2)}_q Q(T_1 - v) Q(T_2) +$$

$$\sum_{v \in L(T_2)} q^{r(T_2,v)} \binom{E(T_1)+E(T_2)-1}{E(T_1), E(T_2)-1}_q Q(T_1) Q(T_2 - v) =$$

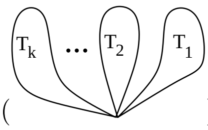
$$Q(T_2) q^{E(T_2)} \binom{E(T_1)+E(T_2)-1}{E(T_1)-1, E(T_2)}_q \sum_{v \in L(T_1)} q^{r(T_1,v)} Q(T_1 - v) +$$

$$Q(T_1) \binom{E(T_1)+E(T_2)-1}{E(T_1), E(T_2)-1}_q \sum_{v \in L(T_2)} q^{r(T_2,v)} Q(T_2 - v) =$$

$$Q(T_1)Q(T_2)(q^{E(T_2)} \binom{E(T_1)+E(T_2)-1}{E(T_1)-1, E(T_2)}_q + \binom{E(T_1)+E(T_2)-1}{E(T_1), E(T_2)-1}_q) =$$

$$Q(T_1)Q(T_2) \binom{E(T_1)+E(T_2)}{E(T_1), E(T_2)}_q \text{ as needed .}$$

Corollary:

(i) If a plane rooted tree is a wedge of k trees (

) and

$$T = T_k \vee ... \vee T_2 \vee T_1, \text{ then}$$

$$Q(T) = \binom{E_k + E_{k-1} + ... + E_1}{E_k, E_{k-1}, ..., E_1}_q Q(T_k) Q(T_{k-1}) \cdots Q(T_1).$$

where $E_i = |E(T_i)|$ is the number of edges in T_i.

(ii) (State product formula)

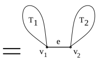
$$Q(T) = \prod_{v \in V(T)} W(v),$$

where $W(v)$ is a weight of a vertex (we can call it a Boltzmann weight) defined by:

$$W(v) = \binom{E(T^v)}{E(T^v_{k_v}), ..., E(T^v_1)}_q,$$

where T^v is a subtree of T with vertex v (part of T above v, in other words growing from v) and T^v can be decomposed into wedge of trees: $T^v = T^v_{k_v} \vee ... \vee T^v_2 \vee T^v_1$.

(iii) (change of a root). Let e be an edge of a tree T with endpoints v_1 and v_2 and E_1 be the number of edges on the v_1 part of the edge, and E_2 the number of edges of T on the v_2 side of e;

Thus $T =$  and then $Q(T, v_1) = \frac{[E_1+1]_q}{[E_2+1]_q} Q(T, v_2)$.

Proof. (i) Formula z (i) follows by using several times the formula

$$Q(T_2 \vee T_1) = \binom{E(T_2)+E(T_1)}{E(T_2), E(T_1)}_q Q(T_2)(Q(T_1),$$

as we have:

$$\binom{a_k + a_{k-1} + ... + a_2 + a_1}{a_k, a_{k-1}, ..., a_2, a_1}_q =$$

$$\binom{a_{k-1} + ... + a_2 + a_1}{a_{k-1}, ..., a_2, a_1}_q \binom{a_k + a_{k-1} + ... + a_2 + a_1}{a_k, a_{k-1} + ... + a_2 + a_1}_q = ...$$

$$= \binom{a_2+a_1}{a_2, a_1}_q \binom{a_3+a_2+a_1}{a_3, a_2+a_1}_q \binom{a_4+a_3+a_2+a_1}{a_4, a_3+a_2+a_1}_q ... \binom{a_k + a_{k-1} + ... + a_2 + a_1}{a_k, a_{k-1} + ... + a_2 + a_1}_q$$

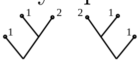
(ii) Formula (ii) follows by using (i) several times. □

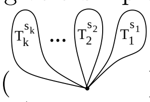
One can propose many modifications and generalizations of the polynomial $Q(T)$, for example, for a graph with a base point we can take the set (or the sum) over all spanning trees of $Q(T)$ but we propose below the one having close relation with knot theory.

Let T be a plane rooted tree and $f : L(T) \to N$ a function from leaves of T to positive integers. We call f a delay function as our intuition is that a leaf with value k cannot be used before kth move. Formally $Q(T, f)$ is defined by recursive relation:

$$Q(T, f) = \sum_{v \in L_1(T)} q^{r(T,v)} Q(T - v, f_v),$$

where $L_1(T)$ is a set of leaves for which f is equal to 1. $f_v(u) = max(1, f(u) - 1)$ if u is also a leaf of T, and it is equal to 1 if it is a new leaf of $T - v$.

Example. For a rooted tree with delay function the polynomial $Q(T)$ is not necessary a product of cyclotomic polynomials, the simplest example is given by trees 

with polynomials equal respectively $q(1+q+2q^2+q^3)$ and $1+2q+q^2+q^3$. There is however one special situation when we can give a simple closed formula: Consider the "delayed" tree $T = T_k^{s_k} \vee ... \vee T_2^{s_2} \vee T_1^{s_1}$ (

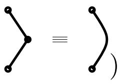
). That is we assume that whole blocks have constant delay function (the block T_i have leaves labelled s_i). We assume also, for convenience, that $s_1 = 1$, $s_1 \leq s_2 \leq E_1+1$, $s_2 \leq s_3 \leq E_2+E_1+1$,..., $s_{k-1} \leq s_k \leq E_{k-1}+...+E_2+E_1+1$ (here $E_i = |E(T_i)|$). Then

$$Q(T) = \binom{E_2+E_1-s_2+1}{E_2, E_1-s_2+1}_q \binom{E_3+E_2+E_1-s_3+1}{E_3, E_2+E_1-s_3+1}_q \cdots \binom{E_k+...+E_1-s_k+1}{E_k, E_{k-1}+...+E_1-s_k+1}_q$$

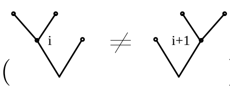
$$Q(T_1)Q(T_2)...Q(T_k).$$

We didn't reach yet relations neither with knot theory nor with distributive structures; these should be left for the next occasion, however we finish the talk with one curious question and related observation. Consider a chain complex over a commutative ring k

$$\mathcal{C} : ... \to C_{n+1} \overset{\partial_{n+1}}{\to} C_n \overset{\partial_n}{\to} C_{n-1} \overset{\partial_{n-1}}{\to} ... \to C_1 \overset{\partial_1}{\to} C_0$$

and assume that $\mathcal{C}$ comes from a presimplicial module $\partial_n = \sum_{i=0}^n (-1)^i d_i$ where $0 \leq i \leq n$, $d_i d_j = d_{j-1} d_i$ for $i < j$. We ask whether it is useful (already used?) to consider q-version: $C_n^q = C_n \otimes_k Z[q]$ and the q-map $\partial_n^q = \sum_{i=0}^n q^i d_i$. Clearly (C_n^q, ∂_n^q) is not generically a chain complex but we can make another use of it. For example, we can identify x with $\partial^q(x)$, that is to consider $(\bigoplus_{n\geq 0} C_n^q)/(x - \partial^q(x))$. Here an example which I learned from JP. Loday is very handy:
Consider presimplicial set (Y_n, d_i) where Y_n is the set of topological rooted trees with n ordered leaves (topological means that 

). We define $d_i(T) = T - v_i$, where v_i is the ith leaf of T. We can also introduce degeneracy maps $s_i : Y_i \to Y_{i+1}$ planting $\bigvee$ on the ith leaf. We check directly that:
(1) $d_i d_j = d_{j-1} d_i$ for $i < j$,
(2') $s_i s_j = s_{j+1} s_i$ for $i < j$,
(3) $d_i s_j = \begin{cases} s_{j-1} d_i & \text{if } i < j \\ s_j d_{i-1} & \text{if } i > j+1 \end{cases}$
((4) $d_i s_i = d_{i+1} s_i = Id$

The condition $s_i s_i = s_{i+1} s_i$ does not hold (

) so (Y_n, d_i, s_i) is not a simplicial set but only an almost simplicial set.

Now consider the quotient of the sum $(\bigoplus_{n\geq 0} Z[q] Y_n)/(x - \partial^q(x))$. It is a free $Z[q]$ module generated by $\bullet$ (tree without edges).We compute inductively that for a tree with n leaves $T = [n]_q!\bullet$. It is not very sophisticated invariant so we can be glad that polynomial $Q(T)$ is more interesting.

Distributivity leads to another "incomplete" simplicial set, this time condition (4) does not hold, but this should be put aside for the next report which will discuss also a generalization of distributive homology: – Yang-Baxter homology.

References

[1] A. Crans, J. H. Przytycki, K. Putyra, *Torsion in one term distributive homology*, *Fundamenta Mathematicae*, **225**, May, 2014, 75-94. e-print: `arXiv:1306.1506 [math.GT]`

[2] M. K. Dabkowski, C. Li, J. H. Przytycki, Catalan states of lattice crossing, preprint, February 2014. (The second part, in which polynomial $Q(T)$ is used, is in preparation.)

[3] J. H. Przytycki, *Distributivity versus associativity in the homology theory of algebraic structures*, *Demonstratio Math.*, **44**(4), December 2011, 823-869; e-print: `http://front.math.ucdavis.edu/1109.4850`

[4] J. H. Przytycki, *Knot Theory and related with knots distributive structures; Thirteen Gdansk Lectures*, Gdansk University Press, in Polish, June, 2012, pp. 115 (second, extended, edition, in preparation).

[5] J. H. Przytycki, K. Putyra, *Homology of distributive lattices*, the *Journal of homotopy and related structures*, Volume **8**(1), 2013, pages 35-65; e-print: `arXiv:1111.4772 [math.GT]`

[6] J. H. Przytycki,, W. Rosicki, *Cocycle invariants of codimension 2 embeddings of manifolds*, *Banach Center Publications*; Recommended for publication, January, 2014; to appear December 2014. e-print: `arXiv:1310.3030 [math.GT]`

[7] J. H. Przytycki, A. S. Sikora, *Distributive Products and Their Homology*, *Communications in Algebra*, **42**(3), 2014, 1258-1269; e-print: `arXiv:1105.3700 [math.GT]`

Diophantine quadratics and 2–spheres in 4–manifolds

Rob Schneiderman

An obstruction theory for representing homotopy classes of surfaces in 4–manifolds by immersions with pairwise disjoint images is developed, using a theory of *non-repeating* Whitney towers. The accompanying higher-order intersection invariants provide a geometric generalization of Milnor's link-homotopy invariants, and can give the complete obstruction to pulling apart 2–spheres in certain families of 4–manifolds. It is also shown that in an arbitrary simply connected 4–manifold any number of parallel copies of an immersed 2–sphere with vanishing self-intersection number can be pulled apart, and that this is not always possible in the non-simply connected setting. The order 1 intersection invariant is shown to be the complete obstruction to pulling apart 2–spheres in any 4–manifold after taking connected sums with finitely many copies of $S^2 \times S^2$; and the order 2 intersection indeterminacies for quadruples of immersed 2–spheres in a simply-connected 4–manifold are shown to lead to interesting number theoretic questions about systems of Diophantine quadratic equations coupled by the intersection form. This is joint work with Peter Teichner [1].

References

[1] R. Schneiderman, P. Teichner, *Pulling apart 2–spheres in 4–manifolds*, (Preprint) http://arxiv.org/abs/1210.5534.

Knot and orbifold groups which are extreme

Andrei Vesnin

For $f, g \in \mathrm{PSL}(2,\mathbb{C})$ denote $\mathcal{J}(f,g) = |\mathrm{tr}^2(f) - 4| + |\mathrm{tr}[f,g] - 2|$. Let $G < \mathrm{PSL}(2,\mathbb{C})$ be a 2-generated non-elementary group. The value

$$\mathcal{J}(G) = \inf_{\langle f,g \rangle = G} \mathcal{J}(f,g),$$

is said to be a *Jørgensen number* of G. Jørgensen numbers originally arise in the following discreteness condition [1]: if non-elementary group G is discrete then $\mathcal{J}(G) \geq 1$. It was shown in [2] that the figure-eight knot complement is the unique hyperbolic 3-manifold whose fundamental group has Jørgensen number equals to one. Jørgensen numbers for some 2-bridge knot groups were calculated in [2].

Let us denote by K the figure-eight knot and by $K(n)$ the hyperbolic 3-orbifold with singular set K and singular angle $2\pi/n$, $n \geq 4$. The knot group has the presentation

$$\pi_1(S^3 \setminus K) = \langle\, f, g \mid [g,f]\, g^{-1} = f\, [g,f] \,\rangle$$

and the orbifold group has the following presentation:

$$\pi^{\mathrm{orb}}(K(n)) = \langle\, f_n, g_n \mid f_n^n = g_n^n = 1,\ [g_n, f_n]\, g_n^{-1} = f_n\, [g_n, f_n] \,\rangle.$$

Both of them have faithful representations in $\mathrm{PSL}(2,\mathbb{C})$.

We will describe behavior of Jørgensen numbers of figure-eight knot orbifold groups.

Theorem 1. [3] *Let $n \geq 4$. Then the following inequalities hold:*

$$1 \leq \mathcal{J}(\pi^{\mathrm{orb}}(K(n))) \leq 4\sin^2(\pi/n) + \sqrt{1 + 4\sin^2(\pi/n)}.$$

Corollary 1. [3] *The following convergence holds:*

$$\lim_{n\to\infty} \mathcal{J}(\pi^{\mathrm{orb}}(K(n))) = \mathcal{J}(\pi_1(S^3 \setminus K)).$$

An analog of a Jørgensen number was introduced in [4] and [5]. For $f, g \in \mathrm{PSL}(2,\mathbb{C})$ such that $\mathrm{tr}[f,g] \neq 1$ denote $\mathcal{G}(f,g) = |\mathrm{tr}^2(f) - 2| + |\mathrm{tr}[f,g] - 1|$. Let $G < \mathrm{PSL}(2,\mathbb{C})$ be a 2-generated group. The value

$$\mathcal{G}(G) = \inf_{\langle f,g \rangle = G} \mathcal{G}(f,g),$$

is said to be a *GMT number* of G. Gehring and Martin [4] and independently Tan [5] proved that if G is discrete then $\mathcal{G}(G) \geq 1$.

The following results demonstrate behavior of GMT numbers of figure-eight knot orbifold groups.

Theorem 2. [3] *The following equality holds: $\mathcal{G}(\pi_1(S^3 \setminus K)) = 3$.*

Theorem 3. [3] *Let $n \geq 4$. Then the following inequalities hold:*

$$1 \leq \mathcal{G}(\pi^{\mathrm{orb}}(K(n))) \leq 3 - 4\sin^2(\pi/n).$$

Corollary 2. [3] *The following equality holds:* $\mathcal{G}(\pi^{\mathrm{orb}}(K(4))) = 1$.

References

[1] T. Jørgensen, *On discrete groups of Mobius transformations*, Amer. J. Math. **93** (1976), 739–749.

[2] J. Callahan, *Jørgensen number and arithmeticity*, Conform. Geom. Dyn. **13** (2009), 160–186.

[3] A. Vesnin, A. Masley, *On Jørgensen numbers and their analogues for groups of figure-eight knot orbifolds*, Siberian Math. J. **55(5)** (2014) (in press).

[4] F. W. Gehring, G. J. Martin, *Stability and extremality in Jørgensen's inequality*, Complex Variables Theory Appl. **12(1-4)** (1989), 277–282.

[5] D. Tan, *On two-generator discrete groups of Mobius transformations*, Proc. AMS **106(3)** (1989), 763–770.

Reporter: Louis Hirsch Kauffman and Vassily Olegovich Manturov

Participants

Prof. Dr. Dror Bar-Natan
Department of Mathematics
University of Toronto
40 St George Street
Toronto, Ont. M5S 2E4
CANADA

Prof. Dr. Stefan A. Bauer
Fakultät für Mathematik
Universität Bielefeld
Postfach 100131
33501 Bielefeld
GERMANY

Prof. Dr. John M. Bryden
Mathematics & Natural Sciences
Prince Mohammad Bin Fahd University
(PMU)
P.O. Box 1664
Al-Khobar 31952
SAUDI ARABIA

Prof. Dr. Jae Choon Cha
Department of Mathematics
POSTECH
77 Cheongam-Ro, Nam-Gu
Pohang Gyungbuk 790-784
KOREA, REPUBLIC OF

Prof. Dr. Micah Chrisman
Department of Mathematics
Monmouth University
West Long Branch, NJ 07764
UNITED STATES

Prof. Dr. Tim D. Cochran
Department of Mathematics
Rice University
P.O. Box 1892
Houston, TX 77005-1892
UNITED STATES

Prof. Dr. Michael Eisermann
Fachbereich Mathematik
Universität Stuttgart
Pfaffenwaldring 57
70569 Stuttgart
GERMANY

Denis A. Fedoseev
Department of Mathematics
M.V.Lomonosov Moscow State
University
Leninskie Gory
Moscow 119 992
RUSSIAN FEDERATION

Prof. Dr. Roger A. Fenn
School of Mathematical Sciences
University of Sussex
Falmer
Brighton BN1 9QH
UNITED KINGDOM

Prof. Dr. Thomas Fiedler
Institut de Mathématiques de Toulouse
Université Paul Sabatier
118, route de Narbonne
31062 Toulouse Cedex 9
FRANCE

Prof. Dr. Cameron M. Gordon
Mathematics Department
The University of Texas at Austin
1 University Station C 1200
Austin TX 78712-0257
UNITED STATES

Prof. Dr. Shelly Harvey
Department of Mathematics
Rice University
P.O. Box 1892
Houston, TX 77005-1892
UNITED STATES

Prof. Dr. Denis P. Ilyutko
Mechanical & Mathematical Faculty
Moscow State University
Department of Mathematics
Moscow 119 991
RUSSIAN FEDERATION

Prof. Dr. Aaron M. Kaestner
North Park University
Dept. of Mathematics and Statistics
3225 W Foster Ave.
Chicago, IL 60625
UNITED STATES

Prof. Dr. Louis H. Kauffman
Dept. of Mathematics, Statistics
and Computer Science, M/C 249
University of Illinois at Chicago
851 South Morgan St.
Chicago, IL 60607-7045
UNITED STATES

Seongjeong Kim
Department of Mathematics
Kyungpook National University
Daegu 702-701
KOREA, REPUBLIC OF

Prof. Dr. Sofia Lambropoulou
Department of Mathematics
Nat. Technical University of Athens
Zografou Campus
15780 Athens
GREECE

Prof. Dr. Vassily O. Manturov
Bauman Moscow State Technical
University
Department of Mathematics
ul. Baumanskaya 2-ya, 5/1
Moscow 105 005
RUSSIAN FEDERATION

Prof. Dr. Sergey V. Matveev
Department of Mathematics
Chelyabinsk State University
Kashirin Brothers St. 129
Chelyabinsk 454 001
RUSSIAN FEDERATION

Prof. Dr. Hugh R. Morton
Dept. of Mathematical Sciences
University of Liverpool
Peach Street
Liverpool L69 7ZL
UNITED KINGDOM

Prof. Dr. Kent Edward Orr
Department of Mathematics
Indiana University at Bloomington
Swain Hall East
Bloomington, IN 47405
UNITED STATES

Dr. Mark Powell
Department of Mathematics
Indiana University at Bloomington
Swain Hall East
Bloomington, IN 47405
UNITED STATES

Dr. Jozef H. Przytycki
Department of Mathematics
George Washington University
Monroe Hall, 2115 G St. NW
Washington, DC 20052
UNITED STATES

Prof. Dr. Robert Schneiderman
Dept. of Mathematics & Computer
Sciences
Lehman College
The City University of New York
250 Bedford Park Blvd.
Bronx, NY 10468-1589
UNITED STATES

Prof. Dr. Andrei Yu. Vesnin
Leading Researcher
Laboratory of Quantum Topology
Chelyabinsk State University
129 Br. Kashirinykh str.
Chelyabinsk 454 001
RUSSIAN FEDERATION