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# Okounkov Bodies and Applications

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Abstract. The theory of Newton-Okounkov bodies, also called Okounkov bodies, is a relatively new connection between algebraic geometry and convex geometry. It generalizes the well-known and extremely rich correspondence between geometry of toric varieties and combinatorics of convex integral polytopes. Following a successful MFO Mini-workshop on this topic in August 2011, the MFO Half-Workshop 1422b, "Okounkov bodies and applications", held in May 2014, explored the development of this area in recent years, with particular attention to applications and relationships to other areas such as number theory and tropical geometry.

*Mathematics Subject Classification (2010):* 14C20, 14M25, 51M20, 14N25 , 14T05.

## Introduction by the Organisers

Okounkov bodies were first introduced by Andrei Okounkov, in a construction motivated by a question of Khovanskii concerning convex bodies govering the multiplicities of representations. Recently, Kaveh-Khovanskii and Lazarsfeld-Mustata have generalized and systematically developed Okounkov's construction, showing the existence of convex bodies which capture much of the asymptotic information about the geometry of $(X, D)$ where $X$ is an algebraic variety and $D$ is a big divisor. This theory of Newton-Okounkov bodies can be viewed as a vast generalization of the well-known theory of toric varieties. The study of Okounkov bodies is a new research area with many open questions, and the purpose of the Half-Workshop 1422b *Okounkov bodies and applications*, organised by Megumi Harada (McMaster), Kiumars Kaveh (Pittsburgh), and Askold Khovanskii (Toronto), was to explore the many recent (and potential new) applications of this theory to other research areas.

The Half-Workshop was well attended with over 20 participants, with broad geographic representation from all continents. The group of participants was a nice blend of researchers with various backgrounds such as tropical geometry, representation theory, toric topology, symplectic topology, integrable systems, and number theory. In addition to the senior participants, there were 2 participants supported through the Oberwolfach Leibniz Graduate Students" program. The workshop consisted of 18 research talks in total.

In the remaining part of this introduction we briefly describe some of the topics discussed at the workshop.

One of the major themes of the workshop was to define functions to and from Newton-Okounkov bodies. Functions *from* Newton-Okounkov bodies were discussed by Alex Kuronya, with a view towards applications in the study of big divisors and positivity of linear series on algebraic varieties. Functions *to* Newton-Okounkov bodies were discussed by David Witt-Nystrom in his talk on joint work with Julius Ross, in which they define a kind of analogue of a 'moment map' to a Newton-Okounkov body. (Witt-Nystrom also gave another talk on transforming metrics of a line bundle which provided some background on his work on moment maps.)

Several of the talks reported on recent progress in the theory of Newton-Okounkov bodies. Victor Lozovanu reported on recent joint work with Kuronya on positivity of linear series on surfaces, and Kiumars Kaveh presented joint work with Khovanskii on the theory of *local* Newton-Okounkov bodies. One of the junior participants Takuya Murata, invited through the Oberwolfach Leibniz Graduate Students program, was given the opportunity to present his Ph.D. thesis results (supervised by one of the organizers, Kiumars Kaveh) on the asymptotic behavior of multiplicities of reductive group actions.

Another major purpose of the workshop was to explore possible connections with other research areas. In this spirit, Huayi Chen gave a talk outlining possible avenues of applications of Newton-Okounkov bodies to arithmetic. Similarly, Sam Payne gave a talk on tropical methods for the study of linear series and Buchstaber gave a presentation on $(2n, k)$-manifolds; in both talks, many themes overlapped with those arising in the study of Newton-Okounkov bodies. Furthermore, Boris Kazarnovskii talked about an extension of the theory of Newton-Okounkov bodies to the non-algebraic setting of *exponential* sums, and about a very surprising relation of this non-algebraic subject to modern algebraic geometry.

Symplectic geometry, symplectic topology, integrable systems, and toric degenerations also played a main role in the workshop. In this direction, both Chris Manon and Johan Martens reported on their recent work on the Vinberg monoid, while Yuichi Nohara gave a talk on toric degenerations of integrable systems on the Grassmannian and an application to the computation of the potential functions arising in symplectic topology. Continuing the theme of symplectic topology, Kaoru Ono gave a talk on Lagrangian tori in $S^2 \times S^2$.

The relation between Newton-Okounkov bodies and Schubert calculus was also a strong theme of the workshop. Valentina Kiritchenko spoke about a 'geometric mitosis' operation on Newton-Okounkov polytopes (associated to flag varieties) which give rise to collections of faces of the polytope representing a Schubert cycle. Vladlen Timorin gave a talk on counting vertices of Gel'fand-Cetlin polytopes, which are a special case of Newton-Okounkov bodies of flag varieties. June Huh presented his results on positivity of Chern classes of Schubert cells and varieties, concluding with open questions in this area which relate to Newton-Okounkov bodies. Finally, in the last talk of the workshop, Dave Anderson spoke on computing the effective cone of Bott-Samelson varieties, which arise naturally in the study of Newton-Okounkov bodies due to their central role in representation theory and the geometry of flag varieties.

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**Workshop: Okounkov Bodies and Applications**

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# Abstracts

## Functions on Newton–Okounkov bodies

Alex Küronya

This is an account of joint work with many people, coauthors will be pointed out in particular cases. Newton–Okounkov bodies serve to capture the behaviour of we all global sections of all multiples of a given big Cartier divisor at the same time. Building on earlier work of Okounkov and many others, in their current form Newton–Okounkov bodies were first studied by Kaveh–Khovanskii [9] and Lazarsfeld–Mustaţă [11]. For explicit examples see [11] and [10] for instance.

Let $X$ be a smooth projective variety of dimension $n$ over the complex number field, $Y_\bullet$ an admissible flag, $D$ a big line bundle on $X$. The choice of the flag $Y_\bullet$ gives rise to a rank $n$ valuation $\nu_{Y_\bullet}$ on the function field $\mathbb{C}(X)$ of $X$, which, evaluated on the global sections of multiples of $D$, yields a convex body $\Delta_{Y_\bullet}(D) \subseteq \mathbb{R}^n$, the Newton–Okounkov body of $D$.

Our purpose here is to go one step further, and study functions on Okounkov bodies coming from some geometric situation. The general yoga comes from complex analysis, and was first formalized by Boucksom and Chen [3]: the fundamental principle is that multiplicative filtrations of the section ring $R(X, D)$ give rise to concave functions on all corresponding Newton–Okounkov bodies $\Delta_{Y_\bullet}(D)$.

For now, we mention to important sources of such filtrations: test configurations (see [14] for further details), and divisorial valuations $v$ of the function field $X$; we will study functions arising from the second class of filtrations. In fact, the actual valuations we intend to study occur naturally in projective geometry, we will consider orders of vanishing along smooth subvarieties $Z$ of $X$. The (already quite interesting) example to keep in mind is the case when $X$ is a smooth projective surface, and $Z$ is simply a point $x \in X$. We often take $x$ to be the point from a given admissible flag $Y_\bullet$.

With some abuse of terminology, we can say that the function $\phi_v\colon \Delta_{Y_\bullet}(D) \to \mathbb{R}_{\geq 0}$ is the concave transform of the filtration $R(X, D)$ associated to the valuation. In the concrete case of $v = \mathrm{ord}_Z$, the filtration parametrized by a real number $t$ is given by

$$F_Z^t H^0(X, \mathcal{O}_X(mD)) = \{s \in H^0(X, \mathcal{O}_X(mD)) \mid \mathrm{ord}_Z s \geq t\} \ .$$

In the case when a point $\alpha \in \Delta_{Y_\bullet}(D)$ arises as an actual normalized valuation vector of a global section, we set

$$\phi_Z(\alpha) = \lim_{m\to\infty} \frac{1}{m} \sup\{t \geq 0 \mid \exists s \in F_Z^{mt} H^0(X, \mathcal{O}_X(mD)) \ , \nu_{Y_\bullet}(s) = \alpha\} \ ,$$

and then form the concave envelope [3, 4, 13].

In the toric case, it follows from work of Donaldson [6] and Wytt-Nyström [14] that the functions arising from a setting where every object is torus-invariant will be piecewise affine linear with rational coefficients with respect to a finite decomposition of the underlying moment polytope.

We have the following results on the formal behaviour of these functions.

**Theorem A.** [4] With notation as above, the following assertions hold.

(1) If $\dim X = 2$, then the functions $\phi_Z$ are always continuous.
(2) There exist examples of $X$, $L$, $Y_\bullet$, $Z$ in dimensions at least three, where $\phi_Z$ is not continuous along the boundary of $\Delta_{Y_\bullet}(L)$.
(3) The function $\phi_Z$ is homogeneous of degree one.
(4) If $L$ and $L'$ are numerically equivalent divisors, then $\phi_Z\colon \Delta_{Y_\bullet}(L) \to \mathbb{R}$ equals $\phi_Z\colon \Delta_{Y_\bullet}(L') \to \mathbb{R}$.

The first statement draws on the fact that Newton–Okounkov bodies on surfaces are finite polygons [10], which is a consequence of results of [2, 11]. Analogously, if $L$ has a finitely generated section ring, then one can pick a suitable flag $Y_\bullet$ such that all Okounkov functions on $\Delta_{Y_\bullet}(L)$ are continuous [1].

Although the Newton–Okounkov body $\Delta_{Y_\bullet}(L)$ varies wildly with $Y_\bullet$ its Euclidean volume remains constant under change of flags. Analogously it is important to find quantities for the functions $\phi_Z$ that are invariant with respect to the choice of the flag $Y_\bullet$. In this direction one can say the following.

**Theorem B.** [3, 5] With notation as above, the numbers

$$I(L;Z) \overset{\text{def}}{=} \frac{1}{\mathrm{vol}_X(L)} \int_{\Delta_{Y_\bullet}(L)} \phi_Z$$

and

$$\max_{\alpha\in\Delta_{Y_\bullet}(L)} \phi_Z(\alpha)$$

are independent of $Y_\bullet$.

The first one of the above observations was made by Boucksom and Chen. It should be pointed out that except in the toric case, the functions $\phi_Z$ appear to be difficult to determine. In the light of this, the next claim is slightly surprising.

**Theorem C.** [5] Let $\pi\colon Y \to X$ be the blow-up of $X$ along the smooth subvariety $Z$ with exceptional divisor $E$. Then

$$I(L;Z) \overset{\text{def}}{=} \frac{1}{\mathrm{vol}_X(L)} \int_{\Delta_{Y_\bullet}(L)} \phi_Z \;=\; \int_0^\infty t\cdot \mathrm{vol}_{Y|E}(\pi^*L - tE)dt\ .$$

Beside its significance in projective geometry, the invariant $I(L;Z)$ plays a role in diophantine approximation, as shown in the work of McKinnon–Roth [12]

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## Okounkov bodies and moment maps

David Witt Nystrom
(joint work with Julius Ross)

We like to think of the Okounkov body as generalizing the moment polytope associated to a polarized toric manifold. Given a torus invariant positive hermitian metric on the toric line bundle we get a moment map, which maps the toric manifold onto the moment polytope. The question is if there exists in general a natural "moment map" from a polarized manifold $(X, L)$ to the Okounkov body $Delta(L)$ given a positive hermitian metric on $L$ (of course this will also have to depend on the admissible flag used to define the Okounkov body).

One approach to this problem was taken by Harada-Kaveh in [2]. Namely when the semigroup $Gamma(L)$ is finitely generated, Anderson showed in [1] that one has a natural degeneration of the polarized manifold $(X, L)$ to the polarized toric variety with associated moment polytope $Delta(L)$. After equipping the deformation space with an appropriate Kahler structure Harada-Kaveh use the gradient-Hamiltonian flow of Ruan (see [5]) to transport the moment map from the central fiber to $X$.

In the talk I described a different approach to the problem due to Julius Ross and myself.

The idea is to first consider the first coordinate of the moment map, and then proceed inductively. The first coordinate of the moment map should only depend on the metric and the codimension one part of the flag $Y_1 =: Y$ (which we assume to be smooth). We consider the natural deformation of $X$ to the normal bundle of $Y$ one gets by blowing up $Y \times \{0\}$ inside $X \times \mathbb{D}$ ($\mathbb{D}$ denotes the unit disc). The deformation space has a canonical weak Kahler structure (i.e. a closed positive $(1,1)$-current $\Omega$) which is invariant under the natural circle action (in particular the restriction to the normal bundle is $S^1$-invariant). One gets this weak Kahler structure as the unique solution to the homogeneous complex Monge-Ampere equation

$$\Omega^{n+1} = 0$$

where the boundary data is given by the Kahler curvature form of the hermitian metric of $L$ (see [3] for details).

Since the weak Kahler structure is degenerate one cannot use Ruan's gradient-Hamiltonian flow, but when the structure is regular (i.e. smooth and the degeneracy is transversal to the fibers) then one can flow along the degenerate directions and it is well-known that this flow is symplectic. In [3] we prove that the canonical weak Kahler structure in fact is regular near the proper transform of $Y \times \mathbb{D}$. This then allows us to transport the symplectic circle action from the normal bundle to a tubular neighbourhood of $Y$. The Hamiltonian of this action is the first coordinate of our moment map.

Using the flow we then identify the associated symplectic quotients with $Y$ equipped with certain Kahler structures (varying with the value of the Hamiltonian). On the other hand, the fiber of the Okounkov body over $x_1 = \lambda$ is naturally identified with a subset of the Okounkov body of $L - \lambda Y$ restricted to Y, which allows us to proceed inductively. Since the regularity only is local we only get a full moment map near the central point of the flag. To extend the map further one would need to extend the symplectic flow to the parts of the deformation space where the weak Kahler structure is not regular.

In the case when $X$ is a curve we do have a full moment map, and the evolution of the sublevel sets of the Hamiltonian turns out to have a fluid flow interpretation as a Hele-Shaw flow (see [3, 4]).

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## $(2n,k)$-manifolds and applications

Victor M. Buchstaber
(joint work with Svjetlana Terzić)

### 1. Introduction.

We consider the class of smooth, closed manifolds $M^{2n}$ with an action of a compact torus $T^k$ which admit an almost moment map whose image is a conve polytope $P^k$ in $\mathbb{R}^k$.

The new in our approach is that for the purpose of description of the combinatorics of the torus action we introduce so-called admissible polytopes which are convex polytopes spanned by some subsets of vertices of the polytope $P^k$.

"Our bodies" are $CW$-complex whose open cells are one parts of admissible polytopes. In terms of the height function for one-dimensional skeleton of this complex we obtain combinatorial description of the Betti numbers of $M^{2n}$ as well as the equivariant cohomology of a manifold $M^{2n}$.

The examples of $(2n,k)$-manifolds are projective toric manifolds, quasitoric manifolds and compact homogeneous spaces $G/H$ of positive Euler characteristic. [1]

### 2. Theory of $(2n,k)$-manifolds.

We assume the following to be given:

- a smooth, closed simply connected manifold $M^{2n}$;
- a smooth, effective action $\theta$ of the torus $T^k$ on $M^{2n}$, where $2 \leq k \leq n$, such that the stabilizer of any point is connected;
- an open $\theta$-equivariant map $\mu : M^{2n} \to \mathbb{R}^k$ whose image is a $k$-dimensional convex polytope, where $\mathbb{R}^k$ is considered with trivial $T^k$ action.

The polytope which is obtained as an image of $\mu$ we denote by $P^k$. The map $\mu$ we call an *almost moment map* for the given $T^k$-action on $M^{2n}$. We say that the triple $(M^{2n}, \theta, \mu)$ is $(2n,k)$-manifold if it satisfies the following axioms.

**Axiom 1.** *There is a smooth atlas $\mathfrak{M} = \{(M_i, \varphi_i)\}_{i\in I}$ with the homeomorphisms $\varphi_i : M_i \to \mathbb{R}^{2n} \approx \mathbb{C}^n$ for the fixed identification $\approx$, such that any chart $M_i$ is $T^k$-invariant, contains exactly one fixed point $x_i$ with $\varphi_i(x_i) = (0,\dots,0)$, and the closure of $M_i$ is $M^{2n}$.*

Denote by $m$ the number of fixed points for $T^k$-action on $M^{2n}$. The charts given by Axiom 1 we enumerate as $(M_1,\varphi_1),\dots,(M_m,\varphi_m)$. The sets $Y_i = M^{2n} - M_i$ are closed and $T^k$-invariant. Since $M_i$ is everywhere dense in $M^{2n}$ we have that $Y_i = \partial M_i$. Define the sets $W_\sigma$, where $\sigma = \{i_1,\dots,i_l\} \subseteq \{1,\dots,m\}$ by :

$$W_\sigma = M_{i_1} \cap \cdots M_{i_l} \cap Y_{i_{l+1}} \cap \cdots Y_{i_m},$$

where $\{i_{l+1},\dots,i_m\} = \{1,\dots,m\} - \{i_1,\dots,i_l\}$.

[1] V. M. Buchstaber is grateful to the Russian Scientific Foundation for the support, Grant 14-11-00414

**Definition 1.** *The non-empty set $W_\sigma$ is called admissible and the corresponding set $\sigma$ is called admissible too..*

**Axiom 2.** *The characteristic function $\chi$ is constant on $W_\sigma$ for any admissible set $W_\sigma$ and if $W_{\sigma'} \subset \overline{W_\sigma}$ then $\chi(W_\sigma) \subset \chi(W_{\sigma'})$.*

We call $W_\sigma$ a stratum if Axiom 2 is satisfied.

**Axiom 3.** *The map $\mu$ gives the bijection between the set of fixed points and the set of vertices of the polytope $P^k$.*

Let $S(P^k)$ be the family of convex polytopes which are spanned by the vertices of the polytope $P^k$ and $\{W_\sigma\}$ the family of all admissible sets. Define the map $s : \{W_\sigma\} \to S(P^k)$ by

$$s(W_\sigma) = P_\sigma, \text{ where } \sigma = \{i_1, \dots, i_l\} \text{ and } P_\sigma = convhull(v_{i_1}, \dots, v_{i_l}),$$

and $v_{i_1}, \dots, v_{i_l}$ are the vertices of the polytope $P^k$ determined by

$$v_{i_j} = \mu(x_{i_j}) \text{ for } x_{i_j} \in M_{i_j} - \text{ the fixed point.}$$

**Definition 2.** *A polytope $P_\sigma \in S(P^k)$ is said to be admissible if it corresponds to an admissible set.*

Denote by $\widehat{\mu} : M^{2n}/T^k \to P^k$ the map induced by the almost moment map $\mu$.

**Axiom 4.** *The almost moment map $\mu$ gives the mapping from $W_\sigma$ to $\overset{\circ}{P}_\sigma$ and induces fiber bundle $\widehat{\mu} : W_\sigma/T^k \to \overset{\circ}{P}_\sigma$.*

Choose $x_\sigma \in \overset{\circ}{P}_\sigma$ and let $F_\sigma = \widehat{\mu}^{-1}(x_\sigma)$.

**Definition 3.** *The set $F_\sigma$ we call the set of parameters of the stratum $W_\sigma$. It is the fiber of the bundle $\widehat{\mu} : W_\sigma/T^k \to \overset{\circ}{P}_\sigma$.*

**Corollary 1.** *The fiber bundle $\widehat{\mu} : W_\sigma/T^k \to \overset{\circ}{P}_\sigma$ is isomorphic to the trivial bundle. Hence $W_\sigma/T^k$ is homeomorphic to $\overset{\circ}{P}_\sigma \times F_\sigma$.*

For a given trivialization $\xi_\sigma : W_\sigma/T^k \to F_\sigma$ and any point $c_\sigma \in F_\sigma$
the leaf $W_\sigma[\xi_\sigma, c_\sigma] \subseteq W_\sigma$ is defined as

$$W_\sigma[\xi_\sigma, c_\sigma] = (\pi^{-1} \circ \xi_\sigma^{-1})(c_\sigma),$$

where $\pi : W_\sigma \to W_\sigma/T^k$ is the projection.

**Axiom 5.** *For any admissible $\sigma$ there exists the trivialization $\xi_\sigma : W_\sigma/T^k \to F_\sigma$ such that for any $c_\sigma \in F_\sigma$ the boundary $\partial W_\sigma[\xi_\sigma, c_\sigma]$ of the leaf $W_\sigma[\xi_\sigma, c_\sigma]$ of the stratum $W_\sigma$ is the union of the leafs $W_{\bar{\sigma}}[\bar{\xi_\sigma}, c_{\bar{\sigma}}]$ for exactly one $c_{\bar{\sigma}} \in F_{\bar{\sigma}}$, where $P_{\bar{\sigma}}$ runs through the admissible faces for $P_\sigma$.*

Define the operator $d$ on the set of admissible polytopes $\mathfrak{S}$ by $dP_\sigma$ is disjoint union of all proper faces of $P_\sigma$.

We obtain CW complex $CW(M^{2n}, P^k)$ : the vertices of this complex are the vertices of $P^k$ and open cells are $\overset{\circ}{P}_\sigma$ for $P_\sigma \in \mathfrak{S}$. We glue them by induction using the operator $d$.

The orbit space $M^{2n}/T^k$ can be described in terms of $CW(M^{2n}, P^k)$, $F_\sigma$ and $\xi_{\sigma,\sigma'}$:

**Theorem 1.**

$$M^{2n}/T^k = \cup P_\sigma \times F_\sigma / \approx,$$

*where* $(x, f_x) \approx (y, f_y)$ *if and only if* $x = y \in P_{\sigma'} \subset P_\sigma$ *and* $f_y = \xi_{\sigma,\sigma'}(f_x)$.

**Corollary 2.** *The complex Grassmann manifold* $G_{4,2}$ *is* $(8,3)$*-manifold. There is the structure of* $(10,3)$*-manifold on* $\mathbb{C}P^5$ *such that the Plücker embedding* $G_{4,2} \to \mathbb{C}P^5$ *is the map beween* $(2n,k)$*-manifold over the hypersimplex* $\Delta_{4,2}$.

**Corollary 3.** $G_{4,2}/T^3 \cong \partial\Delta_{4,2} * \mathbb{C}P^1, \ \mathbb{C}P^5/T^3 \cong \partial\Delta_{4,2} * \mathbb{C}P^2$

**Axiom 6.** *For any chart* $(M_i, \varphi_i)$ *it is given the characteristic homomorphism* $\alpha_i : \mathrm{T}^k \to \mathrm{T}^n$ *such that its weight vectors are pairwise linearly independent and the homeomorphism* $\varphi_i$ *is* $\alpha_i$ *- equivariant:*

$$\varphi_i(tx_i) = \alpha_i(t)\varphi_i(x_i), t \in \mathrm{T}^k, \ x_i \in M_i.$$

**Definition 4.** *A linear map* $h : \mathbb{R}^k \to \mathbb{R}$, $h(x) =< x, \nu >$ *is said to be the height function for* $T^k$*-manifold* $M^{2n}$ *if:* $h(v_i) \neq h(v_j)$ *for any two vertices* $v_i$ *and* $v_j$ *of* $P^k$ *and the composition* $h \circ \mu : M^{2n} \to \mathbb{R}$ *is a Morse function whose critical points coincides with the fixed points for* $T^k$*-action on* $M^{2n}$.

**Axiom 7.** *For a* $(2n,k)$*-manifold there is a height function* $h : \mathbb{R}^k \to \mathbb{R}$ .

**Definition 5.** *Graph* $\Gamma(M^{2n}, P^k)$ *of* $(2n,k)$*-manifold* $M^{2n}$ *is a graph given by the vertices and 1-dimensional admissible polytopes of* $P^k$.

It is 1-skeleton of the complex $CW(M^{2n}, P^k)$. It any vertex of the graph $\Gamma(M^{2n}, P^k)$ there are exactly $n$ edges.

It follows from axioms that to any edge of the graph of $(2n,k)$-manifolds it can be assigned the subgroup of $T^k$ of codimension one and in this way an integer $k$-vector. Using the height function we can define the ordering on the set of vertices of the graph $\Gamma(M^{2n}, P^k)$. In this way it is defined the index of a vertex of $\Gamma(M^{2n}, P^k)$ which is the number of edges of $\Gamma(M^{2n}, P^k)$ incoming into vertex $v$. We denote by $h_q$ the number of vertices of $\Gamma(M^{2n}, P^k)$ having index $q$.

**Theorem 2.** *The number* $h_q$ *is equal to* $2q$*-th Betti number for* $M^{2n}$ *that is* $h_q = b_{2q}(M^{2n}), \ \ q = 0, \ldots, n$.

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## Tropical methods for linear series

Sam Payne
(joint work with Dave Jensen)

Let $X$ be a smooth projective curve of genus $g$ over a valued field $K$, and let $\mathfrak{X}$ be a semistable model over the valuation ring $R \subset K$. If the special fiber $\overline{\mathfrak{X}}$ is of compact type, meaning that its Jacobian is compact or, equivalently, its dual graph is a tree, then the method of limit linear series developed by Eisenbud and Harris in the 1980s gives a powerful method for studying linear series on $X$, as follows.

Let $L$ be a line bundle of degree $d$ and rank $r$ on $X$.

Suppose $\overline{\mathfrak{X}}$ is of compact type, with components $\{\overline{\mathfrak{X}}_j\}$. Then, for each $i$, there is a unique way of extending $L$ to a line bundle $\mathfrak{L}_i$ on $\mathfrak{X}$ such that

$$\deg \mathfrak{L}_{\mathfrak{i}}|_{\overline{\mathfrak{X}}_j} = \begin{cases} d & \text{if } i = j, \\ 0 & \text{otherwise.} \end{cases}$$

Then all of the interesting information about degenerations of sections of $\mathfrak{L}_i$ is concentrated on $\overline{\mathfrak{X}_i}$. In particular, those sections of $\mathfrak{L}_{\mathfrak{i}}|_{\overline{\mathfrak{X}}_i}$ that are limits of sections of $L$ form a linear series of degree $d$ and rank $r$ on $\overline{\mathfrak{X}}_i$. This gives us a collection of $\mathfrak{g}^r_d$s, one on each component of $\overline{\mathfrak{X}}$, which satisfy a natural compatibility condition that can be phrased in terms of vanishing sequences at the nodes where the components intersect. The combinatorics of these compatibility conditions can be combined elegantly and powerfully with the geometry of $\mathfrak{g}^r_d$s on the lower genus components of $\overline{\mathfrak{X}}$ to give proofs of the Brill-Noether and Gieseker-Petri Theorems [EH83, EH86] along with many other fundamental facts about the geometry of curves and their moduli.

Now, suppose $\overline{\mathfrak{X}}$ is not of compact type. Then $L$ may or may not extend to a line bundle $\mathfrak{L}_i$ on $\mathfrak{X}$ such that

$$\deg \mathfrak{L}_{\mathfrak{i}}|_{\overline{\mathfrak{X}}_j} = \begin{cases} d & \text{if } i = j, \\ 0 & \text{otherwise.} \end{cases}$$

If we assume for convenience that $K$ is discretely valued, $X$ has a rational point which we use to identify $\mathrm{Pic}_d(X)$ with $\mathrm{Jac}(X)$, and $\mathfrak{X}$ is regular, then the obstruction to finding such an extension can be measured by the component group of the Néron model of the Jacobian of $X$, which is the Jacobian of the dual graph of $\overline{\mathfrak{X}}$. Tropical geometry systematically studies, refines, and exploits this obstruction theory to understand the degeneration of the complete linear series of $L$. One fundamental tool in this approach is Baker's Specialization Lemma [Bak08], which gives an explicit obstruction for a component of the Néron model of the Jacobian of $X$ to intersect the closure of a line bundle of degree $d$ and rank $r$. When every component is obstructed in this way, we can conclude, based solely on the dual graph of $\overline{\mathfrak{X}}$, that $X$ has no $\mathfrak{g}^r_d$s. This observation is at the heart of the tropical proof of the Brill-Noether Theorem [CDPR12], although an important refinement is required to control the dimension of the space of $\mathfrak{g}^r_d$s when it is nonempty.

The main topic of this talk is joint work with Dave Jensen [JP14] refining this tropical approach to studying linear series on $X$, when $\overline{\mathfrak{X}}$ is not of compact type. In particular, we developed tools for studying degenerations of multiplication maps and found tropical obstructions to the existence of a nonzero kernel in a multiplication map

$$\Gamma(X, L) \otimes \Gamma(X, M) \to \Gamma(X, L \otimes M).$$

We have used these methods to give a new proof of the Gieseker-Petri Theorem via explicit computations on graphs, showing that if the dual graph of $\overline{\mathfrak{X}}$ is a particular chain of loops with bridges, with generic edge lengths, then the adjoing multiplication map

$$\Gamma(X, L) \otimes \Gamma(X, K \otimes L^{-1}) \to \Gamma(X, K)$$

is injective for all $L$. Refinements of this method may be used to control dimensions of kernels of multiplication maps when they are nonempty, and we hope to develop these techniques further in future work.

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## The applications of Okounkov bodies to arithmetic problems

Huayi Chen

Since the seminal works of Okounkov [10], Kaveh and Khovanskii [6], Lazarsfeld and Muastaţă [7], the theory of Okounkov bodies has been shown to be an efficient tool to describe geometric invariants in birational algebraic geometry. The typical example is the volume function on the group of Cartier divisors on an integral projective variety. Recently, the arithmetic analogue of Okounkov bodies has been discovered in the framework of Arakelov geometry, and has led to interesting applications. In some situations, the application of Okounkov bodies in the arithmetic problem is crucial because the arithmetic analogue of classical methods is still missing or is very sophisticated.

From the point of view of Arakelov geometry, the arithmetic varieties should be considered as the analogue of algebraic varieties fibered over a smooth projective curve (the function field setting). In the number theory case, it is $\operatorname{Spec}\mathbb{Z}$ that plays the role of the base curve. By definition, an arithmetic projective variety refers to a projective and flat morphism $\pi : \mathscr{X} \to \operatorname{Spec}\mathbb{Z}$ from an integral scheme $\mathscr{X}$ to $\operatorname{Spec}\mathbb{Z}$. A major obstruction to study such objects is that the base scheme is not "compact". For example, the principal divisor on $\operatorname{Spec}\mathbb{Z}$ need not have degree zero. It is a natural idea to compactify $\operatorname{Spec}\mathbb{Z}$ by the usual absolute value of $\mathbb{Q}$ (called the *infinite place*). Then the situation becomes analogous to the function field case since the closed points of a regular projective curve correspond to the valuations of the function field of the curve whose restriction on the base field is trivial. The compactness of the augmented object is justified by the following product formula

$$\forall\, a \in \mathbb{Q} \setminus \{0\}, \quad |a| \cdot \prod_p |a|_p = 1,$$

where $p$ runs over the set of all prime numbers, and $|.|_p$ is the p-adic absolute value on $\mathbb{Q}$. However, it turns out that no scheme structure can be defined for this augmented object and one cannot find the direct analogue of projective varieties in the arithmetic setting.

The genuine idea of Arakelov is to introduce analytic object to "compactify" an arithmetic variety. Let $\pi : \mathscr{X} \to \operatorname{Spec}\mathbb{Z}$ be an arithmetic projective variety. One can imagine that we attach to the arithmetic projective variety the complex analytic space associated to the $\mathbb{C}$-scheme $\mathscr{X}_{\mathbb{C}}$ as the "fiber over the infinite place". The algebraic objects in the algebraic geometry setting correspond to the similar algebraic objects on the arithmetic variety $\mathscr{X}$ equipped with additional structures (often metrics) on the induced object on the analytic space. For example, the notion of line bundles in the geometric setting corresponds to the notion of hermitian line bundles in the arithmetic framework as follows. Let $\pi : \mathscr{X} \to \operatorname{Spec}\mathbb{Z}$ be an arithmetic projective variety, a *hermitian line bundle* is defined as any couple $\overline{\mathscr{L}} = (\mathscr{L}, \|.\|)$, where $\mathscr{L}$ is a line bundle on the scheme $\mathscr{X}$, and $\|.\|$ is a continuous metric on the pull-back of $\mathscr{L}$ on the analytic space associated to $\mathscr{L}_{\mathbb{C}}$, invariant under the action of the complex conjugation.

Given a hermitian line bundle $\overline{\mathscr{L}}$ on an arithmetic projective variety $\pi : \mathscr{X} \to \operatorname{Spec}\mathbb{Z}$, one can construct a lattice in a normed vector space as follows. We denote by $\pi_*(\mathscr{L})$ the $\mathbb{Z}$-module $H^0(\mathscr{X}, \mathscr{L})$, whose rank identifies with the dimension of the vector space $H^0(\mathscr{X}_{\mathbb{Q}}, \mathscr{L}_{\mathbb{Q}})$ over $\mathbb{Q}$. The vector space $\pi_*(\mathscr{L}) \otimes_{\mathbb{Z}} \mathbb{R}$, which can be considered as a vector subspace of $H^0(\mathscr{X}_{\mathbb{C}}, \mathscr{L}_{\mathbb{C}})$, is naturally equipped with sup norm

$$\forall\, s \in \pi_*(\mathscr{L}) \otimes_{\mathbb{Z}} \mathbb{R}, \quad \|s\|_{\sup} := \sup_{x \in X(\mathbb{C})} \|s\|(x).$$

We shall use the expression $\pi_*(\overline{\mathscr{L}})$ to denote the lattice $(\pi_*(\mathscr{L}), \|.\|_{\sup})$. We say that a section $s \in \pi_*(\mathscr{L})$ is *small* if $\|s\|_{\sup} \le 1$. We denote by $\widehat{H}^0(\overline{\mathscr{L}})$ the set of all small sections of $\overline{\mathscr{L}}$. The set $\widehat{H}^0(\overline{\mathscr{L}})$ is necessarily finite. This notion is analogous to the space of global sections of a line bundle in the algebraic geometry setting. Motivated by this observation, Moriwaki [8, 9] has introduced the notion of arithmetic volume function for hermitian line bundles (or more generally, for arithmetic $\mathbb{R}$-Cartier divisors) as follows

$$\widehat{\mathrm{vol}}(\overline{\mathscr{L}}) := \limsup_{n\to+\infty} \frac{\ln \#\widehat{H}^0(\overline{\mathscr{L}}^{\otimes n})}{n^{\dim(\mathscr{X})}/\dim(\mathscr{X})!}.$$

This function has soon been proved to be quite useful in the arithmetic geometry. Moreover, it shares many good properties as its avatar in algebraic geometry, as shown by the works of Moriwaki mentioned above, and also by the work [11] of Yuan.

Despite the similitude of definitions, the study of the arithmetic volume function is by no means identical to that of the classical volume function and often much more difficult. In fact the small section set $\widehat{H}^0(\overline{\mathscr{L}})$ is not stable by the addition in general. The classical method in the study of graded linear series do not work in the arithmetic setting. Although tools from the complex analytic geometry can be used to remedy the defeat due to the lack of the algebraic structure, the implementation of these tools is often very sophisticated and demand extra hypotheses (smoothness, positivity, etc.) on the metric of the hermitian line bundle.

Under this circumstance, the theory of Okounkov bodies has been applied to the study of the arithmetic volume function and has let to interesting results such as the arithmetic version of Fujita's approximation theorem. There are essentially two approaches on the arithmetic analogue of Okounknov bodies in the literature : the one developed in [12] constructs a convex body attached to the sets $(\widehat{H}^0(\overline{\mathscr{L}}^{\otimes n}))_{n\in\mathbb{N}}$ in a way similar to the classical approach of Okounkov bodies and requires a fine study on this family of sets; the one developed in [2, 3, 1] uses $\mathbb{R}$-filtrations to interpret the arithmetic volume function as the integral of certain level function on the geometric Okounkov body of the generic fiber $L$ and relies on the theory of Okounkov bodies of graded linear series.

In the following, we will give a brief introduction to the $\mathbb{R}$-filtration approach mentioned above. Consider a lattice $\overline{E} = (E, \|.\|)$ in a normed finite dimensional real vector space. Here $E$ denotes a free $\mathbb{Z}$-module of finite rank and $\|.\|$ is a norm on the real vector space $E_{\mathbb{R}} = E \otimes \mathbb{R}$. We can then introduce a decreasing $\mathbb{R}$-filtration $\mathcal{F}$ on $E_{\mathbb{Q}}$ as follows :

$$\forall\, t \in \mathbb{R}, \quad \mathcal{F}^t(E_{\mathbb{Q}}) = \mathrm{Vect}_{\mathbb{Q}}(\{s \in E \,:\, \|s\| \le e^{-t}\}).$$

The jump points of the filtration are nothing but the logarithmic version of the successive minima of the lattice. The Minkowski's second theorem leads to the following estimation

$$\ln \#\widehat{H}^0(\overline{E}) = \int_0^{+\infty} \mathrm{rk}(\mathcal{F}^t(E_{\mathbb{Q}}))\,\mathrm{d}t + O(r\ln(r)),$$

where $\widehat{H}^0(\overline{E}) = \{s \in E \,:\, \|s\| \le 1\}$, $r = \mathrm{rk}_{\mathbb{Z}}(E)$, and the implicit constant is absolute.

We now consider an arithmetic projective variety $\pi : \mathscr{X} \to \mathrm{Spec}\,\mathbb{Z}$ and a hermitian line bundle $\overline{\mathscr{L}}$ on $\mathscr{X}$. We denote by $X = \mathscr{X}_{\mathbb{Q}}$ the generic fiber of $\pi$ and by $L$ the restriction of $\mathscr{L}$ on $X$. We assume that $L$ is big. It turns out that the lattice structure of $\pi_*(\overline{\mathscr{L}}^{\otimes n})$ induces as above an $\mathbb{R}$-filtration $\mathcal{F}$ on the vector space $V_n = H^0(X, L^{\otimes n})$. The fundamental idea of the $\mathbb{R}$-filtration approach is that the direct sum

$$V_\bullet^t = \bigoplus_{n\geq 0} \mathcal{F}^{nt}(V_n)$$

is actually a graded linear series of $L$. The theory of Okounkov bodies then allows to attach to this graded linear series a convex body $\Delta(V_\bullet^t)$ in $\mathbb{R}^d$ (with $d = \dim(X)$) such that

$$\mathrm{vol}(\Delta(V_\bullet^t)) = \lim_{n\to+\infty} \frac{\mathrm{rk}(V_n)}{n^d}.$$

A direct computation shows that

$$\int_0^{+\infty} \mathrm{rk}(\mathcal{F}^t(V_n))\,\mathrm{d}t = n\int_0^{+\infty} \mathrm{rk}(V_n^t) = \left(\int_0^{+\infty} \mathrm{vol}(\Delta(V_\bullet^t))\,\mathrm{d}t\right) n^{d+1} + o(n^{d+1}).$$

Therefore Minkowski's second theorem stated as above leads to

$$\lim_{n\to+\infty} \frac{\ln \#\widehat{H}^0(\overline{\mathscr{L}}^{\otimes n})}{n^{d+1}} = \int_0^{+\infty} \mathrm{vol}(\Delta(V_\bullet^t))\,\mathrm{d}t.$$

In particular, if one denotes by $\widehat{\Delta}(\overline{\mathscr{L}})$ the convex body

$$\{(x,t)\,:\, t\geq 0,\, x\in\Delta(V_\bullet^t)\}\subset\mathbb{R}^{d+1},$$

then one can interpret the arithmetic volume $\widehat{\mathrm{vol}}(\overline{\mathscr{L}})$ as $(d+1)!\mathrm{vol}(\widehat{\Delta}(\overline{\mathscr{L}}))$. One can also introduce a level function $\varphi_{\overline{\mathscr{L}}}$ on the Okounkov body $\Delta(L)$ of the total graded linear series of $L$ with

$$\varphi_{\overline{\mathscr{L}}}(x) = \sup\{t\in\mathbb{R}\,:\, x\in\Delta(V_\bullet^t)\}.$$

Then $\mathrm{vol}(\widehat{\Delta}(\overline{\mathscr{L}}))$ identifies with the the integral of the function $\max(\varphi_{\overline{\mathscr{L}}}, 0)$ on the Okounkov body $\Delta(L)$ with respect to the Lebesgue measure.

The $\mathbb{R}$-filtration approach is very flexible. It allows to separate difficulties arising from different structure of the problems and reduce the problems of divers natures to the study of graded linear series in the classical algebraic geometry setting. We refer the readers to [4, 5] for further applications of this approach in different settings.

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## Toric degenerations of integrable systems on Grassmannians and potential functions

Yuichi Nohara
(joint work with Kazushi Ueda)

Let $\mathrm{Gr}(2,n)$ be the Grassmannian of 2-planes in $\mathbb{C}^n$. The symplectic reduction

$$\mathcal{M}_r = \mathrm{Gr}(2,n)/\!/_r T$$

of $\mathrm{Gr}(2,n)$ under the action of the maximal torus $T \subset U(n)$ is isomorphic to the *polygon space*, which parametrizes $n$-gons in the Euclidean 3-space with fixed side lengths. For each triangulation $\Gamma$ of a planer $n$-gon, Kapovich-Millson [4] and Klyachko [5] constructed a completely integrable system on $\mathcal{M}_r$, which we call the *bending system*.

**Theorem 3.** *For any triangulation $\Gamma$ of an $n$-gon, there exists a completely integrable system*

$$\Phi_\Gamma : \mathrm{Gr}(2,n) \to \mathbb{R}^{2(n-2)}$$

*which induces the bending system on $\mathcal{M}_r$ through the symplectic reduction.*

Toric degenerations of $\mathrm{Gr}(2,n)$ are parametrized by triangulations of a planer $n$-gon (Speyer-Sturmfels [7]). For a triangulation $\Gamma$, let $X_\Gamma$ denote the central fiber of the corresponding toric degeneration.

**Theorem 4.** *For each triangulation $\Gamma$, the moment polytope of the toric variety $X_\Gamma$ coincides with the image $\Delta_\Gamma = \Phi_\Gamma(\mathrm{Gr}(2,n))$ of the integrable system $\Phi_\Gamma$. Moreover, $\Phi_\Gamma$ admits a deformation into the toric moment map on $X_\Gamma$, which induces a toric degeneration of the bending system on the polygon space $\mathcal{M}_r$.*

As an application to Floer theory and mirror symmetry, we compute the potential function in the sense of Fukaya, Oh, Ohta, and Ono [2]. For a Lagrangian submanifold $L$ in a symplectic manifold $X$, the potential function $\mathfrak{PO} = \mathfrak{PO}^L$ is defined by "counting" pseudo-holomorphic disks $v : D^2 \to X$ with Lagrangian boundary condition $v(\partial D^2) \subset L$. In the case where $L$ is a Lagrangian torus orbit in a compact toric manifold $X$, the potential function gives the superpotential of the Landau-Ginzburg mirror of $X$ (Cho-Oh [1], Fukaya-Oh-Ohta-Ono [3]). For Lagrangian torus fibers of $\Phi_\Gamma$, we obtain the following.

**Theorem 5.** *The potential function of Lagrangian torus fibers of $\Phi_\Gamma$ is a Laurent polynomial given by*

$$\mathfrak{PO}_\Gamma = \sum_{triangles} \left( \frac{y(b)y(c)}{y(a)} + \frac{y(a)y(c)}{y(b)} + \frac{y(a)y(b)}{y(c)} \right),$$

*where $y(a)$ is a Laurent monomial associated with an edge $a$ of a triangle, and the sum is taken over all triangles in the triangulation $\Gamma$.*

**Theorem 6.** *For any pair $(\Gamma, \Gamma')$ of triangulations, the potential functions $\mathfrak{PO}_\Gamma$ and $\mathfrak{PO}_{\Gamma'}$ are related by a subtraction-free rational change of variables whose "tropicalization" is a piecewise-linear automorphism*

$$T_{\Gamma,\Gamma'} : \mathbb{R}^{2(n-2)} \to \mathbb{R}^{2(n-2)}$$

*of the affine space such that $T_{\Gamma,\Gamma'}(\Delta_{\Gamma'}) = \Delta_\Gamma$. The map $T_{\Gamma,\Gamma'}$ is defined over $\mathbb{Z}$ if $\Delta_\Gamma$ is an integral polytope.*

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## Positivity of line bundles and Newton–Okounkov bodies


Victor Lozovanu

(joint work with Alex Küronya)


This is an account of a joint unpublished work with Alex Küronya. Newton–Okounkov bodies, which are bounded convex sets, capture the behaviour of all global sections of all multiples of a given big Cartier divisor at the same time. Based on earlier work of Okounkov, Newton–Okounkov bodies were first introduced in their whole generality in the work of Kaveh–Khovanskii [3] and Lazarsfeld–Mustaţă [5]. Many general properties and explicit examples were studied in [5] and [4]. Here we will explore the connections between Newton-Okounkov bodies and positivity of line bundles on projective algebraic surfaces.

For the rest of the talk let $X$ be a smooth projective surface and let $(C, x)$ be an admissible flag on $X$, i.e. $C \subseteq X$ an irreducible curve and $x \in C$ is a smooth point. If $D$ is a big line bundle, then for an effective divisor $D' \in |D|$, we define the following valuation map

$$\nu_{(C,x)}(D') \;=\; (\mathrm{ord}_C(D'), \big((D' - \mathrm{ord}_C(D')C).C\big)_x) \in \mathbb{Z}^2,$$

where the second coordinate is the local intersection number at $x$ of the effective divisors $D' - \mathrm{ord}_C(D')C$ and $C$, having no common support (see [2, Exercise I.5.4] for more detalied definitions). We say that $D' \sim_{\mathrm{lin}}^{\mathbb{Q}} D$, and say that $D'$ is rationally linear equivalent to $D$, if and only if there exists $m > 0$ such that $mD' \in |mD|$. Then the Newton–Okounkov body of $D$ is defined to be

$$\Delta_{(C,x)}(D) := \overline{\nu_{(C,x)}\big(\{D' \mid D' \sim_{\mathrm{lin}}^{\mathbb{Q}} D \text{ effective } \mathbb{Q} - \text{divisor}\}\big)} \subseteq \mathbb{R}^2.$$

On surfaces we also have Zariski decomposition for big divisors. This says that one can write uniquely $D = P(D) + N(D)$, where $P(D)$ is nef, $N(D)$ is effective, $(P(D).N(D)) = 0$ and the associated intersection matrix of the irreducible components of $N(D)$ is negative definite. Denote by $\nu = \mathrm{ord}_C(N(D))$ and by $\mu = \max\{t > 0 | D - tC \text{ is effective }\}$. If $D_t := D - tC = P_t + N_t$ is the Zariski decomposition for any $t \in [\nu, \mu]$, then

$$\Delta_{(C,x)}(D) \;=\; \{(t, y) \mid 0 \leq t \leq \mu, \alpha(t) \leq y \leq \beta(t)\},$$

where $\alpha(t) = \mathrm{ord}_x((N_t)|C)$ and $\beta(t) = \alpha(t) + (P_t.C)$. This presentation was used in [4] to show that in reality all Newton–Okounkov bodies on surfaces are polygons given by almost only rational data. Furthemore, $\Delta_{(C,x)}(D)$ is a numerical invariant. So, based on this, it becomes interesting to know if the shape of the Newton–Okounkov polygons forces certain positivity properties on the divisor $D$.

In the case of surfaces there are two algebraic sets of a big divisor whose complements encode the positive parts of the divisor. These are

$$\mathrm{Neg}(D) = \bigcup_{C \subseteq \mathrm{Supp}(N(D))} C \quad \text{and} \quad \mathrm{Null}(D) = \bigcup_{P(D).C=0} C$$

For exapmple, if $\mathrm{Neg}(D) = \emptyset$, then $D$ is a big and nef divisor, and if $\mathrm{Null}(D) = \emptyset$, then $D$ is ample.

Before stating the main result, for any $\lambda > 0$ denote the simplex of length $\lambda$ by $\Delta_\lambda = \{(t, y) \in \mathbb{R}^2_+ | t + y \leq \lambda\}$. Under these circumstances the first goal of the project was to prove the following theorem:

**Theorem A.** Let $D$ be a big divisor on a smooth projective surface $X$. Then
*(i)* $x \notin \mathrm{Neg}(D)$ if and only if there exists an admissible flag $(C, x)$ such that the Newton–Okounkov polygon $\Delta_{(C,x)}(D)$ contains the origin $(0, 0) \in \mathbb{R}^2$.
*(ii)* $x \notin \mathrm{Null}(D)$ if and only if there exists an admissible flag $(C, x)$ and a positive real number $\lambda > 0$ such that $\Delta_\lambda \subseteq \Delta_{(C,x)}(D)$.

As a consequence, one obtains criteria for nefness and ampleness for big divisors in terms of Newton–Okounkov polygons, in the vein of the classical Seshadri criterion. Another consequence of this theorem describes which points on the boundary of the Newton–Okounkov polygons are given by valuations of $\mathbb{Q}$-divisors rationally equivalent to $D$. For any $\lambda, \lambda' > 0$, denote by

$$\Delta_{\lambda,\lambda'} = \{(t,y) \in \mathbb{R}^2_+ \mid \lambda' t + \lambda y \leq \lambda\lambda'\} ,$$

the triangle with vertices of coordinates $(0,0), (\lambda, 0)$ and $(0, \lambda')$.

**Corollary B.** Let $D$ be a big $\mathbb{Q}$-divisor on $X$, such that $\Delta_{\lambda,\lambda'} \subseteq \Delta_{(C,x)}(D)$ for some $\lambda, \lambda' > 0$. Then all the rational points on the horizontal segment $[0, \lambda) \times \{0\}$ and the vertical one $\{0\} \times [0, \lambda')$ are given by valuations of rational effective divisors rationally linear equivalent to $D$.

The statement for the vertical segment $\{0\} \times [0, \lambda')$, can be obtained as a consequence of [1, Theorem 2.13], whose proof is very complicated and technical. Instead the above corollary can be obtained easily using the theory of Newton-Okounkov polygons and the theorem above.

Another consequence of the above theorem is a criteria computing the Seshadri constants from the Newton–Okounkov polygons. Let $\pi : X' \to X$ be the blow-up of $X$ at the point $x$ and $E$ be the exceptional divisor. The Seshadri constant $\epsilon(D, x) = \max\{\epsilon > 0 | \pi^*(D) - \epsilon E \text{ is ample }\}$. For any $\lambda > 0$, denote by

$$\Delta^{-1}_{\lambda} = \{(t,y) \in \mathbb{R}^2_+ \mid 0 \leq t \leq \lambda, 0 \leq y \leq t\}$$

the triangle with vertices of coordinates $(0,0), (0,\lambda)$ and $(\lambda, \lambda)$ Then we have

**Corollary C.** Let $D$ be a big divisor on $X$. Then

$$\epsilon(D; x) = \sup\{\lambda > 0 \mid \Delta^{-1}_{\lambda} \subseteq \Delta_{(E,y)}(\pi^*(D))\},$$

for any $y \in E$.

There are two remaining goals of this project. First, we would like to generalize Theorem A to higher dimensions in the language of augmented and restricted base loci associated to a big divisor. These loci are numerical invariants and as in the surface case for Neg$(D)$ and Null$(D)$ their complement encodes the posivity of the divisor. The second goal is to apply the theory of Newton-Okounkov bodies in studying many questions related to Seshadri constants.

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**The Vinberg Monoid and Some of its Applications**

Johan Martens
(joint work with Michael Thaddeus)

## 1. The Vinberg monoid

Let $G$ be a connected reductive group, defined over an algebraically closed field $k$, which we will assume to be $\mathbb{C}$ for convenience. The Vinberg monoid $S_G$ is a canonical flat equivariant degeneration of $G$ that picks up extra torus symmetry. Its total space is a reductive monoid, whose group of units $S_G^\times$ is given by $(G \times T_G)/Z_G$, where $T_G$ is a maximal torus of $G$, and $Z_G$ its centre. It can be constructed (in characteristic 0) using the (algebraic) Peter-Weyl theorem for $S_G^\times$:

$$\mathbb{C}[S_G^\times] = \bigoplus_{\lambda \in \mathfrak{X}_+, \mu \in \mathfrak{X}, \lambda - \mu = \sum m_i \alpha_i} \mathbb{C}[G \times T_G]_{\lambda,\mu},$$

with all $m_i \in \mathbb{Z}$, $\alpha_i$ the simple positive roots of $G$, $\mathfrak{X}$ the weights of $G$ and $\mathfrak{X}_+$ the dominant weights, and $k[G \times T_G]_{\lambda,\mu}$ the isotypical component corresponding to $(\lambda, \mu)$. The Vinberg monoid $S_G$ is now defined to be Spec of the subring of $k[S_G^\times]$ given by only keeping those isotypical components with all $m_i$ non-negative.

In [Vin95a] Vinberg introduced $S_G$ for semi-simple $G$ in characteristic zero (Vinberg refers to $S_G$ as the *enveloping semi-group* of $G$). This construction was generalized by Rittatore [Rit01] to semi-simple groups over arbitrary algebraically closed fields, and Brion and Alexeev [AB04] described the construction for arbitrary reductive groups in zero characteristic as above.

Let $\mathbb{A}$ be the affine GIT quotient $S_G /\!\!/ G \times G$. Vinberg shows that $\mathbb{A}$ is an affine space, which is the affine toric variety for the torus $T_G/Z_G$ determined by the positive Weyl chamber for the adjoint group $G/Z_G$. associated to $G$. The morphism $\pi : S_G \to \mathbb{A}$ is flat and has integral fibres, the generic fibre being $G$ itself.

For semi-simple $G$, Vinberg shows that $S_G$ satisfies a universal property in the category of reductive monoids, essentially saying that any reductive monoid (satisfying some minor conditions) whose semi-simple part is isomorphic to $G$ is obtained from $S_G$ by base change by a morphism of an affine toric variety to $\mathbb{A}$.

When thinking of $S_G \to \mathbb{A}$ as a degeneration of $G$, of particular interest is the most degenerate fibre $\pi^{-1}(0)$. Vinberg [Vin95b] calls this the *asymptotic semigroup* of $G$. It can be understood by modifying the ring structure on $\mathbb{C}[G]$: for any $G \times G$ isotypical $f \in \mathbb{C}[G]_\lambda$ and $g \in k[G]_\mu$ , Vinberg defines $f * g$ as the $\lambda + \mu$ isotypical component of $fg$. Alternatively, one can obtain a filtration on $\mathbb{C}[G]$ by choosing an additive morphism $h$ from $\mathfrak{X}(G)$ to $\mathbb{Z}$ such that $h$ is strictly positive on all positive roots, and putting $\mathbb{C}[G]^{(n)} = \oplus_{h(\lambda) \leq n} \mathbb{C}_\lambda$. The asymptotic semigroup of $G$ is Spec of the associated graded algebra. The morphism $h$ also determines a morphism $\mathbb{C} \to A_\Pi$, and Spec of the Rees algebra is then the base change of $S_g \to A_\Pi$ by this morphism.

## 2. Applications

We will outline here some uses of the Vinberg monoid. The first concerns a non-abelian generalisation of the Cox or Delzant constructions of toric varieties or symplectic toric manifolds, in algebraic or symplectic geometry respectively.

Recall that if we have a toric variety for a complex torus $T_{\mathbb{C}}$ given by a fan $\Sigma$, possibly coming from a polyhedral set $P$, these constructions realise the toric variety (or symplectic toric manifold) as a global quotient. The standard approach to this construction assumes that the rays in the fan generate the Lie algebra of the (compact) torus; we shall denote this condition by †.

If † holds, the associated toric variety $X_\Sigma$ can be constructed as a geometric quotient of an open subset $(\mathbb{C}^N)^o$, defined by the combinatorics of the fan $\Sigma$, (where $N$ is the number of rays in $\Sigma$), by the kernel of the homomorphism $(\mathbb{C}^*)^N \to T_{\mathbb{C}}$. If $\Sigma$ is simplicial, this is a geometric quotient, and if $\Sigma$ comes from $P$ this can be understood as a GIT quotient, with the open set being the semistable locus.

Note that the condition † is always satisfied if $\Sigma$ is complete, or if $P$ is a polytope, but not for many other cases. One can however make a small generalisation that makes the condition † redundant: one can simply take the quotient of $(\mathbb{C}^N)^o \times T_{\mathbb{C}}$ by all of $(\mathbb{C}^*)^N$ (see [MT11] or [CLS11, Thm. 5.1.17]).

Besides relaxing the condition †, this variant also has the advantage that it allows for a non-abelian generalization of the construction, with the aid of the Vinberg monoid. Indeed, the so-called *toroidal* compactifications of $G$, equivariant for the action of $G \times G$ are classified by fans supported in a Weyl chamber of $G$. Each such fan (again with $N$ rays) gives rise to a morphism $\mathbb{C}^N \to A_\Pi$. In joint work with Michael Thaddeus we have showed that the toroidal compactification can be constructed as follows:

**Proposition 7** (Cox-Vinberg construction)**.** *With $G$ and $\Sigma$ as above, the corresponding toroidal compactification of $G$ is given as the categorical quotient of $(\mathbb{C}^N)^o \times_{\mathbb{A}} S_G^o$ by $(\mathbb{C}^*)^N$.*

Here $S_G^o$ is a certain open subvariety of $S_G$ defined by Vinberg, with the property that the geometric quotient $S_G^o/T_G$ is the wonderful compactification of the adjoint group $G/Z_G$. In [MT11] it was shown that both the latter quotient, and, if $\Sigma$ comes from a polyhedral set $P$, the quotient in the property above both can be understood as GIT quotients. Note that if $G = T_{\mathbb{C}}$ is abelian, $S_G$ is just $T_{\mathbb{C}}$ and $\mathbb{A}$ is just a point, hence the construction just reduces to the variation of the Cox construction. If $G$ is already adjoint, and $\Sigma$ is just its positive Weyl chamber, the quotient is likewise just the one considered by Vinberg.

In symplectic geometry, one can interpret the construction for toric symplectic manifolds as saying that every toric manifold (subject to condition †) is a symplectic reduction of $\mathbb{C}^N$. The variant that relaxes † can then be understood as saying that every toric manifold is a symplectic cut (à la Lerman) of $T^*T$.

It is a well-known fact that the symplectic cut of a Hamiltonian $T$-manifold $M$ with respect to a polytope $P$ can be obtained as the symplectic reduction by $T$ of the Cartesian product of $M$ and the toric manifold determined by $P$ – one could say that the symplectic toric manifolds are the *universal* symplectic cuts.

This of course begs the question if this symplectic cutting story also has a non-abelian analogue. In joint work with Thaddeus [MT12], we showed that this was indeed the case. The non-abelian version of symplectic cutting was introduced by Woodward and Meinrenken, but one of the first applications (the cut of a co-adjoint orbit) was also observed not to have any compatible equivariant Kaehler structures, which seemed to leave little hope that the non-abelian cut had an algebro-geometric counterpart, as the abelian cut. However, we observed that this was still the case, under the condition that the polytope with respect to which one cuts has all of its outward normal vectors lie in the positive Weyl chamber.

One can understand this in algebraic geometry as follows: let $M$ be any (semi)-projective linearised $G$-variety. Take the GIT-quotient $M \times S_G$ by $G$, which is a (canonical) degeneration of $M$ over $\mathbb{A}$ – we will refer to it as $M * S_G$, following [AB04]. For any fan in the (positive) Weyl chamber of $G$ as above, take the GIT quotient of

$$M * S_G \times_{\mathbb{A}} \mathbb{C}^N$$

by $(\mathbb{C}^*)^N$. This is the algebro-geometric version of the non-abelian symplectic cut.

## 3. Link with toric degenerations

The family $M * S_G$ we considered above also occurred in the work of Alexeev and Brion [AB04], and it is tempting to try to link it to other known degenerations.

It was suggested by Alan Knutson that indeed the toric degeneration of the flag varieties $F$ (at least in type $A$) linked to the Gelfand-Zetlin integrable system can indeed be obtained this way, where one iterates this procedure for the descending chain of groups $G \supset G_1 \supset \dots$, i.e. one considers

$$((F * S_G) * S_{G_1}) * ...$$

Each step in this iteration procedure picks up extra torus symmetry, until finally the special fibre is indeed toric. Each suitable valuation now corresponds to a morphism from $\mathbb{C}$ to the base of the degeneration, and the corresponding base change of this ought to be exactly the toric degeneration over $\mathbb{C}$. This suggestion was also touched upon in the talk by Chris Manon in this workshop.

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## Geometric mitosis and Newton–Okounkov polytopes

Valentina Kiritchenko

In [K], a convex-geometric algorithm was introduced for building new analogs of Gelfand–Zetlin polytopes for arbitrary reductive groups. Conjecturally, these polytopes coincide with the Newton–Okounkov polytopes of flag varieties for a geometric valuation. I outline an algorithm (*geometric mitosis*) for finding collection of faces in these polytopes that represent a given Schubert cycle. For $GL_n$ and Gelfand–Zetlin polytopes, this algorithm reduces to a geometric version of Knutson–Miller mitosis introduced in [KST].

First, recall the *mitosis on parallelepipeds* from [KST, Section 6]. Let $\Pi(\mu,\nu)\subset\mathbb{R}^n$ be a parallelepiped given by inequalities $\mu_i\le x_i\le\nu_i$ for $i=1,\dots,n$. For every face $\Gamma\subset\Pi(\mu,\nu)$, we now define a collection of faces $M(\Gamma)$ called the *mitosis* of $\Gamma$. Let $k$ be the minimal number such that $\Gamma\subseteq\{x_i=\mu_i\}$ for all $i>k$ (in particular, $\Gamma\not\subseteq\{x_k=\mu_k\}$) and $\nu_i\ne\mu_i$ for at least one $i>k$. If no such $k$ exists then $M(\Gamma)=\emptyset$. Under the isomorphism $\mathbb{R}^n\simeq\mathbb{R}^k\times\mathbb{R}^{n-k}$; $(x_1,\dots,x_n)\mapsto(x_1,\dots,x_k)\times(x_{k+1},\dots,x_n)$ the face $\Gamma$ gets mapped to $\Gamma'\times v$ where $v=(\mu_{k+1},\dots,\mu_n)$ is a point and $\Gamma'\subset\mathbb{R}^k$ is a parallelepiped in $\mathbb{R}^k$. Let $E_i\subset\mathbb{R}^{n-k}$ for $i=k+1,\dots,n$ be the segment with vertices $(\mu_{k+1},\dots,\mu_{i-1},\mu_i,\nu_{i+1}\dots,\nu_n)$ and $(\mu_{k+1},\dots,\mu_{i-1},\nu_i,\dots,\nu_n)$ (that is, the union $\bigcup_{i=k+1}^n E_i$ is a broken line that connects points $(\mu_{k+1},\dots,\mu_n)$ and $(\nu_{k+1},\dots,\nu_n)$). Then $M(\Gamma)$ consists of all faces $\Gamma'\times E_i$ for $k+1\le i\le n$ such that $E_i$ is not a single point (in particular, $\dim\Delta=\dim\Gamma+1$ for any $\Delta\in M(\Gamma)$). Definition of $M(\Gamma)$ is motivated by the identity [KST, Proposition 6.8] for a Demazure-type operator applied to an exponential sum over $\Gamma$.

This geometric version of mitosis reduces easily to the combinatorial mitosis of [KnM] as follows. Every face of $\Pi(\mu,\nu)$ can be represented by a $2\times n$ table $(a_{ij})_{i=1,2,\ 1\le j\le n}$ whose cells are either filled with + or empty. Namely, the face satisfies the equality $x_i=\mu_i$ or $x_i=\nu_i$ if and only if $a_{1i}=+$ or $a_{2i}=+$, respectively (in particular, if $\mu_i=\nu_i$ then the $i$-th column has two +). On the level of tables, operation $M$ coincides the mitosis of [KnM] after reflecting our tables in a vertical line.

**Example 1:** If $\Pi(\mu,\nu)\subset\mathbb{R}^4$, where $\mu=(1,1,1,1)$ and $\nu=(2,2,1,2)$ (that is, $\mu_3=\nu_3$), then the vertex $\Gamma=\{x_1=\nu_1,\ x_2=\mu_2,\ x_4=\mu_4\}$ is represented by the table

| | | | |
|---|---|---|---|
| | + | + | + |
| + | | + | |

The set $M(\Gamma)$ consists of two edges represented by the tables

| | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| | + | + | | & | | | + | | |
| + | | + | | | + | | + | + | |

We now briefly recall a construction from [K, Section 3.3]. Let $G$ be a connected reductive group of semisimple rank $r$. Let $\alpha_1,\ldots,\ \alpha_r$ denote simple roots of $G$, and $s_1,\ldots,\ s_r$ the corresponding simple reflections. Fix a reduced decomposition $w_0 = s_{i_1}s_{i_2}\cdots s_{i_d}$ where $w_0$ is the longest element of the Weyl group of $G$. Let $d_i$ be the number of $s_{i_j}$ in this decomposition such that $i_j = i$. Consider the space

$$\mathbb{R}^d = \mathbb{R}^{d_1} \oplus \ldots \oplus \mathbb{R}^{d_r}$$

and choose coordinates $x = (x_1^1,\ldots,x_{d_1}^1;\ldots;x_1^r,\ldots,x_{d_r}^r)$ with respect to this decomposition. Put $\sigma_i(x) = \sum_{j=1}^{d_i} x_j^i$. Define the projection $p$ of $\mathbb{R}^d$ to the real span $\mathbb{R}^r$ of the weight lattice of $G$ by the formula $p(x) = \sigma_1(x)\alpha_1 + \ldots + \sigma_r(x)\alpha_r$. Let $\lambda$ be a dominant weight of $G$. There is an elementary convex-geometric algorithm for constructing a polytope $P_\lambda(i_1,\ldots,i_d) \subset \mathbb{R}^d$ that yields the Weyl character $\chi(V_\lambda)$ of the irreducible $G$-module $V_\lambda$, that is,

$$\chi(V_\lambda) = \sum_{x\in P_\lambda\cap\ \mathbb{Z}^d} e^{p(x)}$$

(see Theorem [K, Theorem 3.6] for more details). The polytope $P_\lambda$ can be used to extend the results of [KST] from $GL_n$ to $G$ since its *polytope ring* is isomorphic to the cohomology ring of the complete flag variety $G/B$ (with rational coefficients).

In particular, if $G = SL_n$ and $w_0 = (s_1)(s_2s_1)(s_3s_2s_1)\ldots(s_{n-1}\ldots s_1)$, then we get the classical Gelfand–Zetlin polytope [K, Theorem 3.4]. However, if $G = Sp_4$ the resulting polytopes seem to be different from *string polytopes* of Berenstein–Littelmann–Zelevinsky.

**Example 2:** Take $G = Sp(4)$ (that is, $d = 4$ and $r = 2$) and $w_0 = s_2s_1s_2s_1$ (here $\alpha_1$ denotes the shorter root and $\alpha_2$ denotes the longer one). Let $\lambda = \lambda_1\omega_1 + \lambda_2\omega_2$ be a strictly dominant weight of $Sp_4$. Choose a point $a_\lambda = (a,b,c,d)$ such that $p(a_\lambda) = w_0\lambda = -\lambda$. Label coordinates in $\mathbb{R}^4$ by $x := x_1^1 - a$, $y := x_2^1 - b$, $z := x_1^2 - c$ and $t := x_2^2 - d$. The polytope $P_\lambda(2,1,2,1)$ is given by inequalities

$$0 \le x \le \lambda_1, \quad z \le x + \lambda_2, \quad y \le 2z,$$

$$y \le z + \lambda_2, \quad 0 \le t \le \lambda_2, \quad t \le \frac{y}{2}$$

(see [K, Example 3.4]). It has 11 vertices, hence, it is not combinatorially equivalent to string polytopes for $Sp_4$ defined in [L].

**Remark:**Let $X = Sp_4/B$ be the complete flag variety for $Sp_4$, and $L_\lambda$ the line bundle on $X$ corresponding to the weight $\lambda$. Recently, I checked that after a unimodular change of coordinates $P_\lambda(2,1,2,1)$ coincides with the Newton–Okounkov polytope $\Delta_v(X, L_\lambda)$ for the lowest term valuation $v$ corresponding to the flag of translated Schubert varieties: $w_0X_{id} \subset s_1s_2s_1X_{s_2} \subset s_1s_2X_{s_1s_2} \subset s_1X_{s_2s_1s_2} \subset X$ (cf. [Ka, Remark 2.3]).

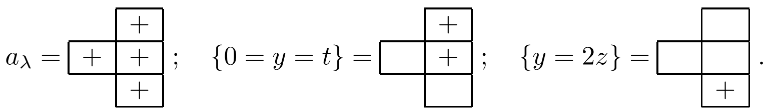
By construction, the intersection of the polytope $P_\lambda := P_\lambda(i_1, \dots, i_d)$ with $(c + \mathbb{R}^{d_i})$ is either a parallelepiped $\Pi(\mu(c), \nu(c))$ or is empty for any $c \in \mathbb{R}^d$. This property of $P_\lambda$ gives $r$ mitosis operations $M_1, \dots, M_r$ corresponding to parallelepipeds $P_\lambda \cap (c + \mathbb{R}^{d_1}), \dots, P_\lambda \cap (c + \mathbb{R}^{d_r})$, respectively. Mitosis on parallelepipeds allows us to produce collections of faces of $P_\lambda$ that represent a given Schubert cycle in $G/B$ (in the sense of [KST, Theorem 5.1]), that is, the exponential sum over the union of these faces yields the Demazure characters. The algorithm is as follows. For an element $w \in W$ of the Weyl group, denote by $[X_w] = [\overline{BwB/B}]$ the Schubert cycle corresponding to $w$. Let $s_{j_1} \dots s_{j_\ell}$ be a reduced decomposition of $w_0 w w_0^{-1}$ such that $(j_1, \dots, j_\ell)$ is a subword of $(i_1, \dots, i_d)$. Then $[X_w]$ is represented by the union of faces produced from a vertex of $P_\lambda$ by applying successively the operations $M_{j_\ell}, \dots, M_{j_1}$. For $G = SL_n$ and $w_0 = (s_1)(s_2 s_1)(s_3 s_2 s_1) \dots (s_{n-1} \dots s_1)$, this algorithm can be described combinatorially using mitosis of Knutson–Miller on pipe-dreams (see [KST]).

For other reductive groups, one can also describe the mitosis algorithm combinatorially using suitable analogs of pipe-dreams.

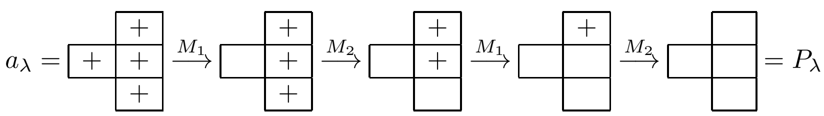
**Example 3:** We continue Example 2. The vertex $a_\lambda$ is the intersection of 4 facets: $0 = x$, $y = 2z$, $0 = t$, $t = \frac{y}{2}$. Let us encode faces that contain $a_\lambda$ by tables using the following rules:

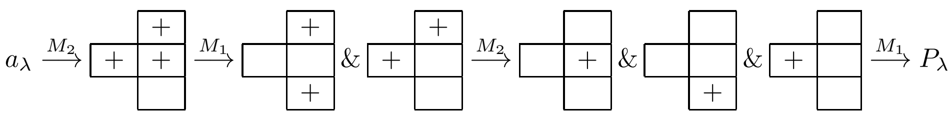
| | |
|---|---|
| | $+ \iff 0 = t$ |
| $+ \iff 0 = x$ | $+ \iff t = \frac{y}{2}$ |
| | $+ \iff y = 2z$ |

Here are three examples:



Every face $\Gamma$ defines two (possibly degenerate) rectangles $\Pi_1(\Gamma) = \Gamma \cap \{z = z_0,\ t = t_0\}$ and $\Pi_2(\Gamma) = \Gamma \cap \{x = x_0,\ y = y_0\}$ (we choose $x_0$, $y_0$, $z_0$ and $t_0$ so that the dimensions of $\Pi_1(\Gamma)$ and $\Pi_2(\Gamma)$ are maximal possible). For instance, the face $\Gamma = \{0 = y = t\}$ defines two segments. Note that $\Pi_i(\Gamma)$ is a face of the rectangle $\Pi_i(P_\lambda)$, and hence, there is a well-defined operation $M_i$ of mitosis on parallelograms for $i = 1, 2$. It is not hard to check that in terms of tables, $M_1$ and $M_2$ do the following:





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## Non-displaceable Lagrangian tori in $S^2 \times S^2$

Kaoru Ono
(joint work with K. Fukaya, Y.-G. Oh and H. Ohta)

I presented a continuum of Hamiltonianly non-displaceable Lagrangian tori in the product of two copies of $(S^2, \omega)$ based on my joint work [5] with K. Fukaya, Y.-G. Oh and H. Ohta. Namely, we showed the following:

**Theorem** There exists a continuum of Lagrangian tori $T(u)$, $u \in (0, 1/2]$, in $(S^2, \omega) \times (S^2 \times \omega)$ with the following properties.
(1) If $u \neq u'$, $T(u)$ is not Hamiltonianly isotopic to $T(u')$.
(2) $T(u)$ is Hamiltonianly non-displaceable. In fact, a certain Floer cohomology of $T(u)$ is isomorphic to the ordinary cohomology of $T^2$ with coefficients in Novikov ring.
(3) If $u \neq u'$, $T(u) \cap T(u') = \emptyset$.
(4) $T(1/2)$ is a monotone Lagrangian torus.
(5) $T(1/2)$ is not Hamiltonianly isotopic to $S^1_{\rm eq} \times S^1_{\rm eq}$, i.e., the product of the equator in $(S^2, \omega)$.

For any closed embedded (relatively) spin Lagrangian sub manifold $L$ in a closed symplectic manifold $(X, \omega)$, we can construct a filtered $A_\infty$-algebra $(H^*(L; \Lambda_0), \{\mathfrak{m}_k\})$. We introduced the universal Novikov ring $\Lambda_0$ in [1], where we denote it by $\Lambda_{0,\rm nov}$. (Forget the formal variable related to the $\mathbf{Z}$-grading, the Novikov ring consists of formal Laurent-type series with not just integer but real exponents.) If $\mathfrak{m}_1 \circ \mathfrak{m}_1 = 0$, we have a cochain complex $(H^*(L; \Lambda_0), \mathfrak{m}_1)$, whose cohomology is the Floer cohomology of the pair $(L, L)$. However, it is not the case, in general. We can rectify it, if there exists a solution or a solution in weak sense of the Maurer-Cartan equation for $b \in H^*(L; \Lambda_+) \oplus H^1(L; \mathbf{C})$[1] of total degree 1 (usual cohomological degree plus the degree on the Novikov ring $\Lambda_0$)

$$\sum_{k=0}^{\infty} \mathfrak{m}_k(b^{\otimes k}) = 0.$$

[1] $\Lambda_+$ is an ideal in $\Lambda_0$ consisting of formal sums with *positive* exponents. The case that $b \in H^*(L; \Lambda_+)$ is due to FOOO and the case that $b \in H^1(L; \mathbf{C})$ is due to C.-H. Cho.

An element $b$ is called a solution in weak sense, the left hand side of the above equation is proportional to the unit (Poincaré dual of the fundamental class of $L$)[2]. We denote by $\mathcal{M}C_{\rm weak}(L)$ the gauge equivalence classes of solutions in weak sense to the Maurer-Cartan equation. We define the potential function $\mathfrak{P}O^L$ : $\mathcal{M}C_{\rm weak}(L) \to \Lambda_+$ by

$$\sum_{k=0}^{\infty} \mathfrak{m}_k(b^{\otimes k}) = \mathfrak{P}O^L(b)PD[L].$$

We can also deform the whole construction using an ambient cohomology class $\mathfrak{b} \in H^*(X;\Lambda_+)$ in $X$ (*bulk deformation*). Then we obtain $\mathfrak{m}_k^{\mathfrak{b}}$, $\mathcal{M}C_{\mathfrak{b},\rm weak}(L)$ and $\mathfrak{P}O^L_{\mathfrak{b}}$.

In a series of papers [2], [3], [4], we studied Lagrangian Floer theory of Lagrangian torus fibers, i.e., Lagrangian tori appearing as inverse images of interior points in the moment polytope under the moment map. Lagrangian Floer theory in such a case is understood in terms of critical point theory of the potential function.

For the proof of Theorem 1, we use the well-known deformation family of Hirzebruch surfaces. We have a holomorphic map $\pi : \mathcal{X} \to \mathbf{C}$ such that the fiber over the origin 0 is isomorphic to $F_2^{\rm sing}$ and other fibers are $F_0$, i.e., $\mathbf{C}P^1 \times \mathbf{C}P^1$ such that it admits a simultaneous resolution $\widetilde{\pi} : \widetilde{\mathcal{X}} \to \mathbf{C}$ (the central fiber is replaced by the Hirzebruch surface $F_2$ of degree 2). Here $F_2^{\rm sing}$ is $F_2$ with the $(-2)$-curve contracted.

Let us take $\epsilon > 0$ and a segment $[0,\epsilon] \in \mathbf{C}$. The Lagrangian tori $T(u)$ is the image of a certain family of Lagrangian tori $L(u)$ in $F_2^{\rm sing}$ under the transportation along characteristics of $\pi^{-1}([0,\epsilon])$. (We may assume the moment polytope of $F_2^{\rm sing}$ is

$$P(0) = \{(u_1,u_2) \in \mathbf{R}^2 | u_1 \geq 0,\ u_2 \geq 0,\ u_1 + 2u_2 \leq 2\}.$$

Then the Lagrangian torus $L(u)$ is the fiber over $(u, 1-u)$, $0 < u \leq 1/2$.) For given $u > 0$, we can regard it as a Lagrangian torus in $F_2$ (not in $F_2^{\rm sing}$) with an appropriate Käbler structure. Namely, the toric Kähler surface $F_2(\alpha)$ corresponding to the trapezoid

$$P(\alpha) = \{(u_1,u_2) \in \mathbf{R}^2 | 0 \leq u_1 \leq 1-\alpha, 0 \leq u_2, u_1 + 2u_2 \leq 2\}$$

for a sufficiently small $\alpha > 0$.

Since $F_2$ is not Fano, but semi-Fano, the potential function may have contributions more than those corresponding to the facets of the moment polytope. Fortunately, we can compute the potential function of $L(u)$ in $F_2(\alpha)$ (there are other methods due to Auroux and Chan-Leung-Lau). We can justify that the potential function of $T(u) \subset (S^2,\omega) \times (S^2\omega)$ is obtained from the potential function of $L(u) \subset F_2(\alpha)$ by taking the limit $\alpha \to 0$. We can also compute the potential function after bulk deformations by the class corresponding to the vanishing cycle, which is a Lagrangian two-sphere and corresponds to the $(-2)$-curve in the central

[2]Here, for simplicity, we forget $\mathbf{Z}$-grading

fiber of $\widetilde{\pi}$. This is the key ingredient of the proof of our Theorem (1) and (2). (The statements (3) and (4) are clear from the construction.)

Finally, I would like to mention that these Lagrangian tori $T(u)$ are superheavy with respect to a certain Calabi quasimorphisms (in the sense of Entov and Polterovich) [6] section 23. The statement (5) follows from this fact. ($S^1_{\mathrm{eq}} \times S^1_{\mathrm{eq}}$ is superheavy with respect to the Calabi quasimorphisms corresponding to all units of four field factors of $QH(S^2 \times S^2)$, while $T(1/2)$ is superheavy with respect to only two of those. The vanishing cycle is superheavy with respect to the rest.) We also have a similar example of continuum of Lagrangian tori in the cubic surface [6] section 24.

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## Symplectic geometry of the Vinberg monoid and branching problems

Chris Manon

In work of Harada-Kaveh, [HK] and Nishinou-Nohura-Ueda [NNU], the theory of Okounkov bodies has found an interesting application in symplectic geometry and Hamiltonian dynamics. These authors roughly show that to a flat toric degeneration $M \Rightarrow M_\Delta$ of a smooth, projective variety $M \subset \mathbb{P}^m$ one can associate the following information.

(1) A dense, open integrable system $M_o \subset M$ with momentum image $\mu(M_o) \subset \Delta$, a dense open subset of the moment polytope $\Delta$ of the toric variety $M_\Delta$.

(2) A continuous, surjective map $\Phi : M \to M_\Delta$ which is a symplectomorphism on $M_o$ and provides a continuous extension $\bar{\mu} : M \to \Delta$ of the momentum map of $M_o$ to $M$.

The map $\Phi$ in this construction, from now on referred to as the contraction map, is built from Ruan's gradient flow technology [R] on hypersurfaces of Kähler manifolds. When the total space of the degeneration $\pi : E \to \mathbb{C}$ is endowed with a Kähler structure, Ruan's theory ensures the existence of a map $\Phi$ from the hypersurface $M = \pi^{-1}(1)$ to the special fiber $M_\Delta = \pi^{-1}(0)$. This construction can also be applied in cases where the degeneration $M \Rightarrow M_\Delta$ is not toric. Indeed one could consider any degeneration where the special fiber comes with extra torus symmetries, and thereby induce a Hamiltonian (not necessarily integrable) torus action on a dense open subspace of $M$ with analogous properties. With this in mind, given a complex variety $X$ with a Kähler structure, and a flat degeneration $X \Rightarrow Y$ to a variety $Y$ with an action of a complex torus $T$, we refer to constructing a surjective, continuous map $\Phi : X \to Y$ with these properties as the problem of finding a contraction map.

A drawback of the gradient-flow method for constructing $\Phi : M \to M_\Delta$ is that this map is not explicitly computable. Also, in its current form, the construction applies only to smooth varieties ( or torus $GIT$ quotients of smooth varieties). We investigate examples of degenerations outside of these restrictions, which nonetheless possess a contraction map that can be computed.

0.1. **Symplectic horospherical contraction.** We fix a connected, reductive complex group $G$, with Weyl chamber $\Delta$ and maximal torus $T$. For any $G-$variety $X$ (say affine or projective), Popov [P] and Grosshans [Gr] have defined a flat, $G-$stable degeneration $X \Rightarrow X^c$ called the horospherical contraction. The coordinate ring $\mathbb{C}[X^c]$ of this variety has the same isotypical decomposition as $\mathbb{C}[X]$.

$$\mathbb{C}[X] = \bigoplus_{\lambda \in \Delta} W_\lambda \otimes V(\lambda) \tag{1}$$

We let $P_X$ be the monoid of dominant weights which appear in this decomposition, and $\Delta_X$ be the real cone spanned by the weights in $P_X$. The contraction $X^c$ comes with an additional action by $T$, which turns this isotypical decomposition into a grading of $\mathbb{C}[X^c]$ by $P_X$.

We let $\Delta^\vee$ be the dual Weyl chamber (itself a Weyl chamber of the Langland's dual group) to $\Delta$. Part of the theory of horospherical contraction associates a $G-$invariant discrete valuation $v_h$ on $\mathbb{C}[X]$ for every integral coweight $h \in \Delta^\vee$. The contraction $X^c$ can be constructed by taking the associated graded ring of $\mathbb{C}[X]$ with respect to $v_h$ when $h$ is in the interior of $\Delta^\vee$.

We let $K \subset G$ be a maximal compact subgroup, with maximal torus $\mathbb{T} \subset T$. A finite dimensional representation $V$ of $G$ can be given its standard $K-$invariant Kähler form $w_V$. This likewise induces the a $K-$invariant Kähler structure on $\mathbb{P}(V)$ by symplectic reduction. A $G-$subvariety $X \subset V$ (resp. $X \subset \mathbb{P}(V)$) can then be given its singular stratification, and each piece of this stratification can be given the structure of a smooth (possibly non-compact) Kähler manifold. We call this the Kähler structure on $X$ inherited from $V$ (resp. $\mathbb{P}(V)$.) The space $X$ likewise inherits a momentum mapping $\mu_X : X \to \mathfrak{k}^*$ from the ambient space.

Using this set up, our first result translates horospherical contraction into symplectic geometry, and provides a contraction map associated to the horospherical contraction degenerations. We make the technical assumption that $G$ is a product of a simply connected semisimple group with a torus.

**Proposition 8.** *For $X \subset V$ (resp. $\mathbb{P}(V)$ a variety with horospherical contraction $X^c$, there is a Kähler structure on $X^c$ which makes it a Hamiltonian $\mathbb{T}-$space with momentum image $\mu_{\mathbb{T}}(X^c) = \Delta_X$. Furthermore, there is a surjective, continuous map $\Phi_X : X \to X^c$ which is a symplectomorphism from a dense, open, smooth subspace $X_o \subset X$ onto a $\mathbb{T}-$stable subspace of $X^c$.*

Note that it follows that $X_o$ inherits the momentum map from $X^c$, which then has a continuous extension to $X$.

The contraction $X^c$ is constructed in the symplectic category by making use of the symplectic implosion operation of Guillemin, Jeffrey, and Sjamaar, [GJS]. This operation is the Hamiltonian analogue of passing from a $G-$representation to its space of highest weights, its input is a Hamiltonian $K-$space $X$, and its output is a Hamiltonian $\mathbb{T}-$space $EX$. As a Hamiltonian $K \times \mathbb{T}$-space, $X^c$ can be constructed as a symplectic reduction at level $0 \in \mathfrak{k}^* \times \mathfrak{t}^*$.

$$X^c \cong K \times \mathbb{T} \backslash_0 [EX \times ET^*(K)] \tag{2}$$

Here $ET^*(K)$ is the imploded cotangent bundle of $K$ (with respect to the right action by $K$).

In order to compute $\Phi_X : X \to X^c$ on a point $p \in X$, one finds an element $k \in K$ such that $\mu_X(k \circ p) \in \Delta$, the image $\Phi_X(p) \in X^c$ is then the $\mathbb{T}$-equivalence class of $([k \circ p], [k^{-1}, \mu_X(k \circ p)]) \in EX \times ET^*(K)$. Particulars of the geometry of symplectic implosion then imply that this map is well-defined, surjective, continuous, and smooth on a subspace $X_o \subset X$.

The result is a computable contraction map for a degeneration of a possibly singular $G-$variety $X$. The construction also makes the subspace $X_o$ explicit, it is the inverse image of the *principal face* of $X$, this is the highest face in $\Delta$ hit by $\mu_X$, under inclusion.

0.2. **Branching degeneration, branching contraction.** We apply both horospherical contraction and its symplectic analogue to branching problems in the representation theory of reductive groups. For a map $\phi : H \to G$ of connected, reductive groups, a resolution of the branching problem defined by $\phi$ is a rule for deciding how many copies of an irreducible representation $V(\eta)$ appears in an irreducible representation $V(\lambda)$ of $G$ as decomposed under the map $\phi$. This problem can be uploaded into the geometry of an affine variety $X(\phi)$ (see e.g. [M]). This variety has a $T_H \times T_G$ action, and its coordinate ring has the following isotypical decomposition.

$$\mathbb{C}[X(\phi)] = \bigoplus_{\eta \in \Delta_H, \lambda \in \Delta_G} Hom_H(V(\eta), V(\lambda)) \tag{3}$$

Finding the dimension of the spaces $Hom_H(V(\eta), V(\lambda))$ can be aided in part by finding a splitting of $\phi$ in the category of connected, reductive groups.

$$H \xrightarrow{\pi} F \xrightarrow{\psi} G$$

Each space $Hom_H(V(\eta), V(\lambda))$ then further decomposes along $\pi, \psi$.

$$(4) \qquad Hom_H(V(\eta), V(\lambda)) = \bigoplus_{\mu \in \Delta_F} Hom_H(V(\eta), V(\mu)) \otimes Hom_F(V(\mu), V(\lambda))$$

Notice that the components of this decomposition are isotypical spaces in the coordinate ring of the variety $X(\pi) \times X(\psi)$. This observation is given geometric meaning by the following application of horospherical contraction.

**Proposition 9.** *For every integral weight $h \in \Delta_F^\vee$, there is a discrete valuation $v_h$ on $\mathbb{C}[X(\phi)]$. If $h$ is chosen in the interior of $\Delta_F$, the associated graded algebra of $v_h$ is the coordinate ring of the variety $[X(\pi)\times X(\psi)]/T_F$, where $T_F$ acts diagonally through the $T_F \times T_F$ action on $X(\pi) \times X(\psi)$.*

The degeneration $[X(\pi) \times X(\psi)]/T_F$ then comes with a residual $T_F$ action. Taking $T_H \times T_G$ $GIT$ quotients at weights $-\eta, -\lambda$ then produces a flat degeneration of the "projective branching variety" $H\backslash[\mathcal{O}(\eta) \times \mathcal{O}(\lambda)]$, where $\mathcal{O}(\lambda)$ is the flag variety of $G$ associated to the highest weight $\lambda$. If $\phi = \psi \circ \pi$ is induced from maps of compact subgroups,

$$L \xrightarrow{\pi} J \xrightarrow{\psi} K$$

the symplectic analogue of horospherical contraction can then be applied to prove the following.

**Proposition 10.** *For $\phi = \psi \circ \pi$ as above, there is a surjective, continuous map $\Phi_{\pi,\psi} : X(\phi) \to [X(\pi) \times X(\psi)]/T_F$, which is a symplectomorphism on a dense, open subspace $X_o(\phi) \subset X(\phi)$.*

The momentum image of the $\mathbb{T}_L \times \mathbb{T}_J \times \mathbb{T}_K$ action on $[X(\pi) \times X(\psi)]/T_F$ is a cone $P(\pi, \psi) \subset \Delta_L \times \Delta_J \times \Delta_K$ obtained as the real span of the triples of weights $\eta, \mu, \lambda$ such that $V(\eta) \subset V(\mu) \subset V(\lambda)$. When applied to $H\backslash[\mathcal{O}(\eta) \times \mathcal{O}(\lambda)]$, this produces the following.

(1) A Hamiltonian $J$-action on a dense open subspace of $H\backslash[\mathcal{O}(\eta) \times \mathcal{O}(\lambda)]$.

(2) A continuous map $\mu_J \circ \Phi_{\eta,\lambda} : H\backslash[\mathcal{O}(\eta) \times \mathcal{O}(\lambda)] \to \Delta_F$.

Similar to the contraction mappings $\Phi_X : X \to X^c$ above, these elements are explicitly computable.

0.3. **An example: The Gel'fand-Tsetlin system.** Our results on branching problems can then be modified and inductively applied to the chain of upper diagonal inclusions of unitary groups.

$$(5) \qquad 1 \subset U(1) \subset U(2) \subset \ldots \subset U(n)$$

The consequences for a $GL_n(\mathbb{C})$ flag variety $\mathcal{O}(\lambda)$ are as follows.

(1) A cone $\Delta_1^\vee \times \ldots \times \Delta_n^\vee$ of valuations on the coordinate rings of $\mathcal{O}(\lambda)$.

(2) A toric degeneration (over $\mathbb{C}$) $\mathcal{O}(\lambda) \Rightarrow X(GT_n(\lambda))$ to the singular toric variety associated to the Gel'fand-Tsetlin polytope $GT_n(\lambda)$.

(3) The dense, open Gel'fand-Tsetlin integrable system $\mathcal{O}_o(\lambda) \subset \mathcal{O}(\lambda)$.

(4) A continuous extension of the momentum map of the Gel'fand-Tsetlin system to $\mathcal{O}(\lambda)$.

This follows much work on the algebraic geometry of the Gel'fand-Tsetlin system, [HK], [NNU].

0.4. **Further questions.**

(1) Determine if the gradient-flow construction corresponding to some valuation $v_h$ coincides with the contraction mapping $\Phi_X$ in known cases. (Joint with Rebecca Goldin and Brent Gorbutt)

(2) Study the relationship between the Hamiltonian systems in $X(\phi)$ corresponding to different factorizations of $\phi$. For example, is there always a collection of factorizations such that any two points in $H\backslash[\mathcal{O}(\eta) \times \mathcal{O}(\lambda)]$ can be connected by the associated torus flows?

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**Transforming metrics of a line bundle to the Okounkov body**

David Witt Nystrom

The main motivation for studying Okounkov bodies has been their connection to the volume of line bundles (or divisors). Recall that the volume of a line bundle $L$ is defined as

$$\mathrm{vol}(L) := \limsup_{k\to\infty} \frac{n!}{k^n}\dim(H^0(kL)),$$

and that $L$ is said to be big if the volume is positive.

Let now $h$ be a continuous hermitian metric on a big line bundle $L$, we then call the pair $(L, h)$ a metrized big line bundle. The notion of a metric volume of a metrized big line bundle was introduced by Berman-Boucksom in [1]. Given a metric $h$ one has a natural norm on the the spaces of holomorphic sections $H^0(kL)$, namely the supremum norm

$$||s||_{h^k,\infty} := \sup\{|s(x)|_{h^k} : x \in X\}.$$

Let $\mathcal{B}^\infty(h^k) \subseteq H^0(kL)$ be the unit ball with respect to this norm.

$H^0(kL)$ is a vector space, thus given a basis we can calculate the volume of $\mathcal{B}^\infty(h^k)$ with respect to the associated Lebesgue measure. This will depend on the choice of basis, but given a reference metric $h_{ref}$ one can compute the quotient

$$\frac{\mathrm{vol}(\mathcal{B}^\infty(h^k))}{\mathrm{vol}(\mathcal{B}^\infty(h^k_{ref}))}$$

and this quantity will be invariant under the change of basis. The metric volume of a metrized big line bundle $(L, h)$, denoted by $\mathrm{vol}(L, h)$, is defined as the limit

$$\mathrm{vol}(L,h) := \lim_{k\to\infty} \frac{n!}{2k^{n+1}} \log\left(\frac{\mathrm{vol}(\mathcal{B}^\infty(h^k))}{\mathrm{vol}(\mathcal{B}^\infty(h^k_{ref}))}\right). \tag{1}$$

The metric volume obviously depends on the choice of $h_{ref}$ as a reference metric but it is easy to see that the difference of metric volumes $\mathrm{vol}(L, h) - \mathrm{vol}(L, h')$ is independent of the choice of reference.

The definition of the metric volume is clearly reminiscent of the definition of the volume of a line bundles. In fact, one easily checks that

$$\mathrm{vol}(L, h) - \mathrm{vol}(L, eh) = \mathrm{vol}(L).$$

In [1] Berman-Boucksom prove that the limit (1) exists. They do this by proving that it actually converges to a certain integral over the space $X$ involving mixed Monge-Ampere measures related to the metrics.

In the talk I explained a different way to prove that the limit exists using Okounkov bodies, which also can be used to prove differentiability properties of the metric volume as the line bundle varies. Given a metrized big line bundle $(L, h)$ I showed how to construct an associated convex function $c[h]$ on the interior of the Okounkov body of $L$ which I call the Chebyshev transform of $h$. The construction can be seen to generalize both the Chebyshev constants in classical potential theory and the Legendre transform of convex functions (and thus the symplectic potential from toric geometry).

First for any point $(\alpha, k) \in Gamma(L)$ we let $A_{\alpha,k}$ denote the affine space of sections in $H^0(kL)$ locally of the form

$$z^\alpha + \text{ higher order terms.}$$

We define the discrete Chebyshev transform $F[h]$ on $Gamma(L)$ as

$$F[h](\alpha, k) := \inf\{\ln ||s||^2_{h^k,\infty} : s \in A_{\alpha,k}\}.$$

A key observation now is that $F[h]$ is subadditive. This allows us to use a multidimensional version of Fekete's lemma to see that for any sequence $(\alpha_i, k_i)$ in $Gamma(L)$ such that $k_i \to \infty$ and $\alpha_i/k_i \to p \in Delta(L)^\circ$, the limit

$$\lim_{k\to\infty} \frac{1}{k_i} F[h](\alpha_i, k_i)$$

exists and only depends on $p$.. We may therefore define the Chebyshev transform $c[h]$ of $h$ by defining $c[h](p)$ be that limit.

This construction is inspired by the work of Zaharjuta, who in [4] used subadditive functions on $\mathbb{N}^n$ when studying directional Chebyshev constants, and also by the article [2] where Bloom-Levenberg extend Zaharjutas results to a more general metrized setting, but still in $\mathbb{C}^n$.

The connection to metric volumes is the following theorem.

**Theorem 11.** *Let $h$ and $h'$ be two continuous metrics on $L$. Then it holds that*

$$vol(L, h) - vol(L, h') = n! \int_{\Delta(L)^\circ} (c[h'] - c[h]) d\lambda, \tag{2}$$

*where $d\lambda$ denotes the Lebesgue measure on $\Delta(L)$.*

The proof of Theorem 11 relies on the fact that one also can use $L^2$-norms instead of supremum norms to compute the Chebyshev transform. Then the right-hand side in equation (2) can be interpreted as a limit of certain Donaldson bifunctionals closely related and asymptotically equal to the ones used in the definition of the metric volume. This then gives a new proof of the fact that the limit (1) exists.

As an application, using the differentiability result of Berman-Boucksom and some pluripotential theory and combining it with the new Okounkov body machinery one can prove that the metric volume is differentiable.

**Theorem 12.** *The metric volume function is $\mathcal{C}^1$ on the open cone of big $\mathbb{R}$-divisors equipped with two continuous metrics.*

Full details are found in [3].

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## Local Newton-Okounkov bodies

Kiumars Kaveh

(joint work with Askold Khovanskii)

We apply techniques from the theory of semigroups of integral points and Newton-Okounkov bodies (for the global case) to local algebraic geometry and commutative algebra to obtain new results as well as new proofs for some previously known results about multiplicities of ideals in local rings. For details see [KKh14].

Let $R = \mathcal{O}_{X,p}$ be the local ring of a point $p$ on an $n$-dimensional irreducible algebraic variety $X$ over an algebraically closed field $\mathbf{k}$. Let $\mathfrak{m}$ denote the maximal ideal of $R$ and let $\mathfrak{a}$ be an $\mathfrak{m}$-primary ideal, i.e. $\mathfrak{a}$ is an ideal containing a power of the maximal ideal $\mathfrak{m}$. Geometrically speaking, $\mathfrak{a}$ is $\mathfrak{m}$-primary if its zero set (around $p$) is the single point $p$ itself. Let $f_1, \ldots, f_n$ be $n$ generic elements in $\mathfrak{a}$. The *multiplicity* $e(\mathfrak{a})$ of the ideal $\mathfrak{a}$ is the intersection multiplicity, at the origin, of the hypersurfaces $H_i = \{x \mid f(x) = 0\}$, $i = 1, \ldots, n$ (it can be shown that this number is independent of the choice of the $f_i$). According to Hilbert-Samuel's theorem, the multiplicity $e(\mathfrak{a})$ is equal to:

$$n! \lim_{k\to\infty} \frac{\dim_{\mathbf{k}}(R/\mathfrak{a}^k)}{k^n}, \tag{1}$$

where $\dim_{\mathbf{k}}$ denotes the dimension as vector space over $\mathbf{k}$. This result is analogous to Hilbert's theorem on the Hilbert function and degree of a projective variety. More generally, let $R$ be an $n$-dimensional Noetherian local domain over $\mathbf{k}$ (where $\mathbf{k}$ is isomorphic to the residue field $R/\mathfrak{m}$ and $\mathfrak{m}$ is the maximal ideal). Let $\mathfrak{a}$ be an $\mathfrak{m}$-primary ideal of $R$. The *Hilbert-Samuel function* of the $\mathfrak{m}$-primary ideal $\mathfrak{a}$ is defined by:

$$H_{\mathfrak{a}}(k) = \dim_{\mathbf{k}}(R/\mathfrak{a}^k).$$

For large values of $k$, $H_{\mathfrak{a}}(k)$ coincides with a polynomial of degree $n$ called the *Hilbert-Samuel polynomial* of $\mathfrak{a}$. The *Samuel multiplicity*, $e(\mathfrak{a})$ of $\mathfrak{a}$ is defined by the limit (1) i.e. the leading coefficient of $H_{\mathfrak{a}}(k)$ multiplied by $n!$.

It is well-known that the Samuel multiplicity satisfies a Brunn-Minkowski inequality [Te77, RS78]. That is, for any two $\mathfrak{m}$-primary ideals $\mathfrak{a}$, $\mathfrak{b} \in R$ we have:

$$e(\mathfrak{a})^{1/n} + e(\mathfrak{b})^{1/n} \geq e(\mathfrak{a}\mathfrak{b})^{1/n}. \tag{2}$$

We generalize the notion of multiplicity to $\mathfrak{m}$*-primary graded sequences of subspaces*. That is, a sequence $\mathfrak{a}_1, \mathfrak{a}_2, \ldots$ of $\mathbf{k}$-subspaces in $R$ such that for all $k, m$ we have $\mathfrak{a}_k\mathfrak{a}_m \subset \mathfrak{a}_{k+m}$, and $\mathfrak{a}_1$ contains a power of the maximal ideal $\mathfrak{m}$. We recall that if $\mathfrak{a}, \mathfrak{b}$ are two $\mathbf{k}$-subspaces of $R$, $\mathfrak{a}\mathfrak{b}$ denotes the $\mathbf{k}$-span of all the $xy$ where $x \in \mathfrak{a}$ and $y \in \mathfrak{b}$. In particular, a graded sequence $\mathfrak{a}_\bullet$ where each $\mathfrak{a}_k$ is an $\mathfrak{m}$-primary ideal, is an $\mathfrak{m}$-primary graded sequence of subspaces.

For an $\mathfrak{m}$-primary graded sequence of subspaces we define multiplicity $e(\mathfrak{a}_\bullet)$ to be:

$$e(\mathfrak{a}_\bullet) = n! \lim_{k\to\infty} \frac{\dim_{\mathbf{k}}(R/\mathfrak{a}_k)}{k^n}. \tag{3}$$

Note that it is not a priori clear that the limit exists.

We use convex geometric arguments to prove the existence of the limit in (3) and a generalization of (2) to $\mathfrak{m}$-primary graded sequences of subspaces, for a large class of local domains $R$.

We briefly discuss the convex geometry part of the story. Let $\mathcal{C}$ be a closed strongly convex cone with apex at the origin (i.e. $\mathcal{C}$ is a convex cone and does not contain any line). We call a closed convex set $\Gamma \subset \mathcal{C}$, a *cobounded $\mathcal{C}$-convex region* if $\Gamma$ is closed and convex and $\mathcal{C} \setminus \Gamma$ is bounded. The set of cobounded $\mathcal{C}$-convex regions is closed under addition (Minkowski sum of convex sets) and multiplication with a positive real number. For a cobounded $\mathcal{C}$-convex region $\Gamma$ we call the volume of the bounded region $\mathcal{C} \setminus \Gamma$ the *covolume of* $\Gamma$ and denote it by $covol(\Gamma)$. Also we refer to $\mathcal{C} \setminus \Gamma$ as a *$\mathcal{C}$-coconvex body.* In [KhT-a, KhT-b], similar to convex bodies and their volumes (and mixed volumes), the authors develop a theory of convex regions and their covolumes (and mixed covolumes). Moreover they prove an analogue of the Alexandrov-Fenchel inequality for mixed covolumes. The usual Alexandrov-Fenchel inequality is an important inequality about mixed volumes of convex bodies in $\mathbb{R}^n$ and generalizes the classical isoperimetric inequality and the Brunn-Minkowski inequality. In a similar way, the result in [KhT-a] implies a Brunn-Minkowski inequality for covolumes, that is, for any two cobounded $\mathcal{C}$-convex regions $\Gamma_1$, $\Gamma_2$ where $\mathcal{C}$ is an $n$-dimensional cone, we have:

$$covol(\Gamma_1)^{1/n} + covol(\Gamma_2)^{1/n} \geq covol(\Gamma_1 + \Gamma_2)^{1/n}. \tag{4}$$

Let $R$ be an $n$-dimensional Noetherian local domain. Given a valuation $v$ with values in $\mathbb{Z}^n$ and certain good properties (which we call a *good valuation*) we associate a convex cone $\mathcal{C} \subset \mathbb{R}^n$ that is the closure of the convex hull of $v(R \setminus \{0\})$. Also to an $\mathfrak{m}$-primary sequence of subspaces $\mathfrak{a}_\bullet$ we associate a cobounded $\mathcal{C}$-convex region $\Gamma(\mathfrak{a}_\bullet)$ by:

$$\Gamma(\mathfrak{a}_\bullet) = \overline{\mathrm{conv}(\bigcup_{k>0} \{v(f)/k \mid f \in \mathfrak{a}_k\})}. \tag{5}$$

If $R$ is an analytically irreducible local domain it has a good valuation (this relies on a deep theorem of Izumi). See [KKh14] for details of definitions.

**Theorem 1.** The limit in (3) exists and:

$$e(\mathfrak{a}_\bullet) = n!\, covol(\Gamma(\mathfrak{a}_\bullet)). \tag{6}$$

**Theorem 2.** Let $R$ be an analytically unramified Noetherian local domain of dimension $n$ over a field $\mathbf{k}$ and let $\mathfrak{a}_\bullet$, $\mathfrak{b}_\bullet$ be $\mathfrak{m}$-primary graded sequences of subspaces then:

$$e(\mathfrak{a}_\bullet)^{1/n} + e(\mathfrak{b}_\bullet)^{1/n} \geq e(\mathfrak{a}_\bullet\mathfrak{b}_\bullet)^{1/n}.$$

The existence of limit (3) has been shown in [Cu-a] and [Fu] with similar methods. Also the Brunn-Minkowski inequality in Theorem 2 also been proved independently in [Cu-b] using similar methods.

The construction of $\Gamma(\mathfrak{a}_\bullet)$ is an analogue of the construction of the Newton-Okounkov body of a linear system on an algebraic variety (see [Ok03], [Ok96], [KKh12], [LM09]). In fact, the approach and results in the present paper are analogous to the approach and results in [KKh12] regarding the asymptotic behavior of Hilbert functions of a general class of graded algebras. On the other hand, the construction of $\Gamma(\mathfrak{a})$ generalizes the notion of the Newton diagram of a power series (see [Ku76] and [AVG85, Section 12.7]). To a monomial ideal in a polynomial ring (or a power series ring), i.e. an ideal generated by monomials, one can associate its (unbounded) *Newton polyhedron.* It is the convex hull of the exponents of the monomials appearing in the ideal.

The Brunn-Minkowski inequality proved in this paper is closely related to the more general Alexandrov-Fenchel inequality for mixed multiplicities. Take $\mathfrak{m}$-primary ideals $\mathfrak{a}_1, \ldots, \mathfrak{a}_n$ in a local ring $R = \mathcal{O}_{X,p}$ of a point $p$ on an $n$-dimensional algebraic variety $X$. The *mixed multiplicity* $e(\mathfrak{a}_1, \ldots, \mathfrak{a}_n)$ is equal to the intersection multiplicity, at the origin, of the hypersurfaces $H_i = \{x \mid f_i(x) = 0\}$, $i = 1, \ldots, n$, where each $f_i$ is a generic function from $\mathfrak{a}_i$. Alternatively one defines the mixed multiplicity as the polarization of the Hilbert-Samuel multiplicity $e(\mathfrak{a})$. The Alexandrov-Fenchel inequality is the following inequality among the mixed multiplicities of the $\mathfrak{a}_i$:

$$e(\mathfrak{a}_1, \mathfrak{a}_1, \mathfrak{a}_3, \ldots, \mathfrak{a}_n)e(\mathfrak{a}_2, \mathfrak{a}_2, \mathfrak{a}_3, \ldots, \mathfrak{a}_n) \geq e(\mathfrak{a}_1, \mathfrak{a}_2, \mathfrak{a}_3, \ldots, \mathfrak{a}_n)^2 \quad (7)$$

When $n = \dim R = 2$ it is easy to see that the Brunn-Minkowski inequality (2) and the Alexandrov-Fenchel inequality (7) are equivalent. By a reduction of dimension theorem for mixed multiplicities one can get a proof of the Alexandrov-Fenchel inequality (7) from the Brunn-Minkowski inequality (2) for $\dim(R) = 2$ (see [Te77], [RS78] and [KKh-a]).

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## Counting vertices in Gelfand–Zetlin polytopes

Vladlen Timorin

(joint work with Pavel Gusev and Valentina Kiritchenko)

This talk is based on results of [1].

Fix a positive integer $k$, and consider all partitions of the form $1^{i_1}\dots k^{i_k}$ (1 with multiplicity $i_1$, …, $k$ with multiplicity $i_k$) and the Gelfand–Zetlin polytopes $GZ(1^{i_1}\dots k^{i_k})$ corresponding to these partitions. Let $E_k$ stand for the exponential generating function for the number $V(1^{i_1}\dots k^{i_k})$ of vertices in the polytope $GZ(1^{i_1}\dots k^{i_k})$, i.e., the formal power series

$$E_k=\sum_{i_1,\dots,i_k\geq 0}V(1^{i_1}\dots k^{i_k})\frac{z_1^{i_1}}{i_1!}\dots\frac{z_k^{i_k}}{i_k!}.$$

We have obtained the following partial differential equation on $E_k$:

$$\left(\frac{\partial^k}{\partial z_1\dots\partial z_k}-\left(\frac{\partial}{\partial z_1}+\frac{\partial}{\partial z_2}\right)\dots\left(\frac{\partial}{\partial z_{k-1}}+\frac{\partial}{\partial z_k}\right)\right)E_k=0.$$

It follows that, for example, $E_1(z_1)=e_1^z$, $E_2(z_1,z_2)=e^{z_1+z_2}I_0\left(2\sqrt{z_1z_2}\right)$, where $I_0$ is the modified Bessel function of the first kind with parameter 0:

$$I_0(t)=\sum_{n=0}^{\infty}\frac{t^{2n}}{n!^2}.$$

It is also interesting to consider the ordinary generating function for the numbers $V(1^{i_1}\dots k^{i_k})$:

$$G_k(y_1,\dots,y_k)=\sum_{i_1,\dots,i_k\geq 0}V(1^{i_1}\dots k^{i_k})y_1^{i_1}\dots y_k^{i_k}.$$

We have obtained difference equations on this function. For every power series $f$ of the variables $y_1$, …, $y_k$, we define the action of the *divided difference operator* $\Delta_i$ on $f$ as

$$\Delta_i(f)=\frac{f-f|_{y_i=0}}{y_i}.$$

The function $G_k$ satisfies the following equation:

$$(\Delta_1 \dots \Delta_k - (\Delta_1 + \Delta_2) \dots (\Delta_{k-1} + \Delta_k))\, G_k = 0.$$

For $k = 1$, 2 and 3, this function can be computed explicitly. It is not hard to see that

$$G_1(y_1) = \frac{1}{1 - y_1}, \quad G_2(y_1, y_2) = \frac{1}{1 - y_1 - y_2}.$$

The function $G_3(x, y, z)$ is equal to

$$\frac{2xz - y(1 - x - z) - y\sqrt{1 - 2(x + z) + (x - z)^2}}{2(1 - x - z)((x + y)(y + z) - y)}.$$

This formula for $G_3$ is deduced from the difference equation. Note, however, that because of the second power of $\Delta_y$ appearing in this difference equation, we have to use the fact that $G_3$ is a power series rather than a function with possible singularities at $(0, 0, 0)$.

The numbers $V_{k,\ell,m} = V(1^k 2^\ell 3^m)$ can be represented as the coefficients of certain polynomials. Namely, the number $V_{k,\ell,m}$ coincides with the coefficient with $x^k z^m$ in the polynomial

$$\frac{1 - xz}{1 + xz} \left( (1 + x)^{k+\ell+m} (1 + z)^{k+\ell+m} - (x + z)^{k+\ell+m} \right).$$

This implies the following explicit formula for the numbers $V_{k,\ell,m}$ $(k, \ell, m > 0)$:

$$V_{k,\ell,m} = \binom{s}{k}\binom{s}{m} + 2\sum_{i=1}^{k} (-1)^i \binom{s}{k - i}\binom{s}{m - i}.$$

Note that the sum $\sum_{i=1}^{k} (-1)^i \binom{s}{k-i}\binom{s}{m-i}$ can be expressed as the value of the generalized hypergeometric function ${}_3F_2$, namely, it is equal to $\binom{s}{k-1}\binom{s}{m-1}\, {}_3F_2(1, 1 - k, 1 - m; 2 + \ell + m, 2 + k + \ell; -1)$.

*Open problems.*

(1) Prove or disprove: the generating function $G_4$ is algebraic. Note that $G_1$ and $G_2$ are rational, and $G_3$ is algebraic.
(2) Deduce differential or difference equations on the generating functions for the $f$-vectors and for the modified $h$-vectors of Gelfand–Zetlin polytopes.
(3) Express the generating function $E_k$ in terms of multidimensional residues.

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**Positivity of Chern classes of Schubert cells and varieties**

June Huh

There is a good theory of Chern classes for singular or noncomplete complex algebraic varieties. If $X^\circ$ is a locally closed subset of a complete variety $X$, then the Chern-Schwartz-MacPherson class of $X^\circ$ is an element in the Chow group

$$c_{SM}(X^\circ) \in A_*(X),$$

which agrees with the total homology Chern class of the tangent bundle of $X$ if $X$ is smooth and $X = X^\circ$. The Chern-Schwartz-MacPherson class satisfies good functorial properties which, together with the normalization for smooth and complete varieties, uniquely determines it.

If $\underline{\alpha}$ is a partition, then there is a corresponding Schubert variety $\mathbb{S}(\underline{\alpha})$ in the Grassmannian of $d$-planes in $E$, parametrizing $d$-planes which satisfy incidence conditions with a flag of subspaces determined by $\underline{\alpha}$. The Schubert variety is a disjoint union of Schubert cells

$$\mathbb{S}(\underline{\alpha}) = \coprod_{\underline{\beta} \leq \underline{\alpha}} \mathbb{S}(\underline{\beta})^\circ.$$

Since each Schubert cell $\mathbb{S}(\underline{\beta})^\circ$ is isomorphic to an affine space, the Chow group of $\mathbb{S}(\underline{\alpha})$ is freely generated by the classes of the closures $\left[\mathbb{S}(\underline{\beta})\right]$. Therefore we may write

$$c_{SM}\big(\mathbb{S}(\underline{\alpha})^\circ\big) = \sum_{\underline{\beta} \leq \underline{\alpha}} \gamma_{\underline{\alpha},\underline{\beta}} \big[\mathbb{S}(\underline{\beta})\big] \in A_*\big(\mathbb{S}(\underline{\alpha})\big)$$

for uniquely determined coefficients $\gamma_{\underline{\alpha},\underline{\beta}} \in \mathbb{Z}$.

Various explicit formulas for these coefficients are obtained by Aluffi and Mihalcea in [AM09]. Based on substantial computer calculations, they conjectured that all $\gamma_{\underline{\alpha},\underline{\beta}}$ are nonnegative [AM09, Conjecture 1]. Since the cone of $k$-dimensional effective cycles in $\mathbb{S}(\underline{\alpha})$ is generated by the classes of $k$-dimensional $\mathbb{S}(\underline{\beta})$ with $\underline{\beta} \leq \underline{\alpha}$, the conjecture is equivalent to the statement that the $k$-dimensional component $c_{SM}\big(\mathbb{S}(\underline{\alpha})^\circ\big)_k$ is effective for all $k$.

The main result of [Huh13] verifies this conjecture.

**Theorem.** *There is a nonempty reduced and irreducible $k$-dimensional subvariety $Z(\underline{\alpha})$ of $\mathbb{S}(\underline{\alpha})$ such that*

$$c_{SM}\big(\mathbb{S}(\underline{\alpha})^\circ\big)_k = \big[Z(\underline{\alpha})\big] \in A_k\big(\mathbb{S}(\underline{\alpha})\big).$$

The proof is based on an explicit description the Chern class of a vector bundle at the level of cycles. This vector bundle lives on a carefully chosen equivariant desingularization of $\mathbb{S}(\underline{\alpha})$, and it is not globally generated in general.

Finding a positive combinatorial formula for $\gamma_{\underline{\alpha},\underline{\beta}}$ remains as a very interesting problem.

**Question** (A)**.** *Is there a combinatorial proof of the nonnegativity?*

It is known that $\gamma_{\underline{\alpha},\underline{\beta}}$ is the number of certain nonintersecting lattice paths joining pairs of points in the plane when $d = 2$ [AM09]. The reader will find useful discussions and numerical tables of $\gamma_{\underline{\alpha},\underline{\beta}}$ in [AM09, Mih07, Jon07, Jon10, Str11, Web12].

Conjecture of Aluffi and Mihalcea was stated and proved for Schubert cells in Grassmannians. One may ask whether the same is true for Schubert cells in generalized flag varieties.

**Question** (B). *Are these numbers nonnegative for Schubert cells in other flag varieties?*

In other words, one asks whether the Chern class of a $B$-orbit in the variety $G/P$ is effective, where $B$ is a Borel subgroup of a connected reductive group $G$ and $P$ is a parabolic subgroup of $G$ containing $B$. I conjecture that, with little evidence, the answer is 'yes'. The first step in testing this conjecture would be to generalize the explicit Chern class formulas in [AM09] to Schubert cells in generalized flag varieties.

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## Algebraic geometry related to the BKK formula for exponential sums

Boris Kazarnovskii

**1**. Let $\Lambda \subset \mathbb{C}^{n*}$ be a finite set, where $\mathbb{C}^{n*}$ is dual to $\mathbb{C}^n$. The function $f$ on $\mathbb{C}^n$

$$f(z) = \sum_{\lambda \in \Lambda, c_\lambda \in \mathbb{C}} c_\lambda \, \mathrm{e}^{\langle z,\lambda \rangle}$$

is called exponential sum (ES) with the support $\Lambda$. The convex hull $\Delta(f)$ of the support $\Lambda$ is called the *Newton polyhedron* of $f$.

Let $S_\Lambda$ be a space of systems of ESs with a fixed set of supports $\Lambda = (\Lambda_1, \cdots, \Lambda_n)$. The zero set of a system $F \in S_\Lambda$ is infinite. The BKK formula (theorem 13) computes its density. Let $N(R)$ be a number of isolated zeroes of a system in the ball

$B(R)$ of radius $R$ centered at 0 and let $\sigma_n$ be a volume of $n$-dimensional ball of radius 1. The limit $\lim_{R\to\infty} \frac{N(R)}{\sigma_n R^n}$ is called the *density of a zero set.*

The subset $D \subset S_\Lambda$ is called a *discriminant set* if

1. theorem 13 holds for $D$
2. the measure of $D$ is zero
3. $S_\Lambda \setminus D$ is connected

**Theorem 13** (BKK type formula)**.** *For a system $F \in S_\Lambda \setminus D$*

$$\lim_{R\to\infty} \frac{N(R)}{\sigma_n R^n} = \frac{1}{(2\pi)^n} n! P(\Delta_1, \cdots, \Delta_n), \tag{1}$$

*where $P(\Delta_1, \cdots, \Delta_n)$ is a mixed pseudo-volume of Newton polyhedra of the system*

The mixed pseudo-volume ([1, 13]) is an analog of a mixed volume depending not only on convexity but on complex structure as well. The support $\Lambda \subset \mathrm{Re}\mathbb{C}^{n*}$ is called *quasialgebraic*. ES with a quasialgebraic support is also called quasialgebraic. For convex bodies contained in $\mathrm{Re}\mathbb{C}^{n*}$ their mixed pseudo-volume coincides with a mixed volume. For quasualgebraic ESs with supports $\Lambda_i \subset \mathbb{Z}^n \subset \mathrm{Re}\mathbb{C}^{n*}$ formula (1) is equivalent to the BKK formula for polynomials.

**Example 1** (1-dimensional binomial $f(z) = \mathrm{e}^{\lambda z} - c$)**.** *The zero set of $f$ is the arithmetic progression $\{\frac{log(c)}{\lambda} + \frac{2\pi i}{\lambda}\mathbb{Z}\}$ and the density equals to $\frac{|\lambda|}{\pi}$.*

The existence of a discriminant set is proved only for the following two cases

1. for a general enough set of supports $\Lambda$ (B. Kazarnovskii, [1])
2. for a quasialgebraic set of supports $\Lambda$ (A. Khovanskii, [2])

There exists the conjectured algebraic version of BKK theorem for ESs. I proved it for quasialgebraic system of two variables. It is related to some well-known statements of algebraic geometry, such as conjectures of Mordell-Lang and Schanuel. This relation is the main subject of the talk.

**2. The algebraic discriminant set.** Let $f_i = \sum_{\lambda\in\Lambda_i} c_\lambda \,\mathrm{e}^{\langle z,\lambda\rangle}$. Consider the system $F = (f_1, f_2)$ of quasialgebraic ESs of 2 variables with the set of supports $\Lambda = (\Lambda_1, \Lambda_2)$ and the Newton polygons $\Delta_1, \Delta_2$. Let $\Delta = \Delta_1 + \Delta_2$ be a Minkowski sum. For a side $\delta$ of $\Delta$ denote by $\delta_1, \delta_2$ the faces-summands of $\delta$ (i.e., $\delta = \delta_1 + \delta_2$ where $\delta_i$ is a face of $\Delta_i$).

**Definition 1** (truncation of a system)**.** *Let $\delta$ be a side of a polygon $\Delta$. Let $f_i^\delta$ denote the* ES *$f_i^\delta = \sum_{\lambda\in\delta_i} c_\lambda \,\mathrm{e}^{\langle z,\lambda\rangle}$. The system $F^\delta = (f_1^\delta, g_2^\delta)$ is called the $\delta$-truncation of the system $F$.*

**Definition 2** (algebraic discriminant set)**.** *We say that $F \in D_{\mathrm{alg}}$ if there exists a side $\delta$ of $\Delta$ such that the* ESs *$f_1^\delta$ and $f_2^\delta$ have a nontrivial common divisor in the ring of* ESs*.*

It's easy to see that $D_{\mathrm{alg}}$ is an **algebraic variety in the space** $S_\Lambda$.

**Theorem 14** (algebraic version of exponential BKK)**.** *The algebraic variety $D_{\mathrm{alg}}$ is a discriminant set.*

The algebraic nondegeneracy of a system does not provide the absence of a non-singleton points of the zero set. The next statement gives some a priori description of non-discrete part of a zero set [8].

**Theorem 15.** *Any non-discrete component of the zero set of $F \in S_\Lambda \setminus D_{\mathrm{alg}}$ is an affine complex line orthogonal to some side of the polygon $\Delta$.*

**Remark 1.** *We can prove (but do not formulate it) that the set of these lines is "small". The stronger statement (corollary 4) follows from conjecture 17.*

**3. Ritt theorem.** The theorems 14, 15 are related to some algebra of ESs discovered by Ritt [3].

**Theorem 16** (Ritt theorem)**.** *Let $f, g$ be* ESs *of one variable. Assume that $f/g$ is an entire function. Then $f/g$ is an* ES.

The theorem 16 has a straightforward multidimensional generalization [4]. There are many publications dedicated to the 1-dimensional case (for example [5, 6, 7]).

**Conjecture 17.** *Let $f, g$ be* ESs *of one variable without non trivial common divisor in the ring of* ES*. Then the set of zeroes of the system $f = g = 0$ is finite.*

It looks like the conjecture 17 was known a long time ago but was never published. The converse statement to the conjecture 17 is obvious (follows from 1-dimensional BKK). Let us state three multidimensional corollaries of the conjecture 17.

**Corollary 4.** *The set of non-discrete components from theorem 15 is finite.*

Let $f, g$ be quasialgebraic ESs without common divisors and let the $Z$ be a set of their imaginary common zeroes. The finiteness of $Z$ follows from conjecture 17 (and is also not proved yet). It is equivalent to the following statement.

**Corollary 5.** *Let $f, g$ be trigonometric polynomials on a real $n$-dimensional torus $T^n$ and let $X$ be a set of their common zeroes. Suppose that* $\operatorname{codim}_{\mathbb{R}} X = 2$*. Then for any dense geodesic line $\xi$ on $T^n$ the set $\xi \cap X$ is finite.*

**Corollary 6.** *Let the $X$ be a set of zeroes of the trigonometric polynomial $g$ and $G$ be a dense cyclic subgroup of the real $n$-dimensional torus $T^n$. Then the set $G \cap X$ is finite.*

The corollary 6 follows from the so-called "Mordell–Lang conjecture" (which in fact is a theorem, in contrast to its title).

**4. Mordell-Lang conjecture.** Recall the formulation of the Mordell-Lang conjecture for a complex torus (for example [12]). An abelian group $G$ is said to be a group of finite rank if exists a finitely generated subgroup $H$ of $G$, such that $G/H$ is a torsion group. Let $A$ be an algebraic subvariety of $(\mathbb{C}^*)^n$. Let $A'$ denote the union of all shifted subtori contained in $A$. Mordell-Lang conjecture: **let $G$ be a subgroup of $(\mathbb{C}^*)^n$ of a finite rank. Then the set $(A \setminus A') \cap G$ is finite.**

Present one more conjecture generalizing both the conjecture 4 and the Mordell–Lang conjecture.

**Definition 3.** *Let $A \subset (\mathbb{C}^*)^n$ be an irreducible algebraic variety and $x \in A$. Let* $\mathrm{T}$ *be a proper subtorus of* $(\mathbb{C}^*)^n$ *of nonzero dimension and* $X_{x,\mathrm{T}}$ *be an irreducible component of* $A \cap (x\,\mathrm{T})$*, contained the point* $x$*. The point* $x$ *is said to be anomalous if* $\exists\,\mathrm{T}\colon\ \dim X_{x,\mathrm{T}} > \max(0, \dim \mathrm{T} + \dim A - n)$.

**Theorem 18** (Bombieri, Masser, Zannier, [11])**.** *The set of anomalous points of* $A$ *is an algebraic subvariety.*

Let $A \subset (\mathbb{C}^*)^n$ be an algebraic variety of codimension $> k$ and let $A'$ be a subvariety of anomalous points of $A$.

**Conjecture 19.** *Let* $L$ *be a* $k$*-dimensional subspace of* $\mathbb{C}^n$ *and let* $\exp(L) \subset G$*, where* $G$ *is a subgroup of* $(\mathbb{C}^*)^n$*, such that the quotient* $G/\exp(L)$ *is a group of finite rank. Then the set* $(A \setminus A') \cap G$ *is finite.*

For $k = 0$ this statement coincides with the Mordell-Lang conjecture. For $k = 1$ this statement gives the strengthening of conjecture 17.

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## On asymptotic behavior of multiplicities of reductive group actions

Takuya Murata

Let $G$ be a connected reductive algebraic group over an algebraically closed field $k$ of characteristic zero, $V$ a finite-dimensional $G$-module and $X = \mathrm{Proj}(A) \subset \mathbf{P}(V) = \mathbf{P}^N$ be a closed invariant subvariety.

Choose a maximal torus and a Borel subgroup $T \subset B$ in $G$ and let $U$ be the unipotent radical of $B$. Let $\Lambda^+$ be the intersection of the character group $\mathrm{Hom}(T, \mathbb{G}_m)$ with the positive closed Weyl chamber (in say $\mathbb{R}^{\dim T}$).

For each $\lambda \in \Lambda^+$, let $V^\lambda$ be the corresponding irreducible module and let (using the Frobenius reciprocity)

$$m_{A_l}(\lambda) = \dim \mathrm{Hom}_G(V^\lambda, A_l) = \dim A^U_{l,\lambda},$$

that is, the number of times $V^\lambda$ appears in the $l$-th degree piece of $A$.

Our work concerns the asymptotic behavior of $m_{A_l}(\lambda)$. More precisely, we have two results: one when $G$ is a torus and the other when $G$ is reductive in general. The first one generalizes [Br91] (except $G$ finite) and the second recovers his.

**Theorem 20.** *Let $R \subset \oplus_{l=0}^\infty H^0(X, \mathcal{O}_{\mathbf{P}^N}(l)|_X)$ be a graded subalgebra (need not be finitely generated). Assume, for large $l$, a generic orbit in* $\operatorname{Spec} R_l$ *is closed and a generic stabilizer in* $\operatorname{Spec} R_l$ *is trivial. ("generic" can be made more precise.) If $m_l(\lambda) \neq 0$ for large $l$, then, for any $\lambda \in \Lambda^+$,*

$$\lim_{l\to\infty} m_{R_l}(\lambda)/l^d = \deg(X//G)/d!$$

*where $d = \dim X//G = \dim \mathrm{Proj}(R^G)$.*

For any $\lambda \in \Lambda^+$, we put $X//_\lambda G = (G/P_\lambda \times X)//G$. As $(V^\lambda)^* = H^0(G/P_\lambda, L_\lambda)$, $L_\lambda$ line bundle with linearlization corresponding to $\lambda$,[AB04] we have (cf. Ch. II, Exercise 5.11. in [Hart77])

$$X//_\lambda G = \mathrm{Proj}(\oplus_{l=0}^\infty (V^{l\lambda^*} \otimes A_l)^G) = \mathrm{Proj}(A^U_{l,l\lambda}).$$

Moreover, if $k = \mathbb{C}$ and if $\lambda$ is a regular value of the moment map for the complex projective space into which $X$ is embedded, then by the Kempf-Ness theorem $X//_\lambda G$ can be identified with the symplectic reduction of $X$ at $\lambda$.

We write $\triangle_Z(L)$ for the Okounkov body of a line bundle $L$ on a projective variety $Z$. Its volume is the self-intersection number of $L$. For simplicity, we also write $\mathrm{vol}_n(Z) = \mathrm{vol}_n(\triangle_Z(\mathcal{O}_Z(1)))$. If $k = \mathbb{C}$, $\mathrm{vol}_n(Z)$ is the symplectic volume of the regular locus of $Z$ (which is a symplectic manifold with the symplectic form restricted from the ambient projective space).

We also let $X^{[s]} = \mathrm{Proj}(\oplus_{l=0}^\infty A_{ls})$ (and use the similar notation for any other projective schemes over $k$.) Let $q = \dim G/P_\lambda$. We need to consider the following continuity condition: as $s \to \infty$

$$(*)\; s^q \,\mathrm{vol}_{d+q}(X^{[s]}//_\lambda G) \sim (\dim V^\lambda)\,\mathrm{vol}_d(X//G).$$

This is not a severe condition. For example, if $k = \mathbb{C}$, it is equivalent to:

$$\frac{\mathrm{vol}_{d+q}(X//_{\lambda/s}G)}{\mathrm{vol}_q(O_{\lambda/s})} \to \mathrm{vol}_d(X//G),$$

which is, if $G$ is a torus, the Duistermaat-Heckman theorem.

With this additional assumption, Theorem 20 continues to hold (which is precisely the one in [Br91].)

**Theorem 21.** *Assume (*) and assume, for large $l$, a generic orbit in* $\operatorname{Spec} A_l$ *is closed and a generic stabilizer in* $\operatorname{Spec} A_l$ *is trivial. If* $m_l(\lambda) \neq 0$ *for large $l$, then, for any* $\lambda \in \Lambda^+$,

$$\lim_{l \to \infty} m_{A_l}(\lambda)/l^d = \dim(V^\lambda) \deg(X//G)/d!.$$

We also note that the theorem says equivalently (under the continuity): $m_{A_l}(\lambda) \sim \operatorname{vol}_{d+q}(X^{[s]}//_\lambda G)$. In this form, it is a sort of the special case of asymptotic Riemann-Roch. (In fact, it may be mentioned that the theorem is also a consequence of Meinrenken's Riemann-Roch for multiplicity.)

## 1. Proofs

The proof of the torus case is based on a nice simple convex-geometry observation: given a convex compact set $\triangle \subset \Lambda \times \mathbb{Z}^d$, let $m_l(\lambda)$ denote the number of integral points lying above $\lambda$ and in $l\triangle$; i.e., $m_l(\lambda)$ is the cardinality of the fiber $S_{l,\lambda} = l\triangle \cap (\lambda + \mathbb{Z}^d)$ of the coordinate projection of $S = l\triangle \cap \mathbb{Z}^N$ over $(l, \lambda)$. If 0 is in the interior of $\triangle$, then, clearly, $m_l(\lambda) \sim \operatorname{vol}_l(l\triangle \cap (\lambda + \mathbb{R}^d)) \sim \operatorname{vol}_l(l\triangle \cap \mathbb{R}^d)$. The boundary case (i.e., 0 is on the boundary) is much tricker.

For the reductive case, the main issue is of dimension. (We also do not know of a nice convex-geometry picture, a counterpart to the torus case.) In his original work, Okounkov considered the map (induced by a linear map) $p : \triangle_X \to P$ where $P$ is the moment polytope of $X$ and $\triangle_X$ is the Okounkov body of $X$. $p$ plays a role of a moment map if our group $G$ is a torus: $P$ lives in the Euclidean space of dimension equal to the dimension of a maximal torus and $\dim \triangle_X = \dim X$. On the other hand, the moment map with respect to a maximal compact subgroup has target in the Euclidean space of dimension equal to the dimension of $G$. (The main role of the continuity hypothesis in the theorem is to correct this dimension discrepancy.)

Given the above, we introduce the semigroup homormphism

$$\beta : \Lambda^{+'} \to \Lambda^+$$

where $\Lambda^{+'} = v'(k[G]^U - 0)$ some valuation $v'$ (we can use the construction at [Ka1X] but for the computational purpose, a less explicit valuation would do.)

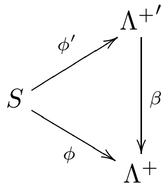
Let $\phi$ be the projection

$$S \overset{\text{def}}{=} \{(l, \lambda, v(f)) | 0 \neq f \in A^U_{l,\lambda}\} \to \Lambda^+.$$

($v$ is a valuation related to $v'$.)

The reductive case then follows (non-trivially) from the next proposition:

**Proposition 22.** *$\phi$ factors through $\beta$:*

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## Divisors on Bott-Samelson varieties

Dave Anderson

The cone of (pseudo)effective divisors on a nonsingular projective variety $X$ plays a key role in the study of the birational geometry of $X$. In general, however, this invariant is quite difficult to compute explicitly. An exception is a situation that occurs frequently in applications to representation theory: if $X$ is equipped with the action of a connected solvable group $B$ which has a dense orbit $U$, then the irreducible components of $X \setminus U$ generate the effective cone $\mathrm{Eff}(X)$. (More generally, $\mathrm{Eff}(X)$ is generated by $B$-invariant divisors, although there may be infinitely many of them. One applies Sumihiro's theorem [6, 4] to linearize any given line bundle, and then the Lie-Kolchin theorem to find a semi-invariant section whenever there is a nonzero section.)

In this talk, I explain how to compute the effective cone of a *Bott-Samelson variety* $X = X(\alpha_1, \ldots, \alpha_d)$ associated to a reductive (or Kac-Moody) group and an arbitrary sequence of simple roots $\alpha_i$. When the sequence is *reduced*, meaning that the corresponding map to the flag variety is birational onto its image, there is a dense $B$-orbit and $X$ falls into the "easy" situation described above; in fact, the generators are quite simple to describe. However, in general there will not be a dense orbit and the problem is nontrivial.

Before stating the theorem, it is worth pointing out that the nef cone of $X$ has a simple description for any sequence of simple roots: one construction of $X$ realizes it as an iterated tower of $\mathbb{P}^1$-bundles, and Lauritzen and Thomsen show that the $d$ universal line bundles $\mathcal{O}(1)$ for these bundles generate both the Picard group and the nef cone [5]. On the other hand, an argument similar to the one given in [1] produces a divisor $\Delta$ such that $(X, \Delta)$ is log Fano, so that one knows $\mathrm{Eff}(X)$ is finitely generated [3].

To describe the theorem, let $G$ be a reductive (or Kac-Moody) group, with Borel subgroup $B$ and maximal torus $T$, and let $W$ be the corresponding Weyl group.

Given any simple root $\alpha$, let $P_\alpha \supset B$ be the corresponding minimal parabolic. Given a sequence of simple roots $\underline{\alpha} = (\alpha_1, \ldots, \alpha_d)$, the Bott-Samelson variety is

$$X(\underline{\alpha}) = P_{\alpha_1} \times^B P_{\alpha_2} \times^B \cdots \times^B P_{\alpha_d}/B,$$

and we can write points of $X = X(\underline{\alpha})$ as $[p_1, \ldots, p_d]$ with $p_i \in P_{\alpha_i}$. For $1 \leq i \leq d$ there are *standard divisors* $X_i$ defined by requiring $p_i = e$; these are also Bott-Samelson varieties, since one evidently has $X_i \cong X(\alpha_1, \ldots, \widehat{\alpha}_i, \ldots, \alpha_d)$. It is not hard to see that the $X_i$ give another set of generators for the Picard group of $X$, and their union forms a normal crossings divisor in $X$.

For each $i$ there are maps $\phi_i \colon X \to G/B$, defined by sending $[p_1, \ldots, p_d]$ to $p_1 \cdots p_i B$. Define $w_i \in W$ to be such that $\phi_i(X) = X(w_i) = \overline{Bw_iB/B} \subseteq G/B$. (Since $\phi_i$ is $B$-equivariant and proper, one knows its image is equal to some Schubert variety.) These elements can also be described as the *Demazure product* $w_i = s_{\alpha_1} \star \cdots \star s_{\alpha_i}$. In particular, each $w_i$ is equal to the greater (in Bruhat order) of $w_{i-1}$ or $w_{i-1}s_{\alpha_i}$.

Finally, there are also maps $\pi_i \colon X \to X(\alpha_1, \ldots, \alpha_{i-1})$, defined by sending $[p_1, \ldots, p_d]$ to $[p_1, \ldots, p_{i-1}]$.

**Theorem.** *The effective cone* $\mathrm{Eff}(X)$ *is generated by* $X_1, \ldots, X_d$*, together with a divisor* $D_i$ *for each* $i$ *such that* $w_i = w_{i-1}$*. This divisor is the unique irreducible component of* $\phi_i^{-1} X(w_i s_{\alpha_i})$ *which* $\pi_i$ *maps surjectively onto* $X(\alpha_1, \ldots, \alpha_{i-1})$*.*

The proof deduces the statement for Bott-Samelson varieties from a more general recipe describing $\mathrm{Eff}(Y)$ in terms of $\mathrm{Eff}(Z)$ in case $Y \to Z$ is a certain type of $\mathbb{P}^1$ bundle.

In addition to what being somewhat interesting in their own right, Bott-Samelson varieties for non-reduced words appear naturally in several situations. They arise in resolving singularities of Richardson varieties (e.g., [2] for the type A case), and even restricting to $G = SL_3$, they are parameter spaces for certain point-line configurations in the projective plane.

Finally, even when one starts with a reduced word $\underline{\alpha}$, most of the standard divisors $X_i \subseteq X(\underline{\alpha})$ are Bott-Samelson varieties for non-reduced words, so to take full advantage of the recursive structure of $X(\underline{\alpha})$ one is forced to consider such words. I envision an inductive approach to computing the global Okounkov cone of a Bott-Samelson variety which uses the above Theorem as a crucial ingredient.

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