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Stochastic Analysis: Around the KPZ Universality Class

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Abstract. The Gaussian distribution is the "universal" distribution arising in a huge variety of contexts that describes the compound effect of the random fluctuations of many independent (or weakly dependent) sources of randomness that are combined in a (close to) additive way. While this has been very well understood for a long time, the last few years have seen an explosion of results around the "KPZ universality class", which contains many systems where strongly interacting individual components are combined in a highly non-linear way. In this class, which is still rather poorly understood from a mathematical perspective, fluctuations typically exhibit scaling exponent 1/3 instead of the exponent 1/2 familiar from the central limit theorem and limiting distributions are of Tracy-Widom type rather than Gaussian.

This workshop brought together outstanding researchers from a variety of mathematical backgrounds whose areas of research are linked to the understanding of the KPZ equation and universality class. While there are strong links between their motivations, the techniques used by these researchers span a large swath of mathematics, ranging from purely algebraic techniques to renormalisation theory, stochastic analysis, random matrix theory, classical probability theory, orthogonal polynomials, the theory of rough paths, etc.

Mathematics Subject Classification (2010): 60xx, 35xx, 82xx.

Introduction by the Organisers

The workshop focused on the latest progresses in the study of the Khardar-Parisi-Zhang (KPZ) equation and its universality classes. In recent years, new important models were shown to belong to this KPZ class thanks in particular to integrable

systems and orthogonal polynomials, approaches to universality florished, in particular in random matrix theory, and the rigorous construction of the solution to KPZ equation was achieved thanks to Malliavin calculus and rough paths analysis.

Most of the conference was devoted to the study of exact models. In the first talk, H. Widom revisited his breakthrough paper with C. Tracy and discussed the analysis of the asymmetric simple exclusion process and how to transform a priori untractable formulas by wise uses of complex analysis and changes of contours of integration. A. Borodin surveyed the range of known exact models in the KPZ class which could be solved, and emphasized the role of Fourier analysis and orthogonal polynomials, in particular those introduced by Mac Donald.

Later in the week, I. Corwin showed in great detail how one of these models, the q-TASEP, could be analyzed either by using MacDonald processes or a new duality approach, both leading to the exact computation of natural observables. V. Gorin talked about new applications of these techniques to the six-vertex model. P. Ferrari showed on the last day how the study of these exact models, and in particular the so-called q-Whittaker process, allowed to get an exact formula for the one point correlation function of the solution to the KPZ equation. In the same vein, A. Hammond described a new approach to KPZ equation as the lowest indexed curve of an N-ensemble of curves, with a property to penalize, but not completely forbid ding crossing: this ensemble is integrable and offers a powerful tool for the analysis of KPZ equation. P. Le Doussal explain the physics approach to these questions by replicas and Bethe ansatz. It was discussed by D. Remenik that putting some of the results discussed by Le Doussal on a firm mathematical ground can be a real challenge, for instance for the model of random growth off a flat surface where justifying some formal critical point analysis is yet to be done. S. Chiita and K. Johansson discussed the two-periodic weighting of the Aztec diamond: its asymptotics were described by Kenyon and Okounkov and the authors undertake the analysis of the fluctuations and the two points correlations functions of this model in the different phases, by exhibiting new exact formulas. Event though pairwise correlations at the solid-liquid and liquid-gas region can be analyzed, the liquid-gas phase remains a challenge due to non-local phenomenon. Other models could be analyzed and shown to be related to the KPZ equation. This is the case of weakly interacting particle systems presented by P. Gonçalves, or directed polymers as shown by T. Seppäläinen. Other universal behaviors also appear for the latter model, as discussed by K. Khanin.

During the first morning, random matrices and universality were introduced by P. Bourgade, who showed how recent techniques based on Dyson Brownian motion and homogenization could be used to prove fixed energy universality of the correlation function. A. Knowles discussed the phase diagram for the mesoscopic phase diagram of d-dimensional random band matrices, the appearance of the Altshuler-Shklovskii and its universality. B. Virág described the Dirac operator, a random first order differential operator, and showed that the local statistics of its spectrum in the bulk converges to the same limit as those of β-ensemble. For $\beta = 2$, the eigenvalues are conjectured to behave similarly to the imaginary

parts of the zeroes of the Riemann ζ-function. B. Valkó presented matrix analogue to Dufresne identities. O. Zaboronski showed that Pfaffian point processes, which usually appear in random matrix models, make a surprise appearance when studying annihilating Brownian motions in one dimension.

On topics related to the well-posedness and interpretation of the KPZ equation itself, P. Friz gave an introduction to rough path analysis and stressed their importance to define and study the KPZ equation. M. Gubinelli described the slightly different approach based on paracontrolled analysis. D. Khoshnevisan discussed stochastic differential equation with a constant or linear diffusivity, presenting intermittency and multi-fractality, and Y. Bakhtin exposed a recent result on the uniqueness of stationary solutions to the forced Burgers equation on the whole real line which is conjectured to belong to the KPZ universality class.

A special occasion, very nice and enjoyable, was also the Oberwolfach prize ceremony that took place on Thursday late afternoon, and was followed by a nice dinner (and special thanks go to the Oberwolfach staff that made us all feel being in a luxury restaurant on that evening). The winner of the prize, Hugo Duminil-Copin who was also a participant to this workshop, gave a talk on the nature of the phase transition of the three-dimensional Ising model (joint work with Aizenmann and Sidoravicius), which is a topic close to the heart of many participants of this workshop (hence also the choice of the week of this workshop to have this ceremony).

The role of the organizers was to keep the number of talks to a fairly low number (22 all together) in order to leave as much time as possible for informal sessions and discussions between participants, and to look at the weather forecast to advance the traditional mid-week hike (this time to Brandenkopf) to Tuesday afternoon instead of the indeed very rainy Wednesday.

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Workshop: Stochastic Analysis: Around the KPZ Universality Class

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Abstracts

An ASEP Operator

Harold Widom
(joint work with Craig A. Tracy)

For the asymmetric simple exclusion process (ASEP) on the integers with step initial condition (particles initially at the positive integers $\mathbb{Z}^+$), there is a formula for the distribution function for $x_m(t)$, the position of the mth particle from the left at time t, which involves a Fredholm determinant [2]. It is

$$\mathbb{P}\left(x_m(t) \le x\right) = \frac{1}{2\pi i} \int \frac{\det(I - \lambda K)}{\prod_{k=0}^{m-1}(1 - \lambda\,\tau^k)} \frac{d\lambda}{\lambda}.$$

Here $\tau = p/q$, where p is the jump probability to the right and $q = 1 - p$ the jump probability to the left, and K is the operator acting on functions on $\mathcal{C}_R$ (the circle with center zero and large radius R) with kernel

$$K(\xi,\, \xi') = q\, \frac{\xi^x\, e^{\,(p/\xi + q\xi - 1)\,t}}{p + q\xi\xi' - \xi}.$$

We assume $\tau < 1$ and the contour of integration encloses all poles of the integrand.

For asymptotics as $m,\, x,\, t \to \infty$, the operator K is not good. First, replacing t by $t/(q-p)$ and changing variables, we replace $K(\xi, \xi')$ by

$$K_2(\eta,\, \eta') = \frac{\varphi(\eta')}{\eta' - \tau\eta},$$

acting on a little circle around $\eta = 1$, where

$$\varphi(\eta) = \left(\frac{1 - \tau\eta}{1 - \eta}\right)^x\, e^{\left[\frac{1}{1-\eta} - \frac{1}{1-\tau\eta}\right] t}.$$

This is still not good, since nothing essentially changed. But then, using two lemmas on stability of Fredholm determinants, we can replace $K_2(\eta, \eta')$ by $K_1(\eta, \eta') - K_2(\eta, \eta')$ acting on $\mathcal{C}_r$ with $1 < r < 1/\tau$, where

$$K_1(\eta,\, \eta') = \frac{\varphi(\tau\eta)}{\eta' - \tau\eta}.$$

From the nice fact

$$\det(I - \lambda K_1) = \prod_{k=0}^{\infty} (1 - \lambda\tau^k)$$

we then obtain

$$\mathbb{P}(x_m(t/(q-p)) \le x) = \frac{1}{2\pi i} \int \prod_{k=m}^{\infty} (1 - \lambda\,\tau^k)\, \det(I + \lambda K_2\,(I + R))\, \frac{d\lambda}{\lambda},$$

where $R = \lambda K_1 (I - \lambda K_1)^{-1}$. The right side may be replaced by

$$\int \prod_{k=0}^{\infty} (1 - \mu \tau^k) \det(I + \mu \tau^{-m} K_2 (I + R)) \frac{d\mu}{\mu},$$

where we set $\lambda = \mu \tau^{-m}$ with $|\mu|$ now fixed. Luckily the kernel of the resolvent oprator R is computable. Using it and the stability lemmas we find that we may replace $\tau^{-m} K_2 (I + R)$ in the integrand by the operator J with kernel

$$J(\eta, \eta') = \frac{1}{2\pi i} \int \frac{\varphi_\infty(\zeta)}{\varphi_\infty(\eta')} \frac{\zeta^m}{{\eta'}^{m+1}} \frac{f(\mu, \zeta/\eta')}{\zeta - \eta} d\zeta$$

acting on $\mathcal{C}_r$ with $\tau < r < 1$, where for the integration $1 < |\zeta| < r/\tau$. Here

$$\varphi_\infty(\eta) = \prod_{n=0}^{\infty} \varphi(\tau^n \eta) = (1 - \eta)^{-x} e^{\frac{\eta}{1-\eta} t}$$

and

$$f(\mu, z) = \sum_{k=-\infty}^{\infty} \frac{\tau^k}{1 - \mu \tau^k} z^k \qquad (1 < |z| < 1/\tau).$$

Now all the parameters m, x, and t are in the function $\varphi_\infty(\zeta)\, \zeta^m$, which appears in both parts of a quotient. This allows a steepest descent analysis that establishes F_2 fluctuations [3]. The result for TASEP had already been known [1].

The same path yields fluctuations for step-Bernoulli initial condition, where sites in $\mathbb{Z}^+$ are initially occupied with probability ρ [4]. The only difference is that the function φ_∞ gets an extra factor, which involves ρ. Depending on the relation between ρ and τ, the fluctuations are either Gaussian, F_1^2, or F_2.

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Homogenization and universality for Wigner matrices

Paul Bourgade

(joint work with L. Erdős, H.-T. Yau, J. Yin)

E. Wigner envisioned that energy levels of large quantum systems remarkably exhibit universality patterns. He introduced the Wigner matrix ensemble (normalized centered entries, independent up to symmetry), and postulated that the statistics of the eigenvalues depend only on the symmetry class of the model and are independent of the details of the ensemble. A few years after, Gaudin, Mehta

and Dyson computed explicitly the eigenvalue correlation functions for the case of Gaussian entries. Later on, Mehta formalized a version of the (Wigner-Dyson-Mehta) universality conjecture [6] by stating that the appropriately rescaled correlation functions for any Wigner ensemble coincide with those for the Gaussian cases as N, the size of the matrix, tends to infinity. This holds for both real symmetric and complex Hermitian ensembles.

One possible topology for the correlation functions is the pointwise convergence. But the convergence in this topology cannot hold for Wigner ensembles with discrete (e.g. Bernoulli) matrix elements. Thus a reasonably strong topology suitable for the universality of the whole class of Wigner matrices is the vague convergence of the local correlation functions, rescaled around a fixed energy E. Instead of fixing the energy E, we can also take weak convergence in E or equivalently, taking some average in the energy. More precisely, the following are two possible universal statements, in which random matrices have subgaussian entries and $\rho_k^{(N)}$ classically means their eigenvalues correlation functions.

Fixed energy universality. For any $k \geq 1$, $F : \mathbb{R}^k \to \mathbb{R}$ continuous and compactly supported and $\kappa > 0$, uniformly in $E \in (-2+\kappa, 2-\kappa)$ we have

$$\lim_{N\to\infty} \frac{1}{\varrho(E)^k} \int \mathrm{d}\mathbf{v}\mathbf{F}(\mathbf{v})\rho_{\mathbf{k}}^{(\mathbf{N})}\left(\mathbf{E} + \frac{\mathbf{v}}{\mathbf{N}\varrho(\mathbf{E})}\right) = \int \mathrm{d}\mathbf{v}\mathbf{F}(\mathbf{v})\rho_{\mathbf{k}}^{(\mathrm{GOE/GUE})}(\mathbf{v})\,.$$

Averaged energy universality. For any $k \geq 1$, $F : \mathbb{R}^k \to \mathbb{R}$ continuous and compactly supported, arbitrarily small $\varepsilon, \kappa > 0$ and $s = N^{-1+\varepsilon}$, uniformly in $E \in (-2+\kappa, 2-\kappa)$ we have

$$\lim_{N\to\infty} \frac{1}{\varrho(E)^k} \int_E^{E+s} \frac{\mathrm{dx}}{s} \int \mathrm{d}\mathbf{v}\mathbf{F}(\mathbf{v})\rho_{\mathbf{k}}^{(\mathbf{N})}\left(\mathbf{x} + \frac{\mathbf{v}}{\mathbf{N}\varrho(\mathbf{E})}\right)\mathrm{d}\mathbf{v} = \int \mathrm{d}\mathbf{v}\mathbf{F}(\mathbf{v})\rho_{\mathbf{k}}^{(\mathrm{GOE/GUE})}(\mathbf{v})\,.$$

The Wigner-Dyson-Mehta conjecture has been open until the recent work [1] where a general scheme to approach it was outlined and carried out for complex Hermitian matrices. The first step is to establish a local version of the semicircle law and use it as an input to control the correlation function asymptotics in the Brezin-Hikami formula (which is related to Harish-Chandra/Itzykson-Zuber formula). This provides universality for the so-called Gaussian divisible models with a very small Gaussian component, or "noise". Previously, this universality was established by Johansson [5] when the noise is of order one. The last step is an approximation of a general Wigner ensemble by Gaussian divisible ones and this leads to the fixed energy universality for Hermitian Wigner matrices whose matrix elements have smooth distributions. The various restrictions on the laws of matrix elements were then greatly relaxed [7, 2]. In particular, using the local semicircle law [3] as a main input, Tao-Vu [7] proved a comparison theorem which provides an approximation result for Wigner matrices satisfying a "four moment matching condition".

For real symmetric matrices, no algebraic formula in the spirit of Brezin-Hikami is known. A completely new method based on relaxation of the Dyson Brownian

Motion (DBM) to local equilibrium was developed in a series of papers including [4]. This approach is very robust but yields only the averaged energy universality.

Our talk explained the proof of the last remaining case of the Wigner-Dyson-Mehta conjecture, i.e. universality of local correlation functions at any fixed energy E in the bulk spectrum for Wigner matrices of any symmetry classes.

Theorem 1. *For symmetric or Hermitian matrices from the Wigner ensemble, fixed energy universality holds in the bulk of the spectrum.*

In particular the following three results hold for real matrices, including the Bernoulli case: (1) existence of the density of states on microscopic scales for Wigner matrices, (2) the extension of the Jimbo-Miwa-Mori-Sato formula of the gap probability to Wigner matrices, (3) the precise distribution of the condition number or the smallest (in absolute value) eigenvalue of Wigner matrices.

The proof relies on a homogenization theory for the discrete parabolic equation with time dependent random coefficients $B_{ij}(t) = (x_i(t) - x_j(t))^{-2}$, where $\mathbf{x}(\mathbf{t})$ is a typical DBM trajectory. By a rigidity property of the DBM trajectories, the random coefficients are close to deterministic ones, $B_{ij}(t) \approx (\gamma_i - \gamma_j)^{-2}$ if $|i-j| \gg 1$ (the typical locations γ_i are the quantiles of the semicircle distribution). The continuous version of the corresponding heat kernel is explicitly known. By coupling two DBM for two different initial conditions $\mathbf{x}(\mathbf{0})$ and $\mathbf{y}(\mathbf{0})$ (one for Wigner, one for a reference Gaussian ensemble), we show that after a sufficiently long time, the difference between $x_i(t)$ and $y_i(t)$ is given by the deterministic heat kernel acting on the difference of the initial data. Due to the scaling properties of the explicit heat kernel, this latter involves only mesoscopic linear statistics of the initial conditions which are more accessible than microscopic ones. Homogenization thus enables us to transfer mesoscopic statistics to microscopic ones.

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KPZ and integrability

Alexei Borodin

The goal of the talk is to survey the emerging field of integrable probability, whose goal is to identify and analyze exactly solvable probabilistic models. The models and results are often easy to describe, yet difficult to find, and they carry essential information about broad universality classes of stochastic processes.

Basics of rough paths

Peter K. Friz

Differential equations are omnipresent in modern pure and applied mathematics; many "pure" disciplines in fact originate in attempts to analyse differential equations from various application areas. Classical ordinary differential equations (ODEs) are of the form $\dot{Y}_t = f(Y_t, t)$; an important sub-class is given by *controlled ODEs* of the form

$$\dot{Y}_t = f_0(Y_t) + f(Y_t)\dot{X}_t \ , \tag{1}$$

where X models the input (taking values in R^d, say), and Y is the output (in R^e, say) of some system modelled by nonlinear functions f_0 and f, and by the initial state Y_0. The need for a non-smooth theory arises naturally when the system is subject to white noise, which can be understood as the scaling limit as $h \to 0$ of the discrete evolution equation

$$Y_{i+1} = Y_i + hf_0(Y_i) + \sqrt{h} f(Y_i)\xi_{i+1} \ , \tag{2}$$

where the (ξ_i) are i.i.d. standard Gaussian random variables. Based on martingale theory, Itô's *stochastic differential equations* (SDEs) have provided a rigorous and extremely useful mathematical framework for all this. And yet, stability is lost in the passage to continuous time: Taking $\dot{X} = \xi$ to be white noise in time (which amounts to say that X is a Brownian motion, say B), the solution map $S\colon B \mapsto Y$ to (1), known as *Itô map*, is a measurable map which in general lacks continuity, whatever norm one uses to equip the space of realisations of B. In a sense, solving SDEs is an analytically ill-posed task! On the other hand, there are well-known probabilistic well-posedness results for SDEs of the form

$$dY_t = f_0(Y_t)dt + f(Y_t) \circ dB_t \ , \tag{3}$$

which imply for instance statements of the following form: Let $\xi_\epsilon = \delta_\epsilon * \xi$ denote the regularisation of white noise in time with a compactly supported smooth mollifier δ_ϵ and let Y^ϵ be the solutions to 1 driven by $\dot{X} = \xi_\epsilon$. Then Y^ϵ converges in probability (uniformly on compact sets). The limiting process does not depend on the choice of mollifier δ_ϵ, and in fact is the Stratonovich solution to (3).

There are many variations on such "Wong–Zakai" results, another popular choice being $\xi_\epsilon = \dot{B}^{(\epsilon)}$ where $B^{(\epsilon)}$ is a piecewise linear approximation (of mesh size $\sim \epsilon$) to Brownian motion. However, as consequence of the aforementioned lack of

continuity of the Itô-map, there are also reasonable approximations to white noise for which the above convergence fails. Perhaps rather surprisingly, it turns out that (analytical) well-posedness is restored via certain iterated integrals which are in fact the only data that is missing to turn S into a continuous map.

Rough path analysis introduced by Terry Lyons in the seminal article [3] and by now exposed in several monographs [4, 5, 1, 2], provides the following remarkable insight: Itô's solution map can be factorised into a measurable "universal" map Ψ and a "nice" solution map $\hat{S}$ as

$$B(\omega) \overset{\Psi}{\mapsto} (B, \mathbb{B})(\omega) \overset{\hat{S}}{\mapsto} Y(\omega). \tag{4}$$

The map Ψ is universal in the sense that it depends neither on the initial condition, nor on the vector fields driving the stochastic differential equation, but merely consists of *enhancing* Brownian motion with iterated integrals of the form

$$\mathbb{B}^{i,j}(s,t) = \int_s^t \big(B^i(r) - B^i(s)\big)\, dB^j(r)\ . \tag{5}$$

At this stage, the choice of stochastic integration (e.g. Itô or Stratonovich) *does* matter and probabilistic techniques are required for the construction of Ψ. Indeed, the map Ψ is only measurable and usually requires the use of some sort of stochastic integration theory.

The solution map $\hat{S}$ on the other hand, the solution map to a *rough differential equation (RDE)*, also known as *Itô–Lyons map* is purely deterministic and only makes use of analytical constructions. More precisely, it allows input signals to be arbitrary *rough paths* which are objects (thought of as *enhanced paths*) of the form $(X, \mathbb{X})$, defined via certain algebraic properties (which mimic the interplay between a path and its iterated integrals) and certain analytical, Hölder-type regularity conditions. These conditions can be seen to hold true a.s. for $(B, \mathbb{B})$; a typical realisation is thus called *Brownian rough path.*

The Itô–Lyons map turns out to be "nice" in the sense that it is a continuous map of both its initial condition and the driving noise $(X, \mathbb{X})$, provided that the dependency on the latter is measured in a suitable "rough path" metric. In other words, rough path analysis allows for a pathwise solution theory for (multidimensional) SDEs i.e. for a fixed realisation of the Brownian rough path. The solution map $\hat{S}$ is however a much richer object than the original Itô map, since its construction is completely independent of the choice of stochastic integral and even of the knowledge that the driving path is Brownian. For example, if we denote by Ψ^I (resp. Ψ^S) the maps $B \mapsto (B, \mathbb{B})$ obtained by Itô (resp. Stratonovich) integration, then we have the almost sure identities

$$S^I = \hat{S} \circ \Psi^I\ , \qquad S^S = \hat{S} \circ \Psi^S\ , \tag{6}$$

where S^I (resp. S^S) denotes the solution to (3) interpreted in the Itô (resp. Stratonovich) sense. As a consequence, the afore-mentioned Wong–Zakai type results are really a deterministic consequence of the probabilistic question whether or not $\Psi^S(B^\epsilon) \to \Psi^S(B)$ in probability and rough path topology, with $\dot{B}^\epsilon = \xi^\epsilon$.

So how is this Itô–Lyons map $\hat{S}$ built? In essence, one has to make sense of the expression

$$\int_0^t f(Y_s)\, dX_s \ , \tag{7}$$

where Y is itself the as yet unknown solution. Here is where the usual pathwise approach breaks down. Actually, if we measure regularity in terms of Hölder exponents, then the integral makes sense as a limit of Riemann sums for X and Y that are arbitrary α-Hölder continuous functions if and only if $\alpha > \frac{1}{2}$. The keyword here is *arbitrary*: in our case the function Y is anything but arbitrary! Actually, one would expect the small-scale fluctuations of Y to look exactly like the small-scale fluctuations of X in the sense that one would expect that

$$Y_{s,t} = f(Y_s)X_{s,t} + R_{s,t}$$

where, for any path F with values in a linear space, we set $F_{s,t} = F_t - F_s$, and where $R_{s,t}$ is some remainder that one would expect to be "of higher order".

Suppose now that X is a "rough path", which is to say that it has been "enhanced" with a two-parameter function $\mathbb{X}$ which should be interpreted as giving the values for

$$\mathbb{X}^{i,j}(s,t) = \int_s^t X^i_{s,r}\, dX^j_r \ . \tag{8}$$

Note here that this identity should be read in the reverse order from what one may be used to: it is the right hand side that is *defined* by the left hand side and not the other way around! The idea here is that if X is too rough, then we do not a priori know how to define the integral of X against itself, so we simply postulate its values. Of course, $\mathbb{X}$ cannot just be anything, but should satisfy a number of natural algebraic identities and analytical bounds.

Anyway, assuming that we are provided with the data $(X, \mathbb{X})$, then we know how to give meaning to the integral of components of X against other components of X: this is precisely what $\mathbb{X}$ encodes. Intuitively, this suggests that if we similarly encode the fact that Y "looks like X at small scales", then one should be able to extend the definition of "rough integrals" to a large enough class of integrands to include solutions to differential equations, even when $\alpha < \frac{1}{2}$. Rough path theory allows us to make this intuition precise.

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Exact formulas for random growth off a flat interface

Daniel Remenik
(joint work with Janosch Ortmann, Jeremy Quastel)

We consider the one-dimensional asymmetric simple exclusion process (ASEP), which is a continuous time Markov process on with state space $\{0,1\}^{\mathbb{Z}}$, the 1's being thought of as particles and the 0's as holes. Each particle has an independent exponential clock which rings at rate one. When it rings, the particle chooses to attempt to jump one site to the right with probability p, or one site to the left with probability $q = 1-p$. However, the jump is only executed if the target site is empty; otherwise the jump is suppressed and the particle must wait for the alarm to ring again. We will assume that $q > p$.

We denote by $\eta_x(t) = 1$ or 0 the presence or absence of a particle at $x \in \mathbb{Z}$ at time t. Alternatively, we can write $\hat{\eta}(x) = 2\eta(x) - 1$, which takes values $\{-1, 1\}$ instead of $\{0, 1\}$. Two quantities of interest for this process are

$$N_x(t) = \sum_{y=-\infty}^{x} \eta_y(t)$$

(that is, the number of particles at or to the left of x, which can take the vaule ∞ if the system has no left-most particle), and the ASEP *height function*, defined by

$$h(t,x) = \begin{cases} 2N_0^{\text{flux}}(t) + \sum_{0<y\leq x} \hat{\eta}(t,y), & x > 0, \\ 2N_0^{\text{flux}}(t), & x = 0, \\ 2N_0^{\text{flux}}(t) - \sum_{x<y\leq 0} \hat{\eta}(t,y), & x < 0, \end{cases}$$

where $N_0^{\text{flux}}(t)$ is defined as the net number of particles which crossed from site 1 to 0 up to time t, meaning that particle jumps $1 \to 0$ are counted as $+1$ and jumps $0 \to 1$ are counted as -1. The problem is to compute the asymptotic fluctuations of these quantities as $t \to \infty$ for certain special initial conditions.

For the totally asymmetric (TASEP) case, where $q = 1 - p = 0$ the fluctuations are completely understood. The first result [5] was for TASEP with *step initial condition* $\eta_0(x) = \mathbf{1}_{x>0}$, and showed that the asymptotic fluctuations for the height function are described by the Tracy-Widom GUE distribution. For the *flat* or *periodic* initial condition $\eta_0(x) = \mathbf{1}_{x\in 2\mathbb{Z}}$ it was later shown in [3] (although it essentially goes back to [1]) that the asymptotic fluctuations are given by the Tracy-Widom GOE distribution. For the *half-flat* case $\eta_0(x) = \mathbf{1}_{x\in 2\mathbb{Z},\, x>0}$ the fluctuations were derived in [4] and are given by a one-parameter family of distributions which interpolate between the Tracy-Widom GUE and GOE distributions. (There are also TASEP results involving the *stationary* initial condition where particles are placed according to a Bernoulli product measure).

For the partially asymmetric case $0 < q < 1$ things are much more complicated due to the lack of the determinantal structure present in ASEP. After a series of papers Tracy and Widom [8] were able to show that, as expected, the ASEP fluctuations in the case step initial condition are still given by the Tracy-Widom

GUE distribution. In a later paper the were able to treat the so-called *step-Bernoulli* initial condition, but nothing is known so far in the flat cases.

In this talk I reported on work in progress where we obtain moment formulas for the ASEP height function in the half-flat and flat cases. The formulas are obtained by solving certain systems of differential equations satisfied by the exponential moments of $N_x(t)$, derived in [2], which arise from duality properties of ASEP. In the half-flat case the formula reads as follows. Let $\tau = p/q \in (0,1)$. Then

$$\mathbb{E}^{\mathrm{half-flat}}\Big[\tau^{mN_x(t)}\Big] = m_\tau! \sum_{k=0}^{m} \tfrac{1}{k!} \sum_{\substack{n_1,\dots,n_k\geq 1\\ n_1+\cdots+n_k=m}} \int_{C^k} \frac{d\vec{w}}{(2\pi i)^k} \det\Big[\tfrac{-1}{w_a\tau^{n_a}-w_b}\Big]_{a,b=1}^{k} \times \prod_a f(w_a;n_a)g(w_a;n_a) \prod_{a<b} \alpha(w_a,w_b;n_a,n_b),$$

where

$$f(w;n) = (1-\tau)^{n_a} e^{t\left[\frac{1}{1+w_a}-\frac{1}{1+\tau^{n_a}w_a}\right]} \Big(\tfrac{1+\tau^{n_a}w_a}{1+w_a}\Big)^{x-1},$$

$$g(w;n) = \tfrac{(-w;\tau)_\infty}{(-\tau^n w;\tau)_\infty} \tfrac{\left(\tau^{2n}w^2;\tau\right)_\infty}{(\tau^n w^2;\tau)_\infty}, \quad \alpha(w_1,w_2;n_1,n_2) = \tfrac{(w_1w_2;\tau)_\infty \left(\tau^{n_1+n_2}w_1w_2;\tau\right)_\infty}{(\tau^{n_1}w_1w_2;\tau)_\infty (\tau^{n_2}w_1w_2;\tau)_\infty}$$

and the integration contour C contains -1 and 0 with $\pm\tau^{-1}$ on the outside. Here the τ-Pochhammer symbol is $(a;\tau)_\infty = \prod_{\ell=0}^{\infty}(1-\tau^\ell a)$ and the τ-factorial $m!_\tau = (1-\tau)^m (\tau;\tau)_\infty / (\tau^m;\tau)_\infty$. This moment formula can then be summed to obtain a certain moment generating function for $N_x(t)$, and one can verify through formal critical point analysis on this formula that the limiting fluctuations for half-flat ASEP coincide with those of TASEP. There are, nevertheless, some serious technical problems in making the limit argument rigorous, and this consitutes works in progress.

For the flat case we use a limiting procedure which involves looking at the half-flat moment formula for $\tau^{N_x(t)}$ and taking $x \to \infty$ (thus probing into the flat region). Since for the half-flat case it holds that $h(t,x) = 2N_x(t) - x$, this yields a moment formula for the exponential moments of the flat ASEP height function. In this case we are able to sum the moments in a certain way which yields a Fredholm Pfaffian. The answer we obtain is described as follows. Define the *symmetric τ-exponential function* $\exp_\tau(x) = \sum_{k=0}^{\infty} \tau^{\frac{k(k-1)}{4}} \frac{x^k}{k!_\tau}$. Then, for $\zeta \in \mathbb{C}$, $|\zeta| < \tau^{1/4}$,

$$\mathbb{E}^{\mathrm{flat}}\Big[\exp_\tau\Big(-\zeta\tau^{\frac{1}{2}h(t,0)}\Big)\Big] = \mathrm{Pf}(J-K)_{L^2[0,\infty)}\,.$$

The *Fredholm Pfaffian* is defined as

$$\mathrm{Pf}(J-K)_{L^2[0,\infty)} = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \int_{[0,\infty)^n} \mathrm{Pf}\Big[K(x_i,x_j)\Big]_{i,j=1}^{n} d\vec{x}, \qquad J = \left(\begin{smallmatrix} 0 & I \\ -I & 0 \end{smallmatrix}\right).$$

Here K is a certain 2×2 explicit kernel which involves expressions similar to the ones appearing in the half-flat case. As in that case, one can perform a formal critical point analysis on this formula to show that, under the right scaling, the Fredholm Pfaffian converges to a limiting Fredholm Pfaffian which can be shown to yield exactly the Tracy-Widom GOE distribution, as expected. Nevertheless, and

again similarly to the half-flat case, the limit computation presents some serious technical obstacles, and also constitutes work in progress.

During the talk I also briefly described some related formulas for the delta Bose gas which can be obtained by the same method, and also appear in the physics literature [6, 7].

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Exact results for the Kardar-Parisi-Zhang equation from the replica Bethe Ansatz and from the sine Gordon field theory

Pierre Le Doussal

The one dimensional KPZ stochastic growth equation for the height field $h(x,t)$ describes many experimental systems of interfaces in planar geometries which grow in time t following a local mechanism: slow combustion, turbulent liquid crystals, bacterial growth, chemical reaction fronts. The KPZ universality class is characterized by self-affine growth at large time $h(x,t) = vt + \delta h(x,t)$ with scaling exponents $\delta h \sim x^{1/2} \sim t^{1/3}$ for fluctuations which have been known for a long time in the physics community. The recent focus is on the probability distribution function (PDF) for $\delta h(x,t)$ for given initial conditions for the interface at time $t = 0$ (flat, droplet, stationary..) Since around 2000 a significant number of discrete models of growth (PNG, TASEP,..) have been shown to share the same universal properties, termed the KPZ class, where $\delta h(x,t)$ at large time converges to a rescaled Tracy Widom (TW) distribution, the distribution of largest eigenvalue of large Gaussian random matrices (GOE,GUE..) depending on the initial condition. It remained to be shown that the continuum KPZ equation belongs to the KPZ class!

In this talk I have shown how to calculate the PDF of $h(x,t)$ at all times (describing the crossover from the Edward Wilkinson to the KPZ class) using methods of physics, from disordered systems (replica) and integrable systems (Bethe

Ansatz). We solve the droplet and flat initial conditions, as well as the crossover between them (the latter only at large time) and also KPZ in a half-space. This work was done in collaboration with P. Calabrese, and also in part with A. Rosso and T. Gueudre and led to 7 publications. Note parallel works by V. Dotsenko also using the replica Bethe ansatz. The results are non-rigorous however they have been confirmed for the droplet initial condition by more rigorous approaches (Amir et al., Sasamoto and Spohn) based on the limit of WASEP. For the flat initial condition there is at present no rigorous result, but there is a work in preparation by Ortmann, Quastel and Remenik.

The method starts by using the Cole-Hopf mapping from the KPZ equation to the problem of a directed path (DP) in a quenched random potential. Denoting Z the partition sum of the directed path, the correspondence is schematically $h = \ln Z$. The aim now is to calculate the moments $E(Z^n)$ (expectation over the random potential) and from them "guess" the Laplace transform of the PDF of Z, $g(u) = E(e^{-uZ})$ using its formal series expansion $g(u) = \sum_n \frac{1}{n!} E(Z^n)(-u)^n$. It is a guess because the moments grow to fast with n to uniquely determine the PDF. The second step uses the mapping from the DP problem to the quantum mechanics of bosons with attractive δ interactions, the so-called Lieb Liniger (LL) model. More precisely any expectation of the product of n partition sums Z obeys the Schrodinger equation with the Hamiltonian H_n of the LL model. This model is integrable by the Bethe Ansatz hence all eigenfunctions and eigenvalues of H_n are known. This allows us to derive, for various initial conditions, a formula for the $E(Z^n)$. This formula is quite complicated and involves summation overs the so-called "string states" which fortunately have a simple structure in the limit of a large system size, related to the partitions of the integer n. After a few technical steps we arrive at an expression for $g(u)$ as a Fredholm determinant (for droplet initial conditions) for all times, involving a kernel containing Airy functions. At large time this kernel converges to the Airy kernel, and the PDF of $h(x,t)$ hence converges to the GUE TW PDF, hereby showing that the KPZ equation belongs to the KPZ class. In the case of the flat initial condition the calculation is much more involved but leads to a Fredholm Pfaffian, with convergence to the GOE TW distribution at large time. A variation of the calculation with half flat initial conditions leads to the crossover PDF from GOE to GUE at large time. Finally, for the half-space problem (DP in presence of a repulsive wall) one finds that $g(u)$ is also a Pfaffian with convergence to the GSE TW distribution.

The talk terminates by a summary of another recent result, with Kormos and Calabrese, which shows that the Sine Gordon field theory (which is an integrable field theory) in the non relativistic limit (NRL) allows to recover the moments $E(Z^n)$ associated to the KPZ equation. This is achieved by showing that the form factors for the breathers (since solitons decouple in that limit) reduce to matrix elements of string states. The explicit NR limit can be taken on the 2 point correlation function of the exponential field $e^{ik\phi}$ of the sine Gordon theory using the Lehman decomposition in elementary excitations, leading to $g(u)$ of KPZ. This

connection raises the hope of obtaining new results for KPZ using the Sine Gordon theory.

Matrix Dufresne identities

Benedek Valkó
(joint work with Brian Rider)

In 1990 Dufresne [3] showed that if $\mu > 0$ and b_t is a standard Brownian motion then the integral $\int_0^\infty e^{2(b_t-\mu t)}dt$ has the same distribution as $(2\gamma_\mu)^{-1}$ where γ_μ is a gamma variable with parameter μ. In the subsequent decade Matsumoto and Yor discovered a number of remarkable identities related to the integral of the geometric Brownian motion (see e.g. the survey papers [4] and [5]). Many of these identities involve the finite time integral

$$A_t^{(\mu)} = \int_0^t e^{2(b_t+\mu t)}dt$$

(defined for all $\mu \in \mathbb{R}$). Among others, they proved the following process level identity for $\mu > 0$:

$$\left(\frac{1}{A_t^{(-\mu)}} - \frac{1}{A_\infty^{(-\mu)}}, t \geq 0\right) \stackrel{d}{=} \left(\frac{1}{A_t^{(\mu)}}, t \geq 0\right) \tag{1}$$

and also showed that the process on the left is independent of the random variable $A_\infty^{(-\mu)} = \int_0^\infty e^{2(b_t-\mu t)}dt$.

We study the matrix versions of the Dufresne (and related) identities. Motivated by certain scaling limits related to the hard edge β-ensembles [8] we consider the following matrix valued process in place of the exponential Brownian motion:

$$dM_t^{(\mu)} = M_t^{(\mu)}(dB + (1/2 + \mu)Idt), \qquad M_0^{(\mu)} = I, \tag{2}$$

where B is an $n \times n$ matrix with i.i.d. standard Brownian motion entries. This is essentially a multiplicative Brownian motion on GL_n. We study the integrals

$$Q_t^{(\mu)} = \int_0^t M_s M_s^T dt, \qquad Q_\infty^{(\mu)} = \int_0^\infty M_s M_s^T dt.$$

We show that for $\mu > 0$ the distribution of $Q_\infty^{(-\mu)}$ has inverse Wishart distribution with parameters $n, n + 2\mu$. Moreover, we prove a process level distributional identity which has the same form as (1).

An interesting Burke-type identity was proved by O'Connell and Yor in [7]. They showed that with $\alpha_t = \log \int_{-\infty}^t e^{2(b_s-b_t+\mu(s-t))}ds$ the process $b_t + \alpha_t - \alpha_0$ is also a standard Brownian motion. They used this result to construct and study a semi discrete directed polymer where the random environment is given by a sequence of independent Brownian motions. This system turned out to be a member of the KPZ universality class. The fluctuation exponents were identified in [9]; in [6] a connection to a system of interacting diffusions was shown (the

so-called quantum Toda lattice), and in [1] and [2] the Tracy-Widom distribution has been derived as a scaling limit.

We describe a matrix version of Burke-type identity of O'Connell and Yor, which could be the first step towards introducing a directed polymer model built from the matrix Brownian motions (2).

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Stochastic quantum integrable systems

Ivan Corwin

The purpose of this talk is to analyze the exactly solvable particle system known as q-TASEP [4, 6]. Let us fix some notation used throughout. For $N \geq 1$ we denote the state of q-TASEP with N particles as $\vec{x}(t) = \big(+\infty \equiv x_0(t) > x_1(t) > x_2(t) > \cdots > x_N(t) \big) \in \mathbb{Z}^N$, where we have fixed a virtual particle at infinity by setting $x_0(t) \equiv +\infty$. The gap between particle i and $i-1$ is denoted $\mathrm{gap}i(t) := x_{i-1}(t) - x_i(t) - 1$. The N-particle q-TASEP with particle rate parameters $a_1, \ldots, a_N > 0$ is an interacting particle system $\vec{x}(t)$ in which for each $1 \leq i \leq N$, $x_i(t)$ moves to $x_i(t) + 1$ at exponential rate $a_i(1 - q^{\mathrm{gap}i(t)})$ (which vanishes when $\mathrm{gap}i(t) = 0$), where q is a parameter in $(0, 1)$. Here we assume that these exponentially distributed jumping events are all independent, and we note that since they occur in continuous time, no two jumps occur simultaneously, almost surely. Also note that the evolution of $x_i(\cdot)$ only depends on those particles with lower indices than i. Step initial condition is defined as setting $x_i(0) = -i$ for $1 \leq i \leq N$.

The continuous time Poisson q-TASEP and the recognition of its exact solvability comes from the work of [4] on Macdonald processes. It was soon after studied in its own right in [6] via the type of many body system approach methods which we explain in this talk. The evolution of $\mathrm{gap}i(t) = x_{i-1}(t) - x_i(t) - 1$ for $1 \leq i \leq N$ is given by a totally asymmetric zero range process with site dependent jump rate

$g_i(k) = a_i(1-q^k)$ and infinite sink and source at the boundary. A variant of this zero range process can be seen as corresponding to a particular representation of the q-Boson Hamiltonian considered in [10]. In [10], integrability (in the form of L and R matrices satisfying Yang-Baxter relations) for this q-Boson Hamiltonian was shown.A stationary (infinite lattice) variant of the above gap zero range process was also discussed immediately after Theorem 2.9 of [1] and an $O(t^{2/3})$ upper and lower bound on the variance of the stationary current is established therein.

When $q \to 0$, continuous time Poisson q-TASEP becomes the well-studied model of continuous time TASEP (and likewise the discrete time versions of q-TASEP we introduce become known discrete time version of TASEP). The configuration of particles in TASEP for a fixed time t can be described as a determinantal point process in which all correlation functions are given by minors of a single correlation kernel. In other terminology, TASEP is free-fermion, or related to Schur processes, or non-interacting line ensembles. Given that structure there exists a clean path to Fredholm determinant formulas for marginal distributions, which in turn allow readily for asymptotic analysis.

The present case, where $q \in (0,1)$, does not appear to enjoy a full-blown determinantal structure, hence new ideas are necessary to study and extract asymptotic distributional information about the processes considered. In [4, 6] it was explained how, instead of the determinantal structure of correlation functions, these non-determinantal systems are exactly solvable due to the existence of concise and exact formulas for expectations of large classes of observables of the particle systems (in fact, in many cases large enough classes to uniquely characterize the fixed time distribution of the particle systems).

A natural question about q-TASEP is to compute the distribution of $x_n(t)$ (which is the location of particle n at time t). There are (presently) two approaches to compute this distribution. The first is through the theory of Macdonald processes [4], and the second is through the many body system (or duality) approach [6]. These methods lead to the computation of nested contour integral formulas for expectations of certain observables of the process. In particular, for step initial condition this shows that (see also [7]) for $n_1 \geq n_2 \geq \cdots \geq n_k > 0$,

$$\mathbb{E}\left[\prod_{i=1}^{k} q^{x_{n_i}(t)+n_i}\right] = \frac{(-1)^k q^{k(k-1)/2}}{(2\pi\iota)^k} \int \cdots \int \times \prod_{1\leq A<B\leq k} \frac{z_A - z_B}{z_A - q z_B} \prod_{j=1}^{k} \left(\prod_{i=1}^{n_j} \frac{a_i}{a_i - z_j}\right) e^{(q-1)tz_j} \frac{dz_j}{z_j}, \tag{1}$$

where the contour of integration for z_A contains $a_1, \ldots, a_N$ as well as $\{qz_B\}_{B>A}$, but not zero.

Since the observables $q^{x_n(t)+n}$ are all in $(0,1)$, their moment problem is well-posed. This means that the above formulas uniquely characterize the joint distribution of q-TASEP at time t (provided it is started from step initial condition at time zero). It remains a challenge to extract manageable (from the perspective of large t and n asymptotics) formulas for joint distributions, though it is understood

how to derive a Fredholm determinant formula which characterizes the one-point distribution of $x_n(t)$.

Theorem 1. *Fix $q \in (0,1)$ and particle rate parameters $a_i \equiv 1$. Then for all $\zeta \in \mathbb{C} \setminus \mathbb{R}_+$*

$$\mathbb{E}^{\ell}\left[\frac{1}{\left(\zeta q^{x_{n(t)+n}};q\right)_{\infty}}\right] = \det(I+K) \tag{2}$$

where $\det(I+K)$ is the Fredholm determinant of $K : L^2(C_1) \to L^2(C_1)$ for C_1 a positively oriented circle containing 1, not containing 0 and small enough so as not to contain its image when multiplied by q. The operator K is defined in terms of its integral kernel

$$K(w,w') = \frac{1}{2\pi\iota}\int_{-\iota\infty+1/2}^{\iota\infty+1/2} \frac{\pi}{\sin(-\pi s)}(-\zeta)^s \frac{h^{\ell}(q^s w)}{h^{\ell}(w)} \frac{1}{q^s w - w'} ds$$

with $h(w) = (w;q)_{\infty}^{n} e^{tw}$.

From this Fredholm determinant [8, 2] were able to prove $t^{1/3}$ and GUE Tracy-Widom scaling limit asymptotics for q-TASEP.

In this talk we follow the following (second) approach to finding moment formulas for q-TASEP is based on the following three observations:

(1) The expectations of observables $\prod_{i=1}^{k} q^{x_{n_i}(t)+n_i}$ for $n_1 \geq \cdots \geq n_k$ evolve according to closed systems of coupled ODEs which we call the *true evolution equation*.

(2) The true evolution equation is almost constant coefficient or separable, except for effects which arise when some of the n_i are close together. Rather than trying to solve the true evolution equation directly, one can look for solutions to the free evolution equation (i.e., the constant coefficient and separable system neglecting the boundary effects) on the larger space $\vec{n} \in \mathbb{Z}^k_{\geq 0}$ which have the right initial data when restricted to $n_1 \geq \cdots \geq n_k$ and which satisfy certain boundary conditions when some of the n_i are all equal. The restriction of such a solution to $n_1 \geq \cdots \geq n_k$ will coincide with the solution to the true evolution equation. Generally, for every possible combination of clusters of $\vec{n}$ (i.e., strings of equal coordinates) there will be additional boundary conditions which must be satisfied. For q-TASEP, it suffices to consider only $k-1$ two-body boundary conditions corresponding to when $n_i = n_{i+1}$. That the two-body boundary conditions imply all many body boundary conditions is the hallmark of *integrability* in the language of (quantum) many body systems. This type of reduction goes back to the 1931 work of on diagonalizing the Heisenberg spin chain [3]. Our reduction, under the scaling limit to the continuum stochastic heat equation mentioned above, coincides with that for the quantum delta Bose gas (also known as the Lieb-Liniger model [9]).

(3) A general class of solutions to the free evolution equation exists since it is separable and constant coefficient. It is not immediately clear how to

combine solutions from this class so as to additionally satisfy the $k-1$ two-body boundary conditions and the initial data. However, for the initial data corresponding to step initial condition, it is also possible to (easily) check that *nested contour integral formulas* (such as in (1)) solve the free evolution equation and satisfy the $k-1$ two-body boundary conditions. These formulas are special cases of the *Bethe ansatz* for diagonalizing this system [5]. From this it is possible to produce solutions corresponding to general initial conditions.

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Random matrices and Dirac operators

Bálint Virág

(joint work with Benedek Valkó)

We describe a random first order differential operator (a Dirac operator) acting on $[0,1] \to \mathbb{C}^2$ functions that has an almost surely discrete spectrum with the same distribution as the Sine_β process. This process is the bulk scaling limit of the Gaussian β-ensemble derived in [5]. For $\beta = 2$ it is conjectured to behave similarly as the imaginary parts of the set of critical zeros of the Riemann ζ-function [3].

The operator is self-adjoint on the appropriate domain which is defined via differentiability, L^2 and boundary conditions. It is built from the following matrix-valued random process:

$$X_t = \frac{1}{\sqrt{1-|b(\tau(t))|^2}} \begin{pmatrix} 1 & b(\tau(t)) \\ \overline{b}(\tau(t)) & 1 \end{pmatrix}, \qquad \tau(t) = -\frac{4}{\beta}\log(1-t).$$

Here $b(t), t \in [0, \infty)$ is the hyperbolic Brownian motion on the Poincaré disk, given by the SDE

$$db = \tfrac{1}{2}(1 - |b|^2)dZ, \qquad b(0) = 0,$$

where $Z = B_1 + iB_2$ is a complex Brownian motion. The operator itself is then given by

$$\mathrm{SINE}_\beta = \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix} X_t^2 \partial_t.$$

A similar representation is described for the hard edge scaling limit of certain β-ensembles (derived in [4]), here one needs to use a real Brownian motion with drift embedded into the hyperbolic plane in place of $b(t)$.

We show how the finite circular β-ensembles can be obtained as the eigenvalues of similar random Dirac operators using the representation of [1] via Verblunsky coefficients. The role of the hyperbolic Brownian motion is now played by a random walk in the hyperbolic lane with rotational invariant jump steps. These random walks converge in distribution to the time changed hyperbolic Brownian motion appearing in X_t. We outline how this convergence implies that the SINE_β operator can be obtained as the norm resolvent limit of Dirac operators connected to the finite circular β-ensembles. From this it also follows that the limit process of these ensembles (derived in [2]) is the same as the Sine_β process.

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KPZ Line Ensemble: a marriage of integrability and probability

Alan Hammond

(joint work with Ivan Corwin)

The KPZ equation, introduced by Kardar, Parisi and Zhang, is a stochastic PDE that models randomly evolving interfaces that are subject to constraining forces such as surface tension. It is anticipated to be a universal object, in the sense that many microscopic models will share the KPZ equation as an accurate asymptotic description of their late time behavior. This view is supported by extensive numerical evidence, recent experimental evidence involving liquid crystal instabilities, and a limited but growing body of mathematically rigorous work.

In recent work [1], we present a new technique for the analysis of the KPZ equation. The solution to the equation is represented as the lowest indexed curve in an N-indexed ensemble of curves, which we call a KPZ line ensemble. Curves within the ensemble enjoy a natural invariance under resampling, the H-Brownian Gibbs property, which property has the effect of energetically penalizing, but not absolutely forbidding, the crossing of adjacently indexed curves. This property is inherited from the O'Connell Yor semi-discrete continuum random polymer ensemble after a limiting procedure is applied.

The H-Brownian Gibbs property is an integrable one, in the sense that the precursor O'Connell-Yor ensemble is known to enjoy it by virtue of this ensemble's algebraic structure. However, it also offers a powerful probabilistic tool for the analysis of the KPZ equation. Since the solution of this equation is embedded in a KPZ line ensemble, we may analyse it using the H-Brownian Gibbs property, and in this way derive significant new estimates regarding the regularity and local structure of the KPZ solution. As I will aim to explain, these new estimates are valid uniformly in the time parameter for the KPZ equation, even after a natural rescaling of the equation is undertaken which accesses the fluctuation behavior of the KPZ evolution.

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Paracontrolled analysis of the KPZ equation

Massimiliano Gubinelli
(joint work with N. Perkowski)

The Kardar–Parisi–Zhang (KPZ) equation [4]

$$\mathscr{L}h(t,x) = (\partial_x h(t,x))^2 + \xi(t,x) \tag{1}$$

is a mesoscopic model which captures the universal features of one dimensional interface growth. In this equation $h(t,x)$ describes the height of the interface, $\mathscr{L} = \partial_t - \partial_x^2$ is the heat operator and ξ is a space–time white noise. Until the work of Hairer [3] this equation had a heuristic content and the correct physical solution was identified as $h(t,x) = \log Z(t,x)$ where Z is the solution of the stochastic linear heat equation (SHE)

$$\mathscr{L}Z(t,x) = Z(t,x)\xi(t,x). \tag{2}$$

Hairer used rough path techniques [5] in order to give a meaning to eq. (1). In the following we rephrase his analysis in the language of paracontrolled distributions introduced in [2]. The KPZ equation is equivalent to a stochastic Burgers equation (SBE) for the unknown $u(t,x) = \partial_x h(t,x)$:

$$\mathscr{L}u(t,x) = \partial_x (u(t,x))^2 + \partial_x \xi(t,x). \tag{3}$$

We place ourselves on the torus $\mathbb{T} = \mathbb{R}/(2\pi\mathbb{Z})$ and consider a local well–posedness theory for the Cauchy problem related to eq. (3) given an initial condition $u(0,\cdot) = u_0(\cdot)$. Solutions to this problem are expected to be distributions with space regularity $-1/2-$. To understand the physical nature of this singular behavior let us introduce a microscopic version of the SBE in the form

$$\mathscr{L}v(t,x) = \varepsilon^{1/2}\partial_x(v(t,x))^2 + \partial_x\eta(t,x)$$

where ε is a small parameter, η is a smooth centered Gaussian field with correlations of order 1 and where the space variable now lives in the large torus $\mathbb{T}_\varepsilon = \mathbb{T}/\varepsilon$. Performing the change of variables $u_\varepsilon(t,x) = \varepsilon^{-1/2}v(t/\varepsilon^2, x/\varepsilon)$ we see that u_ε satisfy the SBE with noise given by $\eta_\varepsilon(t,x) = \varepsilon^{-3/2}\eta(t/\varepsilon^2, x/\varepsilon)$ and in the limit $\varepsilon \to 0$ we expect to obtain the SBE driven by a space–time white noise ξ. Fluctuations of order 1 on spatial scales of order 1 for v become fluctuations of order $\varepsilon^{-1/2}$ on scales of order ε for u_ε which in the limit as $\varepsilon \to 0$ generate a distribution of order $\sim -1/2$. In this regime the non–linear term u^2 is not well defined and there is no well defined meaning for the limiting equation (3).

Following the approach of Hairer we can expand the solution to the SBE around the solution X of the linearized equation: $\mathscr{L}X = \partial_x\xi$. This perturbative expansion generates recursively a set of distributions which depend polynomially on the noise ξ and which we denote collectively by $\mathbb{X} = (X, Q, X^{\mathbf{V}}, X^{\mathbf{V}}, \ldots)$. These distributions are defined as solutions to a triangular system of non–linear equations

$$\mathscr{L}X = \partial_x\xi, \mathscr{L}Q = \partial_x X, \mathscr{L}X^{\mathbf{V}} = \partial_x(XX), \mathscr{L}X^{\mathbf{V}} = \partial_x(XX^{\mathbf{V}}), \quad \cdots$$

Denoting $\mathscr{C}^\alpha = C([0,T]; B^\alpha_{\infty,\infty}(\mathbb{T}))$ we postulate that $X \in \mathscr{C}^{-1/2-}$, $Q \in \mathscr{C}^{1/2-}$, $X^{\mathbf{V}} \in \mathscr{C}^{0-}$, $X^{\mathbf{V}} \in \mathscr{C}^{1/2-}$. These objects identify a subspace of (paracontrolled) distributions denoted $\mathscr{D}_{\mathrm{sbe}}$ which have the form

$$u = X + X^{\mathbf{V}} + 2X^{\mathbf{V}} + u' \prec\!\!\prec Q + u^\sharp$$

for suitable $u^\sharp \in \mathscr{C}^{1-}$ and $u' \in \mathscr{C}^{1/2-}$. Here $\prec\!\!\prec$ denote a modified *paraproduct* (in the sense of Bony, Meyer et al. [1]). It allow to describe the microscopic fluctuations of the distributions in $\mathscr{D}_{\mathrm{sbe}}$ in terms of the explicit object $\mathbb{X}$ which is amenable to direct analysis. Distributions in $\mathscr{D}_{\mathrm{sbe}}$ can be multiplied, in particular, if $u \in \mathscr{D}_{\mathrm{sbe}}$ then

$$u^2 = \underbrace{XX + 2XX^{\mathbf{V}} + X^{\mathbf{V}}X^{\mathbf{V}}}_{A} + \underbrace{X(2X^{\mathbf{V}} + u' \prec\!\!\prec Q)}_{B}$$
$$+ \underbrace{X^{\mathbf{V}}(2X^{\mathbf{V}} + u' \prec\!\!\prec Q) + (2X^{\mathbf{V}} + u' \prec\!\!\prec Q + u^\sharp)^2}_{C}$$

and the square which a–priori is not well-defined since $u \in \mathscr{C}^{-1/2-}$ can be decomposed in a series of terms. We start noting that

$$A = XX + 2XX^{\mathbf{V}} + X^{\mathbf{V}}X^{\mathbf{V}} = \mathscr{L}(X^{\mathbf{V}} + 2X^{\mathbf{V}} + X^{\mathbf{V}})$$

so these terms are under control provided $\mathbb{X}$ contains $X^{\mathbf{V}}, X^{\mathbf{V}}, X^{\mathbf{V}}$. The various product in C are well defined since the sum of the regularities of the factors is

positive. The only remaining problematic term is B since here the sum of the regularities is (barely) negative. In order to put also this term under control we apply a commutator lemma proved in [2] which allows to write

$$B = X(2X^{\mathbf{V}} + u' \prec\!\!\prec Q) = (2X^{\mathbf{V}} + u' \prec\!\!\prec Q) \prec\!\!\prec X + u'(X \circ Q) + 2X^{\mathbf{V}} \circ Q + \mathscr{C}^{0-}.$$

where the operation denoted by $\circ$ is the remainder term in the paraproduct decomposition of the standard product $ab = a \prec b + b \succ a + a \circ b$. This computation shows that we can control also the term B if $\mathbb{X}$ contains also $X \circ Q$ and $X^{\mathbf{V}} \circ Q$ and that in this case the square is a well defined operation in $\mathscr{D}_{\mathrm{sbe}}$ and thus that it is meaningful to talk about solutions to the SBE in $\mathscr{D}_{\mathrm{sbe}}$. A little bit more of analysis shows also that the SBE can be solved locally in time in $\mathscr{D}_{\mathrm{sbe}}$. While sensible, this procedure has to be slightly modified since when ξ is a white noise various terms in $\mathbb{X}$ contains divergent contributions and need to be *renormalized*. Renormalization, in the context of the SBE, amount to the observation that we are allowed to subtract any constant to u^2 since it will disappear in the final equation. If we consider again the limit of u_ε as introduced above, the difficulties with u^2 can be traced back to the fact that it is not u_ε^2 which has a limit but only its deviations from the spatial mean. Indeed the mean of u_ε^2 has size $1/\varepsilon$ and cannot converge to any sensible limit. What is possible to prove is that there exists a constant c_ε such that $u_\varepsilon^2 - c_\varepsilon$ will have a well defined limit in $\mathscr{C}^{-1-}$ as $\varepsilon \to 0$ and define a *renormalized* square $u^{\diamond 2}$. At the level of the paracontrolled structure this renormalization amounts to perform subtractions to various terms and then consider the limit of the new set of renormalized objects $\mathbb{X}_\varepsilon$ defined starting from η_ε to obtain a limiting $\mathbb{X}$ which contains all the informations necessary to describe the effects of the microscopic fluctuations at the macroscopic scale. Proceeding similarly it is possible to introduce paracontrolled structures for the KPZ and for the SHE equations in such a way that if $h \in \mathscr{D}_{\mathrm{kpz}}$ then $\partial_x h \in \mathscr{D}_{\mathrm{sbe}}$ and if $Z \in \mathscr{D}_{\mathrm{she}}$ and $Z > 0$ then $\log Z \in \mathscr{D}_{\mathrm{kpz}}$ and then show rigorously the set of relations between the corresponding equations.

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Intermittency and Multifractality

Davar Khoshnevisan
(joint work with Kunwoo Kin, Yimin Xiao)

Consider the stochastic partial differential equation,

$$\frac{\partial}{\partial t}u(x\,,t) = \frac{\partial^2}{\partial x^2}u(x\,,t) + \sigma(u(x\,,t))\xi(x\,,t),$$

for $x \in \mathbf{R}$ and $t > 0$, in two different extremal cases where: (1) $\sigma(x) \equiv 1$ for all $x \in \mathbf{R}$; and (2) $\sigma(x) = x$ for all $x \in \mathbf{R}$, where ξ denotes space-time white noise.

For the sake of simplicity, we start our stochastic PDE with initial profile $u(x\,,0) := 0$ in Case (1) and $u(x\,,0) := 1$ in Case (2). Consider, for every $c > 0$, the random sets

$$\mathcal{L}(c) := \{x \geq 100 :\ u(x\,,t) \geq g_c(x\,,t)\}\,,$$

for each $t > 0$, where g is the "gauge,"

$$g_c(x\,,t) := \begin{cases} ct^{1/4}[\log(x)]^{1/2} & \text{in Case (1),} \\ \exp\left\{ct^{1/3}[\log(x)]^{2/3}\right\} & \text{in Case (2).} \end{cases}$$

Each random set $\mathcal{L}(c)$ describes the position of "tall peaks" of the solution to the stochastic measured "in scale c."

We show exact a.s.-formulas for the large-scale Hausdorff dimension of $\mathcal{L}(c)$, where large-scale Hausdorff dimension is understood in the sense of M. T. Barlow and S. J. Taylor (1992).

The two-periodic Aztec diamond

Sunil Chhita and Kurt Johansson

An Aztec diamond of order n consists of all squares of a square lattice whose centers (x, y) satisfy $|x| + |y| \leq n$ for $n \in \mathbb{N}$. This shape can be tiled with dominoes which are the union of two adjacent squares, such that each square of the Aztec diamond is covered exactly once by a domino. There are other interesting examples which involve replacing the graph with a different one, such as the square-octagon lattice (giving the so-called diabolo tilings) and the hexagonal mesh (giving lozenge tilings). For each tiling, there is a surface representation where the third co-ordinate, called the height function is in one-to-one correspondence with the tiling, an observation due to Thurston. For random tilings, with probability tending to one, the height function of a randomly tiled large bounded region tends to a deterministic limit shape. This shape is not smooth over the entire region, there exists three types of regions: 'solid' where the measure exhibits deterministic correlations between dominoes; 'liquid' where the measure has correlations between dominoes which show polynomial decay and 'gas' where the measure has correlations between dominoes which have exponential decay. These are technical terms as opposed to physical states of matter [3].

We introduce a two-periodic weighting of the Aztec diamond. This model has specific edge weights chosen so that tilings of large Aztec diamonds feature all three macroscopic regions. Indeed, the limit shape was computed by Kenyon and Okounkov [2]. Standard techniques to compute the correlations associated with the model break down but recently in [1], the authors give a rather complicated and involved expression for correlations of the dominoes in this model. We present a dramatic simplification of this formula which is suitably nice for asymptotic computations. Using these simplified formulas we are able to find the limiting measure in each phase, agreeing with the characterization given in [3]. We are also able to find the pairwise correlations of the dominoes at the solid-liquid and liquid-gas boundary. Finally, we present a possible combinatorial description between the liquid and gas boundaries which are motivated from the long trajectories separating the liquid and gas phases observed from simulations. Since our methods only analyzes local features, we are unable to analyze these paths due to their non-local nature.

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Zero-temperature directed polymers in a product-type random environment

Konstantin Khanin
(joint work with Yuri Bakhtin, Jeremy Voltz)

Statistical properties of directed polymers were a subject of intensive studies in the past decade. Consider discrete time random walks on the lattice $\mathbb{Z}^d$. We shall assume that time t belongs to $\mathbb{Z}^1$. If the position of the random walk at time $t = i$ is given by $S(i) \in \mathbb{Z}^d$, then next moment of time it either jumps at one of the nearest neighbours, or stays at the the same site: $\|S(i+1) - S(i)\| \leq 1$. Condition above corresponds to the so-called "lazy" random walks. The random environment is given by a random potential. We shall assume that every point (x, i) of the space-time $\mathbb{Z}^{d+1}$ is equipped with a random variable $V^\omega(x, i)$, where by ω we denote the whole configuration of the random field $\{V(x, i), (x.i) \in \mathbb{Z}^{d+1}\}$. For a random walk path $S(i), 0 \leq i \leq n$ of length n define its action

$$A_n^\omega(S) = \sum_{i=0}^{n} V^\omega(S(i), i).$$

Notice that A_n is a random variable which depends on a realization ω of the random field $\{V(x, i), (x.i) \in \mathbb{Z}^{d+1}\}$. Now we can define a zero-temperature directed

polymer as an action-minimizing path starting from a fixed point. If the probability distribution of the field $\{V(x,i),(x.i)\in\mathbb{Z}^{d+1}\}$ is continuous then for a fixed starting point a path S minimizing the action $A_n^\omega(S)$ is unique with probability 1. In what follows such action-minimizing paths are called optimal paths and denoted $\bar{S}$. Note that we consider quenched setting where the configuration ω is assumed to be fixed. The most interesting situation which was recently intensively studied corresponds to $d=1$. The case of independent identically distributed (iid) random variables $\{V^\omega(x,i),(x,i)\in\mathbb{Z}^{1+1}\}$ belongs to so-called KPZ (Kardar-Parizi -Zhang) universality class. In this case there are very precise conjectures about asymptotic properties of the polymer measure and optimal paths. In some cases these conjectures has been proved rigorously.

In this talk we shall present results related to statistical properties of optimal paths for other statistical models for the field $\{V^\omega(x,i),(x,i)\in\mathbb{Z}^{1+1}\}$ corresponding to product-type environments. Namely, we consider a random field $\{F(x),x\in\mathbb{Z}^1\}$ of iid random variables, and another field $\{b(i),i\in\mathbb{Z}^1\}$ which is also assumed to be iid. We also assume that the fields $\{F(x),x\in\mathbb{Z}^1\}$ and $\{b(i),i\in\mathbb{Z}^1\}$ are statistically independent. Now, let us define $V(x,i)=F(x)b(i),(x,i)\in\mathbb{Z}^2$. Roughly speaking such potentials correspond to the spatially disordered setting. In addition a polymer is embedded in an external potential fluctuating in time. We shall assume that the probability distribution for V has compact support, and it is given by the density $p(V)$ which vanishes outside of the closed interval $[-C,C]$ and positive everywhere inside $[-C,C]$. In addition we assume that random variables b have zero expectation: $\mathbb{E}(b)=0$. The asymptotic behaviour for this model is different from the KPZ model, but it is also universal. We show that critical exponents of the optimal action and of the polymer endpoint are both equal to $2/3$. And after normalization on $n^{2/3}$ both the probability distributions for the centered optimal action $A_n^\omega(\bar{S}_n)+Cn$ and the polymer end-point $\bar{S}_n(n)$ converge to universal limit laws which can be described explicitly. One can also show that the shape function corresponding to the optimal action of the point-to-point optimal paths has a corner at the origin. This explains why both critical exponents in this model coincide. While one can show that the shape function is non-linear, we conjecture that it has a linear piece near the origin. All the results are based on a joint paper with Yuri Bakhtin and Jeremy Voltz.

We also present numerical results for another model which is a generalization of the previous one. Instead of one field $\{F(x),x\in\mathbb{Z}^1\}$ we consider two statistically independent copies of it, $\{F_1(x),x\in\mathbb{Z}^1\}$ and $\{F_2(x),x\in\mathbb{Z}^1\}$. We also consider two independent copies $\{b_1(i),i\in\mathbb{Z}^1\}$ and $\{b_2(i),i\in\mathbb{Z}^1\}$ of the process $\{b(i),i\in\mathbb{Z}^1\}$. We then define $V(x,i)=F_1(x)b_1(i)+F_2(x)b_2(i),(x,i)\in\mathbb{Z}^2$. This model is substantially different from the previous one. At present there are no mathematically rigorous results for it. The numerical study by Simo Zhang who is an undergraduate student at the Loughborough University indicates that both critical exponents which we discussed above are universal and take values close to $5/6$.

Derivation of the KPZ equation from particle systems

Patrícia Gonçalves
(joint work with Milton Jara and Sunder Sethuraman)

In this work we consider a collection of interacting particle systems whose equilibrium fluctuations are governed by the stochastic Burgers/KPZ equation. The conditions we impose at the level of the particle systems are quite general and are shared by many models as exclusion processes, zero-range processes, kinetically constrained lattice gases, among others. The models we consider have weakly asymmetric rates and cross from the Edwards Wilkinson universality class, which is to say that the fluctuations are ruled by Ornstein-Uhlenbeck processes, to the KPZ universality class, more precisely, the fluctuations are governed by the stochastic Burgers equation. The crossover occurs for a strength of the asymmetry equal to $1/\sqrt{n}$.

The KPZ equation was first proposed in [4] to model the growth of random interfaces, therefore, taking h_t as the height of the interface, the equation reads as $\partial_t h = D\Delta h + a(\nabla h)^2 + \sigma \mathcal{W}_t$, where D, a, σ are constants related to the thermodynamical properties of the interface and $\mathcal{W}_t$ is a space-time white noise. If $\mathcal{Y}_t = \partial_x h_t$, then $\mathcal{Y}_t$ is solution of the stochastic Burgers equation given by $\partial_t \mathcal{Y}_t = D\Delta \mathcal{Y}_y + a\nabla(\mathcal{Y}_t)^2 + \sigma \nabla \mathcal{W}_t$. By using the renormalization group approach the dynamical scaling exponent was established by the physicists as being $z = 3/2$ and corresponds to saying that a non-trivial behavior occurs under the re-scaling $h_n(t,x) = n^{-1/2} h(tn^{3/2}, x/n)$. This corresponds, in our case, to say that in our models, when taking a diffusive time scale, the crossover occurs for a strength of the asymmetry equal to $1/\sqrt{n}$.

1. The microscopic dynamics

We consider weakly asymmetric nearest-neighbor particle systems whose dynamics conserves the total density. These systems are Markov processes that we denote by $\{\eta_t^n : t \geq 0\}$ whose state space can either be $\mathcal{E} = \mathbb{N}_0^{\mathbb{Z}}$ or $\mathcal{E} = \{0,1\}^{\mathbb{Z}}$. Here $\mathbb{N}_0 = \{0, 1, 2, \ldots\}$. For $t \geq 0$, the random variables $\eta_t(x)$ count the number of particles at the site x at time t. Below we describe all the conditions we have to impose on our processes in order to achieve our goals.

1.1. **Infinitesimal generator.** Fix $a \in \mathbb{R}$ and $\gamma > 0$. We consider Markov processes with infinitesimal generator given on local functions f by

$$L_n f(\eta) \;=\; n^2 \sum_{x \in \mathbb{Z}} \Big\{ j^R_{x,x+1}(\eta)\Big(\frac{1}{2} + \frac{a}{2n^\gamma}\Big) + j^L_{x+1,x}(\eta)\Big(\frac{1}{2} - \frac{a}{2n^\gamma}\Big)\Big\}\Big(f(\eta^{x,x+1}) - f(\eta)\Big)$$

where $\eta^{x,y}$ is the configuration obtained from η by taking a particle from x to y.

1.2. **Current as a gradient.** The dynamics is given by a microscopic current which can be written as a gradient of a local function. Systems sharing this property are known in the literature as *gradient systems*. To be more precise, if $j_{x,x+1}$ denotes the difference between the jump rate from x to $x+1$ (here denoted

by $j^R_{x,x+1}$) and the jump rate from $x+1$ to x (here denoted by $j^L_{x,x+1}$), then there exists a non-negative local function $c^n_x = \tau_x c^n$ on $\mathcal{E}$ such that

$$j^R_{x,x+1}(\eta) - j^L_{x+1,x}(\eta) \;=\; c^n_x(\eta) - c^n_{x+1}(\eta).$$

1.3. **Invariant measure and Dirichlet form.** The invariant measure μ is translation invariant, can be parameterized by the density ρ and is a product measure. From now on, we fix a a density ρ and we denote the invariant measure by μ_ρ. Moreover, we assume $j^R_{x,x+1}(\eta^{x+1,x})\frac{d\mu^{x+1,x}_\rho}{d\mu_\rho}(\eta) = j^L_{x+1,x}(\eta)$, so that the Dirichlet form is written as $D_{\mu_\rho}(f) \;=\; \frac{1}{2}\sum_{x\in\mathbb{Z}} E_{\mu_\rho}\big[j^R_{x,x+1}(\eta)\big(f(\eta^{x,x+1}) - f(\eta)\big)^2\big]$.

1.4. **Initial density field.** Let $\mathcal{S}(\mathbb{R})$ be the standard Schwarz space of rapidly decreasing functions. Fix $f \in \mathcal{S}(\mathbb{R})$ and let $\mathcal{Y}^n_t(f) = \frac{1}{\sqrt{n}}\sum_{x\in\mathbb{Z}} f\Big(\frac{x}{n}\Big)(\eta_{tn^2}(x) - \rho)$. We assume, starting from μ_ρ, that $\mathcal{Y}^n_0$ converges weakly, as $n\to\infty$, to a spatial Gaussian process $\mathcal{Y}_0$ with a certain covariance.

1.5. **Spectral gap.** Our models are particle conservative, therefore, their state space can be decomposed by the hyperplanes with configurations with a fixed number of particles. Let ℓ be an integer and Λ_ℓ the box of size $2\ell+1$ centered around 0. For $k \geq 0$, let $\mathcal{H}_{k,\ell}$ be the hyperplane of configurations on Λ_ℓ with k particles. Let $\mu_{k,\ell}$ denote the invariant measure for the process restricted to $\mathcal{H}_{k,\ell}$ and let $\lambda_{k,\ell}$ be the spectral gap and $W(k,\ell)$ its reciprocal. Then, the Poincaré-inequality reads as

$$\mathrm{Var}(f,\nu_{k,\ell}) \;\leq\; W(k,\ell)\mathcal{D}_n(f,\nu_{k,\ell}),$$

where $\mathrm{Var}(f,\nu_{k,\ell,\xi})$ is the variance of f with respect to $\mu_{k,\ell}$. We assume that there exits a constant C such that $E_{\mu_\rho}[W(\sum_{x\in\Lambda_\ell}\eta(x),\ell)^2] \;\leq\; C\ell^4$.

Remark 1. *We notice that one can take more general conditions that the ones presented above. For the interested reader we refer to* [3].

2. Main results

Let $\mathcal{S}'(\mathbb{R})$ be the dual of $\mathcal{S}(\mathbb{R})$, namely the set of tempered distributions in $\mathbb{R}$, endowed with the strong topology. Denote by $D([0,T],\mathcal{S}'(\mathbb{R}))$ (resp. $C([0,T],\mathcal{S}'(\mathbb{R}))$) the space of right continuous functions with left limits (resp. continuous) from $[0,T]$ to $\mathcal{S}'(\mathbb{R})$. Fix $a,\gamma > 0$. For a local function f let $\varphi_j(\rho) = E_{\mu_\rho}[j(\eta)]$. We want to study the asymptotic behavior of the density fluctuation field which is given on functions $f\in\mathcal{S}(\mathbb{R})$ by

$$\mathcal{Y}^{n,\gamma}_t(f) = \frac{1}{\sqrt{n}}\sum_{x\in\mathbb{Z}} f\Big(\frac{x}{n} - \frac{1}{n}\Big\{\frac{a\varphi_j'(\rho)tn^2}{2n^\gamma}\Big\}\Big)(\eta_{tn^2}(x)-\rho).$$

Our goal consists showing that the sequence $\{\mathcal{Y}^{n,\gamma}_t\}_{n\in\mathbb{N}}$ converges, as $n\to\infty$, to $\mathcal{Y}_t$ which is a solution of the Ornstein-Uhlenbeck process or the stochastic Burgers equation, depending on the range of the parameter γ. It is known that for $\gamma = 1$, the sequence $\{\mathcal{Y}^{n,\gamma}_t\}_{n\in\mathbb{N}}$ converges in the uniform topology of $D([0,T],\mathcal{S}'(\mathbb{R}))$ to $\mathcal{Y}_t$, the unique solution of the Ornstein-Uhlenbeck equation

$$\partial_t \mathcal{Y}_t = \frac{1}{2}\varphi_c'(\rho)\Delta\mathcal{Y}_t + \sqrt{\frac{1}{2}\varphi_j(\rho)}\nabla\dot{\mathcal{W}}_t, \tag{1}$$

where $\dot{\mathcal{W}}_t$ is a space-time white noise with unit variance. Our interest is to strengthening the asymmetry of the jump rates, by decreasing the value of γ, to see its impact in the limiting density field. Nevertheless, for $1/2 < \gamma < 1$, there is no effect of the asymmetry in the limit fluctuation field. This is the content of the next result.

Theorem 1. *For $1/2 < \gamma < 1$, the sequence $\{\mathcal{Y}_t^{n,\gamma}\}_{n\in\mathbb{N}}$ converges, in the uniform topology of $D([0,T],\mathcal{S}'(\mathbb{R}))$ to the process $\mathcal{Y}_t$ which is the solution of (1), with initial condition $\mathcal{Y}_0$ given in Section 1.4.*

However, for $\gamma = 1/2$, the limit field $\mathcal{Y}_t$ has a qualitative behavior completely different from the case $\gamma > 1/2$. The precise results is the following.

Theorem 2. *For $\gamma = 1/2$, the sequence $\{\mathcal{Y}_t^{n,\gamma}\}_{n\in\mathbb{N}}$ is tight in the uniform topology of $D([0,T],\mathcal{S}'(\mathbb{R}))$ and any limit point is an energy solution of the stochastic Burgers equation*

$$\partial_t \mathcal{Y}_t = \frac{\varphi_c'(\rho)}{2}\Delta\mathcal{Y}_t + \frac{a}{2}\varphi_b''(\rho)\nabla\mathcal{Y}_t^2 + \sqrt{\frac{1}{2}\varphi_b(\rho)}\nabla\dot{\mathcal{W}}_t, \tag{2}$$

with initial field $\mathcal{Y}_0$ given in Section (1.4). Above φ_b corresponds to the mean wrt μ_ρ of $b_x = j^R_{x,x+1} + j^L_{x+1,x}$.

We refer the reader to [3] for the notion of energy solutions of the stochastic Burgers equation (2).

3. Examples

Here we present some examples for which we can get the results mentioned above.

3.1. **Exclusion processes.** For these models the state space is $\mathcal{E} = \{0,1\}^{\mathbb{Z}}$. We consider a local function $r : \Omega \to \mathbb{R}$ satisfying $\varepsilon_0 < r(\eta) < \varepsilon_0^{-1}$ for any $\eta \in \Omega$ and for some $\varepsilon_0 > 0$; if for any η and ξ with $\eta(x) = \xi(x)$ for $x \neq 0, 1$, then $r(\eta) = r(\xi)$ and, finally, there exists $\omega : \Omega \to \mathbb{R}$ such that $r(\eta)\big(\eta(1)-\eta(0)\big) = \tau_1\omega(\eta)-\omega(\eta)$, for any $\eta \in \Omega$ (*gradient condition*). In this case $j^R_{x,x+1}(\eta) = r(\tau_x\eta)\eta(x)(1-\eta(x+1))$ and $j^L_{x+1,x}(\eta) = r(\tau_x\eta)\eta(x+1)(1-\eta(x))$. For more details we refer to [2].

3.2. **Zero-range processes.** For these models the state space is $\mathcal{E} = \mathbb{N}_0^{\mathbb{Z}}$. The jump rate of a particle at the site x only depends on the number of particles at x and is given by a function $g : \mathbb{N}_0 \to \mathbb{R}_+$ satisfying $g(0) = 0$, $g(k) > 0$ for $k \geq 1$ and $\sup_{k\geq 0}|g(k+1) - g(k)| < \infty$. Therefore, $j^R_{x,x+1}(\eta) = g(\eta(x))$ and $j^L_{x+1,x}(\eta) = g(\eta(x+1))$, We also need to have zero-range models satisfying the condition on the spectral gap given in Section 1.5. Examples of zero-range processes satisfying that condition are given by $g(x) = x^\gamma$ for $0 < \gamma < 1$; of g

such that there exists x_0 and $\varepsilon_0 > 0$ with $g(x + x_0) - g(x) \geq \varepsilon_0$ for all $x \geq 0$ and $g(x) = 1(x \geq 1)$.

3.3. **Kinetically constrained lattice gases.** For these models the spate space is $\mathcal{E} = \{0,1\}^{\mathbb{Z}}$ and $j^R_{x,x+1}(\eta) = c_x(\eta)\eta(x)(1-\eta(x+1))$ and $j^L_{x+1,x}(\eta) = c_x(\eta)\eta(x+1)(1-\eta(x))$, where $c_x(\eta) = \left[\eta(x-1) + \eta(x+2) + \frac{\theta}{2n}\right]$. It was proved in [3] that these models satisfy the spectral gap condition given in Section 1.5.

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Random matrix asymptotics for the six-vertex model

Vadim Gorin

The six-vertex (or "square-ice") model is one of the most well-studied examples of exactly-solvable lattice models of statistical mechanics. The configurations of the 6–vertex model are assignments of one of 6 types of H_2O molecules shown in Figure 1 to the vertices of $N \times N$ square grid in such a way that the O atoms are at the vertices of the grid. To each O atom there are two H atoms attached, so that they are at angles 90° or 180° to each other, along the grid lines, and between any two adjacent O atoms there is exactly one H, see Figure 2 for an example.

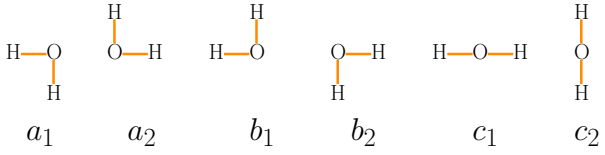

Figure 1. Six types of molecules ("vertices").

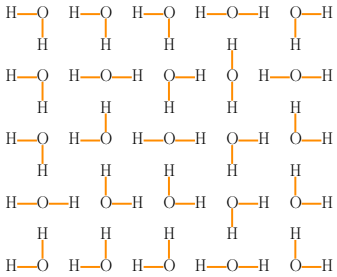

FIGURE 2. A configuration of the six–vertex model with domain–wall boundary conditions.

One typically associates six positive weights to six types of vertices, as shown in Figure 1. We study *Gibbs probability measures* on configurations in finite or infinite domains, i.e. such that the conditional distribution on the configurations in any subdomain assigns to a configuration ω the probability proportional to the product of weights of the vertices in this configuration. For instance, the weight of the configuration shown in Figure 2 should be proportional to $(a_1)^5(a_2)^5(b_1)^4(b_2)^4(c_1)^6c_2$.

In particular, the choice $a_1 = a_2 = b_1 = b_2 = c_1 = c_2 = 1$ leads to the uniform measures. This was the case originally considered by Lieb in 60s, who obtained one of the first mathematical results about the model: Lieb computed the leading asymptotics for the logarithm of the number of configurations of the model on $N \times N$ torus as $N \to \infty$. The answer closely matched the experimental value for the residual entropy of the real–world ice.

The developments of the last 15 years suggest that the asymptotic behavior of the six–vertex model should be governed by the probability distributions of random matrix origin. However, until recently the rigorous mathematical results in this direction were restricted to the so-called free fermion case, which is distinguished by the following relation between the weights:

$$a_1a_2 + b_1b_2 - c_1c_2 = 0. \tag{1}$$

When (1) is satisfied, the model can be analyzed via determinantal point processes; asymptotics of the kernels of these processes yields random–matrix type distributions in the limit. Outside (1) such approach no longer works.

In my talk I discussed two new results, which go beyond the free fermions. The first development deals with uniform measure on the configurations of the model in $N \times N$ square with specific *domain–wall* boundary conditions shown in Figure 2. We prove that for each fixed k as $N \to \infty$ the distribution of the positions of horizontal molecules on the kth horizontal line converges (after proper centering and rescaling) to the eigenvalue distribution of the *Gaussian Unitary Ensemble*. Recall that the Gaussian Unitary Ensemble is the distribution on $k \times k$ Hermitian

matrices with probability density proportional to $\exp\left(-\mathrm{Trace}(X^2/2)\right)$. This is based on the results of [1], [2].

The second result is about the six–vertex model with specific boundary conditions in the quadrant and with weights satisfying the so–called stochastic relation $c_1c_2 = (a_2 - b_1)(a_1 - b_2)$ with $a_2 > b_1$. We prove that the asymptotics of the centered and rescaled *height function* (which can be defined as the number of molecules of types a_2, b_2, and c_2 to the left of a given point) is governed by the *Tracy–Widom distribution* F_2. One way to define F_2 is as the limiting distribution for the largest eigenvalue for $k \times k$ Gaussian Unitary Ensemble as $k \to \infty$. This result will appear in the article [3].

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Pfaffian point processes for interacting particle systems and random matrices

Oleg Zaboronski
(joint work with Roger Tribe)

Annihilating Brownian motions in one dimension (ABM's) is a classical interacting particle system investigated over the years by both mathematics (Bramson, Lebowitz, Griffeath, Kesten,...) and theoretical physics communities (Smoluchowski, Glauber, Doi, Zeldovich, Ovchinnikov, Mikhailov, Peliti, Droz, Derrida, Hakim, Pasteur, ben Avraham, Masser,...)

We study ABM's under the maximal entrance law, which can be viewed as $\lambda \to \infty$ limit of Poisson(λ) initial distribution of particles.

Let $\rho_t^{(n)}(x_1, x_2, \dots, x_n)$ be the Lebesgue intensity for n-particle distribution at time t.

We find [2] that fixed time law of ABM's is a Pfaffian point process:

Theorem 1. *Under the maximal entrance law, ABM's at $t > 0$ is a Pfaffian point process with the kernel $K_t(x, y) = t^{-1/2}K(xt^{-1/2}, yt^{-1/2})$, where*

$$K(x,y) = \begin{pmatrix} -F''(y-x) & -F'(y-x) \\ F'(y-x) & sgn(y-x)F(|y-x|) \end{pmatrix}$$

and $F(x) = erfc(x)$.

A simple corollary of Theorem 1 is the scaling behaviour of the intensities [1]:

Corollary 2. *Under the assumptions of Theorem 1,*

$$\rho_t^{(n)}(x_1, x_2, \dots, x_n) = c_n|\Delta(x)|t^{-n/2-n(n-1)/4}(1 + O(t^{-1/2})),$$

where $c_n > 0$ is independent of t and x and $\Delta(x)$ is the Vandermonde determinant.

Moreover, we find that ABM's as a process can be characterised as an extended Pfaffian point process [3]:

Theorem 3. *Under the maximal entrance law, ABM's at $t > 0$ is an extended Pfaffian point process,*

$$\rho^{(n)}_{t_1 t_2 \dots t_n}(x_1, x_2, \dots, x_n) = Pf(K((t_i, x_i), (t_j, x_j) : i < j)),$$

where for $t > s$ and $i, j \in \{1, 2\}$

$$K^{ij}((t, x); (s, y)) = G_{t-s} K^{ij}_s(y - x) - I_{\{i=1, j=2\}} g_{t-s}(y - x),$$

where $K_t(x, y)$ is the single time kernel defined in Theorem 1.

The kernel which defines the law of ABM's at a fixed time was originally derived in [5],[6] as the $N \to \infty$ bulk limit of the Pfaffian point process for real eigenvalues in the real $N \times N$ Ginibre ensemble. It is therefore natural to ask whether ABM's and the evolution of real eigenvalues for Brownian motions with values in real $N \times N$ matrices are identical as processes in the limit of large N. We refer to the the matrix valued Brownian motion as Ginibre(N) process.

The answer to the above question turns out to be negative as can be seen from the following. Let

$$S_t(x) = (-1)^{N_t(0,x)},$$

where $N_t(0, x)$ is the number of real eigenvalues of the Ginibre(N) process in the interval $(0, x)$. We refer to the random function $S_t(x)$ as 'spin variable'. Note that spin variables can be defined for any simple point process on a line including ABM's. We have [4]:

Theorem 4. *Let $\mathbb{E}_N$ be the expectation value with respect to Ginibre(N) process. Then*

$$\lim_{N \to \infty} \mathbb{E}_N \left(S_{t+\tau}(x) S_t(0) \right) = 0$$

$$\lim_{N \to \infty} \mathbb{E}_N \left(S_{t+\frac{T}{N}}(x) S_t(0) \right) = erfc \left(\sqrt{\frac{x^2}{t} + \frac{T}{2t}} \right),$$

which is different from the behaviour of the corresponding product moment of spin variables for ABM's.

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Continuity of the phase transition of the three dimensional Ising model

Hugo Duminil-Copin
(joint work with Michael Aizenman, Vladas Sidoravicius)

The Ising model is one of the most classical example of a system undergoing a phase transition. The transition it exhibits was found to be of rather broad relevance and the model has provided the testing ground for a large variety of techniques.

We focus here on the ferromagnetic Ising model on $\mathbb{Z}^d$. Associated with the sites are ± 1 valued spin variables, whose configuration is denoted $\sigma = (\sigma_x : x \in \mathbb{Z}^d)$. For a general ferromagnetic pair interaction, the system's Hamiltonian defined for finite subsets $\Lambda \subset \mathbb{Z}^d$ and + boundary conditions is given by the function

$$H^+_\Lambda(\sigma) \;:=\; - \sum_{\{x,y\}\subset\Lambda : x\neq y} J_{x,y}\sigma_x\sigma_y \;- \sum_{x\in\Lambda : y\in\mathbb{Z}^d\setminus\Lambda} J_{x,y}\sigma_x\,,$$

for any $\sigma \in \{-1,1\}^\Lambda$, where $(J_{x,y})_{x,y\in\mathbb{Z}^d}$ is a family of nonnegative *coupling constants*. If $J_{x,y} = 1$ if x and y are neighbors, and 0 otherwise, the model is said to be *nearest neighbor*.

For $\beta \in (0,\infty)$, finite volume Gibbs states with + boundary conditions are given by probability measures on the spaces of configurations in finite subsets $\Lambda \subset \mathbb{Z}^d$ under which the expected values of functions $f : \{-1,1\}^\Lambda \to \mathbb{R}$ are

$$\langle f\rangle^+_{\Lambda,\beta,h} = \sum_{\sigma\in\{-1,1\}^\Lambda} f(\sigma)\frac{e^{-\beta H^+_\Lambda(\sigma)}}{Z^+(\Lambda,\beta,h)}\,,$$

where the sum is normalized by the partition function $Z^+(\Lambda,\beta,h)$ so that $\langle 1\rangle^+_{\Lambda,\beta} = 1$. The finite-volume Gibbs states are known to converge to the corresponding infinite-volume Gibbs measures $\langle\cdot\rangle^+_{\mathbb{Z}^d,\beta}$.

Of particular interest is the model's phase transition, which is reflected in the non vanishing of the symmetry breaking order parameter (which is customarily referred to as the *spontaneous magnetization*): for $d \geq 2$, there exists $\beta_c = \beta_c(d) \in (0,\infty)$ such that $\langle\sigma_0\rangle^+_{\mathbb{Z}^d,\beta} > 0$ if $\beta > \beta_c$ and $\langle\sigma_0\rangle^+_{\mathbb{Z}^d,\beta} = 0$ if $\beta < \beta_c$. The goal of this talk is to present a study of the critical case $\beta = \beta_c$.

Theorem 1 (Aizenman, Duminil-Copin, Sidoravicius [2])**.** *For the nearest-neighbor ferromagnetic Ising model on $\mathbb{Z}^3$, we have that $\langle\sigma_0\rangle^+_{\mathbb{Z}^3,\beta_c} = 0$.*

Theorem 1 is proven by extending the random current representation of [4], to infinite graphs and establishing uniqueness of the infinite cluster for the resulting system of (duplicated) random currents on $\mathbb{Z}^d$. The second ingredient in the proof is the infrared bound, a fact which follows from Reflection Positivity (see e.g. [6]).

The past results on the continuity of the spontaneous magnetization at β_c are naturally split into two distinct classes: *i.* the special low dimensional case of $d = 2$, and *ii.* high dimensions. The earliest results have been derived for the nearest neighbor model in $d = 2$ dimensions, for which the spontaneous magnetization was computed by Yang [7] (using the methods of Onsager), and proved to be equal to 0 at β_c. For high dimensions, the continuity of the spontaneous magnetization

in the nearest neighbor model was established in [3] for $d \geq 4$ through reflection positivity bounds combined with differential inequalities.

Let us finish by mentioning that the continuity of the phase transition fails for closely related models; for instance for the ferromagnetic nearest-neighbor Potts models with Q large enough (see [5] and references therein). The Ising model itself can also have a discontinuous phase transition, as shown by the special case of the one-dimensional model with $J_{x,y} = 1/|x-y|^2$ (see. e.g. [1]).

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Mesoscopic Eigenvalue Statistics for Random Band Matrices

Antti Knowles

(joint work with L. Erdős)

Random band matrices were introduced in the physics literature in 80s to model quantum transport in a disordered system, as an alternative to the Anderson model. They are expected to exhibit the same behaviour as the Anderson model, but are in many cases more amenable to analysis. These two properties make them particularly attractive for establishing rigorous results. Moreover, they contain an additional parameter, the *band width*, which may be used to interpolate between the Anderson model and Wigner matrices.

Let $\mathbb{T} := ([-L/2, L/2) \cap \mathbb{Z})^d$ denote the discrete cube of side length L. A d-dimensional random band matrix is a Hamitonian $H = H^* = (H_{xy})_{x,y \in \mathbb{T}}$ whose upper-triangular entries $(H_{xy} : x \leq y)$ are independent mean-zero random variables. Their magnitudes are given by their variances

$$S_{xy} := \mathbb{E}|H_{xy}|^2 = \frac{1}{W^d} f\left(\frac{x-y}{W}\right), \tag{1}$$

where $W \in [1, L]$ is the *band width* and f is the *profile function*, a fixed symmetric probability density on $\mathbb{R}^d$. If $W = L$ then S_{xy} is essentially independent of x and y, which implies that the profile of the variances of H is flat and hence H is

essentially a Wigner matrices. At the opposite end, if $W = O(1)$, only an order L^d entries of H are nonzero, all of them concentrated around the diagonal; this corresponds roughly to the (tridiagonal) Anderson model.

We are always interested in the limit $W \to \infty$. In this case, it is not hard to check that the macroscopic eigenvalue density of H converges to the semicircle law supported in the interval $[-2, 2]$. Fix an energy $E \in (-2, 2)$ in the bulk of the spectrum, and let $\eta \equiv \eta_W$ denote a possibly W-dependent scale. We are interested in the eigenvalue statistics on the scale η around the energy E. The two most commonly studied scales are the *macroscopic scale*, $\eta \asymp 1$, and the *microscopic scale*, $\eta \asymp L^{-d}$. The macroscopic scale is the scale on which the global eigenvalue density, the semicircle law, is visible. This law is model-dependent (for instance it does not hold for the Anderson model), and has no physical relevance. The distribution of individual eigenvalues and eigenvalue spacings take place on the microscopic scale. For Wigner matrices, the microscopic statistics are known to be governed by sine kernel statistics [7, 8]. On the other hand, for the one-dimensional Anderson model, they are governed by Poisson statistics [10]. Generally, it is expected that the microscopic eigenvalue statistics of a generic random Hamiltonian (including the random band matrices) is governed by one of these two statistics.

A mesoscopic scale is any scale η between the macroscopic and microscopic scales: $L^{-d} \ll \eta \ll 1$. A convenient way to study the eigenvalue statistics around the energy E at any scale η is via the distribution of the linear statistics

$$Y_\phi^\eta(E) \;:=\; \sum_i \phi^\eta(\lambda_i - E)\,, \qquad \phi^\eta(e) \;:=\; \eta^{-1}\phi(e/\eta)\,,$$

where $\{\lambda_i\}$ denote the eigenvalues of H. Here ϕ is some suitable test function. Apart from the independent mathematical interest, an important physical motivation to study the statistics of $Y_\phi^\eta(E)$ is a result from the physics literature by Thouless [12], asserting that the conductance of a sample of linear size L is determined by the energy levels in an energy band of a specific, mesoscopic, width η around the Fermi energy.

The correlations of $\{Y_\phi^\eta(E_i)\}$ may be expressed using the *truncated correlation functions* $p^{(k)}$: for instance

$$\langle Y_\phi^\eta(E_1)\,; Y_\phi^\eta(E_2)\rangle \;=\; \int \mathrm{d}x\,\mathrm{d}y\,\phi^\eta(x - E_1)\,\phi^\eta(y - E_2)\,p^{(2)}(x; y)\,.$$

If the sine kernel held on all mesoscopic scales, we would get, with $\omega := |E_2 - E_1|$,

$$\int_{|e-\omega|\le\eta} \left(\frac{\sin(e/\Delta)}{e/\Delta}\right)^2 \mathrm{d}e \;\sim\; \frac{1}{\omega^2} \qquad (\Delta \ll \eta \ll \omega \ll 1)\,. \tag{2}$$

The extrapolation from $\eta \sim \Delta$ to $\eta \gg \Delta$ looks easy. In fact, (2) has been proved for GUE [3].

However, the extrapolation (2) is in general wrong. In particular, for random band matrices a new type of statistics emerges on mesoscopic scales, which we call the *Altshuler-Shklovskii statistics* after the seminal work [1] in which it was first predicted. We emphasize that the Altshuler-Shklovskii statistics are only expected

to occur on mesoscopic scales. In Figure 1, we give a summary of the expected phase diagram of the mesoscopic eigenvalue statistics of a random band matrix for $d = 1$ and $d = 3$.

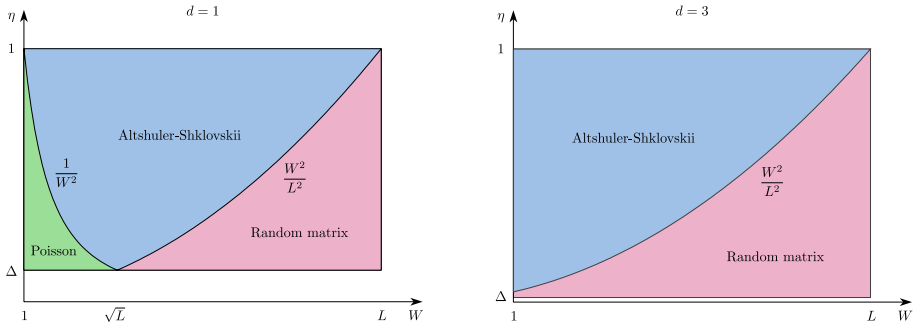

FIGURE 1. The phase diagram for the mesoscopic eigenvalue statistics of d-dimensional random band matrices for $d = 1$ (left) and $d = 3$ (right). Here W is the band width of the matrix and η the scale of the energy window. In general, one expects to observe one of three types of statistics: Poisson, random matrix (exhibited e.g. by GUE), or Altshuler-Shklovskii. The transition from random matrix statistics to Altshuler-Shklovskii statistics occurs at the Thouless energy $\eta_c \sim W^2/L^2$. Here $\Delta := L^{-d}$ is the typical eigenvalue spacing, and hence the *microscopic* scale. Conversely, 1 is the *macroscopic* scale, on which the limiting eigenvalue density of H is supported. The range $\Delta \ll \eta \ll 1$ is the mesoscopic regime.

In fact, there is a sharp transition in the mesoscopic eigenvalue statistics of random band matrices, which is related to the phenomenon of *quantum diffusion* [6]. The critical scale is the so-called *Thouless energy*

$$\eta_0 \;=\; \big(\text{time for diffusion to reach the boundary of } \mathbb{T}\big)^{-1}.$$

For random band matrices the diffusion coefficient is W^2 [6], so that $\eta_0 \sim W^2/L^2$. The physical heuristic is that for $\eta \gg \eta_0$ boundary effects on $\mathbb{T}$ are irrelevant, while for $\eta \ll \eta_0$ the statistics are mean-field.

The Altshuler-Shklovskii formulas may be informally summarized as follows.

- In the *diffusive regime*, $\eta_0 \ll \eta \ll 1$, we have

$$\operatorname{Var} Y_\phi^\eta(E) \;\sim\; (\eta/\eta_0)^{d/2-2} \qquad (d = 1, 2, 3)$$

and

$$\langle Y_\phi^\eta(E + \omega/2)\,; Y_\phi^\eta(E - \omega/2)\rangle \;\sim\; \omega^{d/2-2} \qquad (d = 1, 3)\,,$$

for $\eta \ll \omega \ll 1$. For $d = 2$, the right-hand side of the latter formula vanishes in leading order.

- In the *mean-field regime*, $\eta \ll \eta_0$, the same formulas hold with $d = 0$.

Our results [4, 5] on the meoscopic eigenvalue statistics of d-dimensional random band matrices are valid for mesoscopic scales $\eta \gg W^{-d/3}$. They may be summarized as follows.

(a) We prove the Altshuler-Shklovskii formulas for $d = 1, 2, 3, 4$, and show in particular that they do not depend on the details of H, such as the law of the entries of the shape of the profile function f. This is a universality result of the mesoscopic eigenvalue statistics.
(b) For $d \geq 5$ the universality breaks down, and the behaviour of the linear statistics depends on the shape of f in a complicated, albeit explicit, fashion.
(c) We prove a central limit theorem: the mesoscopic densities $\{Y_\phi^\eta(E)\}_{\phi,E}$ converge to Gaussian process whose covariance is given by the Altshuler-Shklovskii formulas.
(d) For $d = 2$ the correlations are governed by so-called *weak localization corrections*. Our result differs substantially from the prediction [9] in the physics literature.
(e) We investigate a critical band matrix model, defined by $d = 1$ and $S_{xy} = \mathbb{E}|H_{xy}|^2 \sim |x - y|^{-2}$. This model is expected to describe the system at the so-called metal-insulator transition, where the microscopic statistics change from sine kernel to Poisson statistics. Our result agrees with the prediction from [2] on the multifractality of the eigenvectors.
(f) We introduce a large family of random band matrices that interpolates between the real ($\beta = 1$) and complex ($\beta = 2$) symmetry classes, and track the crossover in the mesoscopic eigenvalue statistics.

The starting point of the proofs is a Chebyshev-Fourier expansion of the linear statistics, similar to the ones used in [11, 6]. The main work is to control the resulting sums, which are highly oscillatory. This requires somewhat involved algebra, which is encoded by a graphical language. The oscillations are dealt with by a resummation of certain families of subgraphs, which then have to be computed with high precision before estimates may be applied. The leading term, which in particular yields the Altshuler-Shklovskii formulas, arises from the so-called *one-loop* diagrams.

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Variational formulas for directed polymer and percolation models

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Let $\omega = (\omega_x)_{x\in\mathbb{Z}^2}$ be an IID weight configuration on the planar integer lattice under the probability distribution $\mathbb{P}$. Make the moment assumption $\mathbb{E}\,|\omega_x|^p < \infty$ for some $p > 2$. An admissible directed lattice path $(x_k)_{k=0}^n$ satisfies $x_k - x_{k-1} \in \{e_1, e_2\}$ where $e_1 = (1, 0)$ and $e_2 = (0, 1)$ are the basis vectors. The partition function of the point-to-line directed polymer in 1+1 dimensions is

$$Z_{(N)} = \sum_{(x_k)_{k=0}^N} 2^{-N} \exp\Big\{\beta \sum_{k=0}^{N-1} \omega_{x_k} + \beta h \cdot x_N\Big\},$$

where the sum is over all admissible directed lattice paths of length N starting at $x_0 = 0$, $0 < \beta < \infty$ is an inverse temperature parameter and $h \in \mathbb{R}^2$ is an external field.

The limiting free energy

$$g_{\mathrm{pl}}^{\beta}(h) = \lim_{N\to\infty} N^{-1}\beta^{-1} \log Z_{(N)}$$

exists $\mathbb{P}$-almost surely for subadditivity reasons. The purpose of the talk is to describe a variational formula for $g_{\mathrm{pl}}^{\beta}(h)$ and the connections of this formula to other features of the model.

In order to state the variational formula we need the following definition. Let $(T_x)_{x\in\mathbb{Z}^2}$ denote the group of translations on the space Ω of environments ω. A measurable function $F : \Omega\times\mathbb{Z}^2\times\mathbb{Z}^2 \to \mathbb{R}$ is a *stationary* $L^1(\mathbb{P})$ *cocycle* if it satisfies the following three conditions for all $x, y, z \in \mathbb{Z}^2$ and for $\mathbb{P}$-a.e. ω.

(i) Integrability: $\mathbb{E}|F(x, y)| < \infty$.

(ii) Stationarity: $F(\omega, z + x, z + y) = F(T_z\omega, x, y)$.

(iii) Cocycle property: $F(\omega, x, y) + F(\omega, y, z) = F(\omega, x, z)$.

The cocycles is *centered* if $\mathbb{E}[F(x, y)] = 0$ for all $x, y \in \mathbb{Z}^2$. The space of centered cocycles is denoted by $\mathcal{C}_0$.

In the point-to-line case, the variational formula is stated as follows [1]:

$$g^{\beta}_{\mathrm{pl}}(h) = \inf_{F\in\mathcal{C}_0} \mathbb{P}\text{-ess}\sup_{\omega} \beta^{-1}\log \sum_{i\in\{1,2\}} \tfrac{1}{2} e^{\beta\omega_0+\beta h\cdot e_i+\beta F(\omega,0,e_i)}.$$

A minimizing cocyle F is expected to come from Busemann functions. This can be presently proved for the exactly solvable log-gamma polymer introduced in [2].

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Ergodic theory of stochastic Burgers equation in noncompact setting

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The inviscid Burgers equation is

$$\partial_t u(t,x) + \partial_x\left(\frac{u^2(t,x)}{2}\right) = \partial_x F(t,x). \tag{1}$$

In fluid dynamics terms, $u(t,x)$ is the velocity of the particle located at point $x\in\mathbb{R}$ at time $t\in\mathbb{R}$, and $f(t,x)=\partial_x F(t,x)$ is the external forcing term describing the acceleration of the particle at time t at point x.

We can represent the velocity profile as $u(t,x)=\partial_x U(t,x)$, where the potential $U(t,x)$ is a solution of the Hamilton–Jacobi–Bellman (HJB) equation

$$\partial_t U(t,x) + \frac{(\partial_x U(t,x))^2}{2} = F(t,x), \tag{2}$$

we can write the solution of the Cauchy problem of the latter as

$$U(t,x) = \inf_{\gamma:[t_0,t]\to\mathbb{R}} \left\{ U_0(\gamma(t_0)) + \frac{1}{2}\int_{t_0}^{t} \dot\gamma^2(s)ds + \int_{t_0}^{t} F(s,\gamma(s))ds \right\}, \tag{3}$$

where the infimum is taken over all absolutely continuous curves γ satisfying $\gamma(t)=x$. Then the solution u of the Burgers equation can be found either by $u(t,x)=\partial_x U(t,x)$ or by using the slope of γ^*, the path on which the minimum in (3) is attained: $u(t,x)=\dot\gamma^*(t)$.

We are interested in the ergodic theory of this equation when the forcing is random. This problem was considered first in compact settings, and a complete description of invariant distributions on a circle or, in the multi-dimensional version of the problem, on a torus, was obtained in [8], [14], [9]. Also the existence and uniqueness of an invariant distribution for the Burgers dynamics with random boundary conditions were proved in [1].

In these cases, the existence and uniqueness of an invariant distribution on the set of velocity profiles with a given average followed from the One Force — One Solution Principle (1F1S) that asserts that for any $v\in\mathbb{R}$ and for almost every realization of the forcing in the past, there is a unique velocity profile at

the present that averages to v and is compatible with the history of the forcing. For any v, the collection of those velocity profiles indexed by all times $t \in \mathbb{R}$ forms a global solution that can be understood as a random one-point pullback attractor. Moreover, the distribution of this global solution at any fixed time is then a unique invariant distribution for the Markov semigroup generated by the Burgers equation on velocity profiles averaging to v.

The Burgers equation preserves mean velocity and is invariant with respect to Galilean shear space-time transformations, so without loss of generality one may confine oneself to zero average velocity.

To establish 1F1S, one can use the variational principle and reduce the study of the long term properties of the system to the behavior of minimizers over long time intervals. One can prove, in fact, that minimizers over increasing time intervals in the past converge to limiting trajectories, so called one-sided infinite minimizers. These are paths with infinite history such that every finite restriction of such a path minimizes Lagrangian action over paths with the same endpoints. The entire space-time is foliated by these trajectories, and one can use their slopes to construct a global solution and prove that any global solution has to agree with this field of one-sided minimizers.

For this program to go through one needs an additional property of one-sided minimizers called hyperbolicity. It means that those paths approach each other in the past sufficiently fast. The reason to consider this property is that in order to use (3) in the proof that the velocity profile obtained from slopes of the minimizers is, in fact, a global solution, one has to keep track not only of the velocity, but also of the velocity potential. To find the increment of the velocity potential, we can consider two minimizers approaching each other in reverse time. Although the action corresponding to each of them is infinite, one still can make sense of the difference in action between these paths since there will be a diminishingly small contribution from times in a distant past. This analysis leads to an analogy between the global solution of the HJB equation (2) and Busemann functions in last passage percolation theory.

For 1D periodic (or circle) setting hyperbolicity was first established in [8] and recently a simpler proof exploiting the rigidity of 1D geometry was constructed in [5].

The first attempts of extending this program to noncompact settings, i.e., evolution of velocity profiles on the entire real line with no periodicity or other compactness assumption were [10] and [2].

In this talk we consider two fully noncompact space-time stationary settings. In the first case, the forcing is concentrated at configuration points of a space-time stationary Poisson point process. In the second case, the forcing is of "kick" type, i.e., at each integer time the velocity profile undergoes an instantaneous increment, and between those the evolution follows the unforced Burgers equation. The velocity increments are i.i.d. spatially stationary processes with finite dependence range.

In both cases one can use the last passage percolation approach of [15], [11], [12], [13], [16], [7], [6], although several technical difficulties have to be overcome. The entire program consists of constructing fields of one-sided minimizers for every value of the asymptotic slope, studying their properties, and using them to construct global solutions of the Burgers equation and proving 1F1S.

In [4] where the Poissonian forcing was cosidered, the following strengthening of the hyperbolicity property was instrumental in constructing a global solution, proving its uniqueness and attraction property: with probability 1, any two of these one-sided minimizers coalesce, i.e., they meet at one of the Poissonian points and coincide from that point on. This strengthening of the hyperbolicity property is certainly an artefact of the model where the forcing is concentrated in a discrete set of space-time points.

In the second case, distinct minimizers do not coalesce and it is not known if they are asymptotic to each other, i.e., if hyperbolicity holds true. This poses a serious difficulty. However, in [3], we are able to replace hyperbolicity by a much weaker property. We prove that any two one-sided minimizers approach each other rather closely *along a sequence* (m') of *pairing times*. It turns out that this weak hyperbolicity is sufficient to prove existence and uniqueness of a global solution of the Burgers equation in our model, and to study its domain of attraction. Of course, some limit transitions $m \to -\infty$ are replaced by limits along the pairing sequence of times m'. In particular, our definition of Busemann function is based on partial limits.

The new argument we use to prove the weak hyperbolicity is quite soft and can be applied to other last passage percolation type models and random Lagrangian systems.

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Height fluctuations for the stationary KPZ equation

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(joint work with Alexei Borodin, Ivan Corwin, and Bálint Vető)

In their seminal 1986 paper [13], Kardar, Parisi, and Zhang (KPZ) proposed the stochastic evolution equation for a height function $h(t,x) \in \mathbb{R}$ ($t \in \mathbb{R}_+$ is time and $x \in \mathbb{R}$ is space)

$$\partial_t h(t,x) = \tfrac{1}{2}\partial_x^2 h(t,x) + \tfrac{1}{2}\left(\partial_x h(t,x)\right)^2 + \xi(t,x).$$

The randomness ξ models the deposition mechanism and it is taken to be space-time Gaussian white noise, so that formally $\mathbb{E}[\xi(t,x)\xi(t',x')] = \delta(t-t')\delta(x-x')$. The Laplacian reflects the smoothing mechanism and the non-linearity reflects the slope-dependent growth velocity of the interface. Using earlier physical work of Forster, Nelson and Stephen [10] KPZ predicted that for large time t, the height function $h(t,x)$ has fluctuations around its mean of order $t^{1/3}$ with spatial correlation length of order $t^{2/3}$. For additional background, see the reviews [8, 15, 9, 17].

The physically relevant solution to the KPZ equation [17] is defined indirectly via the well-posed stochastic heat equation (SHE) with multiplicative noise [5, 2, 16],

$$\partial_t \mathcal{Z}(t,x) = \tfrac{1}{2}\partial_x^2 \mathcal{Z}(t,x) + \mathcal{Z}(t,x)\xi(t,x),$$

with initial condition $\mathcal{Z}(0,x) = \mathcal{Z}_0(x) = e^{h(0,x)}$. The SHE is well-posed and we defines $h(t,x) = \ln(\mathcal{Z}(t,x))$. This is called the Cole – Hopf solution of the KPZ equation.

By a version of the Feynman – Kac formula, the solution of the SHE can be written as

$$\mathcal{Z}(t,x) = \mathbb{E}_{t,x}\left[\mathcal{Z}_0(b(0)) \,:\exp:\left(-\int_0^t \xi(b(s),s)ds\right)\right]$$

where the expectation $\mathbb{E}_{t,x}$ is over a Brownian motion $b(\cdot)$ going backwards in time from $b(t) = x$, and where $:\exp:$ is the Wick ordered exponential. This provides

an interpretation for $\mathcal{Z}(t,x)$ as the partition function of the continuum directed random polymer (CDRP) [2, 1].

Let $B(x)$ be a two-sided Brownian motion with $B(0)=0$ and zero drift. Stationary (zero drift) initial data $h(0,x)=B(x)$ for the KPZ equation corresponds with SHE initial data $\mathcal{Z}(0,x)=e^{B(x)}$. This is called stationary because for any later time t, $h(t,\cdot)$ is marginally distributed as $\tilde{B}(\cdot)+h(t,0)$ where $\tilde{B}(\cdot)$ is a two-sided Brownian motion (though not independent of B or $h(t,0)$).

In our work [7] we provide an exact formula for the one-point probability distribution of the stationary solution to the KPZ equation, and a limit theorem for $h(t,x)$, after proper centering and scaling by $t^{1/3}$. This is made by analyzing a semidiscrete directed polymer model, which in its turn is obtained as a limit of the q-Whittaker process [6] with appropriate initial measure. A different expression using replica trick approach was obtained previously in [12]. The equivalence of the two formulas has not been shown so-far.

For simplicity, let us present the results in the case of zero-drift and position $x=0$ only.

Theorem 1. *Let $h(t,x)$ be the stationary (zero drift) solution to the KPZ equation and let K_0 denote the modified Bessel function. Then, for $t>0$, $\sigma=(2/t)^{1/3}$ and $S\in\mathbb{C}$ with positive real part,*

$$\mathbb{E}\left[2\sigma K_0\left(2\sqrt{S\,\exp\left[\tfrac{t}{24}+h(t,0)\right]}\right)\right]=f\left(S,\sigma\right),$$

where the function f is given below.

To define f, define on $\mathbb{R}_+$ the function

$$Q(x)=\frac{1}{2\pi\mathrm{i}}\int_{-\frac{1}{4\sigma}+\mathrm{i}\mathbb{R}}\mathrm{d}w\frac{\sigma\pi S^{-\sigma w}}{\sin(\pi(-\sigma w))}e^{-w^3/3+wx}\frac{\Gamma(\sigma w)}{\Gamma(-\sigma w)},$$

and the kernel

$$K(x,y)=\frac{1}{(2\pi\mathrm{i})^2}\int_{-\frac{1}{4\sigma}+\mathrm{i}\mathbb{R}}\mathrm{d}w\int_{\frac{1}{4\sigma}+\mathrm{i}\mathbb{R}}\mathrm{d}z\frac{\sigma\pi S^{\sigma(z-w)}}{\sin(\sigma\pi(z-w))}\frac{e^{z^3/3-zy}}{e^{w^3/3-wx}}\frac{\Gamma(-\sigma z)}{\Gamma(\sigma z)}\frac{\Gamma(\sigma w)}{\Gamma(-\sigma w)}.$$

Let $\gamma_{\mathrm{E}}=0.577\ldots$ be the Euler constant, $\gamma=\sigma(2\gamma_{\mathrm{E}}+\ln S)$ and define

$$f(S,\sigma)=-\det(\mathbb{1}-K)\Big[\gamma+\big\langle(\mathbb{1}-K)^{-1}(K1+Q),1\big\rangle+\big\langle(\mathbb{1}-K)^{-1}(1+Q),Q\big\rangle\Big].$$

where the determinants and scalar products are all meant in $L^2(\mathbb{R}_+)$.

There is an explicit inversion formula, although this is not needed to get the large time limit.

Corollary 2. *For any $r\in\mathbb{R}$, we have*

$$\begin{aligned}&\mathbb{P}\left(h(t,0)\le-\frac{t}{24}+r(t/2)^{1/3}\right)\\&=\frac{1}{\sigma^2}\frac{1}{2\pi\mathrm{i}}\int_{-\delta+\mathrm{i}\mathbb{R}}\frac{\mathrm{d}\xi}{\Gamma(-\xi)\Gamma(-\xi+1)}\int_{\mathbb{R}}\mathrm{d}x\,e^{x\xi/\sigma}f\left(e^{-\frac{x+r}{\sigma}},\sigma\right)\end{aligned}$$

for any $\delta>0$ and where $\sigma=(2/t)^{1/3}$.

Universality arises in the large time limit. Indeed, we recover the distribution formula obtained before for the totally asymmetric simple exclusion process and for the polynuclear growth model [14, 3, 4, 11].

Corollary 3. *For any $r \in \mathbb{R}$,*

$$\lim_{t\to\infty} \mathbb{P}\left(h(t,0) \leq -\frac{t}{24} + r\left(\frac{t}{2}\right)^{1/3} \right) = F_0(r),$$

where F_0 is given by

$$F_0(r) = \frac{\partial}{\partial r}\left(g(r) \det\left(\mathbb{1} - P_r K_{\mathrm{Ai}} P_r \right)_{L^2(\mathbb{R})} \right),$$

where $P_s(x) = \mathbb{1}_{\{x>s\}}$, $K_{\mathrm{Ai}}(x,y) = \int_0^\infty \mathrm{d}\lambda \mathrm{Ai}(x+\lambda)\mathrm{Ai}(y+\lambda)$ is the Airy kernel, and $g(r)$ is given below.

To define the function g we need some notations. For $s \in \mathbb{R}$, define

$$\mathcal{R} = s + \int_s^\infty \mathrm{d}x \int_0^\infty \mathrm{d}y \mathrm{Ai}(x+y), \quad \Psi(y) = 1 - \int_0^\infty \mathrm{d}x \mathrm{Ai}(x+y),$$

$$\Phi(x) = \int_0^\infty \mathrm{d}\lambda \int_s^\infty dy \mathrm{Ai}(x+\lambda)\mathrm{Ai}(y+\lambda) - \int_0^\infty \mathrm{d}y \mathrm{Ai}(y+x).$$

Then the function g is defined by

$$g(s) = \mathcal{R} - \left\langle (\mathbb{1} - P_s K_{\mathrm{Ai}} P_s)^{-1} P_s \Phi, P_s \Psi \right\rangle.$$

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