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Classical Algebraic Geometry

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Abstract. Progress in algebraic geometry usually comes through the introduction of new tools and ideas to tackle the classical problems of the field. Examples include new invariants that capture some aspect of geometry in a novel way, such as Voisin's "existence of decomposition of the diagonal", and the extension of the class of geometric objects considered to allow constructions not previously possible, such as stacks, tropical geometry, and log structures. Many famous old problems and outstanding conjectures have been resolved in this way over the last 50 years. While the new theories are sometimes studied for their own sake, they are in the end best understood in the context of the classical questions they illuminate. The goal of the workshop was to study new developments in algebraic geometry, in the context of their application to the classical problems.

Mathematics Subject Classification (2010): 14xx.

Introduction by the Organisers

The workshop Classical Algebraic Geometry, organised by Olivier Debarre (Paris), David Eisenbud (Berkeley), Gavril Farkas (Berlin) and Ravi Vakil (Stanford) was held July 5–9, 2014 and was attended by 53 participants from around the world. The participants ranged from senior leaders in the field to young post-doctoral fellows and several advanced PhD students. The program consisted of 17 one hour talks. Most lectures were followed by lively discussions among participants, at times continuing well into the night. The schedule was designed in such a way to allow ample time for discussions. On Tuesday evening, we had eight short presentations from young researchers, who got a chance in this way to introduce themselves and their work. For a flavor of the range of subjects covered, a few of the talks are highlighted below.

Ein-Lazarsfeld's proof of the Gonality Conjecture. One of the most exciting announcements of the workshop was made by Lazarsfeld. The background is this: In 1984 Mark Green initiated a deep study of the geometry of curves through the higher syzygies of their canonical ideals, vastly sharpening and extending the work of Noether, Petri, and many others. The conjectures he made have inspired a great deal of work. Certain cases were settled by Hirschowitz and Ramanan, and finally all the *generic* cases were settled by Voisin, but the full conjecture remains open.

Just after Green's initial work, Lazarsfeld and he considered the invariants that might be accessible through "high degree'" embeddings, and made another bold conjecture, amounting to the statement that the gonality of each curve can re read off the resolution of any sufficiently positive line bundle. Although the field as a whole generated a great deal of activity, essentially no progress on the Gonality conjecture for arbitrary curves was made... until this year, when Ein and Lazarsfeld announced an amazingly simple proof, using some of the ideas of Voisin. At the time of the Classical Algebraic Geometry meeting there was still no preprint, but Lazarsfeld and Ein were both at the meeting, and Lazarsfeld presented the proof, nearly in its entirety.

The result is close to the work and interests of many participants, and is sure to spark further progress in the field. This is part of the inspiration for our proposal of Syzygies as one of the (tentative) focus areas for the next Classical Algebraic Geometry workshop.

Dynamics and algebraic surfaces. Serge Cantat gave a beautiful and very instructive talk on some of his recent results on the dynamics of an automorphism f of a smooth complex projective surface X. Since most of the participants were not specialists in this area of complex geometry, he started off by explaining the basics: f acts linearly on the second complex cohomology group of X and must respect the cone of ample classes. The Hodge Index Theorem implies that if there is an eigenvalue with complex modulus > 1, this eigenvalue is unique and is a real number, denoted by $\lambda(f)$. The number of isolated periodic points of f of period dividing N then grows like $\lambda(f)^N$, and these periodic points equidistribute to an f-invariant probability measure μ_f.

Cantat's result (in collaboration with Dupont) is that if μ_f is absolutely continuous with respect to the Lebesgue measure, f "comes from" an automorphism of an abelian surface.

Although the setting and the objects were well-known to the audience, the point of view and the techniques are quite different and Cantat's talk was a welcome occasion to look at usual (for algebraic geometers) material from a new perspective.

Tautological rings of Jacobians. Qizheng Yin reported on the results of his exciting PhD Thesis in which he showed that, using only relatively elementary properties of the universal Jacobian variety (existence of a Fourier-Mukai transform and that of a $\mathfrak{sl}_2$-action), one can derive all the relations in the tautological ring of the moduli space of curves. In particular, he was able confirm a famous conjecture of Faber's that the tautological ring satisfied Poincare duality for low genus and offer evidence that it fails for high genus. Yin's way of producing relations is surprisingly simple and it remains a challenge to understand why his relations are essentially the same as the so-called Faber-Zagier relations recently established by Pandharipande and Pixton (another set of cohomology relations in moduli, which appears in a totally different shape and which is conjectured to span all the relations in the tautological ring of $\mathcal{M}_g$).

Decomposition of the diagonal and rationality questions. Rationality questions (of how close a variety is to usual affine space) have been central to mathematics since before the time of Diophantus. There are blunt tools, usually cohomological (and more recent than Diophantus!), which certify that a variety is not rational. An ongoing central challenge is to understand the differences (or lack thereof) between these notions of "almost rationality". As an example, it has been known from the early 1970s, by many means, that unirational varieties need not be rational in dimension at least 3. Artin and Mumford's arguments also show that unirational varieties might not even be stably birational. Voisin opened the conference with one of her typically dramatic talks, introducing a useful new invariant under stable birationality: the existence of a Chow-theoretic or cohomological decomposition of the diagonal in the product. As an example of the power of this invariant, she showed that the desingularization of a very general quartic double solid with at most 7 nodes is not stably rational.

The young participants Roland Abuaf, Giulio Codogni, Anand Deopurkar, Michael Kemeny, Mario Kummer, Margherita Lelli-Chiesa, Daniel Litt and Tim Netzer gave short presentations on a wide variety of topics.

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Workshop: Classical Algebraic Geometry

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Abstracts

Decomposition of the diagonal and rationality questions

Claire Voisin

A smooth complex projective variety X is said to be rational if it is birational to projective space and stably rational if $X \times \mathbb{P}^r$ is rational for some r. The Lüroth problem, asking whether a unirational variety, that is, a variety rationally dominated by projective space, is rational, has been solved negatively in dimension 3 by Artin and Mumford [1], Clemens and Griffiths [3] and Iskovskikh-Manin [6]. The criteria used in [3] and [6] do not apply to the study of stable unirationality, while Artin and Mumford exhibit a strong stable birational invariant (the torsion in $H^3(X, \mathbb{Z})$) which is nonzero on some unirational threefolds, obtained by desingularizing certain quartic 10-nodal double solids.

Further stable birational invariants (higher degree unramified cohomology groups with torsion coefficients) have been constructed in [4] but they vanish for unirational threefolds (see [7]). Thus only the Artin-Mumford invariant has been used up to now to detect non stably rational (but unirational or rationally connected) threefolds.

We introduced in [10] a new stably birationally invariant property, which is the existence of a Chow theoretic or cohomological decomposition of the diagonal:

$$\Delta_X = X \times x + Z \text{ in } \mathrm{CH}^n(X \times X),\ n = \dim X$$

where Z is supported on $D \times X$, D being a proper closed algebraic subset of X, resp.

$$[\Delta_X] = [X \times x] + [Z] \text{ in } H^{2n}(X \times X, \mathbb{Z}),\ n = \dim X,$$

Z being as above. These decompositions exist with $\mathbb{Q}$-coefficients once X has trivial CH_0 group (see [2]).

We prove in [9] the following result:

Theorem 0.1. *The desingularization X of a very general quartic double solid with $k \leq 7$ nodes does not admit a cohomological decomposition of the diagonal. Hence it is not stably rational.*

The varieties in this theorem have no torsion in $H^3(X, \mathbb{Z})$ by [5].

In the case where $k = 7$, we can describe more explicitly the obstruction to stable rationality given by the above theorem:

Theorem 0.2. *If X is as above, with exactly 7 nodes, X does not admit a universal codimension 2 cycle.*

Here the universal codimension 2 cycle should be a codimension 2 cycle Z on $J(X) \times X$ where $J(X)$ is the intermediate Jacobian of X. This cycle should satisfy the property that

$$\Phi_Z : J(X) \to J(X),$$
$$t \mapsto \Phi_X(Z_t),$$

is the identity, where Φ_X is the Abel-Jacobi map of X. Note that it is known by Bloch-Srinivas [2] and [8] that for a variety with trivial CH_0 group, the Abel-Jacobi map

$$\Phi_X : \mathrm{CH}^2(X)_{hom} \to J(X)$$

is an isomorphism.

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Mirror symmetry for affine CY manifolds with maximal boundary

Sean Keel

We begin with a simple motivating example:

Let $Y = \mathbb{A}$ be a two dimension affine space. Then $\mathbb{C}$-algebra of regular functions $\mathcal{O}(Y)$ is isomorphic to the polynomial ring $\mathbb{C}[x, y]$, and as such the set of monomials

$$B_Y := \{x^a y^b | a, b \geq 0\}$$

give a vector space basis. This is in no sense canonical – it depends heavily on the choice of coordinates, thus e.g. $\{(x+6)^a(y+12)^b\}$ would give a basis equally good. But now choose two (non-parallel) lines $D \subset B$, and let $U := Y \setminus D$. $U \subset Y$ is isomorphic to $(\mathbb{C} \setminus \{0\})^2 \subset \mathbb{C}^2_{x,y}$ and thus $\mathcal{O}(U) \supset \mathcal{O}(Y)$ is isomorphic to the ring of Laurent polynomials $\mathbb{C}[x, y, x^{-1}, y^{-1}] \supset \mathbb{C}[x, y]$, and as such the set of monomials

$$B_U := \{x^a y^b | (a, b) \in \mathbb{Z}^2\}$$

is a bases. But surprisingly, unlike B_Y this *is* canonical, at least up to scaling, by the following easy result:

Lemma 1. The set of invertible elements in the ring $\mathcal{O}(U)$ are

$$\mathcal{O}(U)^* = \{\lambda x^a y^b | \lambda \in \mathbb{C}^*, x^a y^b \in B_U\}.$$

And now we note that we can canonically recover B_Y from B_U. Observe:

$$B_Y = B_U \cap \mathcal{O}(Y) \subset \mathcal{O}(U),$$

and thus while $\mathcal{O}(Y)$ has no canonical basis, once we make the choice of $D \subset Y$, we obtain a canonical (at least up to scaling) basis. Gross, Hacking and I conjecture that this example has a vast generalisation. For simplicity of exposition, here we will ignore the scaling issue.

Definition 1. *We say a smooth variety U is log Calabi-Yau if it has log Kodaira dimension zero, with log forms generated by an (algebraic) volume form. I.e. there is an element $\omega \in H^0(U, \omega_U)$ such that*

- *ω is a volume form, i.e. nowhere vanishing on U*
- *For any open immersion $U \subset Y$, and any Weil divisor $E \subset Y \setminus U$, $\omega^{\otimes m}$ has at worst an m^{th} order pole along E (note this holds iff it hold for $m = 1$, we state it in this odd way because of the next property)*
- *For all $m \geq 1$, $\omega^{\otimes m} \in H^0(U, \omega^{\otimes m})$ is the unique element, up to scaling, satisfying property (2).*

ω satisfies (2) so long as this holds for some normal crossing compactification $U \subset Y$.

If U is log CY, and ω has a pole along the full boundary $D := Y \setminus U$ for an open immersion $U \subset Y$ (with Y normal), we call $U \subset Y$ a *partial minimal model*. Here are some examples:

Non-Example (up to scaling) the only volume form on $U = \mathbb{A}^1_z$ is dz. This has a double pole at ∞ in $U = \mathbb{A}^1 \subset \mathbb{P}^1 = Y$, and thus $\mathbb{A}^1$ *is not* log CY.

On the other hand

Basic-Example Let $U = \mathbb{G}_m^n$ with coordinates $z_1, \ldots, z_n$. Then U is log CY with volume form $\omega := \frac{dz_1}{z_1} \wedge \frac{dz_2}{z_2} \cdots \wedge \frac{dz_n}{z_n}$. A partial minimal model is the same thing as a toric compactification.

Example By a (generalized) cluster structure on a log CY U we mean an open cover $U = \cup_{s \in S} T_{L,s}$ by copies of the same algebraic torus T_L, where for a free Abelian group $L = \mathbb{Z}^n$ we write $T_L := L \otimes_{\mathbb{Z}} \mathbb{G}_m$ for the algebraic torus with cocharacter lattice N. The cluster varieties of Fomin-Zelevinski, Fock-Goncharov are a special case.

The key definition is the following:

Definition 2. *Let (U, ω) be log CY.*

$$U^t(\mathbb{Z}) := \{(E, m) | E \subset Y, m > 0\}\} \cup \{0\}.$$

Here $E \subset Y$ is a boundary divisor on some partial minimal model (thus ω has a pole on E), m is any positive integer, and we identity two divisor in two (possibly) different partial compactifications, if they define the same discrete valuation of the field of rational functions on U (one can thus alternatively describe $U^t(\mathbb{Z})$ as the set of divisorial discrete valuations on its field of rational functions where ω has a pole). There is also a real version $U^t(\mathbb{Z}) \subset U^t(\mathbb{R})$.

The following easy lemma gives the meaning in a special case:

Lemma 2. $T^t_N(\mathbb{Z}) \subset T^t_N(\mathbb{R})$ is canonically identified with $N \subset N_{\mathbb{R}} := N \otimes_{\mathbb{Z}} \mathbb{R}$.

Proof. This follows from elementary toric geometry. E.g. if we write $v = m \cdot p$ with $p \in N$ primitive, and let Δ be the fan with a single ray $\mathbb{R} \cdot v$. Then the toric variety $\mathrm{TV}(\Delta)$ has a single boundary divisor E_p, and ω has a pole on E_p. We associate (E, m) to v. □

Now observe that if $V \subset U$ is an open subset, and both are log CY, then by uniqueness, $\omega_U|_V = \omega_V$. It then follows immediately from the definition that there is a canonical identification $U^t(\mathbb{Z}) = V^t(\mathbb{Z})$. Thus e.g. in the cluster case, each choice of torus open set $T_{L,s} \subset U$ gives an identification of $U^t(\mathbb{Z}) \subset U^t(\mathbb{R})$ with $T^t_N(\mathbb{Z}) = N \subset N_{\mathbb{R}} = T^t_N(\mathbb{R})$. But *warning* $U^t(\mathbb{Z})$ *is not* a lattice – the addition law depends on the choice of torus open set.

The real dimension of $U^t(\mathbb{R})$ is at most the complex dimension of U. We say U has *maximal boundary* if $U^t(\mathbb{R})$ has real dimension equal to the complex dimension of U. We note this holds e.g. whenever U contains a torus open subset. We view this as saying U is as non-compact as possible.

By the above, $U^t(\mathbb{Z})$ generalizes the notion of the cocharacter lattice of an algebraic torus to an arbitrary log CY variety. Gross, Hacking and I conjecture that there is also a generalisation of the characters:

Conjecture 1 (GHK 2011)**.** *Let U be an affine log CY with maximal boundary. Let A be the vector space with basis $U^t(\mathbb{Z})$. Write $\vartheta_p \in A$ for $p \in U^t(\mathbb{Z})$. Thus $A = \oplus_{p \in U^t(\mathbb{Z})} \mathbb{C} \cdot \vartheta_p$.*

A has a natural finitely generated commutative and associative $\mathbb{C}$-algebra structure, such that the structure constants, $\alpha(p, q, r) \in \mathbb{C}$ defined by

$$\vartheta_p \cdot \vartheta_q = \sum_{r \in U^t(\mathbb{Z})} \alpha(p, q, r) \vartheta_r$$

are integers, given by counts of tropical discs in $U^t\mathbb{R}$ – certain purely combinatorial objects we define (which morally correspond to holomorphic maps of a punctured disc into U with boundary on a Lagrangian fibre of the (conjectural) SYZ fibration).

$U^\vee := \mathrm{Spec}(A)$ is an affine log CY with maximal boundary, the homological mirror symmetry dual to U. Note by construction $\mathcal{O}(U^\vee)$ has a canonical basis parameterized by $U^t(\mathbb{Z})$.

This construction is involutive, i.e. $(U^\vee)^\vee = U$. In particular $\mathcal{O}(U)$ has a canonical basis, $B_U := (U^\vee)^t(\mathbb{Z})$.

For any partial minimal model $U \subset Y$, $B_Y := B_U \cap \mathcal{O}(Y) \subset \mathcal{O}(U)$ is a basis of regular functions on Y.

Gross, Hacking and I prove the theorem for U of dimension two, and together with Kontsevich, for cluster varieties of all dimensions. But we do not prove that $U^\vee$ is the homological mirror dual, rather we identify it with a *known* variety: In dimension two, with U itself, and for cluster varieties, $U = \cup_{s \in S} T_{L,s}$ with the Fock-Goncharov mirror $U^\vee := \cup_{s \in S} T_{L^*,s}$.

Here are some applications:

Example Let $Y = G/[B, B]$ be the basic affine space for a semi-simple group G (with Borel subgroup $B \subset G$). There is a natural open subset $U \subset Y$, the so-called open double Bruhat cell. Berenstein-Fomin-Zelevinski, building on work of Lusztig, haveshown U is a cluster variety, and $U \subset Y$ is a partial minimal model. Thus by our results, $\mathcal{O}(Y)$ has a canonical basis. G acts on Y, and thus on $\mathcal{O}(Y)$. As G-representation, $\mathcal{O}(Y)$ is the direct sum of all irreducible representations of G, each occuring exactly once. Our results now give a canonical basis for every irreducible representation of G. The result is particularly striking, because our construction uses essentially no representation theory – just the fact that U has its special volume form, with a pole along the boundary $U \subset Y$.

Example Let Y' be a smooth projective variety, and $D' \subset |-K_{Y'}|$ an anti-canonical normal crossing divisor, with a zero stratum (a point where D' looks locally analytically like the union of the coordinate hyperplanes planes in affine space). Assume D' supports an ample divisor (e.g. D' ample, i.e. Y' Fano). Let $U' := Y' \setminus D'$. Let $Y \to Y'$ be the *universal torus*, a canonical $T_{\mathrm{Pic}(Y')^*}$ principal bundle, with global functions $\mathcal{O}(Y) = \mathrm{Cox}(Y') := \oplus_{L \in \mathrm{Pic}(Y'} H^0(Y, L)$. We note $\mathrm{Cox}(Y')$ is the most important Mori theoretic invariant of Y' – it controls all the Mori theory of Y'. Let $U \subset Y$ be the inverse image of $U' \subset Y'$. By construction U is an affine log CY with maximal boundary, and $U \subset Y$ is a partial minimal model. Thus our conjecture says that $\mathcal{O}(Y)$, and thus $H^0(Y, L)$ for every $L \in \mathrm{Pic}(Y)$ comes with a canonical basis, with multiplication rule (given by tensor product) determined by counts of tropical discs. Thus once we make the single choice $D' \subset K_{Y'}$, all the birational contractions of the Mori theory of Y' take place with canonical coordinates.

On motivic stable pairs invariants of $K3$ surfaces

Rahul Pandharipande
(joint work with S. Katz and A. Klemm)

The Yau–Zaslow conjecture (1995) links the GW counts of rational curves on $K3$ surfaces to the generating series of Euler characteristics of the Hilbert schemes of points of $K3$ surfaces. Inclusion of the higher genus counts leads to the KKV conjecture (1999) linked to the the generating series of Chi-y genera of the Hilbert schemes of points of $K3$ surfaces. In my talk, I proposed a further extension to the motivic invariants of $K3$ surfaces (sometimes called refined DT invariants) which are then linked to the generating series of Hodge polynomials of the Hilbert schemes of points of $K3$ surfaces (joint work with A. Klemm and S. Katz).

The outcome is a prediction of the motivic invariants (defined via superpotentials and the motivic Milnor fiber following Joyce and collaborators) of all stable pairs moduli spaces on $K3$ fibrations which satisfy suitable Noether-Lefschetz transversality conditions. The refine GV multiple cover formula plays an essential role.

Finally, I hinted at (but did not state) an associated prediction for the motivic invariants of the STU model in fiber classes. These are perhaps the first interesting compact geometries which admit exact solutions for the refined invariants. The precise formulations can be found in [1].

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Ulrich bundles on $K3$ surfaces

Marian Aprodu
(joint work with Gavril Farkas, Angela Ortega)

I report on a joint work with Gavril Farkas and Angela Ortega [2]. The goal is to prove the existence of rank-two Ulrich bundles on polarised $K3$ surfaces that satisfy a Brill-Noether type condition. This condition is realised on the complement of a Noether-Lefschetz locus in the moduli space of polarised $K3$ surfaces.

The notion of Ulrich bundle originates in classical algebraic geometry, being related to the problem of finding, whenever possible, linear determinantal or linear pfaffian descriptions of hypersurfaces in a complex projective space [3]. However, there are several other reasons to study Ulrich bundles. More recently, Ulrich bundles have been linked by D. Eisenbud and F.-O. Schreyer to cones of cohomology tables, and to linear pfaffian presentations of Cayley-Chow-van der Waerden forms [6].

D. Eisenbud, G. Fløystad and F.-O. Schreyer [5] refined the classical Beilison spectral sequence and proved that any coherent sheaf $\mathcal{F}$ on a complex projective space $\mathbb{P}^N$ can be presented as the cohomology of a monad

$$0 \to B_{-N} \to \ldots \to B_{-1} \to B_0 \to B_1 \to \ldots \to B_N \to 0,$$

with $B_k = \oplus_{i=0}^N H^i(\mathcal{F}(k-i)) \otimes \Omega^{i-k}(i-k)$, meaning that the sequence above is exact everywhere except at 0 where the cohomology equals $\mathcal{F}$. In other words, the sheaf $\mathcal{F}$ can be completely recovered from the cohomology of its twists and from the matrices that appear in the sequence. If we collect the dimensions $h^i(\mathcal{F}(k-i))$ and write them in a table on the kth column and labelling the rows by i, we obtain what is called the *cohomology table of* $\mathcal{F}$. When $\mathcal{F}$ varies, the cohomology tables define a cone in $\prod_{-\infty}^{\infty} \mathbb{Q}^{N+1}$, denoted by $C(\mathbb{P}^N, \mathcal{O}_{\mathbb{P}^N}(1))$ and called the *cone of cohomology tables* of the projective space. If X is an n-dimensional smooth subvariety of $\mathbb{P}^N$, the sheaves supported on X define a subcone of $C(\mathbb{P}^N, \mathcal{O}_{\mathbb{P}^N}(1))$, denoted by $C(X, \mathcal{O}_X(1))$, and called the *cone of cohomology tables of* X. It is clear that this cone depends on the embedding. If $\pi : X \to \mathbb{P}^n$ denotes a central projection, then the pushforward map defines an injection $C(X, \mathcal{O}_X(1)) \xrightarrow{\pi_*} C(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(1))$. Eisenbud and Schreyer prove that this map is bijective if and only if there exists a vector bundle E on X whose pushforward is a trivial bundle (of high rank) and this gives the definition:

Definition 1 (Eisenbud-Schreyer [6]). *Notation as above. A vector bundle E on X is called Ulrich if $\pi_*(E)$ is trivial.*

Apparently, this definition depends on the projection. In fact, it does not, being a cohomological property. Specifically, D. Eisenbud and F.-O. Schreyer [6] prove that the following conditions are equivalent a vector bundle E on X is Ulrich if and only if one of the following equivalent conditions is satisfied:

(i) $H^i(E(-i)) = 0$ for all $i > 0$ and $H^i(E(-i-1)) = 0$ for all $i < N$.

(ii) The minimal resolution of the module $\Gamma_*(E) = \oplus_q H^0(E(q))$ over the polynomial ring $S = \mathbb{C}[X_0, \ldots, X_N]$ is of the form

$$0 \to F_{N-n} \to \ldots \to F_1 \to F_0 \to \Gamma_*(E) \to 0,$$

where F_i is generated in degree i for any i.

Having defined a cone associated to a projective variety, a natural question is whether this interacts in any way with the geometry. D. Eisenbud and F.-O. Schreyer conjecture that this should not be the case, more precisely:

Conjecture 2 (Eisenbud-Schreyer [6]). *On every n-dimensional smooth projective variety X there is an Ulrich vector bundle, in particular $C(X, \mathcal{O}_X(1))$ coincides with the cone of cohomology tables of the n-dimensional projective space.*

In the statement of the conjecture, the rank is not important, however, for geometric applications, notably in relation with Cayley-Chow-van der Waerden form, the most interesting cases appear when the rank is one or two.

The relation with the Cayley-Chow-van der Waerden form is the following [6]. For a smooth n-dimensional projective variety $X \subset \mathbb{P}^N$, consider $\mathbb{G} := G(N - n, H^0(\mathcal{O}_X(1))^*)$ the Grassmannian of codimension-$(N-n-1)$ planes in $\mathbb{P}^N$, and $\mathcal{U} \subset H^0(\mathcal{O}_X(1)) \otimes \mathcal{O}_{\mathbb{G}}$ the universal rank-$(n+1)$ subbundle on $\mathbb{G}$. The locus $\mathcal{Z}(X) := \{L \in \mathbb{G} |\ L \cap X \neq \emptyset\}$ is a divisor in $\mathbb{G}$ and hence it is given by a single equation in the Plücker coordinates of X, called the Cayley-Chow-van der Waerden form of X.

If E is a vector bundle on X, then there is a Bellinson-type complex on $\mathbb{G}$

$$U^\bullet(E): \ \cdots \to U^{-1} \to U^0 \to U^1 \to \cdots$$

where $U^k = \oplus_i H^i(E(k-i)) \otimes \wedge^{i-k} \mathcal{U}$, in other words, it is obtained from the Beilinson monad by replacing the twisted sheaves of differentials by the exterior powers of $\mathcal{U}$. D. Eisenbud and F.-O. Schreyer [6] prove that the complex $U^\bullet(E)$ is generically exact and fails to be exact precisely along $\mathcal{Z}(X)$, a remarkable connection between this complex and the Cayley-Chow-van der Waerden form.

Note that if E is Ulrich, then the complex $U^\bullet(E)$ reduces to

$$0 \to U^{-1} \to U^0 \to 0,$$

and $U^{-1} = H^n(E(-n-1)) \otimes \mathcal{O}_{\mathbb{G}}(-1)$ and $U^0 = H^0(E) \otimes \mathcal{O}_{\mathbb{G}}$. Eisenbud and Schreyer prove [6] that the existence of a rank-one or a rank-two Ulrich bundle

E on X implies that the Cayley-Chow-van der Waerden form of X is linear determinantal, respectively linear pfaffian, description obtained from the complex $U^\bullet(E)$.

The case when X is a hypersurface had been previously considered by A. Beauville [3]. He observed that X is linear determinantal if and only if X carries an Ulrich line bundle, and it is linear pfaffian if and only if it carries an Ulrich bundle of rank two, a precursor of Eisenbud-Schreyer's result on the Cayley-Chow-van der Waerden forms. Several existence/nonexistence results can also be found in [3].

In the curve case, Ulrich line bundles always exist, and hence the next interesting case is for surfaces. Applying the definition, we find the following restrictions for an Ulrich bundle on a surface X $\chi(E(-1)) = \chi(E(-2)) = 0$ and, from Riemann-Roch, we obtain:

$$H \cdot \left(c_1(E) - \frac{\mathrm{rk}(E)}{2}(K_X + H) \right) = 0,$$

where H denotes the hyperplane section. The second Chern class can be determined also from Riemann-Roch, using the condition $\chi(E(-1)) = 0$.

For very general $K3$ surfaces, Ulrich line bundles cannot exist, hence it is natural to look at the next best case, rank two. Given a $K3$ surface S, embedded in a projective space $\mathbb{P}^N$ via a complet linear system, we wish to prove the existence of a rank-two Ulrich bundle E, and, in view of the restriction on the first Chern class mentioned above, it is natural to impose the condition $\det(E) = \mathcal{O}_S(3)$. This bundle must have $c_2(E) = 5N - 1$. Our main result is the following:

Theorem 1 ([2])**.** *Let $S \subset \mathbb{P}^N$ be a linearly normal $K3$ surface such that the Clifford index of cubic sections is computed by $\mathcal{O}_S(1)$. The S carries a $(2N+8)$-dimensional familty of stable rank-two Ulrich bundles with determinant $\mathcal{O}_S(3)$.*

For quartic surfaces in $\mathbb{P}^3$ our condition is automatically satisfied. This case has been previously considered by E. Coskun, R. S. Kulkarni and Y. Mustopa [4].

The proof idea is the following. The hypothesis, together with the results of [1] ensure that base-point-free complete g^1_{5N-1}'s exist on general cubic sections. If (C, A) with $C \in |\mathcal{O}_S(3)|$ and $A \in W^1_{5N-1}(C)$ is a general pair (in a component that dominates the linear system), we can consider the associated rank-two Lazarsfeld-Mukai bundle $E = E_{C,A}$ defined by the sequence

$$0 \to E^* \to H^0(C, A) \otimes \mathcal{O}_S \to A \to 0.$$

It has the same invariants as a rank-two Ulrich bundle, moreover, any Ulrich bundle with determinant $\mathcal{O}_S(3)$ must be of this type [6]. The Ulrich condition reduces to verify the vanishing $H^0(E(-1)) = 0$, which is achieved in the following way. In the parameter space of Lazarsfeld-Mukai bundles, we consider the locus $\{E|\ h^0(E(-1)) \neq 0\}$ and evaluate its dimension. We find that this dimension is precisely one less than the dimension of the space of Lazarsfeld-Mukai bundles, and hence we conclude that a general Lazarsfeld-Mukai bundle is Ulrich.

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Equidistribution of periodic points for automorphisms of complex projective surfaces

Serge Cantat
(joint work with Christophe Dupont)

Let X be a complex projective surface and $\mathsf{Aut}(\mathsf{X})$ be the group of holomorphic diffeomorphisms of X. By Gromov-Yomdin theorem, the topological entropy $\mathsf{h}_{top}(f)$ of every $f \in \mathsf{Aut}(\mathsf{X})$ is equal to the logarithm of the spectral radius λ_f of the linear endomorphism

$$f^*\colon H^2(X;\mathbf{Z}) \to H^2(X;\mathbf{Z}),$$

where $H^2(X;\mathbf{Z})$ is the second cohomology group of X. Thus, f has positive entropy if and only if there is an eigenvalue $\lambda \in \mathbf{C}$ of f^* with $|\lambda| > 1$. In fact such an eigenvalue is unique if it exists, and is equal to the spectral radius λ_f.

When the entropy is positive, there is a natural f-invariant probability measure μ_f on X (see [1, 2]). This measure can be characterized by several dynamical properties:

- μ_f is the unique f-invariant probability measure with maximal entropy,
- if μ_n denotes the average on the set of isolated fixed points of f^n, then μ_n converges towards μ_f as n goes to $+\infty$.

Thus μ_f encodes the most interesting features of the dynamics of f; μ_f is called the **measure of maximal entropy** of f.

By ergodicity, μ_f is either **singular** or **absolutely continuous** with respect to Lebesgue measure. It is singular if there exists a Borel subset A of X satisfying $\mu_f(A) = 1$ and $\mathrm{vol(A)} = 0$ (the volume is taken with respect to any smooth volume form on X); it is absolutely continuous if $\mu_f(B) = 0$ for every Borel subset $B \subset X$ such that $\mathrm{vol}(B) = 0$.

Let A be a complex abelian surface and let $\mathrm{vol_A}$ denote the Haar measure on A, normalized by $\mathrm{vol_A(A)} = 1$. Every $f \in \mathsf{Aut}(\mathsf{A})$ preserves $\mathrm{vol_A}$, and the measure μ_f is equal to $\mathrm{vol_A}$ when $\lambda_f > 1$. The simplest example is obtained as follows. Start with an elliptic curve $E = \mathbf{C}/\Lambda_0$ and consider the product $A = E \times E$. The group $\mathsf{GL}_2(\mathbf{Z})$ acts on $\mathbf{C}^2$ linearly, preserving the lattice $\Lambda = \Lambda_0 \times \Lambda_0$; thus, it acts also on the quotient $A = \mathbf{C}^2/\Lambda$. This gives rise to a morphism $M \mapsto f_M$ from $\mathsf{GL}_2(\mathbf{Z})$ to $\mathsf{Aut(A)}$. The spectral radius of $(f_M)^*$ on $H^2(A;\mathbf{Z})$ is equal to the square of the spectral radius of M. In particular, $\lambda_f > 1$ as soon as the trace of M satisfies $|\mathrm{tr}(M)| > 2$.

The center of $\mathsf{GL}_2(\mathbf{Z})$ is generated by the involution $\eta = -\mathrm{Id}$; it acts on A by $\eta(x,y) = (-x,-y)$. The quotient A/η is a singular surface X_0; denote by X this minimal regular model of the quotient space A/η. Since $\mathsf{GL}_2(\mathbf{Z})$ commutes to η, one gets an injective morphism $M \mapsto g_M$ from $\mathsf{PGL}_2(\mathbf{Z})$ to $\mathsf{Aut(X)}$. The topological entropy of g_M (on X) is equal to the topological entropy of f_M (on A). The holomorphic 2-form $\Omega_A = dx \wedge dy$ is η-invariant and determines a non-vanishing holomorphic 2-form Ω_X on X. The volume form $\Omega_X \wedge \overline{\Omega_X}$ is invariant under each automorphism g_M, and the associated probability measure coincides with the measure of maximal entropy μ_{g_M} when $\mathsf{h}_{top}(g_M) > 0$. Hence, again, one gets examples of automorphisms for which the measure of maximal entropy is absolutely continuous.

Definition. *Let X be a complex projective surface and let $f \in \mathsf{Aut(X)}$. The pair (X,f) is a* **Kummer example** *if there exist*

- *a birational morphism $\pi : X \to X_0$ onto an orbifold X_0,*
- *a finite orbifold cover $\epsilon : Y \to X_0$ by a complex torus Y,*
- *an automorphism f_0 of X_0 and an automorphism $\hat{f}$ of Y such that*

$$f_0 \circ \pi = \pi \circ f \quad \text{and} \quad f_0 \circ \epsilon = \epsilon \circ \hat{f}.$$

Main Theorem. *Let X be a smooth complex projective surface and f be an automorphism of X with positive entropy. Let μ_f be the measure of maximal entropy of f. If μ_f is absolutely continuous with respect to Lebesgue measure, then (X,f) is a Kummer example.*

This answers a question raised by the author in his thesis and solves Conjecture 3.31 of Curtis T. McMullen in [3]. A similar result holds for holomorphic endomorphisms g of the projective space $\mathbb{P}^k_{\mathbf{C}}$ of topological degree > 1. In this case, there is also an invariant probability measure μ_g that describes the repartition of periodic points, and if μ_g is absolutely continuous with respect to Lebesgue measure, then g is a **Lattès examples**: it lifts to an endomorphism of an abelian variety via an equivariant ramified cover. This is due to Zdunik for $k = 1$ and to Berteloot, Loeb and the second author for $k \geq 2$.

This theorem provides explicit examples of automorphisms of complex projective surfaces for which μ_f is singular. All previously known examples were constructed on rational surfaces, and we get examples on K3, Enriques, and rational surfaces. A good example to keep in mind is the family of (smooth) surfaces of degree $(2,2,2)$ in $\mathbb{P}^1_{\mathbf{C}} \times \mathbb{P}^1_{\mathbf{C}} \times \mathbb{P}^1_{\mathbf{C}}$. Such a surface X comes with three double covers $X \to \mathbb{P}^1_{\mathbf{C}} \times \mathbb{P}^1_{\mathbf{C}}$, hence with three holomorphic involutions σ_1, σ_2, and σ_3. The composition $f = \sigma_1 \circ \sigma_2 \circ \sigma_3$ is an automorphism of X of positive entropy. The measure μ_f is singular for a generic choice of X but coincides with $\mathrm{vol_X}$ for specific choices.

The proof relies on a renormalization argument along stable and unstable manifolds of f, a new argument involving Montel families of entire curves (the proof of which follows from Hodge index theorem and a result of Dinh and Sibony), an argument of Ghys concerning lamination by holomorphic curves, and the classification by Favre and the first author of foliated surfaces with an infinite group of birational symmetries.

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Syzygies of algebraic curves of large degree

Robert Lazarsfeld
(joint work with Lawrence Ein)

My talk presented joint work with Lawrence Ein [4] showing that a small variant of the methods used by Voisin in her study [8], [9] of canonical curves leads to a surprisingly quick proof of the gonality conjecture of [7], asserting that one can read off the gonality of an algebraic curve C from its syzygies in the embedding defined by any one line bundle of sufficiently large degree. More generally, we established a necessary and sufficient condition for the asymptotic vanishing of the weight one syzygies of the module associated to an arbitrary line bundle on C. The following paragraphs are adapted from the Introduction to [4].

Let C be a smooth complex projective curve of genus $g \geq 2$, and let L be a very ample line bundle of degree d on C defining an embedding

$$C \ \subseteq \ \mathbf{P}H^0(C, L) \ = \ \mathbf{P}^r.$$

Let $S = \mathrm{Sym}\ \mathrm{H}^0(C, L)$ be the homogeneous coordinate ring of $\mathbf{P}^r$, and denote by

$$R = R(L) = \oplus_m H^0(C, mL)$$

the graded S-module associated to L. Consider the minimal graded free resolution $E_\bullet = E_\bullet(L)$ of R over S:

$$0 \longrightarrow E_{r-1} \longrightarrow \dots \longrightarrow E_2 \longrightarrow E_1 \longrightarrow E_0 \longrightarrow R \longrightarrow 0,$$

where $E_p = \oplus S(-a_{p,j})$. We denote by $K_{p,q}(C; L)$ the vector space of minimal generators of E_p in degree $p+q$, so that

$$E_p \ = \ \bigoplus_q K_{p,q}(C; L) \otimes_{\mathbf{C}} S(-p-q).$$

We will be concerned here with investigating the grading of $E_\bullet(L)$ – ie determining which of the $K_{p,q}$ are non-vanishing – when L has very large degree.

For $q \neq 1$ the $K_{p,q}$ are completely controlled, and the known results leave open only the question of when $K_{p,1}(C;L) \neq 0$ for $p \in [r-g, r-1]$. Our first main theorem asserts that this is determined by the gonality $\mathrm{gon}(C)$ of C, ie the least degree of a branched covering $C \to \mathbf{P}^1$.

Theorem A. If $\deg(L) \gg 0$, then

$$K_{p,1}(C;L) \neq 0 \iff 1 \leq p \leq r - \mathrm{gon}(C).$$

Thus one can read off the gonality of a curve from the resolution of the ideal of C in the embedding defined by any one line bundle of sufficiently large degree. The cases $p = r-1, p = r-2$ were established by Green [5], and the general statement was conjectured in [7], where it was observed that if $1 \leq p \leq r - \mathrm{gon}(C)$, then $K_{p,1}(C;L) \neq 0$. Using Voisin's results [8], [9] on syzygies of general canonical curves, Aprodu and Voisin [1], [3] proved the statement of the Theorem for a general curve of each gonality.

Theorem A follows from a more general result concerning the weight one asymptotic syzygies associated to an arbitrary divisor B. Specifically, fix a line bundle B on C, and with L as above consider the $S = \mathrm{Sym}\ H^0(L)$ module

$$R \ = \ R(B;L) \ = \ \bigoplus_m H^0(C, B + mL).$$

One can again form the graded minimal free resolution $E_\bullet(B;L)$ of $R(B;L)$ over S, giving rise to Koszul cohomology groups $K_{p,q}(C,B;L)$. As in the case $B = \mathcal{O}_C$ discussed in the previous paragraphs, the $K_{p,0}$ and the $K_{p,2}$ are completely controlled when $\deg L \gg 0$, and so the issue is to understand the weight one groups $K_{p,1}(C,B;L)$ when L has large degree. Recall that B is said to be *p-very ample* if every effective divisor ξ of degree $(p+1)$ on C imposes independent conditions on the sections of B, i.e. if the natural map

$$H^0(C,B) \longrightarrow H^0(C, B \otimes \mathcal{O}_\xi)$$

is surjective for every $\xi \in C_{p+1} =_{\mathrm{def}} \mathrm{Sym}^{p+1} C$. Our second main result is:

Theorem B. Fix B and $p \geq 0$. Then

$$K_{p,1}(C,B;L) \ = \ 0 \ \text{ for all } L \text{ with } \deg L \gg 0$$

if and only if B is p-very ample.

Serre duality implies that the vector spaces

$$K_{p,q}(C,B;L) \quad \text{and} \quad K_{r-1-p,2-q}(C, K_C - B; L)$$

are naturally dual, K_C being the canonical divisor of C, and one then finds that Theorem A is equivalent to the case $B = K_C$ of Theorem B. While this is arguably the most interesting instance of the result, it will become clear that decoupling B and L is helpful in guiding the argument.

When B fails to be p-very ample, it is natural to introduce the invariant

$$\gamma_p(B) \ = \ \dim \left\{ \xi \in C_{p+1} \,\middle|\, H^0(B) \longrightarrow H^0(B \otimes \mathcal{O}_\xi) \text{ not surjective } \right\}.$$

Theorem C. Let $L_d = dA + E$, where A is an ample line bundle on C and E is arbitrary. Fix B and p, and assume that B is not p-very ample. Then there is a polynomial $P(d)$ of degree $\gamma_p(B)$ in d such that

$$\dim K_{p,1}(C, B; L_d) \;=\; P(d) \quad \text{for } d \gg 0.$$

In some cases, we are also able to compute the leading coefficient of $P(d)$. We note that Yang [10] has recently proven (by somewhat related arguments) that the dimensions of the vector spaces $K_{p,0}$ and $K_{p,1}$ grow polynomially on an arbitrary variety.

Theorems B and C follow in a surprisingly simple manner from a small variant of the Hilbert scheme computations pioneered by Voisin in her proof [8], [9] of Green's conjecture for general canonical curves. The idea in effect is to push down Voisin's computation to a suitable symmetric product of C, where the p-very-amplitude hypothesis is easily visible in terms of a tautological vector bundle associated to B. Then one can deduce Theorem B simply from Serre vanishing.

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The combinatorics and topology of proper toric maps

Mircea Mustaţă

(joint work with Mark A. de Cataldo and Luca Migliorini)

The topology of toric varieties has been extensively studied (see for example [3]). If X is a complete, simplicial, complex toric variety, then there is an explicit formula for the Betti numbers of X in terms of the f-vector of X, a combinatorial invariant

which encodes the number of cones of each dimension in the fan Δ_X defining X. When X is complete, but not-necessarily-simplicial, then the cohomology of X is not a combinatorial invariant; the right topological invariant to consider in this setting is the intersection cohomology of X, whose dimension in each degree only depends on the combinatorics of Δ_X (see [5], [2], and [4]).
We are interested in two related questions. First, we study the cohomology of the fibers of proper toric maps $f\colon X \to Y$, with X simplicial. Note that each fiber is a union of complete, simplicial toric varieties. However, since the fibers are in general reducible, the usual results do not apply. Second, we use the understanding we have on the cohomology of fibers to study the support loci for the Decomposition Theorem in the toric setting.

The following is our main result concerning the cohomology of fibers of toric maps. For the standard terminology and notation in toric geometry, see [3]. All varieties we consider are defined over $\mathbf{C}$. Recall that a proper map $f\colon X \to Y$ is a fibration if it is surjective and has connected fibers (if f is a toric map, this condition is equivalent with the corresponding lattice map being surjective).

Theorem 1. Let $f\colon X \to Y$ be a proper toric map, with X simplicial, and let $y \in Y$.

i) For every q, the cohomology $H^q(f^{-1}(y), \mathbf{Q})$ is pure, of weight q, and of Hodge-Tate type. In particular, this is zero if q is odd.
ii) If f is a fibration and $\tau \in \Delta_Y$ is such that y lies in the orbit $O(\tau)$, then
$$\dim_{\mathbf{Q}} H^{2m}(f^{-1}(y), \mathbf{Q}) = \sum_{\ell \geq m} (-1)^{\ell - m} \binom{\ell}{m} d_\ell(X/\tau),$$
where $d_\ell(\tau)$ is the number of cones $\sigma \in \Delta_X$ with $\operatorname{codim}(\sigma) = \operatorname{codim}(\tau) + \ell$ and such that $f(O(\sigma)) = O(\tau)$. In particular, we have $\chi(f^{-1}(y)) = d_0(X/\tau)$.

For every complex variety W, we denote by IC_W the intersection complex on W. In the toric setting, we show that the Decomposition Theorem of Beilinson-Berstein-Deligne-Gabber [1] takes the following form.

Theorem 2. If X and Y are toric varieties and $f\colon X \to Y$ is a toric fibration, then we have a decomposition

$$Rf_*(IC_X) \simeq \bigoplus_{\tau \in \Delta_Y} \bigoplus_{b \in \mathbf{Z}} IC_{V(\tau)}^{\oplus s_{\tau,b}}[-b], \tag{1}$$

where the nonnegative integers $s_{\tau,b}$ satisfy $s_{\tau,b} = 0$ if $b + \dim(X) - \dim(V(\tau))$ is odd.

We note that for any proper toric map, by taking the Stein factorization, we obtain a version of the statement in Theorem 2. In the setting of this theorem, an irreducible closed subset $V(\tau)$ of Y is a *support* if some $s_{\tau,b}$ is nonzero. Our goal is to determine the supports of toric fibrations, and more precisely, to give a combinatorial description of $\delta_\tau := \sum_b s_{\tau,b}$. The description is especially nice when both X and Y are simplicial.

Theorem 3. If $f\colon X \to Y$ is a toric fibration, with X and Y simplicial toric varieties, then for every cone $\tau \in \Delta_Y$, we have

$$\delta_\tau = \sum_{\sigma\subseteq\tau}(-1)^{\dim(\tau)-\dim(\sigma)} d_0(X/\sigma). \tag{2}$$

In fact, in this case we obtain an explicit formula for each of the numbers $s_{\tau,b}$ in Theorem 2. An interesting consequence of Theorem 3 is that the expression on the right-hand side of (2) is nonnegative. It would be desirable to find a direct combinatorial argument for this fact. When f is birational and $\dim(\tau) \leq 3$, we give a combinatorial description of δ_τ which implies that it is nonnegative, but we don't have such a formula in general.

For arbitrary toric varieties X and Y, the description of the supports is more involved, and we need to introduce some notation. Given a toric variety Y and two cones $\tau \subseteq \sigma$ in the fan Δ_Y defining Y, we put $r_{\tau,\sigma} := \dim_{\mathbf{Q}} \mathcal{H}^*(IC_{V(\tau)})_{x_\sigma}$, where x_σ can be taken to be any point in the orbit $O(\sigma) \subseteq V(\tau)$. It is a consequence of the results in [4] that $r_{\tau,\sigma}$ is a combinatorial invariant. In turn, we then obtain an invariant $\widetilde{r}_{\tau,\sigma}$ defined for cones $\tau \subseteq \sigma$ in Δ_Y, uniquely determined by the property that for every $\tau \subseteq \sigma$, the sum $\sum_{\tau\subseteq\gamma\subseteq\sigma} r_{\tau,\gamma}\cdot\widetilde{r}_{\gamma,\sigma}$ is equal to 1 if $\tau = \sigma$ and it is equal to 0, otherwise. Suppose now that $f\colon X \to Y$ is a toric fibration. For a cone $\sigma \in \Delta_Y$, we put $p_\sigma(f) := \dim_{\mathbf{Q}} \mathcal{H}^*(f^{-1}(x_\sigma), IC_X)$, where again we may take x_σ to be any point in the orbit $O(\sigma)$. The next result gives a description of δ_τ in terms of the above invariants. The second part implies that the invariants $p_\sigma(f)$, hence also the δ_τ, are combinatorial.

Theorem 4. With the above notation, if $f\colon X \to Y$ is a toric fibration, then the following hold:

i) For every cone $\tau \in \Delta_Y$, we have $\delta_\tau = \sum_{\sigma\subseteq\tau}\widetilde{r}_{\sigma,\tau}\cdot p_\sigma(f)$.
ii) For every cone $\sigma \in \Delta_Y$, we have $p_\sigma(f) = \sum_i r_{0,\sigma_i}$, where the σ_i are the cones in the fan Δ_X defining X with the property that $f(O(\sigma_i)) = O(\sigma)$ and $\dim(O(\sigma_i)) = \dim(O(\sigma))$.

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Tropical methods for linear series

Sam Payne
(joint work with Dave Jensen)

Let $\mathfrak{X}$ be a regular semistable curve over $\mathbb{C}[\![t]\!]$, with generic fiber X and special fiber $\overline{\mathfrak{X}}$. Let L be a line bundle of degree d on X. Then L extends to a line bundle on the total space $\mathfrak{X}$, but this extension is not unique.

If the special fiber $\overline{\mathfrak{X}}$ is of compact type, meaning that its Jacobian is compact or, equivalently, its dual graph is a tree, then for each component $\overline{\mathfrak{X}}_i$ there is a unique extension $\mathfrak{L}_i$ such that

$$\deg \mathfrak{L}_{\mathfrak{i}}|_{\overline{\mathfrak{X}}_j} = \begin{cases} d & \text{if } i = j, \\ 0 & \text{otherwise.} \end{cases}$$

Then all of the interesting information about degenerations of sections of $\mathfrak{L}_i$ is concentrated on $\overline{\mathfrak{X}_i}$.

If $W \subset \Gamma(X, L)$ is a linear series of rank r then those sections of $\mathfrak{L}_{\mathfrak{i}}|_{\overline{\mathfrak{X}}_i}$ that are limits of sections of L form a linear series of degree d and rank r on $\overline{\mathfrak{X}}_i$. This yields a collection of $\mathfrak{g}^r_d$s, one on each component of $\overline{\mathfrak{X}}$, which satisfy a natural compatibility condition that can be phrased in terms of vanishing sequences at the nodes where the components intersect. The combinatorics of these compatibility conditions are at the heart of the theory of limit linear series, due to Eisenbud and Harris, which gives proofs of the Brill-Noether and Gieseker-Petri Theorems [EH83, EH86] along with many other fundamental facts about the geometry of curves and their moduli.

Now, suppose $\overline{\mathfrak{X}}$ is not of compact type. Then L may or may not extend to a line bundle $\mathfrak{L}_i$ on $\mathfrak{X}$ such that

$$\deg \mathfrak{L}_{\mathfrak{i}}|_{\overline{\mathfrak{X}}_j} = \begin{cases} d & \text{if } i = j, \\ 0 & \text{otherwise.} \end{cases}$$

The obstruction to finding such an extension can be measured by the component group of the Néron model of the Jacobian of X, which is the Jacobian of the dual graph of $\overline{\mathfrak{X}}$. Tropical geometry systematically studies, refines, and exploits this obstruction theory to understand the degeneration of the complete linear series of L. One fundamental tool in this approach is Baker's Specialization Lemma [Bak08], which gives an explicit obstruction for a component of the Néron model of the Jacobian of X to intersect the closure of a line bundle of degree d and rank r. When every component is obstructed in this way, we can conclude, based solely on the dual graph of $\overline{\mathfrak{X}}$, that X has no $\mathfrak{g}^r_d$s. This observation is at the heart of the tropical proof of the Brill-Noether Theorem [CDPR12], although an important refinement is required to control the dimension of the space of $\mathfrak{g}^r_d$s when it is nonempty.

The main topic of this talk is joint work with Dave Jensen [JP14] refining this tropical approach to studying linear series on X, when $\overline{\mathfrak{X}}$ is not of compact type. In particular, we developed tools for studying degenerations of multiplication maps and found tropical obstructions to the existence of a nonzero kernel in a multiplication map

$$\Gamma(X,L)\otimes\Gamma(X,M)\to\Gamma(X,L\otimes M).$$

We have used these methods to give a new proof of the Gieseker-Petri Theorem via explicit computations on graphs, showing that if the dual graph of $\overline{\mathfrak{X}}$ is a particular chain of loops with bridges, with generic edge lengths, then the adjoing multiplication map

$$\Gamma(X,L)\otimes\Gamma(X,K\otimes L^{-1})\to\Gamma(X,K)$$

is injective for all L. Refinements of this method may be used to control dimensions of kernels of multiplication maps when they are nonempty, and we hope to develop these techniques further in future work.

In the last section of the talk, I discussed computations on Jacobians of random graphs and a Cohen-Lenstra heuristic for these Jacobians, based on joint work with Julien Clancy, Nathan Kaplan, Timothy Leake, and Melanie Wood [CLP13, CKLPW14], several of which are now proved by Wood [Woo14]. Our cyclicity conjecture is still open. We predict that for any fixed $0<q<1$, the probability that the Jacobian of the Erdös-Renyi random graph $G(n,q)$ is cyclic tends to the infinite product $\zeta(3)^{-1}\zeta(5)^{-1}\zeta(7)^{-1}\zeta(9)^{-1}\cdots$, which is roughly .7935.

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Effective divisors on moduli spaces of sheaves on the plane

Izzet Coskun

(joint work with Jack Huizenga, Matthew Woolf)

Let ξ be the Chern character of a stable sheaf F on $\mathbb{P}^2$. In joint work with Jack Huizenga and Matthew Woolf, we determine the effective cone of the moduli spaces $M(\xi)$ of Gieseker semi-stable sheaves on $\mathbb{P}^2$ with Chern character ξ. For simplicity, assume that the rank r of ξ is positive. It is then convenient to record ξ in terms of the slope $\mu = \frac{c_1}{r}$ and the discriminant $\Delta = \frac{\mu^2}{2} - \frac{\mathrm{ch}_2}{r}$.

A sheaf E *satisfies interpolation* with respect to a coherent sheaf F on $\mathbb{P}^2$ if $h^i(E \otimes F) = 0$ for every i (in particular, $\chi(E \otimes F) = 0$). The stable base locus decomposition of $M(\xi)$ is closely tied to the higher rank interpolation problem.

Problem 3 (Higher rank interpolation)**.** Given $F \in M(\xi)$ determine the minimal slope $\mu \in \mathbb{Q}$ with $\mu + \mu(\xi) \geq 0$ for which there exists a vector bundle E of slope μ satisfying interpolation with respect to F.

If E satisfies interpolation with respect to F, then the Brill-Noether divisor

$$D_E := \{G \in M(\xi) | h^1(E \otimes G) \neq 0\}$$

is an effective divisor that does not contain F in its base locus. The interpolation problem in general is very hard, but has been solved in the following cases:

(1) $F = I_Z$, where Z is a complete intersection, zero-dimensional scheme in $\mathbb{P}^2$ [CH].
(2) $F = I_Z$, where Z is a monomial, zero-dimensional scheme in $\mathbb{P}^2$ [CH].
(3) $F = I_Z$, where Z is a general, zero-dimensional scheme in $\mathbb{P}^2$ [H].
(4) $F \in M(\xi)$ is a general stable sheaf [CHW].

These theorems depend on finding a good resolution of F. If F were unstable, then the maximal destabilizing object would yield an exact sequence

$$0 \to A \to F \to B \to 0.$$

The idea is to destabilize F via Bridgeland stability and use the exact sequence arising from the Harder-Narasimhan filtration just past the wall where F is destabilized. If a bundle E satisfies interpolation with respect to A and B, then E satisfies interpolation for F by the long exact sequence for cohomology. One may hope that for an interpolating bundle E with minimal slope, E satisfies interpolation for F because it does so for both A and B. Because the Harder-Narasimhan filtrations of A and B are "simpler", we can try to prove interpolation inductively. This strategy works in all 4 cases listed above.

We first explain the case of monomial schemes. The Grothendieck K group of $\mathbb{P}^2$ is a free abelian group of rank 3. Let $\chi(\xi, \zeta) = \sum_{i=0}^{2} \mathrm{ext}^i(F, E)$, where F and E are sheaves with Chern characters ξ and ζ. Then there is a natural pairing on $K(\mathbb{P}^2)$ given by $(\xi, \zeta) = \chi(\xi^*, \zeta)$. If we scale by the rank, every Chern character ξ (of positive rank) determines a parabola Q_ξ of orthogonal invariants in the (μ, Δ)-plane defined by $P(\mu(\xi) + \mu) - \Delta(\xi) = \Delta$, where $P(m) = \frac{1}{2}(m^2 + 3m + 2)$ is the Hilbert polynomial of $\mathcal{O}_{\mathbb{P}^2}$. The invariants of any sheaf satisfying interpolation with respect to F lie on the parabola Q_ξ.

A monomial scheme Z can be represented by a box diagram D_Z recording the monomials that are nonzero in $\mathbb{C}[x, y]/I_Z$. Let h_i be the number of boxes in the ith row counting from the bottom and let v_i be the number of boxes in the ith column counting from the left. Define

$$\mu_j = -1 + \frac{1}{j}\sum_{i=1}^{j}(h_i + i - 1)\,, \quad \nu_k = -1 + \frac{1}{k}\sum_{i=1}^{k}(v_i + i - 1), \quad \mu_Z = \max_{j,k}(\mu_j, \nu_k).$$

Assume that the maximum is achieved by μ_h, (i.e., $\mu_Z = \mu_h$). Let D_U be the portion of D_Z lying above the hth horizontal line and let D_V be the potion of D_Z lying below this line. The diagrams D_U and D_V correspond to monomial zero-dimensional schemes U, V. We then have the following theorem.

Theorem 1. [CH] *Let Z be a zero-dimensional monomial scheme with Chern character ξ. There exists a vector bundle E of slope $\mu \in \mathbb{Q}$ satisfying interpolation for I_Z if and only if $\mu \geq \mu_Z$. We may take E to be prioritary. Furthermore, if there exists stable bundles of slope μ along Q_ξ, we may take E to be stable.*

The Bridgeland destabilizing sequence is given by

$$0 \to I_U(-h) \to I_Z \to I_{V \subset hL} \to 0,$$

where L is the line defined by $y = 0$. One proves the theorem by inducting on the complexity of Z. In fact, one computes the entire Harder-Narasimhan filtration of I_Z for different Bridgeland stability conditions, inductively decomposing the box diagram of the monomial scheme into pieces until each piece is a rectangle. As a corollary, one determines when monomial schemes are in the stable base loci of linear systems on the Hilbert schemes of points.

We now describe the effective cone of $M(\xi)$ in general. The possible invariants of stable vector bundles on $\mathbb{P}^2$ have been classified by Drézet and Le Potier [DLP], [LP]. First, there are *exceptional bundles* which are stable bundles E such that $\mathrm{Ext}^1(E, E) = 0$. The moduli space of an exceptional bundle is a single isolated point. The slope α of an exceptional bundle E_α is called an *exceptional slope.* The exceptional slopes exhibit remarkable number theoretic properties. For example, the even length continued fraction expansions of exceptional slopes between 0 and $\frac{1}{2}$ are palindromes consisting of 1s and 2s [H].

There is an explicit fractal curve δ in the (μ, Δ)-plane made of pieces of parabolas. For each exceptional slope α, there is an interval $I_\alpha = [\alpha - x_\alpha, \alpha + x_\alpha]$ where over that interval the curve δ is $Q_{-\alpha}$ on $[\alpha - x_\alpha, \alpha]$ and $Q_{-\alpha-3}$ on $[\alpha, \alpha + x_\alpha]$. The complement of these intervals is a Cantor set C with the following property (which plays an essential role in the geometry).

Theorem 2. [CHW] *A point of C is either an end point of an I_α (hence a quadratic irrational) or transcendental.*

Drézet and Le Potier prove that there exists a positive dimensional moduli space of stable bundles with invariants (r, μ, Δ) if and only if $r\mu, r(P(\mu) - \Delta) \in \mathbb{Z}$ and $\Delta \geq \delta(\mu)$ [DLP]. In this case, the moduli space is normal, projective, $\mathbb{Q}$-factorial of dimension $r^2(2\Delta - 1) + 1$ [LP]. The stable bundles with $\Delta = \delta(\mu)$ are called *height zero* bundles and their moduli spaces have Picard rank 1. For moduli spaces $M(\xi)$ with $\Delta > \delta(\mu)$, the Picard group is isomorphic to $\mathbb{Z} \oplus \mathbb{Z}$ naturally identified with $\xi^\perp$ in $K(\mathbb{P}^2)$ with respect to the Euler pairing introduced above.

Since the intersection of Q_ξ with $\Delta = \frac{1}{2}$ is a quadratic irrational, by Theorem 2, Q_ξ intersects $\Delta = \frac{1}{2}$ along some I_α and determines an exceptional bundle E_α. This bundle controls the effective cone of $M(\xi)$. Let ξ_α be the Chern character of E_α. Our main theorem is in terms of the following invariants:

(1) If $\chi(\xi, \xi_\alpha) > 0$, let $(\mu^+, \Delta^+) = Q_\xi \cap Q_{-\alpha}$.
(2) If $\chi(\xi, \xi_\alpha) = 0$, let $(\mu^+, \Delta^+) = (\alpha, \Delta_\alpha)$.
(3) If $\chi(\xi, \xi_\alpha) < 0$, let $(\mu^+, \Delta^+) = Q_\xi \cap Q_{-\alpha-3}$.

Theorem 3. [CHW] *Let F be a general point of $M(\xi)$ and let r^+ be sufficiently large and divisible. Let ζ be the Chern character with rank r^+, slope μ^+ and discriminant Δ^+. Then the general point E of $M(\zeta)$ satisfies interpolation with respect to F. Furthermore, the Brill-Noether divisor D_E spans an extremal ray of the effective cone of $M(\xi)$. If $\chi(\xi, \xi_\alpha) \neq 0$, then D_E also spans an extremal ray of the movable cone.*

The Beilinson spectral sequence for an exceptional collection explicitly determined by α [Dr] yields a canonical two term complex. One thus obtains a good resolution of F that allows one to compute cohomology. (This resolution also coincides with the destabilizing sequence in the sense of Bridgeland.) In particular, one obtains a rational map to a moduli space of Kronecker modules. When $\chi(\xi, \xi_\alpha) \neq 0$, the divisor D_E is the pullback of the ample generator via this map. Using the fact that there are complete curves in the fibers, we deduce that D_E is extremal. When $\chi(\xi, \xi_\alpha) = 0$, the rational map is birational and contracts the divisor D_E.

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Knotty character varieties

Vivek Shende

Let M be a smooth manifold. Recall that its cotangent bundle T^*M is canonically a symplectic manifold, and that the Nadler-Zaslow correspondence[6] asserts an equivalence

$$Fuk(T^*M) \cong Sh(M)$$

where the right hand side denotes the A-infinity category of twisted complexes over the infinitesimally wrapped Fukaya category, and the LHS denotes the DG category of complexes of sheaves on M. It is natural to use a Legendrian Λ at contact infinity – the infinite co-sphere bundle $T^\infty M$ – to impose boundary conditions on this category [8]. The corresponding category $Fuk_\Lambda(T^*M)$ morally consists of objects whose underlying Lagrangians are asymptotic to Λ; more formally, $Sh_\Lambda(M)$ is by definition those complexes of sheaves whose singular support intersects $T^\infty M$ in a subset of Λ. The following results indicate the naturality of this invariant:

Theorem 1. [8] *The category $Sh_\Lambda(M)$ is an invariant of Hamiltonian isotopy*

Theorem 2. [7] *When $M = \mathbb{R}^2$, the category $Sh_\Lambda(M)$ is the representation category of the Chekanov-Eliashberg DGA for the Legendrian Λ.*

We now restrict ourselves to the case when M is two-dimensional. Then there is a 'microlocal monodromy' map $Sh_\Lambda(M) \to Loc(\Lambda)$; we write $\mathcal{M}_1(\Lambda)$ for the moduli space of objects whose microlocal monodromy is a one-dimensional local system. We recall that for a (topological) knot, the HOMFLY polynomial is a certain invariant taking values in $\mathbb{Z}[a^{\pm 1}, (q^{1/2} - q^{-1/2})^{\pm 1}]$.

Theorem 3. [8] *When $M = S^1 \times \mathbb{R}$ and $\Lambda \subset T^\infty M$ is a positive braid closure, the orbifold cardinality $\#\mathcal{M}_1(\Lambda)(\mathbb{F}_q)$ is the highest degree coefficient of 'a' in the HOMFLY polynomial of Λ.*

This coefficient of the HOMFLY polynomial has appeared before in algebraic geometry:

Theorem 4. [5, 4, 3] *Let C be a singular plane curve, and J the compactified Jacobian of its blowup. Then up to a normalization factor, $\sum q^i \chi(\mathrm{Gr}^i_P H^*(J))$ is the highest degree coefficient of 'a' in the HOMFLY polynomial of the link of C. Here, P is the local perverse Leray filtration induced by any smoothing of J into an integrable system.*

As observed in [8], the relation between Theorems 3 and 4 above is reminiscent of the P=W conjecture relating the perverse filtration on the moduli of Higgs bundles to the weight filtration on the corresponding character variety [1]. This suggests that $M_1(\Lambda)$ should admit an interpretation as a wild character variety. In fact, this is a consequence of the wild Riemann-Hilbert correspondence.

Theorem 5. *(Deligne-Malgrange [2]) Let $\widetilde{\mathbb{C}}$ be by taking the real blowup of the complex plane at the origin, and gluing in an annulus inside the puncture. For any given formal type of singularity of ordinary differential equation, there is an extension of $\mathcal{O}_{\mathbb{C}\setminus 0}$ to this annulus such that the solution functor gives an equivalence of categories between differential equations of this formal type and its image in the category of constructible sheaves on the annulus. Moreover, all these sheaves have the same singular support Λ, and the image category is exactly the rank one objects of Sh_Λ. Thus $\mathcal{M}_1(\Lambda)$ is the wild character variety of the given formal type.*

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Stability conditions on threefolds via a conjectural Bogomolov-Gieseker type inequality

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(joint work with A. Bayer and P. Stellari)

The talk was based on the results contained in the work in progress [BMS14]. The main result is the construction of Bridgeland stability conditions on any abelian threefold, as well as on Calabi-Yau threefolds that admit an étale cover by an abelian threefold, or that are constructed as crepant resolutions of singular quotients of abelian threefolds.

The existence of Bridgeland stability conditions [Bri07] on smooth projective three-dimensional varieties in general, and more specifically on Calabi-Yau threefolds, is often considered the biggest open problem in the theory of Bridgeland stability conditions. Until the recent work [MP13a, MP13b], they were only known to exist on threefolds whose derived category admits a complete exceptional collection [Mac14, Sch13]. Possible applications of stability conditions range from modularity properties of generating functions of Donaldson-Thomas invariants [Tod14] to Reider-type theorems for adjoint linear series [BBMT11].

In [BMT14], the first two authors and Yukinobu Toda, also based on discussions with Aaron Bertram, proposed a general approach towards the construction of stability conditions on a smooth projective threefold X. The construction is based on the auxiliary notion of *tilt-stability* of certain two-term complexes, and a conjectural Bogomolov-Gieseker type inequality for the third Chern character of tilt-stable objects. It was shown that this conjecture would imply the existence of Bridgeland stability conditions.

Our first main result is the following, generalizing the result of [MP13a, MP13b] for the case when X has Picard rank one:

Theorem 1. *The Bogomolov-Gieseker type inequality for tilt-stable objects holds when X is an abelian threefold.*

There are Calabi-Yau threefolds that admit an abelian variety as a finite étale cover; these are usually called *Calabi-Yau threefolds of abelian type*. Our result applies similarly in these cases:

Theorem 2. *The Bogomolov-Gieseker type inequality for tilt-stable objects holds when X is a Calabi-Yau threefold of abelian type.*

Combined with the results of [BMT14], these theorems imply the existence of Bridgeland stability conditions in either case. There is one more type of Calabi-Yau threefolds whose derived category is closely related to those of abelian threefolds: namely *Kummer threefolds*, that are obtained as the crepant resolution of the quotient of an abelian threefold X by the action of a finite group G. Using the method of "inducing" stability conditions on the G-equivariant derived category of X and the BKR-equivalence, we can also treat this case.

Theorem 3. *Bridgeland stability conditions exist when X is an abelian threefold, or a Calabi-Yau threefold of abelian type, or a Kummer threefold.*

Approach. Let X be a smooth projective threefold over $\mathbb{C}$. We fix a very ample divisor class $H \in \mathrm{NS}(X)$. The conjectural Bogomolov-Gieseker type inequality depends on the choice of two divisor classes

$$\omega = \alpha H \qquad B = \beta H,$$

for $\alpha > 0$ and $\beta \in \mathbb{R}$.

The first part of our approach is as follows. We reduce the conjectural Bogomolov-type inequality to a statement that only considers objects E that are stable in the limit as $\alpha \to 0$ and the tilt slope $\nu_{\alpha,\beta}(E) \to 0$; if $\overline{\beta} := \lim \beta$, the claim is then

$$\int e^{-\overline{\beta}H} \mathrm{ch}(E) \leq 0.$$

The reduction is based on the methods of [Mac14]: as we approach this limit, either E remains stable, in which case the above inequality is enough to ensure that E satisfies our conjecture everywhere. Otherwise, E will be strictly semistable at some point; we then show that all its Jordan-Hölder factors have strictly smaller *H-discriminant* (which is a variant of the discriminant appearing in the classical Bogomolov-Gieseker inequality). This allows us to proceed by induction.

In the case of an abelian threefold, we then make extensive use of the multiplication by m map $\underline{m}\colon X \to X$; the key observation being that if E is tilt-stable, then so is $\underline{m}^*(E)$.

Via pull-back and tensor product with line bundles, we can then assume that $\beta = 0$. We then have to prove that $\mathrm{ch}_3(E) \leq 0$; in other words, we have to prove an inequality of the Euler characteristic of E. To obtain a contradiction, assume that $\mathrm{ch}_3(E) > 0$, and consider further pull-backs:

$$\chi(\mathcal{O}_X, \underline{m}^*(E)) = \mathrm{ch}_3(\underline{m}^*(E)) = m^6 \mathrm{ch}_3(E) \geq m^6. \qquad (1)$$

However, by stability we have $\mathrm{Hom}(\mathcal{O}_X(H), \underline{m}^*(E)) = 0$; moreover, if $D \in |H|$ is a general element of the linear system of H, classical arguments give a bound of the form

$$h^0(\underline{m}^*(E)) \leq h^0(\underline{m}^*(E)|_D) = O(m^4)$$

Similar bounds for h^2 lead to a contradiction to (1).

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Beauville-Voisin Conjecture for generalized Kummer varieties

Lie Fu

In [1], Beauville and Voisin observe the following property of the Chow rings of projective K3 surfaces.

Theorem 0.3 (Beauville-Voisin)**.** *Let S be a projective K3 surface. Then*
(i) There is a well defined 0-cycle $o \in \mathrm{CH}_0(S)$, which is represented by any point on any rational curve on S. It is called the canonical cycle.
(ii) For any two divisors D, D', the intersection product $D \cdot D'$ is proportional to the canonical cycle o in $\mathrm{CH}_0(S)$.
(iii) $c_2(T_S) = 24o \in \mathrm{CH}_0(S)$.
In particular, for any algebraic cycle which is a polynomial on Chern classes of the tangent bundle T_S and of line bundles on S, it is rationally equivalent to zero if and only if it is numerically equivalent to zero.

The above result is surprising because $\mathrm{CH}_0(S)$ is very huge ('infinite dimensional' in the sense of Mumford [2], *cf.* [3, Chapter 10]). In a subsequent paper [4], Beauville proposed a conjectural explanation for Theorem 0.3 to put it into a larger picture. To explain his idea, let us recall a notion generalizing K3 surfaces to higher dimensions: a smooth projective complex variety X is called *hyperkähler* or *irreducible holomorphic symplectic*, if it is simply connected and $H^{2,0}(X)$ is 1-dimensional and generated by a holomorphic 2-form which is non-degenerate at each point of X. In particular, a hyperkähler variety has trivial canoncial bundle. Here are some basic examples of projective hyperkähler manifolds:

- (Beauville [5]) Let S be a projective K3 surface and $n \in \mathbf{N}$, then $S^{[n]}$, which is the Hilbert scheme of subschemes of dimension 0 and length n, is hyperkähler of dimension $2n$.
- (Beauville [5]) Let A be an abelian surface and $n \in \mathbf{N}$. Let $s : A^{[n+1]} \to A$ be the natural morphism defined by the composition of the Hilbert-Chow morphism $A^{[n+1]} \to A^{(n+1)}$ and the summation $A^{(n+1)} \to A$ using the group law of A. It is clear that s is an isotrivial fibration. Then a fibre $K_n := s^{-1}(O_A)$ is hyperkähler of dimension $2n$, called *generalized Kummer variety.* The name is justified by the fact that K_1 is exactly the Kummer K3 surface associated to A.
- (Beauville-Donagi [6]) Let $X \subset \mathbf{P^5}$ be a smooth cubic fourfold, then its *Fano variety of lines* $F(X) := \left\{ l \in \mathrm{Gr}\left(\mathbf{P^1}, \mathbf{P^5}\right) \mid l \subset X \right\}$ is hyperkähler of dimension 4.

As an attempt to understand Theorem 0.3 in a broader framework, Beauville gives the point of view in [4] that we can regard this result as a 'splitting property' of the conjectural Bloch-Beilinson-Murre filtration on Chow groups (see [7], [8]) for certain varieties with trivial canonical bundle. He suggests to verify the following down-to-earth consequence of this conjectural splitting of the conjectural filtration on Chow groups of hyperkähler varieties. As a first evidence, the special cases when $X = S^{[2]}$ or $S^{[3]}$ for a projective K3 surface S are verified in his paper *loc.cit.*

Conjecture 0.4 (Beauville)**.** *Let X be a projective hyperkähler manifold, and $z \in \mathrm{CH}(X)_{\mathbf{Q}}$ be a polynomial with $\mathbf{Q}$-coefficients of the first Chern classes of line bundles on X. Then z is homologically trivial if and only if z is (rationally equivalent to) zero.*

Voisin pursues the work of Beauville and makes in [9] the following stronger version of Conjecture 0.4, by involving also the Chern classes of the tangent bundle:

Conjecture 0.5 (Beauville-Voisin)**.** *Let X be a projective hyperkähler manifold, and $z \in \mathrm{CH}(X)_{\mathbf{Q}}$ be a polynomial with $\mathbf{Q}$-coefficients of the first Chern classes of line bundles on X and the Chern classes of the tangent bundle of X. Then z is numerically trivial if and only if z is (rationally equivalent to) zero.*

Here we replaced 'homologically trivial' in the original statement in Voisin's paper [9] by 'numerically trivial'. But according to the standard conjecture [10], the homological equivalence and the numerical equivalence are expected to coincide.

In [9], Voisin proves Conjecture 0.5 for the Fano varieties of lines of cubic fourfolds, and for $S^{[n]}$ if S is a projective K3 surface and $n \leq 2b_{2,tr}+4$, where $b_{2,tr}$ is the second Betti number of S minus its Picard number. We remark that here we indeed can replace the homological equivalence by the numerical equivalence since the standard conjecture in these two cases has been verified by Charles and Markman [11].

The main purpose of this talk is to prove the Beauville-Voisin conjecture 0.5 for generalized Kummer varieties.

Theorem 0.6. *Let A be an abelian surface, $n \geq 1$ be a natural number. Denote by K_n the generalized Kummer variety associated to A. Consider any algebraic cycle $z \in \mathrm{CH}(K_n)_{\mathbf{Q}}$ which is a polynomial with rational coefficients of the first Chern classes of line bundles on K_n and the Chern classes of the tangent bundle of K_n, then z is numerically trivial if and only if z is (rationally equivalent to) zero.*

There are two key ingredients in the proof of the above theorem:
Ingredient I: De Cataldo-Migliorini's result. Let S be a projective surface, $n \in \mathbf{N}_+$ and $\mathscr{P}(n)$ be the set of partitions of n. For any such partition $\mu = (\mu_1, \cdots, \mu_l)$, we denote by $l_\mu := l$ its length. Define $S^\mu := S^{l_\mu} = \underbrace{S \times \cdots \times S}_{l_\mu}$, and also a natural morphism from it to the symmetric product:

$$\begin{array}{ccc} S^\mu & \to & S^{(n)} \\ (x_1, \cdots, x_l) & \mapsto & \mu_1 x_1 + \cdots + \mu_l x_l. \end{array}$$

Now define $E_\mu := \left(S^{[n]} \times_{S^{(n)}} S^\mu\right)_{red}$ to be the reduced incidence variety inside $S^{[n]} \times S^\mu$. Then E_μ can be viewed as a correspondence from $S^{[n]}$ to S^μ, and we will write ${}^tE_\mu$ for the *transpose* correspondence, namely the correspondence from S^μ to $S^{[n]}$ defined by the same subvariety E_μ in the product. Let $\mu = (\mu_1, \cdots, \mu_l) = 1^{a_1}2^{a_2}\cdots n^{a_n}$ be a partition of n, we define $m_\mu := (-1)^{n-l}\prod_{j=1}^{l}\mu_j$ and $c_\mu := \frac{1}{m_\mu}\frac{1}{a_1!\cdots a_n!}$.

Theorem 0.7 (De Cataldo-Migliorini [12])**.** *Let S be a projective surface, $n \in \mathbf{N}_+$. For each $\mu \in \mathscr{P}(n)$, let E_μ and ${}^tE_\mu$ be the correspondences defined above. Then the sum of the compositions*

$$\sum_{\mu \in \mathscr{P}(n)} c_\mu \; {}^tE_\mu \circ E_\mu = \Delta_{S^{[n]}}$$

is the identity correspondence of $S^{[n]}$, modulo rational equivalence.

Return to the case where $S = A$ is an abelian surface. We view $A^{[n+1]}$ as a variety over A by the natural summation morphism $s\colon A^{[n+1]} \to A$. Similarly, for each $\mu \in \mathscr{P}(n+1)$ of length l, A^μ also admits a natural morphism to A, namely, the *weighted sum*. Now by taking their fibres over the origin of A, we obtain a correspondence $\Gamma_\mu := \pi_\mu^{-1}(O_A)$ between the generalized Kummer variety $K_n := s^{-1}(O_A)$ and the possibly non-connected abelian variety $B_\mu := \ker(s_\mu\colon A^\mu \to A)$. Theorem 0.7 then implies the following

Corollary 0.8. *For each $\mu \in \mathscr{P}(n+1)$, let Γ_μ be the correspondences between K_n and B_μ defined above. Then for any $\gamma \in \mathrm{CH}\left(A^{[n+1]}\right)$, we have*

$$\sum_{\mu \in \mathscr{P}(n+1)} c_\mu \Gamma_\mu^* \circ \Gamma_{\mu *}(\gamma|_{K_n}) = \gamma|_{K_n} \text{ in } \mathrm{CH}(K_n),$$

where for a partition $\mu = (\mu_1, \cdots, \mu_l) = 1^{a_1}2^{a_2}\cdots(n+1)^{a_{n+1}} \in \mathscr{P}(n+1)$*, the constant* c_μ *is defined as* $\frac{1}{(-1)^{n+1-l}\prod_{j=1}^{l}\mu_j} \cdot \frac{1}{a_1!\cdots a_{n+1}!}$.

Ingredient II: Moonen-O'Sullivan's result. The following consequence of Beauville's conjecture for algebraic cycles on abelian varieties is verified:

Theorem 0.9 (Moonen[13] , O'Sullivan [14])**.** *Let A be an abelian variety. Let $P \in \mathrm{CH}^*(A)$ be a polynomial with rational coefficients in the first Chern classes of symmetric line bundles on A, then P is numerically equivalent to zero if and only if P is (rationally equivalent to) zero.*

Strategy of the proof of main theorem. On the one hand, as in [9], the result of De Cataldo-Migliorini above relates the Chow groups of $A^{[n]}$ to the Chow groups of various products of A. On the other hand, the result on algebraic cycles on abelian varieties due to Moonen and O'Sullivan allows us to upgrade a relation modulo numerical equivalence to a relation modulo rational equivalence. To verify that Moonen and O'Sullivan's theorem indeed applies to our situation, we use a similar technical computation as in [9].

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Tautological cycles on powers of varieties

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We report a number of results obtained in [15] and [16] about tautological cycles on curves, Jacobians and $K3$ surfaces.

Throughout, Chow groups CH are taken with $\mathbb{Q}$-coefficients. Let X be a smooth projective variety over a field k. For $n \geq 1$ and $m \geq 0$, consider maps $T = (T_1, \ldots, T_m)\colon X^n \to X^m$ such that each T_i is a projection of X^n to one of its factors (we set $X^0 = k$). These maps are called tautological maps.

Choose a (usually finite-dimensional) subspace $A \subset \mathrm{CH}(X)$ of classes carrying certain geometric information. The tautological rings of X^n with respect to A, denoted by $\mathscr{R}_A(X^n)$, form the smallest system of $\mathbb{Q}$-subalgebras of $\mathrm{CH}(X^n)$ containing A and closed under pull-backs and push-forwards via all tautological maps. Later we shall drop the subscript $_A$ when the choice of A is clear from the context.

One may also extend this definition to the relative setting, where X is a smooth projective scheme over a smooth connected k-scheme M, and X^n stand for the powers of X relative to M (we set $X^0 = M$). Another generalization is to include subspaces $A \subset \mathrm{CH}(X^n)$ for some $n > 1$.

Curves. We look at the case of curves in the relative (or universal) setting. Consider the universal curve $\mathscr{C}_g$ over the moduli space of smooth genus g curves $\mathscr{M}_g$ (we assume $g \geq 2$). Then by setting $A = \emptyset \subset \mathrm{CH}(\mathscr{C}_g)$, we recover the tautological ring $\mathscr{R}(\mathscr{M}_g)$ introduced by Mumford [9]. One classical problem is to study relations between the generators of $\mathscr{R}(\mathscr{M}_g)$ (and more generally $\mathscr{R}(\mathscr{C}_g^n)$), which is the subject of the Faber conjectures [4].

It is often convenient to work with a variant: the moduli of smooth pointed genus g curves $\mathscr{M}_{g,1}$ and the universal curve $\mathscr{C}_{g,1}$. Denote by $x_0\colon \mathscr{M}_{g,1} \to \mathscr{C}_{g,1}$ the section (marked point). We set $A = \mathbb{Q} \cdot \big[x_0(\mathscr{M}_{g,1})\big]$ and we obtain the tautological rings $\mathscr{R}(\mathscr{C}_{g,1}^n)$. The section x_0 induces the Abel-Jacobi map $\mathscr{C}_{g,1}^n \to \mathscr{J}_{g,1}$, where $\mathscr{J}_{g,1}$ is the universal Jacobian. It factors through the symmetric power $\mathscr{C}_{g,1}^{[n]}$, whose tautological ring $\mathscr{R}(\mathscr{C}_{g,1}^{[n]})$ is identified with the $\mathfrak{S}_n$-invariant part of $\mathscr{R}(\mathscr{C}_{g,1}^n)$.

Although $\mathscr{R}(\mathscr{C}_{g,1}^n)$ becomes more complicated as n increases, we get somewhat better control of $\mathscr{R}(\mathscr{C}_{g,1}^{[n]})$. The reason is simply that when $n \geq 2g-1$, the map $\mathscr{C}_{g,1}^{[n]} \to \mathscr{J}_{g,1}$ is a $\mathbb{P}^{n-g}$-bundle. Then there is an isomorphism of $\mathbb{Q}$-algebras

$$\mathrm{CH}(\mathscr{C}_{g,1}^{[n]}) \simeq \mathrm{CH}(\mathscr{J}_{g,1})[t] \big/ \big\langle P(t) \big\rangle, \tag{1}$$

where $P(t)$ is a polynomial of degree $n-g+1$ with coefficients in $\mathrm{CH}(\mathscr{J}_{g,1})$. We have figured out the Jacobian counterpart of $\mathscr{R}(\mathscr{C}_{g,1}^{[n]})$ under this isomorphism. Interestingly, it was also called tautological ring when introduced by Beauville [1].

Jacobians. With $\mathscr{J}_{g,1}$ being an abelian scheme, its Chow ring carries a second ring structure given by the Pontryagin product. There is also the action of the multiplication $[N]\colon \mathscr{J}_{g,1} \to \mathscr{J}_{g,1}$, for $N \in \mathbb{Z}$. Consider the Abel-Jacobi embedding $\mathscr{C}_{g,1} \hookrightarrow \mathscr{J}_{g,1}$. Following Beauville, we define the tautological ring of $\mathscr{J}_{g,1}$ to be the smallest $\mathbb{Q}$-subspace of $\mathrm{CH}(\mathscr{J}_{g,1})$ containing $[\mathscr{C}_{g,1}]$ and closed under both products as well as the action of $[N]$. We use a different notation $\mathscr{T}(\mathscr{J}_{g,1})$ for this ring.

The study of $\mathscr{T}(\mathscr{J}_{g,1})$ is powered by several structures on the Chow ring of abelian schemes, namely the Beauville decomposition, the Fourier transform and the Lefschetz decomposition (or $\mathfrak{sl}_2$-action). As is shown by Künneman [7], the background is the so-called motivic Lefschetz decomposition. These structures have been studied in details by Polishchuk [11] in the Jacobian context. Using Polishchuk's formulas we obtain the following result, which generalizes previous works of Beauville and himself to the relative (or universal) setting.

Theorem 0.10 ([16], Theorem 3.5)**.** *The ring $\mathscr{T}(\mathscr{J}_{g,1})$ admits an explicit finite set of generators with respect to the intersection product. There is also an explicit description of the $\mathfrak{sl}_2$-action on $\mathscr{T}(\mathscr{J}_{g,1})$ in terms of the generators.*

One consequence is that the ring $\mathscr{R}(\mathscr{M}_{g,1})$ can be identified with a $\mathbb{Q}$-subalgebra of $\mathscr{T}(\mathscr{J}_{g,1})$. With some more effort we prove the following comparison (previously obtained by Moonen and Polishchuk [8] in an easier context).

Theorem 0.11 ([16], Theorem 5.4)**.** *When $n \geq 2g-1$, the isomorphism* (1) *restricts to an isomorphism of $\mathbb{Q}$-algebras*

$$\mathscr{R}(\mathscr{C}_{g,1}^{[n]}) \simeq \mathscr{T}(\mathscr{J}_{g,1})[t] \big/ \big\langle P(t) \big\rangle.$$

Polishchuk [10] also described how to obtain relations between the generators via the $\mathfrak{sl}_2$-action. The recipe is simple yet powerful: we know the maximal weight of $\mathscr{T}(\mathscr{J}_{g,1})$ as an $\mathfrak{sl}_2$-representation. All polynomials in the generators that are beyond the maximal weight vanish for obvious reasons. Then one applies the $\mathfrak{sl}_2$-action to those polynomials to get not-so-obvious relations.

Using these relations we can study (analogues of) Faber's Gorenstein conjecture for various tautological rings. We confirm that $\mathscr{R}(\mathscr{M}_{g,1})$ (resp. $\mathscr{R}(\mathscr{M}_g)$) is Gorenstein for $g \leq 19$ (resp. $g \leq 23$). As far as computation goes, we seem to obtain the same set of relations as the Faber-Zagier relations. Further, we are able to formulate the Gorenstein property for $\mathscr{T}(\mathscr{J}_{g,1})$ and prove the following equivalence.

Corollary 0.12 ([16], Theorem 6.15)**.** *The ring $\mathscr{T}(\mathscr{J}_{g,1})$ is Gorenstein if and only if $\mathscr{R}(\mathscr{C}_{g,1}^{[n]})$ is Gorenstein for all $n \geq 0$.*

Again we confirm these Gorenstein properties for $g \leq 7$, leaving $g = 8$ the ultimate critical case.

As is mentioned above, the relations via $\mathfrak{sl}_2$ are of motivic nature. Polishchuk [10] conjectured that for the generic Jacobian and modulo algebraic equivalence, the $\mathfrak{sl}_2$-action should provide all the relations. Following this idea, we may ask a similar question for $\mathscr{T}(\mathscr{J}_{g,1})$ (and $\mathscr{R}(\mathscr{C}_{g,1}^{[n]})$): are all relations of motivic nature? A positive answer to the question would explain why the Gorenstein properties might not hold in general.

$K3$ **surfaces.** Let S be a $K3$ surface over k. We set $A = \mathrm{Pic}(S)$ and get the tautological rings $\mathscr{R}(S^n)$. Inspired by the Beauville conjecture for the Hilbert schemes of S (see [5]), Voisin [13] conjectured that (the restriction of) the cycle class map $\mathrm{cl}\colon \mathscr{R}(S^n) \to H(S^n)$ is injective. The case $n = 1$ is the classical result of Beauville and Voisin [2] on the existence of a distinguished point c_S. In fact, they proved the following stronger result

$$(2) \qquad [\Delta_{123}] - \big([\Delta_{12} \times c_S] + \text{perm.}\big) + \big([c_S \times c_S \times S] + \text{perm.}\big) = 0 \text{ in } \mathscr{R}(S^3),$$

where Δ_{123} is the small diagonal in S^3.

Voisin's conjecture turns out to be rather strong. Notably it implies the Kimura-O'Sullivan finite-dimensionality conjecture [6] for S, which predicts that

$$(3) \qquad \sum_{\sigma \in \mathfrak{S}_{25}} \prod_{i=1}^{25} \mathrm{sgn}(\sigma)[\Delta_{i,25+\sigma(i)}] = 0 \text{ in } \mathscr{R}(S^{50}),$$

and corresponds to the vanishing $\wedge^{25} H(S) = 0$. The latter is known for example for all $K3$ surfaces dominated by products of curves. One may also consider the conjectural relation in $\mathscr{R}(S^{2(b_{\mathrm{tr}}+1)})$ corresponding to $\wedge^{b_{\mathrm{tr}}+1} H^2_{\mathrm{tr}}(S) = 0$, with $H^2_{\mathrm{tr}}(S)$ the transcendental part of $H^2(S)$ and b_{tr} its Betti number.

Surprisingly, we have the following algebraic result.

Theorem 0.13 ([15], Theorem)**.** *All relations in* $\mathrm{cl}\big(\mathscr{R}(S^n)\big)$ *are generated by* (2) *and the variant of* (3) *corresponding to* $\wedge^{b_{\mathrm{tr}}+1} H^2_{\mathrm{tr}}(S) = 0$*. In particular, Voisin's conjecture holds for* S *if and only if* S *is finite-dimensional.*

As a consequence, we obtain the Beauville-Voisin conjecture for the Hilbert schemes of finite-dimensional S (here k is algebraically closed; see [5] for an introduction). More precisely, let $X = S^{[m]}$ be the m-th Hilbert scheme of S. We define the tautological ring $\mathscr{R}(X)$ (also called the Beauville-Voisin ring) and more generally $\mathscr{R}(X^n)$ by setting $A = \mathrm{Pic}(X) + \mathbb{Q} \cdot \{c_i(X)\}$. By the work of de Cataldo and Migliorini [3], there is a motivic decomposition $h(X) \simeq \oplus_{\nu \in \mathfrak{P}(m)} h(S^{(\nu)})(l(\nu) - m)$, which induces a decomposition of Chow groups

$$(4) \qquad \mathrm{CH}^i(X) \simeq \bigoplus_{\nu \in \mathfrak{P}(m)} \mathrm{CH}^{i+l(\nu)-m}(S^{(\nu)}).$$

Here $\mathfrak{P}(m)$ is the set of partitions of m, $l(\nu)$ is the length of the partition ν and $S^{(\nu)}$ is the quotient of $S^{l(\nu)}$ by the symmetries of ν. One also gets similar decompositions for X^n. The calculation of Voisin [13] shows that under (4), the ring $\mathscr{R}(X^n)$ is always mapped to $\mathscr{R}(S^{n'})$ for various n'. Hence in view of Theorem 0.13, the finite-dimensionality of S implies the injectivity of $\mathrm{cl}\colon \mathscr{R}(X^n) \to H(X^n)$.

More recently, Voisin [14] made an interesting observation. First, the decomposition (4) and the calculation in [13] works for arbitrary surfaces. Moreover, the first statement in Theorem 0.13 holds for any regular surface (*i.e.* of Albanese dimension 0). On the other hand, the conjectural relation (3) depends on the Betti numbers of the surface. The conclusion is that whenever one considers a specific relation in $\mathrm{cl}\big(\mathscr{R}(S^n)\big)$ that is not sensitive to the surface, it has to be generated by (2) and thus holds in $\mathscr{R}(S^n)$ for the $K3$ surface S.

Further, Vial [12] made the following remark. As is discussed in [5], the background of the Beauville-Voisin conjecture is a multiplicative splitting of the conjectural Bloch-Beilinson filtration on the Chow ring of hyper-Kähler manifolds. In the case of Hilbert schemes of $K3$ surfaces $X = S^{[m]}$, the motivic decomposition in [3] induces a Chow-Künneth decomposition $h(X) \simeq \oplus_i h^i(X)$. Then one can formally split a Bloch-Beilinson type filtration on $\mathrm{CH}(X)$ by setting

$$\mathrm{CH}^i_{(j)}(X) := \mathrm{CH}^i\big(h^{2i-j}(X)\big), \text{ so that } \mathrm{CH}^i(X) \simeq \bigoplus_j \mathrm{CH}^i_{(j)}(X). \tag{5}$$

The remark is that whether (5) is multiplicative can be expressed in terms of certain relations in $\mathscr{R}(X^3)$. We know that these relations hold in cohomology and that they are not sensitive to the surface, so they hold in $\mathscr{R}(X^3)$ unconditionally. It follows that we get a multiplicative decomposition of $\mathrm{CH}(X)$ as in (5).

Of course, the Beauville-Voisin conjecture for $X = S^{[m]}$ remains open in general and cannot be deduced from this decomposition. However, one can restate and generalize the conjecture as follows.

Conjecture 0.14. *In* (5)*, we have* $\mathrm{CH}^i_{(j)}(X) = 0$ *for* $j < 0$*. Moreover, the cycle class map* $\mathrm{cl}\colon \mathrm{CH}^i_{(0)}(X) \to H^{2i}(X)$ *is injective.*

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Matrix factorizations and families of curves of genus 15

Frank-Olaf Schreyer

The moduli spaces $\mathcal{M}_g$ of curves of genus g are known to be unirational for $g \leq 14$, [15, 14, 3, 16]. For $g = 22$ or $g \geq 24$ they are known to be of general type [12, 7, 10, 11]. The cases in between are not fully understood: $\mathcal{M}_{23}$ has positive Kodaira dimension [10], $\mathcal{M}_{15}$ is rationally connected [4, 1], and $\mathcal{M}_{16}$ [5, 11] is uniruled. In this paper we are mainly concerned with $\mathcal{M}_{15}$ and an attempt to prove its unirationality.

By Brill-Noether theory, a general curve of genus 15 has a smooth model of degree 16 in $\mathbb{P}^4$. Let

$$\mathcal{H} \subset \mathrm{Hilb}_{16t+1-15}(\mathbb{P}^4)$$

be the component of the Hilbert scheme of curves of degree $d = 16$ and genus $g = 15$ in $\mathbb{P}^4$, which dominates the moduli space $\mathcal{M}_{15}$. Let

$$\widetilde{\mathcal{M}}^4_{15,16} \subset \{(C, L) \mid C \in \mathcal{M}_{15}, L \in W^4_{16}(C)\}$$

be the component which dominates $\mathcal{M}_{15}$. So $\mathcal{H}//PGL(5)$ is birational to $\widetilde{\mathcal{M}}^4_{15,16}$. Our main result connects this moduli space to a moduli space of certain matrix factorizations.

Theorem 1. *The moduli space $\widetilde{\mathcal{M}}^4_{15,16}$ of curves of genus 15 together with a g^4_{16} is birational to a component of the moduli space of matrix factorizations of type $(\psi\colon \mathcal{O}^{18}(-3) \to \mathcal{O}^{15}(-1) \oplus \mathcal{O}^3(-2), \varphi\colon \mathcal{O}^{15}(-1) \oplus \mathcal{O}^3(-2) \to \mathcal{O}^{18})$ of cubic forms on $\mathbb{P}^4$.*

As a corollary of our proof we obtain the dimension statement in

Theorem 2. *A general cubic threefold in $\mathbb{P}^4$ contains a 32-dimensional uniruled family of smooth curves of genus 15 and degree 16.*

Since a general curve in $\mathcal{H}$ lies on a unique cubic threefold, and cubic threefolds depend on 10 parameters up to projectivities, the dimension 32 fits with $\dim \mathcal{M}_{15} = 42$.

Our approach to construct a family of curves of genus 15 builds upon the construction of a matrix factorization on a cubic as a syzygy module of an auxiliary module N. We use Boij-Söderberg theory [8], [2], and the `Macaulay2` package [9] to get a list of candidate Betti tables. In all our cases the sheaf $\mathcal{L} = \widetilde{N}$ will be a line bundle on an auxiliary curve E. The choice of E and $\mathcal{L}$ is motivated by a dimension count and the shape of the Betti table of N. We succeeded to construct altogether 20 families of curves in $\mathcal{H}$, and 17 of the families are unirational. However, the unirational families do not dominate $\mathcal{M}_{15}$ although the number of parameters in the construction exceeds 42. Three of these families have a non-unirational step in their construction. (We need an effective divisor on the auxiliary curve). Precisely, those three families dominate $\mathcal{M}_{15}$. We use one of the non-unirational families to prove

Theorem 3. *The moduli space* $\widetilde{\mathcal{M}}^4_{15,16}$ *is uniruled.*

and the uniruledness in Theorem 2.

The proofs of the Theorems in this article rely on computer algebra. An implementation of all necessary computations can be found on my homepage.

Many of the images of the unirational families have dimension 39. There is one of dimension 41, one of dimension 40, and some of dimension < 39. A good explanation why I failed to prove the unirationality of $\mathcal{M}_{15}$ with this method could be

Conjecture 3. *The maximal rationally connected fibration of* $\widetilde{\mathcal{M}}^4_{15,16}$ *has a three dimensional base.*

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Toward an Effective Theory of GW-Invariants of Quintic CY-Threefolds

Jun Li
(joint work with Huailiang Chang, Weiping Li and Mellisa Liu)

Explicitly solving all genus Gromov-Witten invariants (in short GW invariants) of quintic Calabi-Yau threefolds is one of the major goals in the subject of Mirror Symmetry. The genus zero GW invariants of quintic threefolds has been solved following the work of [Ko, Gi, LLY]. For positive genus, besides the solution of genus one invariants of quintics by the work of [LZ, Zi], no effective theory for higher genus is known, other than that by mathematical physics based on Mirror Symmetry conjecture. In addition, the algorithm proposed in [MP] using the degeneration of GW invariants, though useful in theoretical study, so far has not been implemented.

In a joint work with Huailiang Chang, Weiping Li and Mellisa Liu, we are developing an effective theory towards evaluating all genus GW invariants of quintic Calabi-Yau threefolds.

Our theory builds on a transition between FJRW invariants [FJR1] and GW invariants of stable maps with p-fields [CL]. Recall that the former is the LG theory taking values in $[\mathbb{C}^5/\mu_5]$ (via Spin fields), and the later is the LG theory taking values in the tautological line bundle $K_{\mathbb{P}^4}$ (via P-fields). As pointed out by Witten, $[\mathbb{C}^5/\mu_5]$ and $K_{\mathbb{P}^4}$ are the two GIT quotients of $[\mathbb{C}^6/\mathbb{G}_m]_{(1^5,-5)}$ and these two theories should be connected via a wall-crossing of GIT.

We realize this by intruding the notion of Mixed-Spin-P fields (MSP fields) and its LG theory, providing a geometric theory of the transition between the LG theories of the two GIT quotients of $[\mathbb{C}^6/\mathbb{G}_m]$,

An MSP field is a collection $(\Sigma^{\mathscr{C}}, \mathscr{C}, \mathscr{L}, \mathscr{N}, \varphi, \rho, \nu)$ of twisted curves $\Sigma^{\mathscr{C}} \subset \mathscr{C}$, and a collection of fields (φ, ρ, ν). An MSP field comes with numerical invariants: the genus of $\mathscr{C}$, the monodromies γ, and the bi-degree $\mathbf{d} = (d_0, d_\infty)$. We form the moduli $\mathcal{W}_{g,\gamma,\mathbf{d}}$ of equivalence classes of stable MSP fields of numerical data $(g, \gamma, \mathbf{d})$. It is a separated DM stack, locally of finite type, and with a perfect obstruction theory.

We construct a properly supported virtual cycle of $\mathcal{W}_{g,\gamma,\mathbf{d}}$ by constructing a cosection σ of its obstruction sheaf and applying the theory of cosection localized virtual cycles of [KL]. The degeneracy locus $\mathcal{W}^-_{g,\gamma,\mathbf{d}}$ of the cosection σ is a closed substack of $\mathcal{W}_{g,\gamma,\mathbf{d}}$, proper and of finite type. We then build a $\mathbb{G}_m$ structure on $(\mathcal{W}_{g,\gamma,\mathbf{d}}, \sigma)$, with $\mathbb{G}_m$-equivariant perfect obstruction theory, making the cosection localized virtual cycle $[\mathcal{W}_{g,\gamma,\mathbf{d}}]^{\mathrm{vir}}_\sigma$ an equivariant cycle.

Applying the analogous virtual localization formula of [GP], we obtain vanishings

$$\sum_{\Gamma}\Big[u^{\delta(g,0,\mathbf{d})}\cdot\frac{[W_\Gamma]^{\mathrm{vir}}}{e(N_{W_\Gamma/\mathcal{W}_{g,0,\mathbf{d}}})}\Big]_0=0,$$

which after a detailed study of the fixed locus $(\mathcal{W}_{g,\gamma,\mathbf{d}})^{\mathbb{G}_m}$, provides us polynomial relations among the GW invariants of the quintic threefolds and the FJRW invariants of the Fermat quintic polynomial $\mathfrak{w}_5=x_1^5+\cdots+x_5^5$.

Let $N_g(d)$ be the genus g degree d GW invariants of the quintic threefolds. We prove using the mentioned relations

Theorem 0.15. *The relations provide an effective algorithm to determine the GW invariants $N_g(d)$ provided*
(1). genus g' FJRW invariants of multiple insertions $-\frac{2}{5}$ are known for $g'\le g$;
(2). $N_{g'}(d')$ are known for (g',d') such that $g'<g$, and $d'\le d$;
(3). $N_g(d')$ are known for $d'\le g$.

Using the full set of relations and the moduli of some variations of MSP fields, we conjecture

Conjecture 0.16. *The moduli of MSP-fields will provide an effective algorithm to determine all genus FJRW invariants of the quintic polynomial $\mathfrak{w}_5$ of multiple insertions $-\frac{2}{5}$ and all genus GW invariants $N_g(d)$.*

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Local cohomology wih determinantal support and syzygies

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(joint work with Jerzy Weyman)

This is a report on joint work with Jerzy Weyman on the study of local cohomology modules with support in generic determinantal ideals. We compute the multiplicities of the simple equivariant D-modules that arise in the Jordan–Hölder decomposition of the local cohomology modules, and give an explicit description of the characters of these D-modules. The main technical part of our approach is the calculation of certain Ext modules. This is done using Grothendieck duality, which allows one to translate everything into a computation with homogeneous bundles on a product of partial flag varieties, followed by a direct application of the Borel–Weil–Bott theorem. The calculation of Ext modules has further applications to computing the syzygies of certain equivariant thickenings of generic determinantal ideals. Many of our techniques, just like so many of the successful instances of computing syzygies in an equivariant setting, use geometric ideas that go back to the seminal work of Kempf in the 70s.

One of the early motivations for our investigation of local cohomology was to understand the Cohen–Macaulayness of modules of covariants: when H is a reductive group acting linearly on a polynomial ring S, for which irreducible representations of H is the module of covariants $(S \otimes U)^H$ Cohen–Macaulay? This question originated in the work of Stanley, who formulated a precise conjecture [7], and was studied extensively by Van den Bergh (see [8] for a survey). When U is the trivial representation, the C–M property of the ring of invariants S^H is a classical result of Hochster and Roberts [2]. The general question is equivalent to understanding when U^* appears as a subrepresentation of the local cohomology modules $H_I^\bullet(S)$, where I is the ideal in S generated by the positive degree invariants, so it can be easily answered by knowing the characters of $H_I^\bullet(S)$. For instance, we show:

Theorem [4, Thm. 4.6] *Consider positive integers $m > n$ and let $W = (\mathbb{C}^n)^{\oplus m}$ denote the direct sum of m copies of the standard representation of the group SL_n. Let $S = \mathrm{Sym}(W)$ and consider the irreducible SL_n-representation $U = S_\mu\mathbb{C}^n$ associated to some partition $\mu = (\mu_1 \geq \mu_2 \geq \cdots \geq \mu_n = 0)$. The module of covariants $(S \otimes U)^H$ is Cohen–Macaulay if and only if $\mu_s - \mu_{s+1} < m - n$ for all $s = 1, \cdots, n-1$.*

The vector space of $m \times n$ complex matrices ($m \geq n$) admits the natural action of a larger group $G = \mathrm{GL}_m \times \mathrm{GL}_n$, via row and column operations. The orbits of this action are $O_0, O_1, \cdots, O_n$, where O_i consists of all matrices of rank i. Since the isotropy groups for this action are connected, the only G-equivariant local system on each O_i is the trivial one. The equivariant version of the Riemann–Hilbert correspondence [3, Thm. 11.6.1] yields simple G-equivariant regular holonomic D-modules $D_0, \cdots, D_n$, with D_i supported on the orbit closure $\overline{O_i}$. Their characters, as well as their multiplicities in the composition series of the local cohomology modules $H^{\bullet}_{I_p}(S)$ (where S is the ring of polynomial functions on $m \times n$ matrices, and I_p is the ideal of $p \times p$ minors defining $\overline{O_{p-1}}$) are described in the following:

Theorem [5, Thm. 6.1] *The following equality holds, with w being a formal variable, and $[M]$ denoting the class of a D-module M in the Grothendieck group:*

$$\sum_{j\geq 0}[H^j_{I_p}(S)] \cdot w^j = \sum_{s=0}^{p-1}[D_s] \cdot w^{(n-p+1)^2+(n-s)\cdot(m-n)} \cdot \binom{n-s-1}{p-s-1}(w^2),$$

where

$$\binom{b}{a}(w) = \sum_{b-a\geq t_1\geq\cdots\geq t_a\geq 0} w^{t_1+\cdots+t_a}$$

is a Gauss polynomial. Moreover, we have a G-equivariant decomposition

$$D_s = \bigoplus_{\substack{\lambda=(\lambda_1\geq\cdots\geq\lambda_n)\in\mathbb{Z}^n \\ \lambda_s\geq s-n,\ \lambda_{s+1}\leq s-m}} S_{\lambda(s)}\mathbb{C}^m \otimes S_\lambda \mathbb{C}^n,$$

where for a dominant weight μ, S_μ denotes the associated Schur functor, and

$$\lambda(s) = (\lambda_1, \cdots, \lambda_s, \underbrace{s-n, \cdots, s-n}_{m-n}, \lambda_{s+1} + (m-n), \cdots, \lambda_n + (m-n)).$$

A key step in our proof of the theorem is to compute the modules $\mathrm{Ext}^{\bullet}_S(S/I_\mu, S)$, where $\mu = (\mu_1 \geq \cdots \geq \mu_n \geq 0)$ is a partition and I_μ is the ideal in S generated by the unique copy of the irreducible representation of G isomorphic to $S_\mu\mathbb{C}^m \otimes S_\mu\mathbb{C}^n$ contained in S. We then use the fact that $H^{\bullet}_{I_p}(S) = \varinjlim_d \mathrm{Ext}^{\bullet}_S(S/I_{p\times d}, S)$, where $I_{p\times d} = I_\mu$ with $\mu_1 = \cdots = \mu_p = d$, $\mu_i = 0$ for $i > p$. In contrast with the Ext calculation, the syzygy modules $\mathrm{Tor}^S_{\bullet}(S/I_\mu, \mathbb{C})$ turn out to be much more mysterious. They were understood by Lascoux in the case when $I_\mu = I_p = I_{p\times 1}$ is the ideal of $p\times p$ minors, but very little is known in other situations. Together with Weyman, we are able to compute the syzygies for all the ideals $I_{p\times d}$, as explained below.

Akin and Weyman constructed in [1] linear complexes $X_{p\times d}$, whose homology groups are direct sums of copies of the ideals $I_{(p+q)\times(d+q)}$. Via an iterated mapping cone construction, it is then easy to obtain a free resolution of $I_{p\times d}$ by putting together copies of the complexes $X_{(p+q)\times(d+q)}$. Such a resolution is not minimal in general, but using the G-equivariance we can show that the minimal resolution of

$I_{p\times d}$ has a filtration where the subquotients are the complexes $X_{(p+q)\times(d+q)}$. To compute their multiplicities, as well as the cohomological degrees in which they live, we use the explicit information encoded by the character of $\mathrm{Ext}^{\bullet}_S(S/I_{p\times d}, S)$. We obtain

Theorem [6] *Letting* $\mathrm{Syz}(I_{p\times d})$ *denote the minimal free resolution of* $I_{p\times d}$*, we have with the notation for Gauss polynomials as before,*

$$[\mathrm{Syz}(I_{p\times d})] = \sum_{q=0}^{n-p} [X_{(p+q)\times(d+q)}] \cdot w^{q^2+2q} \cdot \binom{q+\min(p,d)-1}{q}(w^2).$$

In the theorem above, the variable w indexes the homological degree. For a complex $X = X_\bullet$, $[X] := \sum_i [X_i]\cdot w^i$, where as before, given a module M, $[M]$ denotes its class in the Grothendieck group. It would be desirable to extend this result to arbitrary ideals of the form I_μ.

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Rational points of K3 surfaces and derived equivalence

Brendan Hassett
(joint work with Yuri Tschinkel)

Let X and Y be projective K3 surfaces over a field F. The surfaces are said to be *derived equivalent* if there exists an equivalence of bounded derived categories of coherent sheaves

$$D^b(X) \xrightarrow{\sim} D^b(Y),$$

as triangulated categories over F. Such equivalences are induced by *Fourier-Mukai transforms*, i.e., if $p : X \times Y \to X$ and $q : X \times Y \to Y$ are the projections then each bounded complex of locally free sheaves $\mathcal{E}$ on $X \times Y$ induces a transform

$$\begin{array}{rcl} \Phi_{\mathcal{E}} : D^b(X) & \to & D^b(Y) \\ \mathcal{F} & \mapsto & q_*(p^*\mathcal{F} \otimes \mathcal{E}) \end{array}$$

where the relevant operations are taken in the derived sense. Each derived equivalence takes the form of a Fourier-Mukai transform for a suitable $\mathcal{E}$.

A natural extension is to *twisted K3 surfaces*, pairs (X, α) where X is a K3 surface and α is in the Brauer group $\mathrm{Br}(X)$. If $P \to X$ is a Brauer-Severi scheme of relative dimension $n-1$ representing α then we may interpret

$$(X, \alpha) = [P/\,\mathrm{SL}_n],$$

the quotient stack arising from the natural action of the special linear group. A derived equivalence

$$D^b(X, \alpha) \xrightarrow{\sim} D^b(Y, \beta)$$

is interpreted via coherent sheaves on the associated stacks.

We are interested in how derived equivalence interacts with rational points. Note that

$$(X, \alpha)(F) = \{x \in X(F) : \alpha|_x = 0 \in \mathrm{Br}(F)\},$$

the set of points over which the Brauer class vanishes. Thus twisted sheaves are useful in interpreting and analyzing Brauer-Manin type obstructions; see [2, 3] for explicit examples.

Our guiding question is

> Suppose that (X, α) and (Y, β) are derived-equivalent twisted K3 surfaces over a field F. Is
>
> $$(X, \alpha)(F) \neq \emptyset \Leftrightarrow (Y, \beta)(F) \neq \emptyset?$$

It would be interesting to have counterexamples to this statement over some field F. Our main focus is relatively simple fields of arithmetic intersest: finite fields, $\mathbb{R}$, and p-adic fields, as well as closely related 'geometric fields' like $\mathbb{C}((t))$.

For simplicity, we focus on untwisted K3 surfaces. General results include

- invariants like the Brauer group and the stable isomorphism class of the Picard group are derived invariant;
- other attributes, like whether the automorphism group is finite or infinite and whether there are smooth rational curves, are not derived invariant;
- the *index* of a K3 surface, i.e., the greatest common divisor of the degrees of field extensions over which it has a rational point, is a derived invariant;
- for elliptic K3 surfaces, the existence of a rational point is a derived invariant.

For specific fields we have suggestive results:

- Finite fields: Zeta functions are derived invariants [4] so over finite fields the presence of a rational point is as well.
- $\mathbb{R}$: The diffeomorphism type of the real points of a K3 surface is a derived invariants by fundamental work of Nikulin.
- $\mathbb{C}((t))$: K3 surfaces with unipotent monodromy have a rational point by the monodromy classification of Kulikov models; the same holds true for K3 surfaces with ADE singularities in the central fiber; the case of finite monodromy can be addressed using classification results for automorphisms of finite order.

- p-adic fields: For K3 surfaces with good or ADE reduction, having a point over the p-adics is a derived invariant.

Details will be presented in [1].

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