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Algebraische Zahlentheorie

Organised by
Benjamin Howard, Chestnut Hill
Guido Kings, Regensburg
Ramdorai Sujatha, Vancouver
Otmar Venjakob, Heidelberg

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Abstract. The workshop brought together leading experts in Algebraic Number Theory. The talks presented new methods and results that intertwine a multitude of topics ranging from classical diophantine themes to modern arithmetic geometry, modular forms and p-adic aspects in number theory.

Mathematics Subject Classification (2010): 11F, 11G, 11M, 11S, 14C, 14D, 14G, 14L, 19F.

Introduction by the Organisers

The workshop "Algebraische Zahlentheorie" was held at the Mathematisches Foschungs Institut between July 7-14, 2014. The organizers were Ben Howard, Guido Kings, Sujatha Ramdorai and Otmar Venjakob. The workshop covered different aspects of Algebraic Number theory, ranging from classical diophantine themes to modern arithmetic geometry, modular forms and p-adic aspects in number theory.

The conference opened with a talk by Henri Darmon, the Simons Visiting Professor, on progress towards constructing new Euler systems or at least specific global cohomology classes. Kato's Euler system which was introduced by Kato more than two decades back, marks a major milestone in the Iwasawa theory of p-adic Galois representations. Darmon, Massimo Bertolini and Victor Rotger constructed global Galois cohomology classes arising from Gross-Kudla-Schoen diagonal cycles in a tower of triple products of modular curves, which generalize both Kato classes and Beilinson-Flach elements. In his talk, Darmon explained how these constructions lead to new results concerning the generalized Birch and Swinnerton-Dyer (BSD) Conjecture, and how the older results fit within this framework. Victor Rotgers continued to talk on their joint work by illustrating four concrete settings in which progress on the BSD conjecture has been made. In particular he stated an explicit conjecture on the existence of certain p-adic periods which should one allow to express the above mentioned cohomology classes in terms of the determinant of an explicit matrix. This makes these cases of the BSD conjecture accessible to computer calculations and verifications. Recent constructions of Euler systems using Beilinson-Flach elements, due to Guido Kings, David Loeffler and Sarah Zerbes along with the variation of special elements in Hida families were reported in the the talk by Zerbes. She explained how a refinement of the above techniques relying on Kings' Λ-adic sheaves allows one to bound not only the Mordell-Weil group but also the Selmer group attached to an elliptic curve E. In particular, they show that the non-vanishing of the Hasse-Weil L-function $L(E, \rho, 1)$ at 1 of an elliptic curve E without complex multiplication twisted by a two-dimensional odd Artin representation ρ implies the rank-statement of the generalized BSD conjecture concerning the ρ-part of the Mordell-Weil group of E as well as the finiteness of the ρ-part of the p-primary Tate-Shafarevich group attached to E. The p-adic invariant cycle theorem for semi-stable curves was considered in Valentina Di Proietto's talk, in which she spoke about a direct and very geometric proof of a theorem of Mokrane in this special case. The topic of Ashay Burungale was the reduction of generalized Heegner cycles modulo p over the antcyclotomic $\mathbb{Z}_\ell$-extension. His results refine earlier results of Cornut and Vatsal in the case of weight 2 modular forms and rely on a p-adic Waldspurger formula of Bertolini-Darmon-Prasanna.

A recurring theme in the talks was about the geometry of Shimura varieties and connections with automorphic forms. George Pappas spoke about joint work with Mark Kisin on the construction of integral models of Shimura varieties with bad reduction, and about techniques for analyzing the structure of their singularities. Benoit Stroh spoke about joint work with Tom Haines on the construction of integral models of symplectic and unitary Shimura varieties with even worse reduction, and applications to the calculation of nearby cycles. Fabrizio Andreatta spoke about joint work with Adrian Iovita generalizing the all-important Eichler-Shimura isomorphism for modular curves which enables one to define spaces of overconvergent Hilbert and Siegel modular forms, and to construct eigenvarieties for these spaces. Jan Bruinier explained his proof (with Martin Raum) of a remarkable result asserting, roughly, that any formal power series that looks like it could be the Fourier-Jacobi expansion of a Siegel modular form must automatically be one. Arithmetic aspects of other types of varieties were also discussed. Kazuya Kato presented joint work with Spencer Bloch describing asymptotic bounds on the variation of Beilinson-Bloch heights in families of smooth varieties. Héléne Esnault reported on the question of when automorphisms of $K3$ surfaces in characteristic p can be lifted to characteristic 0. The answer, from her joint with Keiji Oguiso: almost never. In his talk on Plectic cohomology, Jan Nekovar speculated about a new framework, in which one might hope to find new Euler systems ('plectic Siegel units') in an adapted arithmetic (absolute) cohomology theory attached to abelian varieties with real multiplication by a totally real number field.

The talk of Veronika Ertl explained the extension of the comparison isomorphism between overconvergent de Rham-Witt cohomology and Monsky-Washnitzer cohomology to the respective cohomology theories with coefficients. The main new ingredient here is the construction of good coefficients for the overconvergent de Rham-Witt complex. Philipp Graf gave a new and purely topological construction of Harder's Eisenstein classes for Hilbert modular varieties, which gives an a priori proof of the rationality of the classes, avoiding multiplicity one arguments as used by Harder. Oliver Wittenberg reported on joint work with Y. Harpaz on a conjecture of Colliot-Thélène-Sansuc and Kato-Saito on the exactness of a sequence involving the Chow groups of zero cycles on a smooth proper variety X over a number field k, and those of X_v as v varies over the finite places of k_v.

Kiran Kedlaya spoke about integrating the theory of (ϕ, Γ)-modules with work of Peter Scholze via the theory of relative (ϕ, Γ)-modules, which leads to finiteness results for the cohomology of these modules via the study of étale $\mathbb{Q}_p$-local systems in a rigid analytic space and is a vast generalization of earlier finiteness results due to Tate, Herr, Liu. Laurent Berger explained the theory of multi-variable (or Lubin-Tate) (ϕ, Γ)-modules, where the theory of locally analytic vectors allows for the construction of Lubin-Tate (ϕ, Γ)-modules over some power series rings in several variables. Specialising to the study of one variable Lubin-Tate (ϕ, Γ)-modules attached to F-analytic vectors, Berger establishes a folklore conjecture of Colmez and Fontaine. Anna Caraiani spoke on her recent work (joint with Emerton, Gee, Geraghty, Paskunas and Shin) that describes an approach to p-adic local Langlands correspondence (LLC) for $\mathrm{GL}_n(F)$ where $F/\mathbb{Q}_p$ is a finite extension, by using global methods and completed cohomology. This leads to an affirmative answer in many cases of a conjecture of Breuil and Schneider. Pierre Colmez spoke about his work which partially extends the p-adic LLC for $\mathrm{GL}_2(\mathbb{Q}_p)$ to (ϕ, Γ)-modules over the Robba ring, and a conjectural extension of this construction to analytic (ϕ, Γ)-modules for Lubin-Tate extensions and analytic representations of $\mathrm{GL}_2(F)$.

Thus the talks in the conference covered a broad range of topics that are at the forefront of current research in Algebraic Number theory. Befitting the Oberwolfach tradition, these were supplemented by stimulating discussions among the participants. The inclement weather aided the interaction among the participants. In conclusion, this workshop reflected the breadth and depth of on-going research in this old and beautiful area of mathematics. The historic World Cup match between Germany and Brazil during the week of the workshop added to the excitement!

Acknowledgement: The MFO and the workshop organizers would like to thank the National Science Foundation for supporting the participation of junior researchers in the workshop by the grant DMS-1049268, "US Junior Oberwolfach Fellows". Moreover, the MFO and the workshop organizers would like to thank the Simons Foundation for supporting Henri René Darmon in the "Simons Visiting Professors" program at the MFO.

Workshop: Algebraische Zahlentheorie

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Abstracts

Euler systems and the Birch and Swinnerton-Dyer conjecture

Henri Darmon
(joint work with Massimo Bertolini, Victor Rotger)

The Birch and Swinnerton-Dyer conjecture for an elliptic curve $E/\mathbb{Q}$ asserts that

$$\operatorname{ord}_{s=1} L(E,s) = \operatorname{rank}(E(\mathbb{Q})), \tag{0.1}$$

where $L(E,s)$ is the Hasse-Weil L-function attached to E. The scope of the conjecture can be broadened somewhat by introducing an Artin representation

$$\varrho : G_{\mathbb{Q}} \longrightarrow \operatorname{Aut}(V_\varrho) \simeq \mathbf{GL}_n(\mathbb{C}), \tag{0.2}$$

and studying the Hasse-Weil-Artin L-function $L(E,\varrho,s)$, namely, the L-function attached to $H^1_{\text{et}}(E_{\bar{\mathbb{Q}}}, \mathbb{Q}_p) \otimes V_\varrho$, viewed as a (compatible system of) p-adic representations. The "equivariant Birch and Swinnerton-Dyer conjecture" states that

$$\operatorname{ord}_{s=1} L(E,\varrho,s) = \dim_{\mathbb{C}} \hom_{G_{\mathbb{Q}}}(V_\varrho, E(H) \otimes \mathbb{C}), \tag{0.3}$$

where H is a finite extension of $\mathbb{Q}$ through which ϱ factors. Denote by $\text{BSD}_r(E,\varrho)$ the assertion that the right-hand side of (0.3) is equal to r when the same is true of the left-hand side. Virtually nothing is known about $\text{BSD}_r(E,\varrho)$ when $r > 1$. For $r \leq 1$, there are the following somewhat fragmentary results, listed in roughly chronological order:

Theorem (Gross-Zagier 1984, Kolyvagin 1989) *If ϱ is induced from a ring class character of an imaginary quadratic field, and $r \leq 1$, then $BSD_r(E,\varrho)$ holds.*

Theorem A (Kato, 1990) *If ϱ is abelian (i.e., corresponds to a Dirichlet character), then $BSD_0(E,\varrho)$ holds.*

Theorem B (Bertolini-Darmon-Rotger, 2011) *If ϱ is an odd, irreducible, two-dimensional representation whose conductor is relatively prime to the conductor of E, then $BSD_0(E,\varrho)$ holds.*

Theorem C (Darmon-Rotger, 2012) *If $\varrho = \varrho_1 \otimes \varrho_2$, where ϱ_1 and ϱ_2 are odd, irreducible, two-dimensional representations of $G_{\mathbb{Q}}$ satisfying:*

(1) $\det(\varrho_1) = \det(\varrho_2)^{-1}$, *so that ϱ is isomorphic to its contragredient representation;*
(2) *ϱ is regular, i.e., there is a $\sigma \in G_{\mathbb{Q}}$ for which $\varrho(\sigma)$ has distinct eigenvalues;*
(3) *the conductor of ϱ is prime to that of E;*

then $BSD_0(E,\varrho)$ holds.

This lecture endeavoured to explain the proofs of Theorems A, B, and C, emphasising the fundamental unity of ideas underlying all three.

The key ingredients are certain global cohomology classes

$$\kappa(f,g,h) \in H^1(\mathbb{Q}, V_f \otimes V_g \otimes V_h(c))$$

attached to triples (f, g, h) of modular forms of respective weights (k, ℓ, m); here V_f, V_h and V_g denote the Serre-Deligne representations attached to f, g and h, and it is assumed that the triple tensor product of Galois representations admits a Kummer-self-dual Tate twist, denoted $V_f \otimes V_g \otimes V_h(c)$. (This is true when the product of nebentype characters associated to f, g and h is trivial.)

When f, g and h are all of weight two and level dividing N, and f is cuspidal, associated to an elliptic curve E, say, the class $\kappa(f, g, h)$ admits a geometric construction via p-adic étale regulators/Abel-Jacobi images of

(1) Beilinson-Kato elements in the higher Chow group $\mathrm{CH}^2(X_1(N), 2)$ of the modular curve $X_1(N)$, when g and h are Eisenstein series of weight two arising as logarithmic derivatives of suitable Siegel units;
(2) Beilinson-Flach elements in the higher Chow group $\mathrm{CH}^2(X_1(N)^2, 1)$ when g is cuspidal and h is an Eisenstein series;
(3) Gross-Kudla-Schoen diagonal cycles in the Chow group $\mathrm{CH}^2(X_1(N)^3)$, when all forms are cuspidal.

When g and h are of weight one rather than two, and hence, are associated to certain (possibly reducible) odd two-dimensional Artin representations, the construction of $\kappa(f, g, h)$ via K-theory and algebraic cycles ceases to be available. The class $\kappa(f, g, h)$ is obtained instead by a process of p-adic analytic continuation, interpolating the geometric constructions at all classical weight two points of Hida families passing through g and h in weight one, and then specialising to this weight. The resulting $\kappa(f, g, h)$ is called the *generalised Kato class* atttached to the triple (f, g, h) of modular forms of weights $(2, 1, 1)$.

The generalised Kato classes arising from (p-adic limits of) Beilinson-Kato elements, Beilinson-Flach elements, and Gross-Kudla-Schoen cycles are germane to the proofs of Theorems A, B and C respectively. The key point in all three proofs is an *explicit reciprocity law* which asserts that the global class $\kappa(f, g, h)$ is *non-cristalline at p* pecisely when the classical central critical value $L(f \otimes g \otimes h, 1) = L(E, \varrho, 1)$ is non-zero. The non-cristalline classes attached to (f, g, h) (of which there are actually four, attached to various choices of ordinary p-stabilisations of g and h) can then be used (by a standard argument involving local and global Tate duality) to conclude that the natural inclusion of $E(H)$ into $E(H \otimes \mathbb{Q}_p)$ becomes zero when restricted to $\varrho_g \otimes \varrho_h$-isotypic components, and hence, that $\hom_{G_{\mathbb{Q}}}(V_\varrho, E(H) \otimes \mathbb{C})$ is trivial when $L(E, \varrho, 1) \neq 0$.

The lecture strived to set the stage for the two that immediately followed, which were both devoted to further developments arising from these ideas:

(1) Victor Rotger's lecture studied the generalised Kato classes $\kappa(f, g, h)$ when $L(f, g, h, 1) = 0$. In that case, they belong to the Selmer group of E/H, and can be viewed as p-adic avatars of $L''(E, \varrho, 1)$;
(2) Sarah Zerbes' lecture reported on [LLZ1], [LLZ2], [KLZ] in which the study of Beilinson-Flach elements undertaken in [BDR] is generalised, extended and refined. By making more systematic use of the Euler system properties of Beilinson-Flach elements, notably the possibility of "tame deformations" at primes $\ell \neq p$, the article [KLZ] is also able to establish strong

finiteness results for the relevant ϱ-isotypic parts of the Shafarevich-Tate group of E over H, in the setting of Theorem B.

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Euler systems and the Birch and Swinnerton-Dyer conjecture II

Victor Rotger

(joint work with Henri Darmon)

Let $E_{/\mathbb{Q}}$ be an elliptic curve over the rational numbers and let

$$\varrho : G_{\mathbb{Q}} \longrightarrow \mathrm{Aut}(V_\varrho) \simeq \mathbf{GL}_n(L) \tag{0.1}$$

be an Artin representation, factoring through the Galois group of a finite extension $H/\mathbb{Q}$ and taking values in the group of linear automorphisms of a vector space V_ϱ of dimension $n \geq 1$ over a finite extension $L/\mathbb{Q}$.

Let $L(E, \varrho, s)$ be the Hasse-Weil-Artin L-function associated to the twist of E by ϱ, that is to say, the L-function attached to $H^1_{\mathrm{et}}(E \times \bar{\mathbb{Q}}, \mathbb{Q}_\ell) \otimes V_\varrho$, viewed as a compatible system of ℓ-adic representations.

Define the ϱ-isotypic component of the Mordell-Weil group of E over H as

$$E(H)^\varrho := \hom_{G_{\mathbb{Q}}}(V_\varrho, E(H) \otimes L).$$

The equivariant Birch and Swinnerton-Dyer conjecture for the pair (E, ϱ) predicts that

$$\mathrm{ord}_{s=1} L(E, \varrho, s) = \dim_L E(H)^\varrho. \tag{0.2}$$

Denote by $\mathrm{BSD}_r(E, \varrho)$ the assertion that the right-hand side of (0.2) is equal to r when the same is true of the left-hand side.

As reported by Henri Darmon in his lecture, $\mathrm{BSD}_0(E, \varrho)$ is known in a number of scenarios, due to the works of Gross-Zagier (1984), Kolyvagin (1989), Kato (1990), Bertolini-Darmon-Rotger (2011) and Darmon-Rotger (2012). We refer to Darmon's report in this workshop for more details about the precise statements and methods of proof.

In my lecture, I focussed on the case $r = 2$ in the setting where

$$\varrho = \varrho_1 \otimes \varrho_2$$

is the tensor product of two odd, irreducible, two-dimensional representations ϱ_1 and ϱ_2 of $G_{\mathbb{Q}}$, subject to the following assumptions:

(1) $\det(\varrho_1) = \det(\varrho_2)^{-1}$;
(2) the conductors N_E and N_ϱ of E and ϱ are relatively prime;
(3) ϱ is regular, i.e., there is a $\sigma \in G_{\mathbb{Q}}$ for which $\varrho(\sigma)$ has distinct eigenvalues;

While hypothesis (1) appears to be crucial, assumptions (2) and (3) can be significantly relaxed, and the results and conjectures that are described below hold in greater generality. Hypothesis (1) implies that ϱ is isomorphic to its contragredient representation, and (2) implies that all local signs of the functional equation of $L(E, \varrho, s)$ are $+1$, and hence $r_{\rm an}(E, \varrho) = {\rm ord}_{s=1} L(E, \varrho, s)$ is even.

As a piece of notation, set $N = N_E \cdot N_\varrho$ and let $\chi : (\mathbb{Z}/N\mathbb{Z})^\times \to L^\times$ denote the determinant of ϱ_1, regarded as a Dirichlet character.

By the results of Wiles, Khare, Wintenberger et al. both the elliptic curve E and the Artin representations ϱ_1 and ϱ_2 are modular. Let $f \in S_2(N)$, $g \in S_1(N, \chi)$ and $h \in S_1(N, \chi^{-1})$ be eigenforms of level N associated to E, ϱ_1 and ϱ_2, respectively.

Fix a prime p not dividing N at which E is ordinary, and let $\underline{g}$ and $\underline{h}$ be Hida families of overconvergent modular forms passing through given ordinary p-stabilizations g_α and h_α of g and h in level Np. For every $\ell \geq 1$, let g_ℓ and h_ℓ denote the classical specialization of $\underline{g}$ and $\underline{h}$ at ℓ, respectively.

Associated to these choices, we can construct a one-variable p-adic family of global cohomology classes

$$\kappa(f, \underline{g}, \underline{h}) \in H^1(\mathbb{Q}, V_{f,\underline{g},\underline{h}})$$

with values on a suitable twist of the tensor product $V_f \otimes V_{\underline{g}} \otimes V_{\underline{h}}$ of Hida-Wiles' Λ-adic representations associated to the Hida families, satisfying the following properties:

(i) For every $\ell \geq 1$, the specialization of $V_{f,\underline{g},\underline{h}}$ at $(2, \ell, \ell)$ is the Kummer self-dual twist of the tensor product $V_f \otimes V_{g_\ell} \otimes V_{h_\ell}$ of the Galois representations associated to f, g_ℓ and h_ℓ, respectively.
(ii) For every $\ell \geq 2$, the specialization $\kappa(f_k, g_\ell, h_\ell)$ of $\kappa(f, \underline{g}, \underline{h})$ at $(2, \ell, \ell)$ is critalline at p and coincides (up to an expicit fudge factor) with the image under the étale Abel-Jacobi map of a generalized diagonal cycle on the product of three Kuga-Sato varieties.
(iii) The image of $\kappa(f, \underline{g}, \underline{h})$ under the Λ-adic regulator of Perrin-Riou and Loeffler-Zerbes ([LZ]) is the triple-product p-adic L-function.
(iv) The specialization $\kappa(f_k, g_\alpha, h_\alpha)$ of $\kappa(f, \underline{g}, \underline{h})$ at $(2, 1, 1)$ is cristalline at p if and only if $L(E, \varrho, 1) = 0$.

We refer to $\kappa(f_k, g_\alpha, h_\alpha)$ as the generalized Kato class associated to the triplet (f, g_α, h_α), because when g and h are taken to be Eisenstein then Kato's construction in [Ka] bears strong analogies with ours.

Assume $L(E, \varrho, 1) = 0$, so that $\kappa(f_k, g_\alpha, h_\alpha)$ lies in the Bloch-Kato Selmer group ${\rm Sel}_p(E, \varrho)$ of the twist of E by ϱ.

Darmon and I conjecture that this class actually belongs to $E(H)^\varrho \otimes \mathbb{Q}_p$, and it is actually trivial as soon as $r_{\rm an}(E, \varrho) > 2$. When $r_{\rm an}(E, \varrho) = 2$, we propose an explicit conjectural recipe describing it, in terms of suitable p-adic periods arising from p-adic Hodge theory.

Together with Alan Lauder, in [DLR] we provide abundant numerical evidence for the logarithm of this conjecture, and prove it in the scenario where Heegner points co-exist.

References

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Euler systems and explicit reciprocity laws

Sarah Livia Zerbes

(joint work with Guido Kings, Antonio Lei, David Loeffler)

This is a brief summary of my recent research with Lei and Loeffler [LLZ14, LLZ13] and with Kings and Loeffler [KLZ14] on Euler systems for p-adic Galois representations and explicit reciprocity laws.

1. Euler systems

One of the central questions in number theory is to understand the cohomology of p-adic Galois representations, and the links between these cohomology groups and the values of L-functions.

The theory of Euler systems (originally due to Kolyvagin, and later greatly extended by Rubin) is a powerful technique for proving conjectures of this kind. If V is a p-adic representation of $\mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q})$, and T is a lattice in V, the machinery of Euler systems allows one to prove the finiteness of $H^2(\mathbf{Q},T)$ and bound its order, using a suitable collection of elements of $H^1(\mathbf{Q}(\mu_n),T)$ for varying n. If V is unramified outside a finite set of primes Σ (including p), Rubin has defined an Euler system for V as a collection of classes

$$z_n \in H^1(\mathbf{Q}(\mu_n),T)$$

satisfying the compatibility condition

$$\mathrm{norm}_{\mathbf{Q}(\mu_n)}^{\mathbf{Q}(\mu_{n\ell})}\left(z_{n\ell}\right) = \begin{cases} z_n & \text{if } \ell \mid n \text{ or } \ell \in \Sigma \\ P_\ell(\sigma_\ell^{-1})z_n & \text{otherwise,} \end{cases}$$

where $P_\ell(X)$ is the local Euler factor of $V^*(1)$ at ℓ.

One expects that the classes in the Euler system should be related to values of the L-function $L(V,s)$, via an *explicit reciprocity law* describing the image of the Euler system classes under the Bloch–Kato dual exponential map.

When V is the representation associated to a modular form, Kato has constructed an Euler system for V using classes in K_2 of modular curves, and proved a reciprocity law for this Euler system; the bounds for Selmer groups implied by these results give one inequality in the Bloch–Kato conjecture for the representation V.

2. A new Euler system

As far back as 1984, it was noted by Beilinson [Beĭ84] that there were natural K-theory classes associated to the product of two copies of the modular curve $Y = Y_1(N)$: these are given by the images of modular units on Y via pushforward along the diagonal embedding $Y \hookrightarrow Y \times Y$. Applying a p-adic etale regulator map to this class (for any prime p) gives a Galois cohomology class living in

$$H^1(\mathbf{Q}, V_p(f)^* \otimes V_p(g)^*)$$

for any weight 2 modular forms f, g, which we shall call the *Beilinson–Flach class.*

The point of departure for our work was a theorem of Bertolini–Darmon–Rotger [BDR12] showing that if p does not divide the level of f and g, the image of the Beilinson–Flach class under the Bloch–Kato logarithm map is related to a special value of Hida's p-adic Rankin–Selberg L-function. This strongly suggests that the Beilinson–Flach class should form part of an Euler system. Constructing such an Euler system was the main result of the paper [LLZ14]:

Theorem (Lei–Loeffler–Zerbes)**.** *There exists an Euler system (z_n) for the Galois representation $V = V_p(f)^* \otimes V_p(g)^*$, where f and g are any two weight 2 modular forms, such that z_1 is the Beilinson–Flach class.*

The applications of this theorem are, however, somewhat limited by the fact that the explicit reciprocity law proved by Bertolini–Darmon–Rotger involves the Bloch–Kato logarithm, rather than the dual exponential (the latter vanishes on the Beilinson–Flach class) and more importantly that the L-value involved is a non-critical value of a p-adic L-function.

3. P-adic variation and explicit reciprocity laws

It is fairly straightforward to show, from the construction of the Euler system of Beilinson–Flach classes, that these classes interpolate in p-adic families: if $\mathbf{f}$ and $\mathbf{g}$ are two Hida families of ordinary modular forms, then one can construct 3-parameter families of cohomology classes interpolating the Beilinson–Flach classes for all weight 2 specializations of the families $\mathbf{f}$ and $\mathbf{g}$, twisted by all Dirichlet characters of p-power conductor.

However, only finitely many specializations (f, g, χ) will have levels coprime to p, so the explicit reciprocity law of Bertolini–Darmon–Rotger only applies at a finite number of points in this family; so, although we can construct by this method cohomology classes for arbitrary specializations of the family, the construction gives no information about the resulting classes.

In [KLZ14], we solve this problem by proving the following theorem:

Theorem. (1) *There exist generalized Beilinson–Flach cohomology classes (arising from K-theory of Kuga–Sato varieties) in*

$$H^1(\mathbf{Q}, V_p(f)^* \otimes V_p(g)^*(-j)),$$

for any modular forms f, g of weights $k+2, k'+2$ and integer j such that $0 \le j \le \min(k, k')$.

(2) *If f and g have level coprime to p, the images of these generalized Beilinson–Flach classes under the Bloch–Kato logarithm are related to non-critical values of the p-adic Rankin–Selberg L-function.*

(3) *If f, g are specializations of Hida families $\mathbf{f}$, $\mathbf{g}$, then the Beilinson–Flach cohomology classes constructed geometrically for f, g coincide with the classes obtained by p-adic deformation from the Beilinson–Flach classes for weight 2 specializations of $\mathbf{f}$ and $\mathbf{g}$.*

The last point – asserting the equality of two *a priori* different cohomology classes for $V_p(f)^* \otimes V_p(g)^*(-j)$, one defined geometrically using Kuga–Sato varieties, and the other defined by p-adic deformation which has no *a priori* reason to be geometric – is the most subtle aspect of the construction.

With this theorem in hand, one obtains a family of cohomology classes which is related to p-adic L-values at a Zariski–dense set of points in its domain; hence the relation holds everywhere in the domain, including the points where the p-adic L-function interpolates a critical value of the corresponding complex L-function. This gives the following result:

Theorem. *If f, g are p-ordinary modular forms of weights ≥ 1 and j is an integer such that $s = 1 + j$ is a critical value of the L-function $L(f, g, s)$, then there is an Euler system (z_n) for $V_p(f)^* \otimes V_p(g)^*(-j)$, whose image under the dual exponential map is related to the algebraic part of the critical L-value $L(f, g, 1 + j)$.*

4. Arithmetic applications

The strong explicit reciprocity law obtained in [KLZ14] has many arithmetic applications. We have concentrated on the case where f corresponds to an elliptic curve E, and the form g is a weight 1 θ-series obtained from a finite-order Grössencharacter Ψ of an imaginary quadratic field K. In this case, the Rankin–Selberg L-value $L(f, g, 1)$ can be expressed as an L-value $L(E/K, \Psi, 1)$ of f over K twisted by Ψ; and the corresponding Selmer group is related to the Ψ-isotypical components of the Mordell–Weil and Tate–Shafarevich groups of E/F, where F/K is the finite abelian extension through which Ψ factors.

In order to apply the Euler system machinery to E/K, one needs to extend the Euler system somewhat, by constructing classes over all abelian extensions of K (not just those which are abelian over $\mathbf{Q}$). This is carried out in [LLZ13]. Combined with the above reciprocity law, one obtains the following result:

Theorem. *In the above setting, if $L(f/K, \Psi, 1) \ne 0$, then the Ψ-isotypical component of the Mordell–Weil group of E/F is finite, and the Ψ-isotypical component* *of the p-part of the Tate–Shafarevich group of E/F is finite for infinitely many primes p.*

The finiteness of the Ψ-component of the Mordell–Weil group in this setting has already been proved by Bertolini–Darmon–Rotger, via a related method (also involving Beilinson–Flach classes); our approach using Euler systems has the advantage of also giving control of the Tate–Shafarevich group.

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A topological construction of Eisenstein cohomology classes for Hilbert-Blumenthal-varieties

Philipp Graf

G. Harder has constructed Eisenstein cohomology classes for GL_2 over any numberfield. We want to give an alternative topological approach, when F is a totally real field.
Let us quickly recall Harder′s situation in [Ha].
One considers the space

$$\mathcal{S}_K := KK_\infty \setminus GL_2(\mathbb{A}_F)/GL_2(F),$$

where $\mathbb{A}_F$ is the ring of adeles over F and

$$K_\infty = Z(F \otimes_{\mathbb{Q}} \mathbb{R}) \prod_{\nu \text{ place of F, } \nu|\infty} SO(2),\ K = ker\left(GL_2(\mathcal{O}_F \otimes \hat{\mathbb{Z}}) \to GL_2(\mathcal{O}_F/N)\right)$$

with $Z \subset GL_2$ the center.
With $\mathcal{S}_K$ there comes a second space

$$\partial\mathcal{S}_K := KK_\infty \setminus GL_2(\mathbb{A}_F)/B(F),$$

where $B \subset GL_2$ denotes the standard Borel of upper-triangular matrices. Recall, that $\partial\mathcal{S}_K$ is actually homotopy eqivalent to a disjoint union of pointed neihghborhoods of the cusps, the boundary of the Borel-Serre compactification of $\mathcal{S}_K$.
The standard $GL_2(F)$-representations on

$$Sym^k_{\mathbb{Q}} F^2,\ k \geq 0$$

induce local systems

$$Sym^k_{\mathbb{Q}}\mathcal{H}_{\mathcal{S}_K},\ k \geq 0$$

on $\mathcal{S}_K$. One has the natural restriction map

$$res : H^p(\mathcal{S}_K, Sym^k_{\mathbb{Q}}\mathcal{H}_{\mathcal{S}_K}) \to H^p(\partial\mathcal{S}_K, Sym^k_{\mathbb{Q}}\mathcal{H}_{\mathcal{S}_K}).$$

Note, that the cohomology of the boundary is supported in degrees $1, ..., 2g-1$, with $g = [F : \mathbb{Q}]$.

The problem of Eisenstein cohomology classes is an explicit description of the subspace

$$H^p_{Eis}(\mathcal{S}_K, Sym^k_{\mathbb{Q}}\mathcal{H}_{\mathcal{S}_K}) \subset H^p(\mathcal{S}_K, Sym^k_{\mathbb{Q}}\mathcal{H}_{\mathcal{S}_K}),$$

with

$$H^p_{Eis}(\mathcal{S}_K, Sym^k_{\mathbb{Q}}\mathcal{H}_{\mathcal{S}_K}) \stackrel{\cong}{\to} im(res),$$

in terms of cohomology classes of the boundary. Harder solves this problem by extending coefficients to $\mathbb{C}$ and constructing an operator

$$Eis : im(res)_{\mathbb{C}} \to H^p(\mathcal{S}_K, Sym^k_{\mathbb{Q}}\mathcal{H}_{\mathcal{S}_K})_{\mathbb{C}}$$

right-inverse to res on the level of de Rham cohomology. The construction of Eis and the description of $im(res)$ go actually hand in hand. If $k \geq 1$, we have explicitly

$$Eis(\omega) = \sum_{\gamma \in GL_2(F)/B(F)} \gamma^*\omega$$

If $k = 0$ Eis has to be defined by analytic continuation of the sum. In order to show, that his operator respects the given $\mathbb{Q}$-structures of the cohomology groups, Harder has to use strong multiplicity one.

We want to take a different route.
$\mathcal{S}_K$ is a finite disjoint union of Hilbert-Blumenthal varieties. Blottiere [Bl] and Kings [Ki] have already shown, that polylogarithmic cohomology classes living on universial abelian schemes above certain Hilbert-Blumenthal varieties may give non trivial Eisenstein cohomology classes in degree $2g-1$ by specialization along the zero-section. Over $\mathcal{S}_K$ there is no universial abelian scheme, as the parametrized abelian varieties have non trivial automorphisms by a congruence subgroup of the units of F. This is exactly reflected by the fact, that

$$Z(K) := Z(F) \cap (Z(F \otimes_{\mathbb{Q}} \mathbb{R})^0 K)$$

acts trivially on

$$KK_\infty \setminus GL_2(\mathbb{A}_F)$$

due to the fact, that we devide by $Z(F \otimes_{\mathbb{Q}} \mathbb{R})$ from the left. We introduce a new topological manifold

$$\mathcal{M}_K := KK^1_\infty \setminus GL_2(\mathbb{A}_F)/GL_2(F),$$

where

$$K^1_\infty = \prod_{\nu \text{ place of F, } \nu|\infty} SO(2).$$

Now $GL_2(F)$ acts properly discontinously and fixpoint free and we may define a universial topological torus

$$\mathcal{T}_K := N\mathcal{O}_F \otimes \hat{\mathbb{Z}} \rtimes KK_\infty^1 \setminus \mathbb{A}_F^2 \rtimes GL_2(\mathbb{A}_F)/F^2 \rtimes GL_2(F) \xrightarrow{\pi} \mathcal{M}_K$$

$\mathcal{T}_K$ is a group object in the category of manifolds over $\mathcal{M}_K$ and its fibres are abelian varieties with real mutiplication. Moreover $\mathcal{T}_K$ has a level-N structure. On such a family of topological tori one has the Logarithm sheaf enabling a topological construction of polylogarithmic Eisenstein cohomology classes. More precisely, consider $D \subset \mathcal{T}_K$ the union of images of all N-torsion sections, which are disjoint from the zero section. One has

$$H^0(D, \mathbb{Q}) = \bigoplus_{0 \neq v \in (\mathcal{O}_F/N\mathcal{O}_F)^2} H^0(\mathcal{M}_K, \mathbb{Q}) \xrightarrow{sum} H^0(\mathcal{M}_K, \mathbb{Q})$$

and for each $f \in ker(sum)$ we get polylogarithmic Eisenstein cohomolology classes

$$(Eis^k(f))_{k\geq 0} \in \prod_{k \geq 0} H^{2g-1}(\mathcal{M}_K, Sym^k\mathcal{H}),$$

where $Sym^k\mathcal{H}$ is the local system on $\mathcal{M}_K$ associated to the $GL_2(F)$-representation $Sym^k_{\mathbb{Q}}F^2$. This construction can be found in [Ki], but the ideas go back to [B-L]. Recall, that on the limit of our spaces we have a

$$G_f := GL_2(\mathbb{A}_{F,f}) \times \pi_0(GL_2(F \otimes_{\mathbb{Q}} \mathbb{R}))$$

action by left-multiplication. This induces an action on the colimit of cohomology groups. We drop the subscript K, when we go to the limit of spaces or the colimit of cohomology groups.
By naturality of the construction above we get a G_f-equivariant operator

$$Eis^k : H^0(\mathcal{M}, \mathbb{Q}) \otimes_{\mathbb{Q}} S(\mathbb{A}_{F,f}^2, \mathbb{Q})^0 \to H^{2g-1}(\mathcal{M}, Sym^k\mathcal{H})$$

Here $S(\mathbb{A}_{F,f}^2, \mathbb{Q})^0$ are $\mathbb{Q}$-valued Schwartz-functions f on $\mathbb{A}_{F,f}^2$ with

$$\int_{v \in \mathbb{A}_{F,f}^2} f(v)dv = f(0) = 0.$$

Now we have to push Eis^k forward to $\mathcal{S}$.
To do so, we consider the canonical map

$$\phi : \mathcal{M}_K \to \mathcal{S}_K$$

This map is a locally trivial fibration with fibre

$$Z(F \otimes_{\mathbb{Q}} \mathbb{R})^0/Z(K).$$

Proposition. *There are cohomology classes in $H^\bullet(\mathcal{M}_K, \mathbb{Q})$ defining a trivialization of the local system $R^\bullet\phi_*(\mathbb{Q})$ via the canonical map*

$$H^\bullet(\mathcal{M}_K, \mathbb{Q}) \to R^\bullet\phi_*(\mathbb{Q})(\mathcal{S}_K).$$

We denote with $H^\bullet(Z)$ *their* $\mathbb{Q}$*-span. One obtains a Leray-Hirsch style isomorphism*

$$\bigoplus_{p+q=n} H^p(Z) \otimes H^q(\mathcal{S}_K, Sym^k \mathcal{H}_{\mathcal{S}_K}) \to H^n(\mathcal{M}_K, Sym^k \mathcal{H})$$

induced by cup-product.

Now we may decompose Eis^k by evaluation on $H^\bullet(Z)$ to obtain operators

$$Eis^k_q : H^0(\mathcal{M}, H^q(Z)^*) \otimes_{\mathbb{Q}} S(\mathbb{A}^2_{F,f}, \mathbb{Q})^0 \to H^{2g-1-q}(\mathcal{S}, Sym^k \mathcal{H}_{\mathcal{S}})$$

Note, that Eis^k_q and $H^q(Z)$ are supported in degrees $0, ..., g-1 =$ rank of units of F. Therefore we get Eisenstein cohomology classes in cohomological degrees $n = g, ..., 2g-1$. Moreover, [Ha] Theorem 2 tells us, that we only miss the $n = 0$ part, which just plays a role, if $k = 0$.

Theorem. *The operator* Eis^k_q *factors as*

$$Eis^k_q : H^0(\mathcal{M}, H^q(Z)^*) \otimes_{\mathbb{Q}} S(\mathbb{A}^2_{F,f}, \mathbb{Q})^0 \to H^{2g-1-q}_{Eis}(\mathcal{S}, Sym^k \mathcal{H}_{\mathcal{S}})$$

It is surjective, if $k \geq 1$*. If* $k = 0$*, the image is desribed in* [Ha] *(4.2.1) Corollary (b).*

The proof relies on Nori′s [No] description of the polylogarithm over $\mathbb{C}$ as a current with values in the Logarithm-sheaf. As the proof is very explicit, no theorems of automorphic representation theory are needed.

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Non-liftability of automorphism groups of a K3 surface in positive characteristic

Hélène Esnault

(joint work with Keiji Oguiso)

A *K3 surface* X over a field k is a smooth geometrically connected 2-dimensional proper scheme X such that $H^1(X, \mathcal{O}_X) = 0$ and the dualizing sheaf ω_X is trivial, so $\omega_X \cong \mathcal{O}_X$. This implies in particular that if k is algebraically closed, the Picard group $\mathrm{Pic}(X)$ is discrete, thus is equal to its Néron-Severi quotient $NS(X)$, and in addition, it is torsion-free. If the characteristic of k is equal to 0, then the *Picard* *rank* of X (that is the rank of $NS(X)$ as a $\mathbb{Z}$-module) is between 1 and 20, and one knows it takes all possible values, while if the characterisitc of k is positive, then it could be $22 = \dim_k H^2(X_{\text{ét}}, \mathbb{Q}_\ell) = \dim_K H^2(X/K)$, where $K = \text{Frac}(W(k))$, the former group is étale cohomology, and the latter group is crystalline cohomology.

If R is a complete discrete valuation ring with residue field k, a *model of X over R* is a flat morphism $X_R \to \text{Spec}(R)$ such that $X_R \otimes_R k$ is isomorphic to X. If X is projective, with polarisation L, and if L lifts to L_R over X_R, then L_R is projective as well, so the model is projective.

If k is the field of complex numbers $\mathbb{C}$, assuming X is projective, we know from Hodge theory and the knowledge of the period domain that in the analytic category there is always a projective model $\mathcal{X} \to \Delta$, where Δ is the unit disc, such that if t is a general point of Δ in the complex sense, that is outside of a countable union of closed subsets of Δ, then the Picard rank of $\mathcal{X}_t$ is equal to 1 ([Og03]).

Theorem 1 ([EO14], Thm.4.2)**.** *Let X be a K3 surface defined over an algebraically closed field k of characteristic $p > 0$, where $p > 2$ if X is Artin-supersingular. Then there is a discrete valuation ring R, finite over the ring of Witt vectors $W(k)$, together with a projective model $X_R \to \text{Spec}R$, such that the Picard rank of $X_{\bar{K}}$ is 1, where $K = \text{Frac}\,(R)$ and $\bar{K} \supset K$ is an algebraic closure.*

The restriction on the characteristic is due to the fact that for the proof, we use the existence of a line bundle on X, the Hodge class of which in $H^1(X, \Omega^1_{X/k})$ does not vanish. For this, we have to use the Tate conjecture [MPe13]. It would be nice if there was a more direct way. We refer to [LieOls11] for another proof, based on the methods of [Ogu83], also in larger characteristic. Our proof of Theorem 1 relies on [Del81] and yields a precise statement on the Gauß-Manin connection of the universal formal family along certain hypersurafaces.

A corollary of Theorem 1 is the following theorem.

Theorem 2 ([EO14], Thm.5.1)**.** *Let X be a K3 surface defined over an algebraic closed field k of characteristic $p > 0$, where $p > 2$ if X is Artin-supersingular.*

1) *Assume that either the Picard number of X is ≥ 2 or that* $\text{Pic}\,(X) = \mathbb{Z} \cdot H$ *and $H^2 \neq 2$. Then there is a DVR R, finite over $W(k)$, together with a projective model $X_R \to \text{Spec}R$ of $X \to \text{Spec}k$ such that no subgroup $G \subset \text{Aut}\,(X)$, except for $G = \{\text{id}_X\}$ is geometrically liftable to $X_R \to \text{Spec}R$;*
2) *Assume that* $\text{Pic}\,(X) = \mathbb{Z}\cdot H$ *and $(H^2) = 2$. Then, for any projective model $X_R \to \text{Spec}R$ with R finite over $W(k)$, the specialization homomorphism* $\text{Aut}(X_R) \to \text{Aut}(X)$ *is an isomorphism, and* $\text{Aut}(X) = \mathbb{Z}/2$.

For a complex projective K3 surface X, for any choice of 5 line bundles $L_i, i = 0, \ldots, 4$, with L_0 ample, there is a projective model $\mathcal{X} \to \Delta$ in the analytic category such that all L_i lift ([Og03]).

We show [EO14, Prop.6.2] that on the Kummer K3 surface $X = \text{Km}(E \times_k E)$, where E is a supersingular elliptic curve over an algebraically closed field k of characteristic 3, there are 3 such line bundles $L_i, i = 0, 1, 2$ with L_0 ample, together with 2 automorphisms τ_i of order 2, such that there is no model $X_R \to \text{Spec}(R)$, with R finite over $W(k)$, such that $L_0, L_1, L_2, \tau_1, \tau_2$ lift. More precisely, L_1, L_2 are nef, big, but not ample, with $h^i(L_1) = h^i(L_2) = 0, h^0(L_1) = h^0(L_1) = 3$, they define projective (non-finite) maps $|L_n| : X \to \mathbb{P}^2$ of degree 2, thus involutions τ_n on X, $n = 1, 2$, L_0 is any ample line bundle.

Pushing the computation further, we show [EO14, Thm.6.4] that this surface admits an automorphism of entropy a Salem number of degree 22. Thus for all possible projective models $X_R \to \mathrm{Spec}(R)$, this automorphism can not lift.

The computation is mildly computer aided and relies on [KS12].

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Local models for $\Gamma_1(p)$-level structures

Benoît Stroh
(joint work with Tom Haines)

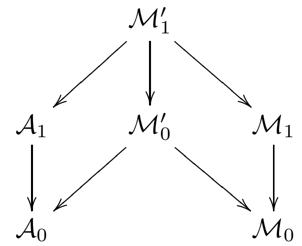
In this talk, we describe local models for $\Gamma_1(p)$-level structures (and their older avatars for $\Gamma_0(p)$-level structures). We use such models to construct many central functions in the pro-p-Iwahori-Hecke algebra.

Let $g \geq 1$ be an integer, p a prime number and $N \geq 3$ an integer not divisible by p. Denote by $\mathcal{A}_0$ the moduli space over $\mathrm{Spec}(\mathbb{Z}_p)$ parametrizing quadruples $(G, H_\bullet, \lambda, \iota)$ where (G, λ) is an abelian variety endowed with a prime-to-p polarization, $\iota : G[N] \simeq (\mathbb{Z}/N)^{2g}$ is a symplectic similitude and $H_\bullet = (H_1 \subset H_2 \subset \cdots \subset H_{2g} = G[p])$ is a flag of finite flat group schemes such that H_i is of rank p^i and $H_{i+g}^\perp = H_{g-i}$ under the Weil pairing for all $1 \leq i \leq g$.

Denote by $K_i = H_i/H_{i-1}$ for all $1 \leq i \leq 2g$ with the convention $H_0 = 0$. Then K_i is an Oort-Tate group scheme associated to an Oort-Tate data $(\mathcal{L}_i, a_i, b_i)$. Here $\mathcal{L}_i$ is a line bundle on $\mathcal{A}_0$ and $a_i \in \Gamma(\mathcal{A}_0, \mathcal{L}_i^{p-1})$, $b_i \in \Gamma(\mathcal{A}_0, \mathcal{L}_i^{1-p})$ with $a_i \cdot b_i = \omega_p$ where $\omega_p \in p \cdot \mathbb{Z}_p^*$ is an explicit element constructed by Oort and Tate. Because K_i is the Cartier dual of K_{2g+1-i} we have $\mathcal{L}_{2g+1-i} = \mathcal{L}_i^{-1}$ and a_{2g+1-i} corresponds to b_i via this identification. In particular, $a_i \cdot a_{2g+1-i}$ is independent of i.

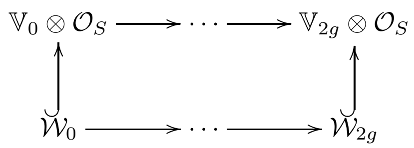
Denote by $\mathcal{A}_1 \to \mathcal{A}_0$ the moduli space of sections $z_i \in \Gamma(\mathcal{L}_i)$ such that $z_i^{p-1} = a_i$ and $z_i \cdot z_{2g+1-i}$ is independent of i. We get a ramified cover with $(\mathbb{F}_p^*)^{g+1}$ acting transitively in the fibers ; over $\mathrm{Spec}(\mathbb{Q}_p)$, this cover is Galois étale.

Theorem 1. *There exists explicit projective schemes $\mathcal{M}_0$ and $\mathcal{M}'_0$ over* $\mathrm{Spec}(\mathbb{Z}_p)$, *an explicit Deligne-Mumford stack $\mathcal{M}_1$ and a commutative diagram*



with smooth representable diagonals arrows of same relative dimension and with $\mathcal{M}'_1 = \mathcal{A}_1 \times_{\mathcal{A}_0} \mathcal{M}'_0$.

Here $\mathcal{M}_0$ is the local model for $\Gamma_0(p)$-level structures, which was constructed by De Jong and Rapoport-Zink, and $\mathcal{M}_1$ the local model for $\Gamma_1(p)$-level structures. Here is a quick definition of $\mathcal{M}_0$ and $\mathcal{M}_1$: denote by $\mathbb{V}_\bullet = (\mathbb{V}_0 \to \cdots \to \mathbb{V}_{2g})$ the standard chain where $\mathbb{V}_i = \mathbb{Z}_p^{2g}$ and $\mathbb{V}_{i-1} \to \mathbb{V}_i$ multiplies the i-th base vector by p and is the identity on the other vectors of the canonical basis. Then $\mathcal{M}_0$ parametrizes on $S \to \mathrm{Spec}(\mathbb{Z}_p)$ the commutative diagrams



where $\mathcal{W}_i \hookrightarrow \mathbb{V}_i \otimes \mathcal{O}_S$ is locally a rank g direct factor and $\mathcal{W}_0$ and $\mathcal{W}_g$ are totally isotropic. Denote by $\delta_i : \mathcal{W}_{i-1} \to \mathcal{W}_i$ the horizontal transition map for all $1 \leq i \leq 2g$. It gives rise to a section $d_i = \det(\delta_i)$ of $\det(\mathcal{W}_i) \otimes (\det(\mathcal{W}_{i-1}))^{-1}$. Moreover $d_i \cdot d_{2g+1-i}$ is naturally identified with a function independent of i. Then $\mathcal{M}_1 \to \mathcal{M}_0$ is the Deligne-Mumford stack parametrizing families (L_i, z_i) for all $1 \leq i \leq 2g$ where L_i is a line bundle endowed with an isomorphism $L_i^{p-1} \simeq \det(\mathcal{W}_i) \otimes (\det(\mathcal{W}_{i-1}))^{-1}$ and $z_i \in \Gamma(L_i)$ is such that z_i^{p-1} maps to d_i under the previous isomorphism. One requires moreover that $L_i = L_{2g+1-i}^{-1}$ and $z_i \cdot z_{2g+1-i}$ is independent of i.

By definition $\mathcal{M}'_0$ parametrizes isomorphisms between $\mathbb{V}_\bullet \otimes \mathcal{O}_S$ and the relative de Rham homology $\mathcal{H}_1^{dR}(G/H_\bullet)$ of the chain of universal abelian varieties $G = G/H_0 \to G/H_1 \to \cdots$ on $\mathcal{A}_0$. By definition we also have $\mathcal{M}'_1 = \mathcal{M}'_0 \times_{\mathcal{A}_0} \mathcal{A}_1$. The main difficulty in the demonstration of the previous theorem is then to construct the map from $\mathcal{M}'_1$ to $\mathcal{M}_1$. This uses the co-Lie complex of Illusie and the theory of det and Div of Mumford-Knudsen.

One can also introduce a rigidification $\mathcal{M}_1^+ \to \mathcal{M}_0^+ \to \mathcal{M}_0$ where all objects are schemes, the first map is a ramified $(\mathbb{F}_p^*)^{g+1}$-cover and the second a $\mathbb{G}_m^{g+1}$-torsor. For this, just denote by $\mathcal{M}_0^+ \to \mathcal{M}_0$ the space of trivializations $\varphi_i : \mathcal{O}_S \simeq \det(\mathcal{W}_i) \otimes (\det(\mathcal{W}_{i-1}))^{-1}$ for all $1 \leq i \leq 2g$ such that $\varphi_i \cdot \varphi_{2g+1-i}$ is independent of i. Under $\varphi_\bullet$, the previous sections $d_\bullet$ corresponds to functions $D_\bullet$. Let finally $\pi : \mathcal{M}_1^+ \to \mathcal{M}_0^+$ denote the space of functions Z_i such that $Z_i^{p-1} = D_i$ for all $1 \leq i \leq 2g$ and $Z_i \cdot Z_{2g+1-i}$ is independent of i.

One would like then to relate the semi-simple trace of Frobenius on

$$\pi_* \mathrm{R}\Psi_{\mathcal{M}_1^+}(\mathbb{Q}_\ell)$$

to some function in a Γ_1-Hecke algebra. Denote for this $I^+(\mathbb{F}_p)$ the subgroup of all $\gamma \in \mathrm{GSp}_{2g}(\mathbb{F}_p[[t]])$ such that γ mod t is strictly upper triangular.

Proposition 2. *There exists a natural embedding*

$$\mathcal{M}_0^+(\mathbb{F}_p) \hookrightarrow \mathrm{GSp}_{2g}(\mathbb{F}_p((t)))/I^+(\mathbb{F}_p)$$

with image stable by the left action of $I^+(\mathbb{F}_p)$ on the target. The semi-simple trace of the geometric Frobenius on $\pi_*\mathrm{R}\Psi_{\mathcal{M}_1^+}(\mathbb{Q}_\ell)$ gives rise thanks to this embedding to a function

$$\tau \ : \ \mathrm{GSp}_{2g}(\mathbb{F}_p((t)))/I^+(\mathbb{F}_p) \longrightarrow \bar{\mathbb{Q}}_\ell$$

which is left-invariant by $I^+(\mathbb{F}_p)$.

Therefore, the function τ naturally belongs to the convolution Hecke algebra $\mathcal{H}$ for the level subgroup $I^+(\mathbb{F}_p)$. The main theorem is the following.

Theorem 3. *The function τ is central and has explicit image under Bernstein and Roche isomorphisms.*

The proof of this theorem uses, as in the Iwahori case, a geometric construction of the convolution product, a proof of its commutation with the nearby cycles functor and a proof a geometric commutativity statement in generic fiber.

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Reductions of Shimura varieties at tame primes

George Pappas
(joint work with Mark Kisin)

We construct integral models of Shimura varieties of abelian type at primes where the group is tamely ramified and the level subgroup is parahoric in the sense of Bruhat-Tits.

Let (G, X, K) be Shimura data: G is a reductive group over the rational numbers $\mathbb{Q}$, X the conjugacy class of a Deligne cocharacter $h : \mathrm{Res}_{\mathbb{C}/\mathbb{R}}\mathbb{G}_m \to G_{\mathbb{R}}$ satisfying Deligne's axioms [1], and $K = \prod_l K_l \subset G(\mathbb{A}_f)$, $K_l \subset G(\mathbb{Q}_l)$, a compact open subgroup of the finite adelic points of G. The Shimura variety

$$Sh_K(G, X) = G(\mathbb{Q})\backslash X \times G(\mathbb{A}_f)/K$$

is an algebraic variety with a canonical model over the reflex field $E = E(G, X) \subset \mathbb{C}$; this is the field of definition of the conjugacy class of the corresponding cocharacter $\mu = \mu_h : \mathbb{G}_{m,\mathbb{C}} \to G_{\mathbb{C}}$.

Suppose that p is an odd prime such that the reductive group $G_{\mathbb{Q}_p}$ splits over a tamely ramified extension of $\mathbb{Q}_p$ and assume that the level subgroup $K_p \subset G(\mathbb{Q}_p)$ is parahoric, *i.e.* it is the connected fixer of a facet in the Bruhat-Tits affine building of $G(\mathbb{Q}_p)$. Let v be a prime of E above p. Denote by E_v the completion of E at v, and let $\mathcal{O}_{(v)}$, $\mathcal{O}_v$, be the localization, resp. completion, of the ring of integers $\mathcal{O}$ of E at v.

Let $\mathrm{M}^{\mathrm{loc}}$ be the "local model" which was constructed starting from the local Shimura data $(G_{\mathbb{Q}_p}, \{\mu\}_v, K_p)$ over v in [3]. This is a flat projective scheme over $\mathrm{Spec}(\mathcal{O}_v)$ whose generic fiber is a homogeneous space for G_{E_v} (the base change of the compact dual of X.) When p does not divide the order of the algebraic fundamental group of the derived group $G^{\mathrm{der}}_{\mathbb{Q}_p}$, the local model $\mathrm{M}^{\mathrm{loc}}$ is normal and has reduced special fiber which can be embedded in a generalized affine Grassmannian. Assume that this condition on p is also satisfied.

Suppose that (G, X) is of abelian type. The main result described in the talk is the construction of a Hecke equivariant normal model $\mathcal{S}_K(G, X)$ of $Sh_K(G, X)$ over $\mathcal{O}_{(v)}$ which, provided $\prod_{l \neq p} K_l$ is sufficiently small, has each of its local rings étale locally isomorphic to a corresponding local ring of $\mathrm{M}^{\mathrm{loc}}$.

The construction is performed in two steps: We first deal with the case where (G, X) is of Hodge type, *i.e.* there is an embedding $(G, X) \hookrightarrow (\mathrm{GSp}, S^{\pm})$ in Siegel Shimura data. Then the model is obtained as the normalization of the Zariski closure of the Shimura variety in a suitable integral model of a Siegel moduli space. The integral model in the general abelian type case is obtained by a quotient construction that uses the formalism of the theory of connected Shimura varieties. Our method uses p-adic Hodge theory as in [2] but also constructions from the Bruhat-Tits theory of buildings for p-adic groups.

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On the non-triviality of generalised Heegner cycles modulo p

Ashay A. Burungale

When a motive over a number field is self-dual with root number -1, the Bloch-Beilinson conjecture implies the existence of a non-trivial null-homologous cycle in a Chow group. For a prime p, one generically expects the non-triviality of the p-adic Abel-Jacobi image of these cycles.

An instructive set up arises from a self-dual Rankin-Selberg convolution of an elliptic Hecke eigenform and a theta series over an imaginary quadratic extension K with root number -1. In this situation, a natural candidate for a non-trivial null-homologous cycle is the generalised Heegner cycle. It lives in a middle dimensional Chow group of a fiber product of a Kuga-Sato variety arising from a modular curve curve and a self product of a CM elliptic curve. In the case of weight two, the cycles coincide with the Heegner points. For a prime l, twists of the theta series by l-power order anticyclotomic characters of K give rise to an Iwasawa theoretic family of generalised Heegner cycles. Under mild hypotheses, we prove the generic non-triviality of the p-adic Abel-Jacobi image of these cycles modulo p. In particular, this implies the generic non-triviality of the cycles in the top graded piece of the coniveau filtration along the $\mathbb{Z}_l$-anticyclotomic extension of K.

In the report, for brevity we mostly restrict to the case of Heegner points.

Let $p > 3$ be an odd prime. We fix two embeddings $\iota_\infty : \overline{\mathbb{Q}} \to \mathbb{C}$ and $\iota_p : \overline{\mathbb{Q}} \to \mathbb{C}_p$. Let v_p be the p-adic valuation induced by ι_p so that $v_p(p) = 1$. Let $\mathfrak{m}_p$ be the maximal ideal of $\overline{\mathbb{Z}}_p$.

Let $K/\mathbb{Q}$ be an imaginary quadratic extension and O the ring of integers. We assume the following:
(ord) p splits in K.

For an integral ideal $\mathfrak{n}$ of K, we fix a decomposition $\mathfrak{n} = \mathfrak{n}^+\mathfrak{n}^-$ where $\mathfrak{n}^+$ (resp. $\mathfrak{n}^-$) is only divisible by split (resp. ramified or inert) primes in $K/\mathbb{Q}$. Let $H_\mathfrak{n}$ be the ring class field of K of conductor $\mathfrak{n}$.

Let N be a positive integer prime to p. Let f be a normalised elliptic Hecke-eigen cuspform of weight 2, level $\Gamma_0(N)$ and neben-character ϵ. Let $N_\epsilon | N$ be the conductor of ϵ. Let E_f be the Hecke field of f. Let $\rho_f : G_\mathbb{Q} \to GL_2(\overline{\mathbb{Z}}_p)$ be the corresponding p-adic Galois representation.

We assume the following Heegner hypothesis:
(Hg) O contains a cyclic ideal $\mathfrak{N}$ of norm N.
From now, we fix such an ideal $\mathfrak{N}$. Let $\mathfrak{N}_\epsilon|\mathfrak{N}$ be the unique ideal of norm N_ϵ.

Let $\mathbf{N}$ denote the norm Hecke character of $\mathbb{Q}$ and $\mathbf{N_K} := \mathbf{N} \circ N_{\mathbb{Q}}^K$ the norm Hecke character of K. For a Hecke character λ of K, let $\mathfrak{f}_\lambda$ (resp. ϵ_λ) denote its conductor (resp. central character i.e. $\lambda|_{\mathbf{A}_{\mathbb{Q}}^\times}$, where $\mathbf{A}_{\mathbb{Q}}$ denotes the adele ring over $\mathbb{Q}$). We say that λ is central critical for f if it is of infinity type (j_1, j_2) with $j_1 + j_2 = 2$ and $\epsilon_\lambda = \epsilon_f \mathbf{N^2}$.

Let b be a positive integer prime to pN. Let $\Sigma_{cc}(b, \mathfrak{N}, \epsilon)$ be the set of Hecke characters λ such that:
(C1) λ is central critical for f,
(C2) $\mathfrak{f}_\lambda = b \cdot \mathfrak{N}_\epsilon$ and
(C3) The local root number $\epsilon_q(f, \lambda^{-1}) = 1$, for all finite primes q.

Let χ be a finite order Hecke character such that $\chi\mathbf{N_K} \in \Sigma_{cc}(b, \mathfrak{N}, \epsilon)$. Let $E_{f,\chi}$ be the finite extension of E_f obtained by adjoining the values of χ.

Let $X_1(N)$ be the modular curve of level $\Gamma_1(N)$, ∞ a cusp of $X_1(N)$ and $J_1(N)$ the corresponding Jacobian. Let B_f be the abelian variety associated to f by the Eichler-Shimura correspondence and $T_f \subset E_f$ an order such that B_f has T_f-endomorphisms. Let $\Phi_f : J_1(N) \to B_f$ be the associated surjective morphism. Let ω_f be the differential form on $X_1(N)$ corresponding to f. We use the same notation for the corresponding one form on $J_1(N)$. Let $\omega_{B_f} \in \Omega^1(B_f/E_{f,\chi})^{T_f}$ be the unique one form such that $\Phi_f^*(\omega_{B_f}) = \omega_f$. Let A_b be an elliptic curve with endomorphism ring $\mathbb{Z} + bO$, defined over the ring class field H_b. Let t be a generator of $A_b[\mathfrak{N}]$. We thus obtain a point $(A_b, A_b[\mathfrak{N}], t) \in X_1(N)(H_{b\mathfrak{N}})$. Let $\Delta_b = [A_b, A_b[\mathfrak{N}], t] - (\infty) \in J_1(N)(H_{b\mathfrak{N}})$ be the corresponding Heegner point on the modular Jacobian. We regard χ as a character $\chi : Gal(H_{b\mathfrak{N}}/K) \to E_{f,\chi}$. Let $G_b = Gal(H_{b\mathfrak{N}}/K)$. Let H_χ be the abelian extension of K cut out by the character χ. To the pair (f, χ), we associate the Heegner point $P_f(\chi)$ given by

$$P_f(\chi) = \sum_{\sigma \in G_b} \chi^{-1}(\sigma)\Phi_f(\Delta_b^\sigma) \in B_f(H_\chi) \otimes_{T_f} E_{f,\chi}.$$

The restriction of the p-adic formal group logarithm gives a homomorphism $\log_{\omega_{B_f}}$: $B_f(H_\chi) \to \mathbb{C}_p$. We extend it to $B_f(H_\chi) \otimes_{T_f} E_{f,\chi}$ by $E_{f,\chi}$-linearity.

We now fix a finite order Hecke character η such that $\eta\mathbf{N_K} \in \Sigma_{cc}(c, \mathfrak{N}, \epsilon)$, for some c. For $v|c^-$, let $\Delta_{\eta,v}$ be the finite group $\eta(O_{K_v}^\times)$. Here O_{K_v} denotes the integer ring of the local field K_v. Let $l \neq p$ be an odd prime unramified in K and prime to cN. Let $H_{c\mathfrak{N}l^\infty} = \bigcup_{n \geq 0} H_{c\mathfrak{N}l^n}$ be the ring class field of conductor $c\mathfrak{N}l^\infty$. Let $\Gamma_n = Gal(K_{c\mathfrak{N}l^n}/K)$ and $\Gamma_l = \varprojlim \Gamma_n$. Let $\mathfrak{X}_l$ be the set of l-power order characters of Γ_l. We consider the non-triviality of $\log_{\omega_{B_f}}(P_f(\eta\nu))/p$ modulo p, as $\nu \in \mathfrak{X}_l$ varies.

Theorem. *Let $f \in S_2(\Gamma_0(N), \epsilon)$ be a Hecke eignform and η a finite order Hecke character such that $\eta\mathbf{N_K} \in \Sigma_{cc}(c, \mathfrak{N}, \epsilon)$, for some c. In addition to the hypotheses (ord) and (Hg), suppose that*

(1). The residual representation $\rho_f|_{G_K} \mod \mathfrak{m}_p$ *is absolutely irreducible and*
(2). $(p, \prod_{v|c^-} \Delta_{\eta,v}) = 1$.
Then, for all but finitely many $\nu \in \mathfrak{X}_l$ *we have*

$$v_p\left(\frac{\log_{\omega_{B_f}}(P_f(\eta\nu))}{p}\right) = 0.$$

In particular, for all but finitely many $\nu \in \mathfrak{X}_l$ *the Heegner points* $P_f(\eta\nu)$ *are non-zero in* $B_f(H_{\eta\nu}) \otimes_{T_f} E_{f,\eta\nu}$.

Note that "In particular" part of the theorem involves only the prime l in its formulation. For an analog of the theorem in the case of generalised Heegner cycles modulo p, we refer to [3, §3.2]. As indicated above, the non-triviality gives an evidence for the refined Bloch-Beilinson conjecture (cf. [2, §2]).

Our approach is modular, based on Bertolini-Darmon-Prasanna's p-adic Waldspurger formula (cf. [1, Thm. 5.13]) and Hida's approach to non-triviality of anticyclotomic toric periods modulo p (cf. [7]). The later fundamentally relies on Chai-Oort rigidity principle that a Hecke stable subvariety of a mod p Shimura variety is a Shimura subvariety (cf. [6]).

"In particular" part of the theorem was conjectured by Mazur and proven independently by Cornut and Vatsal. We give a new approach and as far as we know the theorem is a first result regarding the non-triviality of the p-adic formal group logarithm of Heegner points modulo p. It seems suggestive to compare our approach with the earlier approach. In Vatsal's approach, Jochnowitz congruence is a starting point. It reduces the non-triviality of the Heegner points to the non-triviality of the Gross points on a suitable definite Shimura "variety". The later non-triviality fundamentally relies on Ratner's theorem regarding closures of unipotent flows on p-adic Lie groups. In our approach, the non-triviality is based on the modular curve itself. As indicated above, our approach fundamentally relies on Chai's theory of Hecke stable subvarieties of a mod p Shimura variety. It is rather surprising that we have these quite different approaches for the same characteristic zero non-triviality. Before the p-adic Waldspurger formula, the non-triviality and Hida's approach appeared to be complementary. The formula also allows a rather smooth transition to the higher weight case.

In [4], we consider a (p, p)-analogue of the theorem for generalised Heegner cycles arising fron indefinite Shimura curves over the rationals. In [5], we plan to consider a direct analogue of the theorem for the cycles. Recently, the p-adic Waldspurger formula has been generalised by Liu-Zhang-Zhang to the case of indefinite Shimura curves over a totally real field. In the near future, we hope to consider an analogous non-triviality of generalised Heegner cycles over a CM field.

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Multivariable (φ, Γ)-modules for the Lubin-Tate extension

Laurent Berger

1. Introduction

The goal of my talk was to explain some recent progress concerning (φ, Γ)-modules in the "Lubin-Tate" setting. This work was motivated by the p-adic local Langlands correspondence for $\mathrm{GL}_2(\mathbf{Q}_p)$. This correspondence is a bijection between the set of irreducible 2-dimensional p-adic representations of $\mathrm{Gal}(\overline{\mathbf{Q}}_p/\mathbf{Q}_p)$ and the set of some $\mathbf{Q}_p$-Banach representations of $\mathrm{GL}_2(\mathbf{Q}_p)$.

The construction of the p-adic local Langlands correspondence for $\mathrm{GL}_2(\mathbf{Q}_p)$ (see for instance [Bre10], [Col10] and [Ber11]) uses the theory of (cyclotomic) (φ, Γ)-modules in an essential way. Consider the ring $\mathbf{Z}_p[\![X]\!]$, and endow it with a Frobenius map φ given by $(\varphi f)(X) = f((1+X)^p - 1)$ and with an action of the group $\Gamma = \mathrm{Gal}(\mathbf{Q}_p(\mu_{p^\infty})/\mathbf{Q}_p) \simeq \mathbf{Z}_p^\times$ given by $([a]f)(X) = f((1+X)^a - 1)$ if $a \in \mathbf{Z}_p^\times$. A *(cyclotomic)* (φ, Γ)*-module* is a module D over a ring which contains $\mathbf{Z}_p[\![X]\!]$, and endowed with a semilinear Frobenius map φ and a compatible semilinear action of Γ.

We can package this data into an action of the monoid $\left(\begin{smallmatrix} \mathbf{Z}_p\setminus\{0\} & \mathbf{Z}_p \\ 0 & 1 \end{smallmatrix}\right)$ on D, with φ given by $\left(\begin{smallmatrix} p & \\ & 1 \end{smallmatrix}\right)$, $[a]$ given by $\left(\begin{smallmatrix} a & \\ & 1 \end{smallmatrix}\right)$, and multiplication by $(1+X)^b$ given by $\left(\begin{smallmatrix} 1 & b \\ & 1 \end{smallmatrix}\right)$. Colmez makes a similar definition and then extends the action of $\left(\begin{smallmatrix} \mathbf{Z}_p\setminus\{0\} & \mathbf{Z}_p \\ 0 & 1 \end{smallmatrix}\right)$ on D to an action of $\mathrm{GL}_2(\mathbf{Q}_p)$ on a bigger space.

If we are interested in a p-adic local Langlands correspondence for $\mathrm{GL}_2(F)$, with F a finite extension of $\mathbf{Q}_p$, then the above construction shows that it is possible that we will need (φ, Γ)-modules with $\Gamma \simeq \mathcal{O}_F^\times$, so that instead of working with the cyclotomic extension, we should work with Lubin-Tate extensions.

2. Fontaine's (φ, Γ)-modules

Let F be a finite Galois extension of $\mathbf{Q}_p$ of degree h, let π_F be a uniformizer of $\mathcal{O}_F$, let q be the cardinality of k_F, let $G_F = \mathrm{Gal}(\overline{\mathbf{Q}}_p/F)$, and let $\mathrm{E} = \mathrm{Emb}(F, \overline{\mathbf{Q}}_p)$ be the set of embeddings of F into $\overline{\mathbf{Q}}_p$. Let LT be the Lubin-Tate formal group attached to π_F and choose some variable T for the formal group law. We then have for every $a \in \mathcal{O}_F$ a power series $[a](T) = a \cdot T + \deg \geq 2$ giving the multiplication-by-a map. Let $\chi_F : G_F \to \mathcal{O}_F^\times$ be the Lubin-Tate character and let $H_F = \ker \chi_F$ and $\Gamma_F = G_F/H_F$. If $F = \mathbf{Q}_p$ and $\pi_F = p$, all of this is the usual cyclotomic data.

Let Y be a variable and let $\mathcal{O}_\mathcal{E}(Y)$ be the set of power series $f(Y) = \sum_{i \in \mathbf{Z}} a_i Y^i$ such that $a_i \in \mathcal{O}_F$ for $i \in \mathbf{Z}$ and $a_i \to 0$ as $i \to -\infty$. Let $\mathcal{E}(Y) = \mathcal{O}_\mathcal{E}(Y)[1/\pi_F]$. This is a two-dimensional local field. We endow it with a relative Frobenius map φ_q by $(\varphi_q f)(Y) = f([\pi_F](Y))$, and an action of Γ_F by $(gf)(Y) = f([\chi_F(g)](Y))$.

A *(φ, Γ)-module over $\mathcal{E}(Y)$* is a finite dimensional $\mathcal{E}(Y)$-vector space endowed with a compatible Frobenius map φ_q and a compatible action of Γ_F. We say that it is *étale* if it admits a basis in which $\mathrm{Mat}(\varphi_q) \in \mathrm{GL}_d(\mathcal{O}_\mathcal{E}(Y))$. By a theorem of Kisin-Ren (theorem 1.6 of [KR09]), based on the constructions [Fon90] of Fontaine, there is an equivalence of categories between $\{F$-linear representations of $G_F\}$ and $\{$étale (φ, Γ)-modules over $\mathcal{E}(Y)\}$. Let $\mathrm{D}(V)$ denote the étale (φ, Γ)-module over $\mathcal{E}(Y)$ attached to a representation V.

The (φ, Γ)-module $\mathrm{D}(V)$ is useful if one can relate it to p-adic Hodge theory, in particular the ring $\mathbf{B}_{\mathrm{dR}}$ and its subrings [Fon94]. This is possible if $\mathrm{D}(V)$ is *overconvergent*, that is if it admits a basis in which $\mathrm{Mat}(\varphi_q)$ and $\mathrm{Mat}(g)$, for $g \in \Gamma_F$, belong to $\mathrm{GL}_d(\mathcal{E}^\dagger(Y))$, where $\mathcal{E}^\dagger(Y)$ denotes the subfield of $\mathcal{E}(Y)$ consisting of those power series $f(Y)$ that have a nonempty domain of convergence. We say that V is overconvergent if $\mathrm{D}(V)$ is.

Which representations are overconvergent? If $F = \mathbf{Q}_p$, then all of them are by a theorem of Cherbonnier and Colmez [CC98]. If $F \neq \mathbf{Q}_p$, then not all representations are overconvergent [FX13]. Let us say that an F-linear representation V of G_F is *F-analytic* if for all $\tau \in \mathrm{E} \setminus \{\mathrm{Id}\}$, V is Hodge-Tate with weights 0 "at τ", or in other words if $\mathbf{C}_p \otimes_F^\tau V$ is the trivial $\mathbf{C}_p$-semilinear representation of G_F. For example $F(\chi_F)$ is F-analytic but $F(\chi_{\mathrm{cyc}})$ is not if $F \neq \mathbf{Q}_p$. The following result (theorem 4.2 of [Ber13]) shows that most representations of G_F are not overconvergent if $F \neq \mathbf{Q}_p$: if V is absolutely irreducible and overconvergent, then there is a character $\delta : \Gamma_F \to \mathcal{O}_F^\times$ such that $V(\delta)$ is F-analytic.

Conversely, we have the following theorem [Ber14]: if V is F-analytic, then it is overconvergent. This theorem had been proved for crystalline representations by Kisin and Ren [KR09], and for some reducible representations by Fourquaux and Xie [FX13]. Kisin and Ren had further suggested that in order to have overconvergent (φ, Γ)-modules for all F-representations of G_F, we need rings of power series in $[F : \mathbf{Q}_p]$ variables, one for each $\tau \in \mathrm{E}$. Later on we will see how to achieve this with the variables $\{Y_\tau\}_{\tau \in E}$ where $g(Y_\tau) = [\chi_F(g)]^\tau(Y_\tau)$ (if $f(T) = \sum a_i T^i$ with $a_i \in \mathcal{O}_F$, then $f^\tau(T) = \sum \tau(a_i) T^i$).

3. Construction of overconvergent (φ, Γ)-modules

We start by reviewing overconvergent (φ, Γ)-modules in the cyclotomic setting. Let $F = \mathbf{Q}_p$ and $\pi_F = p$ and let X denote the variable Y above. The *Robba ring* $\mathcal{R}(X)$ is a ring of holomorphic power series, which contains $\mathcal{E}^\dagger(X)$. Let $\widetilde{\mathbf{B}} = \widetilde{\mathbf{B}}^\dagger_{\mathrm{rig}}$ denote one of Fontaine's big rings of periods [Ber02]. It contains the element $\pi = [\varepsilon] - 1$

of p-adic Hodge theory, for which $g(\pi) = (1+\pi)^{\chi_{\mathrm{cyc}}(g)} - 1$ and $\varphi(\pi) = (1+\pi)^p - 1$. There is therefore a φ-and-$G_{\mathbf{Q}_p}$ compatible injection $\mathcal{R}(X) \to \widetilde{\mathbf{B}}$, sending X to π.

Let $\mathrm{D}(V)$ denote the (φ, Γ)-module $\mathrm{D}^{\dagger}_{\mathrm{rig}}(V)$ over the Robba ring attached to V, whose existence follows from the Cherbonnier-Colmez theorem (we drop the decorations to lighten the notation). In order to construct it, we first descend from $\overline{\mathbf{Q}}_p$ to $\mathbf{Q}_p(\mu_{p^\infty})$ by setting $\widetilde{\mathrm{D}}(V) = (\widetilde{\mathbf{B}} \otimes_{\mathbf{Q}_p} V)^{H_{\mathbf{Q}_p}}$. There then exists some analogues of Tate's normalized trace maps [Tat67], $T_n : \widetilde{\mathrm{D}}(V) \to \varphi^{-n}(\mathcal{R}(\pi)) \otimes_{\mathcal{R}} \mathrm{D}(V)$, which allow us to "decomplete" $\widetilde{\mathrm{D}}(V)$. This procedure is analogous to the construction of $\mathrm{D}_{\mathrm{Sen}}(V)$ in Sen theory [Sen81], where one decompletes $(\mathbf{C}_p \otimes_{\mathbf{Q}_p} V)^{H_{\mathbf{Q}_p}}$ using Tate's normalized trace maps. This procedure, descent and decompletion, is how the Cherbonnier-Colmez theorem is proved.

The main idea for our construction of multivariable (φ, Γ)-modules is that there is a different way of decompleting, which is still available in the cases when Tate's normalized trace maps no longer exist (which is the case as soon as $F \neq \mathbf{Q}_p$). If W is an LF space (i.e., an inductive limit of Fréchet spaces), that is endowed with a continuous action of a p-adic Lie group G, then following [ST03], we can consider the *locally analytic vectors* of W. We let W^{la} be the set of vectors of W such that the orbit map $g \mapsto g(w)$ is locally analytic on G.

Let $\widetilde{\mathbf{B}}_{\mathbf{Q}_p} = \widetilde{\mathbf{B}}^{H_{\mathbf{Q}_p}}$. This is an LF space, with an action of $\Gamma_{\mathbf{Q}_p} \simeq \mathbf{Z}_p^\times$. We have [Ber14] $(\widetilde{\mathbf{B}}_{\mathbf{Q}_p})^{\mathrm{la}} = \cup_{n \geq 0} \varphi^{-n}(\mathcal{R}(\pi))$ and $\widetilde{\mathrm{D}}(V)^{\mathrm{la}} = \cup_{n\geq 0} \varphi^{-n}(\mathcal{R}(\pi)) \otimes_{\mathcal{R}} \mathrm{D}(V)$. This gives a powerful alternate way of decompleting $\widetilde{\mathrm{D}}(V)$.

If $F \neq \mathbf{Q}_p$, we proceed in a similar way. Let $F_0 = F \cap \mathbf{Q}_p^{\mathrm{unr}}$, let $\widetilde{\mathbf{B}} = F \otimes_{F_0} \widetilde{\mathbf{B}}^{\dagger}_{\mathrm{rig}}$ and let $\widetilde{\mathrm{D}}(V) = (\widetilde{\mathbf{B}} \otimes_F V)^{H_F}$. Using almost étale descent, it is easy to show that $\widetilde{\mathrm{D}}(V)$ is a free $\widetilde{\mathbf{B}}_F$-module of rank d, stable under φ_q and Γ_F. We then have the following theorem [Ber14]: $\widetilde{\mathrm{D}}(V)^{\mathrm{la}}$ is a free $\widetilde{\mathbf{B}}_F^{\mathrm{la}}$-module of rank d. It is therefore a (φ, Γ)-module over $\widetilde{\mathbf{B}}_F^{\mathrm{la}}$.

4. The structure of $\widetilde{\mathbf{B}}_F^{\mathrm{la}}$

The above theorem is meaningful if we understand the structure of $\widetilde{\mathbf{B}}_F^{\mathrm{la}}$. Using the theory of p-adic periods, we can construct [Col02] for each $\tau \in \mathrm{E}$ an element $y_\tau \in \widetilde{\mathbf{B}}_F$ such that $g(y_\tau) = [\chi_F(g)]^\tau(y_\tau)$ if $g \in \Gamma_F$ and $\varphi_q(y_\tau) = [\pi_F]^\tau(y_\tau)$. This way, we get a (φ, Γ)-equivariant map from the Robba ring $\mathcal{R}(\{Y_\tau\}_{\tau\in E})$ in the h variables alluded to at the end of §2 to $\widetilde{\mathbf{B}}_F$, by sending Y_τ to y_τ. This map is injective. In addition, it extends to a map $\cup_{n\geq 0}\varphi_q^{-n}(\mathcal{R}(\{Y_\tau\}_{\tau\in E})) \to \widetilde{\mathbf{B}}_F$, whose image is then dense in $\widetilde{\mathbf{B}}_F$ for the locally analytic topology [Ber14]. This is why we call (φ, Γ)-modules over $\widetilde{\mathbf{B}}_F^{\mathrm{la}}$ *multivariable* (φ, Γ)*-modules.*

We can ask whether $\widetilde{\mathrm{D}}(V)^{\mathrm{la}}$ descends to a nice subring of $\widetilde{\mathbf{B}}_F^{\mathrm{la}}$. The main result of [Ber13] shows that if V is crystalline, then $\widetilde{\mathrm{D}}(V)^{\mathrm{la}}$ descends to a reflexive coadmissible module over the ring $\mathcal{R}^+(\{Y_\tau\}_{\tau\in E})$ of power series that converge on the open unit polydisk.

In general, since the action of Γ_F on $\widetilde{\mathrm{D}}(V)^{\mathrm{la}}$ is locally analytic, it extends to an action of $\mathrm{Lie}(\Gamma_F)$. For each $\tau \in \mathrm{E}$, there is an element $\nabla_\tau \in F \otimes \mathrm{Lie}(\Gamma_F)$ that is the "derivative in the direction of τ". Let $t_\tau = \log_{\mathrm{LT}}^\tau(y_\tau)$, so that $g(t_\tau) = \chi_F^\tau(g) \cdot t_\tau$. If $f((Y_\sigma)_\sigma) \in \mathcal{R}(\{Y_\tau\}_{\tau \in E})$, then we have $\nabla_\tau f((y_\sigma)_\sigma) = t_\tau \cdot v_\tau \cdot \partial f((y_\sigma)_\sigma)/\partial Y_\tau$ where v_τ is a unit. Using these operators, we can prove the theorem to the effect that F-analytic representations are overconvergent. First, we can relate Sen theory and (φ, Γ)-modules as in the cyclotomic case [Ber02], and we get [Ber14] that V is Hodge-Tate with weights 0 at τ if and only if $\nabla_\tau(\widetilde{\mathrm{D}}(V)^{\mathrm{la}}) \subset t_\tau \cdot \widetilde{\mathrm{D}}(V)^{\mathrm{la}}$. If this is the case, and if $\partial_\tau = t_\tau^{-1}\nabla_\tau$, then $\partial_\tau(\widetilde{\mathrm{D}}(V)^{\mathrm{la}}) \subset \widetilde{\mathrm{D}}(V)^{\mathrm{la}}$, so that if V is F-analytic, then $\widetilde{\mathrm{D}}(V)^{\mathrm{la}}$ is endowed with a system $\{\partial_\tau\}_{\tau \in \mathrm{E} \setminus \{\mathrm{Id}\}}$ of p-adic partial differential operators, as well as a compatible Frobenius map φ_q. A monodromy theorem [Ber14] then allows us to show that $(\widetilde{\mathrm{D}}(V)^{\mathrm{la}})^{\partial_\tau = 0 \text{ for } \tau \in \mathrm{E} \setminus \{\mathrm{Id}\}}$ is free of rank d over $(\widetilde{\mathbf{B}}_F^{\mathrm{la}})^{\partial_\tau = 0 \text{ for } \tau \in \mathrm{E} \setminus \{\mathrm{Id}\}}$. Finally, we show that $(\widetilde{\mathbf{B}}_F^{\mathrm{la}})^{\partial_\tau = 0 \text{ for } \tau \in \mathrm{E} \setminus \{\mathrm{Id}\}} = \cup_{n \geq 0} \varphi_q^{-n}(\mathcal{R}(y_{\mathrm{Id}}))$. This way, we can descend $\widetilde{\mathrm{D}}(V)^{\mathrm{la}}$ to $\mathcal{R}(Y)$ and then finally prove our theorem, using Kedlaya's theory of Frobenius slopes [Ked05].

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Patching and p-adic local Langlands

Ana Caraiani
(joint work with Matthew Emerton, Toby Gee, David Geraghty, Vytautas Paskunas, Sug Woo Shin)

The p-adic local Langlands correspondence is an exciting, recent generalization of the classical Langlands program. For $GL_2(\mathbb{Q}_p)$ it consists of functors between two-dimensional, continuous p-adic representations of $\mathrm{Gal}(\bar{\mathbb{Q}}_p/\mathbb{Q}_p)$ and certain admissible unitary p-adic Banach space representations of $GL_2(\mathbb{Q}_p)$ [4, 9, 5]. The correspondence has several remarkable properties, namely it is compatible with deformations and reduction mod p, with the classical local Langlands correspondence via taking locally algebraic vectors, and with the global p-adic correspondence, i.e. with the completed cohomology of modular curves. These properties led to spectacular applications to the Fontaine-Mazur conjecture for GL_2 over $\mathbb{Q}$ [6, 8].

However, most techniques involved in the construction of the p-adic local Langlands correspondence seem to break down if one tries to move beyond $GL_2(\mathbb{Q}_p)$. For $GL_n(F)$, it is unclear even what the precise conjectures should be, though the best possible scenario would involve all three of the properties listed above. In this talk, I described the construction of a candidate for the p-adic local Langlands correspondence for $GL_n(F)$, where $F/\mathbb{Q}_p$ is a finite extension, using global techniques, specifically the Taylor-Wiles-Kisin patching method applied to completed cohomology [3].

More precisely, we associate to a continuous n-dimensional representation r of $\mathrm{Gal}(\bar{\mathbb{Q}}_p/F)$ an admissible Banach space representation $V(r)$ of $GL_n(F)$, by p-adically interpolating completed cohomology for global definite unitary groups. The method involves working over an unrestricted local deformation ring of the residual $\bar{r}$, finding a global residual Galois representation which is automorphic and restricts to our chosen local representation $\bar{r}$, and gluing corresponding spaces of completed cohomology with varying tame level at so-called Taylor-Wiles primes. The output is a module M_∞ over $R_{\bar{r}}$, which also has an action of $GL_n(F)$ and whose fibers over closed points are admissible, unitary p-adic Banach spaces. We define $V(r)$ to be the fiber of M_∞ over the point of $R_{\bar{r}}$ corresponding to r.

We also show that, when r is de Rham, we can recover the compatibility with classical local Langlands $r \mapsto \pi_{\mathrm{sm}}(r)$ in many situations. More precisely, when r lies on an automorphic component of a local deformation ring, we can compute the locally algebraic vectors in $V(r)$ and show that they have the expected form $\pi_{\mathrm{sm}}(r) \otimes \pi_{\mathrm{alg}}(r)$. This involves first establishing an inertial local Langlands correspondence via the theory of types. The next step is to construct a map from an appropriate Bernstein center to a local deformation ring for a specific inertial type, a map which interpolates classical local Langlands. Finally, we appeal to the automorphy lifting theorems of [1] to guarantee that the locally algebraic vectors we obtain are non-zero. Our control over locally algebraic vectors allows us to prove many new cases of an admissible refinement of the Breuil-Schneider conjecture [2], concerning the existence of certain unitary completions.

Theorem 1. *Suppose that $p > 2$, that $r : G_F \to GL_n(\bar{\mathbb{Q}}_p)$ is de Rham of regular weight, and that r is generic. Suppose further that either*

(1) *$n = 2$, and r is potentially Barsotti–Tate, or*
(2) *$F/\mathbb{Q}_p$ is unramified and r is crystalline with Hodge–Tate weights in the extended Fontaine–Laffaille range, and $n \neq p$.*

Then $\pi_{\mathrm{sm}}(r) \otimes \pi_{\mathrm{alg}}(r)$ admits a nonzero unitary admissible Banach completion.

For example, when $F/\mathbb{Q}_p$ is unramified and p is large, Theorem 1 applies to all unramified principal series representations. Note that this existence is a purely local result, even though it is proved using global, automorphic methods.

Unfortunately, the Taylor-Wiles patching method involves gluing spaces of automorphic forms with varying tame level in a non-canonical way, using a sort of diagonal argument to ensure that their compatibility can always be achieved. Therefore, it is not at all clear that $r \mapsto V(r)$ is a purely local correspondence: it depends on the choice of global residual representation as well as on the choice of a compatible system of Taylor-Wiles primes. If there was a purely local correspondence satisfying all three properties listed in the beginning, then our construction would necessarily recover it. This is the case for $GL_2(\mathbb{Q}_p)$ and, in fact, the six of us are in the process of writing a paper elaborating on this and reproving many properties of the p-adic local Langlands correspondence for $GL_2(\mathbb{Q}_p)$, without making use of Colmez's functors. Our arguments rely heavily instead on the ideas of [9], especially the use of projective envelopes. For $GL_n(F)$, the question of whether our construction is purely local seems quite hard.

However, there is forthcoming work of Scholze, who constructs a purely local functor in the opposite direction: from admissible unitary p-adic Banach space representations of $GL_n(F)$ to admissible representations of $D^\times \times W_F$, where D is a division algebra with center F and invariant $1/n$. His construction uses the cohomology of the Lubin-Tate tower, which is known to realize both classical local Langlands and the Jacquet-Langlands corespondence when $l \neq p$ [7]. This functor satisfies local-global compatibility, in the following sense: if the input is the completed cohomology for a definite unitary group G, split at p, then the output is the completed cohomology of a Shimura variety associated to an inner form J of G which is isomorphic to $D^\times$ at p. Just as one can patch completed cohomology for G, it is also possible to patch completed cohomology for J. Moreover, Scholze can even prove that if one uses our patched module M_∞ as the input for his functor, then the output is the patched object for J. A consequence of this is that, at the very least, it should be possible to recover the Galois representation r from the Banach space $V(r)$.

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Finiteness of cohomology for relative (φ, Γ)-modules

Kiran S. Kedlaya

(joint work with Ruochuan Liu)

Fix a prime number p and a finite extension K of $\mathbb{Q}_p$. For V a finite-dimensional $\mathbb{Q}_p$-vector space equipped with a continuous action of the absolute Galois group G_K of K (for short a *p-adic representation*), it was shown by Tate [17] that the continuous Galois cohomology groups $H^i_{\mathrm{cont}}(G_K, V)$ are finite-dimensional over $\mathbb{Q}_p$. This was later reproved by Herr [9] using Fontaine's equivalence of categories between p-adic representations and *étale* (φ, Γ)*-modules.* As per Fontaine's original formulation [8], (φ, Γ)-modules are finite free modules over a certain ring equipped with semilinear extensions of certain endomorphisms of the base ring; the étale condition posits the existence of a suitable lattice in such a module. Herr showed that the Galois cohomology of a p-adic representation is computed by the continuous cohomology of the (φ, Γ)-module for the topological monoid generated by the endomorphism φ and the group of endomorphisms Γ.

Subsequent work of various authors [2, 3, 5, 10] implies that one can realize equivalent categories of étale (φ, Γ)-modules over various rings, the most convenient of which seems to be the *Robba ring* (a certain ring of convergent Laurent series). For example, Berger used this base ring to explicate Fontaine's period functors [2] and to recover the Colmez-Fontaine characterization of admissible filtered isocrystals [3].

The resulting full embedding of the category of p-adic representations into the category of (not necessarily étale) (φ, Γ)-modules over the Robba ring, originally used by Berger to explicate Fontaine's period functors [2], has proved unexpectedly fruitful: in particular, the fact that an irreducible Galois representation can become reducible in the larger category of (φ, Γ)-modules provides a vast generalization of the triangular decomposition of an ordinary Galois representation. This was originally realized by Colmez in the course of his construction of a p-adic local Langlands correspondence for $\mathrm{GL}_2(\mathbb{Q}_p)$ [6]. With this in mind, Liu [14] showed (by reduction to Herr's theorem) that for D an arbitrary (φ, Γ)-module, the cohomology groups $H^i(D)$ (defined as per Herr) are again finite-dimensional $\mathbb{Q}_p$-vector spaces. In particular, if V is a p-adic representation and D is its associated (φ, Γ)-module, any exact sequence

$$0 \to D_1 \to D \to D_2 \to 0$$

of (φ, Γ)-modules gives rise to a long exact sequence

$$\cdots \to H^i(D_1) \to H^i_{\mathrm{cont}}(G_K, V) = H^i(D) \to H^i(D_2) \to \cdots$$

of finite-dimensional $\mathbb{Q}_p$-vector spaces; in this sense, even if one is only interested in p-adic representations, the larger category of (φ, Γ)-modules makes its presence known via Galois cohomology. This has strong consequences for Iwasawa theory, particularly in the study of Selmer groups of nonordinary representations [15].

In this lecture, we describe a generalization (to appear in [12]) of Liu's theorem to the category of *relative* (φ, Γ)*-modules* associated to a rigid analytic space X over K in [11]. By analogy with the previous discussion, this category admits a full embedding from the category of *étale* $\mathbb{Q}_p$*-local systems* constructed by de Jong [7]. The relative (φ, Γ)-modules over X are modules over a certain sheaf of rings on the *pro-étale site* of X in the sense of Scholze [16] equipped with a semilinear extension of the action of an endomorphism φ on the base sheaf. There is no Γ-action, as this is subsumed by the sheaf property. The cohomology of a relative (φ, Γ)-module $\mathcal{F}$ is defined as the hypercohomology of the complex

$$0 \to \mathcal{F} \stackrel{\varphi - 1}{\to} \mathcal{F} \to 0;$$

we prove that if X is smooth proper, then these cohomology groups are finite-dimensional $\mathbb{Q}_p$-vector spaces.

The proof of the theorem amounts to an application of the Cartan-Serre finiteness criterion [4] as adapted to rigid analytic geometry by Kiehl [13]. When X is reduced to a point, the statement specializes back to Liu's theorem, but the proof is different: one constructs a pair of complexes computing the same (φ, Γ)-cohomology using power series on annuli, one embedded in the other, so that the quasi-isomorphism between the two complexes consists of completely continuous morphisms of Banach spaces over $\mathbb{Q}_p$. In the general case, one proceeds similarly, but there is an important subtlety: the relative (φ, Γ)-modules as constructed above are defined over rings on which φ is bijective, but to use the mechanism of completely continuous morphisms one must "deperfect" the base rings so that φ is only injective. This cannot be done in a global or functorial way; instead, one must work locally by choosing suitable coordinates. This amounts to a moderate generalization of the construction of multivariate (φ, Γ)-modules by Andreatta-Brinon [1].

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Towards a crystalline Eichler-Shimura map

Fabrizio Andreatta
(joint work with Adrian Iovita)

Let K be a complete discrete valuation field of characteristic 0 and perfect residue field k of characteristic $p \geq 3$. We normalize the valuation v on K so that $v(p) = 1$. Let R be an integral normal domain, p-adically complete and separated and flat over $\mathcal{O}_K$.

We say that an elliptic curve $E \to \mathrm{Spec}(R)$ is *close to be ordinary* if the Frobenius morphism $\mathrm{H}^1\big(E, \mathcal{O}_E/p\mathcal{O}_E\big)^{(p)} \to \mathrm{H}^1\big(E, \mathcal{O}_E/p\mathcal{O}_E\big)$ has cokernel annihilated by an element p^w of $\mathcal{O}_K$ of valuation $w < \frac{1}{p}$. Under this assumption E admits a canonical subgroup scheme $C \subset E[p]$. Notice that, if $w = 0$ then E has ordinary reduction and viceversa.

Let $R \subset \overline{R}$ be the union of all finite normal extensions of R, étale after inverting p (contained in a common algebraic closure of the fraction field of R). Let $T_p(E)$

be the p-adic Tate module of E over $\overline{R}[1/p]$ and assume that the Cartier dual of C is constant over $R[1/p]$. Set $v := \frac{w}{p-1}$. It is then proven in [3]

Theorem 1 There exists a unique free R-submodule $\Omega_{E/R}$ of the invariant differentials $\omega_{E/R}$ such that the Hodge-Tate map $\mathrm{HT}\colon T_p(E) \otimes \widehat{\overline{R}} \to \omega_{E/R} \otimes_R \widehat{\overline{R}}$ has $\Omega_{E/R} \otimes_R \widehat{\overline{R}}$ as image. Moreover, the Hodge-Tate map induces an isomorphism $a\colon C^\vee(R) \otimes_{\mathbb{F}_p} R/p^vR \cong \Omega_{E/R}/p^v\Omega_{E/R}$.

Application 1: *Construction of overconvergent elliptic modular forms:*

Fix an integer $N \geq 4$ prime to p. Let $\chi\colon \mathbb{Z}_p^* \to K^*$ be an analytic weight. Following Katz we define an overconvergent modular form of width w and weight χ over $\mathcal{O}_K$ to be a rule associating to E/R as above, a $\Gamma_1(N)$-structure Ψ_N, a generator Ω of $\Omega_{E/R}$ and a generator γ of $C^\vee(R)$ such that $a(\gamma \otimes 1) \equiv \Omega$ an element $f(E/R, \Psi_N, \Omega, \gamma) \in R$ such that for every $\alpha \in \mathbb{Z}_p^*(1 + p^vR)$ we have

$$f(E/R, \Psi_N, \alpha\Omega, \alpha\gamma) = \chi(\alpha)^{-1} f(E/R, \Psi_N, \Omega, \gamma).$$

A variant of this definition provides (families of) sheaves Ω^χ of overconvergent modular forms of arbitrary p-adic weights χ whose sections, after inverting p and taking limits for $w \to 0$, are proven in [3] to coincide with Coleman's notion of (families of) overconvergent modular forms of weight χ.

Application 2: *Construction of eigenvarieties:*

The approach in Application 1 does not make any use of Eisenstein series/families which are at the heart of Coleman's method. In particular it can be extended to other Shimura varieties, such as Siegel and Hilbert modular varieties, allowing the construction of eigenvarieties in those cases. See [1] and [2].

Work in progress with Adrian Iovita and Jacques Tilouine should allow to extend the construction of the overconvergent sheaves to arbitrary Hodge type Shimura varieties under the assumption that the ordinary locus is dense.

Application 3: *The Eichler-Shimura morphism for modular symbols:*

Let Γ be the modular group $\Gamma_0(p) \cap \Gamma_1(N)$, $X(N,p)$ the modular curve of level Γ over K and $X(N,p)(w)$ the neighborhood of the ordinary locus of width w. Let $\mathcal{D}^\chi$ be the K-Banach space of analytic distributions, homogeneous of degree χ constructed by G. Stevens. They p-adically interpolate the natural representations $\mathrm{Sym}^k(K^2)$ for k a positive integer. We then construct in [4] a *Eichler-Shimra map*

$$\mathrm{ES}_\chi\colon \mathrm{H}^1\big(\Gamma, \mathcal{D}^\chi\big)^{(\leq h)} \otimes_K \mathbb{C}_p(1) \longrightarrow \lim_{w \to 0} \mathrm{H}^0\big(X(N,p)(w), \Omega^{\chi+2}\big)^{(\leq h)} \otimes_K \mathbb{C}_p,$$

equivariant for the actions of the Galois group of K and of the Hecke algebra and interpolating the classical Eichler-Shimura morphism constructed by Faltings for $\chi = k$ a positive integer. The superscript $(\leq h)$, with h a non-negative integer, refers to a slope decomposition with respect to the action of the U_p-operator.

Future directions:

It can be shown that the map ES_χ is not surjective in general. In order to understand this phenomenon and to extend Application 2 to Hodge type Shimura varieties, we have studied the lift of the Hodge-Tate map to the crystalline level, constructed and analyzed in general by Faltings and Fargues, under the further assumption that E admits a canonical subgroup. Such lift is a map

$$\alpha_{\mathrm{cris}}\colon T_p(E) \otimes \mathbb{A}_{\mathrm{cris}}(R) \to \mathbb{D}(E) \otimes \mathbb{A}_{\mathrm{cris}}(R)$$

where $\mathbb{A}_{\mathrm{cris}}(R)$ is a suitable crystalline relative period ring for R and $\mathbb{D}(E)$ is the covariant Dieudonné module of E over R/pR. In this direction we can prove in work in progress

Theorem 2 We can find a "modification" $\mathbb{D}'(E)$ of $\mathbb{D}(E)$ and a "modification'" $\mathbb{A}'_{\mathrm{cris}}(R)$ of the period ring $\mathbb{A}_{\mathrm{cris}}(R)$ such that α_{cris} factors via $\mathbb{D}'(E)$ and its matrix, with respect to suitable bases, is $\begin{pmatrix} t & * \\ 0 & 1 \end{pmatrix}$.

Here t is Fontaine's multiplicative period. The module $\mathbb{D}'(E)$ should be thought of as an analogue of the 'modification" $\Omega_{E/R}$ of $\omega_{E/R}$. Notice that if E has ordinary reduction, then the connected-étale decomposition of $T_p(E)$ provides a basis with respect to which α_{cris} has the shape required in the Theorem, even over $\mathbb{A}_{\mathrm{cris}}(R)$. So Theorem 2 can be phrased by saying that, upon passing to a big period ring, if E is close to be ordinary, the crystalline period matrix has the shape of the crystalline period matrix of an ordinary elliptic curve.

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Asymptotic behaviours of heights and regulators

Kazuya Kato

The subject of this talk is that the study of degeneration of Hodge structure has applications to arithmetic.

I collaborated with C. Nakayama and S. Usui on the degeneration of Hodge structures, and worked with S. Bloch to apply this to the asymptotic behaviours of height pairings and regulators. The height pairings and the regulators are described by using the invariant of Hodge structures, and their asymptotic behaviours are understood by using the theory of degeneration of Hodge structures.

Overconvergent de Rham–Witt connections

Veronika Ertl

Let k be a perfect field of positive characteristic p, $W = W(k)$ its ring of p-typical Witt vectors, and K the fraction field of W. Let further X/k be a smooth variety.

In this talk we generalise a definition of Bloch [1] to not necessarily proper varieties and introduce overconvergent de Rham–Witt connections. This provides a tool to extend the comparison isomorphism of Davis, Langer and ZInk [3] (see also [2]) between overconvergent de Rham–Witt cohomology and rigid respectively Monsky–Washnitzer cohomology to coefficients.

There is a well-known equivalence of categories between crystals on X/W, hyper PD-stratifications and integrable, quasinilpotent connections on $\Omega_{X/W}$. By a result of Bloch and Etesse [1], [6] this data allows us to associate to a crystal $\mathcal{E}$ a W_X-module together with an integrable quasinilpotent connection

$$\nabla : \mathcal{E} \to \mathcal{E} \otimes W\Omega_{X/k},$$

where $W\Omega$ denotes the de Rham–Witt complex as defined by Illusie. Bloch shows that this functor provides an equivalence of categories between the category of locally free crystals on X/W and the category of locally free W_X-modules with quasinilpotent, integrable connection [1, Theorem1.1]. We construct a subcategory of the latter category in order to obtain a suitable category of coefficients for the overconvergent de Rham–Witt complex $W^\dagger\Omega$.

Definition 1. An overconvergent de Rham–Witt connection consists of a $W_X^\dagger$-module $\mathcal{E}$ together with a map

$$\nabla : \mathcal{E} \to \mathcal{E} \otimes_{W_X^\dagger} W^\dagger\Omega_{X/k}$$

Which satisfies the Leibniz rule. It is said to be integrable if for the induced map on the complex $\nabla^2 = 0$.

We restrict our attention to locally free $W_X^\dagger$-modules with overconvergent, integrable, quasinilpotent de Rham–Witt connections $(\mathcal{E}, \nabla)$.

Remark 2. In the obvious way, one can define overconvergent F- de Rham–Witt connections. The category of locally free, quasinilpotent integrable overconvergent F-de Rham–Witt connections should be, via Bloch's equivalence of categories, a full subcategory of the category of locally free F-crystals.

To compare this to Monsky–Washnitzer cohomology, let $\overline{B}$ be a smooth k-algebra, B a smooth lift to W and $B^\dagger$ its weak completion. We can now consider the associated Monsky–Washnitzer complex. Coefficients in this case are given by pairs $(\mathcal{E}_{\mathrm{MW}}, \nabla_{\mathrm{MW}})$, where $\mathcal{E}_{\mathrm{MW}}$ is a locally free $B^\dagger$-module and $\nabla_{\mathrm{MW}} : \mathcal{E}_{\mathrm{MW}} \to \mathcal{E}_{\mathrm{MW}} \otimes \Omega^1_{B^\dagger/W}$ an integrable quasinilpotent connection.

For a smooth Frobenius lift F to $B^\dagger$, Davis, Langer and Zink construct a comparison map

$$t_F : \Omega_{B^\dagger/W} \to W^\dagger\Omega_{\overline{B}/k}.$$

Using this map, one can associate to $(E_{\mathrm{MW}}, \nabla_{\mathrm{MW}})$ an overconvergent de Rham–Witt connection by setting $\mathcal{E} = \mathcal{E}_{\mathrm{MW}} \otimes W^{\dagger}(\overline{B})$ and ∇ to be the composition of the two maps

$$\mathcal{E}_{\mathrm{MW}} \otimes W^{\dagger}(\overline{B}) \to \mathcal{E}_{\mathrm{MW}} \otimes \Omega^1_{B^{\dagger}/W} \otimes W^{\dagger}(\overline{B}) \oplus \mathcal{E}_{\mathrm{MW}} \otimes W^{\dagger}\Omega_{\overline{B}/k},$$

given by $\nabla_{\mathrm{MW}} \otimes \mathbf{id} + \mathbf{id} \otimes d$ and

$$\mathcal{E}_{\mathrm{MW}} \otimes \Omega^1_{B^{\dagger}/W} \otimes W^{\dagger}(\overline{B}) \oplus \mathcal{E}_{\mathrm{MW}} \otimes W^{\dagger}\Omega_{\overline{B}/k} \to \mathcal{E}_{\mathrm{MW}} \otimes W^{\dagger}\Omega^1_{\overline{B}/k},$$

given by $\mathbf{id} \otimes t_F + \mathbf{id} \otimes \mathbf{id}$.

One can show that in the derived category this construction and the associated comparison map

$$\mathcal{E}_{\mathrm{MW}} \otimes \Omega_{B^{\dagger}/W} \to \mathcal{E} \otimes W^{\dagger}\Omega_{\overline{B}/k}$$

is in a sense independent of the choice of Frobenius lift F and that this provides a rational isomorphism

$$H^*_{\mathrm{MW}}(\overline{B}/K, \mathcal{E}_{\mathrm{MW}}) \xrightarrow{\sim} \mathbf{H}^*(W^{\dagger}\Omega_{\overline{B}/k}).$$

In order to pass to more general settings, one would like to globalise this result to rigid cohomology. The glueing process however causes some problems. In the case of a smooth quasiprojective verity Davis, Langer and Zink are able to construct a comparison isomorphism

$$H^*_{\mathrm{rig}}(X/K) \xrightarrow{\sim} \mathbf{H}^*(W^{\dagger}\Omega_X/k) \otimes \mathbf{Q}.$$

Considering the results described above, it is natural to follow their argumentation.

In the same way as above we associate to an overconvergent isocrystal $\mathcal{E}$ on $X \subset \overline{X}/K$ a locally free, rational, integrable overconvergent de Rham–Witt connection

$$(\mathcal{E}, \nabla : \mathcal{E} \otimes W^{\dagger}\Omega_{X/k} \otimes \mathbf{Q}).$$

For a suitable cover of X, one finds oneself locally in the Monsky–Washnitzer situation described above, where the desired result was shown. Via Grosse-Klönne's theory of dagger spaces and a spectral sequence argument, one passes from the local to the global situation and so obtains the main result.

Theorem 3. *Let X be a smooth quasi-projective scheme over k, and $\mathcal{E} \in \mathrm{Isoc}^{\dagger}(X \subset \overline{X}/W(k))$ a locally free isocrystal. Then there is a natural quasi-isomorphism*

$$R\Gamma_{rig}(X, \mathcal{E}) \to R\Gamma\left(X, \mathcal{E} \otimes (W^{\dagger}\Omega_{X/k} \otimes \mathbf{Q})\right).$$

Note that once one has the local result for coefficients, the argumentation of Davis, Langer and Zink carries over almost verbatim.

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Kudla's Modularity Conjecture and Formal Fourier Jacobi Series

Jan Hendrik Bruinier
(joint work with Martin Raum)

Starting with the celebrated paper of Hirzebruch and Zagier [5] on intersection numbers of Hirzebruch-Zagier curves on Hilbert modular surfaces, the interplay of the geometry of special cycles on certain Shimura varieties and coefficients of modular forms has been a subject of active research with various applications.

Gross, Kohnen and Zagier proved in connection with their work on height pairings of Heegner points that the generating series of certain Heegner divisors on modular curves $X_0(N)$ is a (vector valued) modular form of weight 3/2 with values in the first Chow group of $X_0(N)$, see [4]. A far-reaching generalization of this result for Shimura varieties associated with orthogonal groups was conjectured by Kudla in [7]. Here we briefly recall the background and report on recent results on the problem [10], [2], [9], [3].

Let (V, Q) be a quadratic space over $\mathbb{Q}$ of signature $(n, 2)$, and write $(\cdot,\cdot)$ for the bilinear form corresponding to Q. The hermitian symmetric space associated with the special orthogonal group $\mathrm{SO}(V)$ of V can be realized as

$$D = \{z \in V \otimes_{\mathbb{Q}} \mathbb{C} : \ (z, z) = 0 \text{ and } (z, \bar{z}) < 0\}/\mathbb{C}^\times.$$

This domain has two connected components. We fix one of them and denote it by D^+. Let $L \subset V$ be an even lattice. For simplicity we assume throughout this exposition that L is unimodular. This simplifies several technical aspects. For the general case we refer to [3]. Let $\Gamma \subset \mathrm{SO}(L)$ be a subgroup of finite index which takes D^+ to itself. The quotient

$$X_\Gamma = \Gamma\backslash D^+$$

has a structure as a quasi-projective algebraic variety of dimension n. It has a canonical model defined over a cyclotomic extension of $\mathbb{Q}$. For instance, if $n = 1$, then $\mathrm{SO}(V) \cong PB^\times$ for a quaternion algebra B over $\mathbb{Q}$ which is split at the archimedian place, D^+ is isomorphic to the upper complex half plane $\mathbb{H}$, and X_Γ is a (connected) Shimura curve.

There is a vast supply of algebraic cycles on X_Γ arising from embedded quadratic spaces $V' \subset V$ of smaller dimension. Let $1 \le g \le n$. For any $\lambda = (\lambda_1, \dots, \lambda_g) \in L^g$ with positive semi-definite inner product matrix $Q(\lambda) = \frac{1}{2}((\lambda_i, \lambda_j))_{i,j}$ there is a special cycle

$$Y_\lambda = \{z \in D^+ : \ (z, \lambda_1) = \dots = (z, \lambda_g) = 0\}$$

on D^+, whose codimension is equal to the rank of $Q(\lambda)$. Its image in X_Γ defines an algebraic cycle, which we also denote by Y_λ. If $T \in \mathrm{Sym}_g(\mathbb{Q})$ is positive semi-definite of rank $r(T)$, we consider the special cycle on X_Γ of codimension $r(T)$ given by

$$Y(T) = \sum_{\substack{\lambda \in L^g/\Gamma \\ Q(\lambda)=T}} Y_\lambda,$$

see [6], [7]. We obtain a class in the Chow group $\mathrm{CH}^g(X_\Gamma)$ of codimension g cycles by taking the intersection pairing

$$Z(T) = Y(T) \cdot (\mathcal{L}^\vee)^{g-r(T)}$$

with a power of the dual of the tautological bundle $\mathcal{L}$ on X_Γ. Since the cycles Y_λ depend only on the orthogonal complement of the span of the vectors $\lambda_1, \dots, \lambda_g$, the cycles $Z(T)$ satisfy the symmetry condition

$$Z(T) = Z(u^t T u) \tag{0.1}$$

for all $u \in \mathrm{GL}_g(\mathbb{Z})$.

The following conjecture [7, Section 3, Problem 3] describes all rational relations among these cycles in an elegant way by means of a generating series on the Siegel upper half plane $\mathbb{H}_g$ of genus g. For $\tau \in \mathbb{H}_g$ we put $q^T = e(\mathrm{tr}(T\tau)) = \exp(2\pi i\, \mathrm{tr}(T\tau))$. The space of Siegel modular forms of weight k for the symplectic group $\mathrm{Sp}_g(\mathbb{Z})$ of genus g is denoted by $\mathrm{M}_k^{(g)}$.

Conjecture 1 (Kudla)**.** *The formal generating series*

$$A_g(\tau) = \sum_{\substack{T \in \mathrm{Sym}_g(\mathbb{Q}) \\ T \geq 0}} Z(T) \cdot q^T$$

is a Siegel modular form of weight $1 + n/2$ for $\mathrm{Sp}_g(\mathbb{Z})$ with values in $\mathrm{CH}^g(X_\Gamma)_\mathbb{C}$. That is, for any linear functional $h : \mathrm{CH}^g(X_\Gamma)_\mathbb{C} \to \mathbb{C}$, the series

$$h(A_g)(\tau) = \sum_{\substack{T \in \mathrm{Sym}_g(\mathbb{Q}) \\ T \geq 0}} h(Z(T)) \cdot q^T$$

is a Siegel modular form in $\mathrm{M}_{1+n/2}^{(g)}$.

The analogous statement for the cohomology classes in $H^{2g}(X_\Gamma)$ of the $Z(T)$ was proved by Kudla and Millson in a series of paper, see e.g. [8]. However, the cycle class map $\mathrm{CH}^g(X_\Gamma) \to H^{2g}(X_\Gamma)$ can have a large kernel. For instance, if $n = g = 1$ then its kernel consists of the subgroup of divisor classes of degree 0, which is arithmetically of great significance.

For modular curves $X_0(N)$, the above conjecture is true essentially by the Theorem of Gross-Kohnen-Zagier. For general n and codimension $g = 1$ (and arbitrary lattices and level) it was proved by Borcherds [1].

In the case of general n and general codimension g, Zhang proved the following partial modularity result. Write the variable $\tau \in \mathbb{H}_g$ and $T \in \mathrm{Sym}_g(\mathbb{Q})$ as block matrices

$$\tau = \begin{pmatrix} \tau_1 & z \\ z^t & \tau_2 \end{pmatrix}, \qquad T = \begin{pmatrix} n & r/2 \\ r^t/2 & m \end{pmatrix},$$

where $\tau_1 \in \mathbb{H}_1$, $\tau_2 \in \mathbb{H}_{g-1}$ and $z \in \mathbb{C}^{1\times(g-1)}$, and analogously $n \in \mathbb{Q}_{\geq 0}$, $m \in \mathrm{Sym}_{g-1}(\mathbb{Q})$ and $r \in \mathbb{Q}^{1\times(g-1)}$. Then we have $q^T = e(n\tau_1 + rz^t + \mathrm{tr}(m\tau_2))$. For fixed $m \in \mathrm{Sym}_{g-1}(\mathbb{Q})$ we consider the partial generating series

$$\phi_m(\tau_1, z) = \sum_{\substack{n\in\mathbb{Q}_{\geq 0} \\ r\in\mathbb{Q}^{1\times(g-1)}}} Z\begin{pmatrix} n & r/2 \\ r^t/2 & m \end{pmatrix} \cdot e(n\tau_1 + rz^t),$$

which can be viewed as the m-th formal Fourier-Jacobi coefficient of A_g. The truth of Kudla's conjecture would imply that $\phi_m(\tau_1, z)$ is a Jacobi form of index m. Zhang established this statement without assuming Kudla's conjecture. We write $J_{k,m}$ for the space of Jacobi forms of weight k and index m for the Jacobi group $\mathrm{SL}_2(\mathbb{Z}) \ltimes \mathbb{Z}^{2\times(g-1)}$.

Theorem 2 (Zhang)**.** *The generating series $\phi_m(\tau_1, z)$ is a Jacobi form in $J_{1+n/2,m}$ with values in $\mathrm{CH}^g(X_\Gamma)_\mathbb{C}$, that is, an element of $J_{1+n/2,m} \otimes_\mathbb{C} \mathrm{CH}^g(X_\Gamma)_\mathbb{C}$.*

The proof of this result is based on the fact that $\phi_m(\tau_1, z)$ can be interpreted as a sum of push forwards of divisor generating series on embedded smaller orthogonal Shimura varieties of codimension $g-1$. The modularity of divisor generating series is known by Borcherds' result.

If we knew that the generating series A_g converged, then Zhang's theorem together with (0.1) would imply Kudla's conjecture, since the symplectic group is generated by translations, the discrete Levi factor $\mathrm{GL}_g(\mathbb{Z})$, and the embedded Jacobi group $\mathrm{SL}_2(\mathbb{Z}) \ltimes \mathbb{Z}^{2\times(g-1)}$. However, there seems to be no direct way to obtain any such convergence result.

In [3] we prove a general modularity result for formal Fourier-Jacobi series, which implies the desired convergence statement. We now describe this.

A formal Fourier-Jacobi series of genus g (and weight k and cogenus $g-1$) is a formal series

$$f(\tau) = \sum_{\substack{m\in\mathrm{Sym}_{g-1}(\mathbb{Q}) \\ m\geq 0}} \phi_m(\tau_1, z) q_2^m,$$

with coefficients $\phi_m \in J_{k,m}$. Here $q_2^m = e(\mathrm{tr}\, m\tau_2)$ and $\tau_1 \in \mathbb{H}$, $z \in \mathbb{C}^{1\times(g-1)}$. We denote the Fourier coefficient of index (n, r) of ϕ_m by $c(\phi_m, n, r)$, and define the formal Fourier coefficient of f of index $T = \left(\begin{smallmatrix} n & r/2 \\ r^t/2 & m \end{smallmatrix}\right)$ by

$$c(f, T) = c(\phi_m, n, r).$$

The formal Fourier-Jacobi series f is called *symmetric*, if

$$c(f, T) = \det(u)^k \cdot c(f, u^t T u), \ \textit{for all } u \in \mathrm{GL}_g(\mathbb{Z}).$$

Theorem 3 (see [3])**.** *Every symmetric formal Fourier-Jacobi series of genus g and weight k converges, that is, it is the Fourier-Jacobi expansion of a Siegel modular form in $\mathrm{M}_k^{(g)}$.*

Corollary 4. *Conjecture 1 is true.*

Proof of the corollary. The result of Zhang shows that the generating series A_g is a formal Fourier-Jacobi series of weight $1+n/2$ and genus g and cogenus $g-1$. It is symmetric because of (0.1). Hence the claim follows from Theorem 3. □

Corollary 5. *The subgroup of $\mathrm{CH}^g(X_\Gamma)$ generated by the classes $Z(T)$ for $T \in \mathrm{Sym}_g(\mathbb{Q})$ positive semi-definite has rank $\leq \dim(\mathrm{M}_{1+n/2}^{(g)}$.*

Note that it is not known in general whether the rank of $\mathrm{CH}^g(X_\Gamma)$ is finite.

Finally, we briefly comment on the idea of the proof of Theorem 3, referring to [3] for details. The space of Siegel modular forms $\mathrm{M}_k^{(g)}$ is a subspace of the space $\mathrm{FM}_k^{(g)}$ of symmetric formal Fourier-Jacobi series of weight k and genus g. In easy special cases (certain cases in genus 2) one can show that the dimensions of the two spaces agree and thereby prove the theorem. However, in general this seems hopeless. Instead, we use the symmetry condition and slope bounds for Siegel modular forms to compare the dimension asymptotics for $k \to \infty$. While $\dim(\mathrm{M}_k^{(g)})$ grows like a positive constant times $k^{\frac{g(g+1)}{2}}$, we can establish the bound

$$\dim(\mathrm{FM}_k^{(g)}) \ll k^{\frac{g(g+1)}{2}}.$$

This implies that any $f \in \mathrm{FM}_k^{(g)}$ satisfies a non-trivial algebraic relation over the graded ring of Siegel modular forms. Now, viewing f as an element of the completion of the local ring $\hat{\mathcal{O}}_a$ at boundary points a of a regular toroidal compactification of X_Γ, one can deduce that f converges in a neighborhood of the boundary. Again using the algebraic relation, it can be shown that f has a holomorphic continuation to the whole domain $\mathbb{H}_g$.

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Plectic cohomology

Jan Nekovář
(joint work with A.J. Scholl)

Let $Sh(G,X)$ be the Shimura stack attached to a Shimura datum (G,X), where $G = R_{F/\mathbf{Q}}(H)$ is the restriction of scalars from a totally real number field F. Let $Aut(F \otimes \overline{\mathbf{Q}}/F)_{[\mu]} \supset Aut(\overline{\mathbf{Q}}/\mathbf{Q})_{[\mu]} = Aut(\overline{\mathbf{Q}}/E)$ be the respective stabilisers of the conjugacy class $[\mu] \in Aut_{AlgGp/\overline{\mathbf{Q}}}(\mathbf{G}_{m,\overline{\mathbf{Q}}}, G_{\overline{\mathbf{Q}}})/int(G(\overline{\mathbf{Q}}))$ of the cocharacter μ attached to (G,X) in the groups $Aut(F \otimes \overline{\mathbf{Q}}/F) \supset Aut(\overline{\mathbf{Q}}/\mathbf{Q})$.

We conjecture that $\mathbf{R}\Gamma_{et}(Sh(G,X) \otimes_E \overline{\mathbf{Q}}, \mathbf{Q}_l) \in D^+(\mathbf{Q}_l[Aut(\overline{\mathbf{Q}}/E)] - Mod)$ admits a canonical lift $\widetilde{\mathbf{R}\Gamma}_{et}(Sh(G,X) \otimes_E \overline{\mathbf{Q}}, \mathbf{Q}_l) \in D^+(\mathbf{Q}_l[Aut(F \otimes \overline{\mathbf{Q}}/F)_{[\mu]}] - Mod)$, compatible with products and the action of $G(\widehat{\mathbf{Q}})$.

If true, then one can define – for certain j – plectic l-adic etale cohomology $\mathbf{R}\Gamma_{pl-et}(Sh(G,X), \mathbf{Q}_l(j)) = \mathbf{R}\Gamma(Aut(F \otimes \overline{\mathbf{Q}}/F)_{[\mu]}, -) \circ \widetilde{\mathbf{R}\Gamma}_{et}(Sh(G,X) \otimes_E \overline{\mathbf{Q}}, \mathbf{Q}_l(j))$.

In the simplest case $H = GL(2)_F$ the corresponding plectic cohomology of Hilbert modular stacks – if it exists – can be used to construct an Euler system of rank $[F : \mathbf{Q}]$ (with coefficients in $\mathbf{Q}_l$) for Hilbert modular forms of weight two.

On a p-adic invariant cycles theorem

Valentina Di Proietto
(joint work with Bruno Chiarellotto, Robert Coleman, Adrian Iovita)

In my talk I explained the results proven in the paper [2]; it is about a p-adic version of the invariant cycles theorem.

We recall what is the invariant cycles theorem in the complex setting, following the works of Clemens, Schmid, Steenbrink, El Zein, Saito and others.

Let Δ be the unit disk in the complex plane. Let $\mathcal{X}$ be a smooth complex variety which is a Kähler manifold; and let $\pi : \mathcal{X} \to \Delta$ be a semistable degeneration, *i.e.* a proper holomorphic, flat map, such that $\mathcal{X}_t := \pi^{-1}(t)$ is smooth for $t \neq 0$ and $\mathcal{X}_0$ is sum of smooth irreducible components which meet transversally. The special fiber $\mathcal{X}_0$ is a strong deformation retract of $\mathcal{X}$, and the retraction induces an isomorphism in cohomology[1]: $H^m := H^m(\mathcal{X}) \cong H^m(\mathcal{X}_0)$. The inclusion of the smooth fiber $i : \mathcal{X}_t \to \mathcal{X}$ induces a map in cohomology $i^* : H^m(\mathcal{X}) \to H^m(\mathcal{X}_t)$. The cohomology of the smooth fiber comes equipped with a monodromy action $N : H^m(\mathcal{X}_t) \to H^m(\mathcal{X}_t)$ induced by the Picard-Lefschetz transformation.

[1] We consider cohomology with rational coefficients

The classical invariant cycles theorem asserts that for all m the following sequence is exact

$$H^m \xrightarrow{i^*} H^m(\mathcal{X}_t) \xrightarrow{N} H^m(\mathcal{X}_t),$$

moreover for $m = 1$ the first arrow is injective. If we want the above sequence to be exact not only as a sequence of vector spaces, but as sequence of mixed Hodge structures, we have to consider instead of $H^m(\mathcal{X}_t)$ and N, the limit cohomology H^m_{lim} with the induced action of the monodromy operator N.

Let k be a perfect field of characteristic $p > 0$, let $W := W(k)$ be the ring of Witt vectors of k and let K be the fraction field of W. Let X be a semistable curve over W, so the generic fiber X_K is smooth and the special fiber X_k is a normal crossing divisor. Moreover we suppose that X_k is connected, with at least two irreducible components and that all the intersection points of the irreducible components of X_k are k-rational.

The p-adic analogue of H^m is Berthelot's rigid cohomology $H^m_{\mathrm{rig}}(X_k)$, while H^m_{lim}, endowed with the monodromy operator N, becomes in the p-adic setting the Hyodo-Kato cohomology $H^m_{\mathrm{log-cris}}(X_k)$ endowed with its monodromy operator N.

The theorem reads as follows

Theorem 1. *The following sequence is exact:*

$$0 \to H^1_{\mathrm{rig}}(X_k) \xrightarrow{i^*} H^1_{\mathrm{log-cris}}(X_k) \xrightarrow{N} H^1_{\mathrm{log-cris}}(X_k)$$

Remark 2. If $|k| < \infty$ Chiarellotto proved in [1] that theorem 1 is a consequence of the weight monodromy conjecture.

Remark 3. We can read theorem 1 as a way to look at rigid cohomology à la Fontaine, indeed if $\overline{K}$ is an algebraic closure of K

$$D_{\mathrm{st}}(H^1_{\text{ét}}(X_K \times \overline{K}, \mathbb{Q}_p)) = H^1_{\mathrm{log-cris}}(X_k) \otimes K$$

$$D_{\mathrm{st}}^{N=0} = D_{\mathrm{cris}},$$

then

$$H^1_{\mathrm{rig}}(X_k) = D_{\mathrm{cris}}(H^1_{\text{ét}}(X_K \times \overline{K}, \mathbb{Q}_p))$$

We give a direct proof of theorem 1, which uses the theory developed by Coleman and Iovita in [3]: using their description of the action of the monodromy operator we are able to translate the study of the exactness of the invariant cycles in terms of a linear algebra problem on the dual graph of the curve.

We look also at the case of cohomologies with non-trivial coefficients.

Question 4. Let E be F-convergent isocrystal, is the following sequence exact?

$$H^1_{\mathrm{rig}}(X_k, E) \xrightarrow{i^*_E} H^1_{\mathrm{log-cris}}(X_k, E) \xrightarrow{N_E} H^1_{\mathrm{log-cris}}(X_k, E) \tag{0.1}$$

Using the description of the monodromy operator N_E given by Coleman and Iovita in [4] we prove the following theorem.

Theorem 5. *Let be the hypothesis as before, then*

- i_E^* *is injective*
- $i_E^*(H^1_{\mathrm{rig}}(X_k, E)) \subset \mathrm{Ker}(N_E)$
- *the sequence* (0.1) *is not necessarily exact*

We analyze in particular the case of E a unipotent F-isocrystal and we give a sufficient condition for the non exactness.

In view of this result, the following natural question arises.

Question 6. What are the coefficients for which the sequence (0.1) is exact?

The expectation is that coefficients that come from geometry should give rise to exact sequences.

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Chow groups of zero-cycles in fibrations

Olivier Wittenberg

(joint work with Yonatan Harpaz)

Let X be a smooth, proper, irreducible variety over a number field k. Denote by $\mathrm{CH}_0(X)$ the Chow group of zero-cycles up to rational equivalence, by $\mathrm{Br}(X)$ the cohomological Brauer group of X, by $\Omega = \Omega_f \sqcup \Omega_\infty$ the set of places of k, and let $\widehat{M} = \varprojlim_{n\geq 1} M/nM$ for any abelian group M. According to the reciprocity law of global class field theory, the local pairings

$$\langle -,-\rangle_v : \mathrm{CH}_0(X\otimes_k k_v)\times \mathrm{Br}(X\otimes_k k_v)\to \mathrm{Br}(k_v)\hookrightarrow \mathbf{Q}/\mathbf{Z}$$

for $v\in\Omega$, characterised by the property that $\langle P,\alpha\rangle_v$ is the local invariant of $\alpha(P)\in \mathrm{Br}(k_v(P))$ whenever P is a closed point of $X\otimes_k k_v$, fit together in a complex

$$\widehat{\mathrm{CH}_0(X)} \longrightarrow \widehat{\mathrm{CH}_{0,\mathbf{A}}(X)} \xrightarrow{\sum_{v\in\Omega}\langle -,-\rangle_v} \mathrm{Hom}(\mathrm{Br}(X),\mathbf{Q}/\mathbf{Z}),$$

where

$$\mathrm{CH}_{0,\mathbf{A}}(X) = \prod_{v\in\Omega_f}\mathrm{CH}_0(X\otimes_k k_v)\times\prod_{v\in\Omega_\infty}\frac{\mathrm{CH}_0(X\otimes_k k_v)}{N_{\bar{k}_v/k_v}(\mathrm{CH}_0(X\otimes_k \bar{k}_v))}.$$

Conjecture (Colliot-Thélène [1], Kato and Saito [7, §7])**.** *The above complex is exact for any smooth, proper, irreducible variety X over k.*

For rational surfaces, a more precise conjecture, which also predicts the kernel of the first map, appears in [3]. Even the case of cubic surfaces over $\mathbf{Q}$ is still widely open.

A notable consequence of the exactness of this complex would be that X possesses a zero-cycle of degree 1 if and only if there exists a family $(z_v)_{v\in\Omega}$ of local zero-cycles of degree 1 whose image in $\mathrm{Hom}(\mathrm{Br}(X), \mathbf{Q}/\mathbf{Z})$ vanishes.

Saito [9] proved the conjecture for curves, under the assumption that the divisible subgroup of the Tate–Shafarevich group of the Jacobian is trivial. (When the curve has a rational point, this assumption is in fact equivalent to the conjecture.) The aim of the talk was to discuss the following fibration theorem and its proof.

Theorem. *Let $f : X \to C$ be a dominant morphism with rationally connected (*e.g.*, geometrically unirational) generic fiber, where C is a smooth, proper, irreducible curve with $\text{Ш}(k, \mathrm{Jac}(C))_{\mathrm{div}} = 0$. If the smooth fibers of f satisfy the conjecture, then so does X.*

The oldest instance of a fibration argument establishing a particular case of the above conjecture is Hasse's proof of the Hasse–Minkowski theorem for quadratic forms in four variables with rational coefficients. It was based on Dirichlet's theorem on primes in arithmetic progressions and on the global reciprocity law. A delicate argument relying on the same two ingredients allowed Salberger [10] to settle the conjecture for conic bundle surfaces over $\mathbf{P}^1_k$. His proof was later generalised in various directions (see [5], [4], [2], [6], [11], [12], [8]). In all of these papers, Dirichlet's theorem on primes in arithmetic progressions for general number fields was used in the following form: given a finite subset $S \subset \Omega_f$ and elements $\xi_v \in k_v^*$ for $v \in S$, there exists $\xi \in k^*$ arbitrarily close to ξ_v for $v \in S$ such that ξ is a unit outside S except at a unique (unspecified) place v_0, at which it is a uniformiser. (Strictly speaking, when k is not totally imaginary, a more general statement which incorporates approximation conditions at the real places, and which builds on results of Waldschmidt in transcendence theory, needs to be used.) Given a finite *abelian* extension L/k from which ξ_v is a local norm for $v \in S$ and which is unramified outside S, it then follows, in view of the global reciprocity law, that ξ is a local norm from L at v_0 too, and hence that v_0 splits in L.

The reciprocity argument we have just described fails when the extension L/k is not abelian. This failure has led to severe restrictions on the fibrations to which the previous methods could apply. In the proof of the above theorem, the following elementary lemma serves as a substitute for Dirichlet's theorem.

Lemma. *Let L/k be a finite Galois extension. Let S be a finite set of places of k. For each $v \in S$, let $\xi_v \in k_v^*$. If ξ_v is a local norm from L/k for each $v \in S$, there exists $\xi \in k^*$ arbitrarily close to ξ_v for $v \in S$, such that ξ is a unit outside S except at places which split in L.*

Proof. Affine space minus any codimension 2 subset satisfies strong approximation off one place. The lemma is a consequence of this fact applied to $(R_{L/k}\mathbf{A}^1_L) \setminus F$, where F denotes the singular locus of the complement of the torus $R_{L/K}\mathbf{G}_{\mathrm{m}}$ in $R_{L/k}\mathbf{A}^1_L$. □

Another ingredient of the proof is an arithmetic duality theorem (obtained in [12, §5]) for a variant of Rosenlicht's relative Picard group, denoted $\mathrm{Pic}_+(C)$, whose definition we briefly recall. Let $M \subset C$ be the set of points over which the fiber of f is singular. For each $m \in M$, fix a finite extension $L_m/k(m)$. Then $\mathrm{Pic}_+(C)$ is defined as the quotient of $\mathrm{Div}(C \setminus M)$ by the subgroup generated by the principal divisors $\mathrm{div}(h)$ such that for each $m \in M$, the function h is invertible at m and $h(m)$ is a norm from L_m.

In the very simple situation of a fibration over $\mathbf{P}^1_k$ with only two singular fibers, above 0 and ∞, each of which possesses an irreducible component of multiplicity 1, if we let $L_0 = L_\infty = L$ and if we are given an adelic point $(P_v)_{v\in\Omega} \in X(\mathbf{A}_k)$ supported outside of $f^{-1}(M)$, it is easy to see that the class of $(f(P_v))_{v\in\Omega}$ belongs to the image of the diagonal map $\mathrm{Pic}_+(C) \to \prod_{v\in\Omega} \mathrm{Pic}_+(C \otimes_k k_v)$ if and only if there exists $c \in k^*$ such that ct_v is a local norm from L for all $v \in \Omega$, where $t_v \in k_v^*$ denotes the coordinate of $f(P_v) \in \mathbf{P}^1(k_v)$. If this condition is satisfied, applying the lemma to $\xi_v = ct_v$ and setting $t = \xi/c$ yields a point $t \in k^* = \mathbf{G}_{\mathrm{m}}(k) \subset \mathbf{P}^1(k)$ such that $X_t(\mathbf{A}_k) \neq \varnothing$, provided L and S were chosen large enough in the first place. It is this argument, which bypasses any abelianness assumption, which forms the core of the proof of the theorem.

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Local-global compatibility in weight 1

Pierre Colmez

I described partial extensions of the p-adic local Langlands correspondence for $GL_2(\mathbb{Q}_p)$ to rank 2 (φ, Γ)-modules, not necessarily étale, over the Robba ring (with a conjectural extension to analytic representations of $GL_2(F)$, for a finite extension F of $\mathbb{Q}_p$). I also explained how to recover the classical Langlands correspondence from the p-adic one in weight 1.

Reporter: Veronika Ertl

Participants

Prof. Dr. Fabrizio Andreatta
Dipartimento di Matematica
"Federigo Enriques"
Universita di Milano
Via Saldini 50
20133 Milano
ITALY

Prof. Dr. Kenichi Bannai
Department of Mathematics
Faculty of Science and Technology
Keio University
3-14-1, Hiyoshi, Kohoku-ku
Yokohama 223-8522
JAPAN

Dr. Rebecca Bellovin
Department of Mathematics
Imperial College of Science,
Technology and Medicine
London SW7 2BZ
UNITED KINGDOM

Prof. Dr. Laurent Berger
Dept. de Mathématiques, U.M.P.A.
École Normale Superieure de Lyon
46, Allee d'Italie
69364 Lyon Cedex 07
FRANCE

Prof. Dr. Massimo Bertolini
Institut f. Experimentelle Mathematik
Universität Essen
Ellernstr. 29
45326 Essen
GERMANY

Prof. Dr. Jan Hendrik Bruinier
Fachbereich Mathematik
TU Darmstadt
Schloßgartenstr. 7
64289 Darmstadt
GERMANY

Dr. Oliver Bültel
Fakultät für Mathematik
Universität Duisburg-Essen
45117 Essen
GERMANY

Ashay A. Burungale
Department of Mathematics
University of California at
Los Angeles
Los Angeles, CA 90095-1555
UNITED STATES

Dr. Ana Caraiani
Department of Mathematics
Princeton University
710 Fine Hall
Washington Road
Princeton NJ 08544-1000
UNITED STATES

Prof. Dr. Pierre Colmez
CNRS / Institut de Mathématiques de
Jussieu
Theorie des Nombres, Case 247
Université de Paris VI
4, Place Jussieu
75252 Paris Cedex
FRANCE

Prof. Dr. Henri René Darmon
Department of Mathematics & Statistics
McGill University
805, Sherbrooke Street West
Montreal, QC H3A 0B9
CANADA

Dr. Valentina Di Proietto
I.R.M.A.
Université de Strasbourg
7, rue Rene Descartes
67084 Strasbourg Cedex
FRANCE

Dr. Daniel Disegni
Department of Mathematics & Statistics
McGill University
805 Sherbrooke Street West
Montreal, QC H3A 0B9
CANADA

Dr. Ellen E. Eischen
Mathematics Department
The University of North Carolina
at Chapel Hill, CB # 3250
Phillips Hall
Chapel Hill, NC 27599-3250
UNITED STATES

Dr. Veronika Ertl
Fakultät für Mathematik
Universität Regensburg
93053 Regensburg
GERMANY

Prof. Dr. Dr. h.c. Hélène Esnault
FB Mathematik und Informatik
Freie Universität Berlin
Arnimallee 3
14195 Berlin
GERMANY

Dr. Olivier Fouquet
Departement de Mathématiques
Université Paris-Sud
Batiment 425
91405 Orsay Cedex
FRANCE

Prof. Dr. Takako Fukaya
Department of Mathematics
The University of Chicago
5734 South University Avenue
Chicago, IL 60637-1514
UNITED STATES

Philipp Graf
NWF I - Mathematik
Universität Regensburg
93053 Regensburg
GERMANY

Dr. Eugen Hellmann
Mathematisches Institut
Universität Bonn
Endenicher Allee 60
53115 Bonn
GERMANY

Prof. Dr. Benjamin V. Howard
Department of Mathematics
Boston College
301 Carney Hall
Chestnut Hill, MA 02467-3806
UNITED STATES

Prof. Dr. Ming-Lun Hsieh
Department of Mathematics
National Taiwan University
New Math. Building; Room 400
No1, Sec. 4, Roosevelt Road
Taipei 10617
TAIWAN

Prof. Dr. Annette Huber-Klawitter
Mathematisches Institut
Universität Freiburg
Eckerstr. 1
79104 Freiburg
GERMANY

Prof. Dr. Uwe Jannsen
Fakultät für Mathematik
Universität Regensburg
93040 Regensburg
GERMANY

Dr. Henri Johnston
Department of Mathematics
University of Exeter
North Park Road
Exeter EX4 4QE
UNITED KINGDOM

Prof. Dr. Kazuya Kato
Department of Mathematics
The University of Chicago
5734 South University Avenue
Chicago, IL 60637-1514
UNITED STATES

Prof. Dr. Kiran S. Kedlaya
Department of Mathematics
University of California, San Diego
9500 Gilman Drive
La Jolla, CA 92093-0112
UNITED STATES

Prof. Dr. Guido Kings
Fakultät für Mathematik
Universität Regensburg
93040 Regensburg
GERMANY

Prof. Dr. Shinichi Kobayashi
Mathematical Institute
Faculty of Science
Tohoku University
Sendai 980-8578
JAPAN

Prof. Dr. Stephen S. Kudla
Department of Mathematics
University of Toronto
Toronto ON M5S 2E4
CANADA

Dr. David A. Loeffler
Mathematics Institute
University of Warwick
Gibbet Hill Road
Coventry CV4 7AL
UNITED KINGDOM

Prof. Dr. Keerthi Madapusi Pera
Department of Mathematics
Harvard University, Science Center
One Oxford Street
Cambridge MA 02138-2901
UNITED STATES

Andreas Mihatsch
Mathematisches Institut
Universität Bonn
Endenicher Allee 60
53115 Bonn
GERMANY

Prof. Dr. Sophie Morel
Department of Mathematics
Princeton University
Fine Hall
Princeton, NJ 08544-1000
UNITED STATES

Prof. Jan Nekovar
Institut de Mathématiques de Jussieu
Theorie des Nombres, Case 247
Université de Paris VI
4, Place Jussieu
75252 Paris
FRANCE

Prof. Dr. Georgios Pappas
Department of Mathematics
Michigan State University
Wells Hall
East Lansing, MI 48824-1027
UNITED STATES

Prof. Dr. Kartik Prasanna
Department of Mathematics
University of Michigan
530 Church Street
Ann Arbor, MI 48109-1043
UNITED STATES

Prof. Dr. Michael Rapoport
Mathematisches Institut
Universität Bonn
Endenicher Allee 60
53115 Bonn
GERMANY

Andreas Riedel
Mathematisches Institut
Universität Heidelberg
Im Neuenheimer Feld 288
69120 Heidelberg
GERMANY

Prof. Dr. Victor Rotger
Facultat de Matematiques
Universitat de Barcelona
Departament d'Algebra i Geometria
Gran Via 585
08007 Barcelona
SPAIN

Prof. Dr. Alexander Schmidt
Mathematisches Institut
Universität Heidelberg
Im Neuenheimer Feld 288
69120 Heidelberg
GERMANY

Prof. Dr. Peter Schneider
Mathematisches Institut
Universität Münster
Einsteinstr. 62
48149 Münster
GERMANY

Dr. Sudhanshu Shekar
Mathematisches Institut
Universität Heidelberg
69120 Heidelberg
GERMANY

Dr. Florian Sprung
School of Mathematics
Institute for Advanced Study
Princeton University
Einstein Drive
Princeton NJ 08540
UNITED STATES

Prof. Dr. Benoit Stroh
Mathématiques
LAGA - CNRS: UMR 7539
Université Paris Nord (Paris XIII)
99, Avenue J.-B. Clement
93430 Villetaneuse
FRANCE

Prof. Dr. Ramdorai Sujatha
Department of Mathematics
University of British Columbia
1984 Mathematics Road
Vancouver B.C. V6T 1Z2
CANADA

Prof. Dr. Eric Urban
Department of Mathematics
Columbia University
2990 Broadway
New York NY 10027
UNITED STATES

Prof. Dr. Otmar Venjakob
Mathematisches Institut
Universität Heidelberg
Im Neuenheimer Feld 288
69120 Heidelberg
GERMANY

Preston Wake
Department of Mathematics
The University of Chicago
5734 South University Avenue
Chicago, IL 60637-1514
UNITED STATES

Dr. Malte Witte
Mathematisches Institut
Universität Heidelberg
Im Neuenheimer Feld 288
69120 Heidelberg
GERMANY

Dr. Sarah Zerbes
Department of Mathematics
University College London
Gower Street
London WC1E 6BT
UNITED KINGDOM

Prof. Dr. Olivier Wittenberg
Dept. de Mathématiques et Applications
École Normale Superieure
45, rue d'Ulm
75230 Paris Cedex 05
FRANCE