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Komplexe Analysis

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Abstract. Complex Analysis is a very active branch of mathematics with applications in many other fields. The central aim of our workshop was to present recent results in several complex variables and complex geometry, and to survey topics that link it to other branches of mathematics.

Mathematics Subject Classification (2010): 32, 32J, 32Q, 32Q15.

Introduction by the Organisers

The workshop *Komplexe Analysis*, organised by Philippe Eyssidieux (Grenoble), J.-M. Hwang (Seoul), S. Kebekus (Freiburg) and M. Păun (Seoul) was well attended in number and quality, with close to 50 participants. We were particularly glad to notice the presence of well-known experts from different backgrounds, who shared generously their ideas, points of view and recent results with young researchers.

The workshop was intended to be articulated on the interaction between global aspects of several complex variables and algebraic geometry. As we will try to highlight next, from this perspective it is beyond any reasonable doubt that our meeting was a success.

A. Höring presented his joint work with F. Campana and T. Peternell in a very precise and understandable manner. The topic was a fundamental problem in algebraic geometry, the so-called abundance conjecture. The main result he has discussed was a solution of this problem for Kähler 3-folds: it is the achievement of their efforts extended over many years. The arguments that Andreas decided to explain involved the classical proof for projective 3-folds (due to Y. Kawamata, Y.

Miyaoka, ...), as well as purely transcendental techniques, specific to the Kähler setting.

Let (X, L) be a smooth, projective manifold endowed with an ample bundle L. The Yau-Tian-Donaldson conjecture predicts that the class $c_1(L)$ contains a metric with constant scalar curvature iff the pair (X, L) is K-stable. Given the spectacular progress in case of Fano manifolds, i.e. for $L = -K_X$, there are many recent works exploring the K-stability. In our meeting we had two beautiful accounts on different aspects of this notion, by R. Berman (joint work with D. Witt-Nyström) and S. Boucksom (joint with T. Hisamoto and M. Jonsson). The presentation of S. Takayama concerned a closely related area: the degeneration of polarized Ricci-flat Kähler manifolds. His main result answers an open question by V. Tosatti, asking for the connection between the finiteness of the Weil-Petersson metric and the singularities of the limits of families of manifolds with trivial canonical class.

From the algebraic geometry side, J. McKernan presented his joint work with P. Cascini, concerning their version of some conjectures of V. Shokurov. It was particularly nice, since abundantly commented and motivated by examples. L. Ein presented the solution of the long-standing gonality conjecture the he has recently obtained in a joint work with R. Lazarsfeld.

We had two talks concerning hyperkähler manifolds. E. Amerik explained us her impressive work with M. Verbitsky about the Kawamata-Morrison conjecture (which predicts that the automorphism group of X has only a finite number of orbits on the set of faces of the Kähler cone). The arguments invoked in the particular cases they are treating involve many techniques from hyperbolic geometry, ergodicity... and it gives strong support for the conjecture. The work that G. Pacienza presented concerns the study of families of rational curves on projective hyperkähler manifolds; it is based on his joint work with F. Charles. They equally show the existence of uniruled divisors on an important family of hyperkähler manifolds.

Given the increasing number of the problems in algebraic geometry which require deep techniques from the dynamics, we have asked N. Sibony to give an overview lecture. Our goal was to offer the participants an introduction to these techniques by one of the experts in the field.

The closing lecture of our meeting was given by Y.-T. Siu. With his usual inspired energy, Siu discussed a possible approach for the deformation invariance of the plurigenera in the Kähler setting. His idea is to build a Hodge decomposition involving mK_X as a summand, for $m \geq 2$; he has explained the case of curves, based on classical works of Bol and Schwarz, going back to the 19^{th} century (AD).

We have asked young researchers to give a very concise presentation of their work, during an informal evening session. This session has generated many discussions and questions between the participants. The overall experience was very positive.

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Workshop: Komplexe Analysis

Table of Contents

Abstracts

Abundance for Kähler threefolds

Andreas Höring
(joint work with Frédéric Campana and Thomas Peternell)

Since the 1990's, the minimal model program for smooth projective threefolds is complete: every such manifold X admits a birational model X', which is $\mathbb{Q}$-factorial with only terminal singularities such that X' either carries a Fano fibration, in particular is uniruled, or the canonical bundle $K_{X'}$ is semi-ample, i.e., some positive multiple $mK_{X'}$ is generated by global sections. There are basically two parts in the program: first to establish the existence of a model X' which is either a Mori fibre space or has nef canonical divisor, and then to show that nefness implies semi-ampleness. This second part, known as "abundance", is established by [Miy87, Kaw92a, Kwc92].

The aim of the talk is to present a series of papers [HP13a, HP13b, CHP14] which fully establish the minimal model program in the category of Kähler threefolds. The first part of the program, i.e., the existence of a bimeromorphic model X' which is a either Mori fibre space or has nef canonical divisor was carried out in the papers [HP13a] and [HP13b]. The focus of this talk is on the final chapter [CHP14]: nefness of the canonical divisor implies semi-ampleness, i.e. abundance holds for Kähler threefolds:

Theorem 1. *Let X be a normal $\mathbb{Q}$-factorial compact Kähler threefold with at most terminal singularities such that K_X is nef. Then K_X is semi-ample, that is some positive multiple mK_X is globally generated.*

The paper [DP03] established the existence of some section in mK_X for non-algebraic minimal models, so the assumption above implies $\kappa(X) \geq 0$. In [Pet01] abundance was shown for non-algebraic minimal models, excluding however the very challenging case when X has no non-constant meromorphic function. The proof of Theorem 1 does not use any assumption on the structure of X and includes the earlier arguments for the projective case. As a corollary, we establish a longstanding conjecture on Kähler threefolds:

Theorem 2. *Let X be smooth compact Kähler threefold. Assume that X is simple, i.e. there is no positive-dimensional proper subvariety through the very general point of X. Then there exists a bimeromorphic morphism $X \to T/G$ where T is a torus and G a finite group acting on T.*

For the proof of an abundance statement it is not sufficient to work with varieties, one has to consider the more general setting of pairs (X, B) where B is an effective divisor on X that is not too singular. More precisely we want to run a minimal model program for certain pairs (X, B) that are log-canonical. A crucial point for the existence of a minimal model program is to construct (rational) curves $C \subset X$ such that $(K_X + B) \cdot C < 0$. While the details of the construction are quite technical, their existence becomes plausible using the following heuristic argument: suppose that $K_X + B$ is pseudoeffective, but not nef. Then there exists a Kähler class ω such that

$$\alpha := K_X + B + \omega$$

is nef and big but not Kähler. By a theorem of Collins and Tosatti [CT13] this implies the existence of a subvariety $S \subset X$ such that $\alpha|_S$ is nef but not big. For simplicity's sake suppose that $S \subset X$ is a smooth surface, and consider the case where $\alpha|_S \neq 0$ but $(\alpha|_S)^2 = 0$. By the Hodge index theorem we obtain

$$(K_X + B)|_S \cdot \alpha|_S = -\omega|_S \cdot \alpha|_S < 0,$$

combining the last inequality with the Boucksom-Zariski decomposition [Bou04] we deduce that

$$S|_S \cdot \alpha|_S < 0.$$

By the adjunction formula this shows that $(K_S + B|_S) \cdot \alpha|_S < 0$, in particular K_S is not pseudoeffective. It now follows from the surface classification that S is covered by rational curves $(C_t)_{t \in T}$ and one can refine the analysis to see that these curve satisfy $(K_X + B) \cdot C_t < 0$.

This elementary computation suggest a two-step approach to producing rational curves on compact Kähler manifolds of arbitrary dimension. First one should prove

Conjecture 3. *Let X be a compact Kähler manifold. Then X is covered by rational curves if and only if K_X is not pseudoeffective.*

This statement is known in the projective case by the seminal work [BDPP13], but the Kähler case is only known in dimension at most three [Bru06]. Using the study of adjoint classes $K_X + \omega$ one can then try to go further:

Conjecture 4. *Let X be a compact Kähler manifold such that K_X is pseudoeffective. If K_X is not nef, there exists a rational curve on X.*

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Degenerations of polarized Ricci-flat Kähler manifolds

Shigeharu Takayama

We discuss relations among various geometric properties along degenerations of smooth projective varieties with trivial canonical bundle, such as the finiteness of the Weil-Petersson distance, the uniform boundedness of diameters with respect to Ricci-flat Kähler metrics, the volume non-collapsing property, and that the limit variety has canonical singularities at worst. Full details are explained in [Ta2].

Set-up 1. *Let X be a normal complex space admitting $f : X \to C$ a projective surjective holomorphic map with connected fibers, to a Riemann surface C with a special point $0 \in C$. Suppose that $X^o := X \setminus f^{-1}(0)$ is smooth, f is smooth on X^o, and that $X_t := f^{-1}(t)$ is of n-dimensional and has the trivial canonical bundle, i.e. $K_{X_t} = \mathcal{O}_{X_t}$ for any $t \in C^o := C \setminus \{0\}$. Let $X_0 := f^*(0)$ be the special/central fiber, which may be non-reduced. The symbol t will also stand for a local coordinate of C centered at 0.*

Let L be a holomorphic line bundle on X which is f-ample, and denote by $L_t = L|_{X_t}$ for $t \in C^o$. According to Yau, there exists a unique Ricci-flat Kähler form ω_t on X_t in the cohomology class $c_1(L_t)$ for $t \in C^o$. □

Definition 2. *There are fundamental geometric intrinsic objects/properties attached to a family of varieties in 1. For our purpose here, we may suppose that C is a disk in $\mathbb{C}$ or an open Riemann surface.*

(1) We consider a smooth $(1,1)$-form

$$\omega_{WP} := \frac{\sqrt{-1}}{2\pi}\overline{\partial}\partial \log \left(\int_{X_t} (-1)^{n^2/2} \Omega_t \wedge \overline{\Omega}_t \right)$$

on C^o, where $\Omega \in H^0(X, K_{X/C})$ is a generator and $\Omega_t = \Omega|_{X_t} \in H^0(X_t, K_{X_t})$. By Griffiths' computation on the curvature of the Hodge (line) bundle $f_* K_{X^o/C^o}$, ω_{WP} is a semi-positive $(1,1)$-form on C^o. This ω_{WP}, or the corresponding (pseudo-)metric tensor, is called the Weil-Petersson (pseudo-)metric on C^o. Thus we can discuss whether or not 0 is at finite distance from C^o with respect to ω_{WP} (from any reference point $q \in C^o$). We will refer as $d_{WP}(C^o, 0) < \infty$ or $d_{WP}(C^o, 0) = \infty$.

(2) Let $B_{\omega_t}(x, r)$ be the geodesic ball of radius r centered at $x \in X_t$ and let $\mathrm{Vol}_{\omega_t} B_{\omega_t}(x, r)$ *be the volume with everything respect to ω_t. Let also* $\mathrm{diam}\,(X_t, \omega_t)$

be the diameter. We consider the following volume non-collapsing property (with respect to L or ω_t) [DS, (1.2)]*: There exists a constant $\alpha > 0$ such that, for any $t \neq 0$, any $x \in X_t$, and any $0 < r \leq \operatorname{diam}(X_t, \omega_t)$, a uniform estimate*

$$\mathrm{Vol}_{\omega_t} B_{\omega_t}(x, r) \geq \alpha r^{2n}$$

holds. This property is in fact equivalent to the following uniform diameter bound: There exists a constant $\alpha > 0$ such that

$$\operatorname{diam}(X_t, \omega_t) \leq \alpha$$

holds for all $t \neq 0$. □

Wang [W1, 2.3] proved that, if X_0 (is normal and) has canonical singularities at worst and $K_{X_0} = \mathcal{O}_{X_0}$, then 0 is at finite Weil-Petersson distance from C^o; $d_{WP}(C^o, 0) < \infty$. He conjectured a kind of converse [W1, 2.4]: the finiteness $d_{WP}(C^o, 0) < \infty$ implies X_0 has canonical singularities at worst and $K_{X_0} = \mathcal{O}_{X_0}$, possibly after a finite base change and a birational modification, and he proved it under a sort of semi-stable relative minimal model conjecture holds [W2, 1.2]. Recently [To, 1.2] proves such a kind of converse using the semi-stable minimal model theory from [Fu]. Our first result is to prove a more precise version of the converse without using the semi-stable minimal model theory.

Theorem 3. *Suppose in 1 that the morphism $f : X \to C$ is log-canonical, and that $K_{X/C} = 0$. Suppose further that 0 is at finite Weil-Petersson distance from C^o; $d_{WP}(C^o, 0) < \infty$. Then X_0 has canonical singularities at worst and $K_{X_0} = \mathcal{O}_{X_0}$.*

In our setting, $f : X \to C$ is log-canonical, if the pair (X, X_0) has log-canonical singularities, for example if X is smooth and X_0 is reduced and (not necessarily simple) normal crossing. If X is smooth with $K_{X/C} = \mathcal{O}_X$ and X_0 is simple normal crossing, 3 is a direct consequence of [W1, 2.1] together with an adjunction formula. To obtain 3, we use, other than [W1, 2.1], (non-)uniruledness criteria of varieties such as [HM] and [Ta1].

Tosatti [To] introduces a new perspective in the study of Wang, related to the work of Donaldson-Sun [DS] on the stability problem in Kähler geometry (Donaldson-Tian-Yau conjecture). He shows that if X_0 has canonical singularities, then the volume non-collapsing property holds, and he asks the converse. We answer his question by proving 3 and the following

Theorem 4. *Suppose in 1 ($f : X \to C$ may not be log-canonical and $K_{X/C}$ may not be trivial) that the volume non-collapsing property with respect to L holds. Then 0 is at finite Weil-Petersson distance from C^o; $d_{WP}(C^o, 0) < \infty$.*

This together with 3 gives an algebro-geometric characterization of volume non-collapsing property for families of Calabi-Yau type manifolds, which was mentioned by Donaldson-Sun [DS, p. 2]. A theorem [DS, 1.2] says, at least in our setting 1, that after embedding these X_t ($t \neq 0$) into $\mathbb{P}^N$ and taking a projective transform of X_t, there exists a limit X_∞ as $t \to 0$ in a Hilbert scheme of varieties in $\mathbb{P}^N$, moreover X_∞ turns out to be a normal projective variety with log-terminal singularities at worst ([DS, 4.15], which actually proves X_∞ has canonical singularities at worst and $K_{X_\infty} = \mathcal{O}_{X_\infty}$). The limit variety X_∞ and our X_0 may be different, however we can compare the period maps of manifolds converging to these two (one is our $f : X^o \to C^o$). For a family converging to X_∞, we can apply [W1, 2.3] and obtain the finiteness of the Weil-Petersson distance. We then deduce the finiteness $d_{WP}(C^o, 0) < \infty$ for $f : X \to C$.

Combining with other formerly known results due to [W1], [To], [RZ], we obtain

Corollary 5. *In Set up 1, the followings are equivalent:*
(1) 0 *is at finite Weil-Petersson distance from* C^o*;* $d_{WP}(C^o, 0) < \infty$.
(2) There is a constant $\alpha > 0$ *such that* $\operatorname{diam}(X_t, \omega_t) \leq \alpha$ *for all* $t \neq 0$.
(2') The volume non-collapsing property with respect to L *holds.*
If we suppose that the morphism $f : X \to C$ *is log-canonical, and that* $K_{X/C} = 0$*, then the following condition is also equivalent to those stated above.*
(3) X_0 *has canonical singularities at worst and* $K_{X_0} = \mathcal{O}_{X_0}$.

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The Prym map revisited

Klaus Hulek

(joint work with Sebastian Casalaina-Martin, Sam Grushevsky, Radu Laza)

To every étale double cover $\pi : \tilde{C} \to C$ one can associate a principally polarized abelian variety (P, Θ_P) where P is the connected component of the kernel of the norm map $\mathrm{Nm} : \mathrm{Jac}^0(\tilde{C}) \to \mathrm{Jac}^0(C)$, $\mathrm{Nm}(\sum n_i P_i) = \sum n_i \pi(P_i)$ and where $2\Theta_P$ is the restriction of the theta divisor of $\tilde{C}$ to P. If the genus of C is $g+1$, then P is of dimension g. This gives rise to a morphism of moduli spaces

$$P_g : \mathcal{R}_{g+1} \to \mathcal{A}_g$$

from the moduli space $\mathcal{R}_{g+1}$ of étale double covers of genus $g+1$ curves to principally polarized abelian varieties $\mathcal{A}_g$ of dimension g. It is natural to ask whether this map extends as a morphism to suitable compactifications of these moduli spaces. For this one has to specify the compctifications one wants to consider. In the case of $\mathcal{R}_{g+1}$ there is the natural compactification $\overline{\mathcal{R}}_{g+1}$ of admissible covers of nodal curves, which is the analogue of the Deligne-Mumford compactification $\overline{\mathcal{M}}_g$ of the moduli space $\mathcal{M}_g$ of curves of genus g. For $\mathcal{A}_g$ there are several natural choices. Apart from the Satake compactification $\mathcal{A}_g^{\mathrm{Sat}}$, which is "minimal", one has toroidal compactifications, which depend on the choice of an admissible cone decomposition of the rational closure of the cone of positive definite symmetric $g \times g$ matrices.Three such decompositions are known, which in turn lead to the *second Voronoi compactification* $\mathcal{A}_g^{\mathrm{Vor}}$, the *first Voronoi* or *perfect cone compactifiation* $\mathcal{A}_g^{\mathrm{Perf}}$ and the *central cone* or *Igusa compactification* $\mathcal{A}_g^{\mathrm{Ctr}}$. The second Voronoi compactification is distinguished by the fact that it has a modular interpretation, i.e. represents a moduli functor, as the work of Alexeev and Olsson shows. Shepherd-Barron has proved that $\mathcal{A}_g^{\mathrm{Perf}}$ is a canonical model in the sense of the minimal model program, and finally the central cone decomposition $\mathcal{A}_g^{\mathrm{Ctr}}$ can, by a result of Namikawa, be identified with the blow-up of the Satake compactification which was constructed by Igusa. Clearly, the answer to the question raised about the extendability of the Prym map P_g can (and will) depend on the target space.

Our renewed interest in this problem has two sources. One is that Alexeev and Brunyate have recently revisited the Torelli map $t_g : \mathcal{M}_g \to \mathcal{A}_g$, which associates to a curve its Jacobian variety. It is a result due to Namikawa and Mumford from the 1970's that this map extends to a morphism $t_g^{\mathrm{Vor}} : \overline{\mathcal{M}}_g \to \mathcal{A}_g^{\mathrm{Vor}}$. Alexeev and Brunyate [2] have shown that this also holds for the map $t_g^{\mathrm{Perf}} : \overline{\mathcal{M}}_g \to \mathcal{A}_g^{\mathrm{Perf}}$, whereas it fails for the central cone decomposition for $g \geq 9$. They also noticed that the image of the Torelli maps t_g^{Vor} and t_g^{Perf} are contained in the so called *matroidal locus* $\mathcal{A}_g^{\mathrm{matr}}$, which is a partial toroidal compactification of $\mathcal{A}_g$ and which, by a result of Melo and Viviani [5], is the biggest partial compactfication which is common to $\mathcal{A}_g^{\mathrm{Vor}}$ and $\mathcal{A}_g^{\mathrm{Perf}}$. Our second motivations stems from studying intermediate Jacobians of cubic threefolds and their degenerations, which are closely linked to the Prym map in genus 5.

The extension of the Prym map was first studied by Friedman and Smith. In an unpublished version of [4] they showed that the Prym map does not extend as a morphism to any "reasonable" toroidal compactification. Their example is the following: let $\tilde{C} = \tilde{C}_1 \cup \tilde{C}_2$ be the union of two smooth curves intersecting in 4 points $\tilde{C}_1 \cap \tilde{C}_2 = \{P_1, P_2, Q_1, Q_2\}$ and let ι be an involution on $\tilde{C}$ which restricts to fixed point free involutions on the components $\tilde{C}_i$ interchanging the points of intersection $\iota(P_i) = Q_i, i = 1, 2$. The quotient $C = \tilde{C}/\langle\iota\rangle$ is a nodal curve $C = C_1 \cup C_2$ with 2 nodes. We refer to an example of this type as an FS_2 example. Natural generalizations of this are the so called FS_n examples where $\tilde{C}_1$ and $\tilde{C}_2$ intersect in $2n$ points. The extension of the Prym map to the second

Voronoi compactification $\mathcal{A}_g^{\mathrm{Vor}}$ was studied in detail by Alexeev, Birkenhake and Hulek and by Vologodsky. Their result is the following

Theorem 1 ([1], [7]). *An admissible cover $\pi : \tilde{C} \to C$ is in the indeterminacy locus of the map $P_g^{\mathrm{Vor}} : \overline{\mathcal{R}}_{g+1} \dashrightarrow \mathcal{A}_g^{\mathrm{Vor}}$ if and only it is a degeneration of an $(FS)_n$ example for $n \geq 2$.*

The approach we take in [3] differs from that in [1]. Using monodromy and the Clemens-Schmid theorem we associate to each (singular) admissible cover $\pi : \tilde{C} \to C$ a monodromy cone $\sigma(\tilde{C}/C)$ in the rational closure $\mathrm{Sym}^2_{\mathrm{rc}}(\mathbb{R}^g)$ of the cone $\mathrm{Sym}^2_{>0}(\mathbb{R}^g)$ of positive definite symmetric $g \times g$ matrices. The Picard-Lefschetz theorem shows that this cone is generated by rank 1 forms. A toroidal compactification $\mathcal{A}_g^{\Sigma}$ is given by an (admissible) rational cone decomposition $\Sigma = \{\sigma\}$ of $\mathrm{Sym}^2_{\mathrm{rc}}(\mathbb{R}^g)$. It then follows from a result due to Namikawa that the Prym map $P_g^{\Sigma} : \overline{\mathcal{R}}_{g+1} \dashrightarrow \mathcal{A}_g^{\Sigma}$ extends to a morphism near $\pi : \tilde{C} \to C$ if and only if there is a cone $\sigma \in \Sigma$ with $\sigma(\tilde{C}/C) \subset \sigma$. This reduces the question of the extendability of the Prym map to a question of whether the monodromy cones are contained in cones of a given admissible decomposition Σ. It must, however, be stressed that the latter question is anything but trivial. Indeed, there are criteria which determine whether cones spanned by rank 1 forms are contained in cones of the second Voronoi, perfect cone or central cone decomposition, but applying these criteria is highly non-trivial. In the case of the second Voronoi compactification this amounts to checking that the monodromy matrix is matroidal, but in the other two cases the criteria require the existence of a quadratic form with certain properties, which is typically hard to find. These questions can be checked for small genus (at most up to 9) by computer, but there are no general results available. Following this approach we can prove

Theorem 2. *The indeterminacy locus of the Prym map $P_g^{\mathrm{Perf}} : \overline{R}_{g+1} \dashrightarrow \mathcal{A}_g^{\mathrm{Perf}}$ satisfies*

$$\overline{FS}_2 \cup \overline{FS}_3 \subseteq \mathrm{Ind}(P_g^{\mathrm{Perf}}) \subseteq \overline{FS}_2 \cup \overline{FS}_3 \cup \partial\overline{FS}_4 \cup \ldots \cup \partial\overline{FS}_g \tag{1}$$

where $\partial\overline{FS}_n = \overline{FS}_n - FS_n$. Moreover,

$$\mathrm{codim}_{\overline{R}_{g+1}} \mathrm{Ind}(P_g^{\mathrm{Perf}}) \setminus \left(\overline{FS}_2 \cup \overline{FS}_3\right) \geq 6.$$

We are also able to recover the results of Theorem 1 with our methods. Motivated by numerous examples we would like to ask the following

Question 3. *Is it true that* $\mathrm{Ind}(P_g^{\mathrm{Perf}}) = \overline{FS}_2 \cup \overline{FS}_3$*?*

Finally, one can apply the same methods also to the central cone compactification. This results is

Theorem 4 (Dutour Sikirić). *For the extension of the Prym map $P_g^{\mathrm{Ctr}} : \overline{R}_{g+1} \dashrightarrow \mathcal{A}_g^{\mathrm{Ctr}}$ to the central cone compactification the following holds:*

(1) $\overline{FS}_2 \cup \overline{FS}_3 \subseteq \mathrm{Ind}(P_g^{\mathrm{Ctr}})$*, and for $n \geq 4$ the strata FS_n are not contained in* $\mathrm{Ind}(P_g^{\mathrm{Ctr}})$.

(2) *If* $g \geq 9$, *the indeterminacy locus* $\operatorname{Ind}(P_g^{\mathrm{Ctr}})$ *contains points that are not contained in* $\cup_{n\geq 1}\overline{FS}_n$.

The last statement of the theorem should be compared to the result of Alexeev and Brunyate which says that the Torelli map to the central cone compactification has points of indeterminacy if $g \geq 9$. Their examples can be readily adopted to give points of indeterminacy for the Prym map.

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Kähler-Ricci solitons and K-stability

Robert J. Berman

(joint work with David Witt-Nyström)

We report on joint work with David Witt-Nyström [2]. Let X be a an n-dimensional Fano manifold, i.e. its anti-canonical line bundle $-K_X$ is ample. A Kähler metric ω on X is said to be a *Kähler-Ricci soliton* if there exists a holomorphic vector field V of type $(1,0)$ such that

$$\operatorname{Ric}\omega = \omega + L_V\omega, \tag{1}$$

where L_V denotes the Lie derivative along V. In particular, ω is in the first Chern class $c_1(X)$ and invariant under the flow of the imiginary part of V and Kähler-Einstein precisely when $V = 0$. We will denote by T the real torus acting on X whose orbits are the closure of the flow of the imaginart part of V. In case X is a singular Fano variety (with log terminal singularities) the equation 1 is assumed to hold on the regular locus X_{reg} of X and the volume of ω on X_{reg} is assumed to coincide with $c_1(X)^n/n!$ (see [1] for the Kähler-Einstein case).

The pluripotential setup on polarized $T-$varities. Writing $\omega = \omega_\phi := dd^c\phi$, where $e^{-\phi}$ defines an Hermitian metric on $-K_X$ the equation 1 equivalently means that

$$(dd^c\phi)^n e^{f_\phi} = e^{-\phi} i^n dz_1 \wedge d\bar{z}_1 \cdots \wedge dz_n \wedge d\bar{z}_n \tag{2}$$

where f_ϕ is the Hamiltonian function determined by (V, ω_ϕ), which is defined in terms of the first derivatives of ϕ. One of the main aims of the work [2] is to give a weak meaning to the previous equation, i.e. valid for singular metrics ϕ on $-K_X$, and to explore the applications to Kähler-Ricci solitions, notably on singular Fano varities. As it turns out the proper geometric setting is that of a *polarized $T-$variety* (X, L), i.e. T is a real torus of rank $m \leq n$ acting on a polarized variety (X, L). Given a smooth positively curved metric ϕ on L we denote by m_ϕ the *moment map* of the $T-$action on the symplectic manifold (X, ω_ϕ) :

$$m_\phi : \; X \to \mathrm{Lie}(T)^* \cong \mathbb{R}^m, \;\; \langle m_\phi, J\xi \rangle := \frac{d}{dt}_{|t=0} \exp(tJ\xi)^*\phi$$

where the image $P := m_\phi(X)$ is the *moment polytope* of the $T-$action. Now, given a, say smooth, function g on P we define the correspondin*g $g-$Monge-Ampère operator* on the space of smooth $T-$invariant metrics ϕ on L with positive curvature, by

$$MA_g(\phi) := (dd^c\phi)^n g(m_\phi) \tag{3}$$

In particular, this construction reproduces the left hand side in the Kähler-Ricci soliton equation 1 when $g(p) := e^{\langle p, \xi_V \rangle}$, where ξ_V is the element in the Lie algebra of T corresponding to the vector field V.

Theorem 1. *There exists a unique extension of the $g-$Monge-Ampère operator MA_g to the space of all $T-$invariant metrics ϕ on L with positive curvature current with the following properties:*

- *MA_g is continuous under decreasing sequences of bounded metrics*
- *The measure $MA_g(\phi)$ does not charge pluripolar subsets of X*
- *The measure $MA_g(\phi)$ is local with respect to the $T-$plurifine topology on X*

This result generalizes the two classical extreme case when T is trivial and T has maximal rank, respectively. The first case is the pluripotential setting of Bedford-Taylor generalized in [3] while the second case appears when (X, L) is toric, in which case the $g-$Monge-Ampère measure may be identified with Alexandrov's real Monge-Ampère measure, defined in terms of the subgradient image of the convex function on $\mathbb{R}^n$ corresponding to ϕ. However, the construction in the general case requires completely new ideas. In a nut shell, the linearity of $MA_g(\phi)$ wrt g is used to reduce the situation to the case when g is a caracteristic function of a half-space, i.e. $g(p) = 1_{\{p>\lambda\}}(p)$ for a given vector $\lambda \in \mathbb{R}^n$. The point is that in the latter case one can represent

$$MA_g(\phi) = MA(P_\lambda\phi),$$

for a certain projection operator P_λ and thus reduce the definition of MA_g and the study of its properties to the setting of the ordinary (non-polar) complex Monge-Ampère operator MA. The projection operator in question may be defined by the following Legendre transform type formula:

$$P_\lambda\phi := \inf_{t\geq 0}\left(\phi_t - \langle t, \lambda\rangle\right), \quad \phi_t := (e^t)^*\phi \tag{4}$$

Applications to Kähler-Ricci solitons and K-stability. One application of the pluripotential setting above is an extension of the Tian-Zhu uniqueness result [6] to the setting of singular Fano varieties:

Theorem 2. *(X, V) admits at most one Kähler-Ricci soliton, modulo the action of $Aut(X, V)_0$.*

Here $\mathrm{Aut}(X, V)_0$ denotes the connected component of the group of all automorphisms of X commuting with the flow of V, i.e. with the corresponding torus T. By the argument in [4] (concerning the Kähler-Einstein case when $V = 0$) the previous uniqueness result implies a generalization of Matsushima's obstruction:

Corollary 3. *If (X, V) admits a Kähler-Ricci soliton then the group $Aut(X, V)_0$ is reductive.*

To get more and (conjectureally even all) obstructions we propose in [2] a modified form of K-polystability: we say that (X, V) *is K-polystable* if Tian-Zhu's modified Futaki invariant $F(X_0, V_0)$ of any $\mathbb{C}^*-$equivariant deformation $(\mathcal{X}, \mathcal{V}) = \{(X_\tau, V_\tau)\}_{\tau\in\mathbb{C}}$ of (X, V) is non-negative with equality if and only if (X_0, V_0) is isomorphic to (X, V) (see also [7] for an extended definition). The invariant $F(X_0, V_0)$ was originally defined by Tian-Zhu [6] using a metric expression, but it may in general be defined algebraically as

$$F(X_0, V_0) := -\lim_{k\to\infty}\frac{1}{k^n}\sum_{l=1}^{N_k}\exp(\frac{v_l^{(k)}}{k})\frac{w_l^{(k)}}{k},$$

where $(v_l^{(k)}, w_l^{(k)})$ are the joint eigenvalues (weights) for the commuting action of the real parts of the holomorphic vector fields V and the generator of the given $\mathbb{C}^*-$action on $H^0(X_0, -kK_{X_0})$, respectively (using the canonical lifts to $-K_X$).

Theorem 4. *If (X, V) admits a Kähler-Ricci soliton then (X, V) is K-polystable*

The proof uses a generalization of the $g-$Monge-Ampère measure to the family $(\mathcal{X}, \mathcal{V})$. As a corollary one gets examples of Fano manifolds admitting non-trivial holomorphic vector fields, but no Kähler-Ricci solitons (for example, $X = Y \times Z$, where Y is $\mathbb{P}^2$ blown up in one point and Z is Tian's deformation of the Mukai threefold). It seems natural to conjecture that the converse to the previous theorem also holds, thus extending the seminal Yau-Tian-Donaldson (YTD) conjecture to the case when $V \neq 0$. It should be stressed that V in Theorem 4 necessarily coincides with the Tian-Zhu extremal vector field V_{TZ}, which is uniquely determined modulo automorphisms [6]. Accordingly, we will say that a Fano variety X is *K-polystably in the modified sense* if (X, V_{TZ}) is K-polystable.

Outlook on GIT stability and moduli spaces of Fano varities. First recall that in the Kähler-Einstein setting (or more generally thet constant scalar curvature setting) there is, as shown by Donaldson and Fujiki, an infinite dimensional version of Geometric Invariant Theory (GIT) motivating the YTD-conjecture. Briefly, fixing a Kähler form ω_0 on the Fano manifold (X, J_0) the space $\mathcal{J}$ of all integrable complex structures can be naturally viewed as an infinite dimensional Kähler manifold with Kähler metric Ω defined by

$$\Omega(\delta J, \overline{\delta J}) := \int_X \|\delta J\|^2_{\omega_0} \omega_0^n / n!, \tag{5}$$

where δJ is a given tangent vector at $J \in \mathcal{J}$, identified with a smooth section of $T^{1,0}(X, J) \otimes T^{*0,1}(X, J)$. The group G of all symplectic diffeomorphisms of (X, ω_0) acts holomorphically and isometrically on $(\mathcal{J}, \Omega)$. We propose the following generalization of this setup. Fix a torus action T on (X_0, J_0, Ω_0) which is extremal in the sense of Tian-Zhu. Let G_T be the centralizer of T in G and define a two-form Ω_T on the complexified orbit $G_T^c J_0$ in V by replacing ω_0^n in formula 5 by $e^f \omega_0^n$, where f is the Hamiltonian function determined by the $T-$action on (X_0, J_0, Ω).

Theorem 5. *The two-form Ω_T defines a Kähler form on the complexified orbit $G_T^c J_0$ in $\mathcal{J}$ and the G_T-action on $(G_T^c J_0, \Omega_T)$ admits a moment map which may be identified with the modified scalar curvature map of the metric $g_J := \omega_0(\cdot, J\cdot)$ (defined in* [6]*). Moreover, the G_T-equivariant Kähler potential of Ω_T may be identified with the modified Mabuchi K-energy functional and its slope at infinity, along a given one-parameter sub-group with limit point J_∞ in $\mathcal{J}$, coincides with the modified Futaki invariant of (X, J_∞, T).*

Hence the modified form of the YTD-conjecture proposed above corresponds to the GIT stability defined wrt G_T and Ω_T. Let us finally discuss a generalization, to the case $V \neq 0$, of a conjecture of Odaka [5] concerning moduli of K-polystable Fano varieties. Consider the moduli functor of families $\pi : \mathcal{X} \to B$ of Fano varieties with log terminal singularities such that there exists a holomorphic vector field $\mathcal{V}$ on $\mathcal{X}$ which is tangential to the fibers such that (X_t, V_t) is K-semistable.

Conjecture 6. *There exists a proper course algebraic moduli space $\mathcal{M}$ for the moduli functor above parametrizing Fano manifold X which are K-polystable in the modified sense. Moreover, each connected component of $\mathcal{M}$ is Kähler.*

Interestingly, any one-parameter group as in Theorem 5 or more generally any equivariant degeneration $(\mathcal{X}, \mathcal{V})$ gives rise to a family $\pi : \mathcal{X} \to \mathbb{P}^1$ in the moduli functor with a sequence of $\mathbb{R}$-lines bundle $\mathcal{L}_{T,k} \to \mathbb{P}^1$ defined as $\mathcal{L}_{T,k} k^{-(n+1)} \sum_{v_i^{(k)}} e^{v_i^{(k)}/k} \det(\pi_* E_i)$, where E_i is the eigenbundle corresponding to $v_i^{(k)}$. Moreover, the limits of the degrees of $\mathcal{L}_{T,k}$ is equal to $F(X_0, V_0)$. We conjecture that the limit of $\mathcal{L}_{T,k}$ as $k \to \infty$ exists, say as a cohomology class, and defines a Kähler class on $\mathcal{M}$, or perhaps even an ample $\mathbb{R}-$line bundle $\mathcal{L}_T \to \mathcal{M}$ (thus generalizing the CM-line bundle). Moreover, generalizing a conjecture of Odaka [5] it seems natural to conjecture that "filling in" a family over a punctured curve $C^*$with a K-polystable (X_0, V_0) correponds to minimizing $\deg(\mathcal{L}_T)$ over all (possibly non K-semistable) Fano families over C.

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Bounded negativity of Shimura curves

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(joint work with Domingo Toledo)

The bounded negativity conjecture claims that for any smooth projective complex surface X there is a constant $b = b(X) \geq 0$, such that for any integral curve $C \subset X$ the self-intersection number C^2 is bounded below by $-b$.

Over fields of positive characteristic this conjecture is obviously false, since Frobenius pullbacks of any curve with negative self-intersection provide counterexamples. Guided by this example, the first attempt to construct counterexamples was by using complex surfaces with endomorphisms that are not automorphisms. However, it was shown in [2] that this approach does not work, using the classification of surfaces admitting such an endomorphisms. The next attempt was to use correspondences rather than endomorphisms to construct a counterexample. It was shown in [2] that Picard modular surfaces (i.e. quotients of $\mathbb{H}^2$ by an aritheoremetic lattice) do not produce counterexamples either. The proof is rather ad hoc, using the logaritheoremic version of the Bogomolov-Miyaoka-Yau inequalitiy. This method does not apply to the other type of Shimura surfaces, that is, quotients of the 2-ball.

Our main result uniformly treats both cases and shows that Shimura curves on Shimura curves do not violate the bounded negativity conjecture.

Theorem 1 ([1])**.** *For any compact smooth Shimura surface X not isogeneous to a product and for any real number M there are only finitely many Shimura curves C on X with $C^2 < M$.*

The proof relies on Ratner's results on orbits of groups generated by unipotent elements, more precisely a version of equidistribution proved by Eskin-Mozes-Shah. While being beyond of the scope of the original bounded negativity conjecture, the theorem holds, suitably interpreted, as well in the non-compact case and in the presence of torsion elements.

We conclude by the remark that the bounded negativity conjecture is currently not known for any surface of general type with Picard group of rank ≥ 3. The preceding theorem does not prove it for Shimura surfaces either, since there could be non-Shimura curves with arbitrarily large negative self-intersection number. The author is however not aware of a non-Shimura curve with negative self-intersection number on a smooth compact Shimura surface.

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Analytic Serre duality on singular complex spaces

Jean Ruppenthal

Classical Serre duality can be formulated as follows: Let X be a complex n-dimensional manifold, let $V \to X$ be a complex vector bundle, and let $\mathcal{E}^{0,q}(X,V)$ and $\mathcal{E}^{n,q}_c(X,V^*)$ be the spaces of global smooth $(0,q)$-form with values in V and global smooth compactly supported (n,q)-forms with values in the dual bundle V^* respectively. Then the following pairing is non-degenerate

$$(1)\quad H^q\big(\mathcal{E}^{0,\bullet}(X,V),\bar\partial\big) \times H^{n-q}\big(\mathcal{E}^{n,\bullet}_c(X,V^*),\bar\partial\big) \to \mathbb{C}, \quad ([\varphi]_{\bar\partial},[\psi]_{\bar\partial}) \mapsto \int_X \varphi\wedge\psi$$

provided that $H^q\big(\mathcal{E}^{0,\bullet}(X,V),\bar\partial\big)$ and $H^{q+1}\big(\mathcal{E}^{0,\bullet}(X,V),\bar\partial\big)$ are Hausdorff topological spaces (e.g., finite dimensional).

If X is allowed to have singularities, then, traditionally, Serre duality takes a more algebraic and much less explicit form (involving Grothendieck's dualizing sheaf in the Cohen-Macaulay case, or a dualizing complex if the space is not Cohen-Macaulay). Here, we will explain two natural generalizations of the smooth $(0,q)$- and (n,q)-forms to the singular setting so that Serre duality has an analytic realization completely analogous to (1). All the following statements are valid for forms with values in a hermitian vector bundle V and its dual V^*, respectively, but we will omit that for ease of notation.

The first approach is via L^2-theory for the $\overline\partial$-operator on singular spaces. Here, we consider $\overline\partial$-complexes of L^2-forms on the regular part of a hermitian singular complex space X of pure dimension n. In [PS], Pardon and Stern proved exactness of the complex of fine sheaves of L^2-forms (in the domain of $\overline\partial_w$)

$$(2)\qquad 0 \to \mathcal{K}_X \hookrightarrow \mathcal{C}^{n,0} \xrightarrow{\overline\partial_w} \mathcal{C}^{n,1} \xrightarrow{\overline\partial_w} \mathcal{C}^{n,2} \xrightarrow{\overline\partial_w} \dots$$

where $\overline\partial_w$ is the $\overline\partial$-operator in the sense of distributions and $\mathcal{K}_X$ is the Grauert–Riemenschneider canonical sheaf of square-integrable holomorphic n-forms ($\overline\partial_w$

stands for the $\overline{\partial}$-operator in a weak sense). Hence,

$$H^q(X,\mathcal{K}_X) \cong H^q\big(\mathcal{C}^{n,\bullet}(X),\overline{\partial}_w\big), \tag{3}$$

which has interesting consequences for the solvability of the L^2-$\overline{\partial}_w$-equation, particularly if X is for example q-complete or q-convex.

In [R1], we introduced another $\overline{\partial}$-operator acting on L^2-forms on singular complex spaces, the so-called $\overline{\partial}_s$-operator which comes with a certain Dirichlet boundary condition at the singular set ($\overline{\partial}_s$ stands for the $\overline{\partial}$-operator in a 'strong' sense). This operator played a crucial role in the determination of certain L^2-cohomology groups on singular complex spaces and the proof of a conjecture of Pardon and Stern (see [R1] and [OV]).

Let us consider now the $\overline{\partial}$-complex

$$0 \to \mathcal{O}_X \hookrightarrow \mathcal{F}^{0,0} \xrightarrow{\overline{\partial}_s} \mathcal{F}^{0,1} \xrightarrow{\overline{\partial}_s} \mathcal{F}^{0,2} \xrightarrow{\overline{\partial}_s} \ldots \tag{4}$$

of fine sheaves of L^2-forms in the domain of the $\overline{\partial}_s$-operator. Then (4) is not necessarily exact on arbitrary singular complex spaces, but it was shown in [R2], Theorem 1.4, that it is a dualizing Dolbeault complex for $\mathcal{K}_X$ in the sense that there is a non-degenerate topological pairing

$$H^q\big(\mathcal{C}^{n,\bullet}(X),\bar{\partial}_w\big) \times H^{n-q}_{cpt}\big(\mathcal{F}^{0,\bullet}(X),\bar{\partial}_s\big) \to \mathbb{C}, \quad ([\varphi]_{\bar{\partial}_w},[\psi]_{\bar{\partial}_s}) \mapsto \int_X \varphi\wedge\psi \tag{5}$$

provided that $H^q(X,\mathcal{K}_X) \cong H^q\big(\mathcal{C}^{n,\bullet}(X),\bar{\partial}_w\big)$ and $H^{q+1}(X,\mathcal{K}_X) \cong H^{q+1}\big(\mathcal{C}^{n,\bullet}(X),\bar{\partial}_w\big)$ are Hausdorff. Analogously, there is another non-degenerate topological pairing

$$H^q_{cpt}\big(\mathcal{C}^{n,\bullet}(X),\bar{\partial}_w\big) \times H^{n-q}\big(\mathcal{F}^{0,\bullet}(X),\bar{\partial}_s\big) \to \mathbb{C}, \quad ([\varphi]_{\bar{\partial}_w},[\psi]_{\bar{\partial}_s}) \mapsto \int_X \varphi\wedge\psi \tag{6}$$

under the corresponding Hausdorff condition. It is also shown in [R2], Theorem 1.5, that the cohomology spaces

$$H^q\big(\mathcal{C}^{n,\bullet}(X),\bar{\partial}_w\big)\ ,\ H^q_{cpt}\big(\mathcal{C}^{n,\bullet}(X),\bar{\partial}_w\big)$$

are Hausdorff for all $q \geq 0$ if X is holomorphically convex.

Let us explain two interesting applications of (5) and (7). First, consider a complex space X of pure dimension $n \geq 2$ which is cohomologically $(n-1)$-complete (or just $(n-1)$-complete in the sense of Grauert). Then (3) and (6) yield that

$$H^1_{cpt}\big(\mathcal{F}^{0,\bullet}(X),\overline{\partial}_s\big) \cong H^{n-1}\big(\mathcal{C}^{n,\bullet}(X),\overline{\partial}_w\big) \cong H^{n-1}(X,\mathcal{K}_X) = 0, \tag{7}$$

i.e., the $\overline{\partial}_s$-equation is solvable for $(0,1)$-forms with compact support. Using this, one can prove Hartogs' extension theorem by the $\overline{\partial}$-technique of Ehrenpreis in its most general form easily on such a space X (see [R2], Theorem 1.7).

A second very interesting application is as follows. Let $\pi : M \to X$ be a resolution of singularities and $\Omega \subset\subset X$ holomorphically convex. Give M any hermitian metric. Then pullback of L^2-(n,q)-forms under π induces an isomorphism

$$\pi^* : H^q_{cpt}\big(\mathcal{C}^{n,\bullet}(\Omega),\overline{\partial}_w\big) \xrightarrow{\cong} H^q_{cpt}\big(\mathcal{C}^{n,\bullet}(\pi^{-1}(\Omega)),\overline{\partial}_w\big) \cong H^q_{cpt}\big(\pi^{-1}(\Omega),\mathcal{K}_M\big) \tag{8}$$

for all $0 \le q \le n$ by use of Pardon–Stern [PS] and the Takegoshi vanishing theorem. Now we can use the L^2-Serre duality (6) and classical Serre duality on the smooth manifold $\pi^{-1}(\Omega)$ to deduce that push-forward of forms under π induces another isomorphism

$$\pi_* : H^{n-q}\big(\pi^{-1}(\Omega), \mathcal{O}_M\big) \xrightarrow{\cong} H^{n-q}\big(\mathcal{F}^{0,\bullet}(\Omega), \overline{\partial}_s\big) \tag{9}$$

for all $0 \le q \le n$ ([R2], Theorem 1.1). This shows that the obstructions to solving the $\overline{\partial}_s$-equation locally for $(0,q)$-forms can be expressed in terms of a resolution of singularities. For the cohomology sheaves of the complex $(\mathcal{F}^{0,\bullet}, \overline{\partial}_s)$, we see that

$$\big(\mathcal{H}^q\big(\mathcal{F}^{0,\bullet}, \overline{\partial}_s\big)\big)_x \cong (R^q\pi_*\mathcal{O}_M)_x$$

in any point $x \in X$ for all $q \ge 0$, i.e., the functions in the kernel of $\overline{\partial}_s$ are precisely the weakly holomorphic functions and the complex (4) is exact in a point $x \in X$ exactly if x is a rational point (see [R2], Theorem 1.3).

We conclude by mentioning another approach to analytic Serre duality on singular spaces which is based on the so-called $\mathcal{A}_{0,q}$-sheaves introduced by Andersson and Samuelsson in [AS]. These are certain sheaves of $(0,q)$-currents on singular complex spaces which are smooth on the regular part of the variety and such that the $\overline{\partial}$-complex

$$0 \to \mathcal{O}_X \hookrightarrow \mathcal{A}_{0,0} \xrightarrow{\overline{\partial}} \mathcal{A}_{0,1} \xrightarrow{\overline{\partial}} \mathcal{A}_{0,2} \longrightarrow \dots \tag{10}$$

is a fine resolution of the structure sheaf. The $\mathcal{A}$-sheaves are defined via Koppelman integral formulas on singular complex spaces.

Analogously, in [RSW], we introduced a $\overline{\partial}$-complex of fine sheaves of (n,q)-currents (smooth on the regular part of the variety)

$$0 \to \omega_X \hookrightarrow \mathcal{A}_{n,0} \xrightarrow{\overline{\partial}} \mathcal{A}_{n,1} \xrightarrow{\overline{\partial}} \mathcal{A}_{n,2} \longrightarrow \dots \tag{11}$$

where X is of pure dimension n and ω_X denotes the Grothendieck dualizing sheaf. The complex (11) is exact only under some additional assumptions, e.g. if X is Cohen-Macaulay. We call $(\mathcal{A}_{n,\bullet}, \overline{\partial})$ a dualizing Dolbeault complex for $\mathcal{O}_X$ because we obtain a non-degenerate topological pairing

$$H^q\big(\mathcal{A}_{0,\bullet}(X), \bar\partial\big) \times H^{n-q}_{cpt}\big(\mathcal{A}_{n,\bullet}(X), \bar\partial\big) \to \mathbb{C}, \quad ([\varphi]_{\bar\partial}, [\psi]_{\bar\partial}) \mapsto \int_X \varphi \wedge \psi \tag{12}$$

provided that $H^q(X, \mathcal{O}_X) \cong H^q\big(\mathcal{A}_{0,\bullet}(X), \bar\partial\big)$ and $H^{q+1}(X, \mathcal{O}_X) \cong H^{q+1}\big(\mathcal{A}_{0,\bullet}(X), \bar\partial\big)$ are Hausdorff topological spaces.

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Density of positive closed currents and dynamics of Hénon-type automorphisms of $\mathbb{C}^k$

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(joint work with Tien-Cuong)

The first aim is to introduce a new approach in order to get several equidistribution properties in complex dynamics in higher dimension. The strategy that we will describe in the case of Hénon-type automorphisms, requires developments of the theory of positive closed currents which are of independent interest.

Let $f : \mathbb{C}^k \to \mathbb{C}^k$ be a polynomial automorphism. We extend it to a birational self-map of the projective space $\mathbb{P}^k$ that we still denote by f. We assume that f is not an automorphism of $\mathbb{P}^k$; otherwise the associated dynamical system is elementary. We say that f is *regular or of Hénon-type* if the indeterminacy sets I_+ and I_- of f and of its inverse f^{-1} satisfy $I_+ \cap I_- = 0$. In dimension 2, Hénon maps satisfy this property and any dynamically interesting automorphism is conjugated to a Hénon map. Consider a Hénon-type map f as above. There is an integer $1 \leq p \leq k-1$ such that $\dim I_+ = k - p - 1$ and $\dim I_- = p - 1$. Let d_+ (resp. d_-) denote the algebraic degrees of f^+ (resp. of f^-), i.e. the maximal degrees of its components which are polynomials in $\mathbb{C}^k$. We have $d_+^p = d_-^{k-p}$ and we denote this integer by d. On can construct for such a map an invariant measure μ with compact support in $\mathbb{C}^k$ which turns out to be the unique measure of maximal entropy $\log d$.

The measure μ is called *the Green measure or the equilibrium measure of* f. It is obtained as the intersection of the main Green current T_+ of f and the one associated to f^{-1}. The authors have shown that T_+ (resp. T_-) is the unique positive closed (p,p)-current (resp. $(k-p, k-p)$-current) of mass 1 supported by the set $\mathscr{K}_+$ (resp. $\mathscr{K}_-$) of points of bounded orbit (resp. backward orbit) in $\mathbb{C}^k$. They are also unique currents having no mass at infinity which are invariant under $d^{-1}f^*$ (resp. $d^{-1}f_*$). Let P_n denote the set of periodic points of period n of f in $\mathbb{C}^k$ and SP_n the set of saddle periodic points of period n in $\mathbb{C}^k$. We have the following result.

Theorem 1. *Let f, d, μ, P_n and SP_n be as above. Then the saddle periodic points of f are asymptotically equidistributed with respect to μ. More precisely, if Q_n denotes P_n or SP_n we have*

$$d^{-n} \sum_{a \in Q_n} \delta_a \to \mu \qquad \text{as} \quad n \to \infty,$$

where δ_a denotes the Dirac mass at a.

We can replace Q_n with other subsets of SP_n which allow us to precise the nature of typical periodic points. For example, given an $\epsilon > 0$, we can take only periodic points a of period n such that the differential Df^n at a admits p eigenvalues of modulus larger than $(\delta - \epsilon)^{n/2}$ and $k - p$ eigenvalues of modulus smaller than $(\delta - \epsilon)^{-n/2}$ with $\delta := \min(d_+, d_-)$.

In dimension 2, the above Theorem and the uniqueness of the maximal entropy measure were obtained by Bedford-Lyubich-Smillie and the uniqueness of $T_\pm$ were obtained by Fornæss and the second author . In order to obtain the equidistribution of periodic points in dimension 2, Bedford-Lyubich-Smillie proved and used that the Green currents T_+, T_- are laminated by Riemann surfaces whose intersections give the measure μ. In the higher dimensional case, we will use another method which also allows us to obtain as a consequence the laminar property of $T_\pm$ and the product structure of μ. The approach, that we describe below, has some advantages. It permits to show for example that if L_+ and L_- are Zariski generic subvarieties of dimension $k - p$ and p respectively, then the points in $f^{-n}(L_+) \cap f^n(L_-)$ are also equidistributed with respect to μ. One can hope that our approach will allow to estimate the speed of convergence in the above equidistribution results.

Let Δ denote the diagonal of $\mathbb{P}^k \times \mathbb{P}^k$ and Γ_n denote the compactification of the graph of f^n in $\mathbb{P}^k \times \mathbb{P}^k$. The set P_n can be identified with the intersection of Γ_n and Δ in $\mathbb{C}^k \times \mathbb{C}^k$. The dynamical system associated to the map $F := (f, f^{-1})$ on $\mathbb{P}^k \times \mathbb{P}^k$ is similar to the ones associated to Hénon-type maps on $\mathbb{P}^k$. It was used by the first author in in order to obtain the exponential mixing of μ on $\mathbb{C}^k$. Observe that Γ_n is the pull-back of Δ or Γ_1 by $F^{n/2}$ or $F^{(n-1)/2}$. So a property similar to the uniqueness of the main Green currents mentioned above implies that the positive closed (k, k)-current $d^{-n}[\Gamma_n]$ converges to the main Green current of F which is equal to $T_+ \otimes T_-$. Therefore, Theorem 1 is equivalent to

$$\lim_{n\to\infty} [\Delta] \wedge d^{-n}[\Gamma_n] = [\Delta] \wedge \lim_{n\to\infty} d^{-n}[\Gamma_n]$$

on $\mathbb{C}^k \times \mathbb{C}^k$ since $\mu = T_+ \wedge T_-$ can be identified with $[\Delta] \wedge (T_+ \otimes T_-)$. So our result requires the development of a good intersection theory in any dimension.

The typical difficulty here is illustrated in the following example. Consider Δ' the unit disc in $\mathbb{C} \times \{0\} \subset \mathbb{C}^2$ and Γ'_n the graph of the function $x \mapsto x^{d^n}$ over Δ'. The current $d^{-n}[\Gamma'_n]$ converges to a current on the boundary of the unit bidisc in $\mathbb{C}^2$ while their intersection with $[\Delta']$ is the Dirac mass at 0. So we have

$$\lim_{n\to\infty} [\Delta'] \wedge d^{-n}[\Gamma'_n] \neq [\Delta'] \wedge \lim_{n\to\infty} d^{-n}[\Gamma'_n]$$

We see in this example that Γ'_n is tangent to Δ' at 0 with maximal order. We can perturb Γ'_n in order to get manifolds with intersect Δ' transversally but the limit of their intersections with Δ' is still equal to the Dirac mass at 0. In fact, this phenomenon is due to the property that some tangent lines to Γ'_n are too close to tangent lines to Δ'.

It is not difficult to construct a map f such that Γ_n is tangent or almost tangent to Δ at some points for every n. In order to handle the main difficulty in our problem, the strategy is to show that the almost tangencies become negligible when n tends to infinity. This property is translated in our study into the fact that a suitable density for positive closed currents vanishes. We explain now the notion of density of currents in the dynamical setting and then develop the theory in the general setting of positive closed currents.

Let $\mathrm{Gr}_k(\mathbb{P}^k \times \mathbb{P}^k)$ denote the Grassmannian bundle over $\mathbb{P}^k \times \mathbb{P}^k$ where each point corresponds to a pair $(x, [v])$ of a point $x \in \mathbb{P}^k \times \mathbb{P}^k$ and of the direction $[v]$ of a k-vector v in the complex tangent space to $\mathbb{P}^k \times \mathbb{P}^k$ at x. Let $\widehat{\Gamma}_n$ denote the lift of Γ_n to $\mathrm{Gr}_k(\mathbb{P}^k \times \mathbb{P}^k)$, i.e. the set of points $(x, [v])$ with $x \in \Gamma_n$ and v tangent to Γ_n at x. Let $\widetilde{\Delta}$ denote the set of points $(x, [v])$ in $\mathrm{Gr}_k(\mathbb{P}^k \times \mathbb{P}^k)$ with $x \in \Delta$ and v non-transverse to Δ. The intersection $\widehat{\Gamma}_n \cap \widetilde{\Delta}$ corresponds to the non-transverse points of intersection between Γ_n and Δ. Note that $\dim \widehat{\Gamma}_n + \dim \widetilde{\Delta}$ is smaller than the dimension of $\mathrm{Gr}_k(\mathbb{P}^k \times \mathbb{P}^k)$ and in general the intersection of subvarieties of such dimensions are often empty. Analogous construction can be done for the manifolds Γ'_n and Δ' given above.

We show that the current $d^{-n}[\widehat{\Gamma}_n]$ converges to some positive closed current $\widehat{\mathbb{T}}$ which is considered as the lift of $\mathbb{T} := T_+ \otimes T_-$ to $\mathrm{Gr}_k(\mathbb{P}^k \times \mathbb{P}^k)$. Using a theorem due to de Thélin on the hyperbolicity of μ we show that the density between $\widehat{\mathbb{T}}$ and $\widetilde{\Delta}$ vanishes. This property corresponds that almost tangencies is negligible when n goes to infinity. The example with Γ'_n and Δ' illustrates typically the opposite situation.

Consider now the general situation on a Kähler manifold X of dimension k. Assume for simplicity that X is compact. We want to introduce a notion of density between two positive closed currents T_1 and T_2 on X of bidegree (p_1, p_1) and (p_2, p_2) respectively. Consider first the case where T_1 and T_2 are given by integration on submanifolds V_1 and V_2 such that $\dim V_1 + \dim V_2 < k$. For generic submanifolds, we have $V_1 \cap V_2 = \varnothing$. However, in general this intersection may be non-empty and the classical theory of intersection of currents does not give a meaning to this intersection for bi-degree reason.

On the other hand, when V_2 is a point, denoted by a, there is a notion of multiplicity of V_1 at a. More generally, if T_1 is a general positive closed current there is a notion of Lelong number $\nu(T_1, a)$ of T_1 at a which represents the density of T_1 at a. Choose a local holomorphic coordinate system x near a such that $a = 0$ in these coordinates. The Lelong number of T_1 at a is the limit of the normalized mass of T_1 on the ball $\mathbb{B}(0, r)$ of center 0 and of radius r when r tends to 0. More precisely, we have

$$\nu(T_1, a) := \lim_{r \to 0} \frac{\|T_1\|_{\mathbb{B}(0,r)}}{(2\pi)^{k-p_1} r^{2k_1 - 2p_1}}.$$

We can represent the Lelong number in another geometric point of view related to Harvey's results . Let $A_\lambda : \mathbb{C}^k \to \mathbb{C}^k$ be defined by $A_\lambda(x) := \lambda x$ with $\lambda \in \mathbb{C}^*$. When λ goes to infinity, the domain of definition of the current $T_{1,\lambda} := (A_\lambda)_*(T_1)$ converges to $\mathbb{C}^k$. This family of currents is relatively compact and any limit current, for $\lambda \to \infty$, is invariant under the action of $\mathbb{C}^*$, i.e. invariant under

$(A_\lambda)_*$. If S is a limit current, we can extend it to $\mathbb{P}^k$ with zero mass on the hyperplane at infinity. Thus, there is a positive closed current S_∞ on $\mathbb{P}^{k-1}$ such that $S = \pi_\infty^*(S_\infty)$ where we identify the hyperplane at infinity with $\mathbb{P}^{k-1}$ and we denote by $\pi_\infty : \mathbb{P}^k \setminus \{0\} \to \mathbb{P}^{k-1}$ the canonical central projection (we do not consider here the case where T_1 is a measure, i.e. $p_1 = k$). The class of S_∞ (resp. of S) in the de Rham cohomology of $\mathbb{P}^{k-1}$ (resp. of $\mathbb{P}^k$) is equal to $\nu(T_1, a)$ times the class of a linear subspace. So these cohomology classes do not depend on the choice of S. Kiselman showed that in general the current S is not unique . We consider now the situation where T_1 is a general positive closed (p_1, p_1)-current and T_2 is given by a submanifold V_2. For simplicity, we will write T, p, V instead of T_1, p_1, V_2 and denote by l the dimension of V. With respect to the above case, the point a is replaced by the manifold V. We want to define a notion of tangent current to T along V that corresponds to currents S above. Let E denote the normal vector bundle to V in X and $\overline{E}$ its canonical compactification. Denote by $A_\lambda : \overline{E} \to \overline{E}$ the map induced by the multiplication by λ on fibers of E with $\lambda \in \mathbb{C}^*$. We identify V with the zero section of E. The tangent currents to T along V will be positive closed (p, p)-currents on $\overline{E}$ which are V-conic, i.e. invariant under the action of A_λ. The first difficulty is that when V has positive dimension, in general, no neighbourhood of V is X is biholomorphic to a neighbourhood of V in E.

Let τ be a diffeomorphism between a neighbourhood of V in X and a neighbourhood of V in E whose restriction to V is identity. We assume that τ is admissible in the sense that the endomorphism of E induced by the differential of τ is identity. It is not difficult to show that such maps exist. Here is the main result in the first part of this paper.

Theorem 2. *Let $X, V, T, E, \overline{E}, A_\lambda$ and τ be as above. Then the family of currents $T_\lambda := (A_\lambda)_* \tau_*(T)$ is relatively compact and any limit current, for $\lambda \to \infty$, is a positive closed (p, p)-current on E which extends by 0 to a positive closed (p, p)-current on $\overline{E}$. Moreover, if S is such a current, it is V-conic, i.e. invariant under $(A_\lambda)_*$, and its de Rham cohomology class in $H^{2p}(\overline{E}, \mathbb{C})$ does not depend on the choice of τ and of S.*

We say that S is a tangent current to T along V. Its class in the de Rham cohomology group is the total tangent class to T along V. Note that this notion generalizes a notion of tangent cone in the algebraic setting where T is also given by a manifold, see Fulton for details. The key point in the dynamical setting considered above is that the tangent currents to $\widehat{\mathbb{T}}$ along $\widetilde{\Delta}$ vanish.

The cohomology ring of $\overline{E}$ is generated by the cohomology ring of V and the tautological $(1, 1)$-class on $\overline{E}$. Therefore, we can decompose the class of S and associate it to classes of different degrees on V. These classes represent different parts of the tangent to T along V.

Consider now arbitrary positive closed currents T_1, T_2 on X and the tensor product $T_1 \otimes T_2$ on $X \times X$. Let Δ denote the diagonal of $X \times X$. We can consider the tangent currents and the total tangent class to $T_1 \otimes T_2$ along Δ. The normal vector bundle to Δ is canonically isomorphic to the tangent bundle of X if we identify Δ with X. The tangent currents and the total tangent class in this case induce the density currents and the total class of density associated to T_1 and T_2.

Assume that $p_1 + p_2 \leq k$ and there is a only one tangent current S to $T_1 \otimes T_2$ along Δ. Assume also that for $j > k - p_1 - p_2$, the current S vanishes on the pull-back of (j, j)-forms by the canonical projection onto X. Then we show that S is the pull-back of a unique positive closed current S^h of bidegree (p_1+p_2, p_1+p_2) on X. In this case, we call S^h the wedge-product of T_1 and T_2 and denote it by $T_1 \curlywedge T_2$. The notion can be extended to a finite number of currents. So the density of currents extends the theory of intersection.

Economical toroidal resolutions

James M$^{\text{c}}$Kernan
(joint work with Paolo Cascini)

"Toric varieties are everywhere dense."

We start with a little bit of background to motivate the main conjecture. In what follows, ACC stands for the ascending chain condition and DCC stands for the descending chain condition.

Conjecture 1. *Fix a positive integer n, and two subsets $I \subset [0,1]$ and $J \subset [0,\infty)$ satisfying the DCC.*

Then there are two finite sets I_0 and J_0 with the following property:

Let (X, Δ) be a log canonical pair, where X has dimension n, the coefficients of Δ belong to I and the log discrepancy of (X, Δ) belongs to J.

Then the coefficients of Δ belong to I_0 and the log discrepancy of (X, Δ) belongs to J_0.

This conjecture is an amalgam of two separate conjectures due to Shokurov:

Conjecture 2 (ACC for the log discrepancy)**.** *Fix a positive integer n, and two subsets $I \subset [0,1]$ and $J \subset [0,\infty)$ satisfying the DCC.*

Then there is a finite set J_0 with the following property:

Let (X, Δ) be a log canonical pair, where X has dimension n, the coefficients of Δ belong to I and the log discrepancy of (X, Δ) belongs to J.

Then the log discrepancy of (X, Δ) belongs to J_0.

Conjecture 3 (ACC for the a-threshold)**.** *Fix a positive integer n, a subset $I \subset [0,1]$ satisfying the DCC and a real number a.*

Then there is a finite set I_0 with the following property:

Let (X, Δ) be a log canonical pair, where X has dimension n, the coefficients of Δ belong to I and the log discrepancy of (X, Δ) is a.

Then the coefficients of Δ belong to I_0.

We know some special cases of these conjectures. Conjecture 3 was recently proved in the special case when $a = 0$:

Theorem 4 (ACC for the log canonical threshold [3])**.** *Fix a positive integer n and a subset $I \subset [0,1]$ satisfying the DCC.*

Then there is a finite set I_0 with the following property:

Let (X,Δ) be a log canonical pair, where X has dimension n, the coefficients of Δ belong to I and the log discrepancy of (X,Δ) is zero.

Then the coefficients of Δ belong to I_0.

We also know some special cases of Conjecture 2:

Theorem 5 (Borisov [2], Lawrence [5])**.** *Conjecture 2 holds if (X,Δ) is toric.*

Theorem 6 (Alexeev [1])**.** *Conjecture 2 holds if X is a surface.*

Finally we know a version of Conjecture 1 for threefolds:

Theorem 7 (Kawakita [4])**.** *Conjecture 1 holds if X is a threefold and $J \subset [1,\infty)$.*

Recall one further conjecture due to Ambro and Shokurov:

Conjecture 8 (Semi-continuity of the log discrepancy [6])**.** *Let (X,Δ) be a log pair.*

The minimal log discrepancy of (X,Δ) is lower semi-continuous.

Theorem 9 ([6])**.** *Assume Conjecture 2 and Conjecture 8.*

If (X,Δ) is a kawamata log terminal pair, where X is a $\mathbb{Q}$-factorial projective variety, then any sequence of flips terminates.

Note that the only version of Conjecture 2 which is known in all dimensions is for toric pairs, cf. Theorem 5. One approach to prove Conjecture 2 would be to try to reduce to this case.

Conjecture 10. *Fix a positive integer n.*

Then there is a positive integer s with the following property:

If (X,Δ) is any log canonical pair, where X has dimension n, then there is projective birational morphism $Y \to X$ such that (Y,Γ) is toroidal, in a neighbourhood of the generic point of every non kawamata log terminal centre of (Y,Γ), where $\Gamma = \tilde{\Delta} + E$ is the sum of the strict transform $\tilde{\Delta}$ of Δ and all of the exceptionals. Moreover there are at most s exceptional divisors and every exceptional divisor has log discrepancy zero.

Note that the economical part of the conjecture is the statement that there are at most s exceptional divisors. Note also that the condition that (Y,Γ) is toroidal at the generic point of every non kawamata log terminal centre is equivalent to the condition that π is a qdlt modification, that is, (Y,Γ) has quotient divisorially log terminal singularities.

It is interesting to consider what happens for surfaces. Alexeev and Kawamata completely classified kawamata log terminal surface singularities. By considering the canonical cover, Kawamata observed that a surface has kawamata log terminal singularities if and only if it has quotient singularities. There are five types of possible quotients, quotient by a cyclic group, quotient by a dihedral group and quotient by one of three exceptional groups. Alexeev derived Kawamata's classification by direct calculation of the resolution graph of the minimal resolution. Every exceptional curve is a copy of $\mathbb{P}^1$. The cyclic case corresponds to a chain, the dihedral case to a chain with two vertices added at one end, corresponding to two -2–curves. The exceptional cases all have one vertex of degree three and they have at most eight vertices.

In terms of Conjecture 10 note that the cyclic case is toric. We may take π to be the identity and so $s = 0$ in this case. In the dihedral case, there is only one vertex of degree three. This corresponds to a curve on the minimal resolution. If we let $\pi : T \to S$ be the birational morphism we get by contracting every curve on the minimal resolution apart from this curve, then π has one exceptional divisor E and (T, E) is toroidal, since there are three chain singularities along E. Thus $s = 1$ works in the dihedral case. There is a similar analysis for the exceptional singularities. Thus $s = 1$ works for surfaces.

We sketch part of a possible argument for threefolds. We work locally about $x \in X$. If x is not a non kawamata log terminal centre we are done by induction on the dimension. For simplicity, we assume that X is $\mathbb{Q}$-factorial.

Write $\Delta = S + B$ where $\lfloor \Delta \rfloor = S$ and $\{\Delta\} = B$. Let m be the number of components of S. Then $m \leq 3$ and if $m = 3$ then $B = 0$ and (X, Δ) is toric. Our aim is to reduce to this case. We proceed by induction on m. At each step we try to increase m by going to a higher model. In fact there are at also most three curves which are non kawamata log terminal centres and so it suffices to increase the number of non kawamata log terminal centres containing x. We sketch the first two steps of the argument and part of the third step.

If $m = 0$ then we may construct a birational projective morphism $\pi : Y \to X$ with one exceptional divisor E of log discrepancy zero such that $\pi(E) = x$. Since X is $\mathbb{Q}$-factorial, E has Picard number one. It follows that there are at most three points on E which are non kawamata log terminal centres for $(Y, \Gamma = \tilde{\Delta} + E)$. Thus we may divide and conquer and treat each point separately.

Replacing (X, Δ) by (Y, Γ) we may assume that $m \geq 1$. This completes the first step. Suppose that $m = 1$. For simplicity we assume that $B = 0$. We reduce to the case when there is a non kawamata log terminal centre which is a curve passing through x. As before we may construct a birational projective morphism $\pi : Y \to X$ with one exceptional divisor E of log discrepancy zero such that $\pi(E) = x$. Let T be the strict transform of S. If y is a non kawamata log terminal centre which belongs to T then it belongs to two components of coefficient one of Γ, E and T. Thus we may assume that there is a non kawamata log terminal centre y on E not on T. By connectedness there is a curve D on E, a non kawamata log terminal centre, connecting the non kawamata log terminal centre y to the non kawamata log terminal centre $T \cap E$.

Replacing (X, Δ) by (Y, Γ) we may assume that $m = 1$ and that there is a non kawamata log terminal centre which is a curve C. This completes the second step. We reduce to the case when S is not normal along C. Cutting by a hyperplane, and using the classification of kawamata log terminal surfaces, we know that if S

is normal along C then the index of $K_X + S$ is two along C, that is, $K_X + S$ is not Cartier but $2(K_X + S)$ is Cartier in a neighbourhood of the generic point of C. It follows that there is a $\mathbb{Z}_2$-cover of $X' \to X$ such that the inverse image of S is not normal along the inverse image of C.

If we can find a birational morphism $Y' \to X'$ such that every non kawamata log terminal centre which is a point is contained in three components of coefficient one, then we can take the $\mathbb{Z}_2$-quotient to get a birational morphism $Y \to X$ with $m \geq 2$.

Replacing (X, Δ) by (X', Δ') we may assume that either $m = 2$ or S is not normal along C. The rest of the argument proceeds in a similar way.

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Uniruled divisors and Chow groups on projective hyperKähler manifolds

Gianluca Pacienza

(joint work with Francois Charles)

I report on a joint on-going project with Francois Charles, in which we study families of rational curves on projective hyperKähler manifolds with applications to the study of their Chow groups of 0-cycles. In particular we prove that a projective hyperKähler manifold X of $K3^{[n]}$-type contains uniruled divisors and that the subgroup that those uniruled divisors determine inside $CH_0(X)$ is always the same.

1. Introduction

The starting point of our project are two “classical” results on complex projective $K3$ surfaces.

Theorem 1 (Bogomolov-Mumford, Mori-Mukai, cf. [MoMu])**.** *Any ample linear system on a complex projective $K3$ surface contains a divisor whose components are rational curves.*

The above result has been used by Beauville and Voisin to obtain, in a rather easy way, a surprising property of Chow groups of complex projective $K3$ surfaces.

Theorem 2 (Beauville-Voisin, cf. [BV])**.** *Let S be a complex projective $K3$ surface.*

(i) *Any point on any rational curve on S determines the same class c_S in $CH_0(S)$;*

(ii) *The image of the intersection product*

$$Pic(S) \otimes Pic(S) \to CH_0(S), \quad L_1 \otimes L_2 \mapsto L_1 \cdot L_2$$

lies in $\mathbb{Z} \cdot c_S$.

The Beauville-Voisin result is somehow unexpected, since, by Mumford's theorem on 0-cycles on surfaces with $p_g \neq 0$ (cf. [V1]), we know that $CH_0(S)$ is "huge", in the sense that it cannot be parametrized in any reasonable way by any quasi-projective variety of any dimension. Together with a third result (saying that $c_2(S) = 24c_S$) it led the authors to a conjectural generalization which we will discuss in further details below.

I will report here on a joint on-going project with François Charles, in which we investigate to which extent the above theorems can be generalized to the higher-dimensional setting. Some of our results are contained in the preprint [CP].

Before stating our results, it is useful to recall the ideas underlying the proofs of the above theorems, with the hope of clarifying the approach we will follow. As for the existence of an ample divisor which is sum of rational curves, the proof relies on the one hand on the fact that "special" $K3$ surfaces (namely Kummer surfaces) are known to have such a property, and, on the other hand, on some deformation theory allowing to propagate that property from a special point in the moduli space of (polarized) $K3$ surfaces to a general one. Concerning the Chow groups, Beauville and Voisin observe that if two rational curves on a $K3$ do not intersect (otherwise we are done), then they can be (rationally) connected using the ample divisor which is sum of rational curves given by Theorem 1. The second item of Theorem 2 follows from the first, by noticing that, again thanks to Theorem 1, the Picard group of a projective $K3$ surface is generated by classes of rational curves.

Our main results are the following.

Theorem 3. *Let X be a projective hyperKähler manifold which is deformation equivalent to $(K3)^{[n]}$. Any ample linear system on X contains a divisor whose components are uniruled.*

Theorem 4. *Let X be a projective hyperKähler manifold which is deformation equivalent to $(K3)^{[n]}$.*

(i) *Any two uniruled divisors D_1, D_2 on X determine the same subgroup in $CH_0(X)_{\mathbb{Q}}$, i.e.*

$$Im\big(CH_0(D_1)_{\mathbb{Q}} \to CH_0(X)_{\mathbb{Q}}\big) = Im\big(CH_0(D_2)_{\mathbb{Q}} \to CH_0(X)_{\mathbb{Q}}\big) =: S^1CH_0(X).$$

(ii) *The image of the intersection product*

$$Pic(S) \otimes CH_1(X)_{\mathbb{Q}} \to CH_0(X)_{\mathbb{Q}}, \quad L \otimes Z \mapsto L \cdot Z$$

lies in $S^1CH_0(X)$.

It is to be noted that hyperKähler manifolds are not hyperbolic. This has been recently proved by Verbitsky, cf. [Ver], using among other things his global Torelli theorem [Ver09]. Nevertheless, no general results on the existence of rational curves on hyperKähler manifolds seemed to be known before our work.

The proof of Theorem 3 goes as follows. First, using Markman's theory of monodromy invariants, cf. [Mar] for a survey, one proves that any primitively polarized hyperKähler manifold (X, h), with $X \sim_{def} (K3)^{[n]}$ can be deformed *together with its polarization* to $(S^{[n]}, L_i)$, where S is a $K3$ surface and $L_1, \ldots, L_n$ are explicit classes in $\mathrm{Pic}(S^{[n]})$. The next step is to show, using classical Brill-Noether theory, that all the linear systems $|L_i|$ contain uniruled divisors. The final ingredient is provided by a geometric deformation-theoretic criterion, for which we need to introduce some notation. Let $\pi : \mathcal{X} \to B$ be a smooth projective morphism of projective hyperKähler manifolds of relative dimension $2n$, and let α be a global section of the local system $R^{4n-2}\pi_*\mathbb{Z}$. Let b_0 be a point of B such that $\mathcal{X}_{b_0} = X$.

Proposition 5. *Let $f : \mathbb{P}^1 \to X$ a non-constant map such that $\alpha_{b_0} = f_*[\mathbb{P}^1]$. Let $\mathcal{M}$ be an irreducible component of the Kontsevich moduli stack of genus zero stable curves $\overline{\mathcal{M}_0}(X, f_*[\mathbb{P}^1])$ containing $[f]$. Let Y be the subscheme of X covered by the deformations of f parametrized by $\mathcal{M}$. If Y is a divisor in X, then:*

(1) *the map $f : \mathbb{P}^1 \to X$ deforms over a finite cover of B;*
(2) *for any point b of B, the fiber $\mathcal{X}_b$ contains a uniruled divisor.*

As for Theorem 4 the key role is played by the Beauville-Bogomolov quadratic form q_X (cf. [B1]). If D_1 and D_2 are two uniruled divisors such that $q_X(D_1, D_2) \neq 0$, then one has that $\Sigma := D_1 \cap D_2$ is not empty and, for $i = 1,\ 2$ we have

$$Im\big(CH_0(D_i)_{\mathbb{Q}} \to CH_0(X)_{\mathbb{Q}}\big) = Im\big(CH_0(\Sigma)_{\mathbb{Q}} \to CH_0(X)_{\mathbb{Q}}\big) =: S^1CH_0(X).$$

If $q_X(D_1, D_2) = 0$, then one uses an primitive ample uniruled divisor D given by Theorem 3. Hence $q_X(D, D_i) \neq 0$ and the previous case allows to conclude. The proof of item (ii) of Theorem 4 uses similar arguments.

So far we have seen a rational curve on a K3 surface as a uniruled divisor on a projective hyperKähler manifold. It can of course equally be regarded, in higher dimension, as a Lagrangian subvariety of maximal dimension which is rationally connected. This leads to the following questions.

Question 6. *Let X be a projective hyperKähler manifold of dimension $2n$.*

(i) *Does there exist a rationally connected subscheme Y of X of pure dimension n?*
(ii) *Do any two points on any two rationally connected n-dimensional subschemes of X determine the same class $c_X \in CH_0(X)$?*

We can answer positively to the first question in dimension 4:

Theorem 7. *Let X be a projective holomorphic symplectic fourfold of $K3^{[2]}$-type. Then X contains a rational Lagrangian surface.*

To put the item (ii) of the question into perspective recall that, inspired by their result, Beauville has stated in [B2] a conjecture, further refined by Voisin in [V2], saying that for a projective hyperKähler manifold X the natural cycle map, from cohomology to the Chow group, is injective when restricted to the subalgebra generated by the 1st Chern classes of line bundles and the Chern classes of X. This conjecture predicts in particular the existence of a canonical 0-cycle $c_X \in CH_0(X)$ of degree 1 such that

$$L_1 \cdot \ldots \cdot L_r \cdot c_2(X)^{e_2} \cdot \ldots \cdot c_{2n}(X)^{e_{2n}} = \deg(L_1 \cdot \ldots \cdot L_r \cdot c_2(X)^{e_2} \cdot \ldots \cdot c_{2n}(X)^{e_{2n}}) \cdot c_X,$$

for all $L_i \in \mathrm{Pic}(X)$ and r, e_{2i} such that $r + \sum 2i \cdot e_{2i} = 2n$. Our approach tries to realize geometrically such a 0-cycle as a point on a half-dimensional rationally connected subvariety of X.

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Representing Analytic Cohomology Groups of Complex Manifolds

László Lempert

Consider a holomorphic vector bundle $L \to X$. Its cohomology groups $H^q(X, L)$ are often represented in terms of open covers $\mathfrak{U} = \{U_a : a \in A\}$ of X and the associated Čech complex $C^\bullet(\mathfrak{U}, L)$, whose elements are collections $(f_{a_0 \ldots a_q})_{a_j \in A}$, with each $f_{a_0 \dots a_q} \in \Gamma(\bigcap_{j=1}^q U_{a_j}, L)$ a holomorphic section of L. If each U_a is Stein, by Cartan's Theorem B and by Leray's theorem $H^q(X, L) \approx H^q(C^\bullet(\mathfrak{U}, L))$.

The notion of Čech cochains $(f_{a_0 \dots a_q})$ is very natural if the cover $\mathfrak{U}$ is indexed by a set A without any structure. However, as Gindikin noted in the 1990s, if A has some structure, then it makes sense to consider cochains that, in their dependence on a_j, reflect this structure. For example, if A is a differential or complex manifold, or a measure space, one can work with the subspaces $C^\bullet_{\text{smooth}}(\mathfrak{U}, L)$, $C^\bullet_{\text{hol}}(\mathfrak{U}, L)$, or $C^\bullet_{\text{meas}}(\mathfrak{U}, L)$ of cochains $(f_{a_0 \dots a_q})$ that depend smoothly, holomorphically, or measurably on $a_0, \dots, a_q$. The main result described in my talk was that under a certain condition the holomorphic Čech complex $C^\bullet_{\text{hol}}(\mathfrak{U}, L)$ and $C^\bullet(\mathfrak{U}, L)$ have isomorphic cohomology groups:

Theorem 1. *Let X, A be complex manifolds, $L \to X$ a holomorphic vector bundle, and $\mathfrak{U} = \{U_a : a \in A\}$ an open cover of X. If the graph*

$$S = \{(x, a) \in X \times A : x \in U_a\} \subset X \times A$$

of the cover is a Stein open subset, then inclusion $C^\bullet_{hol}(\mathfrak{U}, L) \subset C^\bullet(\mathfrak{U}, L)$ induces an isomorphism of cohomology groups.

Covers parametrized by complex manifolds occur in many situations. A natural Stein cover of projective space $\mathbb{P}$ is by complements of hyperplanes. This cover is parametrized by the hyperplanes, i.e., by points of the dual projective space $\mathbb{P}^*$. By restriction, we also obtain a Stein cover of any projective manifold $X \subset \mathbb{P}$, parametrized by $\mathbb{P}^*$. These covers satisfy the assumptions of Theorem 1.1.

The theorem is related to a theorem of Eastwood, Gindikin, and H. Wong, who additionally assume that the sets $\{a \in A : x \in U_a\}$ are contractible. Their conclusion is that $H^q(\mathfrak{U}, L)$, or $H^q(X, L)$, is isomorphic to a certain relative holomorphic De Rham cohomology group.

So far I have been vague about the sort of complex manifolds and vector bundles covered by Theorem 1.1. In fact, while the theorem is new even for finite dimensional L, it holds for a large class of Banach manifolds X, A and Banach bundles $L \to X$; and the isomorphism in the theorem is that of topological vector spaces. Establishing a very special case of this theorem was the first step, in joint work with N. Zhang, of the computation of the first cohomology group of various loop spaces $L\mathbb{P}_1$ of the Riemann sphere (in guise of the Dolbeault group $H^{0,1}(L\mathbb{P}_1)$).

The idea of the proof of Theorem 1.1 is, perhaps predictably, to embed the two complexes $C^\bullet_{\text{hol}}(\mathfrak{U}, L)$ and $C^\bullet(\mathfrak{U}, L)$ in a double complex, and show that all rows and columns of this double complex, except for the two above, are exact.

In the talk I mentioned an application to holomorphic group actions. Suppose a complex Lie group G acts holomorphically on the vector bundle $L \to X$. When X and L are finite dimensional, it is known that the induced action of G on the cohomology groups $H^q(X, L)$ is holomorphic, and the same is expected in infinite dimensions. However, the import of such a statement is dubious, even in finite dimensions. The point is that to talk about holomorphy the topology of cohomology groups must be brought in. This topology in general will not be Hausdorff, and holomorphic (or even differentiable) maps between non–Hausdorff spaces are strange creatures, for example their derivatives are not uniquely determined. Two holomorphic functions may have the same derivatives everywhere, but the difference of the two functions may not be constant.

For this reason one is lead to consider the induced action of G on cochains, rather than cohomology classes. What one gains by this is that the locally convex topological vector spaces $C^q(\mathfrak{U}, L)$ of cochains are Hausdorff (and sequentially compact). One needs to be careful, though: G will not act on $C^q(\mathfrak{U}, L)$ unless $\mathfrak{U}$ is G–invariant, and even if $\mathfrak{U}$ is G–invariant, the action is not going to be holomorphic or even continuous. But, if instead of $C^q(\mathfrak{U}, L)$ one works with $C^q_{\text{hol}}(\mathfrak{U}, L)$, under natural conditions the G–action on this latter will be holomorphic. As a result, for the action of a complex reductive group G one can prove an isotypical decomposition theorem for $C^q_{\text{hol}}(\mathfrak{U}, L)$ and, ultimately, for $H^q(X, L)$.

Application of Pluricanonical Periods to Problem of Schottky-Jung and Differential Equations

Yum-Tong Siu

The motivation is to understand the phenomenon of the deformational invariance of plurigenera in terms of some Hodge decomposition in the pluricanonical setting. The m-genus $\dim_{\mathbb{C}} H^0(X, mK_X)$ of a compact Kähler manifold X is conjectured to remain unchanged when X is holomorphically deformed. The conjecture was proved for the case of projective algebraic X by Siu in 2002. For $m = 1$ such a deformational invariance of $\dim_{\mathbb{C}} H^0(X, K_X)$ for a compact Kähler manifold X is just a direct consequence of the Hodge decomposition. The question is whether the deformational invariance of m-genus for $m \geq 2$ can also be understood in the context of some form of Hodge decomposition with $H^0(X, mK_X)$ as a summand. We discuss in this talk the results and the developments in the study of this problem by starting with the simplest case of compact Riemann surfaces and the work of Bol in 1949, Eichler in 1957, Shimura in 1959, and Gunning in 1960.

For a compact Riemann surface X of genus $g \geq 2$, the global coordinate z of the open unit 1-disk Δ as its universal cover can serve as local coordinates of X, giving X a projective structure in the sense that the coordinate transformations are Möbius transformations. Differentiating $(2m-1)$-times an element of $\mathcal{O}_X((1-m)K_X)$ with respect to the global coordinate z of Δ yields an element of $\mathcal{O}_X(mK_X)$. The exact sequence

$$0 \to \operatorname{Ker} d^{2m-1} \to \mathcal{O}_X((1-m)K_X) \xrightarrow{d^{2m-1}} \mathcal{O}_X(mK_X) \to 0$$

yields the exact sequence

$$(*) \qquad 0 \to H^0\left(X, \mathcal{O}_X(mK_X)\right) \xrightarrow{\Theta_m} H^1\left(X, \operatorname{Ker} d^{2m-1}\right) \to H^1\left(X, \mathcal{O}_X((1-m)K_X)\right) \to 0,$$

where $\operatorname{Ker} d^{2m-1}$, consisting of $P(z)(dz)^{1-m}$ with $P(z)$ being a polynomial of degree $\leq 2m-2$, is a flat $\mathbb{C}$-vector bundle over M of rank $2m-1$ when the coefficients of $P(z)$ are used as fiber coordinates. The $(2m-1)$-vector with components $z^k(dz)^{1-m}$ for $0 \leq k \leq 2m-2$ defines a holomorphic section σ_{m-1} of $\left(\operatorname{Ker} d^{2m-1}\right)^* \otimes ((1-m)K_X)$ over X, where $\left(\operatorname{Ker} d^{2m-1}\right)^*$ is the dual bundle of $\operatorname{Ker} d^{2m-1}$. Since $H^1\left(X, \mathcal{O}_X((1-m)K_X)\right)$ is dual to $H^0\left(X, \mathcal{O}_X(mK_X)\right)$, from $(*)$ we have the *pluricanonical Hodge decomposition*

$$(\dagger) \qquad \begin{aligned} H^1\left(X, \operatorname{Ker} d^{2m-1}\right) &= H^0\left(X, \mathcal{O}_X(mK_X)\right) \oplus H^1\left(X, \mathcal{O}_X((1-m)K_X)\right) \\ &\approx H^0\left(X, \mathcal{O}_X(mK_X)\right) \oplus \overline{H^0\left(X, \mathcal{O}_X(mK_X)\right)}, \end{aligned}$$

which becomes the usual Hodge decomposition when $m=1$ with $\operatorname{Ker} d^{2m-1}$ reduced to the trivial $\mathbb{C}$-line bundle. The map Θ_m is the *m-canonical period map*. The transpose $\Xi_m : H^0\left(X, \mathcal{O}_X(mK_X)\right) \to H^1\left(X, \left(\operatorname{Ker} d^{2m-1}\right)^*\right)$ of the surjective map of $(*)$ is the *dual m-canonical period map* and can be described by the integration of the $\left(\operatorname{Ker} d^{2m-1}\right)^*$-valued 1-form $\sigma_{m-1} f$ over the loops of X for $f \in H^0\left(X, \mathcal{O}_X(mK_X)\right)$. From the second line of $(\dagger)$ the map

$$H^0\left(X, \mathcal{O}_X(mK_X)\right) \times H^0\left(X, \mathcal{O}_X(m'K_X)\right) \to H^0\left(X, \mathcal{O}_X((m+m')K_X)\right)$$

given by mulitplication yields a multiplication formula which produces $(m+m')$-canonical periods from m-canonical periods and m'-canonical periods. This multiplication depends algebraically on X as X varies in the moduli space of all compact Riemann surfaces of genus $g \geq 2$. Since the m-canonical periods satisfy a Riemann relation in the same way as the usual periods (where $m=1$), the multiplication formula applied m times to the usual periods can be applied to provide new algebraic relations for the usual periods in the Schottky-Jung problem. At this point explicit expressions for the multiplication formula are not yet known.

Schwarz in 1873 used (equivariant) periods of certain compact Riemann surfaces as integral representations for solutions of the Gauss hypergeometric differential equation. Another application of the m-canonical Hodge decomposition is to analogously use (equivariant) m-canonical periods as some generalized form of integral representations for solutions of a wider class of differential equations.

When X is replaced by an n-dimensional compact complex manifold *with projective structure*, we can replace $\operatorname{Ker} d^{2m-1}$ by a flat bundle $\mathcal{F}_{m-1}$ consisting of $P(z_1, \cdots, z_n)\left(dz_1 \wedge \cdots \wedge dz_n\right)^{1-m}$ with $P(z_1, \cdots, z_n)$ being a polynomial of degree $\leq (n+1)(m-1)$ in the local coordinates $z_1, \cdots, z_n$ of the projective structure of X. The analogue of the dual m-canonical period map Ξ_{m-1} can be defined, but no analogue of the m-canonical Hodge decomposition $(*)$ and $(\dagger)$ is known. The decomposition

$$H^n\left(X, \mathcal{F}^*_{m-1}\right) = \bigoplus_{p+q=n} H^q\left(X, \mathcal{O}_X\left(\mathcal{F}^*_{m-1} \otimes \wedge^p T^*_X\right)\right)$$

does not hold even in the case of $m \geq 2$ and $n=1$, because for any nonzero element $f \in H^0\left(X, \mathcal{O}_X\left((m-1)K_X\right)\right)$ its exterior differential $d\left(f\sigma_{m-1}\right)$ represents a nonzero element of $H^0\left(X, \mathcal{O}_X\left(\mathcal{F}^*_{m-1}\right) \otimes K_X\right)$ which is mapped to 0 in $H^1\left(X, \mathcal{F}^*_{m-1}\right)$. The reason for this phenomenon is that the flat bundle $\mathcal{F}^*_{m-1}$ is not *unitarily flat* in the sense that it cannot carry a positive definite Hermitian metric which is flat in the flat structure of $\mathcal{F}^*_{m-1}$.

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Deformation of twisted harmonic mappings and the Morse function

Marco Spinaci

Twisted harmonic mappings have proved to be a very useful tool in the study of global aspects of Kaehler manifolds. They represent some "preferred" metrics on flat vector bundles, and, as such, are used to study the moduli space of representations of the fundamental group. In this talk we consider the problem of computing the deformations of such mappings with respect to a variation of the representation, and discuss some consequences to the study of the Morse function on the aforementioned moduli space.

In the following, (X, g) will denote a Riemannian manifold, $\Gamma = \pi_1(X, x_0)$ its fundamental group, $\tilde{X} \to X$ its universal cover, $G < \mathrm{GL}(r, \mathbb{C})$ is a linear group and $\rho\colon \Gamma \to G$ a representation. A twisted (or: ρ-equivariant) map is an $h\colon \tilde{X} \to Y \cong G/K$, where $K < G$ is a maximal compact subgroup, such that $h(\gamma\tilde{x}) = \rho(\gamma)h(\tilde{x})$. Such a map is harmonic if it minimizes $\mathrm{Energy(h)} = \frac{1}{2}\int_X \|\mathrm{d}h'\|^2 \mathrm{dVol}_X$ among all ρ-equivariant maps h'. Remark that ρ-equivariant maps are metrics on $E(\rho) = \tilde{X} \times_\Gamma \mathbb{C}^r$, so the harmonic ones are some "preferred" metrics. By Donaldson–Corlette's theorem [4], such harmonic maps exist if and only if ρ is semisimple. They are unique up to multiplication by an element of the centralizer $Z_G(\rho(\gamma))$.

The representations of the fundamental group admit a moduli space, the G-character variety, which is the GIT quotient $\mathcal{M} = \mathrm{Hom}(\Gamma, G)//G$, which explicitly is the geometric quotient of the subset of semisimple representations by G (acting by conjugation). On this moduli space, the energy functional gives a “Morse–Bott” function (on the smooth points) $f\colon \mathcal{M} \to \mathbb{R}$, by associating to ρ the energy of any harmonic ρ-equivariant map. Our analysis of the derivatives of a harmonic ρ-equivariant h with respect to a variation in ρ will give informations on the derivatives of f.

A first order deformation of a representation $\rho\colon \Gamma \to G$ is a 1-cocycle $c\colon \Gamma \to \mathfrak{g} = \mathrm{Lie}(G)$, i.e. $c(\gamma\eta) = c(\gamma) + \mathrm{Ad}_{\rho(\gamma)} c(\eta)$. If we have a smooth family $\rho_t\colon \Gamma \to G$, then $c(\gamma) = \frac{\partial \rho_t(\gamma)}{\partial t}\big|_{t=0} \rho_0(\gamma)^{-1}$. By Hodge theory, fixing a metric h, to its cohomology class we can associate a harmonic 1-form with values in the adjoint local system $\omega \in \mathcal{H}^1(X, \mathrm{Ad}(\rho))$. The first result is then that every first order deformation $v \in \mathcal{C}^\infty(h^*TY)$ of h that behaves as the first derivative of a family of ρ_t-equivariant harmonic maps (morally: $v = \frac{\partial h_t}{\partial t}\big|_{t=0}$) is obtained by integrating ω (via an explicit formula). This gives a formula for computing the first derivative of f:

$$\frac{\partial f(\rho_t)}{\partial t}\Big|_{t=0} = \int_X \langle \omega, \mathrm{d}h \rangle \mathrm{dVol}_X. \tag{0.1}$$

Here we are slightly abusing notation, since ω takes values (locally) in $\mathfrak{g}$, while $\mathrm{d}h$ in the bundle h^*TY; however, it is standard practice to see the latter bundle as a subset of $Y \times \mathfrak{g}$, via the right trivialization. A consequence of (0.1) is that we can compute critical points of the Morse function. Here we actually *define* a critical point as a point where the expression in (0.1) vanishes for all (possibly obstructed) ω, in order to take singular points into account as well. It turns out that the critical points are exactly the representations underlying a complex variation of Hodge structure (see for example [8] for the definition), and that considering obstructed directions is in fact unnecessary, since being a critical point is equivalent to the vanishing of the derivative along the $\mathbb{C}^*$ action only.

The same problem can be asked for second order deformations. Second order deformations of maps are morally the covariant derivative of $\frac{\partial h_t}{\partial t}$, and second order deformations of a pair (ρ, c) consist of a cocycle for the adjoint action of $\rho + tc$ on $\mathfrak{g} \otimes \mathbb{R}[t]/(t^2)$. However, in this case there are obstructions: Given a family of representations $\rho_t\colon \Gamma \to G$ and a harmonic ρ_0-equivariant map h, there needs not exist a second order deformation of this map (along any second order deformation of (ρ, c)). This happens essentially when ρ_t points toward a non-semisimple direction, or when the dimension of the centralizer of $\rho_t(\Gamma)$ jumps down and we have chosen the “wrong” harmonic map h. Indeed, one can prove that the absence of obstructions along *both* (ρ_0, c) *and* (ρ_0, ic) is equivalent to the existence of a $\mathrm{Ad}(\rho_0)$-invariant 1-form θ such that $\mathrm{d}^*\theta = -\sum_{j,k} g^{jk} [\omega(\frac{\partial}{\partial x_j})^*, \omega(\frac{\partial}{\partial x_k})]$, where ω^* is the adjoint of ω. This is in turn equivalent to ω being a minimum for $\|\cdot\|_{L^2}$ in its $\mathrm{Ad}(Z_G(\rho_0(\Gamma)))$-orbit, and it is true that *every* ρ_0-equivariant harmonic map h is non-obstructed in this sense if and only if $H^0(X, \mathrm{Ad}(\rho_0 + tc))$ is a *flat* $\mathbb{R}[t]/(t^2)$-module (i.e. “the centralizers do not jump”).

This analysis to the second order can be used again to compute the second derivatives of the Morse function. Indeed, we can pick a θ as above such that furthermore $\mathrm{d}\theta = -[\omega, \omega]$ (that this cohomology class should vanish is a necessary, and for X Kaehler also sufficient, condition for the existence of ρ_t, see [6]). Then:

$$\frac{\partial^2 f(\rho_t)}{\partial t^2}\Big|_{t=0} = \int_X \left(\langle \theta, \mathrm{d}h\rangle + \left\|\frac{\omega+\omega^*}{2}\right\|^2\right)\mathrm{dVol}. \tag{0.2}$$

As an immediate application of (0.2), one can prove that the Morse function is strictly plurisubharmonic (and actually, that it is a Kaehler potential). Also, supposing X to be Kaehler, at a critical point (that is, a complex variation of Hodge structure), one can compute the Morse indices: It turns out that the weights of the complex variation of Hodge structure, or more precisely the weights of the Deligne-Hodge structure induced on $H^1(X, \mathrm{Ad}(\rho_0))$, correspond to the eigenvalues of the Hessian of f via the formula:

$$\frac{\partial^2 f(\rho_t)}{\partial t^2}\Big|_{t=0} = \int_X \sum_{\text{even } P} 2P\left\|\omega^{(P,Q)}\right\|^2\mathrm{dVol}. \tag{0.3}$$

Here, $\omega^{(P,Q)}$ is the (P,Q)-term in the Deligne-Hodge decomposition, so that $P + Q = 1$. Furthermore, the version of (0.3) stated here is actually accurate only when ρ is Zariski-dense, but there is exists a similar, more involuted, formula for the general case.

We have also discussed possible applications of these results. Suppose that (Y, ω_Y) is itself Kaehler, for example $G = \mathrm{SU}(p,q)$ or $G = \mathrm{SO}^*(2n)$. Then one can define the Toledo invariant of ρ by:

$$\tau(\rho) = \frac{1}{n!}\int_X h^*\omega_Y \wedge \omega_X^{n-1},$$

where here h is actually any continuous ρ-equivariant map, but from now on we will suppose that ρ is semisimple and h is harmonic. One sees readily that $f(\rho) \geq n|\tau(\rho)|$, with equality if and only if h is (anti)-holomorphic. If X is uniformized by the unit complex ball, there is a Milnor–Wood type inequality (due to Domic–Toledo [5] and Clerc–Orsted [3] in dimension 1 and to Burger–Iozzi [1] in higher dimension): $|\tau(\rho)| \leq \mathrm{rk}(G)\mathrm{Vol}(X)$. Representations satisfying the equality are called maximal. It is conjectured that if $n > 1$, maximal representations are very rigid, essentially reducing to the standard diagonal embedding $U(n,1) \hookrightarrow U(nq,q)$.

When X is a Riemann surface, it is a result by Bradlow, Garca-Prada and Gothen (see [2] and the references therein) states that local minima for f always satisfy the equality $f(\rho) = |\tau(\rho)|$ (with few explicit exceptions). Their proof makes use of the analogous equation to (0.3) in dimension 1, which is due to Hitchin [7]. One can prove that a similar result to theirs in higher dimension would imply the above conjecture (see [10]), but unfortunately their proof does not immediately generalize (and indeed, there are counterexamples to the naively generalized statement).

The first part of this talk is based on the author's Ph.D. thesis, see [9].

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Complex Monge-Ampère equations on quasi-projective manifolds

Eleonora Di Nezza
(joint work with Chinh Hoang Lu)

Let X be an n-dimensional compact Kähler manifold and fix ω an arbitrary Kähler form. If we write locally

$$\omega = \frac{i}{\pi}\sum \omega_{\alpha\beta} dz_\alpha \wedge d\bar{z}_\beta,$$

then the Ricci form of ω is (locally)

$$\mathrm{Ric}(\omega) := -\frac{i}{\pi}\sum \frac{\partial^2 \log(\det \omega_{pq})}{\partial z_\alpha \partial \bar{z}_\beta} dz_\alpha \wedge d\bar{z}_\beta.$$

Observe that $\mathrm{Ric}(\omega)$ is a closed positive $(1,1)$-form on X such that for any other Kähler form ω' on X the following holds globally:

$$\mathrm{Ric}(\omega') = \mathrm{Ric}(\omega) - dd^c \log \frac{\omega'^n}{\omega^n}.$$

Here $d = \partial + \bar{\partial}$ and $d^c = \frac{1}{2i\pi}(\bar{\partial} - \partial)$ are both real operators. In particular $\mathrm{Ric}(\omega')$ and $\mathrm{Ric}(\omega)$ represent the same cohomology class, which turns out to be $c_1(X)$. Conversely, given η a closed differential form representing $c_1(X)$, Calabi asked in [1] whether one can find a Kähler form ω such that

$$\mathrm{Ric}(\omega) = \eta.$$

He showed that if the answer is positive, then the solution is unique and he proposed a continuity method to prove the existence. This problem, known as the Calabi conjecture, remained open for two decades and it was finally solved by Yau in [9]. This result is now known as the Calabi-Yau theorem.

The Calabi conjecture reduces to solving a complex Monge-Ampère equation as we can see here below. Fix $\alpha \in H^{1,1}(X,\mathbb{R})$ a Kähler class, ω a Kähler form in α and $\eta \in c_1(X)$ a smooth form. Since $\mathrm{Ric}(\omega)$ represents $c_1(X)$, it follows from the $\partial\bar{\partial}$-lemma that there exists $h \in C^\infty(X,\mathbb{R})$ such that

$$\mathrm{Ric}(\omega) = \eta + dd^c h.$$

We now seek for $\omega_\varphi := \omega + dd^c\varphi$ a new Kähler form in α such that $\mathrm{Ric}(\omega_\varphi) = \eta$. Since

$$\mathrm{Ric}(\omega_\varphi) = Ric(\omega) - dd^c \log\left(\frac{{\omega_\varphi}^n}{\omega^n}\right),$$

the equation $\mathrm{Ric}(\omega_\varphi) = \eta$ is equivalent to

$$dd^c\left\{h - \log\left(\frac{{\omega_\varphi}^n}{\omega^n}\right)\right\} = 0$$

The function inside the brackets is pluriharmonic, hence constant since X is compact. Shifting initially h by a constant, our problem is equivalent to solving the complex Monge-Ampère equation

$$(\omega + dd^c\varphi)^n = e^h\omega^n. \tag{CY}$$

Note that h necessarily satisfies the normalizing condition

$$\int_X e^h\omega^n = \int_X \omega^n = V.$$

Theorem 1 (Yau78)**.** *The equation (CY) admits a unique (up to constant) solution $\varphi \in C^\infty(X,\mathbb{R})$ such that ω_φ is a Kähler form.*

Yau's proof relies on the *continuity method*, a classical tool to solve non linear PDE's: it consists in deforming the PDE of interest into a simpler one for which we already know the existence of a solution. The goal is to establish various *a priori estimates*: in particular it suffices to prove C^0 and C^2-estimates. Indeed, thanks to Evans-Krylov theory we can deduce an estimate of type $C^{2,\alpha}$ and this suffices to apply Schauder's theorem and a bootstrap argument in order to conclude. The most difficult step are the C^0-estimates and Yau's approach uses Moser's iterative process. After the celebrated paper of Yau [9], Kołodziej [6] generalized the C^0 a priori estimates using pluripotential tools. His uniform estimate can indeed be applied to complex Monge-Ampère equations of the type

$$(\omega + dd^c\varphi) = fdV$$

where $0 \le f \in L^p(dV)$ for some $p > 1$. Kołodziej's idea is to show that the Monge-Ampère capacity of sublevel sets $(\varphi < -t)$ vanishes if $t > 0$ is large enough, by a clever use of the comparison principle. Hence, he proves that the solution φ is bounded on X.

Consider now a complex Monge-Ampère equation of the type

$$(\omega + dd^c\varphi)^n = f\omega^n, \tag{0.1}$$

where $f \in L^1(X)$ is such that $\int_X f\omega^n = \int_X \omega^n$. It is very natural for various geometric reasons to look at the case when f is merely smooth and positive on the complement of a divisor D, e.g. when studying Calabi's conjecture on quasiprojective manifolds (see e.g. [7, 8]). Note that such degenerate equations naturally appear when dealing with the problem of the existence of singular Kähler-Einstein metrics on varieties with mild singularities.
We recall that the existence and the uniqueness of a *weak* solution ($\varphi \in \mathcal{E}(X,\omega)$) of the above equations were proved in the last years by Guedj and Zeriahi [5] and Dinew [2], respectively. Thus the relevant question was about the regularity of the solution φ and its asymptotic behavior near D.

In [3] and [4] Chinh H. Lu and I study such a problem. In this wilder setting classical PDE's methods break down, and we found another approach using pluripotential theory. The first very general result that we were able to prove is the following:

Theorem 2 (Di Nezza - Lu 2014). *Assume that $f \lesssim e^{-\phi}$, for some quasi-plurisubharmonic (qpsh for short) function ϕ. Let $\varphi \in \mathcal{E}(X,\omega)$ be the unique normalised solution of (0.1). Then, for any $a > 0$ small enough (i.e. a is such that $a\phi \in PSH(X,\omega/2)$) there exists $A > 0$ depending only on $\int_X e^{2\varphi/a}\omega^n$ such that*

$$\varphi \geq a\phi - A. \tag{0.2}$$

Here, we want to stress that $\int_X e^{2\varphi/a}\omega^n$ is finite thanks to Skoda's theorem since any qpsh funtion belonging to the class $\mathcal{E}$ has zero Lelong number at each point.
The proof of the above theorem deeply relies on pluripotential methods: we follow Kołodziej approach with various novelties. We should emphasise that in our case the solution is not bounded and therefore a natural idea is to bound the solution from below by a "model" quasi-plurisubharmonic function that can be also very singular. This is the reason why we introduce a new tool in pluripotential theory, the *generalized Monge-Ampère capacities*, defined as

$$\mathrm{Cap}_\psi(E) := \sup\left\{\int_E (\omega + dd^c u)^n \ \mid\ u \in PSH(X,\omega)\ \ \psi - 1 \leq u \leq \psi\right\}, \quad \forall E \subset X,$$

where ψ is a $\omega/2$-psh function.
The idea to prove the above generalized C^0-estimate in (0.2) is then to show that the generalized capacity of sublevel sets $(\varphi < \psi - t)$ vanishes when $t > 0$ is large enough. The lower bound in Theorem 2 is the key step that allows us to prove the following regularity result:

Theorem 3 (Di Nezza - Lu 2014). *Assume that $0 < f \in C^\infty(X \setminus D)$ where D is a closed subset of X. Moreover, assume that f can be written of the form $f = e^{\psi^+ - \psi^-}$, where $\psi^\pm$ are qpsh funtions on X and $\psi^- \in L^\infty_{loc}(X \setminus D)$. Let $\varphi \in \mathcal{E}(X,\omega)$ be the solution of (0.1) normalized such that $\sup_X \varphi = 0$. Then φ is smooth outside D.*

Remark 4. *We are also able to prove the same regularity result when working with degenerate complex Monge-Ampère equations, namely when the reference form is merely semipositive($\theta \geq 0$) and big ($\int_X \theta^n > 0$) rather than Kähler.*

The idea of the proof goes as follows:
Step 1. We use Demailly's regularization theorem to obtain quasi-decreasing sequences of smooth qpsh functions $\psi_\varepsilon^\pm$ converging to $\psi^\pm$. Then we let $\varphi_\varepsilon \in C^\infty(X)$ be the normalized ($\sup_X \varphi_\varepsilon = 0$) soltution of the Monge-Ampère equation

$$(\omega + dd^c\varphi_\varepsilon)^n = c_\varepsilon e^{\psi^+{}_\varepsilon - \psi^-{}_\varepsilon}$$

where c_ε is a normalization constant. Observe that here we use Yau's theorem!
Step 2. The goal is to establish a priori estimates for the sequence of smooth ω-psh functions (φ_ε). As it is well-known by experts, it suffices to establish C^0 and C^2 estimates since one these are in hands Evans-Krylov theory, Schauder's theorem and bootstrap arguments can be applied to get higher order estimates.
Step 3. (C^0-estimates) We use Theorem 2 with $\phi = \psi^-$ to obtain *generalized* C^0-estimates.
Step 4. (C^2-estimates) Thanks to step 3 (the crucial step!) we are able to prove laplacian estimates of type

$$\Delta_\omega \varphi_\varepsilon \leq Ce^{-\psi^-}.$$

In the special case when D is a divisor and the density f has some divisorial singularities we can say more about the *asymptotic behavior* of the solution near D.

Let s be a holomorphic section of the line bundle L_D defined by D and assume that f is such that

$$f = \frac{h}{|s|^2(-\log|s|)^{1+\alpha}},$$

where h is smooth and positive on X and $\alpha > 0$. Note that such densities can be written of the form $e^{\psi^+ - \psi^-}$ with $\psi^+ = \log h - (1+\alpha)\log(-\log|s|)$ and $\psi^- = 2\log|s|$. Thus Theorem 3 applies to these cases. Moreover, using pluripotential methods and the comparison principle, we were able to give precise bounds for the soltution φ. Precisely, we prove the followings:

(i) when $\alpha > 1$, the solution φ is bounded on X.

(ii) when $0 < \alpha < 1$, for any $p \in (0, 1-\alpha)$ and $q \in (1-\alpha, 1)$,

$$-a_1(-\log|s|)^q - A_1 \leq \varphi \leq -a_2(-\log|s|)^q + A_2$$

where a_1, A_1 depends on α, q and a_2, A_2 depends on α, p.

(iii) when $\alpha = 1$ (Poincaré case), for any $p \in (0,1)$

$$-B_1\log(-\log|s| + B_2) - B \leq \varphi \leq -c_2(-\log(-\log|s|))^p + C_2$$

where B_1, B_2 are suitable positive constants and c_2, C_2 depends on p.

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The Gonality Conjecture on Syzygies of Alegebraic Curves of Large Degree

Lawrence Ein
(joint work with Robert Lazarsfeld)

In this note, we give a report on some of my recent joint work with Rob Lazarsfeld on the gonality conjecture on syzygies of algebraic curves of large degree. In the 80's Green and Lazarsfeld began a systematic study of the syzygies of smooth projective curves. On of the main driving problems in this area is the important conjecture of Green which predicts that the behavior of the syzygies of a canonical curve is determined by the Clifford index of the curve. See [5] and [7] for more details. A few years ago, Voisin made an important breakthrough [9] and [10] by proving that Green's conjecture is true for a generic curve of genus g. Combined with the result of Texidor Bigas [8], they show that Green's conjecture is true generic $k-$gonal curves. In 1985, Green and Lazarsfeld made another conjecture asserting the following [7]. Suppose a curve C is embedded into projective space by the complete linear system of a sufficiently large degree line bundle L. Then the shape of the minimal resolution of the coordinate ring is determined by the gonality of the curve.

Let C be a smooth projective curve of genus $g \geq 2$ and L be a very ample line bundle of degree d on C defining an embedding

$$C \subset \mathbf{P}H^0(L) = \mathbf{P}^r.$$

Write $S = \mathrm{Sym} H^0(L)$ for the coordinate ring of $\mathbf{P}^r$ and denote by

$$R = R(L) = \bigoplus_m H^0(mL)$$

the graded $S-$ module associated to L. Consider the minimal graded free resolution $E_\bullet = E(L)_\bullet$ of R over S:

$$0, \longrightarrow E_{r-1} \longrightarrow \ldots \longrightarrow E_1 \longrightarrow E_0 \longrightarrow R \longrightarrow 0.$$

Using the Koszul cohomology of Green, we can write

$$E_p = \bigoplus_q K_{p,q}(C, L) \otimes_{\mathbf{C}} S(-p-q).$$

It is elementary that if $H^1(L) = 0$, then $K_{p,q}(C, L) = 0$ if $q \geq 3$. Moreover work of Green [5] and others show that if $d = \deg L >> 0$, then $r = d - g$ and

$$K_{p,0}(C, L) \neq 0 \leftrightarrow p = 0$$
$$K_{p,2}(C, L) \neq 0 \leftrightarrow r - g \leq p \leq r - 1.$$

It follows that

$$K_{p.1}(C, L) \neq 0 \ \text{ for } 1 \leq p \leq r - 1 - g.$$

These results leave the question when $K_{p,1}(C, L) \neq 0$ for $r - g \leq p \leq r - 1$. Let $\mathrm{gon}(C)$ be the least degree of a branched cover of $C \longrightarrow \mathbf{P}^1$. Using the techniques introduced by Voisin [9] and [10], we study the syzygies using the geometry of the Hilbert schemes of the curves. As predicted by the conjecture of Green and Lazarsfeld, we show the following.

Theorem 1. [4] *If* $\deg L >> 0$*, then*

$$K_{p,1}(C, L) \neq 0 \leftrightarrow 1 \leq p \leq r - \mathrm{gon}(C).$$

Thus one can read off the gonality of a curve from the shape of the minimal resolution of the coordinate ring of $R(C, L)$. More generally, if B is a line bundle on C, we consider the $S-$module

$$M((C, L); B) = \bigoplus_m H^0(B + mL).$$

recall that we say that the line bundle B is $p-$very ample, if for every closed subscheme Z of length p of C, the restriction map

$$H^0(B) \longrightarrow H^0(B|_Z)$$

is surjective. One sees that $\mathrm{gon}(C) > p$ if and only if K_C is $p-$very ample. Denote $K_{p,q}(M((C, L); B))$ by $K_{p,q}((C, L); B)$. By Serre's duality, one sees that the above theorem is equivalent to

$$K_{p,1}((C, L); K_C) = 0$$

if and only K_C is $p-$ very ample.

More generally we prove the following.

Theorem 2. [4] *Fix a line bundle B and assume that* $\deg L >> 0$. *Then*

$$K_{p,1}((C,L);B) = 0$$

if and only if B is $p-$very ample.

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Rationality and growth conditions

Jörg Winkelmann

With Frederic Camapana we proved:

Theorem 1. *Let X be a compact complex Kähler manifold. Let $f : \mathbf{C}^n \to X$ be a differentiably non-degenerate meromorphic map. If the order function (in the sense of Nevanlinna theory) fulfills $\rho < 2$, then X is rationally connected.*

This improves earlier results of Campana-Paun and Noguchi-Winkelmann.

An example of Noguchi and Winkelmann shows that the Kähler assumption is necessary.

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Dynamical degrees of birational transformations of projective surfaces

Jérémy Blanc
(joint work with Serge Cantat)

Introduction. The dynamical degree $\lambda(f)$ of a birational transformation f measures the exponential growth rate of the degree of the formulae that define the n-th iterate of f, and also measures the complexity of the dynamics of f. For instance, over the field of complex numbers, $\log(\lambda(f))$ provides an upper bound for the topological entropy of $f\colon X(\mathbb{C}) \dashrightarrow X(\mathbb{C})$ and is equal to it under natural assumptions (see [1, 5]).

The goal of this talk was to describe the structure of the set of all dynamical degrees $\lambda(f)$, when f runs over the group of all birational transformations $\mathsf{Bir}(X)$ and X over the collection of all projective surfaces. An important feature of our results may be summarised by the following slogan: *Precise knowledge on $\lambda(f)$ provides useful information on the conjugacy class of f.*

Let f be a birational transformation of X defined over an algebraically closed field $\mathbf{k}$. Then f determines an endomorphism $f_*\colon \mathsf{NS}(X) \to \mathsf{NS}(X)$ of $\mathsf{NS}(X)$. The **dynamical degree** $\lambda(f)$ of f is defined as the spectral radius of the sequence of endomorphisms $(f^n)_*$, as n goes to $+\infty$. More precisely, once a norm $\|\cdot\|$ has been chosen on the real vector space $\mathrm{End}(\mathsf{NS}_\mathbb{R}(X))$, one defines

$$\lambda(f) = \lim_{n\to\infty} \| (f^n)_* \|^{1/n} ;$$

this limit exists, and does not depend on the choice of the norm. Moreover, for every ample divisor $D \subset X$

$$\lambda(f) = \lim_{n\to\infty} (D \cdot (f^n)_* D)^{1/n} ,$$

where $C \cdot D$ denotes the intersection number between divisors or divisor classes. For instance, when $X = \mathbb{P}^2$, we have $\lambda(f) = \lim_{n\to\infty}(\deg(f^n))^{1/n}$, where deg is the degree of the polynomials (without common factors) that define a birational map of $\mathbb{P}^2$.

The **dynamical spectrum** of X is defined as the set

$$\Lambda(X) = \{\lambda(f) \mid f \in \mathsf{Bir}(X)\}.$$

Diller and Favre proved in [4] that every birational transformation of a projective surface X is conjugate by a birational morphism $\pi : Y \to X$ to an algebraically stable transformation $g = \pi^{-1} \circ f \circ \pi$ (i.e. to a map g such that $(g^n)_* = (g_*)^n$ for each $n \geq 1$) . From this fact and the Hodge index theorem, they obtained the following result.

Theorem 1 (Diller and Favre). *Let $\mathbf{k}$ be a field and let f be a birational transformation of a projective surface defined over $\mathbf{k}$. If $\lambda(f)$ is different from 1, then $\lambda(f)$ is a Salem or a Pisot number.*

By definition, a **Pisot number** is an algebraic integer $\lambda \in]1, \infty[$ whose other Galois conjugates lie in the open unit disk; Pisot numbers include integers $d \geq 2$ as well as reciprocal quadratic integers $\lambda > 1$. A **Salem number** is an algebraic integer $\lambda \in]1, \infty[$ whose other Galois conjugates are in the closed unit disk, with at least one on the boundary; hence, the minimal polynomial of λ has at least two complex conjugate roots on the unit circle, and the degree of λ is at least 4.

Relation between dynamical degree and conjugation to an automorphism. The dynamical degree of an automorphism, if different from 1, is either a quadratic number or a Salem number (see [4]). Here we prove a converse statement.

Theorem 2. *Let* $\mathbf{k}$ *be an algebraically closed field. Let* f *be a birational transformation of a projective surface* X*, defined over* $\mathbf{k}$*. If* $\lambda(f)$ *is a Salem number, there exists a projective surface* Y *and a birational mapping* $\varphi\colon Y \dashrightarrow X$ *such that* $\varphi^{-1} \circ f \circ \varphi$ *is an automorphism of* Y*.*

Thus, one can decide whether a birational transformation is conjugate to an automorphism by looking at its dynamical degree, except when this degree is 1 or a quadratic integer. There are quadratic integers which are simultaneously realised as dynamical degrees of automorphisms, and of birational transformations that cannot be conjugate to an automorphism.

Once Theorem 2 is proved, three corollaries can be deduced from results of McMullen and the second author (see [6] and [3]). The first corollary is a **spectral gap property** for dynamical degrees: *There is no dynamical degree in the interval* $]1, \lambda_L[$. The second corollary does not seem to be related to values of dynamical degrees, but the simple proof given here makes use of the spectral gap. It asserts that *the quotient of the centraliser, in the group* $\mathsf{Bir}(X)$*, of a loxodromic element* f *by the group* $\langle f \rangle$ *is finite.* The third consequence is an effective and explicit bound for the optimal degree of a conjugacy.

Non-rational surfaces. Non rational surfaces are easily handled with.

Theorem 3. *Let* $\mathbf{k}$ *be an algebraically closed field. Let* X *be a projective surface defined over* $\mathbf{k}$*. If* X *is not rational, then*

(1) $\Lambda(X) = \{1\}$ *if* X *is not birationally equivalent to an abelian surface, a K3 surface, or an Enriques surface;*
(2) $\Lambda(X) \setminus \{1\}$ *is made of quadratic integers and of Salem numbers of degree at most* 6 *(resp.* 22*, resp.* 10*) if* X *is an abelian surface (resp. a K3 surface, resp. an Enriques surface).*

The union of all dynamical spectra $\Lambda(X)$ *where* X *runs over the set of non-rational projective surfaces defined over* $\mathbf{k}$*, and* $\mathbf{k}$ *runs over the set of all fields, is a closed discrete subset of the real line.*

This result shows that the most interesting case is provided by rational surfaces.

Rational surfaces: the dynamical spectrum $\Lambda(\mathbb{P}^2)$. In order to describe the dynamical spectrum of $\mathbb{P}^2$, we need to understand the relation between the mininimal degree, up to conjugation, and the dynamical degree.

Given an element f of $\mathsf{Bir}(\mathbb{P}^2_{\mathbf{k}})$, define the **minimal degree** of f in its conjugacy class as the positive integer

$$\mathsf{mcdeg}(f) = \min \deg(g \circ f \circ g^{-1})$$

where g describes $\mathsf{Bir}(\mathbb{P}^2_{\mathbf{k}})$. The function mcdeg is constant on conjugacy classes, and

$$\lambda(f) \leq \mathsf{mcdeg}(f) \leq \deg(f)$$

for all birational transformations of the plane. One of our main goals is to provide the following reverse inequality. This is obtained in the following result.

Theorem 4. *Let* $\mathbf{k}$ *be an algebraically closed field. Let* f *be a birational transformation of the plane* $\mathbb{P}^2_{\mathbf{k}}$.

(1) *If* $\lambda(f) \geq 10^6$ *then* $\mathsf{mcdeg}(f) \leq 4700\,\lambda(f)^5$.
(2) *If* $\lambda(f) > 1$, *then* $\mathsf{mcdeg}(f) \leq \cosh(110 + 345 \log(\lambda(f)))$.

On the other hand, there are sequences of elements $f_n \in \mathsf{Bir}(\mathbb{P}^2_{\mathbf{k}})$ such that $\mathsf{mcdeg}(f_n)$ goes to $+\infty$ with n while $\lambda_1(f_n) = 1$ for all n.

The set $\Lambda(\mathbb{P}^2_{\mathbf{k}})$ is a subset of $\mathbb{R}_{>0}$ and, as such, is totally ordered. The following statement, which follows from Theorem 4, asserts that $\Lambda(\mathbb{P}^2_{\mathbf{k}})$ is **well ordered**: Every non-empty subset of $\Lambda(\mathbb{P}^2_{\mathbf{k}})$ has a minimum; equivalently, it satisfies the descending chain condition (if $(f_n)_{n\geq 0}$ is a sequence of birational transformations of $\mathbb{P}^2_{\mathbf{k}}$ and the dynamical degrees $\lambda(f_n)$ decrease with n, then $\lambda(f_n)$ becomes eventually constant).

Theorem 5. *Let* $\mathbf{k}$ *be an algebraically closed field. The dynamical spectrum* $\Lambda(\mathbb{P}^2_{\mathbf{k}}) \subset \mathbb{R}$ *is well ordered, and it is closed if* $\mathbf{k}$ *is uncountable.*

The results described in this report can be found in [2].

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The Kähler cone of hyperkähler manifolds

Ekaterina Amerik
(joint work with Misha Verbitsky)

Consider an irreducible holomorphic symplectic (also known as hyperkähler) manifold X, that is, a simply-connected compact Kähler manifold with $H^{2,0}(X)$ generated by a nowhere degenerate (that is, symplectic) form σ. By the results of Beauville, Bogomolov and Fujiki, $H^2(X,\mathbb{Z})$ carries an integral non-degenerate quadratic form q, the Beauville-Bogomolov form, which can be given by an explicit formula involving integration, but which is of topological origin: $q(z)^n = cz^{2n}$, where $dim(X) = 2n$ and $c = c(X)$ is a positive constant. This form is of signature $(3, b_2 - 3)$ on $H^2(X,\mathbb{R})$ and of signature $(1, b_2 - 3)$ on $H^{1,1}_{\mathbb{R}}(X)$. The *positive cone* $\mathcal{P}(X) \subset H^{1,1}_{\mathbb{R}}(X)$ is, by definition, the connected component of the set of classes with positive Beauville-Bogomolov square which contains the Kähler cone $\mathcal{K}(X)$.

The Beauville-Bogomolov form allows us to view the cohomology classes of curves on X as (rational rather then integral, since q is in general not unimodular) $(1,1)$-classes, which we shall do throughout this report.

Huybrechts and Boucksom gave the following description of $\mathcal{K}(X)$ inside $\mathcal{P}(X)$.

Theorem: *A class $z \in \mathcal{P}(X)$ is Kähler if and only if $z \cdot [C] > 0$ for the classes of rational curves $C \subset X$.*

In particular, for X very general one has $\mathcal{K}(X) = \mathcal{P}(X)$ (as there are no curves on X at all).

In [1], we have proposed the following refinement of their result. Following Markman, define the *monodromy group* as a subgroup of $O(H^2(X,\mathbb{Z}))$ generated by parallel transports in families, and the *Hodge monodromy group* as its part preserving the Hodge decomposition. If z is a Beauville-Bogomolov negative integral $(1,1)$-class on X, call z an *MBM (Monodromy Birationally Minimal) class* if for some γ belonging to the Hodge monodromy group Γ^{Hdg}, the hyperplane $\gamma(z)^\perp \subset H^{1,1}_{\mathcal{R}}(X)$ supports a face of the Kähler cone of some birational model of X (note that a face is, by definition, of maximal dimension $h^{1,1}(X) - 1$).

Theorem:

(1) The property of being MBM is deformation-invariant as soon as z stays of type $(1,1)$.
(2) The Kähler cone is a connected component of the complement to the union of the orthogonals to all MBM classes in $\mathcal{P}(X)$.

In other words, one can define a purely topological invariant $\mathcal{M} \subset H^2(M,\mathbb{Z})$, where M is a differential manifold underlying X, and characterize the Kähler cone as the complement in $\mathcal{P}(X)$ to the union of the orthogonals to those elements of $\mathcal{M}$ which happen to be of type $(1,1)$ on X.

The class of an extremal rational curve is obviously MBM; conversely, an MBM class z is the class of an extremal rational curve on a deformation of X (for instance, on such a deformation X' that z generates the space of rational $(1,1)$-classes on X').

The following conjecture is a version of the Kawamata-Morrison cone conjecture formulated in a more general Calabi-Yau setting, but only for projective manifolds:

Conjecture: *For X as above, the group $Aut(X)$ has only finitely many orbits on the set of faces of $\mathcal{K}(X)$.*

Remark: The cone considered by Kawamata and Morrison is actually not $\mathcal{K}(X)$ but the so-called "ample effective cone", see [8], which apriori may have "more" faces than $\mathcal{K}(X)$. In the hyperkähler case, new faces do not appear and the classical cone conjecture is easily deduced from our version, but we cannot enter into details here.

The conjecture is known for K3 surfaces (already since mid-eighties, [7]). The following recent results relate $Aut(X)$ to the monodromy and show that the monodromy group is large, thus indicating a way towards the proof of the conjecture in higher dimension:

Theorem (Markman) *Let γ be an element of the Hodge monodromy group of X. If γ takes some Kähler class on X into a Kähler class, then there exists an $f \in Aut(X)$ such that $\gamma = f^*$.*

Theorem (Verbitsky) *The monodromy group is of finite index in $O(H^2(X, \mathbb{Z}))$.*

From Markman's theorem, one deduces (with some work) that it is sufficient to answer the following question in the affirmative:

Question: *Does the Hodge monodromy group act with finitely many orbits on the set of primitive MBM classes in $H^{1,1}(X)$?*

Consider the following

Boundedness assumption: *Primitive MBM classes on X have bounded square.*

From general results on quadratic forms such as presented in Kneser's book [3], one deduces the affirmative answer to the Question under this assumption. In particular, it is true for manifolds of K3 type or generalized Kummer varieties ([1], [5]). Moreover, the following consequence of boundedness is given by Markman and Yoshioka.

Proposition: *Under the boundedness assumption, the number of hyperkähler birational models of X is finite.*

The main point of the recent paper [2] is the unconditional proof of the cone conjecture with the help of ergodic theory (Ratner and Mozes-Shah theorems). Technically, what we prove is as follows.

Main Theorem: *Let L be a lattice of signature $(1,k)$, $k \geq 3$, and Σ a Γ-invariant union of hyperplanes of the form $z^\perp$ in $L \otimes \mathbb{R}$, where $z \in L$, $z^2 < 0$ and Γ is of finite index in $O(L)$. Then either Σ is a finite union up to Γ-action, or Σ is dense in the positive cone.*

Corollary: *Assume X projective, with Picard number at least 4. Then the Cone conjecture holds for X.*

The Boundedness assumption is thus true for X projective with $\rho(X) \geq 4$. Using the deformation invariance of MBM property and an easy fact that any (irreducible hyperkähler) X can be deformed to a projective X' with Picard number at least 4, in such a way that the $(1,1)$-classes on X remain of type $(1,1)$ of X', we obtain:

Corollary: *The Boundedness assumption holds for any irreducible holomorphic symplectic manifold. In particular, the Cone conjecture is true and the number of hyperkähler birational models is finite.*

Let us give a sketch of the proof of the Main Theorem. By Borel – Harish-Chandra theorem, Γ is a lattice (that is, a discrete subgroup of finite covolume) in $G = SO^+(1,k)$. Consider a Γ-invariant union Σ of hyperplanes and let $H_i,\ i \in I$ be the images of these in the projectivization $\mathbb{P}V^+$ of the positive cone. We claim that there is a subgroup $H \subset G$, isomorphic to $SO^+(1,k-1)$, such that certain H-orbits L_i in G project onto the H_i. It is sufficient to prove that the images of L_i in G/Γ are finitely many or dense.

From the rationality of the hyperplanes of Σ (and Borel – Harish-Chandra theorem) one deduces that those images are of finite volume with respect to the Haar measure on H, and therefore one obtains probability measures μ_i supported on the images of L_i in G/Γ. These are the *algebraic measures* of Ratner theory, ergodic with respect to the action of a subgroup H generated by unipotents. It follows from a theorem by Mozes and Shah [4] (with some arguments from hyperbolic geometry used to exclude the case when μ_i tends to infinity) that if the μ_i form an infinite family, there must be a converging subsequence and the limiting measure is also ergodic. By Ratner theory [6], this measure is of the same type, supported on an orbit of a closed subgroup containing H. But there are no such subgroups except H and G. In fact the first case means that there are only finitely many μ_i, and the second case means that L_i are dense in G.

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Uniform K-stability

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(joint work with Tomoyuki Hisamoto, Mattias Jonsson)

Let (X, L) be a smooth projective complex variety endowed with an ample line bundle. Assuming for simplicity that the automorphism group of (X, L) is finite, the Yau-Tian-Donaldson predicts that the first $c_1(L)$ contains a constant scalar curvature Kähler (cscK for short) metric iff (X, L) is K-stable.

Here K-stability is defined as the positivity of the Donaldson-Futaki invariant $\mathrm{DF}(\mathcal{X}, \mathcal{L})$ of every non-trivial test configuration $(\mathcal{X}, \mathcal{L})$, where the latter are $\mathbb{C}^*$-equivariant 1-parameter degenerations of (X, L) (see [Don02]).

In the Kähler-Einstein case, i.e. when $c_1(X) = \lambda c_1(L)$ for some $\lambda \in \mathbb{R}$, any cscK metric $\omega \in c_1(L)$ is automatically Kähler-Einstein, i.e. satisfies $\mathrm{Ric}(\omega) = \lambda\omega$. Due to the work of Aubin and Yau, a unique Kähler-Einstein metric has been known always to exist in the case $\lambda \leq 0$. In contrast, in the Fano case, where $\lambda > 0$, the YTD conjecture has been proved only recently, by Chen-Donaldson-Sun on the one hand and Tian on the other hand.

In the general case of an arbitrary polarization, K-stability is known to follow from the existence of a cscK metric, thanks to work of Donaldson and Stoppa [Don01, Sto09]. However, it is not quite clear that K-stability should really suffice, and the stronger notion of *uniform K-stability* was therefore introduced by Szekelyhidi [Sze06] in order to modify the formulation of the conjecture.

Uniform K-stability involves the *norm* of a test configuration $\|(\mathcal{X}, \mathcal{L})\|$, a non-negative number which is meant to measure 'how non-trivial' a test configuration is. Indeed, our first main result is as follows.

Theorem 1. *Let (X, L) be a polarized manifold and let $(\mathcal{X}, \mathcal{L})$ be a test configuration with $\mathcal{X}$ normal. Then $\|(\mathcal{X}, \mathcal{L})\| = 0$ iff $(\mathcal{X}, \mathcal{L})$ is the trivial test configuration.*

Building upon techniques introduced in [PRS08, Berm12], we then prove:

Theorem 2. *Let (X, L) be a polarized manifold whose K-energy functional is coercive. Then X is uniformly K-stable, i.e. there exists $\delta > 0$ such that*

$$\mathrm{DF}(\mathcal{X}, \mathcal{L}) \geq \delta \|(\mathcal{X}, \mathcal{L})\|.$$

for every test configuration $(\mathcal{X}, \mathcal{L})$.

Thanks to a deep result of Tian [Tia97, PSSW08], this implies:

Corollary 3. *Uniform K-stability holds in the Kähler-Einstein case.*

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