Mathematisches Forschungsinstitut Oberwolfach

Report No. 46/2014

DOI: 10.4171/OWR/2014/46

Mini-Workshop: Differentiable Ergodic Theory, Dimension Theory and Stable Foliations

Organised by
Eugen Mihailescu, Bucharest
Bernd Stratmann, Bremen

28 September – 4 October 2014

Abstract. The mini-workshop *Differentiable Ergodic Theory, Dimension Theory and Stable Foliations* brought together experts in thermodynamical formalism, hyperbolic dynamics and dimension theory from several countries. The geographic representation was broad, from Europe, USA and Japan. All participants gave interesting 1-hour talks, and there was organized also an open problem session, where directions for future work and many open problems were discussed. Among the topics presented/discussed in the workshop, there were ones related to dimension theory and probability measures on fractals, various types of hyperbolicity, systems with overlaps, complex dynamics and iterated function systems.

Mathematics Subject Classification (2010): 37Dxx, 37Axx, 37Cxx, 28A80, 28A78, 60Bxx, 60Dxx.

Introduction by the Organisers

The mini-workshop *Differentiable Ergodic Theory, Dimension Theory and Stable Foliations*, organised by Eugen Mihailescu (Bucharest) and Bernd Stratmann (Bremen) was attended by participants with broad geographic representation, from USA, France, Romania, Sweden, Germany, Poland, Hungary, Japan, etc.

The workshop was an excellent blend of known researchers, with various and relatively close interests in dynamics and ergodic theory. The participants were at various career stages, from senior researchers to postdocs and finishing doctoral students.

The main topics of the workshop were dimension theory for smooth dynamical systems or iterated function systems, thermodynamic formalism, ergodic theory, hyperbolic dynamics, complex dynamics and probability theory on fractals.

Every participant gave a research talk of approximately 1 hour, and the talks were attended by all participants. Moreover, on Friday we had a very interesting and lively open problem session, where most of the participants proposed or discussed various open problems, in a stimulating and informal atmosphere. Besides the regular talks and the open problem session, the workshop generated many informal discussions between the participants, and the continuation or starting of research projects in dynamical systems and ergodic theory.

The talks which were presented and the open problems discussed in this workshop, covered several new directions at the fringes of current research in dimension theory and thermodynamical formalism.

M. Urbański talked about geometric rigidity in the theory of Kleinian groups, and applications to properties and estimates of the Hausdorff dimension of limit sets of Kleinian groups, and applications to complex dynamics of one variable. J. Schmeling talked about the Fourier dimension, and several properties and applications of modified versions of the Fourier dimension, studied jointly with F. Ekström and T. Persson. A. Zdunik studied jointly with M. Urbański, the (Hölder) continuity of the numerical values of Hausdorff measures for families of conformal systems, namely iterated function systems with Strong Separation Condition, and analytic families of conformal expanding repellers in $\mathbb{C}$. B. Saussol presented joint work with F. Pène about statistical properties and the Poisson law in the non-uniformly hyperbolic setting, and especially when the rate of mixing is polynomial. E. Mihailescu talked about joint work with M. Urbański on asymptotic measure-theoretic degrees and the Hausdorff dimension of stable slices, and about equilibrium measures in non-invertible systems with applications to holomorphic dynamics in $\mathbb{P}^2$. T. Persson talked about the typical behaviour of limsup-sets $E(v)$, obtained by using random independent and uniformly distributed vectors v_i in the torus $\mathbb{T}^d$. Y. Coudène presented results, on multiple mixing from weak hyperbolicity by using the Hopf argument, obtained with B. Hasselblatt and S. Troubetzkoy. M. Kessebohmer presented joint work with B. Stratmann about fractal geometry of limiting symbols for modular subgroups. V. Mayer studied with M. Urbański random dynamics for transcendental functions. M. Gröger presented joint results with B. Hunt about the Lyapunov dimension and the information dimension for physical measures for coupled skinny baker's maps, and studied the Kaplan-Yorke conjecture in these cases. J. Jaerisch reported on joint work with H. Sumi about the pointwise Hölder exponent of the complex analogue of Cantor function and the Takagi function. B. Bárany talked about certain conditions when the Ledrappier-Young formula for self-affine measures is satisfied. A. Soós discussed approximations for the solution of a stochastic differential equation driven by fractional Brownian motion. And A. Zielicz presented relations between the entropy of the geodesic flow in the sense of Sullivan and convex-core entropy.

All the topics presented in the workshop are detailed below.

As a conclusion, we had a very interesting and productive workshop in Oberwolfach, which will generate new ideas and joint projects in the future.

Acknowledgement: The MFO and the workshop organizers would like to thank the National Science Foundation for supporting the participation of junior researchers in the workshop by the grant DMS-1049268, "US Junior Oberwolfach Fellows".

Mini-Workshop: Differentiable Ergodic Theory, Dimension Theory and Stable Foliations

Table of Contents

Abstracts

On the Fourier dimension and a modification

Jörg Schmeling
(joint work with Fredrik Ekström and Tomas Persson)

In the following we report on recent results about the Fourier dimension that we obtained in [7].

Let A be a Borel subset of $\mathbb{R}^d$. One way to prove a lower bound for the Hausdorff dimension of A is to consider integrals of the form

$$I_s(\mu) = \int\int |x-y|^{-s}\mu(x)\mu(y);$$

if μ is a Borel measure such that $\mu(A) > 0$ and $I_s(\mu) < \infty$ for some s, then $\dim_{\mathrm{H}} A \geq s$. For a finite Borel measure μ, the *Fourier transform* is defined as

$$\widehat{\mu}(\xi) = \int e^{-2\pi i \xi\cdot x}\,\mathrm{d}\mu\,(x),$$

where $\xi \in \mathbb{R}^d$ and $\cdot$ denotes the Euclidean inner product. It can be shown [19, Lemma 12.12] that if μ has compact support then

$$I_s(\mu) = \mathrm{const.}(d,s)\int |\widehat{\mu}(\xi)|^2|\xi|^{s-d}\,\mathrm{d}\xi$$

for $0 < s < d$, and thus $I_{s_0}(\mu)$ is finite if $\widehat{\mu}(\xi) \lesssim |\xi|^{-s/2}$ for some $s > s_0$, where $f(\xi)\lesssim g(\xi)$ means that there exists a constant C such that $|f(\xi)| \leq C|g(\xi)|$ for all ξ. This motivates defining the *Fourier dimension* of A as

$$\dim_{\mathrm{F}} A = \sup\left\{s \in [0,d];\ \widehat{\mu}(\xi)\lesssim|\xi|^{-s/2},\ \mu \in \mathcal{P}(A)\right\},$$

where $\mathcal{P}(A)$ denotes the set of Borel probability measures on $\mathbb{R}^d$ that give full measure to A . Thus the Fourier dimension is a lower bound for the Hausdorff dimension. The Fourier dimension of a finite Borel measure μ on $\mathbb{R}^d$ is defined as

$$\dim_{\mathrm{F}}\mu = \sup\left\{s \in [0,d];\ \widehat{\mu}(\xi)\lesssim|\xi|^{-s/2}\right\},$$

so that

$$\dim_{\mathrm{F}} A = \sup\left\{\dim_{\mathrm{F}}\mu;\ \mu \in \mathcal{P}(A)\right\}.$$

If $A \subset B$ then $\mathcal{P}(A) \subset \mathcal{P}(B)$ and hence

$$\dim_{\mathrm{F}}(A) = \sup\{\dim_{\mathrm{F}}\mu;\ \mu \in \mathcal{P}(A)\} \leq \sup\{\dim_{\mathrm{F}}\mu;\ \mu \in \mathcal{P}(B)\} = \dim_{\mathrm{F}}(B),$$

showing that the Fourier dimension is monotone. It seems not to be previously known whether the Fourier dimension is stable under finite or countable unions, that is, whether

$$\dim_{\mathrm{F}}\left(\bigcup_k A_k\right) = \sup_k \dim_{\mathrm{F}} A_k, \tag{1}$$

where $\{A_k\}$ is a finite or countable family of sets. The inequality $\geq$ follows from the monotonicity, but there might be sets for which the inequality is strict.

In [7] we show that (1) holds if for each n the intersection $A_n \cap \overline{\bigcup_{k\neq n} A_k}$ has small "modified Fourier dimension" (defined below), and in particular if all such intersections are countable. We also give an example of a countably infinite family of sets such that (1) does *not* hold.

This still leaves open the question of *finite* stability. The most straightforward approach would be to prove a corresponding stability for the Fourier dimension of measures, namely that

$$\dim_{\mathrm{F}}(\mu+\nu) = \min(\dim_{\mathrm{F}}\mu,\, \dim_{\mathrm{F}}\nu). \tag{2}$$

From this one could derive the finite stability for sets, using that any probability measure on $A \cup B$ is a convex combination of probability measures on A and B. The inequality $\geq$ always holds in (2) since the set of functions that are $\lesssim |\xi|^{-s/2}$ is closed under finite sums, but we give an example showing that strict inequality can occur. We also describe some situations in which (2) *does* hold — this seems to be the typical case.

To achieve countable stability, we consider the following modification of the Fourier dimension.

Definition 1. The *modified Fourier dimension* of a Borel set $A \subset \mathbb{R}^d$ is defined as

$$\dim_{\mathrm{FM}} A = \sup\left\{\dim_{\mathrm{F}}\mu;\ \mu \in \mathcal{P}(\mathbb{R}^d),\ \mu(A) > 0\right\},$$

and the modified Fourier dimension of a finite Borel measure μ is defined as

$$\dim_{\mathrm{FM}}\mu = \sup\left\{\dim_{\mathrm{F}}\nu;\ \nu \in \mathcal{P}(\mathbb{R}^d),\ \mu \ll \nu\right\},$$

where $\ll$ denotes absolute continuity.

Thus

$$\dim_{\mathrm{FM}} A = \sup\left\{\dim_{\mathrm{FM}}\mu;\ \mu \in \mathcal{P}(A)\right\}.$$

We investigate some basic properties of the modified Fourier dimension, and give examples to show that it is different from the usual Fourier dimension and the Hausdorff dimension. Moreover, we show that if μ annihilates all the common null sets for the measures that have modified Fourier dimension greater than or equal to s, then $\dim_{\mathrm{FM}}\mu \geq s$. Other classes of measures that can be characterised by their null sets in this way are the measures that are absolutely continuous to some fixed measure, and, less trivially, the measures $\mu \in \mathcal{P}([0,1])$ such that $\lim_{|\xi|\to\infty} \widehat{\mu}(\xi) = 0$ (see [14]). A necessary condition for such a characterisation to be possible is that the class of measures be a *band*, meaning that any measure that is absolutely continuous to some measure in the class lies in the class. The definition of the modified Fourier dimension is natural from this point of view, since the class of measures that have modified Fourier dimension greater than or equal to s is the smallest band that includes the measures that have (usual) Fourier dimension greater than or equal to s.

Previous work. Here we mention briefly some of the previous work related to the Fourier dimension that we are aware of. A *Salem set* is a set whose Fourier dimension equals its Hausdorff dimension.

Salem [16] showed that for each $s \in (0,1)$ there is a compact Salem subset of $[0,1]$ of dimension s (although this was shown for the restriction of the Fourier transform to the integers). An *explicit* example of a Salem set of any prescribed dimension in $(0,1)$ is given by the set of *α-well approximable numbers*, namely the set

$$E(\alpha) = \bigcap_{n=1}^{\infty} \bigcup_{k=n}^{\infty} \left\{ x \in [0,1];\ \|kx\| \le k^{-(1+\alpha)} \right\},$$

where $\|\cdot\|$ denotes the distance to the nearest integer. By a theorem of Jarník [8] and Besicovitch [1] the set $E(\alpha)$ has Hausdorff dimension $2/(2+\alpha)$ for $\alpha > 0$, and Kaufman [12] showed that there is a measure in $\mathcal{P}(E(\alpha))$ with Fourier dimension $2/(2+\alpha)$ (see also Bluhm's paper [2]).

It was shown by Kaufman [11] that for any C^2-curve Γ in $\mathbb{R}^2$ with positive curvature and any $s \subset (0,1)$, there is a compact Salem set $S \subset \Gamma$ of dimension s. From this it can be deduced [6, Proposition 1.1] that for any $s \in [0,1]$ there is a continuous function $[0,1] \to \mathbb{R}$ whose graph has Fourier dimension s. Fraser, Orponen and Sahlsten [6] proved that the graph of any function $[0,1] \to \mathbb{R}$ has *compact* Fourier dimension (defined below) less than or equal to 1, and that the set of continuous functions $[0,1] \to \mathbb{R}$ whose graphs have Fourier dimension 0 is residual with respect to the supremum norm among all continuous functions $[0,1] \to \mathbb{R}$.

Kahane showed that images of compact sets under Brownian motion and fractional Brownian motion are almost surely Salem sets, see [17]. It was shown by Fouché and Mukeru [5] that the level sets of fractional Brownian motion are almost surely Salem sets (in the special case of Brownian motion this follows from a result of Kahane, see [5, Section 3.2]).

Jordan and Sahlsten [10] showed that Gibbs measures of Hausdorff dimension greater than $1/2$ (satisfying a certain condition on the Gibbs potential) for the Gauss map $x \mapsto 1/x \pmod 1$ have positive Fourier dimension.

Wolff's book [22] about harmonic analysis discusses some applications of the Fourier transform to problems in geometric measure theory.

Some remarks. It is not so difficult to see that the Fourier dimension for measures is invariant under translations and invertible linear transformations, and thus the Fourier dimension and modified Fourier dimension for sets are invariant as well.

For any finite Borel measure μ on $\mathbb{R}^d$,

$$\lim_{T\to\infty} \frac{1}{(2T)^d} \int_{[-T,T]^d} |\widehat{\mu}(\xi)|^2 \,\mathrm{d}\xi = \sum_{x\in\mathbb{R}^d} \mu(\{x\})^2$$

(this is a variant of Wiener's lemma). If μ has an atom it is thus not possible that $\lim_{|\xi|\to\infty} \widehat{\mu}(\xi) = 0$, so $\dim_{\mathrm{F}}\mu = 0$, and also $\dim_{\mathrm{FM}}\mu = 0$ since ν has an atom whenever $\mu \ll \nu$. It follows that $\dim_{\mathrm{F}} A = \dim_{\mathrm{FM}} A = 0$ for any countable set $A \subset \mathbb{R}^d$.

Suppose next that A is a countable union of k-dimensional hyperplanes in $\mathbb{R}^d$ with $k < d$. If μ gives positive measure to A, then there must be a hyperplane P such that $\mu(P) > 0$. But then the projection of μ onto any line L that goes through the origin and is orthogonal to P has an atom, so $\widehat{\mu}$ does not decay along L. This shows that $\dim_{\mathrm{F}} A = \dim_{\mathrm{FM}} A = 0$. Thus for example a line segment in $\mathbb{R}^2$ has Fourier dimension 0 even though an interval in $\mathbb{R}$ has Fourier dimension 1.

From a special case of a theorem by Davenport, Erdős and LeVeque [4], it can be derived [15, Corollary 7.4] that if μ is a probability measure on $\mathbb{R}$ such that $\widehat{\mu}(\xi) \lesssim |\xi|^{-\alpha}$ for some $\alpha > 0$, then μ-a.e. x is normal to any base (meaning that $(b^k x)_{k=0}^{\infty}$ is uniformly distributed mod 1 for any $b \in \{2, 3, \ldots\}$). Thus if $A \subset \mathbb{R}$ does not contain any number that is normal to all bases, then $\dim_{\mathrm{F}} A = \dim_{\mathrm{FM}} A = 0$. In particular this applies to the middle-third Cantor set, since it consists of numbers that do not have any 1 in their ternary decimal expansion and hence are not normal to base 3.

Other variants of the Fourier dimension. One alternative way of defining the Fourier dimension of a Borel set $A \subset \mathbb{R}^d$ is to require the measure in the definition to give full measure to a compact subset of A, rather than to A itself. This variant, which will here be called the *compact Fourier dimension*, is thus defined by

$$\dim_{\mathrm{FC}} A = \sup \left\{ s \in [0, d];\ \widehat{\mu}(\xi) \lesssim |\xi|^{-s/2},\ \mu \in \mathcal{P}(K),\ K \subset A \text{ is compact} \right\}.$$

The anonymous referee of [7] provided an argument showing that the compact Fourier dimension is countably stable whenever all the sets in the union are closed, and pointed out that this can be used to deduce that the Fourier dimension and the compact Fourier dimension are not the same. Inspired by that, we then found an example that shows that the compact Fourier dimension is not in general finitely stable.

The Hausdorff dimension is *inner regular* in the sense that

$$\dim_{\mathrm{H}} A = \sup_{\substack{K \subset A \\ K \text{ compact}}} \dim_{\mathrm{H}} K$$

for any Borel set $A \subset \mathbb{R}^d$ (this follows from [21, Theorem 48]), and the same is true of the modified Fourier dimension by inner regularity of finite Borel measures on $\mathbb{R}^d$. Another way of expressing the fact that $\dim_{\mathrm{FC}}$ is different from $\dim_{\mathrm{F}}$ is to say that $\dim_{\mathrm{F}}$ is not inner regular.

One might consider to define the Fourier dimension and the modified Fourier dimension of any $B \subset \mathbb{R}^d$, by taking the supremum over all measures in $\mathcal{P}(\mathbb{R}^d)$ that give full or positive measure to some Borel set $A \subset B$, but then $\dim_{\mathrm{F}}$ and $\dim_{\mathrm{FM}}$ are not even finitely stable. For there is a construction by Bernstein (using the well ordering theorem for sets with cardinality $\mathfrak{c}$) of a set $B \subset \mathbb{R}$ such that any closed subset of B or B^c is countable [20, Theorem 5.3]. Thus any non-atomic measure $\mu \in \mathcal{P}(\mathbb{R})$ gives measure 0 to any compact subset of B or B^c, and by inner regularity to any Borel subset of B or B^c. It follows that B and B^c have Fourier dimension and modified Fourier dimension 0, but $B \cup B^c = \mathbb{R}$ has dimension 1.

This can be modified slightly to produce Lebesgue measurable sets $C_1, C_2 \subset \mathbb{R}$ that would violate the finite stability. For each natural number n, let A_n be a Salem set of dimension $1 - 1/n$ and let

$$C_1 = B \cap \bigcup_{n=1}^{\infty} A_n, \qquad C_2 = B^c \cap \bigcup_{n=1}^{\infty} A_n.$$

Then C_1 and C_2 are Lebesgue measurable since each A_n has Lebesgue measure 0, and since they are subsets of B and B^c respectively they would have Fourier dimension and modified Fourier dimension 0. On the other hand,

$$\dim_{\mathrm{F}}(C_1 \cup C_2) = \dim_{\mathrm{F}} \left(\bigcup_{n=1}^{\infty} A_n \right) = 1,$$

and thus also $\dim_{\mathrm{FM}}(C_1 \cup C_2) = 1$.

References

[1] A. Besicovitch, *Sets of fractional dimension (IV): on rational approximation to real numbers*, J. London Math. Soc., 1934, **9**, 126–131.

[2] Christian Bluhm, *On a theorem of Kaufman: Cantor-type construction of linear fractal Salem sets*, Ark. Mat., 1998, **36**, 307–316.

[3] Christian Bluhm, *Fourier Asymptotics of Statistically Self-Similar Measures*, J. Fourier Anal. Appl., 1999, **5**, 4, 355–362.

[4] H. Davenport and P. Erdős and W. J. LeVeque, *On Weyl's criterion for uniform distribution*, Michigan Math. J., 1963, **10**, 3, 311–314.

[5] Willem L. Fouché and Safari Mukeru, *On the Fourier structure of the zero set of fractional Brownian motion*, Stat. and Probab. Lett., 2013, **83**, 459–466.

[6] Jonathan M. Fraser and Tuomas Orponen and Tuomas Sahlsten, *On Fourier analytic properties of graphs*, Int. Math. Res. Not., 2014, **2014**, 10, 2730–2745.

[7] Fredrik Ekström and Tomas Persson and Jörg Schmeling", *On the Fourier dimension and a modification*, Preprint: arXiv:1406.1480 [math.FA]. To appear in Journal of Fractal Geometry.

[8] Vojtěc Jarník, *Diophantischen Approximationen und Hausdorffsches Mass*, Matematicheskii Sbornik, 1929, **36**, 3–4, 371–382.

[9] Goullet de Rugy, Alain, *Sur les mesures étrangères*, C. R. Acad. Sci. Paris 1971, **272**, 1, 123–126.

[10] Thomas Jordan and Tuomas Sahlsten, *Fourier transforms of Gibbs measures for the Gauss map*, Preprint: arXiv:1312.3619v2 [math.DS].

[11] R. Kaufman, *Random measures on planar curves*, Ark. Mat., 1976, **14**, 245–250.

[12] R. Kaufman, *On the theorem of Jarník and Besicovitch*, Acta Arith., 1981, **39**, 265–267.

[13] Körner, Thomas William", *Hausdorff and Fourier dimension*, Stud. Math., 2011, 206, 37–50.

[14] Russell Lyons, Seventy Years of Rajchman Measures, J. Fourier Anal. Appl., 1995, 363–377, Kahane Special Issue.

[15] Martine Queffélec and Olivier Ramaré", *Analyse de Fourier des fractions continues à quotients restreints*, Enseign. Math., 2003, **49**, 3–4, 335–356.

[16] R. Salem, *On singular monotonic functions whose spectrum has a given Hausdorff dimension*, Ark. Mat., 1951, 1, 4, 353–365.

[17] Jean-Pierre Kahane, Some random series of functions, Cambridge University Press, 1985, Cambridge studies in advanced mathematics, Second ed., 17–18.
[18] Kenneth Falconer, Fractal Geometry — Mathematical Foundations and Applications, John Wiley & Sons, 1990.
[19] Pertti Mattila, Geometry of Sets and Measures in Euclidean Spaces — Fractals and Rectifiability, Cambridge University Press, 1995.
[20] John C. Oxtoby, Measure and Category, Springer-Verlag, 1980, Second ed.
[21] Rogers, C. A., Hausdorff measures, Cambridge University Press, 1970.
[22] Thomas H. Wolff, Lectures on Harmonic Analysis, AMS Press, 2003, University Lecture Series.

Hölder regularity of the complex analogues of the Cantor function and the Takagi function

Johannes Jaerisch

(joint work with Hiroki Sumi)

1. Introduction and Statement of Results

Recently, H. Sumi introduced complex analogues of the classical Cantor function and the Takagi function ([Sum11, Sum13]). These complex analogues are defined on the Riemann sphere $\widehat{\mathbb{C}}$ and share some similarity with the classical functions defined on the real line. Our main results ([JS13, JS14], see Theorems 1 and 2 below) show that the pointwise Hölder exponent of the complex analogue of the Cantor function and the Takagi function can be investigated by means of the thermodynamic formalism in ergodic theory.

Before we state our main results, let us summarize some of the background and motivation from [Sum11]. Let $f_1, f_2 : \mathbb{R} \to \mathbb{R}$ be given by $f_1(x) := 3x$, $f_2(x) := 3x-2$. The classical Cantor function $\varphi : \mathbb{R} \to [0,1]$ satisfies the functional equation

$$\frac{1}{2}\varphi \circ f_1 + \frac{1}{2}\varphi \circ f_2 = \varphi \tag{1}$$

with the boundary conditions $\varphi_{|(-\infty,0]} = 0$ and $\varphi_{|[1,\infty)} = 1$. The function φ varies on the Cantor set, which is the repellor of the semigroup generated by f_1 and f_2. Moreover, φ is monotone and φ is $(\log 2 / \log 3)$-Hölder continuous.

A complex analogue of φ is obtained as follows. Consider the polynomial functions $g_1, g_2 : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$, given by $g_1(z) := (z^2-1) \circ (z^2-1)$, $g_2(z) := (z^2/4) \circ (z^2/4)$. Then there exists a unique continuous function $T : \widehat{\mathbb{C}} \to \mathbb{C}$ such that T satisfies the following analogue of the functional equation (1) above:

$$\frac{1}{2}T \circ g_1 + \frac{1}{2}T \circ g_2 = T, \quad T(0) = 0, \quad T(\infty) = 1. \tag{2}$$

Moreover, the function T varies on the Julia set $J(G)$ of the semigroup G given by

$$G := \langle g_1, g_2 \rangle := \left\{ g_{\omega_n} \circ g_{\omega_{n-1}} \circ \cdots \circ g_{\omega_1} : n \in \mathbb{N}, (\omega_1, \ldots, \omega_n) \in \{1,2\}^n \right\}$$

and

$$J(G) := \{z \in \widehat{\mathbb{C}} : \text{ there exists no non-empty neighborhood } U \text{ of } z \text{ such that } (g_{|U})_{g\in G} \text{ is normal}\}.$$

Also, the function T is monotone with respect to the surrounding order of compact connected subsets of $\widehat{\mathbb{C}}$ ([Sum11]). Finally, it is shown in ([Sum13]) that T is α-Hölder continuous for some $\alpha \in (0,1)$.

To state our first main result, we need further definitions. For a function $\rho : \widehat{\mathbb{C}} \to \mathbb{C}$ we denote by $\text{Höl}(\rho, \cdot)$ the pointwise Hölder exponent of ρ which is for $z \in \widehat{\mathbb{C}}$ given by

$$\text{Höl}(\rho, z) := \sup \left\{ \beta \in \mathbb{R} : \limsup_{y \to z, y \neq z} \frac{|\rho(y) - \rho(z)|}{d(y,z)^{\beta}} < \infty \right\} \in [0, \infty],$$

where d refers to the spherical distance on $\widehat{\mathbb{C}}$. For $\alpha \in \mathbb{R}$ we define the level sets

$$H(\rho, \alpha) := \left\{ z \in \widehat{\mathbb{C}} : \text{Höl}(\rho, z) = \alpha \right\}.$$

Moreover, we set

$$\alpha_{\min}(\rho) := \inf \{\alpha \in \mathbb{R} : H(\rho, \alpha) \neq \emptyset\} \text{ and } \alpha_{\max}(\rho) := \sup \{\alpha \in \mathbb{R} : H(\rho, \alpha) \neq \emptyset\}.$$

Theorem 1 (JS13b)**.** *For the function T given by (2) above we have $\alpha_{\min}(T) < \alpha_{\max}(T)$ and the dimension function given by $\alpha \mapsto \dim_H(H(T, \alpha))$, $\alpha \in (\alpha_{\min}, \alpha_{\max})$, is a positive, real-analytic and strictly concave function with maximum $\dim_H(J(G))$.*

A complex analogue of the classical Takagi function is obtained as follows. First observe that for each $p \in (0,1)$ there exists a unique continuous function $T_p : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ such that

$$p \cdot T_p \circ g_1 + (1-p) \cdot T_p \circ g_2 = T_p, \quad T_{p|(-\infty,0]} = 0 \quad T_{p|[1,\infty)} = 1.$$

Then it follows from [Sum13] that $\frac{\partial}{\partial p}(T_p(z))$ exists and a complex analogue of the Takagi function is given by

$$C(z) = \frac{\partial}{\partial p} (T_p(z))_{|p=1/2}.$$

Our second main result is the following:

Theorem 2 (JS14)**.** *For every $z \in \widehat{\mathbb{C}}$ we have $\text{Höl}(C, z) = \text{Höl}(T, z)$. In particular, the results of Theorem 1 apply to the complex analogue of the Takagi function C.*

We also give general conditions under which the Theorems 1 and 2 can be generalised to large classes of complex analogues of the Cantor function and the Takagi function. We also explain how our results give rise to a gradation between chaos and order for random complex dynamical systems.

References

[JS13] J. Jaerisch and H. Sumi, *Multifractal formalism for expanding rational semigroups and random complex dynamical systems*, preprint available at http://arxiv.org/abs/1311.6241 (2013).

[JS14] ———, *Hölder regularity of the complex analogue of the Takagi function*, Preprint (2014).

[Sum11] H. Sumi, *Random complex dynamics and semigroups of holomorphic maps*, Proc. London Math. Soc. (1) (2011), no. 102, 50–112.

[Sum13] ———, *Cooperation principle, stability and bifurcation in random complex dynamics*, Adv. Math. **245** (2013), 137–181.

Ledrappier-Young formula for self-affine measures

Balázs Bárány

Ledrappier and Young [6] introduced a relation between entropy, Lyapunov exponents and dimension for invariant measures of diffeomorphisms on compact manifolds. It is a widespread claim that self-affine measures satisfy this formula. The first result on a class of self-affine measures, for which the formula hold, was proven by Przytycki and Urbański [7]. Later, Feng and Hu [3] proved that if the linear parts of the mappings are diagonal matrices then the Ledrappier-Young formula holds. Moreover, Ledrappier [5] proved that the formula is valid for a special family of self-affine measures. The results presented here can be found in [1].

Setup and Results

Let $\mathcal{A} := \{A_1, A_2, \dots, A_N\}$ be a finite set of contracting, non-singular 2×2 matrices, and let $\Phi := \{f_i(\underline{x}) = A_i\underline{x} + \underline{t}_i\}_{i=1}^N$ be an iterated function system on the plane with affine mappings. There exists a unique non-empty compact subset Λ of $\mathbb{R}^2$ such that $\Lambda = \bigcup_{i=1}^N f_i(\Lambda)$. We call the set Λ as the attractor of Φ. We call a measure μ self-affine if its compactly supported with support Λ and there exists a $\underline{p} = (p_1, \dots, p_N)$ probability vector such that

$$\mu = \sum_{i=1}^N p_i \mu \circ f_i^{-1}.$$

Let $\Sigma = \{1, \dots, N\}^{\mathbb{Z}}$ be the two side and let $\Sigma^+ = \{1, \dots, N\}^{\mathbb{N}}$ be the set of one side infinite words. Let $\nu = \{p_1, \dots, p_N\}^{\mathbb{N}}$ be a Bernoulli measure on Σ^+, where $\underline{p} = (p_1, \dots, p_N)$ is a probability vector. Denote the entropy of ν by $h_\nu = -\sum_{i=1}^N p_i \log p_i$. If $\pi_+ : \Sigma^+ \mapsto \Lambda$ denotes the natural projection, i.e. $\pi_+(i_0, i_1, \dots) = \lim_{n\to\infty} f_{i_0} \circ \dots \circ f_{i_n}(\underline{0})$, then $\mu = \nu \circ \pi_+^{-1}$.

Oseledec Multiplicative Ergodic Theorem implies that there exist constants $0 < \chi_\mu^s \le \chi_\mu^{ss}$ such that

$$\lim_{n\to\infty} \frac{1}{n} \log \alpha_1(A_{i_0} \cdots A_{i_{n-1}}) = -\chi_\mu^s \text{ and}$$
$$\lim_{n\to\infty} \frac{1}{n} \log \alpha_2(A_{i_0} \cdots A_{i_{n-1}}) = -\chi_\mu^{ss} \text{ for } \nu\text{-a.e. } \mathbf{i} = (i_0, i_1, \dots) \in \Sigma^+.$$

Definition 1. We say that Φ satisfies the strong separation condition (SSC) if $f_i(\Lambda) \cap f_j(\Lambda) = \emptyset$ for every $i \neq j$.

Definition 2. We say that the set $\mathcal{A}$ of matrices satisfies the dominated splitting if there are constants $C, \delta > 0$ such that for every $n \ge 1$ and $i_0, \dots, i_{n-1} \in \{1, \dots, N\}$

$$\frac{\alpha_1(A_{i_0} \cdots A_{i_{n-1}})}{\alpha_2(A_{i_0} \cdots A_{i_{n-1}})} \ge Ce^{\delta n}.$$

Let $C_+ := \{(x, y) \in \mathbb{R}^2 \backslash \{(0, 0)\} : xy \ge 0\}$ be the standard positive cone. A cone is an image of C_+ by a linear isomorphism and a multicone is a disjoint union of finitely many cones.

Lemma 1 (Avila, Bochi, Guermelon, Rams, Yoccoz)**.** *The set $\mathcal{A}$ of matrices satisfies the dominated splitting then*

(1) *There is a multicone M such that $\bigcup_{i=1}^N A_i(M) \subset M^o$, where M^o denotes the interior of M.*
(2) *For every $\mathbf{i} \in \Sigma$ there are two one-dimensional subspaces $e_{ss}(\mathbf{i}), e_s(\mathbf{i})$ of $\mathbb{R}^2$ such that*
$$e_s(\mathbf{i}) = \bigcap_{n=1}^\infty A_{i_{-1}} \cdots A_{i_{-n}}(M) \text{ and } e_{ss}(\mathbf{i}) = \bigcap_{n=1}^\infty A_{i_0}^{-1} \cdots A_{i_{n-1}}^{-1}(\overline{M^c}),$$
where M^c denotes the complement of M.
(3) *$A_{i_0} e_i(\mathbf{i}) = e_i(\sigma\mathbf{i})$ for every $\mathbf{i} \in \Sigma$ and $i = s, ss$.*
(4) *There exists a constant $C > 0$ such that for every $\mathbf{i} \in \Sigma$*
$$C^{-1}\|A_{i_n} \cdots A_{i_0}|e_s(\mathbf{i})\| \le \alpha_1(A_{i_n} \cdots A_{i_0}) \le C\|A_{i_n} \cdots A_{i_0}|e_s(\mathbf{i})\| \text{ and}$$
$$C^{-1}\|A_{i_n} \cdots A_{i_0}|e_{ss}(\mathbf{i})\| \le \alpha_2(A_{i_n} \cdots A_{i_0}) \le C\|A_{i_n} \cdots A_{i_0}|e_{ss}(\mathbf{i})\|.$$

We call the family of subspaces $e_{ss}(\mathbf{i})$ as strong stable directions.

For example, family of matrices with strictly positive entries satisfies dominated splitting.

Theorem 1. *If $\mathcal{A}$ satisfies the dominated splitting and Φ satisfies the strong separation condition then μ is exact dimensional and*

$$\dim_H \mu = \frac{h_\nu}{\chi_\mu^{ss}} + \left(1 - \frac{\chi_\mu^s}{\chi_\mu^{ss}}\right) \dim_H \mu \circ (\mathrm{proj}_{\mathbf{i}}^{ss})^{-1} \text{ for } \nu\text{-almost every } \mathbf{i} \in \Sigma^+,$$

where $\mathrm{proj}_{\mathbf{i}}^{ss}$ denotes the orthogonal projection from $\mathbb{R}^2$ to the subspace perpendicular to $e_{ss}(\mathbf{i})$.

Let us define the push-down measure of ν by e_{ss} to $\mathbf{P}^1$ projective space as $\nu_{ss} := \nu \circ (e_{ss})^{-1}$.

Theorem 2. *If $\mathcal{A}$ satisfies the dominated splitting and Φ satisfies the strong separation condition, moreover, $\dim_H \nu_{ss} \geq \min\left\{1, \frac{h_\nu}{\chi_\mu^s}\right\}$ then*

$$\dim_H \mu = \min\left\{\frac{h_\nu}{\chi_\mu^s}, 1 + \frac{h_\nu - \chi_\mu^s}{\chi_\mu^{ss}}\right\}.$$

Applications

Falconer [2] introduced the subadditive pressure function and showed that if $\|A_i\| < 1/3$ for every $i = 1, \ldots, N$ then for $\mathcal{L}_{2N}$-almost every $\mathbf{t} = (\underline{t}_1, \ldots, \underline{t}_N) \in \mathbb{R}^{2N}$ the Hausdorff and box dimension coincide and equal to the root of the pressure. The bound was later extended to 1/2 by Solomyak, see [8]. Hueter and Lalley [4] proved that the Hausdorff and box dimension of a self-affine set coincide and equal to the root under some separation conditions.

Theorem 3. *Assume that $\mathcal{A}$ satisfies the dominated splitting and Φ satisfies the strong separation condition. If there exists a multicone M such that $A_i^{-1}(M^o) \subseteq M$ and $A_i^{-1}(M^o) \cap A_j^{-1}(M^o) = \emptyset$ for every $i \neq j$ and for every $i = 1, \ldots, N$, $\alpha_1(A_i)^2 \leq \alpha_2(A_i)$ then*

$$\dim_H \mu = \frac{h_\nu}{\chi_\mu^s} \leq 1.$$

Moreover, $\dim_H \Lambda = \dim_B \Lambda = s \leq 1$, where s is the unique root of the pressure function $P(s)$, defined by

$$P(s) := \lim_{n\to\infty} \frac{1}{n} \log \sum_{i_1,\ldots,i_n=1}^N \phi^s(A_{i_1} \cdots A_{i_n}),$$
$$\text{where } \phi^s(A) := \begin{cases} \alpha_1(A)^s & 0 \leq s \leq 1 \\ \alpha_1(A)\alpha_2(A)^{s-1} & 1 < s \leq 2 \\ (\alpha_1(A)\alpha_2(A))^{s/2} & s > 2. \end{cases}$$

Theorem 4 (B., Rams, in progress)**.** *Let*

$$\mathfrak{M} := \left\{A \in \mathbb{R}_+^{2\times 2} : 0 < \frac{|\det A|}{(\|A\|_1)^2} < \frac{1}{2},\ \|A\| < 1,\ \alpha_1(A)^2 < \alpha_2(A)\right\}.$$

Then for $\mathcal{L}_{4N}$-a.e. $\mathcal{A} = \{A_1, \ldots, A_N\} \in \mathfrak{M}^N$ the following holds: for every $(\underline{t}_1, \ldots, \underline{t}_N) \in \mathbb{R}^{2N}$ such that the IFS $\{A_i\underline{x} + \underline{t}_i\}_{i=1}^N$ satisfies the strong separation condition

$$\dim_H \mu = \min\left\{\frac{h_\nu}{\chi_\mu^s}, 1 + \frac{h_\nu - \chi_\mu^s}{\chi_\mu^{ss}}\right\}.$$

Moreover, $\dim_H \Lambda = \dim_B \Lambda = s$, where s is the unique root of the pressure function $P(s)$.

Let $\Phi_c = \{f_1, \dots, f_6\}$ be a parameterized family of IFSs on the plane given by the functions

$$f_1(\underline{x}) = \begin{bmatrix} \frac{1}{3} & 0 \\ 0 & c \end{bmatrix} \underline{x} + \begin{bmatrix} \frac{1}{3} \\ 0 \end{bmatrix}, \qquad f_2(\underline{x}) = \begin{bmatrix} \frac{1}{3} & 0 \\ 0 & c \end{bmatrix} \underline{x} + \begin{bmatrix} \frac{1}{3} \\ 1-c \end{bmatrix},$$
$$f_3(\underline{x}) = \begin{bmatrix} \frac{1}{3} & 0 \\ \frac{1}{2}-c & c \end{bmatrix} \underline{x} + \begin{bmatrix} 0 \\ \frac{1}{2} \end{bmatrix}, \qquad f_4(\underline{x}) = \begin{bmatrix} \frac{1}{3} & 0 \\ \frac{1}{2}-c & c \end{bmatrix} \underline{x} + \begin{bmatrix} \frac{2}{3} \\ 0 \end{bmatrix},$$
$$f_5(\underline{x}) = \begin{bmatrix} \frac{1}{3} & 0 \\ c-\frac{1}{2} & c \end{bmatrix} \underline{x} + \begin{bmatrix} 0 \\ \frac{1}{2}-c \end{bmatrix}, \qquad f_6(\underline{x}) = \begin{bmatrix} \frac{1}{3} & 0 \\ c-\frac{1}{2} & c \end{bmatrix} \underline{x} + \begin{bmatrix} \frac{2}{3} \\ 1-c \end{bmatrix},$$

where $0 < c < 1/2$. Denote Λ_c the attractor of Φ_c

Theorem 5. *For every* $0 < c < \frac{1}{3}$

$$\dim_H \Lambda_c = \dim_B \Lambda_c = 1 - \frac{\log 2}{\log c},$$

and there exists a set $\mathcal{C} \subseteq (\frac{1}{3}, \frac{1}{2})$ *such that* $\dim_P \mathcal{C} = 0$ *and*

$$\dim_H \Lambda_c = \dim_B \Lambda_c = 2 + \frac{\log 2c}{\log 3} \text{ for every } c \in \left(\frac{1}{3}, \frac{1}{2}\right) \setminus \mathcal{C}.$$

Acknowledgement: The research of Bárány was partially supported by the grant OTKA K104745.

References

[1] B. Bárány: On the Ledrappier-Young formula for self-affine measures, preprint, 2014.

[2] K. Falconer: The Hausdorff dimension of self-affine fractals, *Math. Proc. Camb. Phil. Soc.* **103** (1988), 339–350.

[3] D.-J. Feng and H. Hu: Dimension Theory of Iterated Function Systems, *Comm. Pure Appl. Math.* **62** (2009), no. 11, 1435-1500.

[4] I. Hueter and S. P. Lalley: Falconer's formula for the Hausdorff dimension of a self-affine set in $\mathbb{R}^2$, *Erg. Th. & Dyn. Sys.* **15** (1995), no. 1, 77–97.

[5] F. Ledrappier: On the dimension of some graphs, *Contemp. Math.* **135** (1992), 285–293.

[6] F. Ledrappier and L.-S. Young: The metric entropy of diffeomorphisms. II. Relations between entropy, exponents and dimension. *Ann. of Math. (2)* **122** (1985), no. 3, 540–574.

[7] F. Przytycki and M. Urbanski: On the Hausdorff dimension of some fractal sets, *Studia Math.* **93** (1989), no. 2, 155–186.

[8] B. Solomyak: Measure and dimension for some fractal families, *Math. Proc. Camb. Phil. Soc.* **124**, (1998), no. 3, 531–546.

Geometric rigidity for Kleinian groups

Mariusz Urbański

In the seminal paper [2] R. Bowen has proved that any quasi-Fuchsian group has either Hausdorff dimension > 1 or it is a Fuchsian group, meaning that its limit set is a geometric circle. This begun an extensive research of geometric rigidity in the theory of Kleinian groups and other fields, like for example holomorphic dynamics on $\widehat{\mathbb{C}}$. In the theory of Kleinian groups alone it includes [7], [5], [1], [9], [8], culminating in the work [3] of Kapovich, who proved that if G is a geometrically finite Kleinian group acting on some unit ball $\mathbb{B}_n \subset \mathbb{R}^n$, $n \geq 2$, and the Hausdorff dimension of its limit set $\Lambda(G)$ is equal to the topological dimension of $\Lambda(G)$, then the limit set $\Lambda(G)$ is some k-dimensional geometric sphere in $\overline{\mathbb{B}_n}$. Together with T. Das and D. Simmons we prove more. Namely:

Theorem 1. *Let G be a Kleinian group acting on some unit ball $\mathbb{B}_n \subset \mathbb{R}^n$, $n \geq 2$. Let $\Lambda_r(G)$ denote the subset of $\Lambda(G)$ consisting of all radial (conical) points. Assume that*

$$\mathrm{HD}(\Lambda(G) \setminus \Lambda_r(G)) < \mathrm{HD}(\Lambda(G)).$$

Then the following geometric rigidity holds:

If $HD(\Lambda(G))$, the Hausdorff dimension of its limit set $\Lambda(G)$, is equal to the topological dimension of $\Lambda(G)$, then the limit set $\Lambda(G)$ is equal to some k-dimensional geometric sphere in $\overline{\mathbb{B}_n}$.

Our strategy stems from the paper [4], where an alogous rigidity result was proved for countable alphabet conformal iterated function systems. Let

$$k := \mathrm{HD}(\Lambda(G)).$$

Our approach is based on an entirely differen idea than the existing proofs for Kleinian groups. The first step would be to prove that the limit set $\Lambda(G)$ is not k-purely unrectifiable, the second step invoking ergodicity of the action of G on $\Lambda(G)$ would show that $\Lambda(G)$ is k-rectifiable, and the third, final, step would be given by a zooming procedure on points of $\Lambda_r(G)$. Let me also mention that there is a big variety of Kleinian groups that are not geometrically finite but for which $\mathrm{HD}(\Lambda(G) \setminus \Lambda_r(G)) < \mathrm{HD}(\Lambda(G))$ holds.

We go further, beyond the finite-dimensional case, to the setting of discreete groups acting on the unit ball in a separable Hilbert space. The difficulty then is that there is no analogue of Lebesgue measure on the Hilbert space, (consequently) no such measure on the corresponding Grasmannian, and no known Federer's type projection theorems. However, even then we prover the following.

Theorem 2. *Let G be a Kleinian group acting on the unit ball $\mathbb{B}_\infty \subset \ell_2$. Assume as above that*

$$\mathrm{HD}(\Lambda(G) \setminus \Lambda_r(G)) < \mathrm{HD}(\Lambda(G)).$$

Assume in addtion that the upper box-counting dimension of $\Lambda(G)$ is finite. Then the following geometric rigidity holds:

If $\mathrm{HD}(\Lambda(G))$*, the Hausdorff dimension of its limit set* $\Lambda(G)$*, is equal to the topological dimension of* $\Lambda(G)$*, then the limit set* $\Lambda(G)$ *is equal to some* k*-dimensional geometric sphere in* $\overline{\mathbb{B}_\infty}$*.*

We have also proved the following theorem.

Theorem 3. *Suppose that* $f : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ *is a rational function of degree* $d \geq 2$ *whose limit set is not totally disconnected (in other words its topological dimension is equal to* 1*). Assume that*

$$\mathrm{H}(J(f) \setminus J_r(f)) < \mathrm{HD}(J_f).$$

If $\mathrm{HD}(J(f)) = 1$*, then* $J(T)$ *is a geometric circle.*

Remark. Note that condition $\mathrm{H}(J(f) \setminus J_r(f)) < \mathrm{HD}(J_f)$ is satisfied for all topological Collet–Eckmann rational functions since then the difference $J(f) \setminus J_r(f)$ is a countable set. This in particular comprises all non–recurrent rational functions and all expanding rational functions. It also contains all parabolic rational functions.

References

[1] C. Bishop, P. Jones, *Hausdorff dimension and Kleinian groups*, Acta Math., **179** (1997), 139.

[2] R. Bowen, *Hausdorff dimension of quasi-circles*, Publ. Math. IHES, **50** (1980), 11–25.

[3] M. Kapovich, *Homological dimension and critical exponent of Kleinian groups*, Geom. Funct. Anal. **18** (2009), 2017–2054.

[4] V. Mayer, M. Urbański, *Finer geometric rigidity of limit sets of conformal IFS*, Proc. Amer. Math. Soc. **131** (2003), 36953702.

[5] D. Sullivan, *Entropy, Hausdorff measures old and new, and the limit set of a geometrically finite Kleinian groups*, Acta. Math. **153** (1984), 259–277.

[6] D. Sullivan, *Quasiconformal homeomorphisms in dynamics, topology, and geometry*, Proc. Internat. Congress of Math., Berkeley, Amer. Math. Soc., 1986, 1216–1228.

[7] P. Tukia, The Hausdorff dimension of the limit set of a geometrically finite Kleinian group, Acta Math. **152** (1984) 127–140.

[8] X. Xie, *A Bowen type rigidity theorem for non-compact hyperbolic groups*, Mathematische Zeitschrift **259** (2008), 249–253.

[9] C. Yue, *Dimension and rigidity of quasifuchsian representations*, Ann. of Math., **143** (1996), 331–355.

Dimension of limsup-sets of random covers

Tomas Persson

1. Introduction

Let d be a natural number. We consider the d-dimensional torus $\mathbb{T}^d$, and a sequence of open sets $U_i \subset \mathbb{T}^d$. The random vectors v_i are independent and uniformly distributed on the torus $\mathbb{T}^d$, and are used to translate the sets U_i, hence producing a sequence $V_i(v_i)$ of random sets defined by $V_i(v_i) = U_i + v_i$. We are interested in the typical behaviour of the limsup-set

$$E(v) = \limsup_{i\to\infty} V_i(v_i),$$

that is, the set of points on the torus that are covered by infinitely many sets $V_i(v_i)$.

Limsup-sets often possess a large intersection property, see Falconer [2]. This means that the set belongs, for some $0 < s \leq d$, to the class $\mathscr{G}^s(\mathbb{T}^d)$, where $\mathscr{G}^s(\mathbb{T}^d)$ is the largest collection of G_δ subsets of $\mathbb{T}^d$ with the property that it is closed under countable intersections and images of bi-Lipschitz maps, and any set in $\mathscr{G}^s$ has Hausdorff dimension at least s. For instance, we have $\mathscr{G}^s(\mathbb{T}^d) \subset \mathscr{G}^t(\mathbb{T}^d)$ provided $t < s$, and if $A \in \mathscr{G}^t(\mathbb{T}^d)$ for all $t < s$, then $A \in \mathscr{G}^s(\mathbb{T}^d)$. For more properties of these classes, relevant in this paper, we refer the reader to the paper [6]. Here we shall be concerned with the large intersection properties of typical $E(v)$.

Let λ denote the d-dimensional Lebesgue measure on $\mathbb{T}^d$. For $0 < s < d$ and a set $A \subset \mathbb{T}^d$, we define the s-energy of A as

$$I_s(A) = \iint_{A\times A} |x-y|^{-s}\,\mathrm{d}x\mathrm{d}y,$$

where $|x-y|$ denotes the distance between the points x and y.

The aim of this note is to present the following theorem from [4]. For the background of this and other similar results on random coverings of tori, we refer the reader to [3].

Theorem 1. *The set $E(v)$ is almost surely in the class $\mathscr{G}^s(\mathbb{T}^d)$, where s is defined by*

$$s = \inf\{\, t : \sum_{i=1}^{\infty} \frac{\lambda(U_i)^2}{I_t(U_i)} < \infty \ \text{or}\ t = d \,\}.$$

In the paper [3], Järvenpää, Järvenpää, Koivusalo, Li, and Suomala proved a similar result. They imposed more restrictive assumptions on the sets, and they only proved the dimension result, not the large intersection property. It is not immediately clear if the result in [3] provides the same dimension result that Theorem 1 does, under the extra conditions imposed in [3]. However, a simple estimate on the s-energy of a cube shows that the result of Järvenpää, Järvenpää, Koivusalo, Li, and Suomala follows from Theorem 1, see [4] for more details.

2. Shrinking Targets

If we instead translate the sets U_i with a randomly chosen orbit of a sufficiently mixing dynamical system, results similar to that of Theorem 1 can be proved using similar methods. Then, in the case that the sets U_i are balls with shrinking radius, we get a so called shrinking target result. This has been done together with Michał Rams in [5].

3. How to prove Theorem 1

The proof is based on the following lemma from [6], that gives us a method to determine if a limsup-set belongs to the class $\mathscr{G}^s(\mathbb{T}^d)$. The lemma is only stated and proved for $d = 1$ in [6], but it holds for any d, and only minor changes in the proof are required to make it work for $d > 1$. Also, the statement in [6] is for $[0, 1]$ instead of $\mathbb{T}^1$, but this difference is not substantial.

Lemma 1. *Let E_k be open subsets of $\mathbb{T}^d$, and μ_k Borel probability measures, with support in the closure of E_k, that converge weakly to a measure μ with density h in L^2. Assume that $\mu(I) > 0$ for all cubes $I \subset [0, 1)^d$ with non-empty interior, and assume that for each $\varepsilon > 0$, there is a constant C_ε, such that*

$$|I|^{1+\varepsilon} \|h\chi_I\|_2^2 \leq C_\varepsilon \|h\chi_I\|_1^2 \tag{1}$$

holds for any cube $I \subset \mathbb{T}^d$. If there is a constant C such that

$$\iint |x - y|^{-s} \, \mathrm{d}\mu_k(x) \mathrm{d}\mu_k(y) \leq C \tag{2}$$

holds for all k, then $\limsup E_k$ *is in the class $\mathscr{G}^s(\mathbb{T}^d)$.*

In our application of Lemma 1, the limit measure μ will be the Lebesgue measure, and therefore the assumption (1) will be automatically fulfilled. Note also that the proof of Lemma 1 can be significantly simplified in this case.

Let $E_k(v) = \bigcup_{i=m_k}^{k} V_i(v_i)$, where $m_k < k$ is a sequence increasing to infinity. We then have $\limsup E_k(v) = E(v) = \limsup V_i(v_i)$. Define $\mu_k = \sum_{i=m_k}^{k} c_{i,k} \lambda|_{V_i(v_i)}$, where $c_{i,k}$ are constants that will be specified below, but are such that μ_k are probability measures. In particular, $\sum_{i=m_k}^{k} \sum_{j=m_k}^{k} c_{i,k} c_{j,k} \lambda(U_i) \lambda(U_j) \leq 1$.

Let $s = \inf\{\, t : \sum_i \lambda(U_i)^2 / I_t(U_i) < \infty \,\}$, and pick t with $t < s$ and $t < d$. We need to prove that with probability 1, we have $E(v) \in \mathscr{G}^t(\mathbb{T}^d)$. If we put $c_{i,k} = c_k \lambda(U_i)/I_t(U_i)$, were c_k is defined by $c_k = \bigl(\sum_{i=m_k}^{k} \lambda(U_i)^2/I_t(U_i)\bigr)^{-1}$, then it is easy to check that with probability 1, the condition (2) holds for some constant C. Moreover, we observe that the fact that $c_k \to 0$ as $k \to \infty$, implies that there is a sequence n_k such that μ_{n_k} almost surely converges weakly to the Lebesgue measure. The full details of these arguments can be found in [4]. Lemma 1 now finishes the proof.

References

[1] A. Durand, *On randomly placed arcs on the circle*, Recent developments in fractals and related fields, 343–351, Birkhäuser, Boston, 2010.

[2] K. Falconer, *Sets with large intersection properties*, Journal of the London Mathematical Society 49 (1994), no. 2, 267–280.

[3] E. Järvenpää, M. Järvenpää, H. Koivusalo, B. Li, and V. Suomala, *Hausdorff dimension of affine random covering sets in torus*, to appear in Annales de l'Institut Henri Poincaré Probabilités et Statistiques, arXiv:1207.3615

[4] T. Persson, *A Note on Random Coverings of Tori*, arXiv:1307.1455.

[5] T. Persson, M. Rams *On Shrinking Targets for Piecewise Expanding Interval Maps*, arXiv:1406.6785.

[6] T. Persson, H. Reeve, *A Frostman type lemma for sets with large intersections, and an application to Diophantine approximation*, to appear in Proceedings of the Edinburgh Mathematical Society, arXiv:1302.0954.

Random dynamics of transcendental functions

Volker Mayer

In this talk we present joint work with Mariusz Urbański on random dynamics of hyperbolic entire and meromorphic functions of finite order and whose derivative satisfy some growth condition at infinity. This class contains most of the classical families of transcendental functions: all periodic functions (tangent, sine, exponential and elliptic), functions with polynomial Schwarzian derivative, the cosine-root family and many more (see [11] where the deterministic case has been treated in detail). Based on uniform versions of Nevanlinna's value distribution theory we first build a thermodynamical formalism which, in particular, produces unique geometric and fiberwise invariant Gibbs states. Then we go further and explain a spectral gap property for the associated transfer operator along with exponential decay of correlations and a central limit theorem. All of this is part of our recent preprint [12]. We now explain more in detail the content of this work.

Random dynamics is actually a quite active field. The first work on random rational functions is due to Fornaess and Sibony [7]. Related to this is Rugh's paper on random repellers [16] and Sumi's work on rational semi-groups (see for example [18, 19]). A complete picture including thermodynamics and spectral gap is contained in [9] which concerns a much wider class of distance expanding random maps, a class originally introduced by Ruelle [15]. Recently random dynamics of countable infinite Markov shifts [5, 17] and graph directed Markov systems [14] have been treated. Here we extend the picture to a situation where the maps are also countable infinite – to – one, where the phase space is not compact and where in addition there is no Markov structure.

Given a probability space $(X, \mathcal{F}, m)$ along with an invertible ergodic transformation $\theta : X \to X$, we consider the dynamics of

$$f_x^n = f_{\theta^{n-1}(x)} \circ ... \circ f_x \quad , \quad n \geq 1$$

where $f_x : \mathbb{C} \to \widehat{\mathbb{C}}$, $x \in X$, is a family of transcendental functions depending measurably on $x \in X$. Like in the deterministic case, the normal family behaviour of $(f_x^n)_n$ splits the plane into two parts and one is interested in the chaotic part $\mathcal{J}_x$, called random Julia set. Quite general *transcendental random systems* $f_x : \mathcal{J}_x \to \mathcal{J}_{\theta(x)}$, $x \in X$, are considered in this paper and, as already has been mentioned among the major difficulties one encounters is that the phase space $\mathcal{J}_x$ is unbounded and the functions are of infinite degree.

In the deterministic case, this difficulty has been overcome in [11] The key idea was to replace the Euclidean metric by a metric having an appropriate singularity at infinity. Once this is done, one can use Nevanlinna's value distribution theory to show that the corresponding transfer operator is well defined and bounded.

The present paper treats random dynamics of the families of functions considered in [11]. Again we start with an appropriated choice of metric in order to be able to control the transfer operator. This time we make use of the uniform versions of Nevanlinna's theorems available in Cherry-Ye's book [3].

Then, since we are dealing with random dynamics, measurability of all involved operators, measures and functions has to be checked. This point has sometimes been neglected in the literature (see the discussion in [8]) or is the reason for additional assumptions. Here we take advantage of Crauel's framework [4] and treat measurability very carefully. Moreover, this allows us to have a global, in terms of skew product, approach which, for example, produces directly measurable families of conditional measures.

Having then good behaving transfer operators and measurability, we can proceed with building the thermodynamical formalism. As the result, we prove the existence and uniqueness of fiberwise conformal measures and the existence and uniqueness of invariant densities. This gives rise to the existence and uniqueness of fiberwise invariant measures absolutely continuous with respect to the conformal ones.

In order to get further properties, the method introduced by Birkhoff [1] based on positive cones and the Hilbert distance can be employed in random dynamics. However, the phase spaces in the present work being non-compact, the Hilbert distance is of much less use. Indeed, cones of functions of finite distance are too small since all of its members must be comparable near infinity. Fortunately there is a very nice contraction lemma in Bowen's manuscript [2]. In order to be able to adapt it to the present setting, we first produce, via a delicate construction, non-standard appropriate invariant cones. Once this is done, the Bowen-like argument is quite elementary. In this sense, the present work, incidentally, simplifies the deterministic work [11]. In conclusion, we get a spectral gap property. It then almost immediately implies exponential decay of correlations and a Central Limit Theorem.

Acknowledgement: This research was supported in part through the program "Research in Pairs" (resp. "Oberwolfach Leibniz Fellows") by the Mathematisches Forschungsinstitut Oberwolfach in 2013 and by the Labex CEMPI (ANR-11-LABX-0007-01).

References

[1] Garrett Birkhoff. Extensions of Jentzsch's theorem. *Trans. Amer. Math. Soc.*, 85:219–227, 1957.

[2] Rufus Bowen. *Equilibrium states and the ergodic theory of Anosov diffeomorphisms.* Springer-Verlag, Berlin, 1975. Lecture Notes in Mathematics, Vol. 470.

[3] William Cherry and Zhuan Ye. *Nevanlinna's theory of value distribution.* Springer Monographs in Mathematics. Springer-Verlag, Berlin, 2001. The second main theorem and its error terms.

[4] Hans Crauel. *Random probability measures on Polish spaces*, volume 11 of *Stochastics Monographs.* Taylor & Francis, London, 2002.

[5] Manfred Denker, Yuri Kifer, and Manuel Stadlbauer. Thermodynamic formalism for random countable Markov shifts. *Discrete Contin. Dyn. Syst.*, 22(1-2):131–164, 2008.
[6] Alexandre Eremenko. Ahlfors' contribution to the theory of meromorphic functions. In *Lectures in memory of Lars Ahlfors (Haifa, 1996)*, volume 14 of *Israel Math. Conf. Proc.*, pages 41–63. Bar-Ilan Univ., Ramat Gan, 2000.
[7] John Erik Fornæss and Nessim Sibony. Random iterations of rational functions. *Ergodic Theory Dynam. Systems*, 11(4):687–708, 1991.
[8] Yuri Kifer. Thermodynamic formalism for random transformations revisited. *Stochastics and Dynam.*, 8:77–102, 2008.
[9] Volker Mayer, Bartlomiej Skorulski, and Mariusz Urbanski. *Distance expanding random mappings, thermodynamical formalism, Gibbs measures and fractal geometry*, volume 2036 of *Lecture Notes in Mathematics*. Springer, Heidelberg, 2011.
[10] Volker Mayer and Mariusz Urbański. Geometric thermodynamic formalism and real analyticity for meromorphic functions of finite order. *Ergodic Theory Dynam. Systems*, 28(3):915–946, 2008.
[11] Volker Mayer and Mariusz Urbański. Thermodynamical formalism and multifractal analysis for meromorphic functions of finite order. *Mem. Amer. Math. Soc.*, 203(954):vi+107, 2010.
[12] Volker Mayer and Mariusz Urbański. Random Dynamics of Transcendental Functions. *Oberwolfach Preprint* OWP2014 - 12 and *ArXiv* 1409.7179 .
[13] R. Nevanlinna. *Eindeutige analytische Funktionen*. Springer-Verlag, Berlin, 1974. Zweite Auflage, Reprint, Die Grundlehren der mathematischen Wissenschaften, Band 46.
[14] Mario Roy and Mariusz Urbański. Random graph directed Markov systems. *Discrete Contin. Dyn. Syst.*, 30(1):261–298, 2011.
[15] David Ruelle. *Thermodynamic formalism*, volume 5 of *Encyclopedia of Mathematics and its Applications*. Addison-Wesley Publishing Co., Reading, Mass., 1978. The mathematical structures of classical equilibrium statistical mechanics, With a foreword by Giovanni Gallavotti and Gian-Carlo Rota.
[16] Hans Henrik Rugh. On the dimension of conformal repellors. randomness and parameter dependency. Ann of Math., vol 168, no 3, 695-748 (2008).
[17] Manuel Stadlbauer. On random topological Markov chains with big images and preimages. *Stoch. Dyn.*, 10(1):77–95, 2010.
[18] Hiroki Sumi. On dynamics of hyperbolic rational semigroups. *J. Math. Kyoto Univ.*, 37(4):717–733, 1997.
[19] Hiroki Sumi. Random complex dynamics and semigroups of holomorphic maps. *Proc. Lond. Math. Soc. (3)*, 102(1):50–112, 2011.

Fractal geometry of limiting symbols for modular subgroups

Marc Kesseböhmer
(joint work with Bernd O. Stratmann)

Let $\mathcal{C}_2(G)$ refer to the space of cusp forms of weight 2 for some arbitrary modular subgroup G, i.e. G is a finite index subgroup of the modular group $\Gamma := \mathrm{PSL}_2(\mathbb{Z})$. These cusp forms fulfill the following properties: f is holomorphic on $\mathbb{H}$ as well as in each cusp of G; $f = (g') \cdot (f \circ g)$ for all $g \in G$; f vanishes at each cusp of G. It is well known that there is a dual pairing between $\mathcal{C}_2(G)$ and the first homology group $H_1(M_G, \mathbb{R})$ of the associated compactified cusped Riemann surface M_G of genus $\mathfrak{g}$. That is, we have

$$\langle \cdot , \cdot \rangle : H_1(M_G, \mathbb{R}) \times \mathcal{C}_2(G) \to \mathbb{C}, \text{ where } \langle \gamma, f \rangle := \int_\gamma f(z)\, dz.$$

Each element of $H_1(M_G,\mathbb{R})$ can be represented by integrating the 1-form $f\,dz$ along some smooth path between two points ξ,η in $\mathrm{H}\cup P^1(\mathbb{Q})$, and this determines the modular symbol $\{\xi,\eta\}_G \in H_1(M_G,\mathbb{R})$. This idea goes back to Drinfel′d and Manin who found in [8, 2] that for each $\xi,\eta\in P^1(\mathbb{Q})$, we have

$$\{\xi,\eta\}_{\Gamma_0(N)} \in H_1\left(M_{\Gamma_0(N)},\mathbb{Q}\right).$$

In here, $\Gamma_0(N)$ denotes the congruence subgroups defined by

$$\Gamma_0(N) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma \,|\, c\equiv 0 \mod N \right\} \text{ for } N\in\mathbb{N},$$

A possible way to extend these symbols to the non-cuspital boundary of hyperbolic space, and therefore to give a non-trivial homological meaning to algebraically invisible parts of $H_1(M_G,\mathbb{R})$, has been suggested by Manin and Marcolli [9, 10], see also [1]. They introduced the concept *limiting modular symbol*, which is given for $x\in\mathbb{R}$ (whenever the limit exists)

$$\ell_G(x) := \lim_{t\to\infty}\frac{1}{t}\{i, x+i\exp(-t)\}_G \in H_1(M_G,\mathbb{R}).$$

Note that the limit in the definition of ℓ_G exists if and only if it exists for each 1-form $f\,dz$ with $f\in\mathcal{C}_2(G)$, and hence it is sufficient to compute it for a fixed complex basis $f_1+if_2,\ldots,f_{2\mathfrak{g}-1}+if_{2\mathfrak{g}}$ of $\mathcal{C}_2(G)$. Our aim is to use fractal geometry in order to investigate the level sets which arise naturally from these limiting modular symbols. That is, for $\alpha\in\mathbb{R}^{2\mathfrak{g}}$ we consider

$$\mathcal{F}_\alpha := \{x\in\mathbb{R} : (\langle\ell_G(x),f_1\rangle,\ldots,\langle\ell_G(x),f_{2\mathfrak{g}}\rangle) = \alpha\}.$$

A first analysis of this type of level sets was given in [9] and [11] for modular subgroups which satisfy the there so called 'Red-condition'. There it has been shown that for these groups $t^{-1}\{i,x+i\exp(-t)\}_G$ converges weakly to zero with respect to the Lebesgue measure on the unit interval. Subsequently, this result was improved in [11] by showing that $\ell_G(x)$ is equal to zero Lebesgue-almost everywhere. Besides, these papers obtained "non-vanishing" of limiting modular symbols only for the end points of closed geodesics, that is for quadratic surds. In these cases the limiting modular symbol turns out to be given by integrating along the closed geodesic and then normalising by the hyperbolic length of that geodesic Our main results in [6] are summarised as follows, where $\widehat{\beta}_G:\mathbb{R}^{2\mathfrak{g}}\to\mathbb{R}$ refers to the proper concave (negative) Legendre transform of the proper convex function $\beta_G:\mathbb{R}^{2\mathfrak{g}}\to\mathbb{R}$, given by $\widehat{\beta}_G(\alpha) := \inf_{t\in\mathbb{R}^{2\mathfrak{g}}}(\beta_G(t)-(\alpha|t))$.

For an arbitrary modular subgroup G we have that the Red-condition is fulfilled. Moreover, for $\mathfrak{g}\geq 1$ there exists a strictly convex and differentiable function $\beta_G:\mathbb{R}^{2\mathfrak{g}}\to\mathbb{R}$ such that for each $\alpha\in\nabla\beta_G\left(\mathbb{R}^{2\mathfrak{g}}\right)\subset\mathbb{R}^{2\mathfrak{g}}$,

$$\dim_H(\mathcal{F}_\alpha) = \widehat{\beta}_G(\alpha). \tag{1}$$

In here, we have that $\beta_G(0)=1$, and that β_G has a unique minimum at 0. Also, we in particular have $\ell_G(\mathcal{F}_\alpha)=\{h_\alpha\}$, where $h_\alpha\in H_1(M_G,\mathbb{R})$ is uniquely determined *by* $(\langle h_\alpha, f_1\rangle, \ldots, \langle h_\alpha, f_{2\mathfrak{g}}\rangle) = \alpha$. *Furthermore, we have*

$$\overline{\nabla\beta_G\left(\mathbb{R}^{2\mathfrak{g}}\right)} = \left\{\alpha \in \mathbb{R}^{2\mathfrak{g}} : \mathcal{F}(\alpha) \neq \emptyset\right\}.$$

The methods of proof are mainly taken from [3, 4, 5]; we use of (higher-dimensional) thermodynamic formalism to determine the function β_G and derive its properties. For this we first define a suitable shift space Σ_G of certain pairs using the continued fraction expansion in the first coordinate and the co-sets Γ/G as its second coordinate. Then let $I : \Sigma_G \to \mathbb{R}$ refer to the canonical geometric potential function associated with the Gauss-map $\mathcal{G}$, given by

$$I : ((x_k, e_k))_k \mapsto \log\left|\mathcal{G}'\left([|x_1|, |x_2|, \ldots]\right)\right|.$$

Also, define $J : \Sigma_G \to \mathbb{R}^{2\mathfrak{g}}$ given for $\left((x_k, e_k)\right)_k \in \Sigma_G$ by

$$J\left(\left((x_k, e_k)\right)_k\right) := \left(\langle\{e_1(i\infty), e_1(0)\}_G, f_1\rangle, \ldots, \langle\{e_1(i\infty), e_1(0)\}_G, f_{2\mathfrak{g}}\rangle\right).$$

Finally, the modular pressure function $P : \mathbb{R}^{2\mathfrak{g}} \times (1/2, \infty) \to \mathbb{R}$ associated with J is then defined for $t = (t_1, \ldots, t_{2\mathfrak{g}}) \in \mathbb{R}^{2\mathfrak{g}}$ and $\beta \in (1/2, \infty)$ by (here, $[\![\ ]\!]$ refers to the cylinder set in Σ_G)

$$P(t, \beta) := \lim_{n\to\infty} \frac{1}{n} \log \sum_{\omega \in \Sigma_G^n} \exp S_n \sup_{x \in [\![\omega]\!]} \left((t|J(x)) - \beta I(x)\right).$$

Note that $(t|J) - \beta I$ is acceptable in the sense of Mauldin and Urbański ([12, Def. 2.1.4]), and this implies that P is well-defined. Also, since J is Hölder continuous and bounded, one immediately verifies that $(t|J) - \beta I$ is summable for each $\beta > 1/2$ (for the definition of summability we refer to [12, p. 27]). In particular, this also gives that P is continuous. Since $\lim_{\beta \searrow \frac{1}{2}} P(t, \beta) = \infty$ and $\lim_{\beta\to\infty} P(t, \beta) = -\infty$ and since P is continuous, the function $\beta_G : \mathbb{R}^{2\mathfrak{g}} \to (1/2, \infty)$ is for $t \in \mathbb{R}^{2\mathfrak{g}}$ implicitly given by $P(t, \beta_G(t)) = 0$. Finally, the fact that the spectrum is indeed non-trivial, that is $\nabla\beta_G\left(\mathbb{R}^{2\mathfrak{g}}\right)$ has a non-empty interior, follows from results in [7].

References

[1] A. Connes, M. Marcolli. A walk through the noncommutative garden. In: *An invitation to Noncommutative Geometry*, 1–128, World Scientific, 2008.

[2] V. G. Drinfel′d. Two theorems on modular curves. *Funkcional. Anal. i Priložen.*, 7(2):83–84, 1973.

[3] M. Kesseböhmer, B. O. Stratmann. A multifractal formalism for growth rates and application to geometrically finite Kleinian groups. *Ergod. Th. & Dynam. Sys.* , 24 (1): 141–170, 2004.

[4] M. Kesseböhmer, B. O. Stratmann. Stern-Brocot pressure and multifractal spectra in ergodic theory of numbers. *Stoch. Dyn.*, 4(1):77–84, 2004.

[5] M. Kesseböhmer, B. O. Stratmann. A multifractal analysis for Stern-Brocot intervals, continued fractions and Diophantine growth rates. *J. reine angew. Math.* 605:133–163, 2007.

[6] M. Kesseböhmer, B. O. Stratmann. Homology at infinity; fractal geometry of limiting symbols for modular groups *Topology* 46(5):469–491, 2007.

[7] M. Kesseböhmer, M. Urbański. Higher-dimensional multifractal value sets for infinite graph directed Markov systems. *Nonlinearity* 20(8):1969–1985, 2007.

[8] Yu. I. Manin. Parabolic points and zeta functions of modular curves. *Izv. Akad. Nauk SSSR Ser. Mat.*, 36:19–66, 1972.
[9] Yu. I. Manin, M. Marcolli. Holography principle and arithmetic of algebraic curves. *Adv. Theor. Math. Phys.*, 5(3):617–650, 2001.
[10] Yu. I. Manin, M. Marcolli. Continued fractions, modular symbols, and noncommutative geometry. *Selecta Math. (N.S.)*, 8(3):475–521, 2002.
[11] M. Marcolli. Limiting modular symbols and the Lyapunov spectrum. *J. Number Theory*, 98(2):348–376, 2003.
[12] R. D. Mauldin, M. Urbański. *Graph directed Markov systems, geometry and dynamics of limit sets*. Cambridge Tracts in Mathematics, vol. 148. Cambridge, 2003.

Multiple mixing from weak hyperbolicity by the Hopf argument

Yves Coudène
(joint work with B. Hasselblatt and S. Troubetzkoy)

The Hopf argument is a standard tool in the field of hyperbolic dynamical systems for deriving ergodicity in the absence of an algebraic structure (the alternative tool being the theory of equilibrium states). Our purpose is to show how much more than ergodicity it can produce. Specifically, in its original form the Hopf argument establishes ergodicity when the contracting and expanding partitions of a dynamical system are jointly ergodic. We present a recent refinement originally due to Babillot that directly obtains mixing from joint ergodicity of these 2 partitions. Further, we publicize the observation that the argument produces multiple mixing if the stable partition is ergodic by itself, and we give a simple proof of ergodicity of the stable distribution. Taken together, this gives a simple, self-contained general proof of multiple mixing.

In the following, X is a metric space, $f : X \to X$ a Borel invertible transformation, μ a probability measure invariant by f and $\varphi : X \to \mathbf{R}$ a square integrable function. The stable and unstable sets of $x \in X$ are defined by

$$W^s(x) = \{y \in X \mid d(f^n(x), f^n(y)) \underset{n\to\infty}{\longrightarrow} 0\},$$

$$W^u(x) = \{y \in X \mid d(f^n(x), f^n(y)) \underset{n\to-\infty}{\longrightarrow} 0\}.$$

We say that a Borel function $\varphi : X \to \mathbf{R}$ is *W^s-invariant* if we can find a set $\Omega \subset X$ of full measure such that for all $x, y \in \Omega$, the condition $y \in W^s(x)$ implies $\varphi(y) = \varphi(x)$. The distribution W^s is *ergodic* if the only W^s-invariant functions are constant.

Finally we recall that the transformation f is *mixing* if $\varphi \circ f^n$ converges to $\int \varphi$ in the weak topology on L^2 when n goes to infinity. It is *mixing of all orders* if for all k and $\varphi_1, ...\varphi_k$, the sequence $\varphi_1 \circ f^{n_1}\, \varphi_2 \circ f^{n_1+n_2} \ldots \varphi_k \circ f^{n_1+\ldots+n_k}$ converges to $\int \varphi_1 \ldots \int \varphi_k$ when all the sequences n_i go to infinity.

The following result gives us a first relation between the ergodic properties of the stable distribution W^s and the mixing of the transformation f.

Theorem [1]**.** *All accumulation points of the sequence $\varphi \circ f^n$, with respect to the weak topology on L^2, when n goes to $+\infty$, are W^s-invariant and W^u-invariant.*

We also have the following result when it comes to mixing of all orders.

Theorem. *All accumulation points of the sequence* $\varphi_1 \circ f^{n_1} \varphi_2 \circ f^{n_1+n_2} \dots \varphi_k \circ f^{n_1+\dots+n_k}$ *with respect to the weak topology on* L^2*, when the* n_i *go to* $+\infty$*, are* W^s*-invariant.*

Open question. *Are they also* W^u*-invariant?*

Since we don't know that these accumulation points are W^u-invariant, we need to show that the stable distribution W^s is ergodic, in order to prove mixing of all orders for the transformation f. This can be done if we assume that f satisfies some weak hyperbolicity condition, to be explained in the following definition.

Definition. $V \subset X$ *is a* product set *for* f *if*
- *there exists a measurable partition* W^u_{loc} *of* V *that satisfies*

$$W^u_{loc}(x) \subset W^u(x), \qquad W^u_{loc}(f(x)) \subset f(W^u_{loc}(x)),$$

- *there is a product map* $(x, y) \mapsto [x, y]$ *from* $V \times V$ *to* V*, such that*

$$[x, y] \in W^s(x) \cap W^u_{loc}(y),$$

- *the map* $\pi^y(x) = [x, y]$ *satisfies the following absolute continuity property:*

$$\pi^y_* \mu^u_x \ll \mu^u_y, \text{ for all } x, y \in V,$$

where (μ^u_x) *are the conditional measures of* μ *along* W^u_{loc}.

We can now state our main theorem, which goes from total ergodicity to multiple mixing under the assumption that there is a product set for the transformation.

Theorem [2]**.** *Assume that there is a product set* V *of positive measure and that* f^N *is ergodic for all* $N > 0$*. Then* W^s *is ergodic and* f *is mixing of all orders.*

This result admits a short proof and gives mixing of all orders for Anosov systems, hyperbolic measures, Sinaï billiards and more, without resorting to entropy theory.

References

[1] Yves Coudène, *On invariant distributions and mixing.* Ergodic Theory and Dynamical Systems **27**, no. 1, (2007), 109–112

[2] Y. Coudène, B. Hasselblatt, S. Troubetzkoy, *multiple mixing from weak hyperbolicity by the Hopf argument*, http://arxiv.org/abs/1406.4015

Regularity of Hausdorff measure function

Anna Zdunik

Developing the work of L. Olsen we deal with the question of continuity of the numerical value of Hausdorff measures in families of conformal dynamical systems.

We prove Hölder continuity of the function ascribing to a parameter the numerical value of the Hausdorff measure of the limit set, for naturally parametrized families of both conformal iterated function systems in $\mathbb{R}^k$, $k \geq 3$, and linear iterated function systems consisting of similarities in $\mathbb{R}^k$, $k \geq 1$ both satisfying the Strong Separation Condition, and for analytic families of conformal expanding repellers in the complex plane $\mathbb{C}$. For families of naturally parametrized linear IFSs in $\mathbb{R}$, satisfying the Strong Separation Condition, this function is piecewise real–analytic.

We prove continuity for maps including parabolic rational functions, for example that the parameter 1/4 is such a continuity point for quadratic polynomials $z \mapsto z^2 + c$ for $c \in [0, 1/4]$. We also prove the continuity of the numerical value of Hausdorff measures for parabolic Walters conformal maps and for a class of more general Walters conformal maps.

In the context of continued fractions, we prove that if by $J_n(\mathcal{G})$ we denote the set of all numbers in $[0, 1]$ whose infinite continued fraction expansions have all entries in $\{1, 2, \dots, n\}$, then $\lim_{n\to\infty} \mathrm{H}_{h_n}(J_n(\mathcal{G})) = 1 = \mathrm{H}_1([0, 1])$, where h_n is the Hausdorff dimension of $J_n(\mathcal{G})$, H_{h_n} is the corresponding Hausdorff measure. We also construct a class of infinite iterated function systems $\mathcal{S}$ on $[0, 1]$, consisting of similarities, for which $\overline{\lim}_{F\to E} \mathrm{H}_{h_F}(J_F) < \mathrm{H}_{h_\mathcal{S}}(J_\mathcal{S})$; the upper limit is taken over finite subsets of the countable infinite alphabet E.

References

[1] T. Szarek, M. Urbański, and A. Zdunik, *Hausdorff Measure for Conformal Dynamical Systems*, Discrete & Continuous Dynam. Sys., Series A., (2013) 33 4647–4692.

[2] M. Urbański, A. Zdunik, *Regularity of Hausdorff measure function for conformal dynamical systems*, submitted

[3] M. Urbański, A. Zdunik, *Continuity of the Hausdorff Measure of Continued Fractions and Countable Alphabet Iterated Function Systems*, Journal de Théorie des Nombres de Bordeaux, to appear

Approximation of solutions of SDE driven by fractional Brownian motion

Anna Soós

1. Introduction

The aim of this paper is to approximate the solution of a stochastic differential equation driven by fractional Brownian motion using two series expansion for the noise. We prove that the solution of the approximating equations converge in probability to the solution of the given equation.

Let $\Big(B(t)\Big)_{t\geq 0}$ be a fractional Brownian motion with Hurst parameter H such that $H > \frac{1}{2}$. We investigate stochastic differential equations of the form

$$\begin{aligned} dX(t) &= F(X(t),t)dt + G(X(t),t)dB(t), \\ X(t_0) &= X_0, \end{aligned} \tag{1}$$

where $t_0 \in (0,T]$, X_0 is a random vector in $\mathbb{R}^n$ and the random functions F and G satisfy with probability 1 the following conditions:

(C1) $F \in C(\mathbb{R}^n \times [0,T], \mathbb{R}^n), G \in C^1(\mathbb{R}^n \times [0,T], \mathbb{R}^n)$;

(C2) for each $t \in [0,T]$ the functions $F(\cdot,t), \dfrac{\partial G(\cdot,t)}{\partial x^i}, \dfrac{\partial G(\cdot,t)}{\partial t}$ are locally Lipschitz for each $i \in \{1,\dots,n\}$.

The fractional Brownian motion $\Big(B(t)\Big)_{t\in[0,1]}$ with Hurst index $H \in (0,1)$ we approximate using two type of serie expansion.

1. Series expansion for fractional Brownian motion given in [3]

Let J_ν be the Bessel function of first type of order ν and let $x_1 < x_2 < \dots$ be the positive, real zeros of J_{-H}, while $y_1 < y_2 < \dots$ are the positive, real zeros of J_{1-H}. We consider $(X_n)_{n\in\mathbb{N}}$ and $(Y_n)_{n\in\mathbb{N}}$ to be two independent sequences of centered Gaussian random variables such that for each $n \in \mathbb{N}$ we have

$$\mathrm{Var}X_n = \frac{2c_H^2}{x_n^{2H}J_{1-H}^2(x_n)}, \quad \mathrm{Var}Y_n = \frac{2c_H^2}{y_n^{2H}J_{-H}^2(y_n)},$$

where

$$c_H^2 = \frac{\sin(\pi H)}{\pi}\Gamma(1+2H).$$

Then:

$$B(t) = \sum_{n=1}^{\infty} \frac{\sin(x_n t)}{x_n} X_n + \sum_{n=1}^{\infty} \frac{1-\cos(y_n t)}{y_n} Y_n, \quad t \in [0,1].$$

Let

$$B_N(t) = \sum_{n=1}^{N} \frac{\sin(x_n t)}{x_n} X_n + \sum_{n=1}^{N} \frac{1-\cos(y_n t)}{y_n} Y_n, \quad t \in [0,1], N \in \mathbb{N}.$$

2. Optimal wavelet approximation of the fractional Brownian motion given in [1]

$$B(t) = \sum_{j=-\infty}^{\infty} \sum_{k=-\infty}^{\infty} 2^{-jH}(\Psi(2^j t - k) - \Psi(-k))\epsilon_{j,k}, \tag{2}$$

where Ψ is the mother function of the wavelets approximation, and $\epsilon_{j,k}$ are independent identically distributed $N(0,1)$ random variables, Ψ: $\Psi \in C^1(\mathbb{R})$ and there exists a constant $c > 0$ such that

$$|\Psi(t)| \leq \frac{c}{(2+|t|)^2} \text{ and } |\Psi'(t)| \leq \frac{c}{(2+|t|)^3} \text{ for all } t \in \mathbb{R}. \tag{3}$$

We denote

$$B_N(t) = \sum_{j=0}^{N} \sum_{|k| \le \frac{2^{N+4}}{(N-j+1)^2}} 2^{-jH}(\Psi(2^j t - k) - \Psi(-k))\epsilon_{j,k} + \\ + \sum_{j=-2^{[N/2]}}^{-1} \sum_{|k|\le 2^{[N/2]}} 2^{-jH}(\Psi(2^j t - k) - \Psi(-k))\epsilon_{j,k} \quad \text{for each } t \in [0,1].$$

The main difficulty raised by the fractional Brownian motion is that they are not semimartingales. There exist several ways to define the stochastic integral, in this paper we use the approach of M. Zähle [6].

2. Results

Theorem 1. *The sequence $(B_N)_{N\in\mathbb{N}}$ converges to B almost surely in $\omega \in \Omega$ and uniformly in $t \in [0,1]$, i.e.*

$$\mathbb{P}\Big(\lim_{N\to\infty} \sup_{t\in[0,1]} |B_N(t) - B(t)| = 0\Big) = 1.$$

We approximate for each $N \in \mathbb{N}$ the equation (1) through

$$X_N(t) = X_0 + \int_0^t F(X_N(s), s)ds + \int_0^t G(X_N(s), s)dB_N(s). \tag{4}$$

We will show that the equation (4) has a local solution, which converges in probability to the solution of (1) in the interval, where the solutions exist.

Theorem 2. *Let B be a fractional Brownian motion approximated through the processes B_N given in* (2). *Let $F, G : \mathbb{R}^n \times [0,T] \to \mathbb{R}^n$ be random functions satisfying with probability* 1 *the conditions* (C1) *and* (C2). *Let $t_0 \in (0,T]$ be fixed. Then, each of the stochastic equations*

$$X(t) = X_0 + \int_{t_0}^t F(X(s), s)ds + \int_{t_0}^t G(X(s), s)dB(s),$$

$$X_N(t) = X_0 + \int_{t_0}^t F(X_N(s), s)ds + \int_{t_0}^t G(X_N(s), s)dB_N(s), \quad N \in \mathbb{N}$$

admits almost surely a unique local solution on a common interval (t_1, t_2) (which is independent of N and contains t_0). Moreover, we have the following approximation result

$$P(\lim_{N\to\infty} \sup_{t\in(t_1,t_2)} \|X_N(t) - X(t)\| = 0) = 1.$$

References

[1] A. Ayache and M. S. Taqqu, *Rate optimality of wavelet series approximations of fractional Brownian motion. Jour. Fourier Anal. Appl.* 9 (2003), 451-471.

[2] L. Decreusefond and A.S. Üstünel, *Stochastic Analysis of the Fractional Brownian Motion.* Potential Analysis **10** (1999), 177–214.

[3] K. Dzhaparidze and H. van Zanten, *A Series Expansion of Fractional Brownian Motion.* Probab. Theory Relat. Fields **130** (2004), 39–55.

[4] F. Klingenhöfer and M. Zähle, *Ordinary Differential Equations with Fractal Noise.* Proc. Amer. Math. Soc. **127** (1999), 1021–1028.

[5] H. Lisei, A. Soós, Approximation of Stochastic Differential Equations Driven by Fractional Brownian Motion, *in Proceedings of the Conference of Ascona 2005*, Birhaeuser, 2006.

[6] M. Zähle, *Integration with respect to Fractal Functions and Stochastic Calculus I.* Probab.Theory Relat. Fields **111** (1998), 333–374.

Poisson law for some nonuniformly hyperbolic dynamical systems with polynomial rate of mixing

Benoît Saussol

(joint work with Françoise Pène)

1. Introduction

Many dynamical systems with some hyperbolicity enjoy strong statistical properties. Let us mention a few of them: existence of physical measure, exponential decay of correlations, central limit theorem, large deviation principles, etc. That is, the probabilistic behavior of these systems mimics an i.i.d. process. However, when the hyperbolicity is too weak the situation may be different. This has a visible consequence in some non uniformly hyperbolic systems, for example in the validity of the CLT, which is often related to a summable decay of correlations.

We will consider here a class of systems with some weak form of hyperbolicity, for which the polynomial decay of correlations can be arbitrarily slow. The setting, introduced by Alves and Pinheiro [2] and generalized by Alves and Azevedo [1] is a system modeled by a Young tower on which we have a uniform but polynomial control on the contraction along stable manifolds and backward contraction along unstable manifolds. The full description of the setting is in Section 2.

Let f be an invertible map defined on a riemaniann manifold M giving rise to a metric d. Suppose that μ is an invariant measure for f. Given x in $\mathcal{M}$, we are interested in the statistical behavior, with respect to μ, of the number of occurrences of entrance times in the ball $B(x,r)$. Namely, setting

$$N_t(x,r)(y) = \sharp\left\{n\in\mathbb{N}\colon d(f^ny,x)<r,\, 1\le n\le \frac{t}{\mu(B(x,r))}\right\}$$

we are interested in the limit distribution of $(N_t(x,r))_t$, as r goes to 0. The main result of the paper is that this limit is the Poisson distribution, for μ-a.e. x. This question has been addressed for many different dynamical systems. Our result is a generalization of the recent work by Collet and Chazottes [3] who studied towers with exponential tail of return time to a setting where the tail is only polynomial (We refer to [3] and references therein for details on previous works). Our work applies for example to:

- solenoid with intermittency,
- billiard in stadium,
- Axiom A attractor when the SRB measure has large Hausdorff dimension

During the preparation of this work Freitas, Haydn and Nicol [4] obtained a similar result on Poisson distribution for these above mentioned billiards by a different method (inducing the billiard map on a suitable reference set where the induced map has a tower with exponential tail). We also mention Haydn and Wasilewska [5] whose approach needs a polynomial tail of sufficiently large order.

Our limit theorem relies on precise mixing estimates for sets defined with balls. That is why we need a control on the measure of neighborhood of balls as in (3). Outside absolutely continuous measure this leads to delicate questions. In [3] a general result is obtained for SRB measure with one-dimensional unstable manifold. The generalization to higher dimensional systems or polynomial tower being open, we left the condition (3) as an assumption. We emphasize that this condition is the weakest that one can ask in our setting.

A major step in our proof of the Poisson distribution is to bypass the lengthy and delicate study of short returns by a simple argument based on recurrence rates. Indeed we show that for our systems

$$\min\{n \geq 1\colon d(f^n x, x) < r\} \approx r^{-\dim_H \mu}$$

for μ-a.e. x, where $\dim_H \mu$ stands for the Hausdorff dimension of μ.

2. Assumptions

We consider an invertible transformation $f : \mathcal{M} \to \mathcal{M}$ on a finite-dimensional Riemannian manifold satisfying assumptions of [1]. These assumptions ensure the existence of a f-invariant SRB probability measure μ and that $(\mathcal{M}, f, \mu)$ can be modeled by a Gibbs Markov Young tower with good properties as described briefly below for completeness.

Recall that a stable (resp. unstable) manifold is an embedded disk $\gamma \subset \mathcal{M}$ such that, for every $x, y \in \gamma$, $\mathrm{dist}(f^n x, f^n y) \to 0$ (resp. $\mathrm{dist}(f^{-n}x, f^{-n}y) \to 0$) as n goes to infinity.

Consider two continuous families Γ^u and Γ^s of respectively unstable and stable C^1 manifolds such that there exists $\alpha_{\min} > 0$ so that, for every $(\gamma^s, \gamma^u) \in \Gamma^s \times \Gamma^u$, we have

$$\dim \gamma^s + \dim \gamma^u = \dim \mathcal{M}, \quad \#(\gamma^s \cap \gamma^u) = 1 \quad \text{and} \quad |\angle(\gamma^s, \gamma^u)| \geq \alpha_{\min}.$$

We set $\Lambda := (\bigcup_{\gamma^u \in \Gamma^u} \gamma^u) \cap (\bigcup_{\gamma^s \in \Gamma^s} \gamma^s)$. We assume that f is a differentiable on $\bigcup_{n \geq 0} f^n \Lambda$ and that the following properties hold

(P1) <u>Markov</u>: there exists a family $(\Lambda_i)_{i \geq 1}$ of pairwise disjoint subsets of the form $\Lambda_i = (\bigcup_{\gamma^u \in \Gamma^u} \gamma^u) \cap (\bigcup_{\gamma^s \in \Gamma^s_i} \gamma^s)$ for some family $(\Gamma^s_i)_i$ of pairwise disjoint subsets of Γ^s such that

(a) for some $\gamma \in \Gamma^u$, we have $\mathrm{Leb}_\gamma(\Lambda) > 0$ and $\mathrm{Leb}_\gamma(\Lambda \setminus \bigcup_i \Lambda_i) = 0$;
(b) For every $i \geq 1$, there exists an integer $R_i \geq 1$ such that $f^{R_i}(\Lambda_i) = (\bigcup_{\gamma^u \in \Gamma_i^u} \gamma^u) \cap (\bigcup_{\gamma^s \in \Gamma^s} \gamma^s)$ for some $\Gamma_i^u \subset \Gamma^u$. Moreover, for every $\gamma^s \in \Gamma_i^s$, there exists $\gamma_0^s \in \Gamma^s$ such that $f^{R_i}(\gamma^s) \subset \gamma_0^s$ and, for every $\gamma^u \in \Gamma^u$, there exists $\gamma_0^u \in \Gamma_i^u$ such that $\gamma_0^u \subset f^{R_i}(\gamma^u)$.

This enables us to define a particular return time $R : \Lambda \to \mathbb{N}$ and the associated return map $f^R : \Lambda \to \Lambda$ by setting

$$R_{|\Lambda_i} \equiv R_i \quad \text{and} \quad (f^R)_{|\Lambda_i} \equiv f^{R_i}.$$

We also define a separation time $s : \Lambda \times \Lambda \to \mathbb{N} \cup \{\infty\}$ for the return map as follows:

$$\forall x, y \in \Lambda, \quad s(x,y) := \min\{n \geq 0 : \exists j \geq 1,\ (f^R)^n(x) \in \Lambda_j \text{ and } (f^R)^n(y) \notin \Lambda_j\}.$$

With these notations, we assume that there exist $\alpha > 0$, $\beta \in (0,1)$ and $C > 0$ such that, for every $\gamma_0^u, \gamma_1^u \in \Gamma^u$ and every $\gamma^s \in \Gamma^s$, we have

(P2) Polynomial contraction on stable leaves: for every $x, y \in \gamma^s$ and every $n \geq 1$, we have $\mathrm{dist}(f^n(x), f^n(y)) \leq Cn^{-\alpha}$;
(P3) Backward polynomial contraction on unstable leaves: for every $x, y \in \gamma_0^u$ and every $n \geq 1$, we have $\mathrm{dist}(f^{-n}(x), f^{-n}(y)) \leq Cn^{-\alpha}$;
(P4) Bounded distortion: for every $x, y \in \gamma_0^u \cap \Lambda$, we have

$$\log \frac{\det D(f^R)_u(x)}{\det D(f^R)_u(y)} \leq C\beta^{s(f^R(x), f^R(y))}.$$

(P5) Regularity of the stable foliation: consider the map $\Theta_{\gamma_0^u, \gamma_1^u} : \gamma_0^u \cap \Lambda \to \gamma_1^u \cap \Lambda$ defined by $\Theta_{\gamma_0^u, \gamma_1^u}(x)$ is the unique x' for which there exists $\gamma \in \Gamma^s$ such that $x \in \gamma \cap \gamma_0^u$ and $x' \in \gamma \cap \gamma_1^u$. We assume that
(a) $\Theta_{\gamma_0^u, \gamma_1^u}$ is absolutely continuous and

$$U := \frac{d((\Theta_{\gamma_0^u, \gamma_1^u})_* \mathrm{Leb}_{\gamma_0^u})}{d\,\mathrm{Leb}_{\gamma_1^u}} = \prod_{i \geq 0} \frac{\det Df_u \circ f^i}{\det Df_u \circ f^i \circ \Theta_{\gamma_0^u, \gamma_1^u}^{-1}};$$

(b) for every $x, y \in \gamma_1^u$, we have $\log(U(x)/U(y)) \leq C\beta^{s(x,y)}$.

We assume that $\gcd(R_i, i \geq 1) = 1$. We consider here the case when the return time R has a polynomial tail distribution, more precisely we assume that

$$\mathrm{Leb}_\gamma(R > n) \leq Cn^{-\zeta}, \quad \text{for some } \zeta > 1, \tag{1}$$

which ensures the integrability of R with respect to Leb_γ. Under these conditions, the systems admits a SRB measure μ: the conditional measures on local unstable manifolds are absolutely continuous with respect to the Lebesgue measure on these manifolds. We recall the definition of the Hausdorff dimension of the measure μ as

$$\dim_H \mu = \inf_{\mu(Y)=1} \dim_H Y.$$

We make the standing assumption that

$$\alpha > \frac{1}{\dim_H \mu}. \tag{2}$$

3. Poisson law for the number of entrance to balls

For any $x \in \mathcal{M}$ and $r > 0$, we write $B(x,r)$ for the ball of center x and radius r for the Riemannian distance on $\mathcal{M}$. The main result of the paper states that for typical centers x, the time spent inside the ball $B(x,r)$, up to time $t/\mu(B(x,r))$, follows asymptotically the Poisson law with mean t.

We assume that, for μ-almost every $x \in \mathcal{M}$, there exists $\delta \in (1, \alpha \dim_H \mu)$ such that

$$\mu(B(x,r) \setminus B(x, r - r^\delta)) = o(\mu(B(x,r))). \tag{3}$$

Theorem 1. *Let $(\mathcal{M}, f, \mu)$ be as above. For μ-a.e. $x \in \mathcal{M}$ such that* (3) *holds, $(N_t(x,r))_{t>0}$ converges in distribution (in the Skhorohod space $\mathcal{D}([0,T])$ for every $T > 0$) to a Poisson process of intensity 1 as $r \to 0$. In particular*

$$\lim_{r\to 0} \mu\left(\{N_{t_1}(x,r) = k_1, ..., N_{t_m}(x,r) = k_m\}\right) = \prod_{j=1}^{m} \frac{(t_j - t_{j-1})^{k_j - k_{j-1}}}{(k_j - k_{j-1})!} e^{-(t_j - t_{j-1})},$$

for any $m \geq 1$, any $0 = t_0 \leq t_1 < t_2 < ... < t_m$ and any integers $0 = k_0 \leq k_1 \leq ... \leq k_m$.

The conclusion of this theorem means that the successive intervisit times to $B(x,r)$ (suitably normalized) are asymptotically independent with exponential distribution of parameter 1.

References

[1] J. F. Alves and D. Azevedo, Statistical properties of diffeomorphims with weak invariant manifolds. Preprint ArXiv:1310.2754.

[2] J. F. Alves and V. Pinheiro, Slow rates of mixing for dynamical systems with hyperbolic structure, *J. Stat. Phys.* **131** (2008) 505–534.

[3] J.-R. Chazottes and P. Collet Poisson approximation for the number of visits to balls in nonuniformly hyperbolic dynamical systems, *Erg. Th. Dynam. Sys.* **33**-1 (2013) 49–80.

[4] J. Freitas, N. Haydn and M. Nicol, Convergence of rare events point processes to the poisson for billiards, Preprint arXiv:1311.2649.

[5] K. Wasilewska, Limiting distribution and error terms for the number of visits to balls in non-uniformly hyperbolic dynamical systems, Preprint arXiv:1402.2990

Measure-theoretic asymptotic degrees for non-expanding transformations

Eugen Mihailescu

Let $f : M \to M$ be a smooth endomorphism with a locally maximal set Λ. We assume hyperbolicity of $f|_\Lambda$ as an endomorphism, thus unstable manifolds $W^u_r(\widehat{x})$ depend on full prehistories $\widehat{x}$ of points x, where $\widehat{\Lambda} := \{(x, x_{-1}, \ldots), f(x_{-i}) = x_{-i+1}, i \geq 0\}$ is the inverse limit of (Λ, f). The following **Questions** appear:

- If f is not invertible on Λ, what can we say about the number of f-preimages of points $x \in \Lambda$, that *remain in* Λ?
- Does this number of preimages influence the *Hausdorff dimension* of various *stable sections* through the set Λ?

In general Λ is **not** totally invariant, and f may not be constant-to-1 on Λ. The hyperbolic non-expanding non-invertible case is different from the expanding case, and from the hyperbolic diffeomorphism case. One difficulty is that branches of inverse iterates *do not contract* small balls on Λ. Another difficulty is that there may exist uncountably many local unstable manifolds through points in Λ.

In the joint paper [4] with M. Urbański, we introduced a general notion of *asymptotic degree* with respect to equilibrium measures μ_ϕ of Hölder continuous potentials ϕ on the *saddle set* Λ. In particular, for the measure of maximal entropy μ_0 on Λ, we obtain the average logarithmic growth of the number of n-preimages that remain in Λ, when $n \to \infty$, which can be considered as the "degree" of f over Λ. We obtained also a formula for the Jacobian of μ_ϕ with respect to arbitrary iterates $f^m, m \geq 2$. Using this, we found a formula for the pressure $P(\phi)$ on saddle sets, in terms of preimage sets and folding entropy of μ_ϕ, different from the one in the expanding case. Hence, asymptotic degrees are useful in obtaining:

a) *the rate of growth* of the number of n-preimages remaining in Λ, when $n \to \infty$; and the corresponding measure-theoretic notion with respect to μ_ϕ;
b) *a formula for the pressure* $P(\phi)$ in the saddle non-invertible case, in terms of the n-preimages of x that remain in Λ, for μ_ϕ-a.e point x in Λ;
c) *estimates on Hausdorff dimension* of slices through sets of full μ_ϕ-measure.

Let $f : M \to M$ a continuous and μ an f-invariant probability, and assume f is *essentially countable-to-one*, i.e the canonical measures of μ w.r.t the partition into fibers $f^{-1}(\epsilon)$, are purely atomic. Then by Rohlin, Parry, there exists a measurable partition $A_i, i \geq 0$ s.t $f|_{A_i}$ is injective and one defines the **Jacobian** of μ w.r.t f:

$$J_f(\mu)(x) = \frac{d\mu \circ (f|_{A_i})}{d\mu}(x), \ \ \mu - \text{a.e on } A_i, i \geq 0$$

Definition 1 (Ruelle, [6], [7]). The **folding entropy** $F_f(\mu)$ of μ w.r.t f is:

$$F_f(\mu) := H_\mu(\epsilon | f^{-1}\epsilon),$$

where ϵ is the partition into single points. It follows $F_f(\mu) = \int \log J_f(\mu)(x) d\mu(x)$.

Now, for an f-invariant probability μ on Λ, $\tau > 0$ small, $n \in \mathbb{N}$, ϕ Hölder continuous on Λ and $x \in \Lambda$, let us define the finite set (see [4]):

$$G_n(x, \mu, \tau) := \{y \in f^{-n}(f^n x) \cap \Lambda, \text{ s.t } |\frac{S_n\phi(y)}{n} - \int \phi d\mu| < \tau\} \tag{1}$$

Definition 2. In the above setting, denote by

$$d_n(x, \mu, \tau) := \text{Card } G_n(x, \mu, \tau), \ x \in \Lambda, n > 0, \tau > 0$$

The function $d_n(\cdot, \mu, \tau)$ is measurable and finite on Λ.

We want to relate the number of "good" preimages $d_n(x, \mu_\phi, \tau)$ to $F_f(\mu_\phi)$. For this we need the Jacobians of μ_ϕ with respect to arbitrary iterates $f^n, n \geq 1$.

Theorem 1 (Jacobians of equilibrium measures with respect to iterates of endomorphisms, [4]). *Let f be a $\mathcal{C}^2$ hyperbolic endomorphism on a folded basic set Λ, which has no critical points in Λ; let also ϕ be a Hölder continuous potential on Λ and let μ_ϕ the unique equilibrium measure of ϕ on Λ. Then there exists a comparability constant $C > 0$ independent of $m \geq 2$ and of $x \in \Lambda$, such that for $\mu_\phi - a.e\ x \in \Lambda$, the Jacobian of μ_ϕ with respect to the iterate f^m satisfies:*

$$C^{-1} \cdot \frac{\sum\limits_{\zeta \in f^{-m}(f^m(x)) \cap \Lambda} e^{S_m\phi(\zeta)}}{e^{S_m\phi(x)}} \leq J_{f^m}(\mu_\phi)(x) \leq C \cdot \frac{\sum\limits_{\zeta \in f^{-m}(f^m(x)) \cap \Lambda} e^{S_m\phi(\zeta)}}{e^{S_m\phi(x)}} \tag{2}$$

The above result allowed us to obtain the following formula for the average asymptotic degree:

Theorem 2 (Measure-theoretic asymptotic degree for equilibrium states, [4]). *Let $f : M \to M$ be a $\mathcal{C}^2$ non-invertible map and Λ a basic set for f so that f is hyperbolic on Λ and does not have critical points in Λ. Let also ϕ be a Hölder continuous potential on Λ, and consider μ_ϕ the equilibrium measure associated to ϕ. Then, we have the following formula relating the number of preimages $d_n(\cdot, \mu_\phi, \cdot)$ and the folding entropy $F_f(\mu_\phi)$:*

$$\lim_{\tau \to 0} \lim_{n \to \infty} \frac{1}{n} \int_\Lambda \log d_n(x, \mu_\phi, \tau)\ d\mu_\phi(x) = F_f(\mu_\phi)$$

The Theorem allows us to define the asymptotic degree with respect to μ_ϕ:

Definition 3. In the above setting, we define the **asymptotic degree with respect to the measure** μ_ϕ on Λ, as

$$d_\infty(f, \mu_\phi) := \exp\big(\lim_{\tau \to 0} \lim_{n \to \infty} \frac{1}{n} \int_\Lambda \log d_n(x, \mu_\phi, \tau)\ d\mu_\phi(x)\big)$$

Define in particular $a_l(f, \Lambda) := \lim\limits_{n} \frac{1}{n} \int_\Lambda \log d_n(x) d\mu_0(x)$, where μ_0 is the measure of maximal entropy of $f|_\Lambda$. The **asymptotic degree** of $f|_\Lambda$ is then defined as

$$d_\infty(f, \Lambda) := e^{a_l(f, \Lambda)}$$

For expanding maps we have a well-known formula for pressure, namely $P(\phi) = \lim\limits_{n \to \infty} \frac{1}{n} \log \sum\limits_{y \in f^{-n}(x)} e^{S_n\phi(y)}$. In our *saddle* set case we obtain a different formula:

Theorem 3 (Relation between preimage sets and pressure in the saddle non-invertible case, [4])**.** *In the setting of Proposition 1 and for an arbitrary Hölder continuous potential ϕ on Λ, we have for μ_ϕ-a.e $x \in \Lambda$,*

$$P(\phi) = \lim_{n\to\infty} \frac{1}{n} \log \sum_{y\in f^{-n}(f^n(x))\cap\Lambda} e^{S_n\phi(y)} - \log d_\infty(f,\mu_\phi) + h_{\mu_\phi}$$

Once we have a formula for the pressure on a saddle set, we can obtain the entropy h_μ for **any** f-invariant measure μ on Λ, by a reverse Variational Principle,

$$h_\mu = \inf \big\{ P(\psi) - \int_\Lambda \psi \, d\mu, \ \psi \text{ Hölder continuous on } \Lambda \big\},$$

as the entropy map is upper semi-continuous in our case. By using Jacobians of iterates, we can compute also the μ_ϕ-measure of an arbitrary ball centered on Λ.

Let $\Phi^s(x) := \log |Df_s(x)|$, $x \in \Lambda$. We used in [4] the asymptotic degrees in order to estimate the dimensions of stable sections through sets of full μ_ϕ-measure.

Theorem 4 (Dimension estimates for stable slices of sets of full measure, [4])**.** *Assume that f is conformal on local stable manifolds over the saddle basic set Λ, and that μ_ϕ is the equilibrium measure of a Hölder continuous potential ϕ on Λ. Then there exists a Borel set $\mathcal{K}(\mu_\phi) \subset \Lambda$ such that $\mu_\phi(\mathcal{K}(\mu_\phi)) = 1$, and for every $x \in \Lambda$ we have:*

$$HD\big(W^s_r(x) \cap \mathcal{K}(\mu_\phi)\big) \le t^s_{d_\infty(f,\mu_\phi)},$$

where
$t^s_{d_\infty(f,\mu_\phi)}$ is the unique zero of the pressure function $t \to P(t\Phi^s - \log d_\infty(f,\mu_\phi))$.

There exist also geometric consequences induced by the stable dimension, as proved in the next result, obtained jointly with B.Stratmann:

Theorem 5 ([3])**.** *Let $f : M \to M$ be a smooth endomorphism hyperbolic on a basic saddle set Λ conformal on local stable manifolds over Λ, and assume that there exists $x \in \Lambda$ s.t $\delta^s(x)$ is equal to the unique zero t_1 of $t \mapsto P(t\Phi^s)$. Then, there exists an open dense set of points in Λ where f has precisely one preimage in Λ. Moreover, we have that $\delta^s(y) = t_1$, for all $y \in \Lambda$.*

Examples.

In [2] we studied the following class of maps: Fix $\alpha \in (0,1)$, and intervals $I_1^\alpha, I_2^\alpha \subset I = [0,1]$ s.t I_1^α is contained in $[\frac{1}{2} - \epsilon(\alpha), \frac{1}{2} + \epsilon(\alpha)]$ and I_2^α is contained in $[1-\alpha-\epsilon(\alpha), 1-\alpha+\epsilon(\alpha)]$, for small $\epsilon(\alpha) < \alpha^2$. Let a strictly increasing smooth map $g : I_1^\alpha \cup I_2^\alpha \to I$ s.t $g(I_1^\alpha) = g(I_2^\alpha) = I$, s.t $\exists\ \beta > 1$ with $\beta^2 > g'(x) > \beta >> 1, x \in I_1^\alpha \cup I_2^\alpha$. There exist $I_{11}^\alpha, I_{12}^\alpha \subset I_1^\alpha, I_{21}^\alpha, I_{22}^\alpha \subset I_2^\alpha$ s.t $g(I_{11}^\alpha) = g(I_{21}^\alpha) = I_1^\alpha$, and $g(I_{12}^\alpha) = g(I_{22}^\alpha) = I_2^\alpha$. Let $J^\alpha := I_{11}^\alpha \cup I_{12}^\alpha \cup I_{21}^\alpha \cup I_{22}^\alpha$, and the fractal set $J_*^\alpha := \{x \in J^\alpha, g^i(x) \in J^\alpha, i \ge 0\}$. Then, define the **family of skew-product endomorphisms** $f_\alpha : J_*^\alpha \times I \to J_*^\alpha \times I$, $\ f_\alpha(x,y) = (g(x), h_\alpha(x,y))$, with

$$h_\alpha(x,y) = \begin{cases} \psi_{1,\alpha}(x) + s_{1,\alpha}y, \ x \in I_{11}^\alpha \\ \psi_{2,\alpha}(x) + s_{2,\alpha}y, \ x \in I_{21}^\alpha \\ \psi_{3,\alpha}(x) - s_{3,\alpha}y, \ x \in I_{12}^\alpha \\ s_{4,\alpha}y, \ x \in I_{22}^\alpha \end{cases} \tag{3}$$

where $s_{1,\alpha}, s_{2,\alpha}, s_{3,\alpha}, s_{4,\alpha} \in (0,\infty)$ are ε_0-close to $\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}$ respectively, and $\psi_{1,\alpha}(\cdot), \psi_{2,\alpha}(\cdot), \psi_{3,\alpha}(\cdot)$ are $\mathcal{C}^2$ functions on I, ε_0-close in $\mathcal{C}^1$ to $x \to x$, $x \to 1-x$ and $x \to 1$, respectively. Denote $h_\alpha(x,\cdot) : I \to I$ by $h_{x,\alpha}(\cdot)$, for $x \in J_*^\alpha$. Now define the set

$$\Lambda(\alpha) := \bigcup_{x\in J_*^\alpha} \bigcap_{n\geq 0} \bigcup_{y\in g^{-n}x\cap J_*^\alpha} h_{y,\alpha}^n(I), \tag{4}$$

$$\text{where } h_{y,\alpha}^n := h_{f^{n-1}y,\alpha} \circ \ldots \circ h_{y,\alpha}, n \geq 0.$$

$$\text{For } x \in J_*^\alpha \text{ let } \Lambda_x(\alpha) := \bigcap_{n\geq 0} \bigcup_{y\in g^{-n}x\cap J_*^\alpha} h_{y,\alpha}^n(I).$$

Theorem 6 ([2]). *There exists a function $\vartheta(\alpha) > 0$ defined for positive small α, with $\vartheta(\alpha) \underset{\alpha\to 0}{\to} 0$, such that if f_α is the map defined in (3) whose parameters satisfy:*

$$\max\{|\psi_{1,\alpha}(x) - x|_{\mathcal{C}^1}, |\psi_{2,\alpha}(x) - 1 + x|_{\mathcal{C}^1} |\psi_{3,\alpha}(x) - 1|_{\mathcal{C}^1}\} < \vartheta(\alpha), \tag{5}$$

$$\max\left\{|s_{i,\alpha} - \frac{1}{2}|, i = 1,\ldots,4\right\} < \vartheta(\alpha)$$

Then:

a) *For $x \in J_*^\alpha \cap I_1^\alpha$, there exists a Cantor set $F_x(\alpha) \subset \Lambda_x(\alpha)$, s. t every point of $F_x(\alpha)$ has two different f_α-preimages in $\Lambda(\alpha)$. And if $x \in J_*^\alpha \cap I_2^\alpha$, then there exists a Cantor set $F_x(\alpha) \subset \Lambda_x(\alpha)$ s. t every point of $F_x(\alpha)$ has two different f_α^2-preimages in $\Lambda(\alpha)$.*

b) *f_α is hyperbolic on $\Lambda(\alpha)$.*

c) *If $\widehat{z}, \widehat{z}' \in \widehat{\Lambda}(\alpha)$ are different prehistories of arbitrary $z \in \Lambda(\alpha)$, then $E_{\widehat{z}}^u \neq E_{\widehat{z}'}^u$.*

For stable dimension, no formulas, but we can estimate it using the thickness of intersections of Cantor sets.

Corollary 1. *Let a small $\alpha > 0$ and a function f defined as in (3), s. t the parameters $s_i, \psi_j, i = 1,\ldots,4, j = 1,\ldots,3$ of f satisfy (5). Write Λ as the union $V_1 \cup V_2$, where V_1 is defined as the set of points having only one f-preimage inside Λ and V_2 is the set of points having exactly two f-preimages in Λ.*

a) *Then $\delta^s(z) \in (\frac{\log 2}{\log(2+\frac{1}{\Delta(\alpha)})}, 1), z \in \Lambda$. So if α tends to 0, then δ^s at an arbitrary point of Λ may be made as close as we want to 1.*

b) *V_1 is an open uncountable set in Λ, and V_2 is a closed set in Λ.*

c) *Assume moreover that in (3), the contraction factors $s_i, i = 1,\ldots,4$ are all equal to $\frac{1}{2}$. Then V_2 is uncountable as well.*

Hence this family behaves differently from a homeomorphism on Λ_α, and also from a 2-to-1 map on Λ_α. We can obtain the Jacobian $J_{f_\alpha}(\mu_\phi)$, and the folding entropy $F_{f_\alpha}(\mu_\phi)$ of the equilibrium measure μ_ϕ of a Hölder continuous potential ϕ on Λ_α. Now, the average rate of growth of the number of n-preimages in Λ_α, is given by $d_\infty(f_\alpha, \Lambda_\alpha) = e^{F_{f_\alpha}(\mu_{0,\alpha})}$, where $\mu_{0,\alpha}$ is the measure of maximal entropy on Λ_α. If $F_{f_\alpha}(\mu_{0,\alpha}) = 0$, then $\delta^s(x) \geq t_1$, where t_1 is the zero of $t \to P(t\Phi^s)$.

Since $h_{top}(f_\alpha|_{\Lambda_\alpha}) = \log 2$ and $\Phi^s \equiv \frac{1}{2}$, $t_1 = 1$, then $\delta^s(x) = 1$. In [M., 2011] it was shown however that $\delta^s(x) < 1$, for all $x \in \Lambda_\alpha$. Thus $d_\infty(f_\alpha, \Lambda_\alpha) = e^{F(\mu_{0,\alpha})} > 1$.

Another case, studied jointly with J.E Fornaess, is the **restriction** $f|_\Lambda$ of a **holomorphic endomorphism** $f : \mathbb{P}^2 \to \mathbb{P}^2$ on a saddle set Λ, which is minimal, or more general, a terminal set. We obtained a geometric description of the measure of maximal entropy of $f|_\Lambda$ in terms of positive closed currents.

First recall some properties of the associated positive closed **Green current** T. There exists a continuous plurisubharmonic function G on $\mathbb{C}^3 \setminus \{0\}$ called the *Green function* of f, s.t $G(F(z)) = d \cdot G(z)$ where $F : \mathbb{C}^3 \setminus \{0\} \to \mathbb{C}^3 \setminus \{0\}$ is the lift of f relative to the canonical projection $\pi_2 : \mathbb{C}^3 \to \mathbb{P}^2$. Recall that $\pi^* T = dd^c G$, and that the Green measure $\mu = T \wedge T$ is mixing.

We considered *minimal* saddle basic sets for the ordering $\Lambda_i \succ \Lambda_j$ if $W^u(\widehat{\Lambda}_i) \cap W^s(\Lambda_j) \neq \emptyset$. A related notion is that of a *terminal* set Λ, i.e when the iterates of f form a normal family on $W^u(\widehat{\Lambda}) \setminus \Lambda$. Positive closed currents σ on minimal sets can be constructed by using iterated images of unstable disks D: $\frac{f_\star^n([D])}{d^n} \to \sigma \cdot \int D \wedge T$. Using σ, one obtains then an invariant measure ν on Λ as $\nu = \sigma \wedge T$.

There exist also **transversal measures** $\widehat{\mu}_x^s$ associated to a hyperbolic structure on Λ (Sinai, Ruelle, Sullivan, etc.), given by the Smale space structure on $\widehat{\Lambda}$. One obtains then a system of *transversal measures* $\widehat{\mu}_x^s$ on $\widehat{W}_{loc}^s(x) := \pi^{-1}(W_{loc}^s(x) \cap \Lambda)$, satisfying the following properties:

i) if $\chi_{x,y}^s : \widehat{W}_r^s(x) \to \widehat{W}_r^s(y), \chi_{x,y}^s(\widehat{\xi}) = \widehat{W}_r^u(\widehat{\xi}) \cap \widehat{W}_r^s(y)$ is the holonomy, then $\widehat{\mu}_x^s(A) = \widehat{\mu}_y^s(\chi_{x,y}^s(A))$ for any borelian set A.
ii) $\widehat{f}_\star \widehat{\mu}_x^s = e^{h_{top}(f|_\Lambda)} \widehat{\mu}_{f(x)}^s |_{\widehat{f}(\widehat{W}_r^s(x))}$
iii) supp $\widehat{\mu}_x^s = \widehat{W}^s(x)$.

Unstable transversal measures $\widehat{\mu}_{\widehat{x}}^u$ on $\widehat{W}_r^u(\widehat{x}) := \pi^{-1}(W_{loc}^u(\widehat{x}) \cap \Lambda)$ have similar properties, for $\widehat{x} \in \widehat{\Lambda}$. The measure of maximal entropy on $\widehat{\Lambda}$, $\widehat{\mu}_0$, satisfies for any ϕ on a neighbourhood of $\widehat{x} \in \widehat{\Lambda}$,

$$\widehat{\mu}_0(\phi) = \int_{\widehat{W}_r^s(x)} \left(\int_{\widehat{W}_r^u(\widehat{y})} \phi \; d\widehat{\mu}_{\widehat{y}}^u \right) \, d\widehat{\mu}_x^s(\widehat{y})$$

There is a positive current σ^u given in a neighbd of $x \in \Lambda$ by, $< \sigma^u, \chi >= \int_{\widehat{W}_{loc}^s} (\int_{W_{loc}^u(\widehat{y})} \chi) d\widehat{\mu}_x^s(\widehat{y})$, where $\widehat{\mu}_x^s$ are transversal measures on $\widehat{W}_{loc}^s(x) := \pi^{-1}(W_{loc}^s(x)$. If Λ is terminal, then there is an invariant probability measure ν_i on Λ, $\nu_i = \sigma^u \wedge T$.

Theorem 7 ([1]). *Let $f : \mathbb{P}^2 \to \mathbb{P}^2$ holomorphic and Λ a terminal (or minimal) mixing saddle set of f. Then ν_i is equal to the measure of maximal entropy μ_0 of $f|_\Lambda$.*

In the case of minimal c-hyperbolic sets of maps of degree 2, we also determined in [1] all the possible values of the pointwise dimension of ν.

References

[1] J.E Fornaess and E. Mihailescu, *Equilibrium measures on saddle sets of holomorphic maps on $\mathbb{P}^2$*, Math. Annalen, **356**, 2013, 1471–1491.

[2] E. Mihailescu, *Unstable directions and fractal dimensions for a family of skew products with overlaps*, Math. Zeitschrift, **269**, 2011, 733–750.

[3] E. Mihailescu and B. Stratmann, *Upper estimates for stable dimensions on fractal sets with variable number of foldings*, International Math Res Notices, **23**, 2014, 6474–6496.

[4] E. Mihailescu and M. Urbański, *Measure-theoretic degrees and topological pressure for non-expanding transformations*, J. Functional Analysis, **267**, 2014, 2823–2845.

[5] E. Mihailescu and M. Urbański, *Hausdorff dimension of the limit set of conformal iterated function systems with overlaps*, Proceed. Amer. Math. Soc., **139**, 2011, 2767–2775.

[6] D. Ruelle, *Positivity of entropy production in nonequilibrium statistical mechanics*, J. Statistical Physics **85**, 1/2, 1996, 1–23.

[7] D. Ruelle, *Smooth dynamics and new theoretical ideas in nonequilibrium statistical mechanics*, J. Statistical Physics **95**, 1999, 393–468.

Entropy of the geodesic flow in the sense of Sullivan versus convex-core entropy

Anna Zielicz

D. Sullivan considered in [Sullivan1984] a notion of enropy for the geodesic flow on a hyperbolic manifold $\mathbb{H}^n/\Gamma$, where Γ is a non-elementary Kleinian group. This entropy notion is defined by:

$$h_S(\Gamma) := \sup_{V\in\mathcal{V}} \lim_{\epsilon\to 0} \limsup_{T\to\infty} \frac{\log(sep(T,\epsilon,V))}{T}$$

Here, $\mathcal{V}$ denotes the collection of all $V \subseteq T^1(\mathbb{H}^n/\Gamma)$ which are bounded and satisfy $V \subseteq ((L(\Gamma)\times L(\Gamma) - diag)\times\mathbb{R})/\Gamma$, where $L(\Gamma)$ denotes the limit set of Γ and $diag$ the diagonal. The value of $sep(T,\epsilon,V)$ is defined as:

$$sep(T,\epsilon,V) := \sup\{\#U : U\subseteq V, U \ (T\epsilon)-\text{separated}\}$$

where a set $U \subseteq T^1(\mathbb{H}^n/\Gamma)$ is said to be $(T\epsilon)-$ separated if for any two vectors $u, v \in U$ we have:

$$\sup_{t\in[0,T]} d(\pi_b(g^t u), \pi_b(g^t v)) > \epsilon$$

Here, d denotes the hyperbolic metric of $\mathbb{H}^n/\Gamma$, $\{g^t : t\in\mathbb{R}\}$ the geodesic flow and π_b the map defined as $\pi_b(v) = p$ on each tangent space $T^1_p(\mathbb{H}^n/\Gamma)$.
For $n = 3$ and Γ a geometrically finite group, Sullivan has shown that the entropy h_S is equal to the Poincare exponent $\delta(\Gamma)$ of the group Γ. We asked whether the two values agree for other clases of Kleinian groups than the one considered by Sullivan. This question can be also seen in the context of the results of Handel and Kitchens [HandelKitchens1995], and Otal and Peigne [OtalPeigne2004] who have shown that for any non-elementary Kleinian group Γ:

$$\delta(\Gamma) = h_{top} = \sup_\mu h_\mu = \inf_{\tilde{d}} h_{\tilde{d}}$$

Here, $h_{\tilde{d}}$ denotes the Bowen-Dinaburg entropy for non-compact sets and the infimum is taken over all metrics on $((L(\Gamma) \times L(\Gamma) - diag) \times \mathbb{R})/\Gamma$ which induce the standard topology, h_μ denotes the measure theoretic entropy and the supremum is taken over all finite invariant measures on $((L(\Gamma) \times L(\Gamma) - diag) \times \mathbb{R})/\Gamma$. Otal and Peigne have asked if the infimum can me attained for some metric $\tilde{d}$. One can show that in fact:

$$h_S = h_{\widehat{d}}$$

where $\widehat{d}$ is the metric given by:

$$\widehat{d}(u, v) := \sup_{t \in [0,1]} d(\pi_b(g^t u), \pi_b(g^t v))$$

which is one of the standard metrics on $T^1(\mathbb{H}^n/\Gamma)$. Thus the equality of h_S and the Poincare exponent $\delta(\Gamma)$ would give a positive answer to the question posed by Otal and Peigne. However, we have shown that this is not the case in general. For a large class of groups, containing many geometrically infinite groups, we can show that:

$$h_S(\Gamma) = h_c(\Gamma)$$

where h_c denotes the convex core entropy, which in general might not be equal to $\delta(\Gamma)$. Convex core entropy has been defined by Falk and Matsuzaki in [FalkMatsuzakiP] as:

$$h_c(\Gamma) := \limsup_{R \to \infty} \frac{\log(B(z, R) \cap H_\rho(L(\Gamma)))}{R}$$

where $H_\rho(L(\Gamma))$ denotes the ρ-neighborhood of the convex hull of the limit set $L(\Gamma)$ and z a point in $\mathbb{H}^n$; the definition is independent of the choice of the point z. By a theorem of Bishop and Jones [BishopJones1995] [Stratmann2004] one has that the Pincare exponent of Γ is equal to the Hausdorff dimension of the radial limit set $L_r(\Gamma)$, that is:

$$\delta(\Gamma) = dim_H(L_r(\Gamma))$$

while Falk and Matsuzaki show in [FalkMatsuzakiP] that the convex-core entropy is equal to the upper box-counting dimension of the entire limit set $L(\Gamma)$, that is:

$$h_c(\Gamma) = \overline{dim}_B(L(\Gamma))$$

Thus if the equality $h_S(\Gamma) = h_c(\Gamma)$ is satisfied the existance of the so called dimension gap $dim_H(L_r(\Gamma)) < dim_H(L(\Gamma))$ provides cases where the infimum is not attained by $h_{\widehat{d}}$, while for groups for which $dim_H(L_r(\Gamma)) = dim_B(L(\Gamma))$ the infimum is attained. The class for which we can show that the equality $h_S(\Gamma) = h_c(\Gamma)$ contains both groups with dimension gap and groups satisfying $dim_H(L_r(\Gamma)) = dim_B(L(\Gamma))$. This class consists of those groups Γ for which the injectivity radius of $\mathbb{H}./\Gamma$ is bounded and it is possible to conjugate the group Γ using isometries to a group Γ^* in such a way that $\overline{dim}_B(L(\Gamma^*) = \overline{dim}_B(L(\Gamma^*)$ and the limit set $L(\Gamma^*)$ is a compact subset of the boundary of $\mathbb{H}^n$ in the upper half-space model. Interesting examples in this class with dimension gap can be obtained using a theorem of Brooks-Stadelbauer [Brooks1985] [Stadlbauer2013], namely normal subgroups N of convex co-compact groups G with G/N a free group of order at least two, which can be particularly easily done with G a classical Schottky group. On the other hand any geometrically finite group Γ in our class, or f initely generated if we restrict to $n = 3$, satisfies $dim_H(L_r(\Gamma)) = dim_B(L(\Gamma))$.

References

[BishopJones1995] C.J. Bishop and P.W. Jones, Hausdorff dimension and Kleinian groups, *Acta Mathematica*, vol. 179 (1995), p. 1–39.

[Brooks1985] R. Brooks, The bottom of the spectrum of a Riemannian covering, *Journal fur die reine und angewandte Mathematik*, vol. 357 (1985), p. 101–114.

[FalkMatsuzakiP] K. Falk and K. Matsuzaki, The critical exponent, the Hausdorff dimension of the limit set and the convex core entropy of a Kleinian group, Preprint, arXiv:1106.4409v3 (2012).

[HandelKitchens1995] M. Handel and B. Kitchens, Metrics and entropy for non-compact spaces, *Israel J. Math.*, vol. 91 (1995), p. 253–271.

[OtalPeigne2004] J-P. Otal and M. Peigne, Principe variationnel et groupes Kleiniens, *Duke Math. J.*, vol. 125 (2004), p. 15–44.

[Stadlbauer2013] M. Stadlbauer, An extension of Kesten's criterion for amenability to topological Markov chains, *Advances in Mathematics*, vol. 235 (2013), p. 450–468.

[Stratmann2004] B.O. Stratmann, The Exponent of Convergence of Kleinian Groups; on a Theorem of Bishop and Jones, *Progress in Probability*, vol. 57 (2004), p.93–107.

[Sullivan1984] D. Sullivan, Entropy, Hausdorff measures old and new, and limit sets of geometrically finite Kleinian groups, *Acta Mathematica*, vol. 153, p. 259–277.

Coupled skinny baker's maps and the Kaplan-Yorke conjecture

Maik Gröger

(joint work with Brian R. Hunt)

In its present and most commonly stated form the Kaplan-Yorke conjecture claims that for "typical" dynamical systems with a physical measure, the information dimension and the Lyapunov dimension coincide, see [6], [3] and [4]. The conjecture is broad in the sense that it does not specify a precise class of systems to which it should apply, and it does not specify exactly what "typical" means. In the following we present the definitions that are necessary to specify the conjecture in our setting and to state the main result.

We let $\mathcal{B}(M)$ denote the Borel σ-algebra of a subset $M \subseteq \mathbb{R}^d$, λ^d the d-dimensional Lebesgue measure on $\mathbb{R}^d$, $\mathring{M}$ will denote the interior of M and $\overline{M}$ the closure of M. Furthermore, $|| \cdot ||_\infty$ will denote the supremum norm.

Definition 1. Let $F : M \subseteq R^d \to M$ be a map with M locally compact. Assume there exist finitely many pairwise disjoint connected subsets $U_i \in \mathcal{B}(M)$ such that $M = \bigcup_i U_i$ and the map $F|_{U_i}$ is continuous for each U_i (with respect to the relative topology). Further, we assume that $F|_{\mathring{U}_i}$ is C^1, $\max_i || (F|_{\mathring{U}_i})' ||_\infty < \infty$

and $\lambda^d(M_0) = 0$ where

$$M_0 := \bigcup_{n\in\mathbb{N}_0} F^{-n}\left(M\setminus\bigcup_i \mathring{U}_i\right).$$

Given the above, we call F a *piecewise C^1 dynamical system.*

We have that $M_0 \in \mathcal{B}(\mathbb{R}^d)$ and we require $\lambda^d(M_0) = 0$ to insure that orbits are not mapped into an open subset where the derivative is not defined. Such an open subset could be considered as a hole and therefore could cause positive escape rates which in turn would involve a different definition of the Lyapunov dimension. In the following the basic notions of invariance and ergodicity of a measure are required (for further details see [8]).

Definition 2. Let $F : M \subseteq \mathbb{R}^d \to M$ be a piecewise C^1 dynamical system and let μ be an F-invariant Borel probability measure on M. We call μ a *physical measure* if there exists a set $V \subseteq M$ of positive Lebesgue measure such that for every bounded continuous function $\varphi : M \to \mathbb{R}$,

$$\lim_{N\to\infty} \frac{1}{N}\sum_{n=0}^{N-1} \varphi(F^n(x)) = \int \varphi d\mu,$$

for every $x \in V$.

For the definition of Lyapunov exponents of an invariant measure, see for example [2]. Note that by the assumptions in the next definition the existence of Lyapunov exponents is ensured by Oseledets theorem.

Definition 3 ([1])**.** Let $F : M \subseteq \mathbb{R}^d \to M$ be a piecewise C^1 dynamical system. Assume that F has an ergodic invariant measure μ with $\mu(M_0) = 0$. Let $\chi_1(\mu) \geq \chi_2(\mu) \geq \cdots \geq \chi_d(\mu)$ be the Lyapunov exponents and set $j := \max\{i : \chi_1(\mu) + \cdots + \chi_i(\mu) \geq 0\}$, (respectively, 0 if $\chi_1(\mu) < 0$). We define the *Lyapunov* or *Kaplan-Yorke dimension* as

$$D_L(\mu) := \begin{cases} 0 & \text{if } j = 0 \\ j + \frac{\chi_1(\mu)+\cdots+\chi_j(\mu)}{|\chi_{j+1}(\mu)|} & \text{if } 1 \leq j < d \\ d & \text{if } j = d \end{cases}.$$

Definition 4. The *lower* and *upper information dimension* of a Borel probability measure μ are defined as

$$\underline{D}_1(\mu) := \liminf_{\varepsilon\to 0} \frac{\int \log\mu(B(x,\varepsilon))d\mu(x)}{\log\varepsilon}$$

$$\text{and } \overline{D}_1(\mu) := \limsup_{\varepsilon\to 0} \frac{\int \log\mu(B(x,\varepsilon))d\mu(x)}{\log\varepsilon}.$$

If $\underline{D}_1(\mu) = \overline{D}_1(\mu)$, then we call their common value $D_1(\mu)$ the *information dimension* of μ.

Conjecture 1 (Kaplan-Yorke conjecture). *Given a locally compact subset $M \subseteq \mathbb{R}^d$. For "typical" piecewise C^1 dynamical systems $F : M \to M$ with an ergodic invariant physical measure μ, we have that*

$$D_1(\mu) = D_L(\mu).$$

In the following, we call this conjectured equation the Kaplan-Yorke equality.

A simple class of systems where the Kaplan-Yorke equality typically does not hold consists of systems that can be decomposed into two or more uncoupled subsystems, as can been seen in the following. From here on, we assume $0 < \alpha, \beta < \frac{1}{2}$ and set $M := [0, 1) \times \mathbb{R}$.

Definition 5. The *(2-dimensional) skinny baker's map* is defined as

$$B_\alpha : M \subset \mathbb{R}^2 \to M : (x, y) \mapsto \begin{cases} (2x, \alpha y) & \text{if } 0 \le x < \frac{1}{2} \\ \\ (2x - 1, \alpha y + 1 - \alpha) & \text{if } \frac{1}{2} \le x < 1 \end{cases}.$$

The *uncoupled skinny baker's map* is defined as

$$B : M^2 \subset \mathbb{R}^4 \to M^2 : (x, y, z, w) \mapsto (B_\alpha(x, y), B_\beta(z, w)),$$

where $M^2 = M \times M$.

The uncoupled skinny baker's map is a piecewise C^1 dynamical system and has a unique physical measure, which we denote by μ. Direct calculations show that $D_1(\mu) < D_L(\mu)$ for $\alpha \neq \beta$ and $D_1(\mu) = D_L(\mu)$ for $\alpha = \beta$. That is the Kaplan-Yorke equality fails for Lebesgue a.e. $(\alpha, \beta) \in \left(0, \frac{1}{2}\right)^2$. The question which naturally arises is whether one can find a larger class of dynamical systems that contains the uncoupled skinny baker's map but for which the Kaplan-Yorke equality is typically valid. Due to the independent behavior of the subsystems, coupling seems to be the natural way to find this larger class of dynamical systems. In order to do this, we will need a space of coupling functions. Consider an open subset $U \subseteq \mathbb{R}^d$ and denote by $C_b^1(U)$ the space of all C^1 maps $g : U \to \mathbb{R}$ where g and g' are bounded. Note that $C_b^1(U)$ equipped with the norm $||g||_{1,\infty} := \max\{||g||_\infty, ||g'||_\infty\}$ is a Banach space. Also note that $\mathring{M}$ is convex and therefore $g \in C_b^1(\mathring{M})$ is Lipschitz continuous. Hence, g has a unique continuous extension on $\overline{M}$ and therefore $g(0, y)$ with $y \in \mathbb{R}$ is well-defined (with the convention of using the same symbol for the extension). Accordingly, we will just write $g \in C_b^1(M)$ from here on.

Definition 6. For $g \in C_b^1(M)$, we define the *coupled skinny baker's map* as

$$B_g : M^2 \subset \mathbb{R}^4 \to M^2 : (x, y, z, w) \mapsto (B_\alpha(x, y), B_\beta(z, w) + (0, g(x, y))).$$

The coupled skinny baker's map is a piecewise C^1 dynamical system and has a unique physical measure μ_g. Here we only consider uni-directional coupling because of technical reasons. However, the uni-directional case will allow us to make a conjecture on bi-directional coupling, as we will see at the end of this exposition.

As already mentioned, the word "typical" is not precisely defined in the conjecture. Thereby, one problem is that in infinite dimensional vector spaces there is no natural notion of typical phenomena, in the sense of "Lebesgue almost everywhere", respectively, "Lebesgue measure zero". One way to define it is to use the topological notion based on the category theorem of Baire. Prevalence is another concept to provide an analog of what typical could mean in the context of infinite dimensional vector spaces. In our case this infinite dimensional vector space is the space of all coupling functions $C_b^1(M)$. We refer to [7] and [5] for more general definitions and examples regarding the notion of prevalence.

Definition 7. Let V be a completely metrizable topological vector space. A Borel measure ν is said to be *transverse* to a Borel set $E' \subset V$ if there exists a compact subset $S \subset V$ with $0 < \nu(S) < \infty$ and $\nu(E' + v) = 0$ for all $v \in V$. A subset $E \subset V$ will be called *shy* if there exist a Borel set $E' \subset V$ with $E \subseteq E'$ and a measure ν that is transverse to $E' \subset V$. The complement of a shy set is called a *prevalent* set.

Theorem 1. *If $g \in C_b^1(M)$, then for*

(i) $\alpha > \beta$: $D_1(\mu_g) < D_L(\mu_g)$ *for all* g,
(ii) $\alpha = \beta$: $D_1(\mu_g) = D_L(\mu_g)$ *for all* g,
(iii) $\alpha < \beta$: $D_1(\mu_g) = D_L(\mu_g)$ *for a prevalent set of* g*'s.*

Observe that the treated problem is symmetric, in that we could also consider the following map

$$B_g^* : M^2 \subset \mathbb{R}^4 \to M^2 : (x, y, z, w) \mapsto (B_\alpha(x, y) + (0, g(z, w)), B_\beta(z, w))$$

and we would obtain the analogous result to the last theorem. As a consequence, we conjecture that in the case of bi-directional coupling, the Kaplan-Yorke equality holds for a prevalent set of coupling functions.

Further information as well as the proofs of the stated results can be found in [9].

References

[1] J.C. Alexander and J.A. Yorke, *Fat baker's transformations*, Ergodic Theory and Dynamical Systems, 1984, 4, 1-23.

[2] L. Barreira and Ya.B. Pesin, Lyapunov Exponents and Smooth Ergodic Theory, American Mathematical Society, 2002, **23**, University Lecture Series.

[3] J.D. Farmer and E. Ott and J.A. Yorke, *The dimension of chaotic attractors*, Physica D: Nonlinear Phenomena, 1983, **7**, 153–180, 1-3.

[4] P. Frederickson and J.L. Kaplan and E.D. Yorke and J.A. Yorke, *The Liapunov dimension of strange attractors*, Journal of Differential Equations, 1983, **49**, 185–207.

[5] B.R. Hunt and V.Yu. Kaloshin, *Prevalence*, Handbook of Dynamical Systems, Elsevier Science, 2010, editors H. Broer and F. Takens and B. Hasselblatt, **3**, 43–87.

[6] J.L. Kaplan and J.A. Yorke, *Chaotic behavior of multidimensional difference equations*, Functional Differential Equations and Approximation of Fixed Points, 1979, **730**, Lecture Notes in Mathematics, 204–227, Springer.

[7] W. Ott and J.A. Yorke, *Prevalence*, Bulletin of the American Mathematical Society, 2005, **42**, 263–290.

[8] P. Walters, An Introduction To Ergodic Theory, Spinger, 1982, Graduate Texts in Mathematics **79**.
[9] M. Gröger and B. R. Hunt, *Coupled skinny baker's maps and the Kaplan-Yorke conjecture*, Nonlinearity, 2013, **26**, 2641–2667.

Reporter: Eugen Mihailescu

Participants

Dr. Balázs Bárány
Mathematics Institute
University of Warwick
Zeeman Bldg.
Coventry CV4 7AL
UNITED KINGDOM

Prof. Dr. Yves Coudène
Département de Mathématiques
Université de Bretagne Occidentale
6, Ave. Victor Le Gorgeu
29238 Brest Cedex 3
FRANCE

Dr. Maik Gröger
Fachbereich 3
Mathematik und Informatik
Universität Bremen
Bibliothekstr. 1
28359 Bremen
GERMANY

Dr. Johannes Jaerisch
Department of Mathematics
Graduate School of Science
Osaka University
Machikaneyama 1-1, Toyonaka
Osaka 560-0043
JAPAN

Prof. Dr. Marc Keßeböhmer
Fachbereich 3
Mathematik und Informatik
Universität Bremen
Bibliothekstr. 1
28359 Bremen
GERMANY

Prof. Dr. Volker Mayer
U. F. R. Mathématiques
Université de Lille 1
Bat. M2
59655 Villeneuve d'Ascq Cedex
FRANCE

Prof. Dr. Eugen Mihailescu
Institute of Mathematics "Simion Stoilow"
of the Romanian Academy
P.O. Box 1-764
014 700 Bucharest
ROMANIA

Dr. Tomas Persson
Department of Mathematics
Lund University
Box 118
221 00 Lund
SWEDEN

Prof. Dr. Benoit Saussol
Département de Mathématiques
Université de Bretagne Occidentale
6, Ave. Victor Le Gorgeu
29238 Brest Cedex 3
FRANCE

Prof. Dr. Jörg Schmeling
Department of Mathematics
Lund Institute of Technology
P.O. Box 118
22100 Lund
SWEDEN

Prof. Dr. Anna Soós
Faculty of Mathematics & Computer Science
Babes Bolyai University
VII A - 3 Str. Donath
400 300 Cluj Napoca
ROMANIA

Prof. Dr. Mariusz Urbanski
Department of Mathematics
University of North Texas
P.O.Box 311430
Denton, TX 76203-1430
UNITED STATES

Prof. Dr. Anna Zdunik
Instytut Matematyki
Uniwersytet Warszawski
ul. Banacha 2
02-097 Warszawa
POLAND

Dr. Anna Zielicz
Fachbereich 3
Mathematik und Informatik
Universität Bremen
Bibliotheksstraße 1
28359 Bremen
GERMANY