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Mini-Workshop: Einstein Metrics, Ricci Solitons and Ricci Flow under Symmetry Assumptions

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Abstract. Symmetry reduction methods play an important role in the study of Einstein metrics, Ricci solitons and Ricci flow. The general aim of this mini-workshop was to gather researchers who have expertise in the construction of geometric examples and to survey and discuss the singularity properties of homogeneous Ricci flows and the existence question for Ricci solitons, in light of the known rigidity results and general properties. Particular topics focused on were the Alekseevskii conjecture for noncompact homogeneous Einstein spaces, the homogeneous Ricci flow and shrinking solitons.

Mathematics Subject Classification (2010): 53C25, 53C44, 53C30.

Introduction by the Organisers

The mini-workshop 1440b was organized by Christoph Böhm (University of Münster, Germany), Jorge Lauret (Universidad Nacional de Córdoba, Argentina) and McKenzie Wang (McMaster University, Canada).

The mini-workshop was attended by 15 participants, including four female mathematicians. The wonderful academic environment in Oberwolfach but also, maybe in particular, the seminar character of a mini-workshop, turned out to create a very familiar atmosphere. Many more questions were asked during the talks than usual. Also, due to more closely related topics, most participants were familiar - in one way or another - with the subjects considered in most talks. As a consequence, a much higher percentage of participants was able to interact with each other and share ideas.

In total, twenty talks were given by the participants: five survey talks on Monday and 15 more talks in the rest of the weak. Major topics of the workshop were homogeneous Einstein metrics on solvable Lie groups, homogeneous and cohomogeneity one Ricci solitons and compact homogeneous Einstein spaces. Also talks on homogeneous Ricci and symplectic curvature flows, cohomogeneity one metrics with special holonomy, conformally flat shrinking solitons, asymptotically Poincare-Einstein spaces and helicoidal submanifolds of Euclidean space were given.

The Einstein condition for homogeneous metrics becomes a system of algebraic equations for a homogeneous space. At present, classification of homogeneous Einstein metrics seems to be out of reach, in particular in the compact case. In the noncompact case, there is the Conjecture of Alekseevskii, saying that a noncompact homogeneous Einstein space is diffeomorphic to a Euclidean space, or in sharper form, such a space is isometric to an Einstein solvmanifold.

Using methods from Geometric Invariant Theory, M. Jablonski and C. Gordon were able to show that the unique Einstein metric on a solvable Lie group has maximal symmetry group. This is a quite remarkable result not true for compact homogeneous spaces. They gave an example of a solvable Lie group not admitting a Riemannian metric with maximal symmetry group. New examples of Einstein solvmanifold were presented by M. Kerr. Y. Nikolayevsky presented new constructions of solvmanifolds with negative Ricci curvature. R. Lafuente showed that the Alekseevskii Conjecture holds true in dimensions less than or equal to ten, assuming that the isometry group is not semisimple in dimensions six through ten. He also indicated how a proof of the general Alekseevskii conjecture could be possibly reduced to the semisimple case, which is considered to be most difficult.

In the semisimple case the deep results from GIT, very successfully applied in the case of Einstein solvmanifold by Heber, Lauret and Jablonski and Gordon, do not seem to be applicable. There is hope that one can understand this case by using homogeneous Ricci flow methods. By recent results of C. Böhm, it is known by now that any homogeneous Ricci flow develops either a Type I or a Type III singularity only depending on whether the Ricci flow solution has finite extinction time or is immortal. Moreover, for compact homogeneous spaces and for certain homogeneous metrics on homogeneous spaces with semisimple isometry group, it has been shown by R. Lafuente that the homogeneous Ricci flow always has finite extinction time. If this result could be generalized, a proof of the Alekseevskii Conjecture would follow.

Toward constructing new examples of shrinking Ricci solitons, A. Dancer gave an overview showing why at present no new examples can be constructed by using methods which have proved to be very succesful for constructing cohomogeneity one Einstein metrics and steady Ricci solitons. W. Wylie reported on recent results on shrinking Ricci solitons with certain curvature assumptions.

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Survey on Ricci solitons

Andrew Dancer

A Ricci soliton is a pair (g, X), where g is a metric and X a vector field, satisfying the equation

$$\text{Ric}(g) + \frac{1}{2}\mathsf{L}_X g + \frac{\epsilon}{2} g = 0 \tag{1}$$

where ϵ is a constant. Such a pair defines a solution to Hamilton's Ricci flow

$$\frac{\partial g_\tau}{\partial \tau} = -2\text{Ric}(g_\tau)$$

given by $g_\tau = (1 + \epsilon\tau)\psi_\tau^* g$, where ψ_τ is the 1-parameter group of diffeomorphisms integrating the 1-parameter family of vector fields $Y_\tau = (1 + \epsilon\tau)^{-1}X$. Such a solution evolves by the natural symmetries of the Ricci flow, that is, diffeomorphisms and homothetic rescalings, hence the soliton terminology. Ricci solitons are important in analysing singularities of the Ricci flow, as in Perelman's celebrated work (eg. [Pe]).

An important case (*gradient solitons*) is when X is the gradient of a scalar function u, so the Lie derivative term becomes the Hessian of u.

A Ricci soliton is a generalisation of an Einstein metric in the sense that if $X = 0$ (or more generally is Killing) then equation (1) reduces to the Einstein equation (such solitons are called *trivial*). The trichotomy between steady ($\epsilon = 0$), expanding ($\epsilon > 0$) and shrinking ($\epsilon < 0$) solitons in some respects reflects that betwen Ricci-flat metrics and Einstein metrics with negative and positive Einstein constant. However this analogy should not be taken too literally–in particular there do exist complete non-compact shrinkers.

In order to produce examples, it is natural to look at strategies which have been successful in generating Einstein metrics. Two are:

(1) look for examples with extra geometric structure,

(2) look for examples with symmetries.

Many examples are known of Kähler Ricci solitons. The first examples produced (eg. [Ko] and generalised by many authors) used ansätze of Calabi/Bérard Bergery type to obtain explicit solutions via ODEs (the Hamilton cigar [Ha] is also of this type). PDE methods have also been successful in cases where there is a toric structure [WZ]

Question. *Can we find Hermitian non-Kähler examples using these or related techniques?*

Question. *Can PDE techniques yield existence on more general Fano varieties?*

Other types of special holonomy imply the Einstein condition so do not yield nontrivial solitons.

Question. *Are there other special geometries that give rise to nontrivial solitons?*

Turning to strategy (2), Lauret [L] has found examples of homogeneous non-gradient expanders on some solvable Lie groups. There are no *compact* non-trivial homogeneous examples, however. This is an example of a 'rigidity' phenomenon for solitons.

One may also consider cohomogeneity one methods, where the equations are reduced to ODEs, for example by assuming a group action with generically hypersurface orbits. Kähler examples have been mentioned above. Non-Kähler steady examples have been produced by Bryant and Ivey, and generalised by Dancer-Wang. There are also expanding versions of these solitons (see eg [DW]). The examples produced so far are warped products on multiple factors. There is strong numerical evidence of examples with more complicated geometry but in these latter cases an analytical proof is still lacking.

These examples are all non-compact. Nontrivial compact examples must be shrinkers, and these have proved difficult to find.

Question. *Can we produce compact non-Kähler shrinkers, or complete noncompact non-Kähler shrinkers not of Gaussian type?*

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Survey: Non-compact homogeneous Einstein spaces

Michael Jablonski

The study of Riemannian manifolds with special curvature and symmetry properties has been a central theme throughout the history of modern geometry. And an interest in classifying spaces with a set of prescribed, nice properties is ever-present. Here, we are interested in Riemannian manifolds which are homogeneous and simultaneously have constant Ricci curvature, i.e. homogeneous Einstein spaces.

Question 1. *What are the non-compact, non-flat, homogeneous spaces G/K which admit G-invariant Einstein metrics?*

The first general result on such Einstein spaces appeared nearly 70 years ago. Suppose G/K is Einstein and non-flat, then it has negative scalar curvature if and only if it is non-compact; this is a consequence of Myers's Theorem together with a result of Bochner [Boc46]. In this case, G is non-compact. However, attention to this question waned and little else was known about the structure of G, in general, until very recently.

There was a renewed interest in the problem in the 1970s and it was around this time that D. Alekseevskii made the following conjecture.

> **Alekseevskii Conjecture:** A non-compact, non-flat, homogeneous Einstein space G/K must be diffeomorphic to $\mathbb{R}^n$. Equivalently, K is a maximal compact subgroup of G.

The conjecture is known to be true up to dimension 5 [Nik05] and in dimension 6 if the group G is not semi-simple [AL13].

In the 1980s, Dotti, Leite, and Miatello produced a new collection of examples showing that the conjecture could not hold in the more general setting of negative Ricci curvature [LdM82, DM88]. In the late 1990s, Heber [Heb98] introduced a new tool, Geometric Invariant Theory (GIT), to study the case when G is solvable and so-called standard. Using GIT, Lauret [Lau10] was able to show that all Einstein solvmanifolds are necessarily standard.

To date, all known examples of non-compact, non-flat, homogeneous Einstein spaces are isometric to solvable Lie groups with left-invariant metrics and their isometry groups are linear [Jab13]. We propose the following.

> **Strong Alekseevskii Conjecture:** A non-compact, non-flat, homogeneous Einstein space G/K must be isometric to a solvable Lie group with left-invariant metric. Equivalently, G is linear and K is a maximal compact subgroup of G.

Very recently, an enormous amount of progress has been made by Lafuente and Lauret in determining the structure of G [LL12]. Here we see the use of Geometric Invariant Theory as an essential tool. We briefly recall a portion of their results to help motivate future directions below.

Given the Lie group G, we may consider its Levi decomposition $G = G_1 \ltimes G_2$, where G_1 is semi-simple and G_2 is the solvable radical. If G/K admits an Einstein metric, what can be said about G_1 and G_2? From [LL12], we now know

(1) $G_2 = AN$, where N is the nilradical and A is abelian,
(2) N is equipped with a so-called Ricci soliton metric, and
(3) G_1 and A commute, and K can be chosen to be a subgroup of $U = G_1 A$.

Using their work, it can be deduced that the group G_2 admits an Einstein metric.

Open Question 1. *Equipping G_2 with the geometry it picks up as a subspace of G/K, is this an Einstein space?*

This question is closely related to the question of how A and G_1 fit together. More precisely, at the point $[e] \in G/K$, are the orbits of A and G_1 orthogonal? Motivated by examples coming from Geometric Invariant Theory, we expect this and more.

Open Question 2. *Decompose $G_1 = H_1 \cdots H_k$ as a product of simple subgroups. Are the orbits of the H_i at $[e]$ orthogonal to each other?*

Open Question 3. *What is the nature of the U-action on Lie N? Is this action 'self-adjoint'? i.e. is $Ad(U)|_{Lie\ N}$ closed under transpose?*

In light of the structure results of Lafuente-Lauret, answering these questions would go a long ways towards resolving the Strong Alekseevskii Conjecture. Very recently, in joint work with Peter Petersen, we have used those structure results to make partial progress on the Alekseevskii Conjecture, showing that G_1 may be assumed to contain no compact, simple subgroups [JP14]. Once again, GIT appears as an essential tool.

At this point, we anticipate, or at least hope, that the program of using GIT developed by Heber, Lauret, Nikolayevsky and many others will be used to reduce the problem to the case when G is semi-simple. Aside from the work of Nikonorov [Nik00] and a new result in [JP14], very little is known in this case and a new tool will need to be introduced.

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Some open problems on compact homogeneous Einstein manifolds

Yusuke Sakane

On open problems of compact homogeneous Einstein manifolds, we first deal with homogeneous Einstein metrics of generalized flag manifolds and Stiefel manifolds $V_k\mathbb{R}^n = SO(n)/SO(n-k)$ and we also discuss Einstein metrics on compact simple Lie groups.

For generalized flag manifolds, we have a unique Kähler Einstein metric if we fix a complex structure. It is known that there are finitely many invariant complex structures on a generalized flag manifold and these complex structures have been classified (cf.[AlPe]). But we have more homogeneous Einstein metrics on generalized flag manifolds as a real manifold if these are not hermitian symmetric spaces.

Open problem: Find all homogeneous Einstein metrics on generalized flag manifold G/K.

For a homogeneous space G/K with non-equvariant irreducible components, we have an open problem whether the number of homogeneous Einstein metrics on G/K is finite or not. Homogeneous Einstein metrics on generalized flag manifold are a special case of this open problem. Recently we have classified Einstein metrics on generalized flag manifolds with up to five irreducible components and we have obtained partial results for the case of six irreducible components ([ACS1], [ACS2]).

For Stiefel manifolds $SO(n)/SO(n-k)$, as is well known, we have Jensen's homogeneous Einstein metrics.

Open problem: Find all homogeneous Einstein metrics on Stiefel manifolds $V_k\mathbb{R}^n = SO(n)/SO(n-k)$.

This problem may be hard to solve in general, because we have equivalent irreducible components as $Ad(SO(n-k))$-modules, even we have one dimensional components. We consider homogeneous Einstein metrics on Stiefel manifolds $V_{k_1+k_2}\mathbb{R}^n = SO(k_1+k_2+k_3)/SO(k_3)$ with $n = k_1+k_2+k_3$. By considering $Ad(SO(k_1)\times SO(k_2)\times SO(k_3))$-invariant metrics on Stiefel manifolds $SO(k_1+k_2+k_3)/SO(k_3)$, we obtain the decomposition of non-equvariant irreducible components (number of irreducible components are at most six). Recently we obtain that there exist homogeneous Einstein metrics on Stiefel manifolds $SO(n)/SO(n-4)$ for $n \geq 6$[ASS1], which are different from Jensen's Einstein metrics [Jen].

For compact semi-simple Lie groups we have

Open problem: Find all left invariant Einstein metrics on compact semi-simple Lie groups. In particular, we can ask how many are there. Is it finitely many or not?

Even for $G = SU(3)$, or $G = SU(2)\times SU(2)$, we do not know all left-invariant Einstein metrics on G. (For $SU(2)\times SU(2)$, see a result of Nikonorov and Rodionov[NiRo].)

For a compact semi-simple Lie group G and a closed subgroup H, the group $G\times H$ acts transitively on G by $(g,h)y = gyh^{-1}$ $\;(\,(g,h)\in G\times H,\, y\in G\,)$ and the Lie group G can be expressed as $(G\times H)/\Delta H$, where $\Delta H = \{(h,h)\mid h\in H\}$. D'Atri and Ziller [DAZi] considered naturally reductive metrics on compact semi-simple Lie groups and obtained a characterization of these metrics for compact simple Lie groups. They also obtained many naturally reductive Einstein metrics on compact simple Lie groups G by using irreducible symmetric space G/H of compact type and isotropy irreducible spaces.

D'Atri and Ziller asked a following question:

Is there non-naturally reductive left invariant Einstein metrics on a compact Lie group?

Concerning this problem, Mori[Mo] has found non-naturally reductive Einstein metrics on compact simple Lie groups $SU(n)$ for $n \geq 6$. By using generalized flag manifold G/H with two irreducible components, we have obtained non-naturally reductive Einstein metrics on compact simple Lie groups G where G is either $SO(n)$ ($n \geq 11$), $Sp(n)$ ($n \geq 3$), E_6, E_7 or E_8[ArMoSa]. We also can show there exit non-naturally reductive Einstein metrics on compact simple Lie groups $SO(n)$ $n \geq 7$, by using homogeneous spaces $SO(k_1+k_2+k_3)/(SO(k_1)\times SO(k_2)\times SO(k_3))$[ASS2]. Chen and Liang[ChL] has found non-naturally reductive Einstein metrics on the compact simple Lie group F_4.

Open problem: Find more non-naturally reductive Einstein metrics on compact simple Lie groups G by using homogeneous spaces G/H.

Dickinson and Kerr[DiKe] obtained many homogeneous spaces G/H with exactly two irreducible components in the isotropy representation and these spaces include t generalized flag manifolds with two irreducible components. Thus we may find more non-naturally reductive Einstein metrics on compact simple Lie groups G.

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Three pearls of noncompact homogeneous Einstein geometry

Yuri Nikolayevsky

I will speak on three (not very new) results which in my opinion have shaped our modern understanding of noncompact homogeneous Einstein manifolds.

1. Yurii Nikonorov and Alekseevsky Conjecture

Alekseevsky Conjecture: Einstein homogeneous spaces of negative scalar curvature are solvmanifolds (more precisely, the isotropy subgroup is the maximal compact subgroup).

"Discouraging" evidences:

- Leite, Dotti Miatello 1982: $\mathrm{SL}(n,\mathbb{R})$, $n \geq 3$, admits a metric with $\mathrm{Ric} < 0$ (more on that on Thursday).
- Leite, Dotti Miatello, Miatello 1984: a unimodular Lie group which admits a metric with $\mathrm{Ric} < 0$ is noncompact and semisimple. Constructed such a metric on some complex semisimple Lie groups.
- In all the examples the metric was such that the Cartan decomposition was orthogonal.
- Still don't know whether an Einstein metric exists even on small groups, e.g. $\mathrm{SL}(3,\mathbb{R})$.

Let $M = G/H$ be a homogeneous space, with $H \subset K$, $H \neq K$, where K is the maximal connected compact subgroup in G. Let $\mathfrak{g} = \mathfrak{p}' \oplus \mathfrak{k}$ be a Cartan decomposition and let $\mathfrak{k} = \mathfrak{p}'' \oplus \mathfrak{h}$, where $\mathfrak{k}$ is the Lie algebra of K, $\mathfrak{h}$ is the Lie algebra of H and $\mathfrak{p}', \mathfrak{p}''$ are H-modules. Let $\mathfrak{p}' = \mathfrak{p}_1 \oplus \cdots \oplus \mathfrak{p}_u$, $\mathfrak{p}'' = \mathfrak{p}_{u+1} \oplus \cdots \oplus \mathfrak{p}_v$ be decompositions on irreducible H-modules, so that $T_oM = \mathfrak{p}' \oplus \mathfrak{p}'' = \mathfrak{p}_1 \oplus \cdots \oplus \mathfrak{p}_u \oplus \mathfrak{p}_{u+1} \oplus \cdots \oplus \mathfrak{p}_v$.

Definition 1. *An inner product on T_oM is called* awesome*, if $\mathfrak{p}_i \perp \mathfrak{p}_j$ when $i \neq j$, and for $X, Y \in \mathfrak{p}_i$, we have $\langle X, Y\rangle = x_i \varepsilon_i B(X, Y)$, where $x_i > 0$, B is the Killing form, and $\varepsilon_i = 1$ when $\mathfrak{p}_i \subset \mathfrak{p}'$ and $\varepsilon_i = -1$ when $\mathfrak{p}_i \subset \mathfrak{p}''$. When all the x_i are ones, we denote the metric $\langle \cdot, \cdot\,_1$.*

If $H = \{e\}$, so that M is a group, then irreducible modules are just arbitrary one-dimensional subspaces in $\mathfrak{g}$. An inner product on $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}'$ is awesome exactly when $\mathfrak{k} \perp \mathfrak{p}'$.

Theorem 1 ([N]). *If G is semisimple, no awesome inner product on T_oM is Einstein. In particular, no inner product on $\mathfrak{g}$ for which a Cartan decomposition is orthogonal is Einstein.*

Some spaces: never.

Corollary 2 ([N]). *If $\mathfrak{p}'$ and $\mathfrak{p}''$ have no isomorphic irreducible submodules, then no inner product on T_oM is Einstein.*

Example: no left-invariant Einstein metric on the Stiefel manifold $\mathrm{SO}(n,2)/\mathrm{SO}(n)$, or on the space $\mathrm{SL}(n+1)/\mathrm{SO}(n)$.

Proof: elaborate and masterful work with the components of Ric.

2. Jorge Lauret and Standardness Conjecture

Let $(\mathfrak{g}, \langle\cdot,\cdot\,)$ be a metric solvable Lie algebra with the nilradical $\mathfrak{n}$.

J.Heber's Standardness Conjecture, [H]: Any metric solvable Einstein Lie algebra is *standard*, that is, $\mathfrak{a} := \mathfrak{n}^\perp$ is abelian.

Theorem 3 ([L]). *Yes.*

Proof: Very difficult. Uses Geometric Invariant Theory; stratification of the variety of Lie brackets under the action of $\mathrm{GL}(n)$.

3. Jens Heber and the structure of Einstein solvmanifolds

Theorem 4 ([H]). *Every Einstein solvmanifold is isometric to a one of Iwasawa type.*

A metric Lie algebra is called an algebra of *Iwasawa type* when ad_Y, $Y \in \mathfrak{a}$, are symmetric and commute and ad_H is positive.

The Ricci curvature for Iwasawa type algebras has the form

$$\mathrm{Ric}(Y_1, Y_2) = -\operatorname{Tr} \mathrm{ad}_{Y_1} \mathrm{ad}_{Y_2}, \quad \mathrm{Ric}(Y, X) = 0,$$

$$\mathrm{Ric}(X_1, X_2) = -\langle \mathrm{ad}_H X_1, X_2\rangle + \mathrm{Ric}^{\mathfrak{n}}(X_1, X_2),$$

where $X, X_1, X_2 \in \mathfrak{n}$, $Y, Y_1, Y_2 \in \mathfrak{a}$ and $\mathrm{Ric}^{\mathfrak{n}}$ is the Ricci curvature of $(\mathfrak{n}, \langle\cdot,\cdot\,)$. From this we immediately get the following result.

Theorem 5 (Rank one reduction; [H]). *Let $(\mathfrak{g}, \langle\cdot,\cdot\,)$ be a solvable Einstein metric Lie algebra of Iwasawa type. Then $(\mathbb{R}H \oplus \mathfrak{n}, \langle\cdot,\cdot\,)$ is again a solvable Einstein metric Lie algebra of Iwasawa type, but with the nilradical of codimension one. This construction is invertible.*

So to understand Einstein solvmanifolds we need to "only" look at one-dimensional positive symmetric extensions of nilpotent algebras.

Theorem 6 ([H]). *Let $(\mathfrak{g}, \langle\cdot,\cdot\rangle$ be a solvable Einstein metric Lie algebra of Iwasawa type of rank one. Then up to scaling the eigenvalues of ad_H are natural numbers. So $\mathfrak{n}$ admits a gradation.*

Proof: Decomposition on eigenspaces and the relations between the eigenvalues.

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A survey on homogeneous Ricci flow

Ramiro Lafuente

The aim of this talk is to survey the current state of the art of homogeneous Ricci flow, with a special focus on techniques and open questions.

Hamilton's Ricci flow is the following well-known evolution equation for a curve of Riemannian metrics $g(t)$ on a differentiable manifold:

$$\frac{\partial}{\partial t} g(t) = -2\,\mathrm{Ric}(g(t)).$$

A natural symmetry assumption to consider when studying this flow is that of homogeneity. Indeed, when (M, g_0) is a homogeneous manifold, then the unique complete and of bounded curvature Ricci flow solution $g(t)$ with $g(0) = g_0$ is homogeneous for all t, thus reducing the above PDE to an ODE.

The first articles in the subject deal with the behavior of the flow in low-dimensional homogeneous manifolds (mostly in dimensions 3 and 4). In [IJ92] the authors study the volume-normalized Ricci flow of locally homogeneous geometries on closed 3-manifolds, taking advantage of Milnor frames ([M76]) to diagonalize the metrics simultaneously in the Lie group case. They continue their study in [IJL06] with a large family of 4-dimensional homogeneous metrics. After these works, the long-time behavior of immortal solutions (those defined for $t \in [0, \infty)$) was studied in [Lt07], where by using the more geometric point of view of pointed (or Cheeger-Gromov) topology in the space of Riemannian manifolds, convergence to expanding solitons was obtained. The long-time behavior of 3-dimensional solutions was also studied in [KMcL01, CSC09]. Also, in [GP10] a global picture of the flow behavior in this case was obtained by considering the approach of varying Lie brackets instead of inner products.

This approach of varying Lie brackets instead of metrics was first used, with very satisfactory results, in the study of the Ricci flow of simply connected nilmanifolds in [L11], where the author showed -among other things- that the solution is always immortal and of Type-III (i.e. $\|\mathrm{Riem}(g(t))\| \leq C/t$ for some constant C), and that under a very natural normalization it converges to a Ricci soliton which is also invariant under a (possibly differente) nilpotent group. Using the same methods, similar results were obtained in [Arr13] for the Ricci flow of left-invariant metrics on solvable Lie groups with a codimension-one abelian ideal. The Ricci flow of nilmanifolds was also studied in [Gz07, P10, LW11].

Recently, the above mentioned method was extended for the general homogeneous case in [L13] under the name of *bracket flow*, which is indeed an ODE evolution equation for a curve of Lie brackets that is equivalent in a very precise sense to homogeneous Ricci flow. As a direct application of this tool, it was proved in [Lf14] that the scalar curvature of a homogeneous Ricci flow solution must blow up in presence of a finite-time singularity.

However, unlike the case of nilmanifolds, in the general homogeneous case many important questions remain open:

Question 1 For a general solvable Lie group, does the bracket flow converge (after a suitable normalization) to a soliton? Is this limit unique? When is it non-flat?

Question 2 Is it possible to use the bracket flow apporach to study the rather unexplored (noncompact) semisimple case?

The Ricci flow of compact homogeneous manifolds was studied in [GM09, AC11, Bu13], in some particular cases such as flag manifolds, and specially under the assumption that the isotropy representation has only few irreducile summands (namely, 2 or 3). The authors prove that the solutions always develop finite-time singularities which are of Type-I, and that the normalized flow converges to a soliton whose algebraic structure is closely related to the algebraic structure of the homogeneous solution. This results were very recently shown to hold in the general compact homogeneous case in [B14].

Moreover, in [B14] the author proves that any homogeneous Ricci flow solution with finite exctintion time develops a Type-I singularity, and that any immortal solution develops a Type-III singularity.

We conclude this survey with an important open question. Recall that by [Lf14] any homogeneous Ricci flow solution on a homogeneous space that doesn't admit homogeneous metrics of positive scalar curvature must be necessarily immortal. On the other hand, by [B14] a non-flat homogeneous Ricci flow solution on a compact space must develop a finite-time singularity (thus the scalar curvature must become positive before the singularity). It is intriguing to know how far can this dichotomy be pushed:

Question 3 If a homogeneous space admits homogeneous metrics with positive scalar curvature, is it true then that the scalar curvature of any homogeneous Ricci flow solution on that space must eventually become positive?

An affirmative answer to the above question would imply in particular as an immediate corollary the Alekseevskii conjecture.

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Symplectic curvature flow

Cynthia Will
(joint work with Jorge Lauret)

Let (M, ω, g, J) be an almost-Kähler manifold of dimension $2n$, i.e. an almost-hermitian manifold such that $d\omega = 0$. With Kähler-Ricci flow as a motivation, it is natural to evolve the symplectic structure ω in the direction of the Chern-Ricci form p but, since in general $p \neq p^{1,1}$, one is forced to flow the metric g as well in order to preserve compatibility. The following evolution equation for a one-parameter family $(\omega(t), g(t))$ of almost-Kähler structures has recently been introduced by Streets-Tian in [ST] and is called the *symplectic curvature flow* (or SCF for short):

$$\begin{cases} \frac{\partial}{\partial t}\omega = -2p, \\ \frac{\partial}{\partial t}g = -2p^{1,1}(\cdot, J\cdot) - 2\,\mathrm{Rc}^{2,0+0,2}, \end{cases}$$

where p is the Chern-Ricci form of (ω, g) and Rc is the Ricci tensor of g.

In the case of invariant structures on a quotient $M = G/\Gamma$, where Γ is a cocompact discrete subgroup of a Lie group G (e.g. solvmanifolds and nilmanifolds), all the tensors involved are determined by their value at the identity of the group and therefore the SCF becomes an ODE system. Short-time existence (forward and backward) and uniqueness of the solutions are hence, always guaranteed.

In [LW], we study different aspects of this flow in the locally homogeneous case, including long-time existence, solitons, regularity and convergence (see also [L]). We develop in detail two classes of Lie groups, which are relatively simple from a structural point of view but yet geometrically rich and exotic: solvable Lie groups with a codimension one abelian normal subgroup, called *almost-abelian* Lie groups, and a construction attached to each left symmetric algebra (see [AS]).

It is easy to see that any $2n$-dimensional almost-abelian Lie algebra $\mathfrak{g}$ is determined by a $(2n-1)\times(2n-1)$ matrix, so we can denote the corresponding Lie group by G_A, where $A \in \mathbb{R}^{(2n-1)\times(2n-1)}$. After giving some criteria for the equivalence between these structures, we compute their Chern-Ricci and Ricci tensors in terms of A. We then study the existence, uniqueness and structure of solitons among this class, which turn out to be all expanding if nonflat. We obtained that for a large subfamily of matrices, the Lie group G_A admits a soliton if and only A is either semisimple or nilpotent, and the soliton condition is given by A normal or $[A,[A,A^t]] = -\left(|[A,A^t]|^2/|A|^2\right)A$, respectively. Furthermore, the SCF is equivalent to the ODE for $A = A(t)$ given by

$$A' = -\frac{1}{2}((\mathrm{tr}A)^2 + \mathrm{tr}S(A)^2)A + \frac{1}{2}[A,[A,A^t]] - \tfrac{trA}{2}[A,A^t].$$

This allowed us, by following the lines in [A], to show that any solution is immortal since $|A|$ is non-increasing (and type-III if $\mathrm{tr}A_0^2 \geq 0$) and that the quantity $|[A,A^t]|^2/|A|^4$ is strictly decreasing along the flow, unless it is a soliton.

In order to search for solitons beyond the solvable case, we considered a construction attaching to each n-dimensional left-symmetric algebra (LSA for short) an almost-Kähler structure on a $2n$-dimensional Lie group. More precisely, from a Lie algebra $\mathfrak{g}$ endowed with an LSA structure, one obtains an almost-Kähler Lie algebra $((\mathfrak{g} \ltimes_\theta \mathbb{R}^n), \omega, g)$, where θ is determined by the left-multiplication of the LSA product and $n = \dim \mathfrak{g}$. This construction provides many examples with unexpected behavior. For example, if we consider on $\mathfrak{g} = \mathfrak{u}(2)$ the LSA structure obtained by identifying $\mathfrak{g}$ with the quaternion numbers $\mathbb{H}$, we found a family of ancient solutions which develop a finite time singularity. Moreover, neither their Chern scalar nor their scalar curvature are monotone along the flow and they converge in the pointed sense to a (non-Kähler) shrinking soliton solution on the same Lie group. Note that all this is in clear contrast with the Ricci flow behavior.

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A method for parametrizing varieties of nilpotent Lie algebras

Tracy Payne

Let $\Lambda \subseteq \Theta_n = \{(i,j,k) : 1 \le i < j < k \le n\}$. Let $\mathcal{L}_\Lambda$ denote the set of Lie algebras whose Lie brackets relative to a basis $\{x_i\}_{i=1}^n$ have structure constants indexed up to skew symmetry by Λ; that is, Lie brackets are of form $[x_i, x_j] = \alpha_{ij}^k x_k$ where $\alpha_{ij}^k \neq 0$ and $i < j$ if and only if $(i,j,k) \in \Lambda$. We consider the question: What are the isomorphism classes in $\mathcal{L}_\Lambda$?

This problem arises in the classication of special families of nilpotent Lie algebras. The study of geometric structures on nilpotent Lie algebras can also involve this question. See, for example, [Ver66], [Mil04], [Bur06], [Arr11]. We needed a result of this type for a project in which we have classified the soliton and non-soliton nilpotent Lie algebras in dimensions 7 and 8 for which the Nikolayevsky (or pre-Einstein) derivation has distinct positive eigenvalues ([KP13], [Pay14a], [Pay14b], [Pay14c],[Pay]).

Our strategy is as follows. We first consider the set $\mathcal{S}_\Lambda$ of all n-dimensional anticommutative nonassociative algebras such that the product has structure constants indexed by Λ. We find a compact, semi-algebraic set S_Λ so that each isomorphism class in $\mathcal{S}_\Lambda$ is represented by precisely one algebra in S_Λ. This requires finding a transversal to the natural $GL_n(\mathbb{R})$ action on the space of algebras. We use Hadamard's Global Inverse Function Theorem to show that this transversal meets each orbit exactly once. Second, we parametrize the set of algebras in S_Λ satisfying the Jacobi Identity. This requires parametrizing the set S_Λ so that nontrivial terms of the polynomials encoding the Jacobi Identity relate in a simple way to tangential directions to S_Λ. This nice parametrization is related to a set of integer quadruples (s,t,u,v) associated to the set Λ.

We illustrate the strategy with the simplest non-elementary example, finding S_Λ and solutions to the Jacobi Identity for the set $\Lambda = \{(1,2,4),(1,3,5),(1,5,6),(2,4,6),(2,5,7),(3,4,7)\}$. Here, the quadruple associated to Λ is $(1,6,2,4)$, arising from the pairs $\{(1,2,4),(3,4,7)\}$ and $\{(1,3,5),(2,5,7)\}$. Correspondingly, the single nonvanishing equation in the Jacobi Identity is $\alpha_{12}^4\alpha_{34}^7 - \alpha_{13}^5\alpha_{25}^7 = 0$ and the set S_Λ is parallel to the vector $x_1 + x_6 - x_2 - x_7$.

We must impose two hypotheses. First, isomorphism classes in $\mathcal{S}_\Lambda$ should be orbits of the diagonal subgroup in $GL_n(\mathbb{R})$. Second, the set S_Λ should be null space spanning: the tangent space should be described by vectors defined combinatorially in terms of Λ. The first condition holds for all 8-dimensional nilpotent Lie algebras with simple Nikolayevsky derivation, and the second holds for almost all such Lie algebras.

Using these methods we may prove the following.

Theorem 1 ([Pay14c]). *Let $\Lambda \subseteq \Theta_n$ be null space spanning, and suppose that isomorphism classes $\widetilde{\mathcal{L}}_\Lambda$ in $\mathcal{S}_\Lambda$ are orbits of the diagonal subgroup in $GL_n(\mathbb{R})$.*

(1) *If Λ has one quadruple of multiplicity two, then $\widetilde{\mathcal{L}}_\Lambda$ is finite.*
(2) *If Λ has two quadruples of multiplicity two, then $\widetilde{\mathcal{L}}_\Lambda$ is finite.*
(3) *If Λ has one quadruple of multiplicity three, then $\widetilde{\mathcal{L}}_\Lambda$ is one-dimensional.*

(4) *If Λ has one quadruple of multiplicity three and one quadruple of multiplicity two, and these overlap, then $\widetilde{\mathcal{L}}_\Lambda$ is one-dimensional.*

Furthermore, we describe in terms the combinatorics of Λ how to parametrize S_Λ in each case.

More work must be done to understand the sets $\widetilde{\mathcal{L}}_\Lambda$ in the cases when the two hypotheses do not hold. In addition, the connections to Lie algebra cohomology should be developed.

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Some properties of the Ricci flow on generalized Wallach spaces

Yurii Nikonorov

The study of the normalized Ricci flow equation for a 1-parameter family of Riemannian metrics $\mathbf{g}(t)$ in a Riemannian manifold M^n was originally used by R. Hamilton in [8] and since then it has attracted the interest of many mathematicians (cf. [6]). Recently, there is an increasing interest towards the study of the Ricci flow (normalized or not) on homogeneous spaces and under various perspectives ([4], [5], [7], [9], [10], [15] and references therein).

In this talk, we discuss some properties of the normalized Ricci flow on generalized Wallach spaces. The study of the Ricci flow on these spaces was initiated in the papers [1, 2, 3].

Recall that generalized Wallach spaces are compact homogeneous spaces G/H (with semisimple compact Lie group G) whose isotropy representation decomposes into a direct sum $\mathfrak{p} = \mathfrak{p}_1 \oplus \mathfrak{p}_2 \oplus \mathfrak{p}_3$ of three $\mathrm{Ad}(H)$-invariant irreducible modules satisfying $[\mathfrak{p}_i, \mathfrak{p}_i] \subset \mathfrak{h}$ $(i \in \{1,2,3\})$ ([12], [13], [14]). For a fixed bi-invariant inner product $\langle \cdot, \cdot \rangle$ on the Lie algebra $\mathfrak{g}$ of the Lie group G, any G-invariant Riemannian metric $\mathbf{g}$ on G/H is determined by an $\mathrm{Ad}(H)$-invariant inner product

$$(\cdot,\cdot) = x_1 \langle \cdot,\cdot\rangle|_{\mathfrak{p}_1} + x_2 \langle \cdot,\cdot\rangle|_{\mathfrak{p}_2} + x_3 \langle \cdot,\cdot\rangle|_{\mathfrak{p}_3}, \tag{1}$$

where x_1, x_2, x_3 are positive real numbers (we may suppose w.l.o.g. that $\mathfrak{p}$ is a $\langle\cdot,\cdot\rangle$-orthogonal complement to $\mathfrak{h}$ in $\mathfrak{g}$). Since G is compact and semisimple, we may suppose that $\langle\cdot,\cdot\rangle$ is the minus Killing form of the Lie algebra $\mathfrak{g}$.

By using expressions for the Ricci tensor and the scalar curvature in [14] the normalized Ricci flow equation reduces to a system of ODE's of the form

$$(2)\qquad \frac{dx_1}{dt} = f(x_1,x_2,x_3), \quad \frac{dx_2}{dt} = g(x_1,x_2,x_3), \quad \frac{dx_3}{dt} = h(x_1,x_2,x_3),$$

where $x_i = x_i(t) > 0$ $(i = 1,2,3)$, are parameters of the invariant metric (1) and

$$\begin{aligned}
f(x_1,x_2,x_3) &= -1 - \frac{A}{d_1}x_1\left(\frac{x_1}{x_2x_3} - \frac{x_2}{x_1x_3} - \frac{x_3}{x_1x_2}\right) + 2x_1\frac{S_{\mathbf{g}}}{n},\\
g(x_1,x_2,x_3) &= -1 - \frac{A}{d_2}x_2\left(\frac{x_2}{x_1x_3} - \frac{x_3}{x_1x_2} - \frac{x_1}{x_2x_3}\right) + 2x_2\frac{S_{\mathbf{g}}}{n},\\
h(x_1,x_2,x_3) &= -1 - \frac{A}{d_3}x_3\left(\frac{x_3}{x_1x_2} - \frac{x_1}{x_2x_3} - \frac{x_2}{x_1x_3}\right) + 2x_3\frac{S_{\mathbf{g}}}{n},\\
S_{\mathbf{g}} &= \frac{1}{2}\left(\frac{d_1}{x_1} + \frac{d_2}{x_2} + \frac{d_3}{x_3} - A\left(\frac{x_1}{x_2x_3} + \frac{x_2}{x_1x_3} + \frac{x_3}{x_1x_2}\right)\right).
\end{aligned}$$

Here d_i, $i = 1,2,3$, are the dimensions of the corresponding irreducible modules $\mathfrak{p}_i$, $n = d_1 + d_2 + d_3$ and A is some special nonnegative number (see details in [12, 1, 2]). If $A = 0$, then the space under consideration is (at least locally) a direct product of three compact symmetric irreducible spaces [14]. If $A \neq 0$, then by denoting $a_i := A/d_i > 0$, $i = 1,2,3$, the functions f, g, h can be expressed in a more convenient form (independent of A and d_i) as

$$\begin{aligned}
f(x_1,x_2,x_3) &= -1 - a_1x_1\left(\frac{x_1}{x_2x_3} - \frac{x_2}{x_1x_3} - \frac{x_3}{x_1x_2}\right) + x_1B,\\
g(x_1,x_2,x_3) &= -1 - a_2x_2\left(\frac{x_2}{x_1x_3} - \frac{x_3}{x_1x_2} - \frac{x_1}{x_2x_3}\right) + x_2B,\\
h(x_1,x_2,x_3) &= -1 - a_3x_3\left(\frac{x_3}{x_1x_2} - \frac{x_1}{x_2x_3} - \frac{x_2}{x_1x_3}\right) + x_3B,
\end{aligned}$$

where

$$B := \left(\frac{1}{a_1x_1} + \frac{1}{a_2x_2} + \frac{1}{a_3x_3} - \left(\frac{x_1}{x_2x_3} + \frac{x_2}{x_1x_3} + \frac{x_3}{x_1x_2}\right)\right)\left(\frac{1}{a_1} + \frac{1}{a_2} + \frac{1}{a_3}\right)^{-1}.$$

Since the volume $V = x_1^{1/a_1}x_2^{1/a_2}x_3^{1/a_3}$ is a first integral of the system (2), on the surface

$$(3)\qquad V \equiv 1$$

we can reduce (2) to the system of two differential equations of the type

$$(4)\qquad \frac{dx_1}{dt} = \widetilde{f}(x_1,x_2), \quad \frac{dx_2}{dt} = \widetilde{g}(x_1,x_2),$$

where

$$\begin{aligned}\widetilde{f}(x_1,x_2) &\equiv f(x_1,x_2,\varphi(x_1,x_2)),\\ \widetilde{g}(x_1,x_2) &\equiv g(x_1,x_2,\varphi(x_1,x_2)),\\ \varphi(x_1,x_2) &= x_1^{-\frac{a_3}{a_1}} x_2^{-\frac{a_3}{a_2}}.\end{aligned}$$

It is known ([14]) that every generalized Wallach space admits at least one invariant Einstein metric. Later in [11, 12] a detailed study of invariant Einstein metrics was developed for all generalized Wallach spaces. In particular, it was shown that there are at most four invariant Einstein metrics (up to homothety) for every such space. It should be noted that invariant Einstein metrics with $V = 1$ correspond to singular points of (4), therefore, (x_1^0, x_2^0, x_3^0) is a singular point of the system (2), (3) if and only if (x_1^0, x_2^0) is a singular point of (4).

We discuss some qualitative results on singular points of the normalized Ricci flow on generalized Wallach spaces from the papers [1, 2, 3] and some recent results on the maximal interval of existence for a solution of the normalized Ricci flow on the same spaces.

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The evolution of APEs

Eric Woolgar

APEs are Asymptotically Poincaré-Einstein manifolds. They are conformally compactifiable and asymptotically hyperbolic, so that the sectional curvatures approach -1 on asymptotic ends. They admit a notion of conformal boundary-at-infinity. The conformal metric has a polyhomogeneous expansion in Gaussian normal coordinates on a neighbourhood of that boundary, for which the radial normal coordinate is called a *special defining function* x. Furthermore, for an n-dimensional APE, the Einstein equations for the (not conformally rescaled) metric hold up to, but not including, order x^{n-2}. Up to this order, the leading terms in the Gaussian normal coordinate expansion of the metric are universal, depending only on the conformal class of the metric induced on the boundary-at-infinity. All odd powers of x less than order x^{n-2} vanish. This is called a *(partially) even Fefferman-Graham expansion* [5]. APEs have several invariants, including the *renormalized volume*, the *conformal anomaly* (which vanishes if n is even), and, for certain conformal infinities, the *mass*. APEs can also have a so-called *ambient obstruction tensor*, which vanishes for n even and also for many commonly-studied conformal infinities when n is odd. For purposes of this talk we consider only unobstructed APEs and, in what is somewhat a departure from convention, we use n to denote the bulk dimension.

APEs are of interest in gravitational physics, where they describe asymptotically AdS (anti-de Sitter) black holes. The thermodynamics of these black holes was studied long ago by SW Hawking and DN Page [7], who used the idea that thermal properties of metrics can be studied by Wick-rotating a static Lorentzian-signature black hole metric to obtain a Riemannian metric with a circle action whose period is the inverse temperature. The Wick-rotated Kottler, or AdS-Schwarzschild, black holes are Poincaré-Einstein metrics (and hence APEs) with boundary-at-infinity $S^2 \times S^1$, where the S^2 carries the canonical round metric with unit sectional curvature. The conformal class of the boundary metric is then determined by the length L of the S^1 factor. There is an open interval I such that, for any $L \in I$, there are 3 distinct Poincaré-Einstein metrics that induce the same conformal class on the boundary-at-infinity. Two of these are Kottler black holes, called the small and the large black hole, having topology $S^2 \times \mathbb{R}^2$. The third has topology $\mathbb{R}^3 \times S^1$, constructed from standard hyperbolic space by identifying points under a hyperbolic translation. This is called *thermal hyperbolic space*. Hawking and Page defined a renormalized volume for these metrics, known to them as the regularized action and interpreted as a thermodynamic variable related to Helmholtz free energy. At high temperatures, the large black hole minimizes the regularized action, while at low temperatures thermal hyperbolic space is the minimizer. This shows that the system exhibits a phase transition. The transition is mediated by the small black hole, which always has greatest regularized action, representing an energy barrier for the phase transition which must be overcome.

The Hawking-Page regularized action is not defined in the same way as the renormalized volume [8, 6] used for APEs. Nonetheless, these definitions correspond for 4-dimensional AdS-Schwarzschild black holes and for Kottler metrics with toroidal infinity in all dimensions, though not for 5-dimensional AdS-Schwarzschild black holes.

In this talk, I will consider the flow of APE metrics under the normalized Ricci flow of Bahuaud [1]. We first define several APE invariants such as renormalized volume (in the sense of [8, 6]), the conformal anomaly, and mass. We then recall short-time and long-time existence results for normalized Ricci flow of APE data, including a new long-time existence and convergence result for rotationally symmetric APE flow [3]. We then give formulas for the flow of APE invariants [2, 4]. Finally, we discuss an application of APE flows to the Hawking-Page phase transition, and consider the possible interpretation of renormalized volume as Helmholtz free energy for the Kottler static black hole in a radiation bath described by the canonical ensemble. This is based on joint work [4, 2, 3].

I finish by posing an open problem. Find a complete, conformally compactifiable Poincaré-Einstein manifold whose boundary-at-infinity is a compact connected manifold of constant sectional curvature -1, or show that there is no such Poincaré-Einstein manifold.

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Einstein solvmanifolds are maximally symmetric

Carolyn Gordon
(joint work with Michael Jablonski)

Many relationships exist between the geometry of a left-invariant Riemannian metric on a Lie group G and the Lie group structure of G. For example, if G admits a metric of negative sectional curvature, then G must be solvable. A natural question is whether there exists a "best" metric on a given Lie group and, if so, how does its geometry relate to the Lie group structure? We define a left-invariant Riemannian metric g on a Lie group G to be *maximally symmetric* if the isometry group of any other left-invariant metric h on G is contained in that of g (or, more precisely, that of Φ^*g for some automorphism Φ of G). We prove the following:

Theorem. *If a solvable Lie group admits a left-invariant Einstein metric g of negative Ricci curvature, then g is maximally symmetric.*

The longstanding Alekseevskii conjecture asks whether every homogeneous Einstein manifold M of nonpositive Ricci curvature is diffeomorphic to $\mathbf{R}^n$; a slightly strengthened version asks whether every such M can be realized as a simply-connected solvable Lie group with a left-invariant Riemannian metric. The theorem lends philosophical support to the Alekseevskii conjecture, since any counterexample to the Alekseevskii conjecture would not be maximally symmetric. In fact M. Jablonski and P. Petersen proved that any counterexample with semisimple isometry group would essentially be minimally symmetric!

The proof is a blend of Lie group structure theory and geometric invariant theory and relies on the foundational work of Jens Heber [1] on "standard" Einstein solvmanifolds, Jorge Lauret's result [2] establishing that all Einstein solvmanifolds of negative Ricci curvature are standard, and a technique of Yury Nikolayevsky [3] for identifying the nilradicals of Einstein solvmanifolds.

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Boundary behaviour of Hitchin and hypo flows with left-invariant initial data

Vicente Cortés

This talk is based on joint work with Florin Belgun, Marco Freibert and Oliver Goertsches [1]. Hypo and Hitchin flows constitute a system of first order pdes for the construction of Ricci-flat Riemannian manifolds of dimensions 6, 7 and 8 with holonomy group in $SU(3)$, G_2 and $Spin(7)$, respectively. The initial data for these flows are hypo $SU(2)$-structures, half-flat $SU(3)$-structures and cocalibrated G_2-structures on manifolds of dimension 5, 6 and 7, respectively [2, 3, 4, 5]. Given a left-invariant such structure on a Lie group G, the flow equations can be solved on a maximal interval I, defining a Ricci-flat metric g of cohomogeneity 1 on $U = G \times I$ with holonomy group in $SU(3)$, G_2 and $Spin(7)$, respectively. We would like to understand when (U, g) can be realized as a dense open subset in a complete Riemannian manifold with a parallel $SU(3)$-, G_2- or $Spin(7)$-structure. In the case of non-Abelian split-solvable Lie groups G we show that this is never the case if $\dim G = 5$. Similar results for split-solvable groups of dimensions 6 or 7 are obtained, under additional assumptions. For a different proof in the special case of 5-dimensional nilpotent groups see [6].

On the other hand, there are examples of compact semi-simple Lie groups G such that for certain left-invariant initial data the Ricci-flat manifold (U, g) can be completed in the desired way. In fact, the first example of a complete Riemannian manifold with holonomy G_2, discovered by Bryant and Salamon, is of this type with $G = SU(2) \times SU(2)$, see [7, 2]. As an example of a noncompact semi-simple group we consider $G = SL(2, \mathbb{C})$. In this case we classify all maximally symmetric left-invariant half-flat $SU(3)$-structures and determine the maximal solution of the Hitchin flow for all such initial data. We find that the domain of definition is always a bounded interval $I = (a, b)$. We determine the cases when $(U = G \times I, g)$ can be extended by inserting a singular G-orbit over one of the boundary points of the interval $I = (a, b)$. This yields a family of metrics with holonomy G_2, which turn out to be all homothetic to an incomplete metric on the spinor bundle over hyperbolic 3-space found by Bryant and Salamon [7].

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Twistor-like curvature conditions on Riemannian manifolds

Tillmann Jentsch

Let M^m be a Riemannian manifold with Levi Civita connection ∇ and curvature tensor R. It was shown in [5] that the condition $\nabla^{(k)}R = 0$ already implies that M is locally symmetric. Therefore, we consider the *symmetrized k-th covariant derivative of the curvature tensor* (cf. [4, p.1162]))

$$(1)\quad Y^{(k+2,2)}_{x_3,\dots,x_{k+2}}R(x_1,y_a,y_b,x_2) := \begin{array}{|c|c|c|c|}\hline 1 & 2 & \dots & k+2 \\ \hline a & b \\ \cline{1-2}\end{array}\ \nabla^{(k)}_{x_3,\dots,x_{k+2}}R(x_1,y_a,y_b,x_2)\ .$$

Definition 1. *Suppose there exists some $k \geq 0$ and certain real numbers $c_{k-1}, c_{k-3}, \dots$ such that*

$$(2)\qquad Y^{(k+3,2)}_{x_3,\dots,x_{k+3}}R(x_1,y_a,y_b,x_2) =$$

$$(3)\qquad c_{k-1}\begin{array}{|c|c|c|c|}\hline 1 & 2 & \dots & k+3 \\ \hline a & b \\ \cline{1-2}\end{array}\ \nabla^{(k-1)}_{x_3,\dots,x_{k+1}}R(x_1,y_a,y_b,x_2)\,\langle x_{k+2},x_{k+3}\rangle$$

(4)

$$+c_{k-3}\begin{array}{|c|c|c|c|}\hline 1 & 2 & \dots & k+3 \\ \hline a & b \\ \cline{1-2}\end{array}\ \nabla^{(k-3)}_{x_3,\dots,x_{k-1}}R(x_1,y_a,y_b,x_2)\,\langle x_k,x_{k+1}\rangle\langle x_{k+2},x_{k+3}\rangle + \dots\ .$$

The smallest k with this property will be denoted by k_o (if no such k exists then $k_o := \infty$).

Because of the jet isomorphism theorem of Riemannian geometry (see [3, Theorem 2.6]), the property $k_o < \infty$ coincides with the notion of constant Jacobi osculating rank (see [1]). Further, let $Y^{(k+2,2)}_0 R$ denote the completely traceless part of $Y^{(k+2,2)}R$ (for $k = 0$ this is a multiple of the Weyl curvature tensor and hence the vanishing of $Y^{(2,2)}_0 R$ means conformal flatness of M.)

Definition 2. *Consider the condition*

$$(5)\qquad Y^{(k+3,2)}_0 R = 0\ .$$

The smallest $k \geq 0$ for which (5) *holds will be denoted by k_t (if no such k exists then $k_t := \infty$).*

Remark 1. *Let ρ be some irreducible representation of the orthogonal group* $\mathrm{O}(m)$ *and $E \to M$ be the corresponding vector bundle associated to the bundle of orthonormal frames of M. For example, the dual T^*M of the tangent bundle comes from the standard representation* Id. *Further, there is a decomposition of $E\otimes T^*M$ into parallel subbundles corresponding to a decomposition of $\rho\otimes\mathrm{Id}$ into irreducible summands. Among these there is a distinguished one called the Cartan summand. Let us call a section s of E a twistor section if the projection of ∇s onto the Cartan subbundle vanishes (see* [2]*). In order to understand Equation 5 in this context, suppose for simplicity that $m \geq 5$ and consider the representation of the orthogonal group* $\mathrm{O}(m)$ *usually denoted by* $[k+2,2]$ *(see* [4, Ch. 5.2]*). It is irreducible for our choices of m. Furthermore, we can view $Y^{(k+2,2)}_0 R$ as a section of the corresponding vector bundle over M. Then our condition* (5) *holds if and only if*

$s := Y_0^{(k+2,2)} R$ is a twistor section (hence $k_t = 0$ means that the Weyl curvature tensor is a twistor section of the vector bundle defined by [2, 2]*.)*

It is straightforward that

(6) $$k_t \leq k_o \, .$$

The proof of the following theorem will be published in a forthcoming paper.

Theorem 1. *Let M be a homogeneous Riemannian space with Singer invariant k_S (cf.* [6]*).*

(1) *We have*

(7) $$k_S \leq k_o \, .$$

(2) *Suppose that M is Einstein. Then*

(8) $$k_S \leq k_t \, .$$

Further, suppose additionally that $k_t = 1$. Then $k_o = 1$ if and only if there exists $\lambda \neq 0$ such that R is a λ-eigenvector of the rough Laplacian, i.e.

(9) $$\nabla^* \nabla R = \lambda R \, .$$

Examples. The following homogeneous spaces have $k_o < \infty$ (see [1]):

1. The seven-dimensional Berger manifold $V_1 = \mathrm{Sp}(2)/\mathrm{SU}(2)$ is a normal homogeneous Einstein space with $k_o = 2$ and $k_S = 0$.
2. The seven-dimensional Wilking manifold $V_3 := \mathrm{SO}(3) \times \mathrm{SU}(3)/\mathrm{U}^\bullet(2)$ (with the standard metric) is a normal homogeneous Einstein space with $k_o = 2$ and $k_S = 0$.
3. The complex flag-manifold $M^6 := \mathrm{SU}(3)/\mathrm{T}^2$ is a normal homogeneous (strict) nearly Kähler space (in particular, Einstein) with $k_o = 4$ and $k_S = 1$.
4. Kaplans example of a six-dimensional g.o.-space which is not naturally reductive is a nilmanifold of H-type (in particular, non-Einstein) with $k_o = 4$ and $k_S = 1$.

Some open problems.

- What is the value of k_t for the examples given above?
- One should examine other spaces for k_o, k_t or k_S, say the thirteen-dimensional Berger manifold $V_2 = \mathrm{SU}(5)/\mathrm{Sp}(2) \times \mathrm{S}^1$ or the remaining six-dimensional homogeneous (strict) nearly Kähler manifolds.
- Does every (naturally reductive, g.o. or general) homogeneous Riemannian space satisfy $k_t < \infty$? Is the converse true, i.e. is a (complete) Riemannian manifold with $k_t < \infty$ a homogeneous space?
- Does every homogeneous Einstein manifold have $k_o < \infty$?
- When does equality hold in (6), (7) or (8), respectively? (clearly it does for symmetric spaces.)

- It would be interesting to know whether k_o is always an even number (if it is finite). Suppose for example that $k_t = 1$. What can one say about (9)?
- How do Equation (5) and the invariants k_t , k_o and k_S evolve under the Ricci flow?

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Helical submanifolds of $\mathbb{R}^N$

Jens Heber

(joint work with Martin Scheffel)

An embedded compact submanifold M of Euclidean space is called *helical*, if the following equivalent properties are satisfied:

(1) The Riemannian distance on M is a function of the Euclidean distance.
(2) All geodesics of M have the same constant Frenet curvatures.
(3) All geodesics are pairwise congruent and are closed orbits of one-parameter subgroups of isometries of ambient Euclidean space ("helical").
(4) All geodesics are normal sections of M (defined as in classical surface theory).

Helical submanifolds are known to be *Blaschke manifolds*, that is, their injectivity radius and diameter coincide. The *Blaschke conjecture* asserts that any Blaschke manifold is isometric to a compact rank-one-symmetric space ("CROSS"), see e. g. [ShSpW]. The conjecture has been proved in dimension 2 [G], for Blaschke manifolds diffeomorphic to spheres (M. Berger and J. Kazdan, cf. [Be]) and for Riemannian harmonic Blaschke manifolds [Sz]. It appears to be widely open in the general case.

The geometry of helical submanifolds (including the equivalence of 1–4) has been investigated by B.-Y. Chen, Y. Nikolayevsky, K. Sakamoto, Z. Szabó, K. Tzukada, P. Verheyen [CV, N, Sa82, Sa85, Sa86a, Sa86b, Sz, Tz, V] and other authors, mostly by using methods from classical submanifold geometry (that is, based on Gauss-, Codazzi- and Ricci equations). Another approach is followed in [Sch], based on the study of Lagrange tensors along closed geodesics (i. e. endomorphism valued Lagrangian solutions to the Jacobi equation). Our joint work builds upon these methods.

The only known examples of helical submanifolds are "nice embeddings" of Riemannian *harmonic* manifolds into eigenspaces of their Laplacian Δ on functions (as introduced by A. Besse [Be]). Those have been shown to be CROSSes by Z. Szabó [Sz]. Conversely, any helical submanifold of Euclidean space which is isometric to a CROSS, can be realized as a nice embedding into a product of Δ-eigenspaces [Tz]. Combining [Sa82] and [Sz], any helical *minimal* submanifold of a sphere is a (nicely embedded) CROSS.

For a general helical submanifold M, we reduce the degrees of freedom by exhibiting structural properties of generic Lagrange tensors in terms of their Fourier expansions. We prove that M is a (nicely embedded) CROSS, provided that along every geodesic in M, parallel tangent vector fields have finite Fourier expansions.

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Non-Kähler Ricci solitons

ANDREW DANCER

(joint work with Maria Buzano, Michael Gallaugher and Mckenzie Wang)

The Ricci soliton equation

$$\mathrm{Ric}(g) + \frac{1}{2}\mathsf{L}_X g + \frac{\epsilon}{2} g = 0 \tag{1}$$

for a metric g and vector field X, is a generalisation of the Einstein equation. The soliton is called *steady*, *expanding* or *shrinking* depending on whether the constant ϵ is zero, positive or negative.

It is an interesting problem to try and generalise methods of finding Einstein metrics to the soliton case. However, sometimes the soliton problem shows greater rigidity. For example, compact homogeneous Ricci solitons must in fact be Einstein.

Examples of complete Kähler Ricci solitons are known of all possible types, ie. noncompact steady, noncompact expanding, noncompact shrinking and compact shrinking. Many of these examples use the Calabi/Bérard Bergery ansatz which has been fruitful in the Einstein case. Other constructions use toric geometry and PDE methods.

It has proved harder to find non-Kähler examples. In [DW1], [DW2] noncompact steady and expanding examples were found using cohomogeneity one type methods, where the equations are reduced to a nonlinear dynamical system. These examples generalise earlier examples due to Bryant, Ivey, and Gastel-Kronz. The analysis is greatly aided by the existence of a Lyapunov function in the steady case (and analogous techniques using a pair of functions in the expanding case). Moreover, because the examples are multiple warped products on positive Einstein factors, the Lyapunov is, up to an additive constant, a positive definite quadratic form, so yields coercive estimates.

In [BDW], [BDGW] we showed that one may also obtain solitons if we replace one of the Einstein factors by a Ricci-flat space. In particular, taking this factor to be a circle, this yields steady and expanding solitons on $\mathbb{R}^2 \times M$ where M is a positive Einstein space. In the steady case the asymptotics are a mixture of cigar behaviour and the parabolic behaviour found in the Bryant soliton. Taking M to be a Kervaire sphere, with the Einstein metric found by Boyer-Galicki-Kollar, we obtain solitons on manifolds with exotic smooth structures in dimension $4m+2$ for all but finitely many m. Note that the earlier construction of [DW1],[DW2] would not yield such exotic smooth structures, as once $n \geq 3$, taking the product with $\mathbb{R}^n$ destroys the exotic nature of the smooth structure on the Kervaire sphere.

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Geometric invariant theory meets Riemannian geometry

Michael Jablonski

As a tool for studying homogeneous Riemannian manifolds, Geometric Invariant Theory (GIT) has emerged as a means to help in identifying which Lie groups and homogenous spaces admit special geometries, such as Einstein metrics.

In its classical form, GIT is the study of representations of reductive groups and determining the set of invariant polynomials. The invariant polynomials define a categorical quotient which can be realized geometrically as a usual quotient, but of the set of points with closed orbits, not the quotient of the whole space under the group action.

For this reason, closed orbits of reductive algebraic groups G are of historical interest. However, understanding when a given orbit $G \cdot p$ is closed is of interest in and of itself, and so in the late 1970s a new perspective emerged with the work of Kempf and Ness [KN78]. In their work, it was shown that an orbit is closed if and only if it contains a so-called minimal point, i.e. a point closest to the origin. Furthermore, choosing an inner product $\langle \cdot, \cdot \rangle$ on the representation space compatible with the reducitive group action, the set of minimal points lies on a single orbit of a maximal compact subgroup $K = G \cap O(\langle \cdot, \cdot \rangle)$.

In the right setting (i.e. the setting of a change of basis action of $GL(n, \mathbb{R})$ acting on the space of Lie structures), minimal points correspond to either Einstein or Ricci soliton metrics on solvable and nilpotent Lie groups. Then the fact that minimal points all lie on the same K-orbit can be interpreted as giving uniqueness of Einstein and Ricci soliton metrics on a given solvable or nilpotent Lie group. These ideas have been developed in [Heb98, Lau01, Nik11] and references therein.

Question 1. *What other tools from Geometric Invariant Theory can be applied to glean new information on the geometry of homogeneous spaces?*

In the talk, we illustrate how one can use various tools from GIT to determine when the orbit of a reductive group is closed. These tools are then applied to the 'Key Lemma' described in Carolyn Gordon's talk. That is, we use them to prove the following.

Lemma 1. *Let S be a solvable Lie group admitting an Einstein metric. Assume $S = S_1 \ltimes S_2$, where S_1 is the Iwasawa subgroup of a (non-compact) semi-simple group G_1 and that the action of S_1 on S_2 extends to an action of $G_1 \subset Aut(S_2)$. Then S_2 admits an Einstein metric.*

This is the key technical lemma used in proving that Einstein metrics on solvable Lie groups have the largest possible isometry group [GJ14], a result which is joint work with Carolyn Gordon. Similar results hold for Ricci soliton metrics on nilpotent and unimodular solvable Lie groups [Jab11], but it is unknown to what extent such results can hold in general for solvable Lie groups which admit Ricci soliton metrics.

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A method of computing Ricci tensor of generalized flag manifolds

Yusuke Sakane

We discuss a method of computing Ricci tensor of generalized flag manifolds. Let G be a compact semi-simple Lie group and K a connected closed subgroup of G. Let $\mathfrak{m}$ be the orthogonal complement of $\mathfrak{k}$ in $\mathfrak{g}$ with respect to B (= $-$ Killing form of $\mathfrak{g}$). Then we have $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{m}$, $[\,\mathfrak{k},\, \mathfrak{m}\,] \subset \mathfrak{m}$ and a decomposition of $\mathfrak{m}$ into irreducible $\mathrm{Ad}(K)$-modules: $\mathfrak{m} = \mathfrak{m}_1 \oplus \cdots \oplus \mathfrak{m}_q$. We assume that $\mathrm{Ad}(K)$-modules $\mathfrak{m}_j$ $(j = 1, \cdots, q)$ are mutually non-equivalent. Then a G-invariant metric on G/K can be written as

$$< \,,\, >= x_1 B|_{\mathfrak{m}_1} + \cdots + x_q B|_{\mathfrak{m}_q} \tag{1}$$

for positive real numbers $x_1, \cdots, x_q$. Using notations $\left[{k \atop ij}\right] \geq 0$ (where $\left[{k \atop ij}\right] = \left[{k \atop ji}\right] = \left[{j \atop ki}\right]$) introduced by Wang and Ziller [WaZi], we have (where $d_k = \dim \mathfrak{m}_k$)

Lemma 1. *The components $r_1, \cdots, r_q$ of Ricci tensor r of the metric* (1) *on G/K are given by*

$$r_k = \frac{1}{2x_k} + \frac{1}{4d_k} \sum_{j,i} \frac{x_k}{x_j x_i} \begin{bmatrix} k \\ ji \end{bmatrix} - \frac{1}{2d_k} \sum_{j,i} \frac{x_j}{x_k x_i} \begin{bmatrix} j \\ ki \end{bmatrix} \quad (k = 1,\ \cdots, q) \tag{2}$$

where the sum is taken over $i, j = 1, \cdots, q$.

Let G be a compact semi-simple Lie group and K, L two closed subgroups of G with $K \subset L$. Then we have a natural fibration $\pi : G/K \to G/L$ with fiber L/K. With respect to B (- Killing form of $\mathfrak{g}$), put $\mathfrak{p} = \mathfrak{l}^{\perp}$ in $\mathfrak{g}$: the orthogonal complement of $\mathfrak{l}$ in $\mathfrak{g}$ and put $\mathfrak{n} = \mathfrak{k}^{\perp}$ in $\mathfrak{l}$: the orthogonal complement of $\mathfrak{k}$ in $\mathfrak{l}$. Then we have $\mathfrak{g} = \mathfrak{l} \oplus \mathfrak{p} = \mathfrak{k} \oplus \mathfrak{n} \oplus \mathfrak{p}$. Denote a G-invariant metric $\check{g}$ on G/L defined by an $\mathrm{Ad}_G(L)$-invariant scalar product on $\mathfrak{p}$, an L-invariant metric $\hat{g}$ on L/K defined by an $\mathrm{Ad}_L(K)$-invariant scalar product on $\mathfrak{n}$ and consider a G-invariant metric g on G/K defined by the orthogonal direct sum for these scalar products on $\mathfrak{n} \oplus \mathfrak{p}$. Then we see that the map π is a Riemannian submersion from $(G/K,\, g)$ to $(G/L,\, \check{g})$ with totally geodesic fibers isometric to $(L/K,\, \hat{g})$. We consider a decomposition of $\mathfrak{p}$ into irreducible $\mathrm{Ad}(L)$-modules: $\mathfrak{p} = \mathfrak{p}_1 \oplus \cdots \oplus \mathfrak{p}_\ell$ and a decomposition of $\mathfrak{n}$ into irreducible $\mathrm{Ad}(K)$-modules: $\mathfrak{n} = \mathfrak{n}_1 \oplus \cdots \oplus \mathfrak{n}_s$. Note that each irreducible component $\mathfrak{p}_j$ (as $\mathrm{Ad}(L)$-module) can be decomposed into irreducible $\mathrm{Ad}(K)$-modules. We consider a G-invariant metric on G/K defined by a Riemannian submersion $\pi : (G/K,\, g) \to (G/L,\, \check{g})$ of the form

$$g = y_1 B|_{\mathfrak{p}_1} + \cdots + y_\ell B|_{\mathfrak{p}_\ell} + z_1 B|_{\mathfrak{n}_1} + \cdots + z_s B|_{\mathfrak{n}_s} \tag{3}$$

for positive real numbers $y_1, \cdots, y_\ell, z_1, \cdots, z_s$. We decompose each irreducible component $\mathfrak{p}_j$ into irreducible $\mathrm{Ad}(K)$-modules:

$$\mathfrak{p}_j = \mathfrak{m}_{j,1} \oplus \cdots \oplus \mathfrak{m}_{j,\,k_j}.$$

We assume that $\mathrm{Ad}(K)$-modules $\mathfrak{m}_{j,t}$ $(j = 1, \cdots, \ell,\ t = 1, \cdots, k_j)$ are mutually non-equivalent. Note that the metric of the form (3) can be written as

$$g = y_1 \sum_{t=1}^{k_1} B|_{\mathfrak{m}_{1,t}} + \cdots + y_\ell \sum_{t=1}^{k_\ell} B|_{\mathfrak{m}_{\ell,t}} + z_1 B|_{\mathfrak{n}_1} + \cdots + z_s B|_{\mathfrak{n}_s} \tag{4}$$

and this metric is a special case of the metric of the form (1).

Lemma 2. *Let $d_{j,t} = \dim \mathfrak{m}_{j,t}$. The components $r_{(j,\,t)}$ $(j = 1, \cdots, \ell,\ t = 1, \cdots, k_j)$ of Ricci tensor r for the metric (4) on G/K are given by*

$$r_{(j,\,t)} = \check{r}_j - \frac{1}{2d_{j,\,t}} \sum_i \sum_{j',\,t'} \frac{z_i}{y_j y_{j'}} \begin{bmatrix} & i & \\ (j,t) & & (j',t') \end{bmatrix}, \tag{5}$$

where $\check{r}_j$ are the components of Ricci tensor $\check{r}$ for the metric $\check{g}$ on G/L.

Consider a generalized flag manifold G/K with $r = b_2(G/K) = 2$ and a generalized flag manifold G/L with $b_2(G/L) = 1$ with $K \subset L$. Using Kähler-Einstein metrics on generalized flag manifolds and applying Lemma 2 to a fibration $\pi : G/K \to G/L$ with fiber L/K, we can determine Ricci tensor of generalized flag manifolds with $r = b_2(G/K) = 2$ and $q \leq 6$. (For details, see [ACS1] and [ACS1].)

Open problem: Does this method work for any generalized flag manifolds?

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Eclectic thoughts on the curvature of solvable and nilpotent Lie groups

Yuri Nikolayevsky
(joint work with Yurii Nikonorov)

1. Solvable groups of negative Ricci curvature (joint ongoing project with Yurii Nikonorov)

1.1. **Background.** Which homogeneous manifolds (in particular, which Lie groups) admit a left-invariant metric with the given sign of the curvature?

Sectional curvature is well understood: $K > 0$ [Wallach 1972, Bérard Bergery 1976], $K < 0$ [Heintze 1974, Alekseevsky 1975, Azencott, Wilson 1976], $K = 0$ [Alekseevsky 1975, Bérard Bergery 1976].

Ricci curvature: $\mathrm{Ric} > 0$ [Milnor 1976, Berestovski 1995], $\mathrm{Ric} = 0$ [Alekseevsky, Kimel'fel'd 1975].

Negative: wide open; [Leite, Dotti Miatello 1982]: $\mathrm{SL}(n, \mathbb{R})$, $n \geq 3$, admits a metric with $\mathrm{Ric} < 0$; [Leite, Dotti Miatello, Miatello 1984]: a unimodular Lie group which admits a metric with $\mathrm{Ric} < 0$ is noncompact semisimple. Constructed such a metric on a some complex semisimple Lie groups; particular case: Einstein.

Main question: Which (nonunimodular) solvable Lie groups admit a left-invariant metric with $\mathrm{Ric} < 0$?

1.2. **Results.** Main result: necessary and sufficient conditions for solvable Lie algebras whose nilradical is either abelian or Heisenberg or filiform to admit an inner product with $\mathrm{Ric} < 0$. All of them have the same flavour: "there exists $Y \in \mathfrak{g}$ such that real parts of the restriction of ad_Y to the nilradical $\mathfrak{n}$ satisfy certain linear inequalities (which depend on the particular $\mathfrak{n}$)" [NN, CLN]

Proof: degeneration; Richardson's Theorem; "real" Lie Theorem; possibly the moment map (as discussed with J.Lauret).

1.3. **Open question.** Is the following true? "A solvable Lie algebra $\mathfrak{g}$ with the nilradical $\mathfrak{n}$ admits an inner product of negative Ricci curvature if and only if there exists a vector $X \in \mathfrak{g} \setminus \mathfrak{n}$ such that the real parts of the eigenvalues of the restriction of ad_X to $\mathfrak{n}$ satisfy certain linear inequalities determined solely by the structure of $\mathfrak{n}$ (or, in other words, the restriction of ad_X to $\mathfrak{n}$ (or at least, its semisimple part) belongs to the open convex hull of something in the derivation algebra of $\mathfrak{n}$)."

2. Curvature of nilpotent groups (joint ongoing project with Grant Cairns, Ana Hinić Galić and Marcel Nicolau)

2.1. **Motivation.** Milnor, 1976:

(1) for all X in the centre $\mathfrak{z}$ of $\mathfrak{g}$, the sectional curvatures are $K(X, Y) \geq 0$, $\forall Y \in \mathfrak{g}$,
(2) $\mathrm{Ric}(X) \leq 0$ for all X orthogonal to the derived algebra $\mathfrak{g}' = [\mathfrak{g}, \mathfrak{g}]$.

The impression one obtains is that the positive curvature is typically concentrated "near the center", while the negative curvature is found at the "upper levels of the algebra". Another motivation: [Nikonorov, 2014]: Apart from the two cases, when a metric solvable Lie algebra is flat or almost flat, Ric always has at least two negative eigenvalues.

2.2. **Results.** We classify the subsets of a nilpotent Lie algebra $\mathfrak{g}$, such that for any choice of the inner product on $\mathfrak{g}$, the Ricci curvature $\mathrm{Ric}(X)$ has a particular sign; and similarly, the subsets of the Grassmannian $G(2, \mathfrak{g})$ such that for any choice of the inner product on $\mathfrak{g}$, the sectional curvature $K(X, Y)$ has a particular sign [CHGNN].

We also consider so called Ricci-maximal and Ricci-minimal subsets in the projectivisation $\mathbb{P}\mathfrak{g}$ of a nilpotent Lie algebra $\mathfrak{g}$ and prove that the closure of Ricci-minimal subset is always $\mathbb{P}\mathfrak{g}$, and the closure of Ricci-maximal subset is $\mathbb{P}\mathfrak{g}$, except in the following cases: if $\mathfrak{g}$ is two-step nilpotent, then that closure is $\mathbb{P}\mathfrak{g}'$, and if $\mathfrak{g}$ has a codimension one abelian ideal $\mathfrak{a}$ and is not two-step nilpotent, then that closure is $\mathbb{P}\mathfrak{a}$ [CHGNN].

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On the classification of shrinking gradient Ricci solitons

William Wylie

In this talk we survey results about the classification of shrinking gradient Ricci solitons, including discussing in more detail a new result which is recent joint work with Jia-Yong Wu of Shanghai and Peng Wu of Ithaca [12] as well an older joint result with Peter Petersen of Los Angeles [11].

A gradient shrinking Ricci soliton is a triple (M, g, f) where (M, g) is a Riemannian metric and f is a smooth function on M which satisfies $\mathrm{Ric} + \mathrm{Hess} f = \lambda g$ from some $\lambda \in \mathbb{R}$. The gradient Ricci soliton is called shrinking, steady, or expanding, if $\lambda > 0$, $\lambda = 0$, or $\lambda < 0$, respectively.

There are many results about steady or expanding Ricci solitons, for the purposes of this abstract and due to space constraints we focus only on the results for gradient shrinking Ricci solitons which are essential to explain our results. We are interested in understanding the conditions in dimension ≥ 4 when we can classify gradient shrinking Ricci solitons. Any locally conformal flat gradient shrinking Ricci soliton is a finite quotient of S^n, $S^{n-1} \times \mathbb{R}$, or $\mathbb{R}^n$ as follows from the works [4, 2, 8, 13, 10, 6]. Recall that a Riemannian manifold is locally conformally flat if the Weyl tensor vanishes ($W = 0$). The works [5, 6] also give a classification of gradient shrinking Ricci solitons under the weaker assumption of harmonic Weyl curvature ($\delta W = 0$), showing that any such metric is either Einstein, or a finite quotient of $N^k \times \mathbb{R}^{n-k}$ for $0 \leq k \leq n$, where N^k is a k-dimensional Einstein manifold of positive scalar curvature.

In dimension 4 it is natural to consider self dual or anti-self dual part of Weyl curvature $W^{\pm}$ commonly called the half Weyl curvature. X. Chen and Y. Wang [3] proved that a half conformally flat ($W^{\pm} = 0$) four-dimensional gradient shrinking Ricci soliton is a finite quotient of S^4, $\mathbb{C}P^2$, $S^3 \times \mathbb{R}$, or $\mathbb{R}^4$. In joint work with Wu and Wu, we classify four-dimensional gradient shrinking Ricci solitons with harmonic half Weyl curvature,

Theorem 1. *A compact four-dimensional gradient shrinking Ricci soliton with* $\delta W^{\pm} = 0$ *is Einstein. A noncompact four-dimensional gradient shrinking Ricci soliton with* $\delta W^{\pm} = 0$ *is a finite quotient of* $S^3 \times \mathbb{R}$, $S^2 \times \mathbb{R}^2$, *or* $\mathbb{R}^4$.

Instead of studying Ricci solitons via their Weyl curvature, another approach is to study the classification problem under positivity of curvature assumptions. Naber has classified 4-dimensional gradient Ricci solitons with bounded, nonnegative curvature operator [7]. The compact case is also well understood as, by the work of Böhm-Wilking [1], any compact Ricci soliton with positive curvature operator is a round sphere. In higher dimensions this indicates the following conjecture in the noncompact case.

Conjecture 2. *A gradient shrinking Ricci soliton with bounded positive curvature operator is compact.*

In dimension 3 the conjectire is also true by a result is of Perelman [9] and it follows from Naber's classification in dimension 4. The only partial result the author knows of in higher dimensions is the following earlier result with Petersen.

Theorem 1. [11] *A complete, non-compact, cohomogeneity-one shrinking gradient soliton with bounded nonnegative sectional curvature is the product of an Einstein and Euclidean space.*

We note that the proof of Theorem 1 comes from considering an equation along geodesics that can be derived from the equation $\delta\mathrm{Rm}(\cdot,\cdot,\cdot) = 2Rm(\nabla f,\cdot,\cdot\cdot)$, which is true for any gradient Ricci soliton. Since the appearance of the divergence is also key in the results mentioned above about Weyl curvature, it would be interesting if the techniques used to classify spaces under assumptions on their Weyl curvature can be combined with the techniques used to prove Theorem 1 to shed light on Conjecture 2.

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New examples of Einstein solvmanifolds

Megan Kerr

We describe new examples of non-symmetric Einstein solvmanifolds obtained by combining two techniques. In [T2], H. Tamaru constructs new *attached* solvmanifolds, which are submanifolds of the solvmanifolds corresponding to noncompact symmetric spaces, endowed with a natural metric. Extending this construction, we apply it to *associated* solvmanifolds, described in [GK], obtained by modifying the algebraic structure of the solvable Lie algebras corresponding to noncompact symmetric spaces. Our new examples are Einstein solvmanifolds with nilradicals of high nilpotency, which are geometrically distinct from noncompact symmetric spaces and their submanifolds.

Our spaces provide many explicit new examples of homogeneous Einstein manifolds that are neither the solvmanifolds corresponding to noncompact symmetric spaces, nor their submanifolds. We obtain our examples by extending the method of H. Tamaru in [T2], in which, via parabolic subalgebras of semisimple Lie algebras, he builds solvable subalgebras by choosing a subset Λ' of the set Λ of simple roots. Tamaru proves that the solvable subalgebra of the restricted root system, $\mathfrak{s}_{\Lambda'}$, given a natural inner product, called an *attached* solvmanifold, is in fact an Einstein solvmanifold. We combine this with a method introduced by C. S. Gordon and the author in [GK] to construct Einstein solvmanifolds which are *associated* (but not isometric) to the Einstein solvable Lie groups corresponding to higher rank, irreducible symmetric spaces of noncompact type.

Tamaru's construction yields solvmanifolds which are naturally homogeneous submanifolds of symmetric spaces of noncompact type; however, they generally are *not* totally geodesic subalgebras. When Tamaru's method is extended to construct the Einstein solvmanifolds here, which are *attached* to *associated* solvmanifolds, we get completely new examples. We show, by isometry groups, that they cannot be submanifolds of symmetric spaces. Furthermore, we show that while our examples have negative Einstein constant, they admit two-planes of positive sectional curvature, quite different from the geometry of symmetric spaces.

This work recently appeared [K].

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The Alekseevskii conjecture in low dimensions

Ramiro Lafuente
(joint work with Romina Arroyo)

The Alekseevskii conjecture, which states that a connected homogeneous Einstein space of negative scalar curvature must be diffeomorphic to a Euclidean space, is an open problem since the early 80's, and until last year it was only known to be true up to dimension 5 (a result which follows from a complete classification, see [Je69] and [Nik05] for the cases of dimension 4 and 5, respectively).

In [AL14a], which is a joint work with R. Arroyo, we have proved that the conjecture (and also its generalized version for algebraic solitons) also holds in dimension 6, provided the transitive group is not semisimple. The proof of this result is mainly based in the structure theorems for homogeneous Einstein spaces given in [LL13].

More recently, there have been new developments in the above mentioned structure theory of homogeneous Einstein spaces, see [JP14]. These refinements allowed us to investigate the conjecture in higher dimensions, and indeed as part of an ongoing project with R. Arroyo ([AL14b]) we have proved that under the additional hypothesis that the transitive group is not semisimple, the Alekseevskii conjecture also holds up to dimension 10.

Furthermore, in the semisimple case, we could establish the validity of the conjecture up to dimension 8, excluding the cases of left-invariant metrics on the simple Lie groups $\mathrm{SL}_2(\mathbb{R}) \times \mathrm{SL}_2(\mathbb{R}), \mathrm{SL}_2(\mathbb{C}), \mathrm{SL}_3(\mathbb{R})$ and $\mathrm{SU}(2,1)$. This is based on a case-by-case analysis, where we study every homogeneous space G/K of dimension up to 8 with G semisimple. The non-existence of Einstein metrics in each case can be proved by using [Nik00] in some cases, and some new case-specific analysis of the Ricci curvature in the other cases.

Finally, with regards to the rather unpleasant exceptions of simple Lie groups, we remark that the Einstein equation for left-invariant metrics on the Lie group $\mathrm{SU}(2) \times \mathrm{SU}(2)$ is still not completely solved, even though the compact case has been much more investigated in the literature. Therefore, we believe that there is a strong need for new tools in order to attack these most difficult cases.

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