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Valuation Theory and Its Applications

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Abstract. In recent years, the applications of valuation theory in several areas of mathematics have expanded dramatically. In this workshop, we presented applications related to algebraic geometry, number theory and model theory, as well as advances in the core of valuation theory itself. Areas of particular interest were resolution of singularities and Galois theory.

Mathematics Subject Classification (2010): 12J20.

Introduction by the Organisers

This Oberwolfach workshop on *Valuation Theory and Applications* brought together experts from many areas of valuation theory and from areas where major recent progress has been achieved by using valuation theoretic methods. The hospitality of the staff at the MFO provided for a friendly and pleasant meeting environment, and that contributed a lot to a smooth running of the workshop.

Among the prominent areas of research the talks touched upon were number theory, algebraic geometry (especially resolution of singularities), model theory of valued fields and generalizations of these (e.g., difference valued fields) and motivic integration, first order definability of valuations, anabelian geometry (especially Grothendieck's and Bogomolov's birational anabelian geometry), the section conjecture and extensions of these, local global principles of Hasse type and generalizations thereof, specialization/lifting problems for covers of curves and the related aspects of the algebraic fundamental group. The talks reflected both the role of valuation theory as a powerful tool in other areas of mathematics and the new impulses for the valuation theory stemming from its interaction with them. The talks, roughly grouped together according to the major areas of research, were given by:

- Resolution of singularities: talks by Cossart, Cutkosky, Schoutens.
- Anabelian geometry: talks by Silberstein, Strømmen, Topaz.
- Arithmetic geometry: talks by Cluckers, Halupczok, Loeser, Lütkebohmert, Temkin, Wewers.
- Model theory: talks by Ducros, van den Dries, Macintyre, Point, Scanlon.
- Field Arithmetic: talks by Bary-Soroker, Harbater, Jarden, Obus.
- Valuation theory: talks by Blaszczok, Fehm, Jahnke, Novacoski.

There was also a discussion session on Wednesday night chaired by Spivakovsky with questions from the major research areas of the workshop.

In local uniformization, Cossart reported on his important recent work with Piltant concerning resolution of singularities in dimension 3. This includes the case of arithmetic schemes, i.e., the mixed characteristic case, which they now have reduced to their previous results for the case of equal positive characteristic. Cutkosky presented a generalization of the Abhyankar-Jung Theorem to the associated graded ring along a valuation of a local ring dominated by the valuation. Giving three examples, he illustrated the (not always good) behaviour of the extensions of this graded ring that are induced by extensions of the ring. Schoutens discussed a question that has recently regained interest: the total blow up of a regular local ring of dimension ≥ 3 along a valuation may not be a valuation ring, but what are its algebraic properties, and under which conditions is it a valuation ring?

In anabelian geometry, Silberstein and Topaz presented deep results on a geometric reconstruction of function fields of dimension ≥ 2 over algebraically closed constant fields from 2-step nilpotent pro-ℓ Galois groups. Silberstein's result is the first successful attempt to recover not only the isomorphy type of the function fields under discussion, but to construct explicit models of the function fields. Topaz on the other hand, combined Galois theoretical methods with model theoretical methods toward the very exciting perspective of recovering function fields from their $\mathbb{Z}/\ell$ abelian-by-central Galois group. Finally, Strømmen presented his recent breakthrough on pro-p Galois groups that are Demushkin: most algebraic properties of $\mathbb{Q}_2$ are encoded in its maximal pro-2 Galois group.

In arithmetic geometry, Halupczok's p-adic and motivic version of the Kontsevich–Zagier Conjecture and Clucker's motivic transfer principles from the p-adics to the characteristic p local fields brought interesting new perspectives. Lütkebohmert presented a wide generalization of p-adic uniformization (of Abelian varieties and Jacobians of Mumford curves), which might have applications to other major open questions about covers of curves over p-adic numbers. Temkin presented his results about a metric uniformization of morphisms between Berkovich curves, which both generalize and explain the well known different formula of Kato. Wewers gave an explicit method to compute semi-stable models of curves over valuation rings. Finally, Loeser described some of his work with Hrushovski on a model-theoretic version of Berkovich's analytification of varieties by using the so called strongly dominated types.

In the model theory of fields, Macintyre gave a beautiful answer to a question of Boris Zilber showing that arithmetic is not interpretable in any proper quotients of non-standard models of arithmetic. And Ducros launched interesting discussions by suggesting a new model theoretic proof of the Stability Theorem which states that Abhyankar valuations are defectless. Point gave an overview of her results with Bélair on the model theory of various valued difference fields of positive characteristic, viewed as valued modules. Van den Dries reported on two of the many recent results of his work with Aschenbrenner and van der Hoeven on the valued differential field of transseries, giving sufficient conditions for a field to be differentially henselian, and investigated conditions under which a differential Hensel's Lemma in several variables holds. Scanlon reported on work by Silvain Rideau on the model theory of valued difference fields equipped with an analytic structure and showed how this gives uniform bounds for a Manin-Mumford type result for flat commutative group schemes over Witt vectors in positive characteristic.

In field arithmetic, Bary-Soroker's proof of a function field analogue of the Hardy-Littlewood Conjecture was very well received, and the organizers encouraged him to produce an Oberwolfach Snapshot on this subject. Harbater gave an overview of recent and new results concerning local-global principles for homogeneous spaces over function fields of curves over complete DVR's, and indicated how one can apply this in the theory of quadratic forms and division algebras. Denoting by S a finite set of places of a global field, Jarden presented new results concerning the arithmetic of the fields of totally S-adic numbers and generalizations of these. Finally, Obus reported progress on the generalized Oort conjecture on lifting covers of curves, in particular the progress on classifying the Oort groups.

In pure valuation theory, Blaszczok presented new results on the transcendence degree and the uniqueness of maximal immediate extensions. She produced a stunning example of a valued field with both an algebraic maximal immediate extension and another one of infinite transcendence degree. Novacoski reported the final results of a discussion that has spanned many years and has seen important input from Roquette, van den Dries, Knaf, Kuhlmann and himself. The theory of henselian elements has been developed to show how finite inertial extensions can be generated by elements that satisfy the assumptions of Hensel's Lemma; this can be used in the proof of instances of local uniformization for valued function fields.The other main topic here was definability of valuations, which has recently regained momentum. Eight participants have published papers on this subject during the last year which were all reflected in the two talks by Jahnke and Fehm. It is also believed that these methods might become relevant in eventually getting effective first order results in birational anabelian geometry.

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Workshop: Valuation Theory and Its Applications

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Abstracts

From Field Arithmetic to Analytic Number Theory

Lior Bary-Soroker

In this talk I have presented the ancient connection between the ring of integers $\mathbb{Z}$ and the ring of polynomials $\mathbb{F}_q[T]$ over a finite field $\mathbb{F}_q$ with q elements. I have presented some of the abundance of recent works in the study of number theory over $\mathbb{F}_q[T]$.

In the focus of the talk lied the recent resolution of an analogue of the Hardy-Littlewood prime tuple conjecture. This conjecture says that for a tuple $(a_1, \dots, a_k)$ of distinct integers that do not cover the residues mod p for any p we should have

$$(1) \quad \#\{1 \leq n \leq x : n + a_1, \dots, n + a_k \text{ are all prime}\} \sim \mathfrak{S} \frac{x}{(\log x)^k}, \qquad x \to \infty.$$

Here $\mathfrak{S}$ is a positive constant depending only on the a_i's and the notation $f \sim g$ means that $\lim_{x\to\infty} f(x)/g(x) = 1$.

If we denote by $M_n(q) \subseteq \mathbb{F}_q[T]$ the set of monic polynomials of degree n, then the function field analogue of (1) in the limit of large finite fields can be stated as:

Theorem 1 (Bary-Soroker 2012, Carmon 2014)**.** *For any positive integers n and k there exists a constant $c_{n,k}$ depending only on n and k such that for any prime power q and any distinct $A_1, \dots, A_k \in \mathbb{F}_q[T]$ with $\deg(A_i) < n$ we have*

$$(2) \quad \#\{f \in M_n(q) : f + A_1, \dots, f + A_k \text{ are all prime}\} = \frac{q^n}{n^k} + O\left(c_{n,k} q^{n-\frac{1}{2}}\right).$$

Some remarks are in place.

- In (2), the implied constant is absolute and depends on nothing.
- BS proved the result for odd characteristic and Carmon extended the result for even characteristic building on the work of BS.
- The question *what are the limit objects?* naturally arises. We have discussed it and suggested that these are the pseudo finite fields, and more generally the pseudo algebraically closed fields. The latter class of fields, at least for me, was the starting point of investigations of this type.

The proof of (2) is based on a Galois group calculation and on an explicit Chebotarev theorem for varieties over finite fields.

On immediate extensions of valued fields

Anna Blaszczok
(joint work with Franz-Viktor Kuhlmann)

If (K, v) is a valued field, denote its value group by vK and its residue field by Kv. A valued field extension $(L|K, v)$ is called **immediate** if the natural embeddings of vK into vL and of Kv into Lv are onto.

It was shown already by W. Krull in [6] that every valued field admits a maximal immediate extension. The structure of the maximal immediate extensions carries important information about a valued field. A better understanding of the the structure of such extensions is meaningful for problems connected with valued function fields. The latter ones are of particular meaning for various questions in valuation theory, algebraic geometry (like local uniformization) and the model theory of valued fields. Immediate extensions play there a role that is complementary to the Abhyankar valuations, in which the value group and the residue field extensions carry maximal information about a valued field extension. In particular, the transcendence degree of maximal immediate extensions of a given valued field turned out to be important for the description of the possible extensions of a valuation from this field to a rational function field. The following theorem gives several criteria for a valued field to admit maximal immediate extensions of infinite transcendence degree.

Theorem 1. *Assume that $(L|K, v)$ is a valued field extension of finite transcendence degree ≥ 0, with v nontrivial on L. Suppose moreover that one of the following cases holds:*

valuation-transcendental case: vL/vK is not a torsion group, or $Lv|Kv$ is transcendental;

value-algebraic case: vL/vK contains elements of arbitrarily high order;

residue-algebraic case: Lv contains elements of arbitrarily high degree over Kv;

separable-algebraic case: $L|K$ contains an infinite separable-algebraic subextension $L_0|K$ such that within some henselization of L, the corresponding extension $L_0^h|K^h$ remains infinite.

Then every maximal immediate extension of (L, v) is of infinite transcendence degree over L. If the cofinality of vL is countable, then already the completion of (L, v) has infinite transcendence degree over L.

A valued field which does not admit any proper immediate extensions is called **maximal**. Clearly, every maximal immediate extension is itself a maximal field. Since maximal fields are in particular henselian and defectless, an interesting question is what kinds of field extensions preserve the maximality. It is well known that a finite extension of a maximal field is again maximal. The following theorem shows that in general this is not true for algebraic extensions that are not finite.

Theorem 2. *Assume that (K, v) is a maximal field and take an infinite algebraic extension $(L|K, v)$. Suppose that $L|K$ contains an infinite separable subextension* *or that*

$$(vK : pvK)[Kv : Kv^p] < \infty , \tag{1}$$

where $p = \mathrm{char}K$ *if it is positive and* $p = 1$ *otherwise. Then every maximal immediate extension of* (L, v) *has infinite transcendence degree over* L.

Another important question is the uniqueness of maximal immediate extensions. It plays a significant role in problems of the model theory of valued fields. It was proved by I. Kaplansky in [3] that under a certain condition, called "hypothesis A", a valued field admits maximal immediate extensions which are unique up to valuation preserving isomorphism. G. Whaples proved (cf. Theorem 1 of [7]) that hypothesis A is equivalent to the conjunction of the following three conditions, where $p = \mathrm{char}Kv$:

(K1) if $p > 0$ then the value group vK is p-divisible,

(K2) the residue field Kv is perfect,

(K3) the residue field Kv admits no finite separable extension of degree divisible by p.

Kaplansky gave also an example showing that if hypothesis A is not satisfied, then the uniqueness may not hold. This causes one of the hurdles to solve important problems in the model theory of valued fields. An interesting question is how much the maximal immediate extensions of a given valued field can differ. The separable-algebraic case of Theorem 1 allows us to prove the following:

Theorem 3. *There is a class of valued fields which admit an algebraic maximal immediate extension as well as one of infinite transcendence degree.*

The example of Kaplansky and results presented in [5] show that if a valued field does not satisfy **(K3)**, then it may admit nonisomorphic maximal immediate extensions. However, the next results show that even if we omit condition (K3) in hypothesis A, we can still derive surprising properties of the maximal immediate extensions of the valued field. We will consider valued fields satisfying the following conditions:

(2) (K, v) is a henselian field such that **(K1)** and **(K2)** hold.

Theorem 4. *In addition to the assumptions (2), suppose that* (K, v) *admits a maximal immediate extension of finite transcendence degree. Then all maximal immediate extensions of* (K, v) *are isomorphic over* K*, as valued fields.*

This shows in particular that there are valued fields that are not maximal and do not satisfy hypothesis A but admit unique (up to isomorphism) maximal immediate extensions. Note that maximal fields satisfying conditions **(K1)** and **(K2)** are tame (cf., e.g., [4]) . This allows us to say much more about the structure of maximal immediate extensions of valued fields that satisfy the assumptions of the above theorem.

Theorem 5. *Take a valued field (K,v) satisfying the assumptions (2). Suppose that it admits a valued field extension (M,v) that is maximal and of finite transcendence degree over K. Take $L|K$ to be the maximal separable-algebraic subextension of $M|K$. Then we have:*

a) vM/vK and $Mv|Kv$ are finite,

b) K is a separably tame field and $L|K$ is a finite tame extension,

c) the perfect hull of K is contained in the completion of K.

If in addition, the extension $M|K$ is algebraic, then we also have:

e) M is equal to the perfect hull of L and to the completion of L,

f) the perfect hull of K is equal to the completion of K and is the unique maximal immediate extension of K.

If (K,v) satisfies the assumption of the above theorem, then part b) shows in particular that K admits no separable defect extensions. Immediate consequences of the above theorem are the following facts.

Corollary 6. *In addition to the assumptions (2), assume that (K,v) admits a maximal immediate extension (M,v), algebraic over K.*

a) If $\mathrm{char} K = 0$, then $M = K$, so (K,v) is maximal.

b) Otherwise, M is equal to the perfect hull of K and to the completion of K.

Corollary 7. *Take a valued field (K,v) satisfying the assumptions (2). Assume that it admits a separable-algebraic extension (M,v) which is a maximal field. Then $(M|K,v)$ is a finite tame extension. Furthermore, (K,v) is a maximal and a tame field.*

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Transfer Principles between $\mathbb{Q}_p$ and $\mathbb{F}_p((t))$

Raf Cluckers
(joint work with Julia Gordon, Immanuel Halupczok)

In the talk, after recalling motivic exponential functions in a concrete form, and their stability under taking integral transformations, we explained transfer principles for bounds of motivic exponential functions, and linear combinations of such functions. In this context, *transfer* means switching between local fields with isomorphic residue field. By *concrete*, we mean that we work uniformly in all local fields of large enough residue field characteristic, as opposed to genuinely motivic as done in [4] (this concrete setting is perfectly suited for transfer principles which are about local fields indeed).

Our results relate to previously known transfer principles (from [4], [1], and [7, Appendix B]) as follows. The principle given by Theorem 1 below generalizes both the transfer principle of [4, Proposition 9.2.1], where, one can say, the upper bound was identically zero, and the transfer principle of [7, Theorem B.7], where the case without oscillation is treated. The transfer principles of [1] about e.g. integrability seem to be independent, and in fact, our proofs are closer to the ones of [4] and can avoid the more heavy machinery from [1].

After Theorem 1, we gave some further generalizations which treat $\mathbb{C}$-linear combinations of motivic exponential functions, uniformly in the complex scalars. Specifically, we obtain transfer principles for linear (in-)dependence and for upper bounds of linear combinations of motivic exponential functions (or rather, their specializations for any local field F with large residue field characteristic). These generalizations are not described here.

In the following theorem, Loc_M stands for all non-archimedean locally compact fields of residue field characteristic at least M, and $\mathcal{C}^{\exp}(X)$ consists of (complex valued) oscillatory functions $H_{F,\psi}$ for each F in Loc_M and each additive character ψ on F whose restriction to the valuation ring is specified (and in particular nontrivial). We write $\mathcal{D}_F$ for the collection of such characters on F. The class $\mathcal{C}^{\mathrm{e}}(X)$ is similar but with less oscillatory behaviour: only additive characters on the residue field come up. These classes of functions were introduced in [4] (inspired by [5] in the non-oscillatory case), based on the language of Denef-Pas of [6].

Theorem 1. **[Transfer principle for bounds]** *Let X be a definable set, let H be in $\mathcal{C}^{\exp}(X)$ and let G be in $\mathcal{C}^{\mathrm{e}}(X)$. Then there exist M and N such that, for any $F \in \mathrm{Loc}_M$, the following holds. If*

$$|H_{F,\psi}(x)|_{\mathbb{C}} \leq |G_F(x)|_{\mathbb{C}} \text{ for all } (\psi, x) \in \mathcal{D}_F \times X_F \tag{1}$$

then, for any local field F' with the same residue field as F, one has

$$|H_{F',\psi}(x)|_{\mathbb{C}} \leq N \cdot |G_{F'}(x)|_{\mathbb{C}} \text{ for all } (\psi, x) \in \mathcal{D}_{F'} \times X_{F'}. \tag{2}$$

Moreover, one can take $N = 1$ if H lies in $\mathcal{C}^{\mathrm{e}}(X)$.

A key proof technique that we share with [4] consists in reducing from general motivic exponential functions, namely $\mathcal{C}^{\exp}$, to more easy functions where the oscillation only comes from additive characters on the residue field, namely $\mathcal{C}^{\mathrm{e}}$.

Let us finally mention that the transfer principles of [4] have been applied in [3] to obtain the Fundamental Lemma of the Langlands program in characteristic zero, and the ones of [1] have been used in [2] to show local integrability of Harish-Chandra characters in large positive characteristic.

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Resolution of Singularities of Arithmetic Threefolds: Reduction to characteristic $p > 0$

Vincent Cossart

(joint work with Olivier Piltant)

Resolution of Singularities in its full birational form was to this date restricted to surfaces: see [1][2][13][14][12][11][9], to only mention some contributions. In dimension three, some partial results do exist for algebraic varieties over an algebraically closed field k of positive characteristic $p \geq 7$ ([3][10]). These results extend to all characteristics $p > 0$ and fields k with $[k : k^p] < +\infty$ ([4][5] theorem on p.1839). For arithmetical schemes (unequal residue characteristic), birational resolution of singularities was sofar restricted to surfaces. Resolution of singularities in positive characteristic is open in dimension 4. Several programs in this area are going on:

H. Hironaka, H. Kawanoue, O. Villamayor, J. Wlodarczyk for the resolution by blowing ups with regular centers: *open in dimension* 3;

F.V. Kuhlmann, M. Spivakovsky, B. Teissier, M. Temkin for the local uniformization of valuations: *open in dimension* 4.

1. Main Theorem

In [8], we have proved the following Resolution of Singularities Theorem for arithmetical varieties of dimension three.

Theorem 1. *Let C be an integral Noetherian curve which is excellent and $\mathcal{X}/C$ be a reduced and separated scheme of finite type and dimension at most three. There exists a proper birational morphism $\pi: \mathcal{X}' \to \mathcal{X}$ with the following properties:*

(i) *$\mathcal{X}'$ is everywhere regular;*
(ii) *π induces an isomorphism $\pi^{-1}(\mathrm{Reg}(\mathcal{X})) \simeq \mathrm{Reg}(\mathcal{X})$;*
(iii) *$\pi^{-1}(\mathrm{Sing}(\mathcal{X}))$ is a strict normal crossings divisor on $\mathcal{X}'$.*

2. The method: follow Zariski

There are two steps: local uniformization and patch the local uniformizations.

2.1. **Some reductions.** Following S.S. Abhyankar, we get the first reductions:
Reduction of local uniformization to that of rank 1, residually algebraic valuations.
Reduction of local uniformization to the next theorem.

Theorem 2. *(V. Cossart and O. Piltant) Let (S, m_S, k) be an excellent regular local ring of dimension three, quotient field $K := QF(S)$ and residue characteristic $\mathrm{char}k = p > 0$. Let*

$$h := X^p + f_1X^{p-1} + \cdots + f_p \in S[X],\ f_1, \ldots, f_p \in S \tag{1}$$

be an irreducible polynomial, $\mathcal{X} := \mathrm{Spec}(S[X]/(h))$ and $L := \mathrm{QF}(S[X]/(h))$ be its quotient field.

Assume that h satisfies one of the following assumptions:

(i) *$\mathcal{X}$ is G-invariant, where $\mathrm{Aut}_K(L) = \mathbb{Z}/p =: G$, or*
(ii) *$\mathrm{char}K = p$ and $f_1 = \cdots = f_{p-1} = 0$.*

Let μ be a valuation of L which is centered in m_S. There exists a composition of local Hironaka-permissible blowing ups:

$$(\mathcal{X} =: \mathcal{X}_0, x_0) \leftarrow (\mathcal{X}_1, x_1) \leftarrow \cdots \leftarrow (\mathcal{X}_r, x_r), \tag{2}$$

where $x_i \in \mathcal{X}_i$ is the center of μ, such that $(\mathcal{X}_r, x_r)$ is regular.

When the local multiplicity is $< p$, the proof is written [7]. When the local multiplicity is p, and the initial of h is not a p^{th}-power, the proof is rather easy: use Hironaka's invariants Ridge and Directrix. Else, let D be the discriminant of h w.r.t. X, by an appropriate sequence of Hironaka's permissible blowing ups, we get **condition (E)** following:

Definition 2.1. *Let S, $h \in S[X]$ as above, $\mathcal{X}$ and $E = \mathrm{div}(u_1 \cdots u_e)$ be specified. We say that (S, h, E) satisfies assumption* **(E)** *if $\mathrm{char}(S/m_S) = p > 0$ and one of the following properties hold:*
(i) $D = 0$ and $\eta(\mathrm{Sing}_p\mathcal{X}) \subseteq E$ or
(ii) $D \neq 0$ and $\mathrm{div}(D)_{\mathrm{red}} \subseteq E \subseteq \mathrm{div}(p)_{\mathrm{red}}$.

As we make blowing ups centered in the singular locus, condition **E** *imply that the blow-up centers are of equal characteristic $p > 0$.*

2.2. **Main invariants at the closed point** $x := (X, m_S) \in \mathcal{X} := \mathrm{Spec}(S[X]/h)$**.** η is the projection $\mathcal{X} \longrightarrow \mathrm{Spec}(S)$.

Definition 2.2. *Let* $h := X^p + f_{1,X}X^{p-1} + \cdots + f_{p,X} \in S[X]$, (u_1, u_2, u_3) *a r.s.p. of* S *with* $E \subset \mathrm{div}(u_1u_2u_3)$, $f_{j,X} := \sum_{\text{finite}} \gamma_{abcj}u_1^a u_2^b u_3^c$, $\gamma_{abcj} = 0$ *or invertible.*

We define a rational polyhedron:
$\Delta_S(h; u_1, u_2, u_3; X) := \mathrm{Conv}\left(\bigcup_{\gamma_{abcj}\neq 0}\left\{\left(\frac{a}{j}, \frac{b}{j}, \frac{c}{j}\right)\right\} + \mathbb{R}^3_{\geq 0}\right) \subseteq \mathbb{R}^3_{\geq 0}$.

Theorem 3. *(V. Cossart and O. Piltant)* [6]*. There exists* $\phi_0 \in m_S$ *such that* $\Delta_S(h; u_1, u_2, u_3; X - \phi_0)$ *is minimal for the inclusion among all possible* $\Delta_S(h; u_1, u_2, u_3; X - \phi)$, $\phi \in m_{\hat{S}}$*. This is the* **characteristic polyhedron** *attached to* $(h; u_1, u_2, u_3)$*.*

Definition 2.3. *For* $\Delta_S(h; u_1, u_2, u_3; X)$ *minimal, we define:*
$\delta(h; u_1, u_2, u_3) := \inf(\mathrm{ord}(f_{j,X})/j) = \inf_{\mathbf{x}\in\Delta}(|\mathbf{x}|) \ \in \frac{1}{p!}\mathbb{N}$,
$a(i)(h; u_1, u_2, u_3) := \inf(\mathrm{ord}_{u_i}(f_{j,X})/j) = \inf_{\mathbf{x}\in\Delta}(i\text{-th coor. of } \mathbf{x}) \ \in \frac{1}{p!}\mathbb{N}$,
$\epsilon(x) := p(\delta(x) - \sum_{\mathrm{div}(u_i)\subset E} a(i)) \ \in \frac{1}{(p-1)!}\mathbb{N}$.

Fact: $\delta(h; u_1, u_2, u_3), a(i)(h; u_1, u_2, u_3)$ do not depend upon the choice of the parameters (u_1, u_2, u_3) with $E \subset \mathrm{div}(u_1u_2u_3)$. Consequence: $\epsilon(h; u_1, u_2, u_3)$ also.

Definition 2.4. *With notations as above, let* $\sigma \subset \mathbb{R}^3_{\geq 0}$ *be a face of a minimal* $\Delta_S(h; u_1, u_2, u_3; X)$*, such that* σ *is defined by a weight vector (maybe non unique)* $\alpha = (\alpha_1, \alpha_2, \alpha_3) \in \mathbb{Q}^3_{\geq 0}$ *and:* $\mathbf{x} = (a, b, c) \in \sigma \Leftrightarrow \alpha_1 a + \alpha_2 b + \alpha_3 c = 1$.

This defines a monomial valuation v *on* $S[X]$ *by* $v(u_1^a u_2^b u_3^c X^d) = \alpha_1 a + \alpha_2 b + \alpha_3 c + d$ *and a graded ring*
$\mathrm{gr}_\alpha(S[X]) = \oplus_{\gamma\in\mathbb{Q}_{\geq 0}} I_\gamma/I_{\gamma^+}$*, where*
$I_\gamma = \{s \in S[X] | v(s) \geq \gamma\}$, $I_{\gamma^+} = \{s \in S[X] | v(s) > \gamma\}$.

The initial form $\mathrm{in}_\alpha h$ of h w.r.t. α is the polynomial
$\mathrm{in}_\alpha h := X^p + \sum_{j=1}^p F_{j,X,\alpha}X^{p-i} \in \mathrm{gr}_\alpha(S[X]) \simeq \frac{S}{(u_i, \alpha_i>0)}[X, U_i, \alpha_i > 0]$.

Theorem 4. ***Reduction to characteristic*** $p > 0$***.***

Suppose $\epsilon(x) > 0$*, and* $\alpha_i > 0$, $1 \leq i \leq 3$ *then*
$\delta(x) \in \frac{1}{p}\mathbb{N}$, $a(i) \in \frac{1}{p}\mathbb{N}$, $\epsilon(x) \in \mathbb{N}$,
$\mathrm{in}_\alpha h := X^p - G^{p-1}X + F_{p,\alpha,X} \in \mathrm{gr}_\alpha(S[X]) \simeq \frac{S}{(u_i,\alpha_i>0)}[X, U_i, \alpha_i > 0]$*, with* $G = 0$ *or* $G^{p(p-1)} = \mathrm{in}_\alpha($***discr***$)$*.*

This is the key-point of our proof. The invariants of the characteristic $p > 0$ case were defined over equations of these types, we can define the same invariants in the arithmetical case, and they behave the same way in the computations.

Thanks to this theorem, most of the computations of [5] are valid in the mixed characteristics case.

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Extensions of associated graded rings along a valuation

Steven Dale Cutkosky

This talk is based on the paper [4]. Throughout this talk we will use the following notation. $K \to K^*$ is a finite field extension, ν^* is a valuation of K^* and $\nu = \nu^*|K$. We will consider local rings R of K dominated by ν (we do not require local rings to be Noetherian) and the local ring S of K^* which is the localization of the integral closure of R in K^* at the center of ν^*. This extension $R \to S$ occurs in local uniformization with R regular as a first step in reducing the multiplicity of S.

Let $e = [\Gamma_{\nu^*} : \Gamma_\nu]$ be the reduced ramification index of ν^* over ν and $f = [V_{\nu^*}/m_{\nu^*} : V_\nu/m_\nu]$ be the residue degree of the valuation rings V_{ν^*} and V_ν and $\delta(\nu^*/\nu)$ be the defect of ν^* over ν.

Bernard Teissier has defined the associated graded ring along a valuation of a ring R which is contained in a valuation ring in his work on resolution of singularities [10] as follows. For $\gamma \in \Gamma_\nu$, let $\mathcal{P}_\gamma(R) = \{f \in R \mid \nu(f) \geq \gamma\}$ and

$\mathcal{P}^+_\gamma = \{f \in R \mid \nu(f) > \gamma\}$. The associated graded ring of R along ν is defined as

$$\mathrm{gr}_\nu(R) = \bigoplus_{\gamma \in \Gamma_\nu} \mathcal{P}_\gamma(R)/\mathcal{P}^+_\gamma(R).$$

This ring is almost always non Noetherian. The quotient field $\mathrm{QF}(\mathrm{gr}_\nu(R)) = \mathrm{QF}(\mathrm{gr}_\nu(V_\nu))$ and $[\mathrm{QF}(\mathrm{gr}_{\nu^*}(S)) : \mathrm{QF}(\mathrm{gr}_\nu(R))] = ef$.

We give three examples where R is a regular local ring and ν^* has rank 1, illustrating possible behavior of these extensions. Example 1 is an example over any field k, which shows that it is possible for $\mathrm{gr}_{\nu^*}(S)$ to not be integral over $\mathrm{gr}_\nu(R)$. Example 2 (from [7]), which is also over any field, shows that it is possible for both R and S to be regular and the extension of associated graded rings to be integral but not finite. Example 3 (from [6]) is an example of an immediate extension (so that $\mathrm{gr}_{\nu^*}(S)$ and $\mathrm{gr}_\nu(R)$ have the same quotient fields) in characteristic $p > 0$ such that $\mathrm{gr}_{\nu^*}(S)$ is integral (purely inseparable) but not finite over $\mathrm{gr}_\nu(R)$. In fact, $\mathrm{gr}_{\nu^*}(S)^p \subset \mathrm{gr}_\nu(R)$. In this example the property of being not finite and purely inseparable is stable under sequences of monoidial transforms along ν^*.

The conclusions of the theorem below should be compared with Examples 1-3.

Theorem 1. *Suppose that ν^* has rank 1 and R_0 is an excellent local ring of K which is dominated by ν. Then there exists an excellent normal local ring R' of K which is dominated by ν and dominates R_0 such that if R is a normal excellent local ring of K which is dominated by ν and dominates R', then $\mathrm{gr}_{\nu^*}(S)$ is integral over $\mathrm{gr}_\nu(R)$.*

The conclusions of the following theorem should be compared with Example 3. Example 3 is a tower of two immediate Artin-Schreier extensions, so the following theorem ensures that $\mathrm{gr}_{\nu^*}(S)^{p^2} \subset \mathrm{gr}_\nu(R)$ in Example 3.

Theorem 2. *Suppose that K^* is Galois over K, ν^* has rank 1, ν^* is the unique extension of ν to K^* and ν^*/ν is immediate (so $[K^* : K] = p^n$). Suppose R is a normal local ring of K which is dominated by ν. Then $\mathrm{gr}_{\nu^*}(S)^{p^n} \subset \mathrm{gr}_\nu(R)$.*

Our most difficult result is the following generalization of the Abhyankar Jung Theorem to associated graded rings of valuations.

Theorem 3. *(Generalized Abhyankar Jung Theorem) Suppose that K is an algebraic function field over an algebraically closed field k of characteristic zero, ν^* is a rank 1 valuation, $V_{\nu^*}/m_{\nu^*} = k$ and R' is an algebraic local ring of K which is dominated by ν. Then there exists a sequence of monoidal transforms $R' \to R$ along ν such that R is regular and*

1) $\mathrm{gr}_{\nu^*(S)}$ *is a free* $\mathrm{gr}_\nu(R)$*-module of finite rank* $e = [\Gamma_{\nu^*}/\Gamma_\nu]$.
2) Γ^*/Γ *acts on* $\mathrm{gr}_{\nu^*}(S)$ *with* $\mathrm{gr}_{\nu^*}(S)^{\Gamma_{\nu^*}/\Gamma_\nu} \cong \mathrm{gr}_\nu(R)$ *(a toric algebra over* $\mathrm{gr}_\nu(R)$*).*

Theorem 3 is proven in dimension 2 (and rational rank 1) by Ghezzi, Ha and Kashcheyeva [8] and the conclusions of 1) of the theorem are established by Ghezzi and Kascheyeva [9] for two dimensional defectless extensions of positive characteristic algebraic function fields. The conclusions of 1) of the theorem are established for excellent local domains R' of dimension two under a defectless extension in [3]. Thus it is reasonable to ask if the conclusions of 1) of the theorem are true for defectless extensions (assuming resolution of singularities is true).

The assumption that ν has rank 1 is used in Theorem 1 to avoid some problems with extensions of valuations to the completion of an analytically irreducible local ring. The theorem could be true for arbitrary rank valuations.

Examples 1 and 2 satisfy the classical discrimant condition of the Abhyankar Jung Theorem, but do not satisfy the conclusions of the theorem. However, after some blowing up along the valuation,

$$\begin{array}{ccc} R_1 & \rightarrow & S_1 \quad \subset V_{\nu^*} \\ \uparrow & & \uparrow \\ R & \rightarrow & S \end{array}$$

$R_1 \rightarrow S_1$ must satisfy the conclusions of the theorem (if the characteristic of the ground field k is zero).

Example 3 is much worse. Finiteness of the extension of graded rings never holds after blowing up.

The key point of the proof is to find $R \rightarrow S$ such the $\hat{S} = \bigoplus_{i=1}^{e} w_i \hat{R}$ where $\{\nu^*(w_i)\}$ is a complete set of representatives of the cosets of Γ_ν in Γ_{ν^*}.

In general, $\hat{S}$ is not a free $\hat{R}$-module if R is regular and $\dim R > 2$, although the discriminant condition of the classical Abhyankar-Jung Theorem ensures this.

The proof of Theorem 3 uses my local monomialization theorem [2], an extension in [5] giving a nice extension of the valuation to the completions of the local rings and the classical Abhyankar Jung Theorem [1].

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Stability of Abhyankar valuations

Antoine Ducros

Let K be a field equipped with a Krull valuation $|.|$ (throughout the whole paper we will use the *multiplicative notation*) and let L be a finite extension of K. Let $|.|_1, \ldots, |.|_n$ be the valuations on L extending $|.|$, and for every i, let e_i and f_i be the ramification and inertia indexes of the valued field extension $(K, |.|) \hookrightarrow (L, |.|_i)$. One always has $\sum e_i f_i \leqslant [L : F]$, and the extension L of the valued field $(K, |.|)$ is said to be *defectless* if the equality holds. We say that $(K, |.|)$ is *stable* if every finite extension of it is defectless.

Remark. The product $e_i f_i$ can also be intepreted as the degree of the *graded* residue extension induced by $(K, |.|) \hookrightarrow (L, |.|_i)$.

Examples. Any algebraically closed field is stable; any complete, discretely valued field is stable; the function field of an irreducible smooth algebraic curve, endowed with the discrete valuation associated to a closed point of the curve, is stable; any valued field whose residue characteristic is zero is stable

Let us now give a more involved example. Let G be an ordered abelian group containing $|K^*|$ and let $r = (r_1, \ldots, r_n)$ be a n-uple of elements of G. We denote by $\eta_{K,r}$ the valuation on $K(T_1, \ldots, T_n)$ that sends any polynomial $\sum a_I T^I$ to $\max |a_I| \cdot r^I$ (with $T = (T_1, \ldots, T_n)$).

Theorem. *If the valued field $(K, |.|)$ is stable, so is $(K(T_1, \ldots, T_n), \eta_{K,r})$ for every $r = (r_1, \ldots, r_n)$ as above.*

It has been given several proofs by Gruson, Temkin, Ohm, Kuhlmann, Teissier (see [3], [9], [7], [6], [8]; to the author's knowledge, the first proof working in full generality was that of Kuhlmann, the preceeding proofs requiring some extra-assumptions on K and/or on the r_i's). In what follows we will present a new proof of this theorem, which is based upon model-theoretic tools; it is part of a joint work (which is still at its very beginning) with Ehud Hrushovski and François Loeser.

Step 1. By induction, it is sufficient to prove the theorem for $n = 1$. One then reduces straightforwardly to the case where K is algebraically closed; note that this step requires to understand what happens when K is replaced with one of its finite extensions, and this is here that our stability assumption is used.

Step 2. Now we fix a finite extension F of $K(T)$, and an element r in an ordered group G containing $|K^*|$. We want to prove that F is a defectless extension of $(K(T), \eta_r)$. For that purpose, let us consider a non-trivially valued, algebraically closed extension L of K such that $r \in |L^*|$. Let E_0 be a finite dimensional K-vector subspace of F. Let $\langle . \rangle$ be any extension of $\eta_{L,r}$ to F_L. Using (part of) the seminal work [4] of Haskell, Hrushovski and Macpherson on the elimination of imaginaries in the theory ACVF, together with some further results by Hrushovski and Loeser in[5], one gets the following:

1) the restriction of $\langle . \rangle$ to $L \otimes_K E_0$ is a norm which is definable with parameters in $K \cup \{r\}$;

2) as a consequence of 1), there exists a basis $e_1, \dots, e_d$ of E_0 over K and elements $s_1, \dots, s_d$ of $|K^*| \cdot r^{\mathbf{Q}}$ such that $\langle \sum a_i e_i \rangle = \max |a_i| s_i$ for every d-uple $(a_i) \in L^d$.

The formula given in 2) immediately implies that the *graded* reduction $\widetilde{L \otimes_K E_0}^{\mathrm{gr}}$ is equal to $\widetilde{L}^{\mathrm{gr}} \otimes_{\widetilde{K}^{\mathrm{gr}}} \widetilde{E_0}^{\mathrm{gr}}$. As a consequence, $\widetilde{F_L}^{\mathrm{gr}}$ is nothing but the graded fraction field of $\widetilde{L}^{\mathrm{gr}} \otimes_{\widetilde{K}^{\mathrm{gr}}} \widetilde{F}^{\mathrm{gr}}$. As $\widetilde{L(T)}^{\mathrm{gr}}$ is itself equal by a direct computation to the graded fraction field of $\widetilde{L}^{\mathrm{gr}} \otimes_{\widetilde{K}^{\mathrm{gr}}} \widetilde{K(T)}^{\mathrm{gr}}$, we eventually get

$$\widetilde{F_L}^{\mathrm{gr}} = \widetilde{L(T)}^{\mathrm{gr}} \otimes_{\widetilde{K(T)}^{\mathrm{gr}}} \widetilde{F}^{\mathrm{gr}}.$$

In particular $[\widetilde{F_L}^{\mathrm{gr}} : \widetilde{L(T)}^{\mathrm{gr}}] = [\widetilde{F}^{\mathrm{gr}} : \widetilde{K(T)}^{\mathrm{gr}}]$ (note that here, all graded reductions involved should be understood with respect to $\langle . \rangle$ and its restrictions to the various fields). The author has proved in [1] (a mistake in the latter is corrected in [2]), using also the aformentioned work by Haskell, Hrushovski and Macpherson, that the restriction induces a *bijection* between the set of extensions of $\eta_{L,r}$ to F_L and the set of extensions of $\eta_{K,r}$ to F. It thus follows from the above that F is a defectless extension of $(K(T), \eta_{K,r})$ if and only if F_L is a defectless extension of $(L(T), \eta_{L,r})$. Hence by replacing K with a suitable valued extension, we may and do ssume that $r \in |K^*| \neq \{1\}$ (and that K is still algebraically closed).

Step 3. Let $b \in |K^*|$. Let us choose $\lambda \in K$ such that $|\lambda| = b$ and let τ be the image of T/λ in the residue field k of $(K(T), \eta_{K,b})$; note that $k = \widetilde{K}(\tau)$. Let b^- and b^+ be elements of an ordered group containing $|K^*|$ which are infinitely close to b (with respect to $|K^*|$), with $b^- < b < b^+$. The valuation $\eta_{K,b-}$ (resp. η_{K,b^+}) is the composition of $\eta_{K,b}$ and of the discrete valuation $\langle . \rangle_0$ (resp. $\langle . \rangle_\infty$) of k that corresponds to $\tau = 0$ (resp. $\tau = \infty$), and the extensions of η_{K,b^-} (resp. η_{K,b^+}) to F are compositions of extensions of $\eta_{K,b}$ and of extensions of $\langle . \rangle_0$ (resp. $\langle . \rangle_\infty$). Since $(k, \langle . \rangle_0)$ and $(k, \langle . \rangle_\infty)$ are stable, we see that the following are equivalent:

i) F is a defectless extension of $(K(T), \eta_{K,b^-})$;
ii) F is a defectless extension of $(K(T), \eta_{K,b})$;
iii) F is a defectless extension of $(K(T), \eta_{K,b^+})$.

In the same spirit, let ε be an element of an ordered group containing $|K^*|$ which is infinitely close to zero with respect to $|K^*|$. The valuation $\eta_{K,\varepsilon}$ is the composition of the discrete valuation $\langle . \rangle_0'$ of $K(T)$ corresponding to the closed point $T = 0$ and of the valuation of K. Since both $(K, |.|)$ and $(K(T), \langle . \rangle_0')$ are stable, $(K(T), \eta_{K,\varepsilon})$ is stable; in particular, F is a defectless extension of $(K(T), \eta_{K,\varepsilon})$.

Step 4. We will now use the theory of "stable completions" introduced by Hrushovski and Loeser in [5] as kind of a model-theoretic avatar of Berkovich spaces. Let X be an irreducible, smooth, projective curve over K whose function field is isomorphic to F, and such that $K(T) \hookrightarrow F$ is induced by a finite map $f \colon X \to \mathbf{P}^1_K$; the latter induces a map $\hat{f} \colon \hat{X} \to \hat{\mathbf{P}}^1_K$, where the "hat" denotes the stable completion. Let M be the class of algebraically closed valued extension of K. For every $L \in \mathsf{M}$, and any $s \in |L^*|$, the valuation $\eta_{L,s}$ appears as a point of $\hat{\mathbf{P}}^1(L)$, whose pre-images on $\hat{X}(L)$ correspond to the extensions of $\eta_{L,s}$ to F_L. We denote by Δ_L the set of $s \in |L^*|$ such that the extension F_L of $(L(T), \eta_s)$ is defectless.

A fundamental result by Hrushovski and Loeser asserts that $\hat{X}$ is definable (this is specific to the one-dimensional case). This leads, together with the "o-minimality of the value group", to the following fact: there exist finitely many disjoint non-empty intervals $I_1 < I_2 < \ldots < I_m$ of $|K^*|$ with endpoints in $|K| \cup \{+\infty\}$ (and with at least one element of $|K^*|$ lying between I_j and I_{j+1} for every $j < n$) such that for every $L \in \mathsf{M}$ one has $\Delta_L = \coprod I_{j,L}$, where we denote by $I_{j,L}$ the interval of $|L^*|$ with the same definition as that of I_j.

Step 5. Let $L \in \mathsf{M}$ such that there exist ε as in step 3 in the value group $|L^*|$. By step 3, the extension F of $(K(T), \eta_{K,\varepsilon})$ is defectless. By step 2, this implies that the extension F_L of $(L, \eta_{L,\varepsilon})$ is defectless. Therefore $\varepsilon \in \Delta_L$. This implies that $m \geqslant 1$ and that the lower bound of I_1 is equal to zero: indeed, if it were an element $c \in |K^*|$, then $\Delta_L = \coprod I_{j,L}$ would be contained in $[c; +\infty[_L$, contradicting the fact that $\varepsilon < c$.

The interval I_1 is thus of the form $]0; b[$, $[0; b]$ or $]0; +\infty[$. We will exclude $]0; b[$ and $]0; b]$. This will show that $\Delta_K = |K^*|$ and will end the proof.

Step 6. *Assume that* I_1 *is equal to* $]0; b[$ *or* $]0; b]$ *with* $b \in |K^*|$. Choose $L \in \mathsf{M}$ such that there exists elements b^- and b^+ as in step 3 in the value group $|L^*|$. Since $b^- \in]0; b[_L \subset I_{1,L}$, the extension F_L of $(L(T), \eta_{L,b^-})$ is defectless. By step 2, this implies that F is a defectless extension of $(K(T), \eta_{K,b^-})$ as well. By step 3, F is then a defectless extension of $(K(T), \eta_{K,b})$; therefore $b \in \Delta_K$ and $I_1 =]0; b]$. This implies the existence of $c > b$ such that $I_j \subset]c; +\infty[$ for every $j \geqslant 2$.

Since F is a defectless extension of $(K(T), \eta_{K,b})$, by using again step 3, we see that F is a defectless extension of $(K(T), \eta_{K,b^+})$. By step 2, the extension F_L of $(L(T), \eta_{L,b^+})$ is defectless. Hence $b^+ \in \Delta_L$, but the latter consists of $]0; b]_L$ and of elements of $|L^*|$ which belong to $I_{j,L}$ for $j \geqslant 2$, hence are greater than c; since $b < b^+ < c$, we get a contradiction.

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Differential-Henselian Fields

Lou van den Dries
(joint work with Matthias Aschenbrenner, Joris van der Hoeven)

After some twenty years of study we established in Spring 2014 some decisive results about the model theory of the valued differential field $\mathbb{T}$ of transseries, similar to what Tarski achieved for the field of real numbers in the 1940s, and Ax, Kochen, Ersov, and Macintyre in the 1960s and 1970s for henselian valued fields like $\mathbb{C}((t))$ and the p-adic fields. We finished the program outlined in [1], and hope to make the results available soon. Some of our work deals with general valued differential fields with continuous derivation. Below we discuss the notion of differential-henselianity in this setting.

Let K be a *valued differential field*: a field K with a valuation $v : K^\times \to \Gamma$ whose residue field $\mathbf{k} := \mathcal{O}/\mathfrak{m}$ has characteristic zero, and also equipped with a derivation $\partial : K \to K$. Here $\Gamma = v(K^\times)$ is the value group, $\mathcal{O} = \mathcal{O}_K$ is the valuation ring of v, and $\mathfrak{m}$ is the maximal ideal of $\mathcal{O}$. Also $C = C_K := \{f \in K : \partial(f) = 0\}$ is the constant field of the differential field K. We often write f' for $\partial(f)$.

We focus on the case that ∂ is continuous (for the valuation topology). If the derivation is *small* in the sense that $\partial(\mathfrak{m}) \subseteq \mathfrak{m}$, then ∂ is continuous. As a partial converse, if ∂ is continuous, then some multiple $a\partial$ with $a \in K^\times$ is small. *From now on we assume that our derivation ∂ on K is small.* This has the effect that also $\partial(\mathcal{O}) \subseteq \mathcal{O}$, and so ∂ induces a derivation on the residue field; we view $\mathbf{k}$ below as equipped with this induced derivation. Examples of such K include:

(i) $\mathbf{k}(t)$ and $\mathbf{k}((t))$ with the t-adic valuation and $\partial = t\frac{d}{dt}$, with $\mathbf{k}$ any field of characteristic zero;
(ii) Hardy fields, with $\mathcal{O} = \{\text{germs in } K \text{ of bounded functions}\}$; see [3];
(iii) $\mathbb{T}$, the valued differential field of transseries; see [5, 1];
(iv) $\mathbf{k}((t^\Gamma))$ where $\mathbf{k}$ is any differential field of characteristic zero, Γ any ordered abelian group, and $\partial(\sum_\gamma a_\gamma t^\gamma) := \sum_\gamma a'_\gamma t^\gamma$; see [9].

We say that K has *few constants* if v is trivial on C, that is, $v(C^\times) = \{0\}$, and that K has *many constants* if $v(C^\times) = \Gamma$. In (i), (ii), (iii) we have few constants, and in (iv) we have $C = C_{\mathbf{k}}((t^\Gamma))$: many constants. If K has many constants, then K is *monotone* in the sense of [4], that is, $v(f) \leq v(f')$ for all $f \in K$. But the examples in (i) are also monotone.

Asymptotic K are defined by another interaction of valuation and derivation:

$$0 < v(f) \leq v(g) \implies v(f') \leq v(g').$$

It implies few constants, and holds in (i), (ii), and (iii) above. Asymptotic valued differential fields include Rosenlicht's differential-valued fields [8].

It is worth mentioning that if L is any *algebraic* valued field extension of K, then the unique derivation on L that extends ∂ is small. By *extension of* K we mean below a valued field extension of K equipped with a small derivation extending ∂. Here is a "differential" analogue of a result due to Krull [7]:

Theorem 1.1. *If the derivation of* $\mathbf{k}$ *is nontrivial, then* K *has an immediate spherically complete extension.*

We can derive the same conclusion under various other assumptions but these are more complicated to state, and the conclusion harder to prove. Sofar we only know uniqueness-up-to-isomorphism over K of such extensions under rather restrictive conditions. Theorem 1.1 can be obtained along the lines of similar results for certain kinds of valued *difference* fields in [2].

The derivation of $\mathbf{k}$ being nontrivial is an optimal situation, but precarious: if ∂ is replaced by $a\partial$, $a \in K^{\times}$, then for $v(a) < 0$ the derivation is no longer small, while for $v(a) > 0$ the derivation is still small, but the induced derivation on $\mathbf{k}$ becomes trivial. For $\mathbb{T}$ we are not in this optimal situation, but coarsening by any non-trivial convex subgroup of the value group and changing ∂ to $a\partial$ for suitable a gets us there. In this way the results discussed here apply indirectly to $\mathbb{T}$.

We say that K is *differential-henselian* (for short: d-henselian) if every differential polynomial $P \in \mathcal{O}\{Y\} = \mathcal{O}[Y, Y', Y'', \dots]$ whose reduction $\overline{P} \in \mathbf{k}\{Y\}$ has total degree 1 has a zero in $\mathcal{O}$. (Note that restricting this requirement to ordinary polynomials $P \in \mathcal{O}[Y]$ would define the usual notion of a henselian valued field, that is, a valued field whose valuation ring is henselian.)

If K is d-henselian, then its differential residue field is clearly *linearly surjective*: any linear differential equation $y^{(n)} + a_{n-1}y^{(n-1)} + \cdots + a_0 y = b$ with coefficients $a_i, b \in \mathbf{k}$ has a solution in $\mathbf{k}$. This is a key constraint on our notion of d-henselianity. (Accordingly, for $K = \mathbb{T}$ we coarsen the valuation and adjust the derivation to make this notion applicable.) It is easy to show that if K is d-henselian, then the differential residue field can be lifted to a differential subfield of K. If in example (iv) above $\mathbf{k}$ is linearly surjective, then K is d-henselian. More generally, we have the following differential analogue of Hensel's Lemma:

Proposition 1.2. *If* $\mathbf{k}$ *is linearly surjective and* K *is spherically complete, then* K *is* d-*henselian.*

For valued fields of equicharacteristic zero, henselianity is equivalent to algebraic maximality, that is, not having any proper immediate algebraic extension. (This fact plays a key role in the model-theoretic treatment of henselian valued fields stemming from Ax, Kochen, and Ersov.) As to the natural differential analogue, one direction is immediate from Theorem 1.1 and Proposition 1.2: if $\mathbf{k}$ is linearly surjective and K is d-algebraically maximal (that is, has no proper immediate differentially algebraic extension), then K is d-henselian. The more interesting direction fails: there is a d-henselian K with *many* constants that has a proper immediate differentially algebraic extension. We conjecture, however, that if K is d-henselian with *few constants*, then K is d-algebraically maximal, and proved this for monotone K.

Any asymptotic K with linearly surjective $\mathbf{k}$ has a minimal immediate d-henselian differentially algebraic extension, and we hope that such an extension is necessarily a d-henselization in the sense of embedding over K into any d-henselian extension of K. (A d-henselization would be unique-up-to-isomorphism over K, though not in general unique-up-to-unique-isomorphism over K.)

The definiton of "henselian" refers only to solving equations in one variable, but implies a several variable version. We don't know if the natural analogue of this holds for d-henselian K. It does hold for d-algebraically maximal K with linearly surjective $\mathbf{k}$, as we now explain. Let $Y = (Y_1, \dots, Y_n)$ be a tuple of n indeterminates (instead of just one indeterminate as before), and let $A_1, \dots, A_m \in \mathbf{k}\{Y\}$ be homogeneous of (total) degree 1. Then we say that $A_1, \dots, A_m$ are d-independent if the family of derivatives $(A_i^{(j)})_{i=1,\dots,m, j=0,1,2,..}$ is linearly independent over $\mathbf{k}$.

Theorem 1.3. *Suppose K is* d-*algebraically maximal and $\mathbf{k}$ is linearly surjective. Let $P_1, \dots, P_n \in \mathcal{O}\{Y\}$ be such that $P_1(0), \dots, P_n(0) \in \mathfrak{m}$ and $A_1, \dots, A_n$ are* d-*independent, where $A_i \in \mathbf{k}\{Y\}$ is the reduction of the homogeneous part of degree 1 of P_i. Then $P_1(y) = \cdots = P_n(y) = 0$ for some $y = (y_1, \dots, y_n) \in \mathfrak{m} \times \cdots \times \mathfrak{m}$.*

Our proof goes as follows. For spherically complete K with Γ of rank 1, we get the desired result by approximation arguments involving also diagonalizing $n \times n$ matrices with entries in the noncommutative ring $\mathbf{k}[\partial]$ of linear differential operators. For arbitrary spherically complete K we can reduce to the rank 1 case by a careful transfinitely iterated coarsening plus specialization. Thus in general we can find an infinitesimal solution y in some spherically complete immediate extension of K. It remains to appeal to a theorem of Joseph Johnson [6] in "commutative differential algebra" to conclude that any such y must be d-algebraic over K, and thus in K by the assumption that K is d-algebraically maximal.

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Existentially definable henselian valuation rings

Arno Fehm
(joint work with Will Anscombe, Alexander Prestel)

In this talk I gave a survey on some of the recent results on existential definability of henselian valuation rings inside their quotient fields.

1. Existential definability in $\mathcal{L}_{\text{ring}}$

The first part of the talk focused on parameter-free existential definability in the ring language $\mathcal{L}_{\text{ring}} = \{+, \cdot, 0, 1\}$, that is, the question which henselian valuation rings are diophantine in the sense that they are projections of zero sets of polynomials $f_1, \dots, f_r \in \mathbb{Z}[X_1, \dots, X_n]$.

It was observed by Julia Robinson that the henselian valuation ring of $\mathbb{Q}_p$ is diophantine. This was generalized to finite extensions of $\mathbb{Q}_p$ in [3]. The case of positive characteristic local fields $\mathbb{F}_q((t))$ was treated in [2]. In [4], the speaker had used ideas from [2] to generalize all of this to henselian valuation rings with residue field finite or PAC without an algebraically closed subfield, which had motivated Prestel [6] to give (among other things) a general but abstract criterion for uniform existential definability of valuation rings.

In joint work with Anscombe [1], we use Prestel's criterion and F.-V. Kuhlmann's work on the model theory of tame fields to show that the question whether a henselian valuation ring is diophantine depends only on the residue field, and we isolate a suitable condition on the residue field:

Theorem 1 (Anscombe-F.)**.** *Let F be a field.*

(1) *If* some *equicharacteristic henselian valuation ring with residue field F is diophantine, then* every *henselian valuation ring with residue field F is diophantine.*
(2) *An equicharacteristic henselian valuation ring with residue field F is diophantine iff for every $F_1, F_2 \equiv F$ and non-trivial valuation v on F_1, the residue field of v cannot be embedded into F_2.*

This allows to easily reprove the cases mentioned above and leads to new negative and positive definability results.

2. Existential definability in $\mathcal{L}_{\text{Mac}}$

The second part focused on results about uniform parameter-free existential definability in the Macintyre language $\mathcal{L}_{\text{Mac}} = \mathcal{L}_{\text{ring}} \cup \{P_n : n \in \mathbb{N}\}$, where P_n is a predicate for the n-th powers.

It was proven in [3] that there is an $\exists$-$\emptyset$-formula in $\mathcal{L}_{\text{Mac}}$ that defines every henselian valuation ring with residue field of characteristic not 2 which is finite or pseudo-finite.

In joint work with Prestel [5], we use the theory of p-henselian valuations to give a structural proof of a more general result:

Theorem 2 (F.-Prestel)**.** *There is an* $\exists$-$\emptyset$*-formula in* $\mathcal{L}_{\mathrm{Mac}}$ *defining all henselian valuation rings with residue field of characteristic not* 2 *which is finite, PAC and not* 2*-closed, or Hilbertian.*

We also give sufficient conditions for a henselian valuation ring to be *not* definable by an $\exists$-$\emptyset$-formula in $\mathcal{L}_{\mathrm{Mac}}$. For example, we can classify all henselian valuations on higher dimensional local fields like $\mathbb{Q}_p((t))((s))$ according to their quantifier complexity in $\mathcal{L}_{\mathrm{Mac}}$.

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A valued field version of the Kontsevich–Zagier Conjecture about periods

Immanuel Halupczok
(joint work with Raf Cluckers)

The theme of the talk is the following question:

Question 1. Suppose that $\int_{X_1} f_1 = \int_{X_2} f_2$. Does this imply that one of the integrals can be transformed into the other one using some standard integral transformation rules?

To make this precise, one has to specify what kinds of sets X_i and functions f_i one considers and what transformation rules are allowed. Depending on this, the difficulty of the question ranges from trivially true to very hard. At the very hard end, there is a conjecture by Kontsevich–Zagier [3, Conjecture 1] in $\mathbb{R}$. This talk is mainly about a much easier p-adic analogue of that conjecture. That p-adic analogue is closely linked to a version of the question for motivic integration, which was our initial motivation to think about this and which will be explained in the last section.

1. Question 1 in $\mathbb{R}$ – The Kontsevich–Zagier Conjecture

The Conjecture by Kontsevich–Zagier is that the answer to the following instance of Question 1 is "yes".

The sets X_1 and X_2 are semi-algebraic subsets of $\mathbb{R}^n$, i.e., boolean combinations of inequalities of the form $g(x_1, \dots, x_n) \geq 0$, where $g \in \mathbb{Q}[x_1, \dots, x_n]$.

As functions f_i, we allow semi-algebraic functions $X_i \to \mathbb{R}$, i.e., functions whose graphs are semi-algebraic. This is not exactly the same as what Kontsevich and Zagier allowed, but note that the functions do not play an important role, since one could replace an integral $\int_X f$ by the measure of the set below the graph of f. The reason to consider functions at all is that they arise in intermediate step when transforming one integral into the other.

To finish making the conjecture by Kontsevich–Zagier precise, I would have to specify what transformations of integrals they allow. Let me only do this rougly: one is allowed to use additivity of the integral, semi-algebraic changes of variables and the Stokes formula. I will make this more precise in the p-adic setting.

From a model theoretic point of view, working with semi-algebraic sets and functions is very natural: these are exactly sets and functions "definable in the ring language". Let me explain this a bit. The class of sets definable in the ring language (which, in the following, I will call "definable sets", for short) is, by definition, the smallest class of sets with the following properties:

- It contains all zero sets $\{(x_1, \dots, x_n) \in \mathbb{R}^n \mid g(x_1, \dots, x_n) = 0\}$ of polynomials $g \in \mathbb{Q}[x_1, \dots, x_n]$.
- It is closed under boolean combinations and under coordinate projections.

(Note that we simultaneously define subsets of $\mathbb{R}^n$ for all n.) One can show that the class of definable sets is closed under many other natural operations. In particular, one defines a "definable map" to be a map whose graph is definable (i.e., those maps which I called "semi-algebraic" above); using this, the image and the preimage of a definable set under a definable map is again definable.

2. Question 1 in $\mathbb{Q}_p$ – a Theorem

As in $\mathbb{R}$, we consider sets $X_1, X_2 \subset \mathbb{Q}_p^n$ which are definable in the ring language. (Macintyre [4] proved that as in $\mathbb{R}$, there is a more direct description of definable sets: they are boolean combintations of sets given by conditions of the form "$g(x_1, \dots, x_n)$ is a k-th power" for $g \in \mathbb{Q}[x_1, \dots, x_n]$ and $k \in \mathbb{N}$.)

On $\mathbb{Q}_p$, we will integrate with respect to the Haar measure of the additive group $(\mathbb{Q}_p, +)$, normalized such that $\mathbb{Z}_p$ has measure 1. This means that our integrals will take values in $\mathbb{R}$, and hence also our functions f_i should take values in $\mathbb{R}$.[1] This means that it does not make sense to ask for the f_i to be definable (since definable functions would take values in $\mathbb{Q}_p$). Instead, we will consider functions

[1] One could also consider Question 1 for integrals taking values in $\mathbb{Q}_p$; however, it is integration with values in $\mathbb{R}$ which is most closely related to our original goal of understanding motivic integration.

of the form $f(x) = |g(x)|_p$, where g is definable and $|\cdot|_p$ is the p-adic norm. These are the functions which will arise in our integral transformations.

When integrating over sets $X \subset \mathbb{Q}_p^n$, we will use a measure corresponding to the dimension of X; there is a canonical way to define such a measure. If $\dim X = n$, it is simply the restriction of the Haar measure on $\mathbb{Q}_p^n$ to X.

To make precise which transformations of integrals we allow, we introduce a "Grothendieck group modulo integral transformations": Let R be the free abelian group generated by pairs $[X, f]$ (where $X \subset \mathbb{Q}_p^n$ is definable and $f\colon X \to \mathbb{R}$ is a function of the above form), modulo the following relations; let $[X, f]$ and $[Y, g]$ be two such generators.

(1) (Additivity) If X and Y are disjoint then $[X \cup Y, h] = [X, f] + [Y, g]$, where $h|_X = f$ and $h|_Y = g$.
(2) (Change of variables) If there exists a definable C^1-bijection $\phi\colon X \to Y$ such that $f = |\operatorname{Jac}\phi|_p \cdot (g \circ \phi)$, then $[X, f] = [Y, g]$.
(3) (Removing subsets of lower dimension) $X \subset Y$, $f = g|_X$ and $\dim(Y \setminus X) < \dim Y$, then $[X, f] = [Y, g]$.
(4) (Product with $\mathbb{Z}_p$) If $Y = X \times \mathbb{Z}_p$ and $g(x, z) = f(x)$, then $[X, f] = [Y, g]$.

(One can define a natural multiplication on R, though it will not be needed in this talk.) Now we can formulate our (not yet published) result precisely:

Theorem 1 (Cluckers, H.). *For X_1, f_1, X_2, f_2 as above, we have $\int_{X_1} f_1 = \int_{X_2} f_2$ if $[X_1, f_1] = [X_2, f_2]$ in R.*

That this result is much easier than the Kontsevich–Zagier conjecture is due to the fact that $\mathbb{Q}_p$ is ultra-metric, which makes measuring sets much simpler. Indeed, the value of an arbitrary integral $\int_X f$ of the form considered here is always rational, and there are algorithms to compute such integrals. In contrast, in the reals, one can easily obtain transcendental numbers like π, and one hope is that the Kontsevich–Zagier conjecture would help determining whether two integrals have the same value.

3. Question 1 for Motivic Integration – Work in Progress

There are different notions of motivic integration. The one we consider here is the one by Cluckers–Loeser [1]. Let me give a vague description. The "sets" X one integrates over are certain abstract geometric objects, the "functions" f on X are abstract objects, too, and motivic integration associates to f the value of an "integral" $\int_X f$, which is an element of some ring $\mathcal{C}^0$.

To fill the previous sentence with a bit of meaning, let me mention that motivic integration "specializes" to p-adic integration for big p: Suppose that K is a local field (i.e., a finite extension of $\mathbb{Q}_p$ or $\mathbb{F}_p((t))$). Then X yields an actual set $X(K) \subset K^n$ for some n (as an example, X could be an affine algebraic variety defined over $\mathbb{Z}$), f yields a function $f_K\colon X(K) \to \mathbb{R}$, and any element $a \in \mathcal{C}^0$ yields a value $a_K \in \mathbb{R}$. This fits together in such a way that for every X and f, and for every big enough p (depending on X and f), one has

$$\int_{X(\mathbb{Q}_p)} f_{\mathbb{Q}_p} = \left(\int_X f\right)_{\mathbb{Q}_p}$$

(and similarly for other local fields of characteristic p). In fact, motivic integration can be considered as carrying out p-adic integration in a field-independent way. This does not only allow to recover p-adic integration for almost all p, but one also obtains an abstract notion of integration in other discretely valued fields like $\mathbb{C}((t))$. This point of view shows that Theorem 1 is closely related to a corresponding question about motivic integration.

The reason we would like to answer Question 1 (affirmatively) for motivic integration is that this means that motivic integration "does not loose any information". Indeed, motivic integration is supposed to satisfy integral transformation rules (like the ones from Section 2). That Question 1 is true means that the definition of motivic integration is optimal in the sense that two integrals are equal only when they have to be equal according to those rules.

As a final remark, let me mention that there is also Hrushovski–Kazhdan motivic integration [2], which, by its very definition, does not loose information. However, it lives in a slightly different context, and the point of our present work is to understand the corresponding result for Cluckers–Loeser motivic integration.

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Local-global principles in dimension two

David Harbater

(joint work with Julia Hartmann, Daniel Krashen)

A *local-global principle* over a field F classically has the following form: One gives an algebraic structure X over F, and some property that X could have. The principle then states that X satisfies the given property over F if and only if it satisfies the property over F_v for each completion of F at an absolute value v.

For example, the structure could be an F-variety V, with the property of having an F-rational point. A key case is where V is a G-torsor over F. With $G = \mathrm{O}(n)$, the local-global principle is for isotropy of quadratic forms; and with $G = \mathrm{PGL}_d$, it is for splitness of central simple F-algebras. In the case of a global field F, these hold by the Hasse-Minkowski theorem and the theorem of Albert-Brauer-Hasse-Noether. These respectively imply that the u-invariant of a characteristic p

global field equals 4, and that period equals index over a global field. Not all local-global principles hold classically; e.g. for G an elliptic curve, where the obstruction $Ш = \ker\big(H^1(F,G) \to \prod_v H^1(F_v,G)\big)$ is conjectured to be finite.

In this talk, we instead consider local-global principles for fields that arise in two-dimensional situations. Specifically, we treat these three cases:

(1) We may take a one-variable function field F over a complete discretely valued field K. If $T = O_K$, then there is a regular projective model $\mathcal{X}$ of F over T; this T-curve is a two-dimensional scheme. We call F a *semi-global field*. For example, take $K = k((t))$, and take $F = K(x)$, the fraction field of $k[[t]][x]$. This is the function field of $\mathbb{P}^1_T$.

(2) We may take a field like the fraction field of $k[x][[t]]$ (which is larger than the field considered above). This field corresponds to the meromorphic functions on the t-adic closed unit disc around the origin in $\mathbb{P}^1_K$, with $K = k((t))$. We may also take a finite separable extension of this field F, or a mixed characteristic analog.

(3) We may take a field like $k((x,t))$, which corresponds to the meromorphic functions on the t-adic open unit disc around the origin in $\mathbb{P}^1_K$, with $K = k((t))$. More generally, we may take the fraction field of any two-dimensional Noetherian local domain (or equivalently, by a strong form of the Cohen Structure Theorem, a finite separable extension of the fraction field of $T[[x]]$, where T is any complete discrete valuation ring).

We begin by discussing case 1. There, one can consider local-global principles with respect to the discrete valuations on F (either *all* discrete valuations, or those that are *geometric*, i.e. which are induced by codimension one points on regular projective models of F over T).

In this two-dimensional situation, there is also the following alternative: Choose a regular model $\mathcal{X}$ of F over T, with closed fiber X. For $P \in X$, let F_P be the fraction field of the complete local ring $\hat{O}_{\mathcal{X},P}$. We can then consider local-global principles with respect to the overfields F_P, rather than with respect to the overfields F_v. In the case of G-torsors for some algebraic group G, the obstructions in these two cases are $Ш(F,G) := \ker(H^1(F,G) \to \prod_v H^1(F_v,G))$ and $Ш_X(F,G) := \ker(H^1(F,G) \to \prod_P H^1(F_P,G))$. Here $Ш_X(F,G) \subseteq Ш(F,G)$ because every F_v contains an F_P. So a local-global principle with respect to discrete valuations implies a local-global principle with respect to points, in this case.

An advantage of considering $Ш_X(F,G)$ is that it can often be compute explicitly, using patching. In particular, it is trivial if G is a rational connected linear algebraic group [HHK09] such as $\mathrm{SO}(n)$ or PGL_n. As a consequence, we obtain a local-global principle in this form for quadratic forms q over F and for central simple algebras over F. (In the former case, it is necessary to assume that $\dim(q) \neq 2$. We also need to stay away from bad residue characteristic.) Sometimes it can be shown that $Ш(F,G) = Ш_X(F,G)$, e.g. if the residue field k of T is algebraically closed of characteristic zero ([HHK11], using [BKG04]).

Using either $Ш(F,G)$ or $Ш_X(F,G)$, one can obtain applications of the above results that are analogous to those obtained classically for quadratic forms and central simple algebras, concerning the u-invariant and the period-index relationship ([HHK09], [CPS12]). For example, if $F = \mathbb{Q}_p(x)$, the above assertions imply that $u(F) = 8$ ([HHK09]; also proven in other ways in [PS10] and [Lee13], the latter even for $p = 2$); and that the index of a central simple algebra of period prime to p divides the square of the period ([HHK09]; previously proven in another way in [Sal97].) The same approach via local-global principles yields related results for other choices of K. For example, taking $K = \mathbb{Q}_p((t))$, we obtain that $u(K(x)) = 16$ and that the index divides the cube of the period over $K(x)$ (the latter assertion also following from [Lie11]).

The above local-global principle for torsors under rational connected groups, with respect to the fields F_P for $P \in X$, can be shown via patching. This is done by deducing it from a related local-global principle with respect to a *finite* set of overfields of F. Specifically, choose a finite set $\mathcal{P}$ of closed points of X that meets each irreducible component of X. For each $P \in \mathcal{P}$, take the field F_P. Let $\mathcal{U}$ be the (finite) set of connected components of the complement of $\mathcal{P}$ in X. Thus X is partitioned as $\mathcal{P} \cup \mathcal{U}$. For $U \in \mathcal{P}$, there is similarly an associated field F_U. One can then consider the local-global principle with respect to the finitely many fields F_P (for $P \in \mathcal{P}$) and F_U (for $U \in \mathcal{U}$). Using a matrix factorization theorem for rational connected groups (analogous to Cartan's Lemma for the group GL_n), it can be shown that this local-global principle holds for such groups; i.e. the corresponding obstruction $Ш_{\mathcal{P}}(F, G)$ is trivial. Using Artin Approximation it follows that the local-global principle with respect to all the points $P \in X$ also holds.

More generally, we can ask what happens if G is not a rational connected group. If G is not assumed connected but it is rational (i.e. each connected component is rational), then $Ш_X(F, G)$ is finite. An explicit description is given in [HHK11], which in particular shows that the corresponding local-global principle holds if and only if either G is connected or the reduction graph of a semistable model is a tree. On the other hand, if G is connected but not rational, then one can still often obtain a local-global principle using cohomological invariants such as the Rost invariant; see [HHK14]. But an example in [CPS13] shows that local-global principles do not *always* hold. This raises the question of which class of connected groups satisfy a local-global principle.

We now turn to the more local cases (2) and (3) above. Case (2) includes fields of the form F_U (these being closed under finite separable extension, by [HHK14a]). Case (3) includes in particular fields of the form F_P and their finite separable extensions. In these cases we can again obtain local-global principles and use them to get applications to quadratic forms and central simple algebras. The idea is to deduce these from the semi-global case, via a *refinement principle*.

In case (2), we have an open subset U of the closed fiber X of $\mathcal{X}$. We may pick a finite set $\mathcal{P} \subset X$ such that U is in the set $\mathcal{U}$ of components of $X - \mathcal{P}$. Given a finite subset $\mathcal{P}' \subset U$, let $\mathcal{U}'$ be the set of components of $U - \mathcal{P}'$. By the semi-global case, $Ш_{\mathcal{P}}(F, G)$ and $Ш_{\mathcal{P} \cup \mathcal{P}'}(F, G)$ are trivial for G rational connected. By the refinement principle shown in [HHK14a], we may "subtract" these two situations from each other; i.e. $Ш_{\mathcal{P}'}(F_U, G)$ is also trivial.

Case (3) can be handled similarly, but first blowing up $\mathcal{X}$ at P, which replaces P by an exceptional divisor, on which we pick a finite set $\mathcal{P}'$.

For a field E in either of these two cases, we obtain as an application that $u(E) = \sup_v u(E_v)$, where v ranges over the discrete valuations of E. We can use this to compute u-invariants explicitly. For example,

(1) If $R = \mathbb{Z}_p[[x]]$ or the p-adic completion of $\mathbb{Z}_p[x]$ (for any prime p), and E is a finite extension of the fraction field of R, then $u(E) = 8$.

(2) Say k is a field, and E a finite extension of the fraction field of $k[x][[t]]$. Then $u(E) = 4$ if k is algebraically closed; $u(E) = 8$ if k is finite or the field of Laurent series over an algebraically closed field; $u(E) = 16$ if k is $\mathbb{Q}_p$ or the field of Laurent series over a finite field; and $u(E) = 32$ if k is the field of rational function over $\mathbb{Q}_p$ or the field of Laurent series over $\mathbb{Q}_p$.

As an application to central simple algebras, given a partition $\mathcal{P}' \cup \mathcal{U}'$ associated to F_U or F_P as above, and A a central simple algebra over E, $\mathrm{ind}(A)$ divides $\mathrm{per}(A)^d$ if and only if $\mathrm{ind}(A_\xi)$ divides $\mathrm{per}(A_\xi)^d$ for all $\xi \in \mathcal{P}' \cup \mathcal{U}'$. This yields explicit values of the period-index bound d. For example:

(1) If $R = \mathbb{Z}_p[[x]]$ or the p-adic completion of $\mathbb{Z}_p[x]$, and E is finite over the fraction field of R, then $\mathrm{ind}(A)$ divides $\mathrm{per}(A)^2$ for every central simple algebra A over E.

(2) If R is the p-adic completion of $\mathbb{Z}_p^{\mathrm{ur}}[x]$ or of $\mathbb{Z}_p^{\mathrm{ur}}[[x]]$, and E is finite over the fraction field of R, then the same conclusion holds. Moreover per = ind if p does not divide the period.

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Uniformly defining p-henselian valuations

Franziska Jahnke
(joint work with Jochen Koenigsmann)

1. p-henselian valuations

Admitting a non-trivial p-henselian valuation is a weaker assumption on a field than admitting a non-trivial henselian valuation. Unlike henselianity, p-henselianity is an elementary property in the language of rings. We are interested in the question when a field admits a non-trivial $\emptyset$-definable p-henselian valuation (in the language of rings). These $\emptyset$-definable p-henselian valuations can then often be used to obtain parameter-free definitions of non-trivial henselian valuations. The aim of this talk is give a classification of elementary classes of fields in which the canonical p-henselian valuation is uniformly $\emptyset$-definable. As our definitions involve Beth's Definability Theorem, they require a careful analysis of when a p-henselian valuation coincides with the canonical p-henselian valuation.

Let K be a field and p a prime.

Definition. We define $K(p)$ to be the compositum of all Galois extensions of K of p-power degree. A valuation v on K is called *p-henselian* if v extends uniquely to $K(p)$. We call K *p-henselian* if K admits a non-trivial p-henselian valuation.

Clearly, this definition only imposes a condition on v if K admits Galois extensions of p-power degree. Every henselian valuation is p-henselian for any prime p, but there are p-henselian non-henselian valuations. Any p-henselian valuation also satisfies an analogue of Hensel's lemma ([2, Proposition 1.2]) Note that p-henselianity is an elementary property of a valued field (K, v) in the language $\mathcal{L}_{val} = \mathcal{L}_{ring} \cup \{\mathcal{O}\}$, where $\mathcal{O}$ is a unary predicate interpreted as the valuation ring of v ([2, Theorem 1.5]). Furthermore, being p-henselian is an elementary property of K in $\mathcal{L}_{ring}$, provided that K contains a primitive pth root of unity ζ_p in case $\mathrm{char}(K) \neq p$ and that K is not Euclidean in case $p = 2$ ([2, Corollary 2.2]).

Assume that $K \neq K(p)$. We divide the class of p-henselian valuations on K into two subclasses,

$$H_1^p(K) = \{v \text{ } p\text{-henselian on } K | Kv \neq Kv(p)\}$$

and

$$H_2^p(K) = \{v \text{ } p\text{-henselian on } K | Kv = Kv(p)\}.$$

One can show that any valuation $v_2 \in H_2^p(K)$ is *finer* than any $v_1 \in H_1^p(K)$, i.e. $\mathcal{O}_{v_2} \subsetneq \mathcal{O}_{v_1}$, and that any two valuations in $H_1^p(K)$ are comparable. Furthermore, if $H_2^p(K)$ is non-empty, then there exists a unique coarsest valuation v_K^p in $H_2^p(K)$; otherwise there exists a unique finest valuation $v_K^p \in H_1^p(K)$. In either case, v_K^p is called the *canonical p-henselian valuation.* If K is p-henselian then v_K^p is non-trivial.

2. Defining the canonical p-henselian valuation

We want to find a uniform definition of the canonical p-henselian valuation. As p-henselianity is an $\mathcal{L}_{ring}$-elementary property, any sufficiently uniform definition of v_K^p on some field K will also define the canonical p-henselian valuation in any field elementarily equivalent to K. This motivates the following

Definition. Let K be a field, assume that $K \neq K(p)$ and that $\zeta_p \in K$ in case $\mathrm{char}(K) \neq p$. We say that v_K^p is *$\emptyset$-definable as such* if there is a parameter-free $\mathcal{L}_{\mathrm{ring}}$-formula $\phi_p(x)$ such that

$$\phi_p(L) = \mathcal{O}_{v_L^p}$$

holds in any $L \equiv K$.

Recall that a field F is called Euclidean if $[F(2) : F] = 2$. This is an elementary property in $\mathcal{L}_{\mathrm{ring}}$: Every Euclidean field is uniquely ordered, the positive elements being exactly the squares. Note that Euclidean fields are the only fields for which $F(p)$ can be a proper finite extension of F.

We are now in a position to state our main theorem:

Main Theorem (Main Theorem in [1])**.** *Fix a prime p. There exists a parameter-free $\mathcal{L}_{ring}$-formula $\phi_p(x)$ such that for any field K with either* $\mathrm{char}(K) = p$ *or $\zeta_p \in K$ the following are equivalent:*

(1) *ϕ_p defines v_K^p as such.*
(2) *v_K^p is $\emptyset$-definable as such.*
(3) *$p \neq 2$ or Kv_K^p is not Euclidean.*

Note that it may well happen that v_K^p is definable, but not definable as such (see the example given in [1, p. 3]). The proof of the theorem uses Beth's Definability Theorem and a complete 1st-order characterization of v_K^p as described in [1] (see Lemmas 4.1, 4.3 and 4.5 as well as Corollary 4.2).

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Strong Approximation Theorem for Absolutely Integral Varieties over PSC Galois Extensions of Global Fields

Moshe Jarden

(joint work with Wulf-Dieter Geyer and Aharon Razon)

1. Motivation and Goal

We are working over a fixed global field K. It satisfies the classical strong approximation theorem: If $\mathcal{S}$ is a finite set of primes of K, $\mathfrak{q}$ is a prime of K away from $\mathcal{S}$, for each $\mathfrak{p} \in \mathcal{S}$ we are given $a_\mathfrak{p} \in K$, and $\epsilon > 0$, then there exists $x \in K$ such that $|x - a_\mathfrak{p}|_\mathfrak{p} < \epsilon$ for each $\mathfrak{p} \in \mathcal{S}$ and $|x|_\mathfrak{p} \leq 1$ for all primes $\mathfrak{p} \notin \mathcal{S} \cup \{\mathfrak{q}\}$. The field K also satisfies the Hasse-Minkowski local-global principle for quadratic forms with coefficients in K. That principle says that if such a form has a $\mathfrak{p}$-adic non-trivial zero for each prime $\mathfrak{p}$, then it has a non-trivial zero with coordinates in K. Both results fail for arbitrary varieties V, because $V(K)$ is in general too small.

Field Arithmetic shifts the point of view of the Theory of Algebraic Numbers and Arithmetic Geometry from specific varieties over global fields to families of large algebraic extensions of global fields over which every absolutely integral variety V has enough points such that the above results hold for V (and much more).

2. Notation

2.1 *Fields and Galois groups.* We denote the algebraic closure of K by $\tilde{K}$, the separable closure of K in $\tilde{K}$ by K_s, and set $\mathrm{Gal}(K) = \mathrm{Gal}(K_s/K)$ to be the absolute Galois group of K. For each non-negative integer e the group $\mathrm{Gal}(K)^e$ is equipped with its unique normalized Haar measure. Given $\boldsymbol{\sigma} = (\sigma_1, \dots, \sigma_e) \in \mathrm{Gal}(K)^e$, we let $K_s(\boldsymbol{\sigma})$ be the fixed field of $\sigma_1, \dots, \sigma_e$ in K_s and set $K_s[\boldsymbol{\sigma}]$ as the maximal Galois extension of K in $K_s(\boldsymbol{\sigma})$. We shall often use the close "for almost all $\boldsymbol{\sigma} \in \mathrm{Gal}(K)^e$" rather than "for all $\boldsymbol{\sigma}$ in $\mathrm{Gal}(K)^e$ but a set of Haar measure 0".

2.2 *Arithmetical objects.* Let $\mathbf{P}_K$ be the set of all primes of K, $\mathbf{P}_{K,\mathrm{fin}}$ the set of all i.e. non-archimedean primes, and $\mathbf{P}_{K,\mathrm{inf}}$ the set of all archimedean primes.

We fix a proper subset $\mathcal{V}$ of $\mathbf{P}_K$, let $\mathcal{T}$ be a finite subset of $\mathcal{V}$ with $\mathcal{V} \smallsetminus \mathcal{T} \subseteq \mathbf{P}_{K,\mathrm{fin}}$, and $\mathcal{S}$ be a finite subset of $\mathcal{T}$.

For each $\mathfrak{p} \in \mathbf{P}$ we set $K_\mathfrak{p}$ to be a fixed Henselian closure if $\mathfrak{p} \in \mathbb{P}_{K,\mathrm{fin}}$, a fixed real closure at $\mathfrak{p}$ if $\mathfrak{p} \in \mathbf{P}_{K,\mathrm{inf}}$ is real, and as K_s if $\mathfrak{p} \in \mathbf{P}_{K,\mathrm{inf}}$ is complex. Then, we let $|\ |_\mathfrak{p}$ be the $\mathfrak{p}$-adic absolute value of $K_\mathfrak{p}$ and extend $|\ |_\mathfrak{p}$ to an absolute value of $\tilde{K}$ in the unique possible way.

Our central object is the maximal Galois extension

$$K_{\mathrm{tot},\mathcal{S}} = \bigcap_{\mathfrak{p} \in \mathcal{S}} \bigcap_{\tau \in \mathrm{Gal}(K)} K_\mathfrak{p}^\tau$$

of K in which each $\mathfrak{p} \in \mathcal{S}$ totally splits. For each field extension M of K in $\tilde{K}$ and for each subset $\mathcal{U}$ of $\mathcal{V} \cap \mathbf{P}_{\mathrm{fin}}$, we consider the following ring of **$\mathcal{U}$-integers** in M:

$$\mathcal{O}_{M,\mathcal{U}} = \{x \in M \mid |x^\tau|_\mathfrak{p} \leq 1 \text{ for all } \mathfrak{p} \in \mathcal{U} \text{ and all } \tau \in \mathrm{Gal}(K)\}.$$

Finally, for each $\boldsymbol{\sigma} \in \mathrm{Gal}(K)^e$ we let $K_{\mathrm{tot},\mathcal{S}}(\boldsymbol{\sigma}) = K_{\mathrm{tot},\mathcal{S}} \cap K_s(\boldsymbol{\sigma})$ and $K_{\mathrm{tot},\mathcal{S}}[\boldsymbol{\sigma}] = K_{\mathrm{tot},\mathcal{S}} \cap K[\boldsymbol{\sigma}]$. Our main result is:

2.3 **Strong approximation theorem:** *For almost all $\boldsymbol{\sigma} \in \mathrm{Gal}(K)^e$, each extension M of $K_{\mathrm{tot},\mathcal{S}}[\boldsymbol{\sigma}]$ in $K_{\mathrm{tot},\mathcal{S}}$ has the following property:*

- *Let V be an affine absolutely integral variety over K. For each $\mathfrak{p} \in \mathcal{S}$ let $\Omega_\mathfrak{p}$ be a non-empty $\mathfrak{p}$-open subset of $V_{\mathrm{simp}}(K_\mathfrak{p})$. For each $\mathfrak{p} \in \mathcal{T} \smallsetminus \mathcal{S}$ let $\Omega_\mathfrak{p}$ be a non-empty $\mathfrak{p}$-open subset of $V(\tilde{K})$, invariant under the action of $\mathrm{Gal}(K_\mathfrak{p})$. Finally, for each $\mathfrak{p} \in \mathcal{V} \smallsetminus \mathcal{T}$ we assume that $V(\mathcal{O}_{\tilde{K},\mathfrak{p}}) \neq \emptyset$. Then:*

$$V(\mathcal{O}_{M,\mathcal{V}\smallsetminus\mathcal{T}}) \cap \bigcap_{\mathfrak{p}\in\mathcal{T}} \bigcap_{\tau\in\mathrm{Gal}(K)} \Omega_\mathfrak{p}^\tau \neq \emptyset.$$

The local-global principle is a special case of the strong approximation theorem. To express that case we introduce the following notation: For each field L between K and $\tilde{K}$ we define $D_{L,\mathfrak{p}}$ to be the set of all $x \in L$ such that $|x|_\mathfrak{p} \leq 1$ if $\mathfrak{p}$ is finite and $|x|_\mathfrak{p} < 1$ if $\mathfrak{p}$ is infinite.

2.4 **Local-global principle:** *For almost all $\boldsymbol{\sigma} \in \mathrm{Gal}(K)^e$ each extension M of $K_{\mathrm{tot},\mathcal{S}}[\boldsymbol{\sigma}]$ in $K_{\mathrm{tot},\mathcal{S}}$ has the following property:*

Let V be an affine absolutely integral variety over K. Suppose that for each $\mathfrak{p} \in \mathcal{S}$ there exists $\mathbf{z} \in V_{\mathrm{simp}}(D_{K_\mathfrak{p},\mathfrak{p}})$ and for each $\mathfrak{p} \in \mathcal{V} \smallsetminus \mathcal{S}$ there exists $\mathbf{z} \in V(D_{\tilde{K},\mathfrak{p}})$. Then, $\bigcap_{\mathfrak{p}\in\mathcal{V}} \bigcap_{\tau\in\mathrm{Gal}(K)} V(D_{M,\mathfrak{p}}^\tau) \neq \emptyset$.

3. **Special Cases**

The strong approximation theorem encompasses many theorems about large algebraic fields that were proven since the 1970'ies.

Theorem 3.1 ([3]): *For almost all $\boldsymbol{\sigma} \in \mathrm{Gal}(K)^e$, the field $K_s(\boldsymbol{\sigma})$ is* **PAC**. *Thus, every non-void absolutely integral variety over $K_s(\boldsymbol{\sigma})$ has a $K_s(\boldsymbol{\sigma})$-rational point (take $\mathcal{S} = \mathcal{T} = \emptyset$ and $M = K_s(\boldsymbol{\sigma})$).*

Theorem 3.2 ([4]): *For almost all $\boldsymbol{\sigma} \in \mathrm{Gal}(K)^e$, the field $K_s[\boldsymbol{\sigma}]$ is PAC (take $\mathcal{S} = \mathcal{T} = \emptyset$ and $M = K_s[\boldsymbol{\sigma}]$).*

Theorem 3.3 ([7]): *The strong approximation theorem holds for $M = \tilde{\mathbb{Q}}$ (take $e = 0$ and $\mathcal{S} = \emptyset$).*

Theorem 3.4 ([6]): *The strong approximation theorem holds for $M = K_{\mathrm{tot},\mathcal{S}}$ (take $e = 0$).*

Theorem 3.5 ([2]): *The local-global principle holds for $K_{\mathrm{tot},\mathcal{S}}$.*

Theorem 3.6 ([5]): *The strong approximation theorem holds for $K_{\mathrm{tot},\mathcal{S}}(\boldsymbol{\sigma})$ for almost all $\boldsymbol{\sigma} \in \mathrm{Gal}(K)^e$.*

4. **Ingredients of the proof of our strong approximation theorem**

We fix $\mathcal{S}$ and $\mathcal{V}$ and prove that each field M between K and $K_{\mathrm{tot},\mathcal{S}}$ which is "weakly K-stably PSC over $\mathcal{O}_{K,\mathcal{V}}$" satisfies the strong approximation theorem.

The idea is to reduce the theorem to a smooth absolutely integral affine curve C over K. We embed C in an absolutely integral projective scheme $\bar{X}$ which is smooth over a principal ideal domain R with quotient field K. Then we define a birational morphism of $\bar{X}_K$ onto a projective curve Y such that $Y_{\tilde{K}}$ is a "q-curve" for some large prime number q. This gives a "symmetrically stabilizing element" t for the function field F of $\bar{X}$ over K. We use t to produce a global section s of Y such that one of the irreducible components of $\mathrm{div}(s)$ yields the desired M-rational point.

Then, we note that by [1], for almost all $\boldsymbol{\sigma} \in \mathrm{Gal}(K)^e$, each field M between $K_{\mathrm{tot},\mathcal{S}}[\boldsymbol{\sigma}]$ and $K_{\mathrm{tot},\mathcal{S}}$ is "weakly K-stably PSC over $\mathcal{O}_{K,\mathcal{V}}$".

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Abhyankar-type points and o-minimal geometry

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(joint work with E. Hrushovski)

Let val : $K \to \Gamma_\infty$ be a valued field. Here $\Gamma_\infty = \Gamma \cup \{\infty\}$ with Γ an ordered abelian group (no restriction on the rank of Γ is assumed). Let V be an algebraic variety over K. In [2] we introduced the stable completion $\widehat{V}$ of V, which is a model-theoretic version of the Berkovich analytification. Points in $\widehat{V}$ are definable types on V that are dominated by their stable part. $\widehat{V}$ is naturally endowed with a topology coming from the order topology on Γ. A key feature of $\widehat{V}$ is that it is *pro-definable* in the geometric language of [1]. A subset of $\widehat{V}$ is called *iso-definable* resp. *iso-definable* Γ*-internal* if it is pro-definably isomorphic to a definable set, resp. to a definable subset of Γ^n, for some n.

An important role is played by those types in $\widehat{V}$ that satisfy a form of Abhyankar equality, namely those definable types p on V such that there exists a definable map $f : V \to W$ with W defined over the residue field and such that the Zariski dimension of the support of p and of $f_*(p)$ are equal. We call such types *strongly* *stably dominated* and denote the set of those types by $V^\#$. When $\dim(V) \leq 1$, $V^\# = \widehat{V}$, but the inclusion is strict as soon as $\dim(V) \geq 2$ as shown by the next example. An important property of $V^\#$ is that it naturally endowed with the structure of an *ind-definable* space.

Example: Take $K = F((t))$ with F trivially valued and $\mathrm{val}(t) = 1$. Consider a non-algebraic power series $\varphi(x) = \sum_{i\geq 0} a_i x^i$, with $a_i \in F$, and for any non negative integer n, set $\varphi_n(x) = \sum_{0\leq i\leq n} a_i x^i$. For $\gamma \in \Gamma_\infty$ consider the complete type p_γ in (x, y) generated by the generic type of the closed ball $\mathrm{val}(x) \geq 1$ and the formulas

$$\mathrm{val}(y - \varphi_n(x))) \geq \min(n + 1, \gamma).$$

One can check that p_γ belongs to $(\mathbb{A}^2)^\#$ if and only γ is finite, i.e. smaller than n_0 for some integer n_0. Furthermore, the mapping $g : \Gamma_\infty \to \widehat{\mathbb{A}^2}$ sending γ to p_γ is continuous and pro-definable but its image is **not** iso-definable in $\widehat{\mathbb{A}^2}$.

By a *generalized interval* we mean a definable set which is obtained by glueing end-to-end a finite number of intervals in Γ_∞. We say an iso-definable Γ-internal subset $\widehat{V}$ is *topologically* Γ-internal if it is pro-definably homeomorphic to a definable subset of Γ_∞^n, for some n.

Call a subset $\Upsilon \subset \widehat{V}$ a *skeleton* if Υ is topologically Γ-internal, is contained in $V^\#$ and for any irreducible component V_i of V, $\Upsilon \cap \widehat{V_i}$ if of o-minimal dimension $\dim(V_i)$ everywhere. The main result in [2] is the following theorem:

Theorem 1. *Let V be a quasi-projective variety over a valued field. There exists a continuous prodefinable map $h : I \times \widehat{V} \to \widehat{V}$, with I a generalized interval, which is a strong deformation retraction onto a subset $\Upsilon \subset \widehat{V}$ with Υ a skeleton. Furthermore, given a finite number of definable functions $\alpha_i : V \to \Gamma_\infty$ one may require h to respect the α_i.*

A first connection between $V^\#$ and o-minimal geometry is provided by the following proposition:

Proposition 1. *Let V be a variety of dimension n, and let $W \subset \widehat{V}$ be iso-definable Γ-internal. If W is of pure o-minimal dimension n, then $W \subset V^\#$.*

In view of the following rigidity statement it explains the importance of the space $V^\#$ in the proof of Theorem 1:

Proposition 2 (rigidity). *Let V be a variety of dimension n, and let $W \subset \widehat{V}$ be iso-definable Γ-internal. If W is of pure o-minimal dimension n, and $\alpha : \widehat{V} \to \Gamma_\infty$ is pro-definable and finite-to-one on W, then any $h : I \times \widehat{V} \to \widehat{V}$, continuous pro-definable with I a generalized interval respecting α, fixes pointwise W.*

We ended the talk by sketching the proof of the following recent result which is included in the latest versions of [2].

Theorem 2. *Let V be a quasi-projective variety over a valued field. Then $V^\#$ is exactly the union of all skeleta inside $\widehat{V}$.*

Sketch of proof: One has to prove that any point p in $V^\#$ belongs to a skeleton. By increasing the basis, one reduces to the case p is a realized type (recall $V^\#$ is ind-definable). The proof then proceeds by descending induction on the o-minimal dimension, the case of maximal dimension being consequence of Proposition 2.

A striking consequence of Theorem 2 is the following topological characterisation of the points satisfying the equality in the Abhyankar inequality:

Corollary. *The set $V^\#$ is exactly the locus in $\widehat{V}$ of points having local o-minimal dimension the Zariski dimension of V; e.g., if V is of pure dimension n, $V^\#$ is the locus of points of o-minimal dimension n, namely those contained in an iso-definable Γ-internal set of o-minimal dimension n.*

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A p-adic Version of Riemann's Vanishing Theorem

Werner Lütkebohmert

Let X be a smooth projective curve of genus $g \geq 1$ over a field K. Fix a rational point $x_0 \in X$ and let $j : X \to \text{Jac}(X)$, $x \longmapsto [x - x_0]$, be the canonical morphism which maps a point $x \in X$ to the divisor class $[x - x_0]$ in the Jacobian variety $\text{J}ac(X)$ of X. Let $\Theta := j^{(g-1)}\big(X^{(g-1)}\big) \subset \text{J}ac(X)$ be the image of the symmetric product. Usually Θ is called the theta divisor.
The classical theorem of Riemann states that the first Chern class of the divisor Θ is the intersection form on the cycle group $H_1(X, \mathbb{Z})$. Let $(a_1, \ldots, a_g, b_1, \ldots, b_g)$ be a standard basis von $H_1(X, \mathbb{Z})$; i.e., the number of intersection points is given by $\#(a_\alpha, a_\beta) = \#(b_\alpha, b_\beta) = 0$ and $\#(a_\alpha, b_\beta) = \delta_{\alpha,\beta}$. Let $(\omega_1, \ldots, \omega_g)$ be a basis of the vector space $H^0(X, \Omega^1_X)$ of global differential forms satisfying $\int_{a_\alpha} \omega_\beta = \delta_{\alpha,\beta}$. Then, the Jacobian variety $\text{Jac}(X)$ of X can be represented in the form

$$J := \text{Jac}(X) = H^0(X, \Omega^1_X)'/H_1(X, \mathbb{Z}) = \mathbb{C}^g/M \ ,$$

where $H^0(X, \Omega^1_X)'$ is the dual of $H^0(X, \Omega^1_X)$ and where $M = \mathbb{Z}^g \oplus Z \cdot \mathbb{Z}^g$ is a lattice, $Z = Z^t \in M(g, \mathbb{C})$ a symmetric matrix with positive definite $Y := \text{Im}(Z) > 0$. In principle Riemann's vanishing theorem is proved by showing that a translation of Θ is the vanishing locus of Riemann's theta function

$$\theta(z) = \sum_{n \in \mathbb{Z}^g} e^{\pi\imath\langle n, Zn\rangle} \cdot e^{2\pi\imath\langle n, z\rangle} \ .$$

The main topic of my talk is the analogous version of Riemann's theorem over a non-archimedean field K; i.e., a complete rank-1-valued field K. For simplicity, we also suppose that K is algebraically closed. In this case, the structure of the Jacobian variety is much more complicated. Due to the phenomena of good and multiplicative reduction which in general occur in a twisted way, the period relations reflect this behavior. A crucial role is played by the semi-stable reduction $\tilde{X}$ of X and, hence, by the cycle group $H := H_1(\tilde{X}, \mathbb{Z})$ of $\tilde{X}$. The generalized Jacobian $\tilde{J} := \mathrm{Jac}(\tilde{X})$ of $\tilde{X}$ is a torus extension $\tilde{T} \to \tilde{J} \to \tilde{B}$ where $\tilde{B}$ is the Jacobian of the normalization $\tilde{X}'$ of $\tilde{X}$. This extension is associated to a group homomorphism $\tilde{\phi} : H \to \tilde{B}'$ from H to the dual variety $\tilde{B}'$ of $\tilde{B}$ and, hence, H is canonically isomorphic to the group $\mathbb{X}(\tilde{T})$ of characters of $\tilde{T}$. It is known that $\tilde{\phi}$ is induced by the theta divisor of $\tilde{B}$. Over a non-archimedean field K, the Jacobian variety of a smooth projective curve X is a quotient

$$\mathrm{Jac}(X) = \widehat{J}/M$$

of a torus extension $\widehat{J}$ of an abelian variety B by a lattice; the rank of the lattice is equal to the rank of the torus T. The torus extension $\widehat{J}$ is determined by a group homomorphism $\phi : \mathbb{X}(T) \to B'$. The groups of characters $\mathbb{X}(T) = X(\tilde{T})$ are canonically identified. Thus, one obtains a canonical map $h' : H \to M' := \mathbb{X}(T)$. Moreover, there is a universal (rigid analytic) covering $\widehat{X} \to X$ such that the maximal abelian quotient of the deck transformation group Γ is H. The canonical map $j : X \to \mathrm{Jac}(X)$ has a lifting $\widehat{j} : \widehat{X} \to \widehat{J}$ after fixing a point $\widehat{x}_0$ above x_0. Thus, one obtains a canonical isomorphism $h : H \to M$ by sending an element $\gamma \in \Gamma$ to $\widehat{j} \circ \gamma(\widehat{x}_0)$. So, there are two canonical isomorphisms $h : H \to M$ and $h' : H \to M' := \mathbb{X}(T)$ and, hence, a canonical isomorphism $\lambda_c := h' \circ h : M \to M'$. In particular, this gives rise to a canonical pairing

$$H \times H \longrightarrow P_{B \times B'} \,;\, (\alpha, \beta) \longmapsto \langle h(\alpha), h'(\beta) \rangle := h'(\beta)\big(h(\alpha)\big) \;.$$

Let $L := L(\Theta)$ be the line bundle associated to the theta divisor; i.e., the sheaf of sections of L/J is the invertible sheaf $\mathcal{O}_J(-\Theta)$. Using M-linearizations of cubical structures, it can be shown that $L(\Theta)$ gives rise to a pair (N, λ_Θ) where N is a line bundle on B and $\lambda_\Theta : M \to M'$ is a homomorphism. The p-adic version of Riemann's vanishing theorem is the following result:

Theorem. *Keep the situation of above. Then the following holds:*
1. $\lambda_\Theta = \lambda_c$
2. The line bundle N induces a canonical map $\varphi_N : B \to B' \,;\, b \longmapsto \tau_b^* N \otimes N^{-1}$ from B to its dual B'. The map φ_N reduces to $\varphi_{\tilde{\Theta}} : \tilde{B} \to \tilde{B}'$ where $\tilde{\Theta}$ is the theta divisor of $\tilde{B} = \prod_{i=1}^{n} \mathrm{Jac}(\tilde{X}_i)$. Hereby $\tilde{X}_1, \ldots, \tilde{X}_n$ are the irreducible components of the normalization $\tilde{X}'$ of $\tilde{X}$.

As a corollary, one sees that λ_c is symmetric and positive definite in a certain sense. In my talk, a survey on Mumford curves was given as an introduction to curves over non-archimedean fields. In this case the abelian part B is always trivial and, hence, the universal covering $\widehat{J}$ of $\mathrm{Jac}(X)$ is an affine torus T and M is a multiplicative lattice in T of full rank.

For the prerequisites of this talk see to [1]. The main result of this talk will be published as a section in the forthcoming book [3]. It generalizes results of Drinfeld and Manin [2]

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A problem of Zilber about residue rings of models of arithmetic

Angus Macintyre

1. Boris Zilber recently wrote a paper *The semantics of the canonical commutation relations*, connecting model theory and theoretical physics, and involving, inter alia, ultraproducts of Lie algebras. In such structures nonstandard integers naturally occur, and this led Zilber to ask me the following question:

Suppose $\mathcal{M}$ is a nonstandard model of true arithmetic, and μ is an infinite element of $\mathcal{M}$ with $\mu \equiv 1$ mod n for each standard integer n. Can one interpret arithmetic in the ring $\mathcal{M}/\mu\mathcal{M}$?

I chose to generalise this problem to allow $\mathcal{M}$ to be any nonstandard model of first-order Peano arithmetic (PA) and μ to be any nonstandard element of $\mathcal{M}$.

The resulting research is joint with Paola d'Aquino. We show that no such $\mathcal{M}/\mu\mathcal{M}$ interprets arithmetic, via an analysis of the definable relations in such structures. However, some of these structures (but not all) are undecidable.

2. There are several different methods involved:

(a) The method of Rabin, Ryll-Nardezwski and Tennenbaum, from around 1960, nowadays seen as depending on the weak recursive saturation of arbitrary nonstandard models of PA;
(b) Methods deriving form Ax's fundamental work on the theory of finite fields (late 1960's);
(c) Methods deriving from Ax-Kochen and Ershov, on model theory of Henselian fields (mid 1960's);
(d) Internalization to nonstandard products of the classic Feferman-Vaught analysis (late 1950's) of the elementary theory of products.

3. In $\mathcal{M}$ the notions of irreducible, prime and maximal elements coincide. A standard prime is just a classical prime p, and then

$$\mathcal{M}/p\mathcal{M} \cong \mathbb{Z}/p\mathbb{Z} = \mathbb{F}_p.$$

In 1978 Macintyre showed, using Bombieri's elementary proof of the Riemann Hypothesis for curves (based on Stepanov's method), that if μ is a nonstandard prime then $\mathcal{M}/\mu\mathcal{M}$ is a pseudofinite field of characteristic 0. Indeed, every complete theory of pseudofinite fields of characteristic 0 occurs as the theory of $\mathcal{M}/\mu\mathcal{M}$, for some μ (but not in general for all nonstandard $\mathcal{M}$, though for some complete extensions all $\mathcal{M}$ will work).

In all nonstandard $\mathcal{M}$ there is a prime μ satisfying Zilber's congruence conditions (this is proved by doing Dirichlet's Theorem in PA, and using the recursive saturation result). Moreover, there is always a a prime μ satisfying Zilber's congruence conditions such that $\mathcal{M}/\mu\mathcal{M}$ contains all algebraic numbers (and there is a unique theory of such a pseudofinite field of characteristic 0 which is furthermore decidable). However, contrary to a careless remark I made in my talk at Oberwolfach, there are many complete theories of $\mathcal{M}/\mu\mathcal{M}$ where μ satisfies Zilber's congruence conditions. I am grateful to Moshe Jarden for detecting my error, and showing that the possible theories of such $\mathcal{M}/\mu\mathcal{M}$ are exactly the theories whose absolute numbers are of the form $Fix(\sigma)$, where σ is an element of the absolute Galois group of the maximal abelian extension of $\mathbb{Q}$.

In every nonstandard $\mathcal{M}$ there is a prime μ with $\mathcal{M}/\mu\mathcal{M}$ undecidable. This is proved using Dirichlet and recursive inseparability. In fact, using Jarden's technique, and recursive inseparability, one can show that in every nonstandard model there are undecidable examples with μ satisfying Zilber's congruence conditions. For Zilber's problem, mere undecidability is not significant. Various model-theoretic analyses of pseudofinite fields, but particularly that of Chatzidakis, van den Dries and Macintyre(1990), analyzing counting and measure in pseudofinite fields, leading to Hrushovski's work on the simplicity of pseudofinite fields, show that there is no interpretation even of bounded arithmetic, in any $\mathcal{M}/\mu\mathcal{M}$ with μ prime.

However, the situation for composite μ is much trickier.

4. Prime Powers. Now μ is p^k, where p is a prime and k is a positive element of $\mathcal{M}$. If both p and k are standard,

$$\mathcal{M}/\mu\mathcal{M} \cong \mathbb{Z}/p^k\mathbb{Z}.$$

In general, $\mathcal{M}/\mu\mathcal{M}$ is now a local ring, with residue field $\mathcal{M}/p\mathcal{M}$, and the latter is understood by our previous work. We show that $\mathcal{M}/\mu\mathcal{M}$ is a henselian local ring.

Using ideas from our 2013 paper on density in henselizations (for models of arithmetic) we show that for each $\mathcal{M}/\mu\mathcal{M}$ as above there is a henselian valued field K, of characteristic 0 and unramified, with residue field $\mathcal{M}/p\mathcal{M}$ and value group a $\mathbb{Z}$-group,and an element y so that

$$\mathcal{M}/\mu\mathcal{M} \cong \frac{\{x : \nu(x) \geq 0\}}{\{x : \nu(x) \geq \nu(y)\}}$$

Thus all $\mathcal{M}/\mu\mathcal{M}$ are interpretable in some henselian valued field of characteristic 0, with value group a $\mathbb{Z}$-group, and residue field either a standard $\mathbb{F}_p$ or a pseudofinite field of characteristic 0.

Thus, firstly:

Theorem 1. *If p is standard, $\mathcal{M}/\mu\mathcal{M}$ is NIP and so does not interpret arithmetic.*

Note, however, that even for p standard $\mathcal{M}/\mu\mathcal{M}$ can be undecidable. This follows from an important general observation. Any nonnegative $k \in \mathcal{M}$ determines an element $\eta(k) \in \hat{\mathbb{Z}}$ via the remainders of k modulo each standard integer n. It can be shown that the theory of $\mathcal{M}/\mu\mathcal{M}$ codes $\eta(k)$ and conversely. If $\eta(k)$ is not computable, then $\mathcal{M}/\mu\mathcal{M}$ is undecidable.

If p is not standard, $\mathcal{M}/p\mathcal{M}$ is not NIP ,and so $\mathcal{M}/\mu\mathcal{M}$ is not NIP. However, by an important recent result of Chernikov, Kaplan and Simon, in this case the theory of any henselian field with residue field pseudofinite of characteristic 0, and value group a $\mathbb{Z}$-group, is NTP_2. It follows by the preceding that $\mathcal{M}/\mu\mathcal{M}$ is NTP_2, and so does not interpret arithmetic.

Note too that any henselian field of characteristic 0, unramified,and with residue field a prime finite field or pseudofinite of characteristic 0, is elementarily equivalent to an ultraproduct of standard $p-adic$ fields, and it follows easily that any residue ring

$$\mathcal{M}/\mu\mathcal{M} \cong \frac{\{x : \nu(x) \geq 0\}}{\{x : \nu(x) \geq \nu(y)\}}$$

of such a field is elementarily equivalent to some $\mathcal{M}/\mu\mathcal{M}$ .In particular, the theory of the class of *all* $\mathcal{M}/\mu\mathcal{M}$ is decidable.

5. General μ. Now we have to appeal to the yoga of nonstandard models. μ may have infinitely many prime power divisors,but still it is a Δ_0 - product of prime powers, essentially uniquely. In a similar spirit, $\mathcal{M}/\mu\mathcal{M}$ is canonically a Δ_0 - product of various $\mathcal{M}/\mu_j\mathcal{M}$, with the μ_j prime powers. It turns out that one can modify the classical Feferman-Vaught-Mostowski model theory of products to give a thorough analysis of definitions, enough to show noninterpretability of arithmetic. There is a close link to work of Derakhshan- Macintyre on the adeles. Note that neither the adeles nor those $\mathcal{M}/\mu\mathcal{M}$ are NTP_2.

6. Pseudofinite Model Theory. All this connects to the thriving subject of pseudofinite model theory. Scanlon pointed out that nonnterpretability should be deducible simply from pseudofiniteness,and it is clear that all the $\mathcal{M}/\mu\mathcal{M}$ satisfy the standard "injective = surjective " schemata. But our goal is rather to analyze definability in the $\mathcal{M}/\mu\mathcal{M}$, and find axioms with no reference to occurrence in models of PA, the prototype being the analysis of pseudofinite fields of characteristic 0.

Henselian elements

Josnei Novacoski
(joint work with Franz-Viktor Kuhlmann)

For an extension $A \subseteq B$ of rings with unity, an element $b \in B$ is called a **henselian element** over A if there exists a polynomial $h(X) \in A[X]$ (not necessarily monic) such that $h(b) = 0$ and $h'(b)$ is a unit of B. A valued function field $(F|K, v)$ is said to admit **local uniformization** if for every finite set $Z \subseteq \mathcal{O}_F$ there exists an affine model V of $(F|K,v)$ such that the center $\mathfrak{p}$ of v on V is a regular point and $Z \subseteq \mathcal{O}_{V,\mathfrak{p}}$.

Theorem. *Let $(F|K,v)$ be a valued function field such that v is trivial on K. Assume that there exists a transcendence basis T of $F|K$ such that $(K(T)|K,v)$ admits local uniformization and F lies in the absolute inertia field of $K(T)$. Then $(F|K,v)$ admits local uniformization.*

In my talk I discussed how henselian elements can be used to prove this theorem and how it was used by Knaf and Kuhlmann to prove important results on local uniformization.

Take a valued field extension $(F|L,v)$ such that the field F lies in the absolute inertia field (L^i,v) of L and $[F:L]<\infty$ (for short we will call this **a finite valued inertial extension**). In order to prove the theorem above we need to answer positively the following question: is $\mathcal{O}_F$ generated as an $\mathcal{O}_L$-algebra by henselian elements? This question was posted by Kuhlmann in the Valuation Theory Home Page (see [3]) and answered by Roquette (see [4]) and van den Dries (see [5]) in the same web page. I my talk I summarized those answers and discussed how Kuhlmann and I generalize them in [2].

An interesting question is whether $\mathcal{O}_F$ is generated as an $\mathcal{O}_L$-algebra by finitely many henselian elements. We showed that this condition is satisfied if, for instance, the rank of (L,ν) is well-ordered. Since this condition is satisfied for a wide range of valued fields, one is led to believe that this is always true. However, in Theorem 1.5 of [2] we prove that there exists a finite inertial extension $(F|L,v)$ for which $\mathcal{O}_F$ is not a finitely generated $\mathcal{O}_L$-algebra.

I also presented two open problems related to the talk. The first problem is how we can characterize finite inertial extensions $(F|L,v)$ for which $\mathcal{O}_F$ is a finitely generated $\mathcal{O}_L$-algebra. Consider a valued field (L,v) with the following properties:

$$\begin{cases} (L,v) \text{ is not henselian;} \\ \text{the chain of prime ideals of } \mathcal{O}_L \text{ does not admit minimum non-zero element;} \\ \text{the residue field } (Lv_{\mathfrak{p}},\overline{v}_{\mathfrak{p}}) \text{ is henselian for every nonzero prime ideal } \mathfrak{p} \text{ of } \mathcal{O}_L. \end{cases}$$

Theorem 1.5 of [2] shows that if (L,v) satisfies the conditions above, then (L,v) admits a finite inertial extension (F,v) such that $\mathcal{O}_F$ is not generated over $\mathcal{O}_L$ by finitely many (henselian) elements. One could ask whether this characterizes such valued fields.

The second question asks when is $\mathcal{O}_F$ an "essentially finitely generated" extension of $\mathcal{O}_L$. More explicitly, for a finite valued field extension $(F|L,v)$ (not necessarily inertial), does there exist $a_1,\ldots,a_r\in\mathcal{O}_F$ (not necessarily henselian over $\mathcal{O}_L$) such that $\mathcal{O}_F=\mathcal{O}_L[a_1,\ldots,a_r]_{\mathfrak{p}}$ where $\mathfrak{p}=\mathcal{O}_L[a_1,\ldots,a_r]\cap\mathfrak{m}_F$?

This question was presented to me by Hagen Knaf together with a partial answer. Consider all the extensions $v=v_1,\ldots,v_r$ of $v|_L$ to F. For each i,

$1 \le i \le r$, we set

$$\begin{cases} e_i := (v_iF : v_iL), \\ \epsilon_i := |\{\alpha \in v_iF^{\ge 0} \mid \alpha < v_iL^{>0}\}|; \text{ and} \\ d_i := \dfrac{[F : L]}{(v_iF : v_iL) \cdot [Fv_i : Lv_i]}. \end{cases}$$

A necessary condition for $\mathcal{O}_F$ to be essentially finitely generated over $\mathcal{O}_L$ is that $e_1 = \epsilon_1$ and $d_1 = 1$.

This answer can be improved for some cases. The following result can be found in [1].

Theorem. *Take a finite valued field extension $(F|L, v)$. Let $v = v_1, \ldots, v_r$ be all the extensions of $v|_L$ to F. Then the integral closure D of $\mathcal{O}_L$ in F is finitely generated over $\mathcal{O}_L$ if and only if*

$$e_i = \epsilon_i \text{ and } d_i = 1, \text{ for every } i, 1 \le i \le r. \tag{1}$$

Since $\mathcal{O}_F = D_{\mathfrak{m}_F \cap D}$ the theorem above says that if condition (1) is satisfied, then $\mathcal{O}_F$ is essentially finitely generated over $\mathcal{O}_L$. This means that if (L, v) is henselian (and hence $r = 1$) or if $F|L$ is a normal extension (in which case $\epsilon_1 = \epsilon_i$, $e_1 = e_i$ and $d_1 = d_i$ for each i, $1 \le i \le r$), then the condition given by Knaf is also sufficient.

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A generalization of the Oort conjecture: Reduction to a finite computation

Andrew Obus
(joint work with Stefan Wewers)

The *lifting problem for branched covers of curves* asks whether a branched Galois cover of smooth projective curves in characteristic p lifts to characteristic zero. While this question appears to be global, it has in fact been shown to be strictly local. That is, it is sufficient to show that every germ of the cover lifts to characteristic zero. This reduces the lifting problem for branched covers of curves to the following *local lifting problem:*

Problem 1 (Local lifting problem). Let k be an algebraically closed field of characteristic p, and let $k[[z]]/k[[t]]$ be a G-Galois extension. Does there exist a complete discrete valuation ring (R, π) of characteristic 0 with residue field k and a G-Galois extension $R[[Z]]/R[[T]]$ whose reduction modulo π is $k[[z]]/k[[t]]$? If so, can we say anything about R?

1. The Oort conjecture and its proof

The *Oort conjecture* (now a theorem of Obus-Wewers and Pop) states that the local lifting problem always has a solution when G is cyclic. If $p^3 \nmid |G|$, it is further known that one can take $R = W(k)(\zeta_{|G|})$. For other cyclic groups G, this is expected to hold, but is an open question.

For the proof of the Oort conjecture, one first easily reduces to the case $G \cong \mathbb{Z}/p^n$, for some $n \geq 1$. The proof is divided into two parts. In order to state these parts more precisely, we introduce the concept (due to Pop) of *essential ramification.* A cyclic $\mathbb{Z}/p^n$-extension of $k[[t]]$ has n jumps $(u_1, \ldots, u_n)$ in the higher ramification filtration for the upper numbering, the so-called *upper jumps.* By the Hasse-Arf theorem, these numbers are all integers. In fact, one can show that $u_{i+1} \geq pu_i$ for all $i < n$. A $\mathbb{Z}/p^n$-extension of $k[[t]]$ with upper jumps $(u_1, \ldots, u_n)$ is said to have *no essential ramification* if, for every $1 \leq i \leq n-1$, we have $u_{i+1} < pu_i + p$.

Theorem 2 (Obus-Wewers, [OW12])**.** If a $\mathbb{Z}/p^n$-extension of $k[[t]]$ has no essential ramification, then it can be lifted to characteristic zero.

In Pop's paper [Pop12], it is shown that any $\mathbb{Z}/p^n$-extension of $k[[t]]$ has an equicharacteristic deformation to an extension of $k[[t, s]]$, whose generic fiber has no essential ramification (in this case, since the generic fiber is an extension of $k[[t, s]][s^{-1}]$, "no essential ramification" means no essential ramification over each ramified maximal ideal). Pop is then able to use this deformation, together with Theorem 2, to prove

Theorem 3 (Pop, [Pop12])**.** The Oort conjecture holds.

One way of going about this proof is to construct a $\mathbb{Z}/p^n$-cover of $\mathbb{P}^1_k$ totally ramified at one point, where the germ above the branch point is the original extension of $k[[t]]$ (this technique is due to Katz-Gabber-Harbater). Then, using the equicharacteristic deformation and Theorem 2, it can be shown that this cover lifts over a rank 2, characteristic zero valuation ring $\mathcal{R}$ with residue field k. An application of Robinson's theorem then shows that the lifting can be accomplished over a finite extension $R/W(k)$. Taking the relevant germ of this cover gives a lift of the original extension over R.

2. The generalization

If G is a group for which the local lifting problem has a solution for all G-extensions, then G is known as a *local Oort group.* By Theorem 3, all cyclic groups are local Oort groups. Determining the list of local Oort groups is a difficult open problem.

Work of Chinburg-Guralnick-Harbater and Brewis-Wewers has shown that the only possible Local Oort groups are the cyclic groups, the dihedral groups of order $2p^n$ for some n, and the alternating group A_4 for $p = 2$. It has been asserted by Bouw that A_4 is, in fact, an Oort group. We conjecture:

Conjecture 4. The dihedral groups D_{p^n} of order $2p^n$ for *odd* p are Oort groups.

More generally, for metacyclic groups of the form $G = \mathbb{Z}/p^n \rtimes \mathbb{Z}/m$, where $p \nmid m$, there is a known obstruction to lifting, called the *Bertin obstruction*, which involves the higher ramification groups of the given extension. If the upper jumps of the $\mathbb{Z}/p^n$-subextension are $(u_1, \ldots, u_n)$, then the Bertin obstruction vanishes if and only if G is abelian or center-free, and if each $u_i \equiv -1 \pmod{m}$ when G is center-free. This obstruction vanishes when $m = 2$, as all upper jumps must be odd. The following conjecture generalizes Conjecture 4:

Conjecture 5. For groups of the form $G := \mathbb{Z}/p^n \rtimes \mathbb{Z}/m$, where $p \nmid m$, the Bertin obstruction is the only obstruction to the local lifting problem.

Remark 6. In the previous literature, to the best of our knowledge, there is not a single non-cyclic $\mathbb{Z}/p^n \rtimes \mathbb{Z}/m$-extension with $p \nmid m$ and vanishing Bertin obstruction which is known either to lift or not to lift!

Our idea for proving Conjecture 5 parallels the proof of the Oort conjecture. Suppose G as in Conjecture 5 is center-free, and that the G-extension $k[[z]]/k[[t]]$ has vanishing Bertin obstruction. Then, the upper jumps of the $\mathbb{Z}/p^n$-subextension $k[[z]]/k[[x]]$ are $(u_1, \ldots, u_n)$, with all $u_i \equiv -1 \pmod{m}$.

Definition 7. A $G := \mathbb{Z}/p^n \rtimes \mathbb{Z}/m$-extension of $k[[z]]/k[[t]]$ whose $\mathbb{Z}/p^n$-subextension has upper jumps $(u_1, \ldots, u_n)$ is said to have *no essential ramification* if $u_1 < mp$, and, for every $1 \leq i \leq n-1$, we have $u_{i+1} < pu_i + mp$.

As in the case of the Oort conjecture, one can make an equicharacteristic deformation of $k[[z]]/k[[t]]$ such that the generic fiber has no essential ramification. Also as in that case, one can reduce Conjecture 5 to the case of no essential ramification. Note that, for given values of p and m, there are only finitely many possible ramification jump sequences having no essential ramification.

We state one more definition, and then our main theorem:

Definition 8. Let $(p, m, \tilde{u}, N) \in \mathbb{N}^4$ be such that p is prime and $m|N$. Let u be the prime-to-p part of $\tilde{u}$. Then $(p, m, \tilde{u}, N)$ *satisfies the differential data criterion* (with respect to an algebraically closed field k of characteristic p) if there exists a polynomial $f(t) \in k[t]$ of degree exactly N, all of whose terms have degree divisible by m, such that the meromorphic differential form

$$\omega := \frac{dt}{f(t)t^{\tilde{u}+1}} \in \Omega^1_{k(t)/k}$$

satisfies

$$\mathcal{C}(\omega) = \omega + \frac{u}{t^{\tilde{u}+1}} dt.$$

Here $\mathcal{C}$ is the Cartier operator on differential forms. Furthermore, $(p, m, \tilde{u}, N)$ satisfies the *isolated differential data criterion* if the solution $f(t)$ is in some sense not infinitesimally deformable (this amounts to the non-vanishing of a certain determinant in the coefficients of f).

Theorem 9. *Suppose $k[[z]]/k[[t]]$ is a G-extension with no essential ramification as in Definition 7. Suppose that, for all $1 < i \leq n$, the quadruple (p, m, u_{i-1}, N) satisfies the isolated differential data criterion, where $N = (p-1)u_{i-1}$ if $p|u_{i-1}$, and $N = (p-1)u_{i-1} - m$ otherwise. Then the extension lifts to characteristic zero.*

Remark 10. Computational evidence suggests that the isolated differential data criterion can be realized by a polynomial $f(t)$ with coefficients in $\mathbb{F}_p$. If this is true, then to solve the local lifting problem for a particular group $\mathbb{Z}/p^n \rtimes \mathbb{Z}/m$, one need only do finitely many searches (corresponding to the possible sequences of upper jumps) over finite sets of polynomials in order to verify this criterion. Using this method, we can prove the following corollary.

Corollary 11. *D_9 is a local Oort group.*

More efficient algorithms for realizing the isolated differential data criterion should make it feasible to prove many more cases of Conjecture 5, and hopefully shed light on a general solution.

3. Other potential local Oort groups

The case of dihedral groups of 2-power order appears to be much more complicated. However, Pagot has shown that $\mathbb{Z}/2 \times \mathbb{Z}/2$ is an Oort group, and Brewis has exhibited an example of a D_4-extension in characteristic 2 that lifts to characteristic zero.

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Separably closed fields viewed as contractive valued modules

Françoise Point

(joint work with Luc Bélair)

We consider the theory of certain valued fields endowed with a distinguished endomorphism, in weaker (valued) modules formalisms. In order to do so, we consider them as modules over a skew polynomial ring ([3]), endowed with a chain of subgroups. We follow the approach of T. Rowher ([7]) who considered the field of Laurent series over $\mathbb{F}_p$ (or its algebraic closure $\tilde{\mathbb{F}}_p$) and proved model-completeness of its theory of modules over the ring of the additive p-polynomials ([6]), enriched by the chain of subgroups consisting of the elements of valuation bigger than z, $z \in \mathbb{Z}$.

Let (K, v, σ) be a valued field endowed with a distinguish endomorphism σ; let K^σ the subfield of K consisting of the image of K under σ. We may consider K as a K^σ vector-space and we fix a basis $\mathcal{C}$. We assume that $\mathcal{C}$ is finite ($\mathcal{C} = \{c_j; 0 \leq j < n\}$). Let A be the skew polynomial ring $K[t;\sigma]$, where the commutation law is given by $k.t = t.k^\sigma$ with $k \in K$. We consider the A-modules M which have a direct sum decomposition as follows : $M = \oplus_{j=0}^{n-1} M.t.c_j$. We add new unary function symbols λ_j, $j \in n = \{0, \cdots, n-1\}$ to the usual language of A-modules in order to ensure the existence of such decomposition. These functions are additive and so we stay in the setting of abelian structures which enables us to use the Baur-Monk result of positive-primitive quantifier elimination.

Note that the theory of separably closed valued fields of characteristic p and imperfection degree e, $e \in \mathbb{N} \cup \{+\infty\}$, admits quantifier elimination in the language of fields augmented by these λ-functions and a predicate for the valuation ring (see the theses of F. Delon (1983) and J. Hong ([4])). The model theory of separably closed fields with a distinguished endomorphism has been developed by C. Chatzidakis and E. Hrushovski (2004).

Then we consider the additional structure induced by a valuation on K. Let $\mathcal{O}_K$ be the valuation ring of K, and $(\Gamma, +, \leq)$ its value group. We will assume that (K, v, σ) is a valued field endowed with a *contractive* endomorphism σ i.e. $v(\sigma(x)) > v(x)$ for all $v(x) > 0$ with the compatibility between v and σ: $v(a) \leq v(b) \to v(\sigma(a)) \leq v(\sigma(b))$. In particular $\sigma(\mathcal{O}_K) \subset \mathcal{O}_K$ and it induces an endomorphism on Γ defined by $\sigma_\Gamma(v(a)) := v(\sigma(a))$. (In previous works we did consider the case where σ was an isometry and the same framework allowed us to deal also with a continuous derivation [1]. Valued vector spaces have been well-studied (by for instance P. Conrad (1953), I. Fleischer (1958), and L. van den Dries (1981)) and more recently the model theory of valued modules was developed, for instance, in the theses of T. Rohwer ([7]) and G. Onay ([5]).)

One considers a class of two-sorted structures (M, w, Δ) where M is an A-module and w a map to an ordered set $(\Delta \cup \{+\infty\}, \leq)$ endowed with an action of Γ and an action of a map τ defined by $\tau(w(m)) := w(m.t)$. In order to express that the action of τ on Δ comes from a contractive σ, we add a distinguished element 0_Δ on Δ, and impose conditions like: if $\delta > 0_\Delta$, then $\tau(\delta) > \delta$. We call such class, a class of contractive valued A-modules and we look for properties of separably closed valued fields which can be expressed in that weaker formalism.

For instance, letting $\mathcal{I} := \{\sum_{i=0}^d t^i a_i \in A : \; min_i \, v(a_i) = 0\}$, O. Ore proved that if K is separably closed, then any element of $\mathcal{I}$ factors into linear terms belonging to $\mathcal{I}$ ([6]). Together with the fact that the value group of a separably closed field is p-divisible (even divisible), this enables us to express a divisibility property in our class of contractive modules. We say that the ordered set Δ: $(\Delta, \leq, 0_\Delta, \tau, +\gamma; \gamma \in \Gamma)$ is *ordered linearly closed* (*o.l.c.*) if given any finite subset $\{\gamma_i \in \Gamma; 0 \leq i \leq d\}$, for any $\delta \in \Delta$, there exists $\mu \in \Delta$ such that $\delta = \min_{0 \leq i \leq d}\{\tau^i(\mu) + \gamma_i\}$. Given $q(t) = \sum_{i=0}^d t^i a_i \in A$, given δ, we denote by $\Upsilon(q(t), \delta)$ the element μ such that

$\delta = \min\{\tau^i(\mu) + \gamma_i : \ 0 \le i \le d\}$ where $\gamma_i := v(a_i)$, $0 \le i \le d$. The class of A-modules that we consider, have the following divisibility property:

(1) $\forall n\ (w(n) \ge 0_\Delta \to (\exists m\ (w(m) \ge 0_\Delta \,\&\, m \cdot q(t) = n)))$, for all $q(t) \in \mathcal{I}$, with $q(0) \neq 0$. This property together with Δ o.l.c. implies that for any $n \in M$ with $w(n) = \delta$, there exists $m \in M$ such that $m \cdot q(t) = n$ & $w(m) = \Upsilon(q, \delta)$.

(2) The second property that we translate in our setting is that the separable closure of a valued field of characteristic p is dense in its algebraic closure. We work under the further assumption that σ_Γ is 2-contracting on Γ, namely if $\forall \gamma \in \Gamma^+$, then one has $\sigma_\Gamma(\gamma) \ge \gamma + \gamma$ and we express that $M.t$ is dense in M.

In the class of contractive valued modules with these two properties (1), (2), we show an effective quantifier elimination result up to index sentences in the language of A-modules enriched by a chain of subgroups and λ-functions. Recall that index sentences in particular tell us the sizes of the annihilators (of the separable) polynomials (the size of the set of solutions of an additive polynomial).

Finally, we show that within that class, on one hand two torsion-free elements are elementarily equivalent and on the other hand that for any $q(t) \in \mathcal{I}$ of degree d, there is a finite subset $F_{q(t)} \subset \Delta$ of cardinality at most 2^{d-1} such that if $m.q(t) = 0$ and $m \neq 0$, then $w(m) \in F_{q(t)}$. Moreover, we know how to determine $F_{q(t)}$. This analysis enables us to determine all the completions of the theories of contractive valued A-modules satisfying properties (1) and (2).

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The total blow-up along a valuation

Hans Schoutens

(joint work with Alan Loper, Bruce Olberding, William Heinzer)

By a *valued pair* (R, v), we mean a regular local ring $(R, \mathfrak{m})$ together with a valuation v on its field of fractions K which is centered on R (meaning that $v(R) \ge 0$ and $v(\mathfrak{m} > 0)$. Fix a regular system of parameters $x = (x_1, \dots, x_d)$ of R, where $d := \dim R$; the key example to have in mind is when R is either the localization of a polynomial ring $k[x]$ at the ideal generated by the variables, or the power series ring $k[[x]]$. Renumber so that $v(x_1)$ is minimal among all $v(x_i)$. The blow-up $R_1 := \mathbf{B}(R, v)$ of this pair is then given by inverting in $R[\frac{x_2}{x_1}, \dots, \frac{x_d}{x_1}] \subseteq K$ all elements of value zero. It is again a valued pair (R_1, v), and so we can continue the procedure. The union of all these blow-ups is the *total blow-up* of (R, v), denoted $\mathbf{B}_\infty(R, v)$. In [1], Abhyankar shows that if $d = 2$, then $\mathbf{B}_\infty(R, v)$ is just the valuation ring of v. However, his student Shannon gave examples in [3] showing that this no longer holds if $d \geq 3$. This then put a serious damper on Zariski's program of proving resolution of singularities through valuation theory, and so interest in this approach faded.

Only recently, there has been some renewed activity in this topic. Indeed, two major unanswered questions emerge from Shannon's counterexamples:

(1) under which additional conditions is $\mathbf{B}_\infty(R, v)$ a valuation ring?
(2) if not a valuation ring, what good properties does $\mathbf{B}_\infty(R, v)$ then have?

Ad (1), Shannon himself showed that this is the case if v is a prime divisor. Recently, Granja [2] has given an answer to (1). However, it is somehow unsatisfactory, since the criterion is in terms of all R_n. We would like to find instead a 'finite' criterion.

As far as we know, not much has been done towards (2), and so I will concentrate in my talk on this part of the project. Valuation rings can be thought of as non-Noetherian generalizations of regular local rings, with Abhyankar's result connecting the two notions in a strong way: the valuation ring is a direct limit of regular local rings if $d = 2$. In general, we can show that $S := \mathbf{B}_\infty(R, v)$ is regular on its punctured spectrum: $S_\mathfrak{p}$ is a regular local ring for any non-maximal prime ideal $\mathfrak{p}$. So, what vestige of regularity carries a general total blow-up ring S? We would like to show, for instance, that S can be realized as the intersection of some valuation subring and some regular local subring of K. Another open question is whether the complete integral closure of S is a valuation ring. We do not yet have too many solid answers, especially since being non-Noetherian rings, total blow-up rings defy any direct approach using commutative algebra tools, and therefore, I mostly discussed examples.

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Analytic Difference Fields

Thomas Scanlon

We discussed a generalization of valued difference fields known as analytic difference fields. We defined a notion of σ-henselianity for such fields using the notion of Hensel configurations, and presented result by Rideau on the relative completeness and relative quantifier elimination for the theory of σ-henselian analytic fields, generalizing previous work by the speaker. We showed how to apply this to deduce a Manin-Mumford type result for flat, commutative group schemes over the Witt vectors $W(k)$, where k is algebraically closed of characteristic p.

Geometric Reconstruction of Absolute Galois Groups in the 2-Step Nilpotent, pro-ℓ Setting

Aaron Michael Silberstein

The "yoga" of anabelian geometry, first described by Alexander Grothendieck in 1984 in his famous letter to G. Faltings [1], is a synthesis of three ideas:

(1) **Rigidity in differential geometry**.
   (a) Selberg [2] proved in 1960 that discrete, co-compact subgroups Γ of high-rank Lie groups G are algebraic, and that every continuous deformation of the subgroup in the Lie group arises from conjugation: in this sense, these subgroups are **rigid**.
   (b) Weil [3] and Calabi and Vesentini [4], around the same time, proved new rigidity theorems, re-imagining the framework of Selberg's work by replacing the pair of the discrete subgroup Γ of the Lie group G with the double-quotient manifold $\Gamma\backslash G/K$ (where, for instance, K is a maximal compact subgroup). In this approach, Γ becomes the **fundamental group** of $\Gamma\backslash G/K$, and the question becomes one about how the fundamental group of a manifold determines the type of Riemannian geometry it can admit.
   (c) G. Mostow [5], etc. proved that the canonical map

$$\text{Isometries}(M, N) \to \text{Out}(\pi_1(M), \pi_1(N))$$

   between the set of isometries between M and N and the isomorphisms, up to composition with an inner automorphism (the "outer isomorphisms"), between their fundamental groups, is a bijection whenever M and N are hyperbolic, of finite volume, and of dimension ≥ 3.

(2) **Faltings' theorem**: J. Tate [6], in trying to understand algebraic cycles on varieties defined over finite fields, conjectured that for finitely-generated fields k and Abelian varieties A and B defined over K, the canonical map

$$\text{Hom}_k(A, B) \otimes \mathbb{Z}_\ell \to \text{Hom}_{G_k}(T_\ell(A), T_\ell(B))$$

   between the k-homomorphisms between A and B (tensored with the ℓ-adic integers) and the G_k (that is, absolute Galois group of k)-invariant homomorphisms between their ℓ-adic Tate modules is an isomorphism. Tate (*loc. cit.*) proved this when k is finite, and Faltings [7] proved this for k a number field, proving Mordell's conjecture in the process!

(3) The work of J. Neukirch [8], K. Uchida [9], G. İkeda [10], and K. Iwasawa (unpublished) gives a proof that, if K and L are global fields, then the canonical map

$$\text{Isom}^i(L, K) \to \text{Out}(G_K, G_L)$$

from the set of isomorphisms between (in char. p, the inseparable closures of) L and K (in char. p, modulo the action of Frobenius) and the continuous isomorphisms between the absolute Galois groups of K and L, up to composition by an inner automorphism, is a bijection. (As notation: we will only consider *continuous* homomorphisms between Galois groups).

Grothendieck's synthesis was to conjecture that there is a class of varieties $\mathcal{A}$ defined over finitely-generated fields, called **anabelian varieties**, which satisfy the following properties:

(1) There are a set of conditions on varieties over algebraically closed fields so that $V \in \mathcal{A}$ if and only if the basechange to any algebraically closed field containing the field of definition of V satisfies one of these conditions.

(2) Successive fibrations of anabelian varieties are likewise anabelian, and both (geometrically) hyperbolic, normal curves (complete or not) and moduli spaces of marked curves are anabelian.

(3) **The Biregular Anabelian Conjecture**. For anabelian varieties X, Y over a finitely-generated field K, the canonical map

$$\mathrm{Isom}_k(X, Y) \to \mathrm{Out}(\pi_1^{\acute{e}t}(X), \pi_1^{\acute{e}t}(Y)) \tag{1}$$

is an isomorphism.

(4) **The Birational Anabelian Conjecture**. Generic points of varieties are anabelian: if X_1 and X_2 are varieties of dimension ≥ 1 over a finitely-generated field K, then the canonical maps

$$\mathrm{Isom}^i_K(K(Y), K(X)) \to \mathrm{BiratIsom}_k(X, Y) \to$$
$$\mathrm{Out}_{G_K}(\pi_1^{\acute{e}t}(\eta_X), \pi_1^{\acute{e}t}(\eta_Y)) \to \mathrm{Out}_{G(K)}(G_{K(X)}, G_{K(Y)})$$

are all isomorphisms.

This last assertion naturally fits in with the first three, as every variety is covered by Zariski opens which are successive fibrations of hyperbolic curves [13]. We now restrict ourselves to this last assertion. The original approach of Neukirch led to the solution of the birational anabelian conjecture by Pop [17, 16]. He follows the approach of Neukirch, *et al*, dividing the proof into steps:

(1) **Detect valuations**: Associated to each valuation v on a field K are two (conjugacy classes) of subgroups $I_v \trianglelefteq D_v$, the **inertia** and **decomposition** of v. In order to define them, fix a lift $\tilde{v}$ of v to $\tilde{K}$, the separable closure of K, and define:

$$D_v := \left\{ \sigma \in G_K \mid \forall\, x \in \tilde{K}, \tilde{v}(\sigma x) = v(x) \right\}$$

and

$$I_v := \{\sigma \in D_v \mid \sigma\,|_{\tilde{K}\tilde{v}} = \mathrm{id}\}$$

where $\tilde{K}\tilde{v}$ is the residue field of $\tilde{v}$. Neukirch's insight is that there is a *purely group-theoretic characterization* of the groups I_v, D_v inside of G_K for K a global field. Pop extends this insight to all infinite, finitely-generated fields. An important point is that one cannot and should not characterize inertia and decomposition of all valuations: one needs to identify the valuations with "geometric meaning."

(2) **Detect the field.** Using embedding properties of Galois groups of finitely-generated fields, prove that an automorphism which preserves the inertia and decomposition groups of suitable valuations must preserve the field. This can be done not with the full group G_K, but with a suitable characteristic quotient.

F. Bogomolov, in 1991 [11] had the fundamental insight that Grothendieck was too conservative in his birational conjectures in three important ways:

(1) If K is finitely-generated over a finite extension of a prime field k with k alg. cl. in K, and we denote by $\tilde{k}$ the separable closure of k, and set $F := K \otimes_k \tilde{k}$, then there is a short exact sequence

$$1 \longrightarrow G_F \longrightarrow G_K \longrightarrow G_k \longrightarrow 1$$

When we deal with finitely generated fields, we deal with the entire short exact sequence. Bogomolov's idea was that *when the transcendence degree of F over $\tilde{k} \geq 2$*, one should be able to recover this full short exact sequence *completely group-theoretically* from G_F.

(2) Moreover, instead of using the entire absolute Galois group G_F, one should be able to recover the field F from the maximal 2-step nilpotent, pro-ℓ quotient $G_F^{c,\ell}$, for any $\ell \neq \operatorname{char} F$. As evidence of this, Bogomolov gave a recipe to recover the geometrically significant valuations of F from $G_F^{c,\ell}$.

(3) It should be possible to reconstruct the field F from $G_F^{c,\ell}$ explicitly.

The basic idea [20] is that one can use of Hilbert's Fundamental Theorem of Projective Geometry ([12]): Let $\mathbb{P}(V_1), \mathbb{P}(V_2)$ be projective spaces of dim ≥ 2 over any field k, considered as sets. Then any set-theoretic bijection

$$f : \mathbb{P}(V_1) \to \mathbb{P}(V_2)$$

which takes lines to lines is induced by a semilinear isomorphism

$$\tilde{f} : V_1 \to V_2.$$

In the case where k is algebraically closed, this becomes relevant, as Kummer theory gives an isomorphism

$$F^{\times,\ell} \xrightarrow{\sim} G_F^{a,\ell}$$

from the ℓ-adic completion of the multiplicative group of F to the ℓ-adic completion of the abelianization of G_F. The inclusion $F^\times \to F^{\times,\ell}$ is dense, and the kernel is precisely the set of infinitely-divisible elements of $F^\times$ — that is, $\tilde{k}^\times$. This means the image of $F^\times$ in $G_F^{a,\ell}$ is the projectivization of the k-vector space F!

The valuation detection methods are deep and intricate, and I won't go into detail here, except to state the final result.

Theorem 1 (Bogomolov-Tschinkel for $\tilde{k} = \tilde{\mathbb{F}}_p$ [14], [15], Pop for $\tilde{k} = \tilde{\mathbb{Q}}, \tilde{\mathbb{F}}_p$ [18], [19])**.** *Given $G_F^{a,\ell}$ and the map $G_F^{c,\ell} \twoheadrightarrow G_F^{a,\ell}$, there is a group-theoretic characterization of the subgroups I_v and D_v in $G_F^{a,\ell}$ for v Parshin chains.*

A **Parshin chain** is a valuation associated to a flag of prime divisors on a normal model of F. The main difficulty here — and the main stumbling block in extending these results to higher transcendence degree — is that detecting less-geometric valuations called **quasi-Parshin chains** is doable, but selecting the Parshin chains from among them is much more difficult.

In these cases, we have:

Theorem 2 ([14], [15], [18] for $\tilde{k} = \tilde{\mathbb{F}}_p$; [19] for $\tilde{k} = \tilde{\mathbb{Q}}$, transcendence degree of F over $\tilde{k} \geq 3$; and [21] for $\tilde{k} = \tilde{\mathbb{Q}}$, transcendence degree of F over $\tilde{k} = 2$)**.** *When $\tilde{k}$ is the algebraic closure of a prime field of characteristic prime to ℓ, we may functorially reconstruct F from $G_F^{c;\ell}$, and the canonical map* $\mathrm{Isom}^i(F_1, F_2) \to \mathrm{Isom}^c(G_{F_2}^{a,\ell}, G_{F_1}^{a;\ell})$ *from the isomorphisms between inseparable closures of two finitely-generated fields of transcendence degree ≥ 2 over algebraic closures of prime fields to the isomorphisms between ℓ-adically completed abelianizations of absolute Galois groups which lift to 2-step nilpotent pro-ℓ completions is an isomorphism, up to constant multiplication on the right by an element of $\mathbb{Z}_\ell^\times$.*

In order to prove the last open case — that with transcendence degree 2 over $\tilde{\mathbb{Q}}$ — I introduced the notion of **geometric reconstruction**, in which I reconstruct the category of all varieties in the birational equivalence class defined by the function field, rather than reconstructing the function field directly.

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Recovering valuations on Demushkin fields: the wild case

Kristian Strømmen
(joint work with Jochen Koenigsmann)

Given a field K, much effort has been put into understanding what field-theoretic data is encoded in small quotients of the absolute Galois group $G_K = Gal(K^{sep}/K)$ of K. Of particular interest are the maximal pro-p quotients $G_K(p)$ of G_K, since when K is a p-adic field, these quotients are well understood. Indeed, if l is a prime, $l \neq p$, then $G_{\mathbb{Q}_l(\zeta_p)}(p) \simeq \mathbb{Z}_p \rtimes \mathbb{Z}_p$ with cyclotomic action, where ζ_p is a primitive p-th root of unity. Koenigsmann showed ([2]) that if K is any field containing ζ_p, then $G_K(p) \simeq \mathbb{Z}_p \rtimes \mathbb{Z}_p$ if and only if K admits a p-henselian valuation, tamely branching at p. The valuation is recovered using the theory of 'rigid elements' via a combinatorial argument.

Much more mysterious is the 'wild' case $l = p$. Here the structure of $G_{\mathbb{Q}_p(\zeta_p)}(p)$ is known due to work of Demushkin, Labute and Serre: it is a pro-p Demushkin group given by generators and relations which can be specified. In this talk, we presented the following precise conjecture on what these groups are expected to encode:

Conjecture 1 (Koenigsmann, S) Let $F/\mathbb{Q}_p$ be a finite extension with $\zeta_p \in F$. If K is a field with $\zeta_p \in K$, and suppose

$$G_F(p) \simeq G_K(p).$$

Then K has characteristic 0. Furthermore, there is a finite extension $F'/\mathbb{Q}_p$ containing ζ_p with p-adic valuation w, such that $G_K(p) \simeq G_{F'}(p)$, and K admits a non-trivial valuation v such that

- v is p-henselian;
- $F'w = Kv$;
- $[\Gamma_v : p\Gamma_v] = p$;

- There is a uniformizer π of (F', w) such that $\pi \in K \cap \overline{\mathbb{Q}}$ and $v(\pi)$ is a minimal positive element in Γ_v. In particular, v is a discrete valuation.

Here Kv and Γ_v denote the residue field and value group respectively.

Up until this point, only partial results on this Conjecture were known (see e.g. [3]). In this talk, we announced a partial proof in the special case that $F = \mathbb{Q}_2$:

Theorem 1. (Koenigsmann, S.) ([1]) *If K is (almost) any field with $G_K(2) \simeq G_{\mathbb{Q}_2}(2)$, then K has characteristic 0, and admits a 2-henselian valuation v with residue field $\mathbb{F}_2$, $v(2)$ a minimal positive element, and $[\Gamma_v : 2\Gamma_v] = 2$.*

The key part is to recover, from the group-theoretic structure of $G_K(2)$, the 'lattice' of norm-subgroups of $K^\times$ of index p. Then one shows that this lattice, which a priori only carries multiplicative information, actually carries additive data. This is done by direct computation. Finally, an adaptation of the rigid element construction is used to define the valuation, and one checks directly that it satisfies the required conditions.

The missing case in the proof is related to the possible bad behaviour of $k := K \cap \overline{\mathbb{Q}}$. We summarize this bad case in a question:

Question 1: Does there exist a field K of characteristic 0 and transcendence degree 1 over $k := K \cap \overline{\mathbb{Q}}$, such that $G_K(2) \simeq G_{\mathbb{Q}_2}(2)$ and k admits both a nontrivial ordering and a p-adic valuation with p congruent to 1 modulo 8?

If the answer is 'No', then we can show Theorem 1 is unconditional.

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Metric uniformization of morphisms of Berkovich curves

Michael Temkin

Let k be an algebraically closed complete non-archimedean real-valued field. By a nice Berkovich curve we mean a compact separated quasi-smooth strictly k-analytic curve. Such objects play a central role in a variety of recent papers (e.g., [3], [4], [7], [8], [9]), and their "combinatorial" structure is well understood for a long time: it is adequately described by the semistable reduction theorem. The situation with morphisms between nice curves is more complicated: the simultaneous semistable reduction theorem completely describes their combinatorial structure only when a morphism $f : Y \to X$ is topologically tame (i.e. $[\mathcal{H}(y) : \mathcal{H}(f(y))]$ is invertible in the residue field $\widetilde{k}$ for any $y \in Y$). The aim of the two fresh preprints [5] and [6] is to fill in this gap.

The main invariant that measures wildness of non-archimedean field extensions L/K is the different $\delta_{L/K}$, so it is quit natural to consider it when studying topologically wild covers $f : Y \to X$ of Berkovich curves. Nevertheless, it seems that the different was not used in the literature devoted to Berkovich spaces, although it did show up in the adjacent areas of rigid (see [10]) and, especially, formal geometries, see [2] and [1]. In [5], we associate to f the *different function* $\delta_f : Y \to [0,1]$ by $\delta_f(y) = \delta_{\mathcal{H}(y)/\mathcal{H}(f(y))}$ and make a thorough study of δ_f. In particular, we show that it is piecewise monomial with zeros at the wildly ramified rigid points of f, see [5, Corollary 4.1.8 and Theorem 4.6.4] and establish a balancing condition on the slopes of δ_f at type 2 points [5, Theorem 4.5.4], which is very similar to the classical Riemann-Hurwitz formula, but cannot be obtained from the latter by passing to reductions. As an application of this study, we give a constructive description of the minimal skeleton $\Gamma_f = (\Gamma_Y, \Gamma_X)$ of f: it is the minimal connected graph so that the different behaves "trivially" outside of it, see [5, Theorem 6.3.4]. In addition, if the degree of f is $p = \mathrm{char}(\widetilde{k})$ then we completely describe the topological ramification locus in terms of the different and prove that the locus of multiplicity p is a radial set along a subgraph Δ of the skeleton Γ_Y with the radius depending piecewise monomially on the point $y \in \Delta$, see [5, Theorem 7.1.4]. We illustrate this with a very concrete description of two-sheet covers $E \to \mathbf{P}^1_k$, where E is an elliptic curve.

The main information about a morphism f that cannot be extracted from the different is the structure of the sets $N_{f,\geq d}$ consisting of points $y \in Y$ with $n_f(y) := [\mathcal{H}(y) : \mathcal{H}(x)] \geq d$. This question is solved in the sequel paper [6], and it turns out that the whole higher ramification theory is relevant here. More concretely, we show that for a large enough skeleton $\Gamma = (\Gamma_Y, \Gamma_X)$ of f, the sets $N_{f,\geq d}$ are radial around Γ_Y with the radius changing piecewise monomially along Γ_Y, see [6, Theorems 3.3.11 and 3.4.8]. In this case, for any interval $l = [z, y] \subset Y$ connecting a rigid point z to the skeleton, the restriction $f|_l$ gives rise to a *profile* piecewise monomial function $\varphi_y : [0,1] \to [0,1]$ that depends only on the type 2 point $y \in \Gamma_Y$. In particular, the metric structure of f is determined by Γ and the family of the profile functions $\{\varphi_y\}$ with $y \in \Gamma_Y^{(2)}$. We prove that this family is piecewise monomial in y and naturally extends to the whole Y, see [6, Theorem 3.4.8]. In addition, we extend the theory of higher ramification groups to arbitrary real-valued fields and show that φ_y coincides with the Herbrand's function of $\mathcal{H}(y)/\mathcal{H}(f(y))$ [6, Theorems 4.5.2 and 4.5.4]. This gives a curious geometric interpretation of the Herbrand's function, which applies also to non-normal and even inseparable extensions.

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On Milnor K-groups of Function Fields

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In the early 1990's, Bogomolov [Bog91] introduced a program whose ultimate goal is to reconstruct function fields of higher transcendence degree over algebraically closed fields from their 2-step nilpotent pro-ℓ Galois groups. Pop [Pop12a] [Pop12b] later formulated a precise *functorial* form of Bogomolov's conjecture. While the Bogomolov-Pop conjecture is still wide open in full generality, it has been proven in two important special cases: over the algebraic closure of a finite field by Bogomolov-Tschinkel [BT08] [BT11] and by Pop [Pop03] [Pop12a], and over the algebraic closure of $\mathbb{Q}$ by Pop [Pop11b] in transcendence degree ≥ 3 and by Silberstein in transcendence degree 2. In this note, we consider the mod-ℓ analogue of the Bogomolov-Pop conjecture, which can be formulated in terms of mod-ℓ Milnor K-groups, as follows.

Fix a prime ℓ. Let $K|k$ and $L|l$ be two function fields over algebraically closed field k, l of characteristic $\neq \ell$. We let $\mathrm{Isom}^i(K, L)$ denote the set of isomorphisms $K^i \xrightarrow{\cong} L^i$, where K^i resp. L^i denotes the perfect closure of K resp. L. The Frobenius automorphism Frob_K of K (defined as the identity if $\mathrm{char}\, K = 0$) acts on $\mathrm{Isom}^i(K, L)$, and we let $\mathrm{Isom}^i_F(K, L)$ denote the orbits under this action.

Recall that the 2nd Milnor K-group of a field F is defined as follows:

$$\mathrm{K}_2^{\mathrm{M}}(F) = \frac{F^\times \otimes_{\mathbb{Z}} F^\times}{\langle x \otimes (1-x) \rangle}.$$

We let $\mathrm{Isom}^{\mathrm{M}}(K^\times/\ell, L^\times/\ell)$ denote the set of isomorphisms $\phi : K^\times/\ell \xrightarrow{\cong} L^\times/\ell$ such that $\phi^{\otimes 2}$ descends to an isomorphism $\mathrm{K}_2^{\mathrm{M}}(K)/\ell \xrightarrow{\cong} \mathrm{K}_2^{\mathrm{M}}(L)/\ell$. One has a canonical action of $(\mathbb{Z}/\ell)^\times$ on $\mathrm{Isom}^{\mathrm{M}}(K^\times/\ell, L^\times/\ell)$, and we let $\underline{\mathrm{Isom}}^{\mathrm{M}}(K^\times/\ell, L^\times/\ell)$ denote the orbits of this action. Since the inclusion $K \to K^i$ induces isomorphisms

$K^\times/\ell \cong (K^i)^\times/\ell$ and $\mathrm{K}_2^{\mathrm{M}}(K)/\ell \cong \mathrm{K}_2^{\mathrm{M}}(K^i)/\ell$ (and similarly for L), we obtain a canonical map:

$$\mathrm{Isom}^i_F(K, L) \to \underline{\mathrm{Isom}}^{\mathrm{M}}(K^\times/\ell, L^\times/\ell).$$

The following is the mod-ℓ analogue of the Bogomolov-Pop conjecture.

Conjecture 1. In the notation above, assume that $\mathrm{tr.deg}(K|k) \geq 2$. Then the canonical map

$$\mathrm{Isom}^i_F(K, L) \to \underline{\mathrm{Isom}}^{\mathrm{M}}(K^\times/\ell, L^\times/\ell)$$

is a bijection.

Moreover, the proof of this conjecture should yield an isomorphism-invariant group-theoretical recipe which constructs K^i as a field from the following data: $K^\times/\ell$ and the multiplication map $(K^\times/\ell)^{\otimes 2} \to \mathrm{K}_2^{\mathrm{M}}(K)/\ell$.

Motivated by the strategy for the original pro-ℓ version of the Bogomolov-Pop conjecture, it is likely that the proof of Conjecture 1 would involve two main steps: the *local theory* and the *global theory*. While the mod-ℓ local theory (see [Pop11a], [Top13] and [Top12]) is understood just as well as the pro-ℓ local theory developed in [BT02] and [Pop10], the mod-ℓ global theory poses a much bigger problem. The general strategy of the pro-ℓ global theory proceeds as follows (see Pop [Pop12b]):

(1) First, one must reconstruct the set $K^\times/k^\times$ as a subset of
$$\varprojlim_n K^\times/\ell^n =: \widehat{K^\times}$$
by identifying the image of the ℓ-adic completion map $K^\times \to \widehat{K^\times}$.

(2) When considering $K^\times/k^\times$ as an infinite dimensional projective space over k, one must next reconstruct the set of lines on this projective space.

(3) Finally, one applies the *fundamental theorem of projective geometry* [Art57] to reconstruct the field k and $(K, +)$ as a k-vector space, while the multiplicative structure of K is then obtained from the group structure of $K^\times/k^\times$.

In order to carry out this strategy as in Pop [Pop12b], one must have access to the collection of so-called *rational subgroups* of $\widehat{K^\times}$ (see below for definition of a mod-ℓ rational subgroup). Thus, loc.cit. reduces the pro-ℓ Bogomolov-Pop conjecture to the problem of recovering these rational subgroups.

However, this strategy fails from the very beginning in the mod-ℓ setting, since $K^\times/\ell$ contains no appearant geometric object over k on which one can apply the fundamental theorem of projective geometry. The main theorem of this note describes a viable mod-ℓ global theory, which is a direct mod-ℓ analogue (and generalization) of the global theory developed by Pop [Pop12b].

To state the main theorem, we must introduce the notion of a *rational subgroup*. We say that a subgroup A of $K^\times/\ell$ is a **rational subgroup** provided that there exists some $t \in K \smallsetminus k$ such that the following two conditions hold:

(1) $k(t)$ is relatively algebraically closed in K.

(2) A is the image of the canonical map $k(t)^\times \to K^\times/\ell$.

We let $\mathrm{Isom}^{\mathrm{M}}_{\mathrm{rat}}(K^\times/\ell, L^\times/\ell)$ denote the set of all $\phi \in \mathrm{Isom}^{\mathrm{M}}(K^\times/\ell, L^\times/\ell)$ such that ϕ induces a bijection between the set of rational subgroups of $K^\times/\ell$ and $L^\times/\ell$. Finally, we let $\underline{\mathrm{Isom}}^{\mathrm{M}}_{\mathrm{rat}}(K^\times/\ell, L^\times/\ell)$ denote the quotient of $\mathrm{Isom}^{\mathrm{M}}_{\mathrm{rat}}(K^\times/\ell, L^\times/\ell)$ by the action of $(\mathbb{Z}/\ell)^\times$. Similarly to the above, we obtain a canonical map

$$\mathrm{Isom}^i_F(K, L) \to \underline{\mathrm{Isom}}^{\mathrm{M}}_{\mathrm{rat}}(K^\times/\ell, L^\times/\ell).$$

We are now prepared to state the main theorem.

Theorem 1 ([Top14] Theorem B). *In the notation above, assume furthermore that* $\mathrm{tr.deg}(K|k) \geq 5$. *Then the following hold:*

(1) *There is a group-theoretical recipe which constructs $K^i|k$ as fields from the following data:*
   (a) *$K^\times/\ell$, endowed with the collection of its rational subgroups.*
   (b) *The multiplication map $(K^\times/\ell)^{\otimes 2} \to \mathrm{K}^{\mathrm{M}}_2(K)/\ell$.*

(2) *This group-theoretical recipe is functorial with respect to isomorphisms, i.e. the canonical map*

$$\mathrm{Isom}^i_F(K, L) \to \underline{\mathrm{Isom}}^{\mathrm{M}}_{\mathrm{rat}}(K^\times/\ell, L^\times/\ell)$$

is a bijection.

The main idea in the proof of Theorem 1 is to use a modern analogue of the fundamental theory of projective geometry, which stems from the group-configuration theorem in geometric stability theory. More precisely, let $\mathbb{G}(K|k)$ denote the lattice of relatively algebraically closed subextensions of $K|k$. Then the main results of Evans-Hrushovski [EH91] [EH95] and Gismatullin [Gis08] show that K^i is (uniformly) interpretable in $\mathbb{G}(K|k)$. The main strategy for the proof of Theorem 1 is to find an isomorphic copy of $\mathbb{G}(K|k)$ inside the lattice of subgroups of $K^\times/\ell$, by using the collection of rational subgroups and the given K-theoretic data. The reconstruction of this lattice relies on the fact that there are "many" elements of $K^\times/\ell$ which have non-trivial products in the higher mod-ℓ Milnor K-groups of K. Such technical results concerning mod-ℓ Milnor K-groups of function fields use tame symbols and the (birational) geometry of valuations which arise from Weil prime-divisors on normal models of $K|k$.

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Semistable reduction and Epp's theorem

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(joint work with Kai Arzdorf)

Let K be a field which is complete with respect to a discrete valuation v. We denote the valuation ring of K with respect to v by R and the residue field by k. We also let Y denote a smooth, projective and absolutely irreducible curve over K.

An *integral model* of Y is a flat and proper R-scheme $\mathcal{Y}$ with generic fiber Y, i.e. $\mathcal{Y} \otimes_R K = Y$. An integral model $\mathcal{Y}$ is called *semistable* if its special fiber $\bar{Y} := \mathcal{Y} \otimes_R k$ is reduced and has at most ordinary double points as singularities. We say that the curve Y has *semistable reduction* if it has a semistable integral model.

The goal of the talk was to sketch a new proof of the celebrated *Semistable Reduction Theorem*, given in [2]

Theorem: There exists a finite extension L/K such that the curve $Y_L := Y \otimes_K L$ has semistable reduction.

The first complete proof of this theorem was given by Deligne and Mumford in 1969, see [4]. Since then, many more proofs have been given, e.g. by Artin and Winters, Bosch and Lütkebohmert, van der Put, and T. Saito.

Our motivation for working out yet another proof comes from our desire to do explicit computations, see e.g. [3]. The proofs cited above give little insight how to explicitly compute the semistable reduction of a given curve Y. Our proof is based on ideas of Lehr and Matignon ([7], [6]) who solved this problem for p-cyclic covers of the projective line (where p is the residue characteristic of K), under some mild extra condition.

An important ingredient of our proof is Epp's theorem on elimination of wild ramification ([5]). An immediate consequence is that we may assume that the special fiber $\bar{Y}$ of any given model $\mathcal{Y}$ is reduced (after replacing K by a suitable finite extension). We call integral models $\mathcal{Y}$ with reduced special fiber *permanent*, because they retain this property after further base change. Moreover, the special fiber $\bar{Y}$ of a permament model is 'stable' under change of the base field K in the sense that $\bar{Y}$ gets replaced by its base change to the induced extension of the residue field k.

Our proof can be formulated purely in terms of valuation theory (however, in [2] we use the language of rigid geometry). It is an ongoing project to make all the steps in our proof accessible to explict computations. See e.g. the thesis of Kai Arzdorf ([1]) and the forthcoming thesis of Julian Rüth ([8]).

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