Mathematisches Forschungsinstitut Oberwolfach

Report No. 51/2014

DOI: 10.4171/OWR/2014/51

Combinatorial Optimization

Organised by
Gerard Cornuejols, Pittsburgh
Fritz Eisenbrand, Lausanne
Bruce Shepherd, Montreal

9 November – 15 November 2014

Abstract. Combinatorial Optimization is an area of mathematics that thrives from a continual influx of new questions and problems from practice. Attacking these problems has required the development and combination of ideas and techniques from different mathematical areas including graph theory, matroids and combinatorics, convex and nonlinear optimization, discrete and convex geometry, algebraic and topological methods. We continued a tradition of triannual Oberwolfach workshops, bringing together the best international researchers with younger talent to discover new connections with a particular emphasis on emerging breakthrough areas.

Mathematics Subject Classification (2010): 90C27 (Combinatorial Optimization); secondary: 90C57, 90C10, 90C11, 90C22, 90C06, 90C59, 90C90.

Introduction by the Organisers

There has been a tradition of triannual Oberwolfach workshops in Combinatorial Optimization and the 2014 edition was again a great success.

The Oberwolfach workshops have truly played a key role for our field, which cannot be substituted by any other workshop or conference. This success is due to Oberwolfach's reputation for excellence, and its outstanding research conditions as well as the unique format of the workshop.

As in the past, the program consisted of five one-hour focus lectures planned in advance (one on each day of the workshop), and shorter presentations scheduled during the workshop. As in the past, we made sure to leave sufficient time for discussions and research in small groups. We also continued the last workshop's successful micro-presentations (5 minutes, strictly timed) so that all participants had an opportunity to present their hottest recent result or burning open question.

The focus lectures covered topics where recent progress has been most intense, and future progress seems very promising:

Geometric Approach to Cutting Planes (Michele Conforti)
Semidefinite Extended Formulations (Rekha Thomas)
Algebraic Geometry (Jean Lasserre)
Lattice Algorithms (Damien Stehlé)
Complexity of the Union of Polyhedra (Juan Pablo Vielma)

During the workshop, we were excited to see great advances in several new directions. The synergy between discrete and continuous models is emerging as a thriving area of importance, and new techniques from pure mathematics continue to transform combinatorial optimization, such as the use of tools from *algebraic geometry*. This is the case for instance in the theory of *integer programming*, which is under rapid development using techniques from several fields, such as *lattice algorithms* and *semidefinite (SDP) bounds*. There was an exciting advance in this area just prior to the workshop due to Lee-Raghavendra-Steurer: a first super-polynomial lower bound on the complexity of SDP projections yielding certain combinatorial polytopes (such as the travelling salesman polytope). The solution was presented at the workshop by James Lee. Finally matroids and structural *graph theory* have been at the heart of many important advances in combinatorial optimization. We were fortunate to hear from Gyula Pap about one of two independent (and distinct) solutions (the second due to Satoru Iwata) that has been announced to the longstanding open problem of determining whether weighted linear matroid matching is polytime solvable.

We would like to thank all participants for their carefully prepared contributions and the many exciting discussions. Last but certainly not least, we thank the Oberwolfach Research Institute and its members for providing the outstanding meeting and working conditions and the unique inspiring Oberwolfach atmosphere.

Acknowledgement: The MFO and the workshop organizers would like to thank the National Science Foundation for supporting the participation of junior researchers in the workshop by the grant DMS-1049268, “US Junior Oberwolfach Fellows”.

Workshop: Combinatorial Optimization

Table of Contents

Abstracts

On the relationship between different types of cuts

Egon Balas
(joint work with Tamás Kis)

We discuss the relationship between standard intersection cuts (SIC's), generalized intersection cuts (GIC's) and lift-and-project (L&P) cuts.

It is known that (L&P) cuts from split disjunctions are equivalent to (SIC's) and (GIC's). We show that this equivalence does not hold for L&P cuts from other types of disjunctions (multiple-term or non-split). Our main findings are as follows.

GIC's and L&P cuts. The family of GIC's from a given lattice-free polyhedron S is equivalent to the family of L&P cuts from a Cut Generating Linear Program (CGLP) based on a disjunction whose terms are the weak complements of the inequalities defining S.

The family of L&P cuts from a disjunction D with multiple inequalities per term is equivalent to the family of GIC's from the lattice-free polyhedra obtained from the disjunction D by combining the inequalities of each term of D, and taking the weak complements of the resulting inequalities to obtain the polyhedra.

SIC's and L&P cuts from multiple-term or non-split disjunctions. Each SIC is equivalent to a L&P cut from a CGLP solution with a certain property P. The property P of the CGLP solution is necessary and sufficient for the resulting L&P cut to be equivalent to a SIC. CGLP solutions with property P (and associated L&P cuts) are called *regular*, those without the property are called *irregular*.

If the CGLP solution that maximizes the depth of the L&P cut relative to the LP optimum is irregular, then the associated L&P cut cuts off the LP optimum by more than any intersection cut, and is not dominated by any intersection cut.

L&P cuts and corner polyhedra. It has recently been established that all the facets of corner polyhedra are defined by SIC's. We show that irregular L&P cuts cut off integer points of corner polyhedra. Moreover, we give an example where the L&P cut obtained from the irregular CGLP solution that maximizes the cut depth relative to the LP optimum cuts off an integer point of every corner polyhedron associated with the vertices adjacent to the LP optimum.

Irregular CGLP solutions are not exceptional, their frequency is comparable with that of regular solutions, and increases with the number of terms in the disjunction underlying the CGLP.

Finding maximum independent sets in sparse graphs using hierarchies

Nikhil Bansal

(joint work with Anupam Gupta and Guru Guruganesh)

Given a graph $G = (V, E)$, an independent set is a subset of vertices S such that no two vertices in S are adjacent. In general graphs on n vertices, the problem is notoriously hard to approximate and the best known algorithm achieves an approximation ratio of $\widetilde{O}(n/\log^3 n)$. On the hardness side, a result of Håstad [7] shows that no $n^{1-\epsilon}$ approximation exists for any constant $\epsilon > 0$, assuming NP $\not\subseteq$ ZPP.

Here we will consider bounded-degree graphs, with maximum degree d. Recall that the naïve algorithm (that repeatedly picks an arbitrary vertex v and deletes its neighborhood) produces an independent set of size at least $n/(d+1)$, and hence is a $d+1$-approximation. The best known result is an $O(d\,\frac{\log\log d}{\log d})$-approximation[1, 6] based on rounding the natural SDP for the problem. On the negative side, Austrin, Khot and Safra [2] showed an $\Omega(d/\log^2 d)$ hardness of approximation, assuming the Unique Games Conjecture. Assuming P $\neq$ NP, a hardness of $d/\log^4 d$ was recently shown by Chan [5].

We will describe several new results for the problem.

1) The $O(\log^4 d)$-level SA+ relaxation has an integrality gap of $\widetilde{O}(d/\log^2 d)$, where $\widetilde{O}(\cdot)$ suppresses some $\log\log d$ factors [4].

The main observation behind this result is that the SA+ relaxation specifies a local distribution on independent sets, and if the relaxation has high objective value then it must be that any polylog(d) size subset of vertices X most contain a large independent subset. One can then use a result Alon [3], which is turn in based on an elegant entropy-based approach of Shearer [9], to show that such graphs have non-trivially large independents sets. However, this argument is non-algorithmic. Next we give an algorithmic version at the expense of higher running time.

2) There is an $\widetilde{O}(d/\log^2 d)$-approximation algorithm with running time poly$(n) \cdot 2^{O(d)}$, based on rounding a d-level SA+ semidefinite relaxation.

As previously, we observe that if the d-level SA+ relaxation has objective value at least $n/\log^2 d)$, then the neighborhood of every vertex graph is k-colorable for $k = s \cdot \text{polylog}(d)$. However, instead of using [3] which relies on Shearer's entropy based approach, and is not known to be constructive, we use an ingenious and remarkable (and stronger) unpublished result of Johansson [8], who shows that the list chromatic number of such "locally-colorable" graphs is $\chi_\ell(G) = O(d\frac{\log k}{\log d})$.

3) On graphs with maximum degree d, the standard ϑ-function-based SDP formulation for the independent set problem has an integrality gap of $\widetilde{O}(d/\log^{3/2} d)$.

The proof of the above result is non-constructive and is based on the following new Ramsey-type result about the existence of large independent sets in K_r-free graphs.

4) For any $r > 0$, if G is a K_r-free graph with maximum degree d then

$$\alpha(G) = \Omega\left(\frac{n}{d} \cdot \max\left(\frac{\log d}{r \log\log d}, \left(\frac{\log d}{\log r}\right)^{1/2}\right)\right).$$

Previously, the best known bound for K_r-free graphs was $\Omega(\frac{n}{d}\frac{\log d}{r \log\log d})$ given by Shearer [9]. However, this result does not give anything better than the trivial n/d bound when we are only guaranteed to exclude very large cliques, e.g., when $r \geq \frac{\log d}{\log\log d}$. It is in this range of $r \geq \log d$ that the second term in the maximization starts to perform better and give a non-trivial improvement.

References

[1] Noga Alon and Nabil Kahale. Approximating the independence number via the ϑ-function. *Math. Programming*, 80(3, Ser. A):253–264, 1998.

[2] Per Austrin, Subhash Khot, and Muli Safra. Inapproximability of vertex cover and independent set in bounded degree graphs. *Theory of Computing*, 7(1):27–43, 2011.

[3] Noga Alon. Independence numbers of locally sparse graphs and a Ramsey type problem. *Random Struct. Algorithms*, 9(3):271–278, 1996.

[4] Nikhil Bansal. Approximating independent sets in sparse graphs. In *SODA*, 2015.

[5] Siu On Chan. Approximation resistance from pairwise independent subgroups. In *STOC*, pages 447–456, 2013.

[6] Eran Halperin. Improved approximation algorithms for the vertex cover problem in graphs and hypergraphs. *SIAM J. Comput.*, 31(5):1608–1623, 2002.

[7] Johan Håstad. Clique is hard to approximate within $n^{1-\epsilon}$. In *FOCS*, pages 627–636, 1996.

[8] Anders Johansson. The choice number of sparse graphs. preprint, August 1996.

[9] James B. Shearer. On the independence number of sparse graphs. *Random Struct. Algorithms*, 7(3):269–272, 1995.

On the covering property of the lifting region

Amitabh Basu

(joint work with Gennadiy Averkov and Joseph Paat)

Cut-Generating Pairs. Cut-generating functions are a means to have "a priori" formulas for generating cutting planes for general mixed-integer optimization problems. Let S be a closed subset of $\mathbb{R}^n$ with $0 \notin S$. Consider the following set, parametrized by matrices R, P:

$$X_S(R,P) := \left\{(s,y) \in \mathbb{R}^k_+ \times \mathbb{Z}^\ell_+ \,:\, Rs + Py \in S\right\}, \tag{1}$$

where $k, \ell \in \mathbb{Z}_+, n \in \mathbb{N}$, $R \in \mathbb{R}^{n\times k}$ and $P \in \mathbb{R}^{n\times \ell}$ are matrices. Denote the columns of matrices R and P by $r_1, \ldots, r_k$ and $p_1, \ldots, p_\ell$, respectively. We allow the possibility that $k = 0$ or $\ell = 0$ (but not both). This general model contains as special cases classical optimization models such as *mixed-integer linear optimization and mixed-integer convex optimization.*

Given $n \in \mathbb{N}$ and a closed subset $S \subseteq \mathbb{R}^n$ such that $0 \notin S$, a *cut-generating pair* (ψ, π) for S is a pair of functions $\psi, \pi : \mathbb{R}^n \to \mathbb{R}$ such that

$$\sum_{i=1}^{k} \psi(r_i)s_i + \sum_{j=1}^{\ell} \pi(p_j)y_j \geq 1 \tag{2}$$

is a valid inequality (also called a *cut*) for the set $X_S(R,P)$ for every choice of $k, \ell \in \mathbb{Z}_+$ and for all matrices $R \in \mathbb{R}^{n\times k}$ and $P \in \mathbb{R}^{n\times\ell}$. Cut-generating pairs thus provide cuts that separate 0 from the set $X_S(R,P)$. We emphasize that cut-generating pairs depend on n and S and do *not* depend on k, ℓ, R and P. There is a natural partial order on the set of cut generating pairs; namely, $(\psi', \pi') \leq (\psi, \pi)$ if and only if $\psi' \leq \psi$ and $\pi' \leq \pi$. The minimal elements under this partial ordering are called *minimal cut-generating pairs.*

Efficient procedures for cut-generating pairs. Several deep structural results were obtained by Johnson [10] about minimal cut-generating functions for S when S is a translated lattice, i.e., $S = b + \mathbb{Z}^n$ for some $b \in \mathbb{R}^n \setminus \mathbb{Z}^n$. However, a major drawback is that the theory developed is abstract and difficult to use from a computational perspective. A recent approach has been to restrict attention to a specific class of minimal cut-generating pairs for which we can give computational procedures to compute the values $\psi(r_i)$ and $\pi(p_j)$. We show how this is done when S is a translated lattice intersected with a polyhedron, i.e., $S = (b + \mathbb{Z}^n) \cap Q$ for some vector $b \in \mathbb{R}^n \setminus \mathbb{Z}^n$ and some rational polyhedron Q.

Given such a set $S \subseteq \mathbb{R}^n$, define $W_S := \mathbb{Z}^n \cap \mathrm{lin}(\mathrm{conv}(S))$. A convex set B is called S-free if $\mathrm{int}(B) \cap S = \emptyset$. A *maximal* S-free set is an S-free convex set that is inclusion wise maximal. It was shown in [8, 5] that any maximal S-free polyhedron B containing the origin in its interior is given by

$$B = \{r \in \mathbb{R}^n : a_i \cdot r \leq 1 \;\; i \in I\}. \tag{3}$$

Define the following pair of functions associated with B:

$$\psi_B(r) = \max_{i\in I} a_i \cdot r, \quad \pi_B(r) = \inf_{w\in W_S} \psi(r+w) \tag{4}$$

It can be checked that the above pair is a valid cut-generating pair and so for every maximal S-free convex set B, (4) gives formulas to compute with the corresponding cut-generating pair (ψ_B, π_B). Moreover, the pair is “partially” minimal: for every cut-generating pair $(\psi, \pi) \leq (\psi_B, \pi_B)$, we must have $\psi = \psi_B$. However, it may be the case that there exists a pair (ψ, π) with $\pi \leq \pi_B$ and $\pi(r) < \pi_B(r)$ for some $r \in \mathbb{R}^n$. The main question of this talk is:

QUESTION: Let $S = (b + \mathbb{Z}^n) \cap Q$ with $b \in \mathbb{R}^n \setminus \mathbb{Z}^n$ and a rational polyhedron Q. Given a maximal S-free convex set B (3), decide if (ψ_B, π_B) is minimal.

This approach to obtaining cut-generating pairs was pioneered by Dey and Wolsey in [9, 8] which is very much inspired by earlier work of Balas [2] and Balas and Jeroslow [3].

Statement of Results. Let S be a translated lattice intersected by a polyhedron Q, i.e., $S = (b+\mathbb{Z}^n) \cap Q$. Let B be a maximal S-free convex set given by (3). For each $s \in B \cap S$, define the *spindle* $R(s)$ in the following way. Let $k \in I$ such that $a_k \cdot s = 1$. Define $R(s) := \{r \in \mathbb{R}^n : (a_i - a_k)\cdot r \leq 0, \;\; (a_i - a_k)\cdot(s-r) \leq 0 \quad \forall i \in I\}$, and

$$R(S,B) := \bigcup_{s \in B \cap S} R(s).$$

It was shown in [4] that (ψ_B, π_B) is a minimal cut-generating pair if $R(S,B) + W_S = \mathbb{R}^n$, in which case we say that B has the *covering property.*

(1) Our first result is

Theorem 1 (Translation Invariance Theorem [7])**.** *$R(S,B) + W_S = \mathbb{R}^n$ if and only if $R(S+t, B+t) + W_{S+t} = \mathbb{R}^n$ for all $t \in \mathbb{R}^n$ such that $B+t$ also contains the origin in its interior.*

In other words, the covering property is preserved under translations. This result was first proved for the case when S is a translated lattice and B is a maximal S-free *simplicial* polytope [6], and later generalized to all maximal S-free sets when S is a translated lattice in [1]. The proofs in [6] and [1] are based on volume arguments, whereas the proof from [7] is based on a completely different topological argument. Besides achieving greater generality for S, the proof is cleaner, albeit at the expense of using more sophisticated topological tools like the *Invariance of Domain* theorem.

(2) Given two polyhedra B_1 and B_2 of the form (3), we define the *coproduct* $B_1 \diamond B_2$ which is a new polytope that has nice properties in terms of the covering property. More precisely, let $n = n_1 + n_2$. For $i \in \{1,2\}$, let $S_i = Q_i \cap (b_i + \mathbb{Z}^{n_i})$, where $Q_i \subseteq \mathbb{R}^{n_i}$ is a rational polyhedron.

Theorem 2. *For $i \in \{1,2\}$, let $B_i \subseteq \mathbb{R}^{n_h}$ be maximal S_i-free polyhedra with the covering property. Let $\mu \in (0,1)$. Then $\frac{B_1}{\mu} \diamond \frac{B_2}{1-\mu}$ is a maximal $S_1 \times S_2$-free set with the covering property.*

(3) We also show that if a sequence of maximal S-free sets all having the covering property, converges to a maximal S-free set (in a precise mathematical sense), then the "limit" set also has the covering property.

(4) We next characterize pyramids with the covering property.

Theorem 3. *Let $S = b + \mathbb{Z}^n$ for some $b \notin \mathbb{Z}^n$, and let B be a maximal lattice-free pyramid in $\mathbb{R}^n$ ($n \geq 2$) such that every facet of B contains exactly one integer point in its relative interior. K has the covering property if and only if B is the image of* $\mathrm{conv}\{0, ne^1, \ldots, ne^n\}$ *under an affine unimodular transformation.*

The proof of this theorem uses deep results from geometry of numbers, such as the *Venkov-Alexandrov-McMullen theorem*, *McMullen's characterization of zonotopes* and the *Minkowski-Hajós theorem.*

The importance of these results in terms of cutting planes is the following. Using 4) above, we can have a "base set" of maximal S-free sets with the covering property. By iteratively applying the three operations stated in 1), 2) and 3) above, we can then build a vast (infinite) list of maximal S-free sets (in arbitrarily high dimensions) with the covering property, enlarging this "base set". Not only does this recover all the previously known sets with the covering property, it vastly expands this list. This makes a contribution in the modern thrust on obtaining efficiently computable formulas for computing cutting planes, by giving a much wider class of *provably minimal* cut-generating pairs that can be computed via (4).

References

[1] Gennadiy Averkov and Amitabh Basu. Lifting properties of maximal lattice-free polyhedra. `http://arxiv.org/abs/1404.7421`.

[2] Egon Balas. Intersection cuts – a new type of cutting planes for integer programming. *Operations Research*, 19:19–39, 1971.

[3] Egon Balas and Robert G. Jeroslow. Strengthening cuts for mixed integer programs. *European Journal of Operational Research*, 4(4):224–234, 1980.

[4] Amitabh Basu, Manoel Campêlo, Michele Conforti, Gérard Cornuéjols, and Giacomo Zambelli. Unique lifting of integer variables in minimal inequalities. *Math. Program.*, 141(1-2, Ser. A):561–576, 2013.

[5] Amitabh Basu, Michele Conforti, Gérard Cornuéjols, and Giacomo Zambelli. Minimal inequalities for an infinite relaxation of integer programs. *SIAM Journal on Discrete Mathematics*, 24:158–168, February 2010.

[6] Amitabh Basu, Gérard Cornuéjols, and Matthias Köppe. Unique minimal liftings for simplicial polytopes. *Mathematics of Operations Research*, 37(2):346–355, 2012.

[7] Amitabh Basu and Joe Paat. Lifting properties for maximal S-free convex sets. 2014.

[8] Santanu S. Dey and Laurence A. Wolsey. Constrained infinite group relaxations of mips. *SIAM Journal on Optimization*, 20(6):2890–2912, 2010.

[9] Santanu S. Dey and Laurence A. Wolsey. Two row mixed-integer cuts via lifting. *Mathematical Programming*, 124:143–174, 2010.

[10] Ellis L. Johnson. On the group problem for mixed integer programming. *Mathematical Programming Study*, 2:137–179, 1974.

A geometric approach to cutting planes

Michele Conforti

The cutting-plane approach to integer programming was initiated more that 40 years ago: Gomory introduced the corner polyhedron as a relaxation of a mixed integer set in tableau form and Balas introduced intersection cuts for the corner polyhedron. This line of research was left dormant for several decades and did not have an impact in computations until relatively recently.

A paper of Andersen, Louveaux, Weismantel and Wolsey [1] has generated a renewed interest in the corner polyhedron and intersection cuts. Recent developments rely heavily on tools drawn from convex analysis, geometry, number theory, and constitute an elegant bridge between these areas and integer programming.

1. The model

For fixed $n \in \mathbb{N}$, let S be a closed subset of $\mathbb{R}^n$ that does not contain the origin 0. We consider subsets of the following form:

$$C_S(R) := \left\{ s \in \mathbb{R}^k_+ \,:\, Rs \in S \right\},$$

where $k \geq 1$. We address the following

Separation problem: *Find a closed half-space that contains $X_S(R,P)$ but not the origin.*

The fact that S is closed and $0 \notin S$ implies 0 is not in the closed convex hull of $C_S(R)$ [4, Lemma 2.1]. Hence such a half-space always exists.

This problem arises typically when one wants to design a cutting-plane method to optimize a (linear) function over $C_S(R)$ and has on hand a solution (the origin 0) to a relaxation of the problem.

We develop a theory that for fixed S addresses the separation problem *independently of R* by introducing valid functions.

A function $\psi : \mathbb{R}^n \to \mathbb{R}$ is a *valid function* for S if

$$\sum \psi(r) s_r \geq 1$$

is an inequality separating 0 from $C_S(R)$ for every k and R. We use the convention that the above sum is taken over the columns r of R.

There is a natural partial order on the set of valid functions, namely $\psi' \leq \psi$ if and only if $\psi'(r) \leq \psi(r)$ for every $r \in \mathbb{R}^n$. Since $\{s : \sum \psi'(r)s_r \geq 1, s \geq 0\} \subseteq \{s : \sum \psi(r)s_r + \geq 1, s \geq 0\}$ whenever $\psi' \leq \psi$, all the cuts obtained from ψ are dominated by those obtained from ψ'. The minimal elements under this partial order are called *minimal valid functions.* An application of Zorn's lemma shows that every valid function is dominated by a minimal valid function. Thus one can concentrate on the minimal valid functions.

2. Minimal valid functions and maximal S-free convex sets.

Given $S \subset \mathbb{R}^n$, a closed, convex set K is *S-free* if $\text{int}(F) \cap \mathbb{Z}^n = \emptyset$. An S-free set in *maximal* if k is not properly contained in another S-free set. ($\int(\cdot)$ is the interior)

A function $g : \mathbb{R}^n \to \mathbb{R}$ is *subadditive* if $g(r^1) + g(r^2) \geq g(r^1 + r^2)$ for all $r^1, r^2 \in \mathbb{R}^n$. The function g is *positively homogeneous* if $g(\lambda r) = \lambda g(r)$ for every $r \in \mathbb{R}^n$ and every $\lambda > 0$. The function g is *sublinear* if it is both subadditive and positively homogeneous.

Conforti, Cornuéjols, Daniilidis, Lemaréchal and Malick [4] studied the link between minimal valid functions and maximal S-free convex sets.

Theorem 1. *Given a closed set $S \subseteq \mathbb{R}^n \setminus \{0\}$, let $\psi : \mathbb{R}^n \to \mathbb{R}$ be a valid function for S, and let ψ' be defined as*

$$\psi'(\hat{r}) = \inf \left\{ \sum \psi(r)s_r : \sum r s_r = \hat{r}, s_r \geq 0 \right\} \text{ for every } \hat{r} \in \mathbb{R}^n.$$

Then ψ' is a valid function $\mathbb{R}^n \to \mathbb{R}$ which is sublinear.

If ψ and ψ' are as in Theorem 1, then $\psi' \leq \psi$ by definition. Therefore to characterize minimal valid functions, one can concentrate on sublinear functions.

Given a sublinear function ρ, let

$$V_\rho := \left\{ r \in \mathbb{R}^n : \rho(r) \leqslant 1 \right\}.$$

Then V_ρ is a closed convex set and $0 \in \operatorname{int}(V_\rho)$. Conversely, given a closed convex set V with $0 \in \operatorname{int}(V)$, a sublinear function ρ such that $V = V_\rho$ is a *representation* of V.

Theorem 2. *Let $S \subseteq \mathbb{R}^n \setminus \{0\}$ be a closed set, let ρ be a sublinear function, and let V_ρ be defined as above. Then ρ is a valid function for S if and only V_ρ is S-free.*

In view of Theorems 1 and 2, to characterize minimal valid functions for S one has to study representations of S-free convex sets, which are in general not unique. However, these representations satisfy the following:

Theorem 3. *Let $V \subseteq \mathbb{R}^n$ be a closed convex set with $0 \in \operatorname{int}(V)$ and let ρ be a representation of V. Then*

$$\rho(r) \leq 0 \iff r \in \operatorname{rec}(V), \text{ and } \rho(r) < 0 \implies r \in \operatorname{int}(\operatorname{rec}(V)).$$

Furthermore all representations of V coincide in $V \setminus \operatorname{int}(\operatorname{rec}(V))$. ($\operatorname{rec}(\cdot)$ *recession cone).*

The *polar* of a set $V \subseteq \mathbb{R}^n$ is the set $V^\circ = \{r \in \mathbb{R}^n : rd \leq 1 \text{ for all } d \in V\}$. A set $G \subseteq \mathbb{R}^n$ is a *prepolar* of V if $G^\circ = V$. If V is a closed convex set and $0 \in \operatorname{int}(V)$, then V° is a bounded set and $(V^\circ)^\circ = V$. Therefore in this case the polar of V is itself a prepolar, but V may have other prepolars.

The *support function* of a set $G \subset \mathbb{R}^n$ is

$$\sigma_G(r) := \sup_{d \in G} dr. \tag{1}$$

The support function is sublinear, and remains unchanged if G is replaced by its closed convex hull: $\sigma_G = \sigma_{\overline{\operatorname{conv}}(G)}$. Conversely, any sublinear function σ is the support function of a closed convex set, defined by

$$G_\rho := \left\{ d \in \mathbb{R}^n : dr \leqslant \sigma(r) \text{ for all } r \in \mathbb{R}^n \right\}.$$

Theorem 4. *Let V be a closed convex set with $0 \in \operatorname{int}(V)$. Then V admits an inclusion-wise smallest prepolar. The smallest representation of V is the support function of the smallest prepolar of V.*

If V is a polyhedron K and $0 \in \operatorname{int}(K)$, the support function of the smallest prepolar of V can be computed as follows.

Theorem 5. *Let $a_i x \leq 1, i \in I$ be an irredundant representation of a polyhedron K with $0 \in \operatorname{int}(K)$. Then $\{a_i,\, i \in I\}$ is the smallest prepolar of K. Hence the smallest representation of K is the function μ_K defined as*

$$\mu_K(r) = \max_{i \in I} a_i r. \tag{2}$$

So it is important to know when maximal S-free sets are polyhedra. In the next section we show that when $S = \mathbb{Z}^n$, this is always the case.

2.0.1. *Maximal lattice-free convex sets.* In this subsection we give a characterization of the maximal $\mathbb{Z}^n$-free convex sets in $\mathbb{R}^n$. A *lattice* of dimension t is a set of the type $\{x \in \mathbb{R}^n : \; x = \lambda_1 a_1 + \cdots + \lambda_t a_t;\, \lambda_1, \ldots, \lambda_t \in \mathbb{Z}\}$, where $a_1, \ldots, a_t$ are linearly independent vectors in $\mathbb{R}^n$. It follows from this definition that $\mathbb{Z}^n$ is a lattice. We call a $\mathbb{Z}^n$-free convex set *lattice-free.*

Theorem 6. *A set $K \subseteq \mathbb{R}^n$ is a maximal lattice-free convex set if and only if it satisfies one of the following conditions:*

(a) *$K = a + L$, where $a \in \mathbb{R}^n$ and L is a subspace of dimension $n-1$ that is not a lattice subspace.*
(b) *K is an n-dimensional polyhedron of the form $K = Q + L$, where L is a lattice subspace of dimension r $(0 \leq r < n)$, Q is a polytope of dimension $n - r$, and the relative interior of every facet of K contains an integer point.*

References

[1] Kent Andersen, Quentin Louveaux, Robert Weismantel, and Laurence Wolsey. Inequalities from two rows of a simplex tableau. In Matteo Fischetti and David Williamson, editors, *Integer Programming and Combinatorial Optimization. 12th International IPCO Conference, Ithaca, NY, USA, June 25–27, 2007. Proceedings*, volume 4513 of *Lecture Notes in Computer Science*, pages 1–15. Springer Berlin / Heidelberg, 2007.

[2] Amitabh Basu, Michele Conforti, Gérard Cornuéjols, and Giacomo Zambelli. Maximal lattice-free convex sets in linear subspaces. *Mathematics of Operations Research*, 35:704–720, 2010.

[3] Amitabh Basu, Michele Conforti, Gérard Cornuéjols, and Giacomo Zambelli. Minimal inequalities for an infinite relaxation of integer programs. *SIAM Journal on Discrete Mathematics*, 24:158–168, February 2010.

[4] Michele Conforti, Gérard Cornuéjols, Aris Daniilidis, Claude Lemaréchal, and Jérôme Malick. Cut-generating functions. In *Integer Programming and Combinatorial Optimization*, pages 123–132. Springer, 2013.

[5] Michele Conforti, Gérard Cornuéjols, and Giacomo Zambelli. *Integer Programming.* Graduate Textes in Mathematics, Springer, 2015.

Approximations of polytopes using sparse inequalities

SANTANU S. DEY

(joint work with Andres Iroume, Marco Molinaro and Qianyi Wang)

We study how well one can approximate arbitrary polytopes using sparse inequalities. Our motivation comes from the use of sparse cutting-planes in mixed-integer programing (MIP) solvers, since they help in solving the linear programs encountered during branch-and-bound more efficiently. However, how well can we approximate the integer hull by just using sparse cutting-planes? In order to understand this question better, given a polyope P (e.g. the integer hull of a MIP), let P^k be its best approximation using valid inequalities for P that have at most k non-zero coefficients. We consider $d(P, P^k) = \max_{x \in P^k} (\min_{y \in P} \|x - y\|)$ as a measure of the quality of sparse cuts. We note here that $d(P, P^k)$ may be considered as the worst-case additive error between optimizing over P and P^k. In particular, let

$$gap_P^k(c) = \max_{x \in P^k} cx - \max_{x \in P} cx.$$

Then it is straighforward to verify that: For every polytope $P \subseteq \mathbb{R}^n$,

$$d(P, P^k) = \max_{c:||c||=1} gap_P^k(c).$$

Observe also that for polytopes in the $[0,1]^n$ hypercube $d(P, P^k) \leq \sqrt{n}$.

In our first result, we present general upper bounds on $d(P, P^k)$ which depend on the number of vertices in the polytope.

Theorem 1 ([1]). *Let n be a natural number greater than 1. Let $P \subseteq [0,1]^n$ be the convex hull of points $\{p^1, \ldots, p^t\}$. Then*

(1) $d(P, P^k) \leq 4 \max \left\{ \frac{n^{1/4}}{\sqrt{k}} \sqrt{8 \max_{i \in [t]} \|p^i\|} \sqrt{\log 4tn}, \frac{8\sqrt{n}}{3k} \log 4tn \right\}$

(2) $d(P, P^k) \leq 2\sqrt{n} \left(\frac{n}{k} - 1 \right)$.

These upper bounds imply that if P has polynomially many vertices, then using constant factor sparsity already approximates it very well.

We present a lower bound on $d(P, P^k)$ for random polytopes that show that the upper bounds are quite tight.

Theorem 2 ([1]). *Let $k, t, n \in \mathbb{Z}_{++}$ satisfying $64 \leq k \leq n$ and $(0.5k^2 \log n + 2k + 1)^2 \leq t \leq e^n$. Let $X^1, X^2, \ldots, X^t$ be independent uniformly random points in $\{0,1\}^n$ and let $P = \mathrm{conv}(X^1, X^2, \ldots, X^t)$. Then with probability at least $1/4$ we have that*

$$d(P, P^k) \geq \min \left\{ \frac{\sqrt{n}}{\sqrt{k}} \frac{\sqrt{\log t}}{110\sqrt{\log n}}, \frac{\sqrt{n}}{8} \right\} \left(\frac{1}{2} - \frac{1}{k^{3/2}} \right) - 3\sqrt{\log t}.$$

We are able to establish that for a class of hard packing IPs, sparse cutting-planes do not approximate the integer hull well, that is $d(P, P^k)$ is large for such instances unless k is very close to n. Formally, given parameters $n, m, M \in \mathbb{Z}_{++}$, the convex hull of the packing IP is given by

$$P = \mathrm{conv}\left(\{x \in \{0,1\}^n : A^j x \leq \frac{\sum_{i=1}^n A_i^j}{2}, \ \forall j \in \{1, ..., m\}\} \right),$$

where the A_i^j's are chosen independently and uniformly in the set $\{0, 1, \ldots, M\}$. Let (n, m, M)-PIP denote the distribution over the generated P's.

Theorem 3 ([1]). *Consider $n, m, M \in \mathbb{Z}_{++}$ such that $n \geq 50$ and $8 \log 8n \leq m \leq n$. Let P be sampled from the distribution (n, m, M)-PIP. Then with probability at least $1/2$, $d(P, P^k) \geq \frac{\sqrt{n}}{2}\left(\frac{2}{\max\{\alpha,1\}}(1-\epsilon)^2 - (1+\epsilon')\right)$, where $c = k/n$ and*

$$\frac{1}{\alpha} = \frac{M}{2(M+1)}\left[\frac{n - 2\sqrt{n \log 8m}}{c((2-c)n+1) + 2\sqrt{10cnm}}\right],\ \epsilon = \frac{24\sqrt{\log 4n^2 m}}{\sqrt{n}},$$
$$\epsilon' = \frac{4\sqrt{\log 8n}}{\sqrt{m} - 2\sqrt{\log 8n}}.$$

Next we study the effect of rotation of polytopes on $d(P, P^k)$. We are able to establish a negative result.

Theorem 4 (D., Iroume, Molinaro). *Let $n \in \mathbb{Z}_{++}$. There exists a family of polytopes $Q_n \in [-1,\ 1]^n$ such that for every rotation $R : \mathbb{R}^n \to \mathbb{R}^n$ we have*

$$d(R[Q_n], (R[Q_n])^k) = \Omega(\sqrt{n}),$$

where $k = \frac{\sqrt{n}}{100}$.

All the results up till now consider the measure $d(P, P^k)$. We now consider the measure $gap_P^k(c)$. Can this measure be bad in all directions? We present a family of polytopes where such a negative result holds.

Theorem 5 (D., Iroume, Molinaro). *Let $n \in \mathbb{Z}_{++}$. There exists a family of polytopes $Q_n \in [-1,\ 1]^n$ such that (for sufficiently large n): If $C \in \mathbb{R}^n$ is a random direction uniform on the unit sphere, then for $k = \frac{n}{10}$,*

$$\Pr\left(gap_{Q_n}^k(C) \geq \frac{\sqrt{n}}{20}\right) \geq 1 - \frac{4}{n}.$$

Next we consider the following question: Suppose we allow a few dense inequalities in the approximation of the polytope. In particular, we are interested in knowing if adding a polynomial number of dense inequalities can significantly improve over the quality of approximation of P^k. We are able to establish a negative result.

Theorem 6 (D., Iroume, Molinaro). *Let $n \in \mathbb{Z}_{++}$. There exists a family of polytopes $Q_n \in [-1,\ 1]^n$ such that (for sufficiently large n): For $k = \frac{n}{100}$, $c > 1$ and $c \leq \exp(k)$, we have that:*

$$d(Q_n, (Q_n)^k \text{ intersected with any } c \text{ valid inequalities for } Q_n) \geq \frac{1}{6}\sqrt{n}.$$

Finally, we show that using sparse inequalities in extended formulations is at least as good as using them in the original polyhedron, and give an example where the former is actually much better.

Theorem 7 ([1]). *Let $n \in \mathbb{Z}_{++}$. Consider a polyhedron $P \subseteq \mathbb{R}^n$ and an extension $Q \subseteq \mathbb{R}^n \times \mathbb{R}^m$ for it. Then $\mathrm{proj}_x(Q^k) \subseteq (\mathrm{proj}_x(Q))^k = P^k$.*

Moreover, assume n is a power of 2. Then there is a polytope $P_n \subseteq [0,1]^n$ such that:

(1) $d(P_n, P_n^k) = \sqrt{n/2}$ for all $k \leq n/2$.
(2) There is an extension $Q_n \subseteq \mathbb{R}^n \times \mathbb{R}^{2n-1}$ of P_n such that $\mathrm{proj}_x(Q_n^3) = P_n$.

References

[1] S.S. Dey, M. Molinaro, Q. Wang, *How Good Are Sparse Cutting-Planes?*, IPCO **17** (2014), 261–272.

Recent applications of supermodular functions

András Frank
(joint work with Kristóf Bérczi)

In this talk three apparently independent problems are considered.

1. A classic result of Edmonds states that in a digraph $D = (V, A)$ with a specified root-node r_0, there are k disjoint spanning arborescences of root r_0 if and only if the digraph is rooted k-edge-connected, that is, the in-degree $\varrho(X)$ of every non-empty subset X of $V - r_0$ is at least k. Here we provide a necessary and sufficient condition for the existence of k such arborescences when the i' arborescence, in addition, is required to have c_i root-edges, where $c_1, c_2, \ldots, c_k$ are specified positive integers. (A root-edge is an edge with tail r_0).

2. Ryser proved a min-max formula for the maximum term-rank of a $(0,1)$-matrix with specified row-sums and column-sums. The term-rank of a matrix is the maximum number of independent 1-s. Suppose that there is a matroid on the set of columns and there is a matroid on the set of rows. By the matroidal term-rank of a matrix, we mean the maximum number of independent 1-s for which the set of columns containing such an entry is independent in the first matroid and the set of rows containing such an entry is independent in the second matroid. (A theorem of Kőnig characterizes the term-rank of a given matrix, while a theorem of Brualdi characterizes the matroidal term-rank of a given matrix.) As an extension of Ryser's theorem, we developed a min-max formula for the maximum matroidal term-rank of a $(0,1)$-matrix with specified row-sums and column-sums.

3. We describe a necessary and sufficient condition for two specified sequences $\{m_o(v_1), \ldots, m_o(v_n)\}$ and $\{m_i(v_1), \ldots, m_i(v_n)\}$ to admit a simple k-connected digraph on node-set $\{v_1, \ldots, v_n\}$ for which $\varrho(v_j) = m_i(v_j)$ and $\delta(v_j) = m_o(v_j)$ for $j = 1, \ldots, n$, where $\varrho(v)$ and $\delta(v)$, respectively, denotes the in-degree and the out-degree of node v.

Each of the three results is derived with the help of a theorem of Frank and Jordán on covering a supermodular function by a minimum number of directed edges.

Polynomiality for bin packing with a fixed number of item types

Michel X. Goemans
(joint work with Thomas Rothvoß)

In the *bin packing problem*, we are given a set of items, each with a certain size, and the goal is to assign them to the smallest possible number of bins such that the total size of the items assigned to any bin is at most 1. We consider here the *1-dimensional cutting stock* version of the problem, in which there might be several items of the same size. We assume we are given d item types, and type i (for $i = 1, \cdots, d$) has a given size $s_i \in [0, 1]$ and a given multiplicity $a_i \in \mathbb{Z}_{\geq 0}$. All these numbers are given in binary encoding. The study of the cutting stock problem goes back to the classical paper of Gilmore and Gomory [3]. We can represent the set of feasible patterns (or item type multiplicities) that fit a single bin by $\mathcal{P} := \{x \in \mathbb{Z}^d_{\geq 0} \mid s^T x \leq 1\}$. The smallest number of bins is therefore given by

$$\min\Big\{ \sum_{x\in\mathcal{P}} \lambda_x \mid \sum_{x\in\mathcal{P}} \lambda_x \cdot x = a;\ \ \lambda \in \mathbb{Z}^{\mathcal{P}}_{\geq 0} \Big\}. \tag{1}$$

For general d (and even unit multiplicities), the problem is known to be strongly NP-hard [4]. Observe that with multiplicities given in binary, the smallest number of bins can be exponential in the input size, and therefore one needs to consider compact encoding of the output bins, such as by providing λ_x for a polynomial number of patterns $x \in \mathcal{P}$. Eisenbrand and Shmonin [1] have shown that such a compact encoding exists, thereby showing that bin packing is in NP.

In this work, we present a polynomial-time algorithm for the bin packing problem when d is constant. This was an open problem for $d \geq 3$, see [6, 1, 2]. Our algorithm actually solves a more general problem in polynomial-time for constant d: given two d-dimensional polytopes $P \subseteq \mathbb{R}^d$ and $Q \subseteq \mathbb{R}^d$, find the smallest number of integer points in P whose sum lies in Q. This result allows us to tackle a host of other combinatorial optimization problems, such as *high multiplicity* scheduling problems in which the number of copies of each job type is given in binary encoding and each type comes with certain parameters such as release dates, processing times and deadlines. We show that a variety of high multiplicity scheduling problems can be solved in polynomial time if the number of job types is constant.

Our key ingredient is the following. For any polytope P in $\mathbb{R}^d$, consider the set of its integral conic combinations

$$C = \{ \sum_{x\in P\cap\mathbb{Z}^d} \lambda_x x | \lambda_x \in \mathbb{Z}_{\geq 0}\ \forall x\}.$$

We prove that we can (efficiently) find a set $X \subseteq P \cap \mathbb{Z}^n$ of polynomial size (for constant d) such that for any $a \in C$, we can express a as $\sum_{x\in P\cap\mathbb{Z}^d} \lambda_x x$ with

- $\lambda_x \in \{0, 1\}$ for all $x \in (P \cap \mathbb{Z}^d) \setminus X$,
- $\lambda_x \in \mathbb{Z}$ for all $x \in X$,

- $|\,supp(\lambda) \cap X| \leq 2^{2d}$ and
- $|\,supp(\lambda) \setminus X| \leq 2^{2d}$.

Once we have proved this structure theorem, the algorithm follows fairly easily by guessing the (constant number of) points in $supp(\lambda) \cap X$ and using Lenstra's result [5] for integer linear programming in fixed dimension.

References

[1] F. Eisenbrand and G. Shmonin, *Carathéodory bounds for integer cones*, Operations Research Letters, **34** (2000), 564–568.

[2] C. Filippi, *On the bin packing problem with a fixed number of object weights*, European Journal of Operational Research **181** (2007), 117–126.

[3] P.C. Gilmore and R.E. Gomory, *A linear programming approach to the cutting stock problem*, Opereations Research **9** (1961), 849–859.

[4] D.S. Johnson, *The NP-completeness column, an ongoing guide: The tale of the second prover*, Journal of Algorithms **13** (1992), 502–524.

[5] H.W. Lenstra, *Integer programming with a fixed number of variables*, Mathematics of Operations Research **8** (1983), 538–548.

[6] S.T. McCormick, S.R. Smallwood and F.C.R. Spieksma, *A polynomial algorithm for multiprocessor scheduling with two job lengths*, Mathematics of Operations Research **26** (2001), 36–49.

Generalized power diagrams, balanced k-means, and the representation of polycrystals

Peter Gritzmann

Based on a discrete convex maximization model we give an efficient algorithm for computing feasible generalized power diagams with near-optimal separation properties.

Further, we show how this approach can be used to generalize the classical k-means algorithms from data analysis so that it becomes capable of handling prescribed lower and upper bounds on the cluster sizes for weighted point sets. (This part is, generally, based on [2] and [3], and reports on recent joint work with S. Borgwardt and A. Brieden, [1].)

Also we indicate how to handle the discrete inverse problem from material science to compute grain maps i.e., representations of polycrystals, based only on measured data on the volume, center and, possibly, moments of their grains. (This part is ongoing joint work with A. Alpers, A. Brieden, A. Lyckegaard and H. Poulsen, [4].)

References

[1] S. Borgwardt, A. Brieden and P. Gritzmann, *A balanced k-means algorithm for weighted point sets*, 2014, submitted.

[2] A. Brieden and P. Gritzmann, *On clustering bodies: Geometry and polyhedral approximation*, Discrete Computational Geometry **44** (2010), 508–534.

[3] A. Brieden and P. Gritzmann, *On optimal weighted balanced clusterings: Gravity bodies and power diagrams*, SIAM J. Discrete Mathematics **26** (2012), 415–434.

[4] A. Alpers, A. Brieden, P. Gritzmann, A. Lyckegaard and H. Poulsen, *Generalized power diagrams for 3D representations of polycrystals*, 2014, manuscript.

Packing odd T-joins with at most two terminals

Bertrand Guenin
(joint work with Ahmad Abdi)

1. The result

A *signed graph* is a pair (G, Σ) where G is a graph and $\Sigma \subseteq E(G)$. A subset S of the edges is *odd* (resp. *even*) if $|S \cap \Sigma|$ is odd (resp. even). A *graft* is a pair (G, T) where G is a graph, $T \subseteq V(G)$ and $|T|$ is even. Vertices in T are *terminal* vertices. A T*-join* is an edge subset that induces a subgraph of G with the odd degree vertices equal to T. A *signed graft* is a triple (G, Σ, T) where (G, Σ) is a signed graph and (G, T) is a graft. Thus an *odd* T*-join* of (G, Σ, T) is a T-join of G that contains an odd number of edges of Σ. When $T = \emptyset$ an (inclusion-wise) minimal odd T-join, is an odd circuit. When $T = \{s, t\}$ a minimal odd T-join is either an odd st-path, or it is the union of an even st-path P and an odd circuit C where P and C share at most one vertex.

The maximum number of pairwise (edge) disjoint odd T-joins in (G, Σ, T) is denoted $\nu(G, \Sigma, T)$. The minimum number of edges needed to intersect all odd T-joins is denoted $\tau(G, \Sigma, T)$. A signed graft *packs* if $\tau(G, \Sigma, T) = \nu(G, \Sigma, T)$. Our result will give sufficient conditions for a signed graft with at most two terminals to pack. We denote by *odd-*K_5 the signed graft $(K_5, E(K_5), \emptyset)$. Since odd T-joins in that case are odd circuits we have $\tau = 4 > \nu = 2$, thus it does not pack. We denote by $\mathcal{L}_7$ the signed graft obtained as follows:

(a) start with a circuit with four even edges;
(b) for three of these edges add a parallel edge that is odd;
(c) choose the terminals to be equal to the endpoints of the even edge that is not parallel to an odd edge. In can be readily checked in this case that the odd T-joins correspond to the lines of the Fano matroid. Hence, $\tau = 3 > \nu = 1$ in that case.

We say that a signed graft is *Eulerian* if every non-terminal vertex has even degree and either: every terminal has odd degree and the signature has an odd number of edges; or every terminal has even degree and the signature has an even number of edges.

We can now state our main result which is a special case of Seymour's Cycling conjecture [8] (see also [7]),

Theorem 1. [1, 2] *If an Eulerian signed graft has at most two terminals and it does not contain odd-*K_5 *and does not contain* $\mathcal{L}_7$ *then it packs.*

It remains to describe what we mean by "contain" in this setting.

Consider a signed graft (G, Σ, T). (G, Γ, T) is obtained by *resigning* (G, Σ, T) if $\Gamma = \Sigma \triangle \delta(U)$ for some cut $\delta(U)$ where $|T \cap U|$ is even. For $e \in E(G)$, we say that $(G \setminus e, \Sigma - \{e\}, T)$ is obtained by *deleting* e. For $e = uv \in E(G) - \Sigma$, we say that $(G/e, \Sigma, T')$ is obtained by *contracting* e where $T' = T - \{u, v\}$ if both or none of u, v are in T and $T' = T - \{u, v\} \cup \{w\}$ if exactly one of u, v is in T where w is the vertex obtained from e by contracting e. A signed graft is a *minor* of (G, Σ, T) if it is obtained by sequentially deleting/contracting edges and resigning.

2. Some corollaries

We say that a graph H is an *odd-minor* of a graph G if H is obtained from G by first deleting edges and then contracting *all* edges on a cut. Theorem 1 for the case where we have no terminals implies,

Corollary 1 (Geelen and Guenin [4]). Let G be a graph that does not contain K_5 as an odd minor and where every vertex has even degree. Then the minimum number of edges needed to intersect all odd circuits is equal to the maximum number of pairwise disjoint odd circuits.

In the next proposition we indicate a number of classes of signed grafts that do not contain odd-K_5 and do not contain $\mathcal{L}_7$. In particular, for each of these classes, as long as the Eulerian condition holds, the minimum number of edges needed to intersect all odd T-joins is equal to the maximum number of pairwise disjoint odd T-joins. A *blocking vertex* (resp. *blocking pair*) of a signed graph is a vertex (resp. pair of vertices) that intersects every odd circuit.

Proposition 1. Let (G, Σ, T) be a signed graft where $T = \{s, t\}$. If any of (1)-(6) hold then (G, Σ, T) does not contain as a minor odd-K_5 or $\mathcal{L}_7$,

(1) There exists a blocking vertex,
(2) s, t is a blocking pair,
(3) Every inclusion-wise minimal odd T-join is connected,
(4) G is a plane graph with at most two odd faces,
(5) G is a plane graph with a blocking pair u, v where s, u, t, v appear on a facial cycle in this order,
(6) G has an embedding on the projective plane where every face is even and s, t are connected by an odd edge.

As a corollary of Theorem 1 and Proposition 1(1) we obtain,

Corollary 2. Let (H, T) be a graft with $|T| \leq 4$. Suppose that every vertex of H not in T has even degree and that all the vertices in T have degrees of the same parity. Then the maximum number of pairwise disjoint T-joins is equal to the minimum size of a T-cut.

In fact this result holds as long as $|T| \leq 8$ [3].

As a corollary of Theorem 1 and Proposition 1(2) we obtain,

Corollary 3 (Hu [5], Rothschild and Whinston [6]). Let H be a graph with vertices s_1, s_2, t_1, t_2 where $s_1 \neq t_1$, $s_2 \neq t_2$, all of s_1, t_1, s_2, t_2 have the same parity, and all the other vertices have even degree. Then the maximum number of pairwise disjoint paths that are between s_i and t_i for some $i = 1, 2$, is equal to the minimum size of an edge subset whose deletion removes all s_1t_1- and s_2t_2-paths.

Consider G obtained as follows:

($\star$) start from a plane graph with exactly two faces of odd length and distinct vertices s and t, and identify s and t.

As a corollary of Theorem 1 and Proposition 1(3) we obtain,

Corollary 4. Let H be a graph as in ($\star$) and suppose that the length of the shortest odd circuit is k. Then there exists cuts $B_1, \ldots, B_k$ such that every edge e is in at least $k-1$ of $B_1, \ldots, B_k$.

References

[1] A. Abdi and B. Guenin. *The Cycling property for the clutter of odd st-walks.* IPCO, Lecture Notes in Computer Science, 8494, 1–12, (2014).

[2] A. Abdi, B. Guenin. *Packing odd T-joins with at most two terminals.* Manuscript, arxiv.org/abs/1410.7423 (98 pages), (2014).

[3] J. Cohen, C. Lucchesi. *Minimax relations for T -join packing problems.* Proceedings of the Fifth Israeli Symposium on Theory of Computing and Systems (ISTCS 97), 3844, (1997).

[4] J.F. Geelen and B. Guenin. *Packing odd-circuits in Eulerian graphs.* J. Comb. Theory B. 86(2): 280295, (2002).

[5] T. C. Hu. *Multicommodity Network Flows.* Operations Research, volume 11, pages 344-360, (1963).

[6] B. Rothschild, A. Whinston. *Feasibility of two-commodity network flows.* Oper. Res. **14**, 1121–1129, (1966).

[7] A. Schrijver. *Combinatorial optimization. Polyhedra and efficiency.* Springer, 1408–1409 (2003).

[8] P.D. Seymour. *Matroids and multicommodity flows.* Europ. J. Combinatorics 2, 257–290 (1981).

Greedy algorithms for Steiner Forest

Anupam Gupta

(joint work with Amit Kumar)

In the Steiner forest problem, given a metric space and a set of source-sink pairs $\{s_i, t_i\}_{i=1}^K$, a feasible solution is a forest such that each source-sink pair lies in the same tree in this forest. The goal is to minimize the cost, i.e., the total length of edges in the forest. This problem is a generalization of the Steiner tree problem, and hence APX-hard. The constant-factor approximation algorithms currently known for it are all based on linear programming techniques. The first such result was an influential primal-dual 2-approximation due to Agrawal, Klein, and Ravi [AKR95]; this was simplified by Goemans and Williamson [GW95] and extended to many "constrained forest" network design problems. The primal-dual analysis also bounds integrality gap of the the natural LP relaxation (based on covering cuts) by a factor of 2. Other approximation algorithms for Steiner forest based on the same LP, and achieving the same factor of 2, are obtained using the iterative rounding technique of Jain, or the integer decomposition techniques of Chekuri and Shepherd [CS09]. A stronger LP relaxation was proposed by Könemann, Leonardi, and Schäfer, but it also has an integrality gap of 2.

However, it has remained an interesting open problem whether constant-factor approximations are known based on "purely combinatorial" techniques. For the special case of Steiner tree, where all the sources s_i are co-located (at some point s, say), we have long known that the minimum spanning tree (MST) heuristic gives a factor-2 approximation; hence running Prim's or Kruskal's algorithm on the terminals (ignoring the non-terminal, or Steiner vertices) would give a tree whose cost is within a factor 2 of the optimal Steiner tree.

Some natural algorithms have been proposed, but these have defied analysis for the most part. The simplest is the *paired greedy algorithm* that repeatedly connects the yet-unconnected s_i-t_i pair at minimum mutual distance; this can be viewed as an analog of Prim's algorithm in the Steiner forest setting. Unfortunately, it is known by now that this is no better than an $\Omega(\log n)$-factor approximation (see Chen et al.). Even greedier is the so-called *gluttonous algorithm* that connects the closest two yet-unsatisfied terminals regardless of whether they were "mates"; this can be viewed as the extension of Kruskal's algorithm for Steiner forests. The performance of this algorithm has been a long-standing open question. The main result of this talk settles this question in the affirmative: The gluttonous algorithm is a constant-factor approximation for Steiner Forest.

We use this result to obtain a simple combinatorial approximation algorithm for the *two-stage stochastic version* of the Steiner forest problem. In this problem, we are given a probability distribution π defined over subsets of demands. In the first stage, we can buy some set E_1 of edges. Then in the second stage, the demand set is revealed (drawn from π), and we can extend the set E_1 to a feasible solution for this demand set. However, these edges now cost $\sigma > 1$ times more than in the first stage. The goal is to minimize the total expected cost. It suffices to specify the set E_1—once the actual demands are known, we can augment using our favorite approximation algorithm for Steiner forest. Our simple algorithm is the following: sample $\lceil \sigma \rceil$ times from the distribution π, and let E_1 be the Steiner forest constructed by (a slight variant of) the gluttonous algorithm on union of these $\lceil \sigma \rceil$ demand sets sampled from π. The properties of the gluttonous algorithm for Steiner forest can be used to give certain "strict" cost shares that show that this random-sampling algorithm is a constant-factor approximation algorithm for the stochastic Steiner forest problem.

References

[AKR95] Ajit Agrawal, Philip Klein, and R. Ravi. When trees collide: an approximation algorithm for the generalized Steiner problem on networks. *SIAM J. Comput.*, 24(3):440–456, 1995.

[CRV10] Ho-Lin Chen, Tim Roughgarden, and Gregory Valiant. Designing network protocols for good equilibria. *SIAM J. Comput.*, 39(5):1799–1832, 2010.

[CS09] Chandra Chekuri and F. Bruce Shepherd. Approximate integer decompositions for undirected network design problems. *SIAM J. Discrete Math.*, 23(1):163–177, 2008/09.

[GW95] Michel X. Goemans and David P. Williamson. A general approximation technique for constrained forest problems. *SIAM J. Comput.*, 24(2):296–317, 1995.
[GK14] Anupam Gupta and Amit Kumar. Greedy Algorithms for Steiner Forest. manuscript, 2014.
[KLS05] Jochen Könemann, Stefano Leonardi, and Guido Schäfer. A group-strategyproof mechanism for steiner forests. In *SODA*, pages 612–619, 2005.

A quadratic upper bound on the extension complexities of the independence polytopes of regular matroids

Volker Kaibel
(joint work with Jon Lee, Matthias Walter and Stefan Weltge)

The extension complexity of a polytope P is the smallest number of facets of any polytope Q whose image under a linear map is P. Independence polytopes of graphic (and cographic) matroids have extension complexities that are bounded from above by $O(|V|\cdot|E|)$ if $G = (V, E)$ is the underlying graph (due to Martin [1]); if G is planar, the bound can even be improved to $O(|V|)$ (due to Williams [4]). On the other hand, Rothvoss [2] showed that it is not true that the extension complexities of the independence polytopes of all matroids can be bounded by a polynomial in the sizes of their ground sets.

In this talk, we prove that when one restricts to the class of regular matroids, then the corresponding extension complexities can be bounded quadratically in the sizes of the ground sets. The proof relies on Seymour's decomposition theorem [3] for regular matroids using 1-, 2-, and 3-sums in order to construct all regular matroids from graphic and cographic matroids as well one particular special regular matroid. By analyzing the sum-operations in terms of matrices over the field with two elements, we derive characterizations of the indepent sets of such sums of two matroids that allow us to construct extended formulations of the indepence polytopes of the resulting matroids from independence polytopes of the summands. In case all graphs arising in the decomposition are planar, the bound on the extension complexity of the independence polytope of the regular matroid is even linear.

References

[1] R. Kipp Martin, *Using separation algorithms to generate mixed integer model reformulations*, Oper. Res. Lett. **10** (1991), no. 3, 119–128.
[2] Thomas Rothvoß, *Some 0/1 polytopes need exponential size extended formulations*, Math. Program. Ser. A (2012), 1–14.
[3] P. D. Seymour, *Decomposition of regular matroids*, J. Combin. Theory Ser. B **28** (1980), no. 3, 305–359.
[4] Justin C. Williams, *A linear-size zero—one programming model for the minimum spanning tree problem in planar graphs*, Networks **39** (2002), no. 1, 53–60.

Optimization on polyhedra defined by submodular functions on diamonds

Tamás Király

(joint work with Satoru Fujishige, Kazuhisa Makino, Kenjiro Takazawa and Shin-ichi Tanigawa)

A set function $f : 2^V \to \mathbb{Z}$ is **submodular** if $f(X)+f(Y) \geq f(X\cup Y)+f(X\cap Y)$ for every $X, Y \subseteq V$. In the submodular function minimization problem, given an evaluation oracle for a submodular function f, we are asked to find a minimizer of f. For this problem, our goal is to find an algorithm with running time polynomial in $|V|$ and $\log \max_{X\subseteq V}\{|f(X)|\}$ that returns $X \in \mathrm{argmin}(f)$, assuming that the algorithm has access to an oracle that for any given X outputs $f(X)$.

It follows from the work of Grötschel, Lovász and Schrijver [4] on the equivalence of separation and optimization that such an algorithm can be obtained by using the ellipsoid method. Combinatorial strongly polynomial algorithms have only been obtained much later, independently by Schrijver [13] and by Iwata, Fleischer and Fujishige [6].

The generalization that we consider in this talk concerns submodular functions on lattices. Given a finite lattice L, a function $f : L \to \mathbb{Z}$ is **submodular on** L if $f(x) + f(y) \geq f(x \vee y) + f(x \wedge y)$ for every $x, y \in L$. For modular lattices, such functions naturally arise when extending the Dulmage-Mendelsohn decompositions of generic matrices to generic partitioned matrices [8], and it was posed as an open problem in [6] to give a polynomial-time algorithm for minimizing submodular functions on modular lattices.

As observed in [6, 13], one can reduce the problem to the standard submodular function minimization if the underlying lattice is distributive. Krokhin and Larose [9] showed that certain lattice operations preserve the tractability of the corresponding minimization problem in the value oracle model, and as a corollary they showed that the submodular function minimization on the product of the copies of the pentagon, a smallest non-distributive lattice, can be reduced to the standard submodular function minimization.

In this talk we consider the submodular function minimization problem on the product of diamonds, which is the remaining smallest non-distributive case and has an application to the Dulmage-Mendelsohn type decompositions of generic partitioned matrices consisting of two-by-two blocks [7]. A **diamond** is a lattice consisting of a minimal element, a maximal element, and an arbitrary finite number of pairwise incomparable **middle elements**: the meet (resp. join) of any two middle elements is the minimal (resp. maximal) element. A submodular function on the direct product of given diamonds $U_1, \dots, U_n$ is called a **submodular function on diamonds**. Note that if the diamonds have no middle elements, then we have a standard submodular set function. If the diamonds have at most two middle elements, then the lattice is distributive, thus we can use the standard submodular function minimization algorithm in this case too. However, a diamond with more than two middle elements is modular but not distributive, and hence we cannot directly apply the standard algorithms. A pseudo-polynomial algorithm for the minimization of submodular functions on diamonds was given by Kuivinen [10]. Our main result is the first polynomial-time algorithm.

Theorem. *Let f be a submodular function on the direct product of a finite number of diamonds $U_1, \dots, U_n$. A minimizer of f can be computed in a polynomial number of arithmetic steps and function evaluations in m and $\log M$, where $m = \sum_{i=1}^n |U_i|$ and M is the maximum absolute function value.*

Let $U = \bigcup_{i=1}^n U_i$, and call $T \subseteq U$ a **transversal** if $|T \cap U_i| = 1$ for every $i \in [n]$, where $[n]$ denotes the set of integers $\{1, \dots, n\}$. We denote by $\mathcal{T}$ the set of transversals and by $T_\mathbf{0}$ the transversal consisting of the minimal elements. There is a natural one-to-one correspondence between transversals and elements of the direct product lattice, which also defines operations $\wedge$ and $\vee$ on pairs of transversals. We assume $f(T_\mathbf{0}) = 0$.

For a transversal $T \in \mathcal{T}$, let $a(T)_i \in \{0, 1, 2\}$ denote the rank of the unique element $T \cap U_i$ in the lattice U_i. We consider the optimization problem

$$\max\{cx : x \in \mathbb{R}^n,\ a(T)x \le f(T)\ \forall T \in \mathcal{T}\}. \tag{1}$$

We give a combinatorial algorithm with polynomial running time for this problem. By the results of Grötschel, Lovász and Schrijver [4], this implies that the minimization of submodular functions on diamonds can be solved in polynomial time using the ellipsoid method.

When f is derived from a matroid rank function, the polytope describing (1) coincides with the **fractional matroid matching polytope** introduced by Vande Vate [14], and the corresponding optimization problem (1) is known as the **weighted fractional matroid matching problem**, which was solved by Gijswijt and Pap [3]. The main restriction compared to our generalized problem is that the lattice function corresponding to fractional matroid matching is derived from a matroid rank function, and hence it is monotone nondecreasing and has maximum value at most $2n$. Nevertheless, our algorithm makes use of several ideas from the Gijswijt-Pap paper.

A different extension of standard submodular minimization is the minimization of bisubmodular functions by Qi [12], Fujishige and Iwata [1], and Fujishige and McCormick [11]. Min-max theorems (without polynomial algorithms) were also given for the minimization of k-submodular functions, which is a common generalization of bisubmodular functions and multimatroid rank functions, by Huber and Kolmogorov [5], and for the more general class of transversal submodular functions by Fujishige and Tanigawa [2]. One of the exciting open problems is whether k-submodular functions can be minimized in polynomial time.

References

[1] S. Fujishige and S. Iwata, *Bisubmodular function minimization*, SIAM J. Discrete Math. **19** (2006), 1065–1073.

[2] S. Fujishige and S. Tanigawa, *A Min-max theorem for transversal submodular functions and its implications*, SIAM J. Discrete Math. **28** (2013), 1855–1875.

[3] D. Gijswijt and G. Pap, *An algorithm for weighted fractional matroid matching*, J. Combin. Theory, Ser. B **103** (2013), 509–520.
[4] M. Grötschel, L. Lovász and A. Schrijver, *The ellipsoid method and its consequences in combinatorial optimization*, Combinatorica **1** (1981), 169–197.
[5] A. Huber and V. Kolmogorov, *Towards minimizing k-Submodular functions*, in: Proc. 2nd International Symposium on Combinatorial Optimization (2012), 451–462.
[6] S. Iwata, L. Fleischer and S. Fujishige, *A combinatorial strongly polynomial algorithm for minimizing submodular functions*, J. ACM **48** (2001), 761–777.
[7] S. Iwata and K. Murota, *A minimax theorem and a Dulmage–Mendelsohn type decomposition for a class of generic partitioned matrices*, SIAM J. Matrix Anal. Appl. **16** (1995), 719–734.
[8] H. Ito, S. Iwata and K. Murota, *Block-triangularizations of partitioned matrices under similarity/equivalence transformations*, SIAM J. Matrix Anal. Appl. **15** (1994), 1226–1255.
[9] A. Krokhin and B. Larose, *Maximizing supermodular functions on product lattices, with application to maximum constraint satisfaction*, SIAM J. Discrete Math. **22** (2008), 312–328.
[10] F. Kuivinen, *On the complexity of submodular function minimisation on diamonds*, Discrete Optimization **8** (2011), 459–477.
[11] S.T. McCormick and S. Fujishige, *Strongly polynomial and fully combinatorial algorithms for bisubmodular function minimization*, Math. Program. **122** (2010), 87–120.
[12] L. Qi, *Directed submodularity, ditroids and directed submodular flows*, Math. Program. **42** (1988), 579–599.
[13] A. Schrijver, *A combinatorial algorithm minimizing submodular functions in strongly polynomial time*, J. Combin. Theory, Ser. B **80** (2000), 346–355.
[14] J. Vande Vate, *Fractional matroid matchings*, J. Combin. Theory, Ser. B **55** (1992), 133–145.

The Gomory–Johnson infinite group problem: A 42-year update

Matthias Köppe

(joint work with Amitabh Basu, Robert Hildebrand, Reuben La Haye, Quentin Louveaux, Marco Molinaro and Yuan Zhou)

The infinite group problem was introduced 42 years ago by Ralph Gomory and Ellis Johnson in their groundbreaking papers titled *Some continuous functions related to corner polyhedra I, II* [7, 8]. The technique, investigating strong relaxations of integer linear programs by embedding them in an infinite-dimensional master problem (and thus by convexity in a function space), has at times been dismissed as “esoteric.”

The valid inequalities for the master problems, after a standard normalization, are described by real-valued functions, called *valid functions*. The valid functions that are not pointwise dominated by other valid functions are called *minimal*. By a theorem of Gomory–Johnson, they are classified as subadditive periodic functions that satisfy a certain normalization condition. The goal is to study the strongest minimal inequalities for the master problem, which are the analogues to facet-defining inequalities in the finite-dimensional case. They are described by *extreme functions*.

The interest in Gomory–Johnson's work was renewed in the 2000s by various authors. Highlights included:

- New families of extreme functions from various constructions.
- A systematic investigation of limits of extreme functions, leading to the definition of families of discontinuous piecewise linear extreme functions.
- First constructions of extreme functions for multi-row infinite group problems, in particular by the sequential-merge procedure.
- Various computational experiments, including the so-called shooting experiments.

An excellent survey by Richard and Dey [10] reports on these developments.

Due to this work, combined with the recent interest in the more general framework of *cut generating functions* [6], we now recognize the infinite group problem as a technique which may be a key to solving today's pressing need for stronger, multi-row cutting plane approaches.

I present a new comprehensive survey, titled *Light on the infinite group problem* (with A. Basu, R. Hildebrand) [4], which reports on the latest developments since 2010. The highlights regarding the single-row relaxations include:

- The construction of non–piecewise linear continuous extreme functions [1].
- New insights on the question of non-negativity of coefficients of valid inequalities.
- A discussion of cases in which the notions of extreme functions, facets, and weak facets (all of which are analogues of facets in the finite-dimensional case) coincide.
- A systematic study of the space of perturbations of a minimal valid function, which leads to the **first algorithms for testing extremality** in [2]. These algorithms have pseudo-polynomial running time, and it is an open question whether this is best possible.
- The **completion of Gomory–Johnson's program** regarding the relation of the infinite group problem and embedded finite group problems in the single-row case [2].

  Theorem. *Let $m \in \mathbb{Z}_{\geq 3}$ be fixed. Let π be a continuous piecewise linear minimal valid function for the single-row infinite group problem $R_f(\mathbb{R}, \mathbb{Z})$ with breakpoints in $\frac{1}{q}\mathbb{Z}$ and suppose that the right-hand side value f also lies in $\frac{1}{q}\mathbb{Z}$. The following are equivalent:*
  (1) π is a facet for $R_f(\mathbb{R}, \mathbb{Z})$,
  (2) π is extreme for $R_f(\mathbb{R}, \mathbb{Z})$,
  (3) The restriction $\pi|_{\frac{1}{mq}\mathbb{Z}}$ using the oversampling factor of m is extreme for the finite group problem $R_f(\frac{1}{mq}\mathbb{Z}, \mathbb{Z})$.

- The discovery of a new extremality principle based on the density of infinite restricted orbits generated by numbers that are linearly independent over the rationals [2].
- New insights and conjectures on analytic-topological questions regarding the set of extreme functions.

Proposition. *There exists a sequence of continuous extreme functions (of type* `bhk_irrational`*) that converges uniformly to a continuous non-extreme function of the same type.*

The survey also reports on the significant progress that has been made regarding multi-row relaxations (with k rows), including:

- A systematic study of locally finite, periodic polyhedral complexes underlying piecewise linear minimal valid functions.
- A k-dimensional generalization of Gomory–Johnson's so-called Interval Lemma, in other words, a theorem on the regular solutions to Cauchy's additive functional equation in the bounded case [3].

  Theorem. *Let $\bar\pi\colon \mathbb{R}^k \to \mathbb{R}$ be a bounded function. Let $F \subseteq \mathbb{R}^k \times \mathbb{R}^k$ be a convex set such that $\bar\pi(\mathbf{u}) + \bar\pi(\mathbf{v}) = \bar\pi(\mathbf{u}+\mathbf{v})$ for all $(\mathbf{u},\mathbf{v}) \in F$. Let L be a linear subspace of $\mathbb{R}^k$ such that $(L \times L) + F \subseteq \operatorname{aff}(F)$. Let $(\mathbf{u}^0, \mathbf{v}^0) \in \operatorname{relint}(F)$. Then $\bar\pi$ is affine-linear with the same gradient over:*
  - $\operatorname{int}_L((\mathbf{u}^0 + L) \cap p_1(F))$,
  - $\operatorname{int}_L((\mathbf{v}^0 + L) \cap p_2(F))$,
  - $\operatorname{int}_L((\mathbf{u}^0 + \mathbf{v}^0 + L) \cap p_3(F))$.

  (Here $\operatorname{int}_L(U)$ denotes the interior of U in the relative topology of $L + \mathbf{u}$.)

- The $(k+1)$-slope theorem [5], a sufficient condition for extremality in the case of k rows, which generalizes Gomory–Johnson's celebrated 2-slope theorem.
- First results regarding the relation of the infinite group problem and embedded finite group problems in the two-row case [3].

An interactive companion program [9] allows experimentation with valid functions for the single-row infinite group problem and provides an updated compendium of known extreme functions.

I also present ongoing work (with R. Hildebrand, R. La Haye, Q. Louveaux, Y. Zhou) to complete the algorithmic problem of testing extremality for piecewise linear minimal functions with irrational algebraic data.

References

[1] A. Basu, M. Conforti, G. Cornuéjols, and G. Zambelli, *A counterexample to a conjecture of Gomory and Johnson*, Mathematical Programming Ser. A **133** (2012), no. 1–2, 25–38, `doi:10.1007/s10107-010-0407-1`.

[2] A. Basu, R. Hildebrand, and M. Köppe, *Equivariant perturbation in Gomory and Johnson's infinite group problem. I. The one-dimensional case*, Mathematics of Operations Research (2014), `doi:10.1287/moor.2014.0660`.

[3] ———, *Equivariant perturbation in Gomory and Johnson's infinite group problem. III. Foundations for the k-dimensional case and applications to $k = 2$*, eprint arXiv:1403.4628 [math.OC], 2014.

[4] A. Basu, R. Hildebrand, and M. Köppe, *Light on the infinite group relaxation*, eprint arXiv: 1410.8584 [math.OC], 2014.

[5] A. Basu, R. Hildebrand, M. Köppe, and M. Molinaro, *A $(k+1)$-slope theorem for the k-dimensional infinite group relaxation*, SIAM Journal on Optimization **23** (2013), no. 2, 1021–1040, doi:10.1137/110848608.
[6] M. Conforti, G. Cornuéjols, A. Daniilidis, C. Lemaréchal, and J. Malick, *Cut-generating functions*, Integer Programming and Combinatorial Optimization, Springer, 2013, pp. 123–132.
[7] R. E. Gomory and E. L. Johnson, *Some continuous functions related to corner polyhedra, I*, Mathematical Programming **3** (1972), 23–85, doi:10.1007/BF01585008.
[8] ———, *Some continuous functions related to corner polyhedra, II*, Mathematical Programming **3** (1972), 359–389, doi:10.1007/BF01585008.
[9] C. Y. Hong, M. Köppe, and Y. Zhou, *Sage program for computation and experimentation with the 1-dimensional Gomory–Johnson infinite group problem*, 2014, available from https://github.com/mkoeppe/infinite-group-relaxation-code.
[10] J.-P. P. Richard and S. S. Dey, *The group-theoretic approach in mixed integer programming*, 50 Years of Integer Programming 1958–2008 (M. Jünger, T. M. Liebling, D. Naddef, G. L. Nemhauser, W. R. Pulleyblank, G. Reinelt, G. Rinaldi, and L. A. Wolsey, eds.), Springer Berlin Heidelberg, 2010, pp. 727–801, doi:10.1007/978-3-540-68279-0_19.

The moment-LP and moment-SOS approaches in polynomial optimization

Jean B. Lasserre

We discuss the optimization problem $\mathbf{P} : \inf \{ f(x) : x \in \mathbf{K} \}$ where f is a polynomial and $\mathbf{K} \subset \mathbb{R}^n$ is the basic closed semi-algebraic set (assumed to be compact)

$$\mathbf{K} := \{ x \in \mathbb{R}^n : \quad g_j(x) \geq 0, \quad j = 1, \ldots, m \}$$

for some polynomial g_j, $j = 1, \ldots, m$. When one focuses on the *global* minimum f^* (as opposed to a local optimum), problem $\mathbf{P}$ can be written as:

$$f^* = \sup \{ \lambda : \quad f(x) - \lambda \geq 0, \quad \forall x \in \mathbf{K} \}.$$

When f is a polynomial and $\mathbf{K}$ is a compact basic semi-algebraic set, powerful positivity certificates of Real Algebraic Geometry allow to express the difficult positivity constraint "$f(x) \geq 0$ for all $x \in \mathbf{K}$" in a way that can be exploited for efficient numerical computation. Indeed one may then define a hierarchy of *convex relaxations* $(\mathbf{P}_k)$, $k \in \mathbb{N}$, of $\mathbf{P}$ which provides a monotone sequence of upper bounds $(f_k^* \geq f^*)$ such that $f_k^* \to f^*$ as $k \to \infty$. When using a positivity certificate due Krivine, Handelman and Vasilescu one ends up with solving a hierarchy of *LP-relaxations* whereas if one uses a positivity certificate due to Schmüdgen (and later refined by Putinar) one ends up with solving a hierarchy of *SDP-relaxations* (or semidefinite relaxations). In both cases the resulting convex relaxation $\mathbf{P}_k$ becomes more and more difficult to solve as its size increases with k.

We then discuss the relative merits and drawbacks of both hierarchies of relaxations and their impact not only in optimization (and particularly combinatorial optimization) but also in many areas for solving instances of the so-called *Generalized Problem of Moments* (GMP) with polynomial data (of which Global Polynomial Optimization is in fact the simplest instance).

Finally, we also introduce another characterization of nonnegativity on a closed set $\mathbf{K} \subset \mathbb{R}^n$ which can be also exploited to now define a monotone non increasing sequence of upper bounds (f_k^*), $k \in \mathbb{N}$, that converges to the global minimum f^* of $\mathbf{P}$ as $k \to \infty$. Computing each upper bound f_k^* now boils down to solving a generalized eigenvalue problem associated with some pair of real symmetric matrices whose size increases with k.

References

[1] J.B. Lasserre, *Moments, Positive Polynomials and Their Applications*, Imperial College Press, London, 2010.
[2] J.B. Lasserre, *An introduction to Polynomial and Semi-Algebraic Optimization*, Cambridge University Press, Cambridge, in Press.
[3] J.B. Lasserre, *Global optimization with polynomials and the problem of moments*, SIAM J. Optim. **11** (2001), 796–817.

Lower bounds on spectrahedral lifts and semidefinite relaxations

James Lee
(joint work with Prasad Raghavendra and David Steurer)

We introduce a method for proving lower bounds on the efficacy of SDP relaxations for combinatorial problems. In particular, we show that the cut, TSP, and stable set polytopes on n-vertex graphs are not the linear image of the feasible region of any SDP (i.e., any spectrahedron) of dimension less than 2^{n^c} for some $c > 0$. The results follow from a general technique for proving lower bounds on the positive semidefinite rank of a matrix. In order to do this, we establish close connections between arbitrary SDPs and those arising from the sum-of-squares SDP hierarchy.

Solving network problems including physical transport

Alexander Martin
(joint work with B. Geißler, A. Morsi and L. Schewe)

We consider a typical minimal cost (single) network flow problem. We are given a network $G = (V, E)$ with edge capacities $c \in \mathbb{R}^A$ and weights $w \in \mathbb{R}^A$ and a demand vector $b \in \mathbb{R}^V$ with $\sum_{v \in V} b_v = 0$. The only difference to this classical situation which is taught in class is that the flow we have to model is gas or water. The mathematical modeling of gas or water leaves the world of linear optimization and enters the field of partial differential equations. For instance, the 1-dimensional flow of gas in some pipe is described by the so-called isothermal Euler equations:

$$\begin{aligned} \frac{\partial \rho}{\partial t} + \frac{\partial (\rho v)}{\partial x} &= 0 \\ \frac{\partial (\rho v)}{\partial t} + \frac{\partial (\rho v^2 + p)}{\partial x} + g\rho \frac{\partial h}{\partial x} + \frac{\lambda}{2D} \rho |v| v &= 0 \end{aligned}$$

In addition, active elements are present such as valves, which can be opened or closed to (re)direct the flow of gas, or compressors, which may increase the pressure of gas. In other words we must deal with a problem inheriting binary decision variables and partial differential equations at the same time. For the solution of such problems in its general, that is dynamical, setting no appropriate methods and algorithms are available. Instead, we consider the stationary case and dispense with all time dependent terms. Then, the Euler equations reduce to ordinary differential equations. Assuming the real gas factor to be constant along a pipe and that the ram pressure term in the momentum equation can be neglected, these can even be solved in closed form. We end up with a mixed integer nonlinear optimization problem (MINLP). The commonly used approach to solve MINLPs is by outer approximation and spatial branching. We propose a different method by exploiting the strength of mixed integer linear programming solvers. The idea is as follows:

We build relaxations of the feasible set of the MINLP in terms of mixed-integer linear constraints only. To this end we first construct a piecewise linear approximation of each nonlinear expression such that the resulting approximation satisfies an a priori given error bound. This is achieved by an adaptive approximation algorithm based on convex underestimators for multivariate nonlinear expressions [7, 10] and by piecewise linear minimax approximations for the univariate terms [11]. In a second step we extend the so-called incremental method for piecewise linear functions [1] to a MILP model for piecewise polyhedral outer approximations of the same tightness as the initially constructed approximation. Thus, the resulting MILP model is a proper relaxation of the underlying MINLP that incorporates all combinatorial constraints exactly and reflects any nonlinear constraint up to a predefined error bound. Similar techniques have already been applied to variants of the related problem of gas transport energy cost minimization [2, 4, 3, 5, 6, 9]. However, the MILP models used therein only yield approximations and not relaxations of the underlying nonlinear models and are thus not appropriate to disprove feasibility. For a detailed look at the physical model and results on real-world gas networks, we refer to the forthcoming book [15].

In [12] these techniques have been applied to compute energy cost minimal solutions for gas transport problems on networks with several hundred nodes and arcs. The problem to decide, whether there is a feasible control for a gas network (and if yes, which) for a given demand vector b has recently been solved even for networks with several thousand nodes and arcs and several hundred switchable facilities [14], see Figure 1. For smaller gas networks similar results have already been proposed in [13, 15, 16]. Results for energy cost minimal operation of water supply networks have been reported in [8, 10, 11] using a similar approach.

In summary, we are able to solve MINLPs with MIP techniques if the involved nonlinear functions are of low dimension. This is in particular the case for network problems including physical transport. And in this case the computations show that this approach is even by far superior. However, what remains open is the dynamical situation. As mentioned, up to now no appropriate method is around

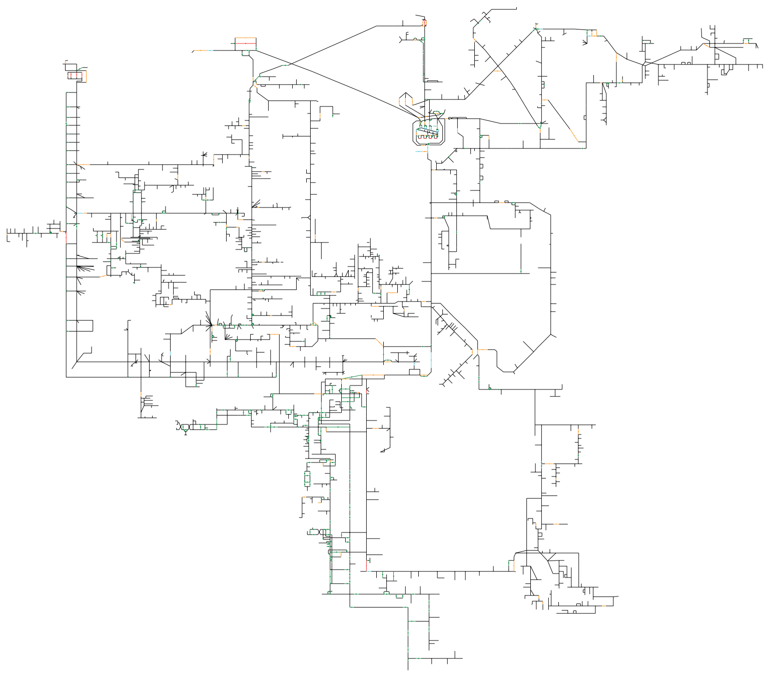

Figure 1. Germany's largest network for low-calorific gas transport.

and it remains a great challenge to attack such kind of problems. It is our current comprehension that we must dispense with the approach of remodeling every function in the MIP world and instead find a way of including the broad and deep knowledge for the solution of partial difference equations in a black box manner in the MIP context.

References

[1] H.M. Markowitz, A.S. Manne, *On the solution of discrete programming problems*, Econometrica **25** (1957), 84–110.

[2] M. Möller, *Mixed Integer Models for the Optimisation of Gas Networks in the Stationary Case*, PhD thesis, Technische Universität Darmstadt (2004).

[3] S. Moritz, *A Mixed Integer Approach for the Transient Case of Gas Network Optimization*, PhD thesis, Technische Universität Darmstadt (2006).

[4] A. Martin, M. Möller, S. Moritz, *Mixed integer models for the stationary case of gas network optimization* Math. Program. **105(2)** (2006), 563–582.

[5] D. Mahlke, A. Martin, S. Moritz, *A simulated annealing algorithm for transient optimization in gas networks*, Math. Methods Oper. Res. **66(1)** (2007), 99–116.

[6] D. Mahlke, A. Martin, S. Moritz, *A mixed integer approach for time-dependent gas network optimization*, Opt. Methods Softw. **25(4)** (2010), 625–644.

[7] B. Geißler, *Towards Globally Optimal Solutions for MINLPs by Discretization Techniques with Applications in Gas Network Optimization*, PhD thesis, Friedrich-Alexander-Universität Erlangen-Nürnberg (2011).

[8] B. Geißler, O. Kolb, J. Lang, G. Leugering, A. Martin, A. Morsi, *Mixed Integer Linear Models for the Optimization of Dynamical Transport Networks*, Mathematical Methods of Operations Research **23(4)** (2011), 339–362.
[9] P. Domschke, B. Geißler, O. Kolb, J. Lang, A. Martin, A. Morsi, *Combination of Nonlinear and Linear Optimization of Transient Gas Networks*, INFORMS Journal on Computing **23(4)** (2011), 605–617.
[10] B. Geißler, A. Martin, A. Morsi, L. Schewe, *Using piecewise linear functions for solving MINLPs*, In *Mixed Integer Nonlinear Programming*, ed. by J. Lee, S. Leyffer, volume **154** of The IMA Volumes in Mathematics and its Applications (2012), 287–314.
[11] A. Morsi, *Solving MINLPs on Loosely-Coupled Networks with Applications in Water and Gas Network Optimization*, PhD thesis, Friedrich-Alexander-Universität Erlangen-Nürnberg (2013).
[12] B. Geißler, A. Morsi, L. Schewe. *A New Algorithm for MINLP Applied to Gas Transport Energy Cost Minimization*, In: *Facets of Combinatorial Optimization*, ed. by M. Jünger, G. Reinelt. Springer (2013), 321–353.
[13] A. Fügenschuh, B. Geißler, R. Gollmer, C. Hayn, R. Henrion, B. Hiller, J. Humpola, T. Koch, T. Lehmann, A. Martin, R. Mirkov, A. Morsi, W. Römisch, J. Rövekamp, L. Schewe, M. Schmidt, R. Schultz, R. Schwarz, J. Schweiger, C. Stangl, M.C. Steinbach, B.M. Willert, *Mathematical optimization for challenging network planning problems in unbundled liberalized gas markets*, Energy Systems **5(3)** (2013), 449–473.
[14] B. Geißler, A. Morsi, L. Schewe, M. Schmidt, *Solving Power-Constrained Gas Transportation Problems using an MIP-based Alternating Direction Method*, Preprint available online: `http://www.optimization-online.org/DB_HTML/2014/11/4660.html` (2014).
[15] T. Koch, B. Hiller, M. Pfetsch, L. Schewe (eds.), *Evaluating Gas Network Capacities*, MOS-SIAM Series on Optimization, 21. Society for Industrial and Applied Mathematics (SIAM), Philadelphia, PA; Mathematical Optimization Society, Philadelphia, PA, (2014), forthcoming.
[16] M.E. Pfetsch, A. Fügenschuh, B. Geißler, N. Geißler, R. Gollmer, B. Hiller, J. Humpola, T. Koch, T. Lehmann, A. Martin, A. Morsi, J. Rövekamp, L. Schewe, M. Schmidt, R. Schultz, R. Schwarz, J. Schweiger, C. Stangl, M.C. Steinbach, S. Vigerske, B.M. Willert, *Validation of nominations in gas network optimization: models, methods, and solutions*, Optimization Methods and Software **30(1)** (2015), 15–53.

Discrete DC programming by discrete convex analysis

Kazuo Murota
(joint work with Takanori Maehara)

The theory of DC functions (difference of two convex functions) and DC programming is one of the most successful areas of non-convex optimization [2, 9]. The DC theory is based on a basic non-convex duality theorem, called Toland-Singer duality [7, 8]:

$$\inf_{x \in \mathbb{R}^n} \{g(x) - h(x)\} = \inf_{p \in \mathbb{R}^n} \{h^*(p) - g^*(p)\}.$$

A DC program is hard to solve in general, but when the objective function has a nice DC representation, there are some practical algorithms based on the Toland-Singer duality. We establish a discrete analogue of the theory of DC programming. using discrete convex analysis [1, 3, 4, 5].

In discrete convex analysis, two convexity notions, M$^\natural$-convexity and L$^\natural$-convexity, are distinguished: M$^\natural$-convexity is a generalization of matroid property and L$^\natural$-convexity is a generalization of submodularity on subsets. Conjugacy between M$^\natural$-convex functions and L$^\natural$-convex functions under discrete Legendre-Fenchel transformation is a distinctive feature of discrete convex analysis. Fundamental results in continuous convex analysis, in particular biconjugacy and subdifferentiability, have corresponding statements in discrete convex analysis. Furthermore, efficient algorithms are available for minimizing discrete convex functions.

We define a discrete DC function as a difference of two discrete convex functions. Since there are two classes of discrete convex functions (M$^\natural$-convex functions and L$^\natural$-convex functions), there are four types of discrete DC functions (an M$^\natural$-convex function minus an M$^\natural$-convex function, an M$^\natural$-convex function minus an L$^\natural$-convex function, and so on). These types of functions contain many functions appearing in practice: a difference of submodular functions is an L$^\natural$−L$^\natural$ DC function, a supermodular function that is restricted to a matroid is an M$^\natural$−L$^\natural$ DC function, and so on.

We propose discrete DC programming problems as optimization problems of discrete DC functions:

$$\text{minimize } (g(x) - h(x)).$$

Since there are two conjugate classes (M$^\natural$ and L$^\natural$) of discrete convex functions, there are four types of discrete DC programs. We prove the discrete version of the Toland-Singer duality for discrete DC programs. The discrete Toland-Singer duality establishes the relation of four types of discrete DC programs, which is a main feature of discrete DC programming.

We also propose algorithms for discrete DC programming. These algorithms are obtained by combining the general discrete DC algorithm, which is a straightforward adaption of the continuous case, and the polyhedral structure of discrete convex functions. The algorithms decrease the function value strictly in each iteration and hence terminate in a finite number of iterations. Furthermore, when the algorithms terminate, the obtained solutions satisfy a local optimality condition.

Narasimhan and Bilmes [6] considered minimization problems of a difference of two submodular set functions (DS programs) and propose an algorithm, named submodular-supermodular procedure. The DS programming is a special case of our discrete DC programming (since submodular set functions coincide exactly with L$^\natural$-convex functions on $\{0,1\}^n$), and their algorithm is a special case of our general discrete DC algorithm.

References

[1] S. Fujishige: Submodular Functions and Optimization. 2nd ed., Annals of Discrete Mathematics, vol. 58, Elsevier, Amsterdam, 2005.

[2] R. Horst, N. V. Thoai: DC Programming: Overview. Journal of Optimization Theory and Applications, **103** (1999), 1–43.

[3] K. Murota: Discrete convex analysis. Mathematical Programming, **83** (1998), 313–371.

[4] K. Murota: Discrete Convex Analysis. Society for Industrial and Applied Mathematics, Philadelphia, 2003.
[5] K. Murota: Recent developments in discrete convex analysis. In: W. Cook., L. Lovász, J. Vygen, eds., Research Trends in Combinatorial Optimization, Springer, Berlin, Chapter 11, pp. 219–260, 2009.
[6] M. Narasimhan and J. Bilmes: A submodular-supermodular procedure with applications to discriminative structure learning. In Proceedings of the 21st Conference on Uncertainty in Artificial Intelligence, pp. 404–412, 2005.
[7] I. Singer: A Fenchel-Rockafellar type duality theorem for maximization. Bulletin of the Australian Mathematical Society, **20** (1979), 193–198.
[8] J. F. Toland: A duality principle for non-convex optimisation and the calculus of variations. Archive for Rational Mechanics and Analysis, **71** (1979), 41–61.
[9] H. Tuy: D.C. optimization: Theory, methods and algorithms. In: R. Horst and P. M. Pardalos, eds., Handbook of Global Optimization, Kluwer Academic Publishers, Dordrecht, pp. 149–216, 1995.

Weighted linear matroid matching

Gyula Pap

The sketch of a polynomial time algorithm is presented in the talk to solve the weighted linear matroid matching problem. A completely different approach is due to Satoru Iwata, both approaches have been discovered independently, and apparently at the same time.

Let V be a vectorspace over a given field Γ. A line is a linear subspace of rank 2. Let E be a set of lines. A subset $M \subseteq E$ is called a matching if $r(\bigcup M) = 2|M|$. For a weight function $w : E \to \mathbb{R}_+$, the weight of a matching M is defined by $w(M) := \sum_{e \in M} w(e)$. The maximum weight linear matroid matching problem takes the input V, E, w, and asks us to determine the maximum weight of a matching. This maximum weight is denoted by $\nu(V, E, w)$.

1. Extended formulation

We consider a linear programming formulation of linear matroid matching, in which we associate vectors x^M and y^M with every matching M. For an arbitrary matching $M \subseteq E$, we define vectors x and y such that $x^M \in \mathbb{R}^E$ with $x^M(l) := \chi_M(l) := 1$ if $l \in M$ and 0 otherwise. For $K < L < V, F \subseteq E$, we define $y^M_{K,L}(F) := r(K \wedge sp(M \cap F)) - r(L \wedge sp(M \cap F))$. Thus we map a matching into this very high dimensional space $M \mapsto (x, y)$.

Let $\mathcal{D} = \{D_i : 0 \le i \le k\}$ denote a chain of subspaces in V such that $\{0\} = D_0 < D_1 < \cdots < D_k = V$, and let $\delta_i \ge 0$, $i = 1, 2 \cdots, k$. For all $F \in \mathcal{L}$ and $1 \le i \le k$, let $D_{i-1} < S^F_i < T^F_i < D_i$. Define $T^F_i := D_{i-1} \vee (sp(F) \wedge D_i)$. When all these properties hold, then we may write $\mathcal{D}, S_i, T_i, F \sim (*)$. We denote $T^e_i := D_{i-1} \vee (e \cap D_i)$, and $I_e := \{i : 1 \le i \le k, (e \cap D_i) - D_{i-1} \ne \emptyset\}$. When $e \in E$ and chain $\mathcal{D}$ are like these ones above, then we put $\mathcal{D}, e \sim (**)$.

Claim 1. *$x = x^M, y = y^M$ satisfies the following inequalities.*

$$
\begin{aligned}
x(F) - \sum_{i=1}^{k} y_{S_i,T_i}(F) &\le \left\lfloor \frac{1}{2}\sum_{i=1}^{k}(r(S_i) - r(D_{i-1}))\right\rfloor && \mathcal{D}, S_i, T_i, F \sim (*)\\
2x(e) &\le \sum_{i\in I_e} y_{D_{i-1},T_i^e}(\{e\}) && \mathcal{D}, e \sim (**)\\
y_{K,L}(F) + y_{K,L}(F') &\le y_{K,L}(F\cup F') && K < L, F\cap F' = \emptyset\\
y_{K,L}(F) + y_{K',L'}(F) &\le y_{K,L'}(F) && K < L < K' < L'\\
y_{K,L}(F) &\le y_{K',L'}(F) && K' < K < L < L'\\
y_{K,L}(F) &\le r(sp(F)) && K < L < V.
\end{aligned}
$$

2. Dual compositions

We define a dual structure that is used in the algorithm as an upper bound, maintained and altered until we find a complementary matching. The dual composition is defined so that we can construct a dual feasible solution of the extended formulation, and thus provide an upper bound on the weight of a matching.

Definition 1. $\beta, \mathcal{D}, \delta, \mathcal{L}, \lambda, S_i^F, T_i^F$ is called a *dual composition* if each of the following properties holds.

(1) For all $e \in E$ let $\beta(e) \ge 0$. $\mathcal{D} = \{D_i : 0 \le i \le k\}$ is a chain of subspaces in V such that $\{0\} = D_0 < D_1 < \cdots < D_k = V$. For all $i = 1, 2\cdots, k$ let $\delta_i \ge 0$. $\mathcal{L}$ is a laminar family of subsets of E. For all $F \in \mathcal{L}$ let $\lambda_F \ge 0$. Denote $T_i^F = (sp(F) \wedge D_i) \vee D_{i-1} = (sp(F) \vee D_{i-1}) \wedge D_i$ and $T_i^e := (e \cap D_{i-1}^i$ for all $1 \le i \le k, F \in \mathcal{L}, e \in E$. For all $F \in \mathcal{L}$ and $1 \le i \le k$, let $D_{i-1} < S_i^F < T_i^F < D_i$ where $F \subseteq F'$ implies that $S_i^F < S_i^{F'}$ and $T_i^F < T_i^{F'}$.

(2) For $1 \le j \le k$ and a chain $\mathcal{C} = \{F_1, F_2, \cdots, F_m\} \subseteq \mathcal{L}$ such that $F_1 \subsetneq F_2 \subsetneq \cdots \subsetneq F_m \subseteq E$, and $T_j^{F_1} - S_j^{F_m} \ne \emptyset$, then the following inequality holds.

$$\sum_{i\ge j} \delta_i \ge \sum_{F\in\mathcal{C}} \lambda_F$$

(3) For $e \in E$, $1 \le j \le k$ and a chain $\mathcal{C} = \{F_1, F_2, \cdots, F_m\} \subseteq \mathcal{L}$ such that $e \in F_1 \subsetneq F_2 \subsetneq \cdots \subsetneq F_m \subseteq E$, $T_j^e - S_j^{F_m} \ne \emptyset$, then the following inequality holds.

$$\sum_{i\ge j} \delta_i \ge \beta_e + \sum_{F\in\mathcal{C}} \lambda_F$$

(4) For all $e \in E$, the following inequality holds.

$$w(e) \le 2\beta_e + \sum_{F: e\in F\in\mathcal{L}} \lambda_F.$$

From a dual composition is defined we can construct a dual feasible solution of the extended formulation, and thus the following weak duality holds.

$$w(M) \leq val(\beta, \mathcal{D}, \delta, \mathcal{L}, \lambda, S_i^F, T_i^F) := \sum_i \delta_i r(D_i) + \sum_{F \in \mathcal{L}} \lambda_F \left\lfloor \frac{1}{2} \sum_{i=1}^{k} (r(S_i) - r(D_i)) \right\rfloor$$

Actually, the dual compositions provide for a tight upper bound on the maximum weight of a matching, as stated by the following theorem. This is proved by an algorithm that maintains a dual composition until a complementary matching is found, to certify optimality.

Theorem 1. *Let V be a vectorspace, let E be a set of lines, and $w : E \to \mathbb{R}_+$ a weight function. Then*

$$\nu(V, E, w) = \min val(\beta, \mathcal{D}, \delta, \mathcal{L}, \lambda, S_i^F, T_i^F),$$

where the minimum is taken over dual compositions.

Inapproximability of combinatorial problems via small LPs and SDPs

Sebastian Pokutta

(joint work with Gábor Braun and Daniel Zink)

In this talk we provide an alternative view on extended formulations, both simplifying and generalizing the previous theory. As a consequence, we establish a strong reduction mechanism for approximate LP and SDP formulations, leading to new LP inapproximability results for non-0/1 CSPs (e.g., VertexCover, Max-MULTICUT, and bounded degree MaximumIndependentSet).

Our framework is motivated by the earlier approach in [4] to capture uniform linear programming formulations independent of the specific linear encoding and it generalizes the polyhedral pair approach in [2, 3]. In particular, there is no need to encode the combinatorial optimization problem first as a polytope, a polyhedral pair, or a linear program, not only simplifying the setup but also enabling the notion of LP or SDP complexity of a *problem* and not just one of its encodings. The resulting notion of *formulation complexity* can be understood as the minimum extension complexity over all possible linear encodings of the considered optimization problem. This independence of encoding addresses previous concerns that the obtained lower bounds are polytope-specific and hence encoding-specific and alternative linear encodings (i.e., LPs to start from) of the same problems might admit smaller formulations: we show that this is not the case. As a consequence we can define a sound reduction mechanism for LP and SDP formulations of problems with approximations in mind.

The key element in the analysis of extended formulations is Yannakakis's celebrated Factorization Theorem (see [10, 9]) and its generalizations (see e.g., [6, 2, 3, 4]) equating the minimal size of an extended formulation with a property of a slack matrix, e.g., in the linear case the nonnegative rank. We provide an abstract factorization theorem acting directly on the optimization problem (and not just on one of its representations). This allows us to characterize the minimum size of any LP or SDP capturing an optimization problem. Moreover, from an optimal factorization we can also explicitly reconstruct an optimal encoding as a linear program or semidefinite program. Combining our framework with [4], we obtain strong inapproximability for various combinatorial problems of interest.

LP and SDP inapproximability of specific problems. By [8, 4] it is known that Max-k-XOR (for $k \geq 2$) cannot be approximated within a factor better than $\frac{1}{2}$ by a linear program with a polynomial number of constraints. In the case of SDPs, so far an unconditionally SDP-hard base problem has been missing and we formulate conditional SDP inapproxamibility factors under the assumption that the Goemans-Williams SDP for MaxCUT is optimal, which is compatible with the Unique Games Conjecture.

Combining these base problems with our reduction mechanism we obtain LP inapproximability and conditional SDP inapproximability for several problems that are not 0/1 CSPs as shown in the table below. In particular, we answer an open question regarding the inapproximability of VertexCover (see [4]) and we answer a weak version of our sparse graph conjecture posed in [1].

	Inapproximability			Approximability
Problem	(LP)	(SDP)	(PCP)	(LP)
VertexCover	$\frac{3}{2} - \varepsilon$	$1.12144 - \varepsilon$	$1.361 - \varepsilon$	2
Max-k-MULTICUT	$\frac{2c(k)+1}{2c(k)+2} + \varepsilon$	$\frac{c(k)+c_{GW}}{c(k)+1}$	$1 - \frac{1}{34k} + \varepsilon$	$\frac{1}{2(1-1/k)}$
bdd MaxIndep	$\frac{1}{2} + \varepsilon$	$0.87856 + \varepsilon$	$O\left(\frac{\log^4 \Delta}{\Delta}\right)$	—

Here $c_{GW} \approx 0.87856$ is the approximation factor of the algorithm for MaxCUT from [5], and $c(k)$ is a constant depending on k.

Note: At the same workshop a $7/8 + \varepsilon$ inapproximability for Max-3-SAT for polynomial size SDPs was presented [7]. Combining this result with our reduction mechanism provides unconditional inapproximability results for SDPs with inapproximability factors mostly matching those from the classical PCP approaches.

References

[1] G. Braun, S. Fiorini, and S. Pokutta. Average case polyhedral complexity of the maximum stable set problem. *Proceedings of RANDOM / arXiv:1311.4001*, 2014.

[2] G. Braun, S. Fiorini, S. Pokutta, and D. Steurer. Approximation limits of linear programs (beyond hierarchies). In *53rd IEEE Symp. on Foundations of Computer Science (FOCS 2012)*, pages 480–489, 2012.

[3] G. Braun, S. Fiorini, S. Pokutta, and D. Steurer. Approximation limits of linear programs (beyond hierarchies). In *To appear in Mathematics of Operations Research*, 2014.

[4] S.O. Chan, J.R. Lee, P. Raghavendra, and D. Steurer. Approximate constraint satisfaction requires large LP relaxations. In *IEEE 54th Annual Symp. on Foundations of Computer Science (FOCS 2013)*, pages 350–359, 2013.

[5] M.X. Goemans, D. Williamson. Improved approximation algorithms for maximum cut and satisfiability problems using semidefinite programming *Journal of the ACM*, 42(6):1115–1145, 1995.

[6] J. Gouveia, P. A. Parrilo, and R. Thomas. Lifts of convex sets and cone factorizations. *Mathematics of Operations Research*, 38(2):248–264, 2013.

[7] J.R. Lee, P. Raghavendra, and D. Steurer. Lower bounds on the size of semidefinite programming formulations. presented at Oberwolfach Workshop 11/2014, 2014.

[8] G. Schoenebeck. Linear level Lasserre lower bounds for certain k-CSPs. In *Proc. FOCS 2008*, pages 593–602, October 2008.

[9] M. Yannakakis. Expressing combinatorial optimization problems by linear programs (extended abstract). In *Proc. STOC 1988*, pages 223–228, 1988.

[10] M. Yannakakis. Expressing combinatorial optimization problems by linear programs. *J. Comput. System Sci.*, 43(3):441–466, 1991.

Constructive discrepancy minimization for convex sets

Thomas Rothvoss

Discrepancy theory deals with finding a *bi-coloring* $\chi : \{1,\dots,n\} \to \{\pm 1\}$ of a set system $S_1,\dots,S_m \subseteq \{1,\dots,n\}$ so that the worst imbalance $\max_{i=1,\dots,m} |\chi(S_i)|$ of a set is minimized, where we denote $\chi(S_i) := \sum_{j\in S_i} \chi(j)$. A seminal result of Spencer [Spe85] says that there is always a coloring χ so that $|\chi(S_i)| \leq O(\sqrt{n})$ if $m = n$. The result is in particular interesting since it beats the random coloring which has discrepancy $\Theta(\sqrt{n\log n})$. Spencer's technique, which was first used by Beck in 1981 [Bec81] is usually called the *partial coloring method* and is based on the argument that due to the pigeonhole principle many of the 2^n many colorings χ, χ' must satisfy $|\chi(S_i) - \chi'(S_i)| \leq O(\sqrt{n})$ for all sets S_i. Then one can take the *difference* between such a pair of colorings with $|\{j \mid \chi(j) \neq \chi'(j)\}| \geq \frac{n}{2}$ to obtain a *partial coloring* of low discrepancy. Iterating the argument $\log n$ times provides a full coloring.

Few years later and on the other side of the iron curtain, Gluskin [Glu89] obtained the same result using convex geometry arguments. In a paraphrased form, Gluskin's result showed the following:

Theorem 1 (Gluskin [Glu89], Giannopoulos [Gia97])**.** *For a small constant $\delta > 0$, let $K \subseteq \mathbb{R}^n$ be a symmetric convex set with Gaussian measure $\gamma_n(K) \geq e^{-\delta n}$ and $v_1,\dots,v_m \in \mathbb{R}^n$ vectors of length $\|v_i\|_2 \leq \delta$. Then there are partial signs $y_1,\dots,y_m \in \{-1,0,1\}$ with $|supp(y)| \geq \frac{m}{2}$ so that $\sum_{i=1}^m y_i v_i \in 2K$.*

While this theorem is non-constructive, Bansal [Ban10] showed that a random walk, guided by the solution of an SDP can find the coloring for Spencer's Theorem in polynomial time. However, the approach needs a very careful choice of parameters and the feasibility of the SDP still relies on the non-constructive argument. A simpler and truly constructive approach was provided by Lovett and Meka [LM12]. Still that approach did not apply for arbitrary convex sets with large enough Gaussian measure.

Our contribution

Our main contribution is the following:

Theorem 2. *There is a polynomial time algorithm, which for any symmetric convex set $K \subseteq \mathbb{R}^n$ with Gaussian measure at least $e^{-n/500}$ finds a point $y \in K \cap [-1,1]^n$ with $y_i \in \{-1,1\}$ for at least $\frac{n}{9000}$ many coordinates. Here it suffices if a polynomial time separation oracle for the set K exists.*

In fact, our method is extremely simple:

(1) take a random Gaussian vector $x^* \sim N^n(0,1)$
(2) compute the point $y^* = \text{argmin}\{\|x^* - y\|_2 \mid y \in K \cap [-1,1]^n\}$
(3) return y^*

References

[Ban10] N. Bansal. Constructive algorithms for discrepancy minimization. In *FOCS*, pages 3–10, 2010.

[Bec81] J. Beck. Roth's estimate of the discrepancy of integer sequences is nearly sharp. *Combinatorica*, 1(4):319–325, 1981.

[Gia97] A. Giannopoulos. On some vector balancing problems. *Studia Mathematica*, 122(3):225–234, 1997.

[Glu89] E. D. Gluskin. Extremal properties of orthogonal parallelepipeds and their applications to the geometry of banach spaces. *Mathematics of the USSR-Sbornik*, 64(1):85, 1989.

[LM12] S. Lovett and R. Meka. Constructive discrepancy minimization by walking on the edges. In *FOCS*, pages 61–67, 2012.

[Spe85] J. Spencer. Six standard deviations suffice. *Transactions of the American Mathematical Society*, 289(2):679–706, 1985.

Problems about uniform covers, with tours and detours

András Sebő

(joint work with Yohann Benchetrit and Matěj Stehlík)

Is $\underline{1}$ a linear or integer combination of some combinatorially interesting vectors? Some examples, with detours:

1. Tours

A *tour* in the graph $G = (V, E)$ is an Eulerian $0-1-2$ function on the edges (even on stars, connected support). We adapt Wolsey's argument [16] to prove:

Fact: *If G is 3-edge-connected, the all 1 function $\underline{1}$ is in the convex hull of tours.*

Proof. $\underline{2}/3$ dominates a point in the spanning tree polytope (satisfies subtour elimination) ; $\underline{1}/3$ dominates a point in the T-join polyhedron, for all T. It is then easy to see that $\underline{1} = \underline{2}/3 + \underline{1}/3$ is in the convex hull of trees + an edge-set for each tree correcting the parities of its degrees. □

The same holds for T-tours, that is, connected T-joins, in particular $\{s,t\}$-tours.

Problem 1: Can this bound be improved for tours?

The answer is probably yes: by the '4/3 integrality gap conjecture' [8] $\underline{4}/3 \times \underline{2}/3 = \underline{8}/9$ is in the convex hull of tours. For $\{s,t\}$-tours $\underline{3}/2 \times \underline{2}/3 = 1$.

We make now a detour to a lower bound that is in some cases better than linear programming. The more there are degree 2 vertices the better it is.

Let $G = (V,E)$ be a graph, $m := |E|$, $n := |V|$. There is a unique graph $G^* = (V^*, E^*)$, $m^* := |E^*|$, $n^* := |V^*|$ without degree 2 vertices of which G is a subdivision. Let T_G be the set of odd degree vertices of G, τ the minimum size of a T_G-join, and OPT the minimum size of a tour.

Inequality: *Let G be a 2-edge-connected graph. Then $m+\tau-2k \le OPT \le m+\tau$, where $k = m-n+1 = m^*-n^*+1$ is the number of ears in an ear-decomposition.*

Proof. Consider a tour in $G = (V,E)$, and let F be the set of edges of multiplicity 2 or 0, and $F^* \subseteq F$ those of multiplicity 0; F is a T_G-join.

Since $E \setminus F^*$ is connected, $|E \setminus F^*| \ge n-1$, that is, $|F^*| \le m-n+1 = m^*-n^*+1 = k$. The tour length is: $|E|+|F|-2|F^*| \ge m+\tau-2k$. □

Note that the upper bound is just the minimum of the Chinese Postman trail; F^* contains at most one edge of each series class; the inequality and its proof can be straightforwardly generalized to weights.

Corollary : *For the subdivisions of a given graph the solution of the Chinese Postman problem has a constant additive error for the smallest tour.*

Problem 2: When the lower bound is bad (k is large), the upper bound can also be replaced by a much smaller value! How to improve the bounds in a useful way?

2. H-perfect graphs

Given a graph G and a non-negative rational λ, the *fractional chromatic number* χ_f is the minimum of λ such that $\underline{1}/\lambda$ is in the stable set polytope. For *t-perfect* graphs [13] the maximum of $\underline{1}$ on $\{x \in \mathbb{R}^{V(G)} : x(S) \le 1, \text{for all stable } S, x \ge 0\}$ is at most 3, so the optimum of the dual, $\chi_f \le 3$.

Shepherd conjectured that the same is true for the chromatic number χ.

Laurent and Seymour [13] realized that the complement of the line graph of the prism (a prism is the complement of C_6) is a counterexample. This graph is the "t-minor" of a 3-colorable t-perfect graph, contradicting the integer round-up property of 3-colorable t-perfect graphs, conjectured by Shepherd [15]. It is then natural to conjecture 4-colorability. Actually more could be true:

Conjecture 3: Every h-perfect graph is $\omega + 1$ - colorable ($\omega :=$ clique-number).
Theorem: *If this conjecture is true for $\omega = 2$, then it is true in general.*

Proof. If $\omega > 2$, the optimal face is that of the ω-cliques so any stable set active in an optimal dual solution meets all ω-cliques. □

Benchetrit [1] found that the complement of the line graph of a 5-wheel is also a counterexample to Shepherd's conjecture. In some sense the two counterxamples are the only obstacles to the integer round-up property [1].

We make now a detour to the maximum number, β, of starting odd ears in an ear decomposition [3], related to h-perfect graphs, rounding, the matching polytope; expressing the complexity of the latter. This is joint work with Yohann Benchetrit.

Question 4: What is the complexity of computing β?

A θ is a subgraph consisting of three edge-disjoint paths, two of which are odd, and one even, between two fixed vertices of a graph. A basis of the cycle space (over GF(2)) of a graph that consists only of odd cycles will be called an *odd cycle-basis*. The existence of an odd cycle basis of a non-bipartite graph immediately follows from the open ear-decomposition of 2-connected graphs, and the following easy and well-known fact [11]: *in a 2-vertex-connected non-bipartite graph there exist between any two vertices both an even and an odd path.*

Theorem *Let G be a 2-vertex-connected graph. The following are equivalent:*

(i) *There exists no θ in G.*
(ii) $\beta(G) \leq 1$.
(iii) *Any two simple odd cycles have an odd number of common edges.*
(iv) *In each odd cycle basis, any two cycles meet in an odd number of edges.*
(v) *There exists an odd cycle basis with the property stated in (iii).*

Proof. Any of (i) or (iii) imply (ii), since an odd cycle C completed by an open odd ear P is a θ, and contradicts (iii). These are known from [5], [6], the rest is from [3]. Supposing (ii) the proof of (iii) is a graph-theory exercise: if two cycles, Q_1 and Q_2 do not satisfy (iii) and $|V(Q_1) \cap V(Q_2)| \geq 2$, then $|E(Q_1) \setminus E(Q_2)|$ is odd, easily contradicting (ii). Otherwise Q_1 and Q_2 are edge-disjoint and one concludes using Menger's theorem.

Two implications are straightforward: (iv) is just a special case of (iii), and (v) is a special case of (iv). Last, but not least, if (v) holds, then any odd cycle is the mod 2 sum of an odd number of cycles, and then knowing (iii) for the basis, it follows for any pair of odd cycles. □

3. Hereditary hypergraphs

This section reports about joint work with Matěj Stehlík [14]. Let $H = (V, E)$ be a *hereditary hypergraph*: if $e \in E$ all subsets of e are in E.

Closed Problems:

1. Is $\underline{1}$ an integer sum of incidence vectors of $e \in E$, $|e| \geq 2$?
2. Compute the minimum size ρ of a cover of V by members of E.
3. Compute the maximum size μ of a set that can be partitioned into $e \in E$, $|e| \geq 2$. Such a set is called a *μ-matching.*

Theorem: *Problem 2. is NP-hard* (SET COVER) *but 1. and 3. are polynomially solvable. Furthermore, there exists a cover of size ρ containing a μ-matching.*

The polynomial algorithms are easy consequences of vertex-packing edges and triangles [7], whereas the last sentence follows from [11][Exercise 9.4], originating from Gallai's work [10]. Yet the connections provide a new insight into packing and covering: the difficult theorem of Gallai [10] is in fact equivalent to a much simpler theorem in [9], and relevant information is smuggled in about the NP-hard problem of minimum covers, and by transposition, about minimum transversals [14].

Problem 5: Study some conjectures about packing, covering and minimum transversals bearing in mind the connections mentioned above.

4. Triangles

Problem 6: [12] Characterize the graphs for which $\underline{1}$ is a nonnegative combination of triangles as edge-sets. In other words, can the system of linear inequalities describing the cone of triangles of a graph be described ?

The origins of this problem are in regular covers of edges by triangles, see [12].

References

[1] Y. Benchetrit, *PhD Thesis*, in preparation
[2] Y. Benchetrit, Integer round-up for some h-perfect graphs, submitted
[3] Y. Benchetrit, A. Sebő, *On the complexity of matching polytopes,* in preparation
[4] H. Bruhn, M. Stein, Mathematical Programming, series A, **133** (1-2), 461–480 (2012)
[5] D. Cao, Topics in node packing and coloring, PhD thesis, Georgia Institute of Technology
[6] D. Cao, G. Nemhauser, *Polyhedra and perfection of line graphs*, DAM, **81**, 141–154 (1998)
[7] G. Cornuéjols, D. Hartvigsen, Pulleyblank, *Packing Subgraphs in a Graph*, Operations Research Letters, **4**, 139–143 (1982)
[8] R. Carr, S. Vempala, *On the Held-Karp relaxation for the asymmetric and symmetric traveling salesman problems*, Mathematical Programming, series A, **100**, 569–587 (2004)
[9] T. Gallai, *Neuer Beweis eines Tutte'schen Satzes* A Magyar Tudományos Akadémia — Matematikai Kutató Intézetének Közleményei, **8**, 135–139, 1963.
[10] T. Gallai, Kritische Graphen II, –”–, 373–395, 1963.
[11] L. Lovász, Combinatorial Problems and Exercises,
[12] S. Poljak, A. Sebő, P. Seymour, *Open Problem 8*, Graph Minors, Robertson, Seymour, p.680
[13] A. Schrijver, Combinatorial Optimization, North Holland (2002)
[14] A. Sebő, Stehlík, *Matching and Covering in Hereditary Hypergraphs*, in preparation
[15] B. Shepherd, *Applying Lehman's theorem to packing,* Math. Prog. **71**, 353-367 (1995)
[16] L. Wolsey, *Heuristic analysis, LP, and branch and bound*, Math. Prog. S. **13** (1980), 121–134

Computability of maximum entropy distributions and counting problems

Mohit Singh

(joint work with Nisheeth Vishnoi)

Given a polytope P and a point x in P, there can be many ways to write x as a convex combination of vertices of P. Interpreting any convex combination as a probability distribution over vertices of P, the distribution that maximizes entropy has received considerable interest. Interest in such distributions arises due to their applicability in areas such as statistical physics, economics, biology, information theory, machine learning, combinatorics and, more recently, approximation algorithms. In this talk, I will discuss the computability of maximum entropy distributions. A key difficulty in computing max-entropy distributions has been to show that they have polynomially-sized descriptions. We show that such descriptions exist under general conditions. Subsequently, we show how algorithms for (approximately) counting the vertices of P can be translated into efficient algorithms to (approximately) compute max-entropy distributions. In the reverse direction, we show how access to algorithms that compute max-entropy distributions can be used to count, which establishes an equivalence between counting and computing max-entropy distributions.

References

[1] Mohit Singh, Nisheeth K. Vishnoi, *Entropy, optimization and counting*, STOC 2014: 50-59.

A note on the Ring Loading Problem

Martin Skutella

An instance of the Ring Loading Problem is given by an undirected ring (cycle) on node set $V = \{1, 2, \dots, n\}$ (numbered clockwise along the ring) with demands $d_{i,j} \geq 0$ for each pair of nodes $i < j$ in V. The task is to route all demands unsplittably, that is, each demand $d_{i,j}$ needs to be routed from node i to node j either in clockwise direction on the path $i, i+1, \dots, j$ or in counterclockwise direction on the path $i, i-1, \dots, j$. The objective is to minimize the maximum load on an edge of the ring. The optimum solution value is denoted by L.

This optimization problem arose in the early 1990ies as a crucial subproblem in the design of survivable telecommunication networks utilizing fiber-optic-based technologies; it was first studied by Cosares and Saniee [1] who also introduced the name Ring Loading Problem.

Known results. Cosares and Saniee [1] prove by a reduction from the problem Partition that the Ring Loading Problem is weakly NP-hard. It is not known whether the problem is even strongly NP-hard or can be solved in pseudo-polynomial time. If all non-zero demands are equal, the problem can be solved in polynomial time; this follows from the work of Frank [2] and is based on a theorem of Okamura and Seymour [5].

The hardness of the general problem motivates the consideration of the relaxed version of the Ring Loading Problem where demands may be split, i. e., a demand can be sent partly clockwise, partly counterclockwise. The optimum ring load L^* of a split routing is obviously a lower bound on the optimum load L of an unsplittable routing. Myung, Kim, and Tcha [4] show how to compute L^* in time $O(nk)$, where k is the number of nonzero demands.

Schrijver, Seymour, and Winkler, in a landmark paper [6], present a clever analysis for a simple greedy algorithm that turns any split routing into an unsplittable routing while increasing the load on any edge by at most $\frac{3}{2}D$, where D is the maximum demand value $\max_{i,j} d_{i,j}$. Their result thus implies $L \leq L^* + \frac{3}{2}D$. On the other hand, they exhibit an instance of the Ring Loading Problem together with a carefully chosen split routing that cannot be turned into an unsplittable routing without increasing the load on some edge by at least $\frac{101}{100}D$. This observation, however, does not immediately imply a gap strictly larger than D between the optimum values of split and unsplittable routings. In the conclusion of their paper, Schrijver, Seymour, and Winkler write:

> "...even though the mathematics refuses to cooperate, *we* guarantee $L \leq L^* + D$."

In the excellent survey [7], Shepherd restates this 'guarantee' as a conjecture.

Our contribution. Our main result is the following theorem.

Theorem 1. *Any split routing solution to the* Ring Loading Problem *can be turned into an unsplittable routing while increasing the load on any edge by no more than* $\frac{19}{14}D$.

In particular, this result implies $L \leq L^* + \frac{19}{14}D$. Our algorithm runs in linear time and combines pairs of solutions obtained by the greedy algorithm of Schrijver, Seymour, and Winkler [6] using a clean crossover operation.

We also exhibit a relatively simple instance of the Ring Loading Problem together with particular split routings that cannot be turned into unsplittable routings without increasing the load on some edge by at least $\frac{11}{10}D$. Our results are the first improvements on the classical results of Schrijver, Seymour, and Winkler [6] (upper bound $\frac{3}{2}D$ and lower bound $\frac{101}{100}D$). Last but not least, we present an instance with $L = L^* + \frac{11}{10}D$, thus disproving Schrijver et al.'s long-standing conjecture $L \leq L^* + D$. Our extensive yet unsuccessful search for instances yielding a larger lower bound gives us serious doubts as to whether there exist any.

Conjecture 1. $L \leq L^* + \frac{11}{10}D$ *for all instances of the* Ring Loading Problem.

We have serious doubts as to whether the algorithmic techniques and analytic tools discussed in this paper are powerful enough to close the remaining gap. But we hope that, 16 years after the publication of Schrijver et al.'s landmark paper [6], the presented progress will stimulate further research and new ideas on this fine and challenging problem.

For further details we refer to [8].

Acknowledgements. The author would like to thank Bruce Shepherd for giving the impetus to this work as well as for numerous inspiring conversations and valuable comments. A very special thanks is due to Tom McCormick for the hospitality at UBC without which this work could never have been done.

References

[1] S. Cosares and I. Saniee. An optimization problem related to balancing loads on SONET rings. *Telecommunications Systems*, 3:165–181, 1994.

[2] A. Frank. Edge-disjoint paths in planar graphs. *Journal of Combinatorial Theory, Series B*, 38:164–178, 1985.

[3] S. Khanna. A polynomial-time approximation scheme for the SONET ring loading problem. *Bell Labs Technical Journal*, 2:36–41, 1997.

[4] Y.-S. Myung, H.-G. Kim, and D.-W. Tcha. Optimal load balancing on SONET bidirectional rings. *Operations Research*, 45:148–152, 1997.

[5] H. Okamura and P. D. Seymour. Multicommodity flow in planar graphs. *Journal of Combinatorial Theory, Series B*, 31:75–81, 1981.

[6] A. Schrijver, P. Seymour, and P. Winkler. The ring loading problem. *SIAM Journal on Discrete Mathematics*, 11:1–14, 1998.
[7] F. B. Shepherd. Single-sink multicommodity flow with side constraints. In W. Cook, L. Lovász, and J. Vygen, editors, *Research Trends in Combinatorial Optimization*. Springer, 2009.
[8] M. Skutella. A note on the ring loading problem. In P. Indyk, editor, *Proceedings of the 26th Annual ACM–SIAM Symposium on Discrete Algorithms*. SIAM, 2015. To appear.

The Learning With Errors problem: from lattices to cryptography

Damien Stehlé

The search variant of the Learning With Errors problem (LWE) is to recover $\mathbf{s} \in (\mathbb{Z}/q\mathbb{Z})^n$ from arbitrarily many samples of the form $(\mathbf{a}_i, < \mathbf{a}_i, \mathbf{s} > + e_i \bmod q)$, where the $\mathbf{a}_i$ are chosen uniformly from $(\mathbb{Z}/q\mathbb{Z})^n$, and the "errors" $e_i \in \mathbb{Z}$ are sampled from some distribution supported on small numbers, typically an integer Gaussian distribution with standard deviation parameter αq for $\alpha = o(1)$.

The decision counterpart of LWE is to tell from which of the following distributions are sampled arbitrarily many given elements of $(\mathbb{Z}/q\mathbb{Z})^{n+1}$, with some non-negligible probability (when $n \log q$ tends to infinity) over the uniform choice of $\mathbf{s} \in (\mathbb{Z}/q\mathbb{Z})^n$ and with some probability non-negligibly larger than 1/2:

$$(\mathbf{a}_i, \langle \mathbf{a}_i, \mathbf{s} \rangle + \ e_i \bmod q)_i \quad \text{and} \quad (\mathbf{a}_i, u_i)_i \ .$$

Here the $\mathbf{a}_i$ and e_i are as above, and the u_i are chosen uniformly from $\mathbb{Z}/q\mathbb{Z}$. There exist polynomial-time reductions between these search and decision variants [6].

Since its introduction by Oded Regev, the presumed hardness of LWE has served as a security foundation of numerous cryptographic primitives: public-key encryption [6], fully homomorphic encryption [2], attribute-based encryption [3], among many others.

In the typical situation where $1/\alpha$ is polynomially bounded in $n \log q$, the best known algorithm for solving LWE runs in time exponential in $n \log q$. That algorithm views LWE as a problem on Euclidean lattices, and consists in calling the Block Korkine Zolotarev lattice reduction algorithm [7, 4]. LWE seems to remain exponentially hard to solve even if one is allowed to rely on quantum computations.

The link between LWE and lattices is in fact very deep. Regev [6] showed a quantum polynomial-time reduction from the approximation variant of the Shortest Vector Problem in dimension n to LWE, assuming that q is prime and polynomially bounded in n, and that α satisfies $\alpha q \geq 2\sqrt{n}$. The Shortest Vector Problem with approximation parameter γ (GapSVP$_\gamma$) consists in assessing whether a given lattice has a first minimum smaller than 1 or larger than γ. In the latter reduction, one can take γ as small as $n(\log n)^{O(1)}/\alpha$. The reduction is quantum in that the reduction algorithm uses the quantum Fourier transform. A classical polynomial-time reduction from GapSVP$_\gamma$ to LWE was recently proposed by Brakerski *et al.* [1], but the dimension of the lattice problem is only $O(\sqrt{n})$. The classical reduction does not require q to be prime and achieves a comparable approximation factor γ. Removing or explaining the GapSVP dimension discrepancy between the quantum and classical reductions is an exciting open problem.

In this talk, we presented LWE, described how it can be used for public-key encryption, and stressed its links with standard algorithmic problems on Euclidean lattices.

References

[1] Zvika Brakerski, Adeline Langlois, Chris Peikert, Oded Regev, Damien Stehlé: Classical hardness of learning with errors. STOC 2013: 575-584.

[2] Zvika Brakerski, Vinod Vaikuntanathan: Efficient Fully Homomorphic Encryption from (Standard) LWE. FOCS 2011: 97-106.

[3] Sergey Gorbunov, Vinod Vaikuntanathan, Hoeteck Wee: Attribute-based encryption for circuits. STOC 2013: 545-554.

[4] Guillaume Hanrot, Xavier Pujol, Damien Stehlé: Analyzing Blockwise Lattice Algorithms Using Dynamical Systems. CRYPTO 2011: 447-464.

[5] Daniele Micciancio, Panagiotis Voulgaris: *A Deterministic Single Exponential Time Algorithm for Most Lattice Problems Based on Voronoi Cell Computations.* SIAM J. Comput. 42(3): 1364-1391 (2013).

[6] Oded Regev: *On lattices, learning with errors, random linear codes, and cryptography.* J. ACM 56(6) (2009).

[7] Claus-Peter Schnorr, M. Euchner: *Lattice basis reduction: Improved practical algorithms and solving subset sum problems.* Math. Program. 66: 181-199 (1994).

LP-based algorithms for capacitated facility location

Ola Svensson

(joint work with Hyung-Chan An and Mohit Singh)

We consider the metric capacitated facility location (Cfl) problem which together with the metric uncapacitated facility location (Ufl) problem is the most classical and widely studied variant of facility location. In Cfl, we are given a single metric on the set of *facilities* and *clients*, and every facility has an associated *opening cost* and *capacity*. The problem asks us to choose a subset of facilities to open and assign every client to one of these open facilities, while ensuring that no facility is assigned more clients than its capacity. Our aim is then to find a set of open facilities and an assignment that minimize the cost, where the cost is defined as the sum of opening costs of each open facility and the distance between each client and the facility it is assigned to. Ufl is the special case of Cfl obtained by dropping the capacity constraints, or equivalently setting each capacity to ∞.

In spite of the similarities in the problem definitions of Ufl and Cfl, current techniques give a considerably better understanding of the uncapacitated version. One prominent reason for this discrepancy is that a standard linear programming (LP) relaxation gives close-to-tight bounds for Ufl, whereas no good relaxation was known in the presence of capacities. For Ufl, on the one hand, the standard LP formulation has been used in combination with most LP-based techniques, such as filtering [10], randomized rounding [4, 11], primal-dual framework [7], and dual fitting [5, 6], to obtain a fine-grained understanding of the problem resulting in a nearly tight approximation ratio [9].

For Cfl, on the other hand, it has remained a major open problem to find a relaxation based algorithms with *any* constant performance guarantee, also highlighted as Open Problem 5 in the list of ten open problems selected by the recent textbook on approximation algorithms of Williamson and Shmoys [12]. However, formulating one for the capacitated facility location problem has turned out to be non-trivial. Aardal et al. [1] made a comprehensive study of valid inequalities (such as various adaptations of flow cover inequalities) for capacitated facility location problem and proposed further generalizations; the strength of the obtained formulations was left as an open problem. Many of these formulations were, however, recently proven to be insufficient for obtaining a constant integrality gap by Kolliopoulos and Moysoglou [8]. In the same paper it is also shown that applying the Sherali–Adams hierarchy to the standard LP formulation will not close the integrality gap.

Our contributions. Our main contribution is a strong linear programming relaxation which has a constant integrality gap for the capacitated facility location problem. We prove its constant integrality gap by presenting a polynomial time approximation algorithm which rounds the LP solution.

Theorem 1. *There is an algorithmically amenable linear programming relaxation for the capacitated facility location problem that has a constant integrality gap. Moreover, there exists a polynomial-time algorithm that finds a solution to the capacitated facility location problem whose cost is no more than a constant (288) factor times the LP optimum.*

Our relaxation is formulated based on multi-commodity flows and is inspired by the general idea of flow cover inequalities. The overall idea is as follows. In an integral solution, each client needs to send one unit of flow to an opened facility with available capacity in a certain multi-commodity flow problem. Writing down the constraints for this problem results in the standard LP which unfortunately has a large integrality gap. To strengthen this formulation we consider partial assignments of clients. In a partial assignment, some clients are already assigned to facilities. Using this concept we strengthen the linear program by, for each partial assignment, enforcing that the unassigned clients are still able to send one unit of flow to opened facilities with available capacity where the capacity of a facility now is reduced by the number of clients that were preassigned to it. Due to space constraints, we refer the reader to the full paper [2] for the formal description of the relaxation.

Open questions. One natural question that arises is characterizing the exact integrality gap of our relaxation. While we prioritized ease of reading over a better ratio in the choice of parameters in our paper, it appears that the current analysis is not likely to give any approximation ratio better than 5, the best ratio given by the local search algorithms [3]. On the other hand, the best lower bound known on the integrality gap of our relaxation is 2, and the question remains open whether we can obtain an approximation algorithm with a ratio smaller than 5 based on our relaxation.

Open Question. Determine the integrality gap of our LP relaxation.

Another interesting question is whether there exists a compact linear programming relaxation for Cfl with constant integrality gap. Our current relaxation has exponentially many constraints and it would be very interesting to decide if there is a relaxation of similar strength but of polynomial size.

Open Question. Determine if there is a polynomial size LP relaxation for the capacitated facility location problem with a constant integrality gap.

References

[1] K. Aardal, Y. Pochet, and L. A. Wolsey, *Capacitated facility location: valid inequalities and facets*, Mathematics of Operations Research, 20 (1995), pp. 562–582.

[2] H.-C. An, M. Singh, and O. Svensson, *LP-Based Algorithms for Capacitated Facility Location*, in FOCS, 2014, pp. 256–265. Full version available at *http://arxiv.org/abs/1407.3263*.

[3] M. Bansal, N. Garg, and N. Gupta, *A 5-approximation for capacitated facility location*, in ESA, 2012, pp. 133–144.

[4] F. A. Chudak and D. B. Shmoys, *Improved approximation algorithms for the uncapacitated facility location problem*, SIAM J. Comput., 33 (2004), pp. 1–25.

[5] K. Jain, M. Mahdian, E. Markakis, A. Saberi, and V. V. Vazirani, *Greedy facility location algorithms analyzed using dual fitting with factor-revealing LP*, J. ACM, 50 (2003), pp. 795–824.

[6] K. Jain, M. Mahdian, and A. Saberi, *A new greedy approach for facility location problems*, in Proceedings of the thiry-fourth annual ACM symposium on Theory of computing, STOC '02, 2002, pp. 731–740.

[7] K Jain and V. V. Vazirani, *Approximation algorithms for metric facility location and k-median problems using the primal-dual schema and Lagrangian relaxation*, J. ACM, 48 (2001), pp. 274–296.

[8] Stavros G. Kolliopoulos and Yannis Moysoglou, *Sherali-Adams gaps, flow-cover inequalities and generalized configurations for capacity-constrained facility location*, CoRR, abs/1312.0722 (2013).

[9] S. Li, *A 1.488 approximation algorithm for the uncapacitated facility location problem*, in Automata, Languages and Programming - 38th International Colloquium (ICALP), 2011, pp. 77–88.

[10] J. Lin and J. S. Vitter, *Approximation algorithms for geometric median problems*, Inf. Process. Lett., 44 (1992), pp. 245–249.

[11] D. B. Shmoys, E. Tardos, and K. Aardal, *Approximation algorithms for facility location problems (extended abstract)*, in STOC '97: Proceedings of the twenty-ninth annual ACM symposium on Theory of computing, 1997, pp. 265–274.

[12] D. P. Williamson and D. B. Shmoys, *The Design of Approximation Algorithms*, Cambridge University Press, 2010.

Positive semidefinite rank

Rekha R. Thomas

The *positive semidefinite (psd) rank* of a matrix $M \in \mathbb{R}_+^{p\times q}$ is the smallest integer k such that there exist psd matrices $A_1, \dots, A_p$ and $B_1, \dots, B_q$ of size $k \times k$ such that $M_{ij} = \langle A_i, B_j \rangle := \text{Trace}(A_i B_j)$. This is an example of a *cone rank* of a nonnegative matrix wherein we have a "closed" family of closed convex cones $\mathcal{C} = \{C_k\}$, and the rank of M with respect to $\mathcal{C}$ is defined as $\text{rank}_{\mathcal{C}}(M) := \min \{k : \exists\, a_1, \dots, a_p \in C_k,\ b_1, \dots, b_q \in (C_k)^* \text{ s.t. } M_{ij} = \langle a_i, b_j \rangle \ \forall\ i, j\}$. Here C_k^* is the dual cone of C_k, and a closed family of cones is one in which every face of every cone in the family is also in the family. The cone rank of M with respect to the family of positive orthants is the *nonnegative rank* of M, while the cone rank of M with respect to the family of psd cones is the psd rank of M.

The notion of psd rank was introduced in [2] to study semidefinite representations of convex sets. For instance, given a full-dimensional polytope $P \subset \mathbb{R}^n$ with vertices $p_1, \dots, p_v$, and facet inequalities $a_j^\top x \le \beta_j$ for $j = 1, \dots, f$, its *slack matrix* S_P is the $v \times f$ matrix whose (i,j)-entry is $\beta_j - a_j^\top p_i$, the slack of the ith vertex in the jth facet inequality. Let $\mathcal{S}_+^k$ denote the cone of $k \times k$ psd matrices. We say that P has a *psd lift* of size k if there exists a linear map π and an affine space L such that $P = \pi(\mathcal{S}_+^k \cap L)$. The set $\mathcal{S}_+^k \cap L$ is called a psd-lift of P. It was shown in [2] that the psd rank of S_P is the size of the smallest psd lift of P. The theorem is in fact more general; a convex set $P \subset \mathbb{R}^n$ has a lift into a closed convex cone C if and only if the *slack operator* of P has a factorization through C and its dual. This factorization theorem generalizes a result of Yannakakis [3] that says that the nonnegative rank of S_P is the size of the smallest polyhedral lift of P, i.e., a lift of the form $\mathbb{R}_+^k \cap L$ for an affine space L.

Several results have been obtained for psd rank since its definition. One of the most active areas has been its use in the factorization theorem for polytopes. It has been shown by Briët, Dadush and Pokutta that there are $0,1$-polytopes in $\mathbb{R}^n$ whose psd rank cannot be polynomial in n. Very recently, Lee, Raghavendra and Steurer have announced that for cut, stable set and TSP polytopes on n-vertex graphs, psd rank is again super polynomial in n. On the positive side, Gouveia, Robinson and myself have characterized polytopes in $\mathbb{R}^n$ whose psd rank is the minimum possible – namely $n + 1$.

The psd rank has many further features and properties beyond its use in understanding psd lifts of convex sets. My talk was based on a recent survey article [1] coauthored with Hamza Fawzi, João Gouveia, Pablo Parrilo and Richard Robinson on the topic of psd rank, with the aim of bringing this invariant to the attention of the broader mathematical community. We examine several directions in which the study of psd rank can be developed and survey the known results thus far. Citations of all results mentioned above can be found in [1].

References

[1] H. Fawi, J. Gouveia, P.A. Parrilo, R.Z. Robinson and R.R. Thomas, *Positive semidefinite rank*, arXiv:1407.4095.
[2] J. Gouveia, P.A. Parrilo and R.R. Thomas, *Lifts of convex sets and cone factorizations*, Math. Operations Research, **38**(2) (2013), 248-264.
[3] M. Yannakakis, *Expressing combinatorial optimization problems by linear programs*, J. Comput. System Sci., **43**(3) (1991), 441-466.

Embedding formulations and complexity for unions of polyhedra

Juan Pablo Vielma

We consider strong Mixed Integer Programming (MIP) formulations for a disjunctive constraint of the form

$$x \in \bigcup_{i=1}^{n} P_i \tag{1}$$

where $\{P_i\}_{i=1}^n \subseteq \mathbb{R}^d$ is a finite family of rational polyhedra, which for simplicity of exposition we assume are bounded. Classical polynomially sized MIP formulations for (1) can be divided into two classes depending on their strength and types of auxiliary variables (e.g. [4]). The first class includes integral formulations by Balas, Jeroslow and Lowe, which use both integer constrained and continuous auxiliary variables. The second class includes formulations by Balas, Blair, Jeroslow, Lee and Wilson, which exclude continuous auxiliary, but can fail to be integral. A common feature of both classes is the use of n non-negative integer variables that are constrained to add up to one. In this talk we show how alternate uses of integer variables can lead to small integral formulations that do not use continuous auxiliary variables. For this we introduce the following generic class of MIP formulations for (1).

Definition 1 (Embedding Formulation)**.** For any $\mathcal{H} := \left\{h^i\right\}_{i=1}^n \subseteq \{0,1\}^k$ such that $h^i \neq h^j$ for all $i \neq j$, we let

$$Q(\mathcal{H}) := \operatorname{conv}\left(\bigcup_{i=1}^{n} P_i \times \left\{h^i\right\}\right).$$

For any $\mathcal{H}$ we have that $Q(\mathcal{H})$ is a rational polyhedron and $\left(x, h^i\right) \in Q(\mathcal{H})$ if and only if $x \in P_i$. Hence, $(x,y) \in Q(\mathcal{H}) \cap \left(\mathbb{R}^d \times \{0,1\}^k\right)$ is a MIP formulation for (1). If $k = n$ and $h^i = e^i$, the i-th unit vector, we obtain a formulation with the traditional use of integer variables and if $n = 2^k$ and $\mathcal{H} = \{0,1\}^k$ we obtain a so-called logarithmic formulation (e.g. [4]). Following [5], we refer to these two choices of $\mathcal{H}$ as unary and binary encodings respectively as they can be interpreted as the corresponding encoding of the choice among the P_i. Furthermore, because for the unary encoding $\bigcup_{i=1}^n P_i \times \left\{h^i\right\}$ and $Q(\mathcal{H})$ are usually denoted the Cayley Embedding and the Cayley polytope of $\{P_i\}_{i=1}^n$ (e.g. [1]), we refer to $Q(\mathcal{H}) \cap$

$\left(\mathbb{R}^d \times \{0,1\}^k\right)$ as an Embedding Formulation of (1). Finally, we note that, by construction, every embedding formulation is integral.

While binary encoded formulations can provide a computational advantage when solved by a branch-and-bound based solver (e.g. [4, 5]), it is not immediately clear which encoding yields the smallest formulation (i.e. the one for which $Q(\mathcal{H})$ has the fewest facets). For instance, $Q(\mathcal{H})$ always contains $\operatorname{conv}\left(\bigcup_{i=1}^n P_i\right)$ trough projection. Hence, the larger number of auxiliary variables of the unary encoded formulation may provide an advantage. However, it is well known that the Cayley polytope also contains the Minkowski sum of $\{P_i\}_{i=1}^n$ through an appropriate affine section (e.g. [1]). Hence, the unary encoded formulation may be large even if $\operatorname{conv}\left(\bigcup_{i=1}^n P_i\right)$ is small. In contrast, the binary encoded formulation only seems to contain partial Minkowski sums of about $\log_2 n$ polytopes. This suggests a potential advantage for the binary encoded formulations.

Using results from [2, 3], we show that for certain classes of disjunctive constraints related to the modeling of piecewise-linear functions, the binary encoding can indeed yield significantly smaller embedding formulations. In particular, for bi-variate piecewise-linear functions the size of the binary encoded formulation is linear in n while the size of the unary encoded formulation is superpolynomial in n. However, we also show that the advantage of the binary encoded formulation is strongly dependent on the specific binary encoding selected (i.e. on the order of $\{0,1\}^k$ induced by $\mathcal{H}$).

References

[1] M. I. Karavelas, C. Konaxis, and E. Tzanaki. *The maximum number of faces of the Minkowski sum of three convex polytopes*, In *Proceedings of the 29th Annual Symposium on Computational Geometry*, pages 187–196. ACM, 2013.

[2] J. Lee and D. Wilson, *Polyhedral methods for piecewise-linear functions I: the lambda method*, Discrete Applied Mathematics **108** (2001), 269–285.

[3] J. P. Vielma and S. Ahmed and G. L. Nemhauser, *Modeling disjunctive constraints with a logarithmic number of binary variables and constraints*, Mathematical Programming **128** (2011), 49–72.

[4] J. P. Vielma, *Mixed Integer Linear Programming Formulation Techniques*, To appear in SIAM Review, doi:10.1137/130915303, 2014. Available at `http://www.optimization-online.org/DB_HTML/2012/07/3539.html`.

[5] S. Yıldız and J. P. Vielma, *Incremental and encoding formulations for mixed integer programming*, Operations Research Letters **41** (2013), 654–658.

Duality for mixed-integer convex minimization

Robert Weismantel

(joint work with Michel Baes and Timm Oertel)

Several attempts have been made in the past to formally define a dual of a linear integer or mixed integer programming problem. Let us first mention some important developments in this direction.

One idea to define a dual program associated with a binary linear integer programming problem is to encode the given 0/1-problem in form of a linear program in an extended space such that the new variables correspond to linearizations of products of original variables. This allows us to apply linear programming duality that can then be reinterpreted in terms of the original variables. This concept of duality has its origins in the work of [7] and is closely connected with work of Lovasz and Schrijver as well as with earlier work of Balas on disjunctive optimization. It also provides us an interesting link to the theory of polynomial optimization including duality results associated with hierarchies of semidefinite programming problems, see [3].

A second import development in integer optimization is based on the connections between valid inequalities and subadditive functions. This leads to a formalism that allows us to establish a subadditive dual of a general mixed integer linear optimization problem, see [6] for a treatment of the subject and references. Since then many papers have dealt with the question of deriving relaxations of an integer optimization problem by means of generating superadditive functions. Recently, a strong subadditive dual for conic mixed integer optimization has been established in [4].

There are several other special cases for which the dual of a mixed integer optimization problem has been derived. One such example is based on the theory of discrete convexity established in [5]. Here, an explicit dual is constructed for L-convex and M-convex functions.

A third general approach to develop duality in several subfields of optimization is based on the Lagrangean relaxation method. The latter method is broadly applicable and – among others – leads to a formalism of duality in convex optimization. This is our point of departure. We will show that optimality certificates and duality in convex optimization have a very natural mixed-integer analogue. A duality theory in Euclidean space follows from a precise interplay between points – that are viewed as primal objects – and hyperplanes interpreted as dual objects. It turns out that there is a similiar interplay in the mixed-integer setting. Here, the primal objects are sets of points, whereas the dual objects are lattice-free polyhedra. Our motivation for studying optimality certificates and a mixed integer convex dual comes from the important developments in convex optimization in the past decade. As a first step towards new mixed-integer convex algorithms it seems natural to make an attempt of extending some of the basic convex optimization tools to the mixed-integer setting.

Let $f : \mathrm{dom}(f) \mapsto \mathbb{R}$ be a continuous, convex function. In order to simplify our exposition we may assume w.l.o.g. here that $\mathrm{dom}(f) = \mathbb{R}^n$. Assume that f has a, not necessarily unique, minimizer $x^\star$. Then a necessary and sufficient certificate for $x^\star$ being a minimizer of f is that $0 \in \partial f(x^\star)$, i.e. the zero-function is in the subdifferential of f at $x^\star$. Hence

$$x^\star = \operatorname*{argmin}_{x \in \mathbb{R}^n} f(x) \Longleftrightarrow 0 \in \partial f(x^\star).$$

The question emerges how to obtain a certificate that a point $x^\star \in \mathbb{Z}^n \times \mathbb{R}^d$ solves the corresponding mixed integer convex problem

$$x^\star = \operatorname*{argmin}_{x \in \mathbb{Z}^n \times \mathbb{R}^d} f(x)?$$

Theorem 1. *Let $f : \mathbb{R}^{n+d} \mapsto \mathbb{R}$ be a continuous convex function. $x^\star = \operatorname*{argmin}_{x \in \mathbb{Z}^n \times \mathbb{R}^d} f(x)$ if and only if there exist $k \leq 2^n$ points $x_1 = x^\star, x_2, \ldots, x_k \in \mathbb{Z}^n \times \mathbb{R}^d$ and vectors $h_i \in \partial f(x_i)$ such that the following conditions hold:*

(a) $f(x_1) \leq \ldots \leq f(x_k)$,
(b) $\{x \in \mathbb{R}^{n+d} \mid h_i^\intercal (x - x_i) < 0$ *for all* $i\} \cap (\mathbb{Z}^n \times \mathbb{R}^d) = \emptyset$,
(c) $h_i \in \mathbb{R}^n \times \{0\}^d$ *for* $i = 1, \ldots, k$.

More generally, let $g_1, \ldots, g_m : \mathrm{dom}(f) \mapsto \mathbb{R}$ be continuous, convex functions. Again we may assume w.l.o.g. that $\mathrm{dom}(g_j) = \mathbb{R}^n$ for all j. By $g(x)$ we denote the vector of components $g_1(x), \ldots, g_m(x)$. Let us first discuss the continous convex optimization problem

$$x^\star = \operatorname*{argmin}_{\substack{x \in \mathbb{R}^n, \\ g(x) \leq 0}} f(x).$$

Assume that there exists a point y fulfilling the so-called Slater condition, that is, $g(y) < 0$. Under this assumption the Karush-Kuhn-Tucker conditions (e.g. [1, 2]) provide necessary and sufficient optimality conditions. Namely, the point $x^\star$ such that $g(x^\star) \leq 0$ attains the optimal continuous solution if and only if there exist $h_f \in \partial f(x^\star)$, $h_{g_i} \in \partial g_i(x^\star)$, for $i = 1, \ldots, m$ and non-negative λ_i, $i = 1, \ldots, m$, such that

$$h_f + \sum_{i=1}^{m} \lambda_i h_{g_i} = 0 \text{ and } \lambda_i g_i(x^\star) = 0 \ \forall i.$$

As a second result we generalize the KKT theorem to the mixed-integer setting

$$x^\star = \operatorname*{argmin}_{\substack{x \in \mathbb{Z}^n \times \mathbb{R}^d, \\ g(x) \leq 0}} f(x). \tag{1}$$

We first generalize the Slater condition.

Definition 1. We say that the constraints $g(x) \leq 0$ fulfill the *mixed-integer Slater condition* if for every point $(y, z) \in \mathbb{Z}^n \times \mathbb{R}^d$ with $g((y, z)) \leq 0$ there exists a $z' \in \mathbb{R}^d$ such that $g((y, z')) < 0$.

Theorem 2. *Let g fulfill the mixed-integer Slater condition. A point $x^\star \in \mathbb{Z}^n \times \mathbb{R}^d$ is optimal with respect to* (1) *if and only if $g(x^\star) \leq 0$ and there exist $k \leq 2^n$ points $x_1 = x^\star, x_2, \ldots, x_k \in \mathbb{Z}^n \times \mathbb{R}^d$ and k vectors $u_1, \ldots, u_k \in \mathbb{R}_+^{m+1}$ with corresponding $h_{i,m+1} \in \partial f(x_i)$, and $h_{i,j} \in \partial g_j(x_i)$ for $j = 1, \ldots, m$ such that the following five conditions hold:*

(a) *If* $g(x_i) \leq 0$ *then* $f(x_i) \geq f(x_1)$, $u_{i,m+1} > 0$ *and* $u_{i,j} g_j(x_i) = 0$ *for* $j = 1, \ldots, m$,
(b) *If* $g(x_i) \not\leq 0$ *then* $u_{i,m+1} = 0$ *and* $u_{i,k}(g_k(x_i) - g_l(x_i)) \geq 0$ *for all* $k, l = 1, \ldots, m$,

(c) $1 \le |\operatorname{supp}(u_i)| \le d+1$ *for* $i = 1, \dots, k$,
(d) $\{x \in \mathbb{R}^{n+d} \mid \sum_{j=1}^{m+1} u_{i,j} h_{i,j}^{\mathsf{T}}(x - x_i) < 0 \text{ for all } i\} \cap (\mathbb{Z}^n \times \mathbb{R}^d) = \emptyset$,
(e) $\sum_{j=1}^{m+1} u_{i,j} h_{i,j} \in \mathbb{R}^n \times \{0\}^d$ *for* $i = 1, \dots, k$.

This result shows that the optimality of a mixed integer point can be verified in polynomial time, provided that the number of integer variables is constant.

In the purely continuous setting it is not too difficult to apply the KKT-theorem in order to show a duality theorem. Provided that the Slater condition holds, that all functions f and $g_j,\ j = 1, \dots, m$ are continuous and convex and that the primal and dual feasible sets are nonempty, one has

$$f^\star = \min_{x \in \mathbb{R}^n} \{ f(x) \mid g(x) \le 0\} = \max_{\alpha, u \in \mathbb{R}^m_+} \{\alpha \mid \alpha \le f(x) + u^T g(x) \forall x \in \mathbb{R}^n\}.$$

In the same vein we use Theorem 2 in order to derive a mixed integer dual.

Theorem 3. *Let* $f : \mathbb{R}^{n+d} \mapsto \mathbb{R}$ *and* $g : \mathbb{R}^{n+d} \mapsto \mathbb{R}^m$ *be convex functions, s.t.* $\{x \in \mathbb{Z}^n \times \mathbb{R}^d \mid g(x) \le 0\}$ *is non-empty, compact and contained in the domain of* f*. Further, let* g *fulfill the mixed-integer Slater condition. Then,*

$$\begin{aligned} & \min_{x \in \mathbb{Z}^n \times \mathbb{R}^d} \{ f(x) \mid g(x) \le 0\} \\ = & \max_{\substack{\alpha \in \mathbb{R} \\ U \in \mathbb{R}_+^{2^n \times m}}} \{ \alpha \mid \exists\, \pi : \mathbb{Z}^n \times \mathbb{R}^d \mapsto \{1, \dots, 2^n\} \text{ s.t.} \\ & \forall x \in \mathbb{Z}^n \times \mathbb{R}^d \ \alpha \le f(x) + U_{\pi(x)} g(x) \text{ or } 1 \le U_{\pi(x)} g(x)\}, \end{aligned}$$

where U_i *denotes the* i*-th row of* U*.*

References

[1] W. Karush. *Minima of Functions of Several Variables with Inequalities as Side Constraints.* M.Sc. Dissertation, Univ. of Chicago (1939).

[2] H.W. Kuhn, A.W. Tucker, *Nonlinear Programming.* Proceedings of the Second Berkeley Symposium on Mathematical Statistics and Probability (1951), 481–492.

[3] J.B. Lasserre, *A Lagrangian relaxation view of linear and semidefinite hierarchies*, SIAM Journal on Optimization **23** (2013), 1742–1756.

[4] D.A. Morán R., S. Dey, J. Vielma, *A Strong Dual for Conic Mixed-Integer Programs*, SIAM Journal on Optimization **22** (2012), 1136–1150.

[5] K. Murota, *Discrete convex analysis*, SIAM Monographs on Discrete Mathematics (2003).

[6] G. Nemhauser, L. Wolsey, *Integer and combinatorial optimization*, Wiley (1988).

[7] H.D. Sherali, W.P. Adams, *A hierarchy of relaxations between the continous and convex hull representations for zero-one programming problems*, SIAM Journal on Discrete Mathematics **3** (1990), 411–430.

Reporter: Neil Olver

Participants

Prof. Dr. Egon Balas
Carnegie Mellon University
Tepper School of Business
Schenley Park
Pittsburgh, PA 15213
UNITED STATES

Prof. Dr. Nikhil Bansal
Dept. of Mathematics & Computer Science
Eindhoven University of Technology
Postbus 513
5600 MB Eindhoven
NETHERLANDS

Prof. Dr. Amitabh Basu
Johns Hopkins University
Dept. of Applied Mathematics & Statistics
100 Whitehead Hall
3400 N. Charles Street
Baltimore, MD 21218
UNITED STATES

Prof. Dr. Chandra Chekuri
Department of Computer Science
University of Illinois at Urbana-Champaign
1304 W. Springfield Ave.
Urbana, IL 61801-2987
UNITED STATES

Prof. Dr. Michele Conforti
Dipto di Matematica Pura ed Applicata
Universita di Padova
Via Trieste
35131 Padova
ITALY

Prof. Dr. William J. Cook
Dept. of Combinatorics & Optimization
University of Waterloo
Waterloo, Ont. N2L 3G1
CANADA

Prof. Dr. Gérard P. Cornuéjols
Tepper School of Business
Carnegie Mellon University
5000 Forbes Avenue
Pittsburgh, PA 15213-3890
UNITED STATES

Prof. Dr. Santanu S. Dey
H. Milton Stewart School of
Industrial and Systems Engineering
Georgia Institute of Technology
765 Ferst Dr.
Atlanta GA 30332
UNITED STATES

Prof. Dr. Friedrich Eisenbrand
EPFL SB MATH AA
MA C1 573
Station 8
1015 Lausanne
SWITZERLAND

Prof. Dr. Samuel Fiorini
Département de Mathématiques
Université Libre de Bruxelles
CP 216 Campus Plaine
Bd. du Triomphe
1050 Bruxelles
BELGIUM

Prof. Dr. András Frank
Department of Operations Research
Eötvös Lorand University
ELTE TTK
Pazmany Peter setany 1/C
1117 Budapest
HUNGARY

Prof. Dr. Satoru Fujishige
RIMS
Kyoto University
Sakyo-ku
Kyoto 606-8502
JAPAN

Prof. Dr. Michel X. Goemans
Department of Mathematics
Massachusetts Institute of Technology
77 Massachusetts Avenue
Cambridge, MA 02139-4307
UNITED STATES

Prof. Dr. Peter Gritzmann
Zentrum Mathematik
Angewandte Geometrie & Diskrete Mathematik
Technische Universität München
Boltzmannstr. 3
85747 Garching bei München
GERMANY

Prof. Dr. Martin Grötschel
Konrad-Zuse-Zentrum für Informationstechnik Berlin (ZIB)
Takustr. 7
14195 Berlin
GERMANY

Prof. Dr. Bertrand Guenin
Dept. of Combinatorics & Optimization
University of Waterloo
200 University Ave.
Waterloo Ont. N2L 3G1
CANADA

Prof. Dr. Anupam Gupta
Department of Computer Science
Carnegie Mellon University
7203 Gates Building
Pittsburgh PA 15213-3890
UNITED STATES

Prof. Dr. Satoru Iwata
Department of Mathematical Informatics
University of Tokyo
Tokyo 113-8656
JAPAN

Prof. Dr. Volker Kaibel
Fakultät für Mathematik
Otto-von-Guericke-Universität Magdeburg
Universitätsplatz 2
39106 Magdeburg
GERMANY

Prof. Dr. Tamas Kiraly
Department of Operations Research
Eötvös Lorand University
ELTE TTK
Pazmany Peter setany 1/C
1117 Budapest
HUNGARY

Prof. Dr. Matthias Koeppe
Department of Mathematics
University of California, Davis
1, Shields Avenue
Davis, CA 95616-8633
UNITED STATES

Prof. Dr. Dr.h.c. Bernhard Korte
Forschungsinstitut f. Diskrete Mathematik
Universität Bonn
Lennestr. 2
53113 Bonn
GERMANY

Dr. Jean Bernard Lasserre
LAAS - CNRS
7, Avenue du Colonel Roche
31077 Toulouse Cedex 4
FRANCE

Prof. Dr. Monique Laurent
Centrum Wiskunde & Informatica
(CWI)
Postbus 94079
1090 GB Amsterdam
NETHERLANDS

Dr. James R. Lee
Dept. of Computer Science &
Engineering
University of Washington
Box 352350
Seattle WA 98195-2350
UNITED STATES

Prof. Dr. Alexander Martin
Universität Erlangen-Nürnberg
Lehrstuhl für Wirtschaftsmathematik
Cauerstr. 11
91058 Erlangen
GERMANY

Prof. Dr. Rolf H. Möhring
Institut für Mathematik
Fakultät II, Sekr. MA 5-1
Technische Universität Berlin
Straße des 17. Juni 136
10623 Berlin
GERMANY

Prof. Dr. Kazuo Murota
School of Information Science & Techn.
University of Tokyo
Hongo 7-3-1, Bunkyo-ku
Tokyo 113-8656
JAPAN

Prof. Dr. George L. Nemhauser
Industrial and Systems Engineering
Georgia Institute of Technology
Atlanta, GA 30332-0205
UNITED STATES

Dr. Neil K. Olver
Department of Mathematics, FEW
Vrije Universiteit Amsterdam
De Boelelaan 1105
1081 HV Amsterdam
NETHERLANDS

Prof. Dr. Gyula Pap
Eötvös University
Department of Operations Research
Egervary Research Group
Pazmany P. s. 1/C
1117 Budapest
HUNGARY

Prof. Dr. Britta Peis
RWTH Aachen
Fakultät für Wirtschaftswissenschaften
Lehrstuhl für Management Science
Kackertstr. 7
52072 Aachen
GERMANY

Prof. Dr. Sebastian Pokutta
School of Industrial & Systems
Engineering
Georgia Institute of Technology
765 Ferst Dr.
Atlanta GA 30332-0205
UNITED STATES

Prof. Dr. Franz Rendl
Institut für Mathematik
Universität Klagenfurt
Universitätsstr. 65-67
9020 Klagenfurt
AUSTRIA

Prof. Dr. Thomas Rothvoss
Department of Mathematics & CSE
University of Washington
Padelford Hall
Seattle WA 98195-4350
UNITED STATES

Dr. Laura Sanità
Dept. of Combinatorics & Optimization
University of Waterloo
Waterloo, Ont. N2L 3G1
CANADA

Prof. Dr. Alexander Schrijver
University of Amsterdam
Centrum voor Wiskunde en Informatica
Science Park 123
1098 XG Amsterdam
NETHERLANDS

Prof. Dr. Andras Sebö
Institut National Polytechnique de
Grenoble, Laboratoire G-SCOP
46, Avenue Felix Viallet
38031 Grenoble Cedex 1
FRANCE

Prof. Dr. Bruce Shepherd
Dept. of Mathematics and Statistics
McGill University
805, Sherbrooke Street West
Montreal, P.Q. H3A 2K6
CANADA

Prof. Dr. Mohit Singh
Microsoft Research Group
Microsoft Corporation
1 Microsoft Way
Redmond WA 98052-8300
UNITED STATES

Prof. Dr. Martin Skutella
Institut für Mathematik
Fakultät II, Sekr. MA 5-2
Technische Universität Berlin
Straße des 17. Juni 136
10623 Berlin
GERMANY

Dr. Damien Stehlé
ENS Lyon
L.I.P. Laboratory
46, Allee d'Italie
69007 Lyon Cedex
FRANCE

Prof. Dr. Ola Nils A. Svensson
EPFL IC IIF THL2
INJ 112 (Batiment INJ)
Station 14
1015 Lausanne
SWITZERLAND

Dr. Rekha R. Thomas
Department of Mathematics
University of Washington
Padelford Hall
Box 354350
Seattle, WA 98195-4350
UNITED STATES

Prof. Dr. Juan Pablo Vielma
Department of Mathematics
MIT, E62-561
77 Massachusetts Avenue
Cambridge, MA 02139-4307
UNITED STATES

Prof. Dr. Jens Vygen
Forschungsinstitut f. Diskrete
Mathematik
Universität Bonn
Lennestr. 2
53113 Bonn
GERMANY

Prof. Dr. Robert Weismantel
Institute for Operations Research
ETH Zürich - HG G 21.5
Rämistr. 101
8092 Zürich
SWITZERLAND

Prof. Dr. Laurence A. Wolsey
Center f. Operations Research &
Econometrics
Université de Louvain
34 Voie du Roman Pays
1348 Louvain-la-Neuve
BELGIUM

Dr. Rico Zenklusen
Mathematik Department
ETH Zürich - HG G 21.3
Institute for Operations Research
Rämistr. 101
8092 Zürich
SWITZERLAND