Mathematisches Forschungsinstitut Oberwolfach

Report No. 1/2015

DOI: 10.4171/OWR/2015/1

Spectral Theory and Weyl Functions

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4 January – 10 January 2015

Abstract. The focus of the workshop was on the interplay between spectral theory of differential operators, system theory, inverse problems and properties of associated classes of analytic functions, among them the classical Titchmarsh-Weyl m-function from singular Sturm-Liouville theory, the Dirichlet-to-Neumann map from the theory of elliptic partial differential equations, and more abstract transfer, Weyl and Q-functions from infinite dimensional system and operator theory.

Mathematics Subject Classification (2010): 34xx, 35xx, 46xx, 47xx, 81xx, 93xx.

Introduction by the Organisers

The main objective of the workshop was to discuss and investigate the multifaceted connections and fruitful interplay between spectral theory of ordinary and partial differential operators, system theory, inverse problems and properties of associated classes of complex, matrix and operator valued analytic functions. The aim was to bring together and to initiate a closer collaboration of four different groups of mathematicians working in the related areas

1. Titchmarsh-Weyl m-functions and ordinary differential operators
2. Dirichlet-to-Neumann maps and elliptic partial differential operators
3. Abstract Weyl functions and Q-functions of selfadjoint operators
4. Transfer functions and system theory

1. The classical *Titchmarsh-Weyl m-function* in singular Sturm-Liouville theory was introduced by H. Weyl more than a century ago. The implications for the spectral theory of the underlying differential operators and the connection with complex analysis were later explored by E.C. Titchmarsh. Roughly speaking the complete spectral data of the corresponding selfadjoint Sturm-Liouville differential operators are reflected in the singularity structure of the m-function. The Titchmarsh-Weyl m-function is also used for solving direct and inverse spectral problems for ordinary differential and difference equations, and plays an important role in the analysis of some nonlinear differential equations, e.g. the Korteweg–de Vries and Camassa–Holm equation. In the workshop recent developments and extensions of the classical Titchmarsh-Weyl m-function to such topics as quantum graphs, non-selfadjoint spectral problems and resonance location were addressed.

2. More recently the *Dirichlet-to-Neumann map* from the theory of elliptic differential equations was interpreted as a multidimensional analogue of the classical Titchmarsh-Weyl m-function, and it was used for spectral analysis of partial differential operators. Similarly as in the case of ordinary differential equations the Dirichlet-to-Neumann map plays a central role in inverse problems for elliptic partial differential equations, e.g. Calderóns inverse problem from electrical impedance tomography, the Gelfand inverse boundary spectral problem, and inverse problems for dynamical Maxwell systems. For the solution of the latter an algebraic operator Riccati equation for the Neumann-to-Dirichlet map is a useful tool. During the meeting inverse problems with partial data, transmission eigenvalues, Titchmarsh-Weyl theory for elliptic operators, and connections to infinite dimensional Hamiltonian systems and algebraic operator Riccati equations were discusssed.

3. From a more global point of view both the Titchmarsh-Weyl m-function and the Dirichlet-to-Neumann map appear to be examples of so-called *Weyl functions* corresponding to an underlying symmetric operator and certain boundary maps defined on the domain of the adjoint operator. This modern concept of boundary triples and Weyl functions which naturally arises in the extension theory of symmetric operators is closely connected to the classical notion of *Q-functions* introduced and studied by M.G. Krein and H. Langer. Boundary triples and Weyl functions/Q-functions are useful and efficient tools in general spectral analysis. The main advantage of this abstract approach is that it may be applied to ordinary and partial differential equations, singular perturbations and boundary value problems, as well as the most recent and highly topical area of quantum graphs.

4. Another important type of abstract Weyl functions are *transfer functions* in the analysis of finite and infinite dimensional linear systems which relate input and output data of time invariant systems. The transfer function reflecs and influences the underlying system in a similar form as the Weyl or Q-function reflects the spectral properties of the corresponding selfadjoint operator. The connections between Dirac structures from the Hamiltonian approach to system theory, boundary control and conservative state/signal system nodes, and boundary triples from extension theory of symmetric operators have become explicit recently. The deeper understanding of Weyl functions, Q-functions, characteristic and transfer functions, their interplay with the intrinsic properties of the corresponding operator and system realizations, and the corresponding inverse problem of determining and realizing the operators or systems from the given abstract boundary data is of essential importance in various applications, e.g. spectral, coupling and recovery problems for difference equations, ordinary and partial differential equations.

The workshop was attended by more than 50 participants and 27 scientific lectures were presented. To stimulate discussions among participants coming from different backgrounds, several shorter presentations on spectral geometry, asymptotic stability of fluid flows, quantum graphs and spectral approximation problems were scheduled on two evenings.

The scientific programme was accompanied by lively discussions after the talks and in the coffee breaks. The longer breaks and free evenings were used for work in groups on current research papers, but also to initiate future activities such as research projects and workshop meetings between the different groups of participants. Informal information obtained from participants indicates that the workshop was a great success. In particular it allowed experts from different facets of spectral theory to have the chance to understand more of the subject from different perspectives.

On behalf of all participants, the organisers wish to thank the staff of MFO in making the workshop a success in all respects.

Acknowledgement: The MFO and the workshop organizers would like to thank the National Science Foundation for supporting the participation of junior researchers in the workshop by the grant DMS-1049268, “US Junior Oberwolfach Fellows”. Moreover, the MFO and the workshop organizers would like to thank the Simons Foundation for supporting Tom A.F.M. ter Elst in the “Simons Visiting Professors” program at the MFO.

Workshop: Spectral Theory and Weyl Functions

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Abstracts

Transition functions for Sturm-Liouville operators, canonical systems and strings

Heinz Langer

1. Consider the Sturm-Liouville problem

(1) $-y''(x) + q(x)y(x) - zy(x) = 0,\ x \in [0, \ell),\ z \in \mathbb{C}, \quad y'(0) - y(0)\cot\alpha = 0,$

where $0 < \ell \le \infty$, $q \in L^1_{loc}([0, \ell)), 0 \le \alpha < \pi$. Denote by $\varphi(x; z), \psi(x; z)$ the solutions of the differential equation in (1) satisfying the initial conditions

$$\varphi(0; z) = \sin\alpha,\ \varphi'(0; z) = \cos\alpha, \quad \psi(0; z) = -\cos\alpha,\ \psi'(0; z) = \sin\alpha.$$

that is, $\varphi(\cdot; z)$ is the solution of the differential equation that satisfies the given boundary condition at $x = 0$.

The *Fourier transformation* U of the problem (1) is defined by the relation

$$(Uf)(\lambda) := \int_0^\ell f(x)\varphi(x; \lambda)\mathrm{d}x, \quad \lambda \in \mathbb{R},\ f \in \mathcal{L}_0,$$

where $\mathcal{L}_0$ is the set of all functions $f \in L^2(0, \ell)$ which vanish identically near ℓ. The measure τ on $\mathbb{R}$ is a *spectral measure* of the problem (1) if

$$\int_0^\ell |f(x)|^2 \mathrm{d}x = \int_{\mathbb{R}} |(Uf)(\lambda)|^2\, \mathrm{d}\tau(\lambda), \quad f \in \mathcal{L}_0,$$

that is U is an isometry from $(\mathcal{L}_0 \subset) L^2(0, \ell)$ into $L^2_\tau(\mathbb{R})$, and hence it can be extended by continuity to all of $L^2(0, \ell)$. The spectral measure τ is called *orthogonal* if the mapping U is onto $L^2(0, \ell)$. For any $0 < b \le \ell$ the set of all spectral measures of the problem (1) with ℓ replaced by b is denoted by $\mathcal{S}_b$.

If $0 < b < \ell$, the spectral measures can be described by the corresponding Weyl-Titchmarsh functions m as follows (see [3, Theorem 14.1]); here $\mathbf{N}$ denotes the set of all Nevanlinna functions (this is the set of all functions which are holomorphic in the upper and lower half-planes $\mathbb{C}^\pm$ and map $\mathbb{C}^\pm$ into $\mathbb{C}^\pm \cup \mathbb{R}$), and we set $\widetilde{\mathbf{N}} := \mathbf{N} \cup \{\infty\}$.

1°. *If* $0 < b < \ell$ *, then the relation*

$$m_\gamma(z) := \int_{\mathbb{R}} \frac{\mathrm{d}\tau_\gamma(\lambda)}{\lambda - z} = \frac{\psi'(b; z) - \psi(b; z)\gamma(z)}{\varphi'(b; z) - \varphi(b; z)\gamma(z)}, \quad z \in \mathbb{C} \setminus \mathbb{R},$$

establishes a bijective correspondence between all $\gamma \in \widetilde{\mathbf{N}}$ *and all spectral measures* $\tau = \tau_\gamma \in \mathcal{S}_b$. *The spectral measure* τ_γ *is orthogonal if and only if* γ *is a real constant or* ∞.

An essential role in M.G. Kreĭn's investigations of Sturm-Liouville problems is played by the *transition functions* Φ_τ, defined for all $\tau \in \mathcal{S}_b$ by

$$\Phi_\tau(t) := \int_{\mathbb{R}} \frac{1 - \cos\left(\sqrt{\lambda}t\right)}{\lambda}\, \mathrm{d}\tau(\lambda), \quad 0 \le \mathrm{t} < 2\mathrm{b}.$$

They have the following property (see [5]):
2°. *If $0 < b < \ell$ and $\tau \in \mathcal{S}_b$ then the values $\Phi_\tau(t)$, $0 \le t \le 2b$, do not depend on $\tau \in \mathcal{S}_b$.*

Denote this common value by $\Phi^b(t)$: $\Phi^b(t) := \Phi_\tau(t)$, $t \in [0, 2b]$, $\tau \in \mathcal{S}_b$.

The transition function has the following physical meaning. Consider the string with elastic damping, corresponding to (1), e.g. with Neumann boundary conditions at $x = 0$ ($\alpha = \frac{\pi}{2}$). Suppose a force $\delta_0(x)$ starts acting at $t = 0$ constantly and orthogonally to the string. Then $\Phi^b(t)$ describes the position of the left endpoint of the string at time t, $0 \le t \le 2b$.

With the transition function Φ on $[0, 2b]$ there is associated the hermitian kernel

$$K_\Phi(s, t) := \Phi(s + t) - \Phi(|s - t|), \quad 0 \le s, t < b.$$

It is positive definite, that is, for all $n \in \mathbb{N}$, $s_i \in [0, b]$, $\xi_i \in \mathbb{C}$, $i = 1, 2, \ldots, n$, it holds

$$\sum_{i,j=1}^{n} K_\Phi(s_i, s_j)\xi_i\overline{\xi_j} \ge 0.$$

The following statement was proved by M.G.Kreĭn in [4]. Given a continuous function Ψ on $[0, 2b]$, such that the kernel K_Ψ on $[0, b]$ is positive definite. Then there exists (at least one) measure τ on $\mathbb{R}$ such that

$$\Psi(t) = \int_{\mathbb{R}} \frac{1 - \cos(\sqrt{\lambda}t)}{\lambda} \, \mathrm{d}\tau(\lambda), \quad t \in [0, 2b]. \tag{2}$$

It turns out (see [5]) that for the function $\Psi = \Phi^b$ the set of all measures τ in the representation (2) coincides with the set $\mathcal{S}_b$ of all spectral measures. A crucial tool in the proof of these statements is the method of directing functionals.

2. Consider the canonical system

$$-J\mathbf{y}'(x) = z\,H(x)\mathbf{y}(x), \ x \in [0, \ell), \quad y_2(0) = 0, \tag{3}$$

where $\mathbf{y} = (y_1\ y_2)^{\mathrm{t}}$, H is a real symmetric non-negative measurable 2×2-matrix function, $\operatorname{tr} H(x) = 1$, $x \in [0, \ell)$ a.e.,

$$\int_0^\varepsilon h_{22}(x)\,\mathrm{d}x > 0 \text{ for all } \varepsilon > 0, \quad J := \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

Let $W(x; z) = \begin{pmatrix} w_{11}(x; z) & w_{12}(x; z) \\ w_{21}(x; z) & w_{22}(x; z) \end{pmatrix}$ be the solution of the initial problem

$$\frac{\mathrm{d}W(x; z)}{\mathrm{d}x} J = z\,W(x; z)H(x), \quad W(0; z) = I_2, \quad 0 \le x < \ell, \ z \in \mathbb{C};$$

for $\ell < \infty$ also $x = \ell$ is allowed. If $0 < b \le \ell$, $b < \infty$ then for $\gamma \in \widetilde{\mathbf{N}}$ the function

$$m_\gamma(z) := \frac{w_{11}(b; z)\gamma(z) + w_{12}(b; z)}{w_{21}(b; z)\gamma(z) + w_{22}(b; z)}$$

belongs to the class $\mathbf{N}$, and it admits a representation

$$m_\gamma(z) = \alpha + \int_{-\infty}^{\infty} \left(\frac{1}{\lambda - z} - \frac{\lambda}{1 + \lambda^2} \right) \mathrm{d}\tau_\gamma(\lambda), \quad z \in \mathbb{C} \setminus \mathbb{R},$$

with some $\alpha \in \mathbb{R}$. The measures τ_γ, $\gamma \in \widetilde{\mathbf{N}}$, are the *spectral measures* of the problem (3) on $[0,b]$; denote this set by $\mathcal{T}_b$. The spectral measures can also be defined through a Fourier transformation. If $\tau \in \mathcal{T}_b$ the *transition function* for the canonical system is now defined as follows:

$$g_\tau(t) := \int_{-\infty}^{\infty} \left(\mathrm{e}^{\mathrm{i}\lambda t} - 1 - \frac{\mathrm{i}\lambda t}{1+\lambda^2} \right) \frac{\mathrm{d}\tau(\lambda)}{\lambda^2}, \quad t \in \mathbb{R}.$$

For $0 < b \le \ell$ we set $t_b := \int_0^b \sqrt{\det H(x)}\,\mathrm{d}x$. Then the following statement holds.

Theorem 1. *Suppose that* $0 < b < \ell$ *and* $t_b > 0$. *If* $\tau_1, \tau_2 \in \mathcal{T}_b$ *then for any two functions* g_{τ_1}, g_{τ_2} *we have*

$$g_{\tau_1}(t) - g_{\tau_2}(t) = \mathrm{i}\beta t, \quad t \in [0, 2t_b],$$

with some $\beta \in \mathbb{R}$.

For any transition function g of the system (3) the kernel

$$G_g(s,t) := g(t-s) - g(t) - \overline{g(s)} + g(0), \quad s,t \in \mathbb{R},$$

is positive definite. Clearly, for the restriction g^b to $(-2t_b, 2t_b)$ instead of g this holds for $s, t \in (-t_b, t_b)$.
3°. *Let* g *be a continuous function on* $(-2T, 2T)$, $0 < T \le \infty$, *with* $g(-t) = \overline{g(t)}$, $t \in (-2T, 2T)$. *The kernel* G_g *is positive definite on* $(-T, T)$ *if and only if* g *admits a representation*

$$g(t) = \alpha + \mathrm{i}\beta t + \int_{-\infty}^{\infty} \left(\mathrm{e}^{\mathrm{i}\lambda t} - 1 - \frac{\mathrm{i}\lambda t}{1+\lambda^2} \right) \frac{\mathrm{d}\tau(\lambda)}{\lambda^2}, \quad t \in (-2T, 2T),$$

with $\alpha \in \mathbb{R}$, $\beta \ge 0$, *and a measure* τ *on* $\mathbb{R}$ *such that* $\int_{\mathbb{R}} \frac{\mathrm{d}\tau(\lambda)}{1+\lambda^2} < \infty$.

Denote this class of functions g by $\mathcal{G}_T$.

Theorem 2. *Let* $0 < b \le \ell$, $b < \infty$, *such that for all* $\varepsilon > 0$, $\int_{b-\varepsilon}^b \sqrt{\det H(x)}\,\mathrm{d}x > 0$, *and let* g^b *be the restriction of a transition function* g_τ, $\tau \in \mathcal{S}_b$, *to* $[-2t_b, 2t_b]$. *Then the set of all spectral measures* $\tau \in \mathcal{T}_b$ *coincides with the set of all measures* τ *such that*

$$g^b(t) = \mathrm{i}\beta t + \int_{-\infty}^{\infty} \left(\mathrm{e}^{\mathrm{i}\lambda t} - 1 - \frac{\mathrm{i}\lambda t}{1+\lambda^2} \right) \frac{\mathrm{d}\tau(\lambda)}{\lambda^2}, \quad t \in [-2t_b, 2t_b], \tag{4}$$

for some $\beta \in \mathbb{R}$.

The integral on the right hand side of (4) defines a function $g \in \mathcal{G}_\infty$ which is a continuation of g^b on $[-2t_b, 2t_b]$ to $\mathbb{R}$. Therefore the spectral functions of the problem (3) are in a bijective correspondence with all the continuations of g^b to $\mathbb{R}$ in the class $\mathcal{G}_\infty$ (see [7]).

A Dirac-Kreĭn system is a canonical system of the form

$$- J\,\mathbf{y}'(x) = z\,\mathbf{y}(x) + \begin{pmatrix} b(x) & a(x) \\ a(x) & -b(x) \end{pmatrix} \mathbf{y}(x),\ x \in [0,\ell), \quad y_1(0) = 0; \tag{5}$$

we suppose that the functions a, b of the potential are continuous on $[0,\ell]$, $0 < \ell \leq \infty$. Let g be a transition function of (5), which is defined by means of the spectral functions as for general canonical systems. A characteristic property of the transition functions of a Dirac- Kreĭn system is, roughly speaking, that they have an *accelerant* h, that is they admit a representation

$$g(t) = -\frac{1}{2}|t| + \int_0^t (t-s)h(s)\,\mathrm{d}s, \quad t \in (-2\ell, 2\ell),$$

with a continuous function h. For more details see [6].

3. Let M be a non-decreasing function on $[0,\ell)$, $M(0) = 0$, $M(x) > 0$ if $x > 0$. The corresponding string is described by the initial value problem

$$-\frac{\mathrm{d}^2 y(x)}{\mathrm{d}M\,\mathrm{d}x} = z\,y(x),\ x \in [0,\ell), \quad y'(0-) = 0. \tag{6}$$

Let φ, ψ be the solutions of the differential equation in (6) that satisfy the initial conditions

$$\varphi(0;z) = 1,\ \varphi'(0-;z) = 0; \quad \psi(0,z) = 0,\ \psi'(0-;z) = 1.$$

That is, $\varphi(x,z)$, $\psi(x;z)$ are the solutions of the integral equations

$$\varphi(x;z) = 1 - z\int_{0-}^x (x-s)\varphi(s;z)\,\mathrm{d}M(s), \quad \psi(x;z) = x - z\int_{0-}^x (x-s)\psi(s;z)\,\mathrm{d}M(s).$$

If $b < \ell$, the set of all *spectral measures* τ of the regular string $S[b,M]$ (regular means $b + M(b-) < \infty$) can be defined by the relation

$$\int_0^\infty \frac{\mathrm{d}\tau_\gamma(\lambda)}{\lambda - z} = \frac{\psi'(\ell;z)\gamma(z) + \psi(\ell;z)}{\varphi'(\ell;z)\gamma(z) + \varphi(\ell;z)},$$

if γ runs through the class $\mathbf{S}$ of all *Stieltjes functions*; recall that by definition $\gamma \in \mathbf{S}$ if γ is holomorphic in $\mathbb{C} \setminus [0,\infty)$ and $\gamma, \widehat{\gamma} \in \mathbf{N}$, where $\widehat{\gamma}(z) := z\gamma(z)$.

The *transition function* corresponding to the spectral measure τ is now the function

$$\Psi(t) = \int_0^\infty \frac{\cos(\sqrt{\lambda}t) - 1}{\lambda}\,\mathrm{d}\tau(\lambda), \quad \lambda \in \mathbb{R}.$$

This transition function has the characteristic property that the kernel

$$\Psi(t-s) - \Psi(t) - \Psi(s), \quad s,t \in \mathbb{R},$$

is positive definite and Ψ is real. Now similar statements as for transition functions of Sturm-Liouville problems and canonical systems can be formulated. For more details see [7].

Finally we mention, that transition functions can be used to give local versions of inverse spectral results as in papers by B. Simon [9], F. Gesztesy/B. Simon [2], and Bennewitz [1] for Sturm-Liouville equations and by M. Langer/H. Woracek [8] for canonical systems.

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Reflections on Herglotz functions, Hill's operator, metric graphs and self-adjoint extensions

Konstantin Pankrashkin

If $Q : \mathbb{R} \to \mathbb{R}$ is continuous and 1-periodic, then the associated *discriminant* $\Delta : \mathbb{C} \to \mathbb{C}$ is defined by $\Delta(\lambda) := s'(1;\lambda)+c(1;\lambda)$, where s and c are the solutions of the Hill's differential equation $-u''+Qu = \lambda u$ satisfying $s(0;\lambda) = c'(0;\lambda) = 0$ and $s'(0;\lambda) = c(0;\lambda) = 1$. If one denotes by μ_n, $n \geq 1$, the sequence of the Dirichlet eigenvalues of the operator $u \mapsto -u'' + Qu$ on $(0, 1)$, then the discriminant enjoys well-known oscillation properties:

- For any $\lambda \in \mathbb{R}$ with $\big|\Delta(\lambda)\big| < 2$ one has $\Delta'(\lambda) \neq 0$.
- If $|\Delta(\lambda)| = 2$ and $\Delta'(\lambda) = 0$, then λ coincides with one of the Dirichlet eigenvalues,
- For any $n \in \mathbb{N}$ one has: *either* $\Delta(\mu_n) \leq -2$ and $\Delta(\mu_{n+1}) \geq 2$ *or* $\Delta(\mu_n) \geq 2$ and $\Delta(\mu_{n+1}) \leq -2$.
- If $\Delta(\mu_n) = 2$ and $\Delta'(\mu_n) = 0$ for some n, then $\Delta''(\mu_n) < 0$.
- If $\Delta(\mu_n) = -2$ and $\Delta'(\mu_n) = 0$ for some n, then $\Delta''(\mu_n) > 0$.

These properties are crucial for the understanding of the structure of the set $\Delta^{-1}\big([-2, 2]\big)$, which is exactly the spectrum of the Hill operator $-d^2/dx^2 + Q$ in $L^2(\mathbb{R})$. In the available literature, the above propeties are usually deduced either using the fact that Δ belongs to some class of entire functions, which follows the orignal approach by Lyapunov [4], or using rather involved computations based on the variation of constants for the differential equation, see e.g. Section VIII.3 in [1].

The aim of the talk is to show that the above proprties can be deduced from the elementary analysis of Herglotz functions, and that objects similar to the discriminant appear when analyzing Weyl functions of self-adjoint extensions.

Recall that a holomorphic function $h : \mathbb{C}\backslash\mathbb{R} \to \mathbb{C}$ is a *Herglotz* one if $h(\overline{z}\,) = \overline{h(z)}$ and $\Im z \cdot \Im h(z) \geq 0$. The following Lemma is proved in [5, Lemma 2].

Lemma 1. *Let $I \subset \mathbb{R}$ be a non-empty open interval, two functions m, n be holomorphic in $(\mathbb{C} \setminus \mathbb{R}) \cup I$ and take real values on I, $n \neq \text{const}$. Assume that there exist $a, b \in \mathbb{R}$ with $a < b$ such that the functions*

$$h_a(z) := \frac{m(z) - a}{n(z)}, \quad h_b(z) := \frac{m(z) - b}{n(z)} \tag{1}$$

are Herglotz and non-constant, then:

- *(a) The zeros of n in I are simple.*
- *(b) If $\lambda \in I$ is such that $m(\lambda) \in (a, b)$, then $n(\lambda)m'(\lambda) > 0$.*
- *(c) If $n(\lambda) = 0$ for some $\lambda \in I$, then $m(\lambda) \notin (a, b)$.*
- *(d) Let μ and ν be successive zeros of n in I, then* either *$m(\mu) \leq a$ and $m(\nu) \geq b$* or *$m(\mu) \geq b$ and $m(\nu) \leq a$.*
- *(e) If $\lambda \in I$ is such that $m'(\lambda) = 0$, then:*
  - *(i) if $m(\lambda)$ coincides with a or b, then $n(\lambda) = 0$.*
  - *(ii) if $m(\lambda) = a$, then $m''(\lambda) > 0$.*
  - *(iii) if $m(\lambda) = b$, then $m''(\lambda) < 0$.*

One can show easily that the functions $z \mapsto h_\pm(z) := -\big(\Delta(z) \pm 2\big)/s(1; z)$ are Herglotz and non-constant, cf. [5, Section 3], and the above properties of the discriminant follow from the Lemma.

Now we are going to show how the functions $h_\pm$ and the associated dicriminant appear in the theory of self-adjoint extensions. We recall first some definitions, see e.g. the monograph [3]. Let $\mathcal{H}$ be a Hilbert space and S be a closed densely defined symmetric operator in $\mathcal{H}$. It is known that S has self-adjoint extensions iff one can construct a *boundary triple*, which consisits of an auxiliary Hilbert space $\mathcal{G}$ and two linear maps $\Gamma, \Gamma' : D(S^*) \to \mathcal{G}$ with the following two properties: (i) $\langle S^* f, g\rangle - \langle f, S^* g\rangle = \langle \Gamma' f, \Gamma g\rangle - \langle \Gamma f, \Gamma' g\rangle$ for all $f, g \in D(S^*)$, and (ii) the mapping $D(S^*) \ni f \mapsto (\Gamma f, \Gamma' f) \in \mathcal{G} \times \mathcal{G}$ is surjective. Given a boundary triple, one can show that the operators H^0 and H defined as the restictions S^* on $\ker \Gamma$ and $\ker \Gamma'$, respectively, are self-adjoint extensions of S. A special role in their spectral analysis is played by the associated Weyl function $M(z) := \Gamma'(\Gamma|_{\ker(S^* - z)})^{-1}$, $z \notin \operatorname{spec} H^0$, see [2]. In particular, for $z \notin \operatorname{spec} H^0$ the condition $z \in \operatorname{spec} H$ is equivalent to $0 \in \operatorname{spec} M(z)$.

Remark that if the Weyl function is of the special form $M(z) = \big(m(z) - T\big)/n(z)$, where T is a bounded self-adjoint operator in $\mathcal{G}$ and m and n are scalar functions, then the condition $0 \in \operatorname{spec} M(z)$ is equivalent to $m(z) \in \operatorname{spec} T$, and, moreover, it was shown in [6] that in this case one can prove a unitary equivalence between the spectral projectors of T and H in some intervals. Furthermore, if

$a := \min \operatorname{spec} T < \max \operatorname{spec} T =: b$, then the functions h_a and h_b defined as in (1) are Herglotz and non-constant, hence, the above Lemma gives some information on the behavior of the function m which can then be viewed as an abstract discriminant. Numerous examples in which Weyl functions are of the above form can be found e.g. in [7, Section 3]: these include the above Hill operator, a class of differential operators on equilateral metric graphs and some more abstract cases.

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Eigenvalue estimates for operators with finitely many negative squares

Carsten Trunk

(joint work with Jussi Behrndt, Roland Möws)

Let A and B be selfadjoint operators and assume that the resolvent difference of A and B is of rank one. Then the continuous spectra of the two operators coincide. The number of eigenvalues in a gap of the continuous spectrum is the same or it differs by one. This is a well-known fact for selfadjoint operators in Hilbert spaces.

The same question for selfadjoint operators in Krein spaces is more delicate and allows a variety of answers. This question arises naturally in the study of singular indefinite Sturm-Liouville problems. Often, a one (or two) dimensional perturbation of a singular indefinite Sturm-Liouville operator leads to an operator which is the direct sum of two (or more) selfadjoint Sturm-Liouville operators in L_2-Hilbert spaces with well-known spectra.

Hence, such an approach can be used to describe the spectrum of the singular indefinite Sturm-Liouville operator. Obviously, the continuous (essential) spectra of the singular indefinite Sturm-Liouville operator and its one dimensional perturbation coincide. With the help of the Weyl function, eigenvalue estimates in gaps of the continuous spectrum can be obtained: The unperturbed and the perturbed operator are considered as two different selfadjoint extensions of a fixed minimal symmetric operator. By standard methods from extension theory, one introduces a Weyl function which, similar to the case of Hilbert spaces, contains all spectral information. If the unperturbed operator belongs to one of the well studied subclasses of selfadjoint operators in Krein spaces (like non-negative operators, operators with finitely many negative squares, definitizable or locally definitizable operators), then this is reflected in the properties of the Weyl function. These properties of the Weyl function can be used to prove eigenvalue estimates and results for accumulation of eigenvalues.

In the recent years we used the above scheme (together with other techniques) to investigate indefinite Sturm-Louville operators defined on the real line. We described the location of the essential spectrum, [9], derived bounds for the number of non-real eigenvalues, [2, 6, 7], obtained eigenvalue estimates in gaps of the continuous spectrum for different classes of operators, [3, 4], characterized non-real accumulation of eigenvalues [1], and obtained bounds for the location of non-real eigenvalues of perturbations of non-negative operators in Krein spaces [5].

Currently (together with Jussi Behrndt (Graz) and Roland Möws (Berlin)) we study operators with finitely many negative squares. A selfadjoint operator A in a Krein space $(\mathcal{K}, [\cdot,\cdot])$ with $\rho(A) \neq \emptyset$ is said to have κ *negative squares* if for some $\kappa \in \mathbb{N}$ the hermitian form $\langle\cdot,\cdot\rangle$ on $\operatorname{dom} A$, defined by

$$\langle f, g\rangle := [Af, g], \qquad f, g \in \operatorname{dom} A,$$

has κ negative squares, that is, there exists a κ-dimensional subspace $\mathcal{M}$ in $\operatorname{dom} A$ such that $\langle v, v\rangle < 0$ if $v \in \mathcal{M}$, $v \neq 0$, but no $\kappa + 1$ dimensional subspace with this property. Selfadjoint operators with finitely many negative squares belong to the class of definitizable operators introduced and comprehensively studied by H. Langer in [11, 12] and are used in the description of the spectral properties of indefinite Sturm-Liouville problems, see, e.g., [7, 8, 10, 13].

For an operator A with κ *negative squares* we obtain a sharp estimate for the number of eigenvalues in a gap of the essential spectrum under rank one perturbations. More precisely, let B be another selfadjoint operator in the same Krein space with

$$\dim\operatorname{ran}\big((A-\lambda_0)^{-1} - (B-\lambda_0)^{-1}\big) = 1$$

and let I be an open interval with $\rho(B) \cap I \neq \emptyset$ such that $\sigma(A) \cap I$ consists only of isolated eigenvalues. Denote the numbers of distinct eigenvalues of A and B in I by $n_A(I)$ and $n_B(I)$, respectively, and denote the number of common eigenvalues in I by

$$n_{A,B}(I) = \sharp\big\{\lambda : \lambda \in I \cap \sigma_p(A) \cap \sigma_p(B)\big\}.$$

Then the number of distinct eigenvalues of B in I can be estimated in terms of the number of distinct eigenvalues of A in I and some correction terms. These correction terms consist of $n_{A,B}(I)$ and the number $\mathfrak{K}_A$ ($\mathfrak{K}_B$) of eigenvalues of A (resp. B) of a specific sign characteristic in I. The number $\mathfrak{K}_A$ ($\mathfrak{K}_B$) is always smaller than the number of negatives squares of A (resp. B),

$$\mathfrak{K}_A \leq \kappa \quad \text{and} \quad \mathfrak{K}_B \leq \kappa + 1.$$

Roughly speaking $\mathfrak{K}_A$ (resp. $\mathfrak{K}_B$) counts the number of negative squares in I and has a somehow localized character. The precise estimates are presented in the following theorem. We emphasize that these estimates are sharp.

Theorem 1. *The following statements are true.*

(i) *If* $n_A(I) < \infty$ *and* $0 \notin I$ *then*

$$n_A(I) - n_{A,B}(I) - 2\mathfrak{K}_A - 1 \le n_B(I) \le n_A(I) + n_{A,B}(I) + 2\mathfrak{K}_B + 1,$$

where $\mathfrak{K}_A$ ($\mathfrak{K}_B$) *denotes the number of all eigenvalues* λ *of* A (*resp.* B) *in* $I \cap \rho(B) \setminus \{0\}$ (*resp.* $I \cap \rho(A) \setminus \{0\}$) *with the property that there exists an eigenvector* x *corresponding to* λ *with* $\lambda[x,x] \le 0$. *We have* $\mathfrak{K}_A \le \kappa$ *and* $\mathfrak{K}_B \le \kappa + 1$.

(ii) *If* $n_A(I) < \infty$ *and* $0 \in I$ *then*

$$n_B(I) \ge n_A(I) - n_{A,B}(I) - 2\mathfrak{K}_A - \begin{cases} 3 & \text{if } 0 \in \rho(B), \\ 1 & \text{if } 0 \in \sigma(B). \end{cases}$$

and

$$n_B(I) \le n_A(I) + n_{A,B}(I) + 2\mathfrak{K}_B + \begin{cases} 3 & \text{if } 0 \in \rho(A), \\ 1 & \text{if } 0 \in \sigma(A), \end{cases}$$

where $\mathfrak{K}_A$ *and* $\mathfrak{K}_B$ *are defined as in* (i).

(iii) *Each of the estimates in* (i) *and* (ii) *is sharp.*

(iv) *We have* $n_A(I) = \infty$ *if and only if* $n_B(I) = \infty$.

We remark that the estimates in (ii) can be slightly improved if one takes into account whether 0 is a critical point of B (for the notion of critical points see [12]).

This gives also, as described above, eigenvalue estimates for singular indefinite Sturm-Liouville problems.

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Analysis of the Dirichlet-to-Neumann operator on nonsmooth domains

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(joint work with W. Arendt)

Let Ω be a bounded open set in $\mathbb{R}^d$ with Lipschitz boundary. The Dirichlet-to-Neumann operator D_0 on the boundary $\Gamma = \partial\Omega$ is defined as follows. Let $\varphi, \psi \in L_2(\Gamma)$. Then $\varphi \in \operatorname{dom}(D_0)$ and $D_0\varphi = \psi$ if and only if there exists a $u \in H^1(\Omega)$ such that $\operatorname{Tr} u = \varphi$ and $\partial_\nu u = \psi$, where ∂_ν is the normal derivative. It turns out that D_0 is a self-adjoint operator with compact resolvent. The aim is to extend the above to more general domains and to study the operator.

We assume from now one that Ω is an open bounded connected set in $\mathbb{R}^d$ with $d \geq 2$. Write $\Gamma = \partial\Omega$ and provide Γ with the $(d-1)$-dimensional Hausdorff measure σ. We assume that $\sigma(\Gamma) < \infty$.

Let $u \in H^1(\Omega)$ and $\varphi \in L_2(\Gamma)$. We say that φ is **a trace** of u if there exist $u_1, u_2, \ldots \in H^1(\Omega) \cap C(\overline{\Omega})$ such that

$$\lim_{n\to\infty} u_n = u \text{ in } H^1(\Omega) \quad \text{and} \quad \lim_{n\to\infty} u_n|_\Gamma = \varphi \text{ in } L_2(\Gamma).$$

Next, let $u \in H^1(\Omega)$ be such that $\Delta u \in L_2(\Omega)$ as distribution. Then we say that u has a **weak normal derivative in** $L_2(\Gamma)$ if there exists a $\psi \in L_2(\Gamma)$ such that

$$\int_\Omega (\Delta u)\, v + \int_\Omega \nabla u \cdot \nabla v = \int_\Gamma \psi\, v\, d\sigma \tag{1}$$

for all $v \in H^1(\Omega) \cap C(\overline{\Omega})$. In that case ψ is uniquely determined by (1), we write $\frac{\partial u}{\partial \nu} := \psi$ and call ψ the **(weak) normal derivative** of u.

Using form methods [2] we proved the next theorem.

Theorem 1. *There exists a unique self-adjoint (single valued) operator D_0 in $L_2(\Gamma)$ such that for all $\varphi, \psi \in L_2(\Gamma)$ one has*

$$\varphi \in \operatorname{dom}(D_0) \textit{ and } D_0\varphi = \psi$$

if and only if

$$\textit{there exists a } u \in H^1(\Omega) \textit{ such that } \left[\begin{array}{l} \Delta u = 0 \textit{ as distribution on } \Omega \\ \varphi \textit{ is a trace of } u \\ \dfrac{\partial u}{\partial \nu} = \psi. \end{array}\right.$$

Let D_0 be as in Theorem 1. It turns out that D_0 is a positive operator. Let S be the semigroup generated by $-D_0$, so $S_t = e^{-tD_0}$ for all $t > 0$. Obviously $S_t \mathbb{1}_\Gamma = \mathbb{1}_\Gamma$ for all $t > 0$. Actually, S is a Markov semigroup.

It is in general possible that there exists a $\varphi \in L_2(\Gamma)$ with $\varphi \neq 0$ and φ is a trace of $0 \in H^1(\Omega)$. Hence in general an element of $H^1(\Omega)$ can have more than one function as trace. Uniqueness of trace can be characterised.

Theorem 2. *The following are equivalent.*

- *Every element of $H^1(\Omega)$ has at most one trace.*
- $\dim(\ker D_0) = 1$.
- *S is irreducible.*
- $\lim_{t\to\infty} S_t\varphi = P\varphi$ *for all* $\varphi \in L_2(\Gamma)$, *where P is the projection onto the constants.*
- *The form $\mathfrak{a}_R$ is closable, where*

$$\mathfrak{a}_R(u, v) = \int_\Omega \nabla u \cdot \overline{\nabla v} + \int_\Gamma u\,\overline{v}$$

  with $D(\mathfrak{a}_R) = H^1(\Omega) \cap C(\overline{\Omega})$.

Let $\widetilde{H}^1(\Omega)$ be the closure of $H^1(\Omega)\cap C(\overline{\Omega})$ in $H^1(\Omega)$. In general $\widetilde{H}^1(\Omega) \neq H^1(\Omega)$, but if Ω has a continuous boundary, then $\widetilde{H}^1(\Omega) = H^1(\Omega)$. Clearly if $u \in H^1(\Omega)$ and there exists a $\varphi \in L_2(\Gamma)$ such that φ is a trace of u, then $u \in \widetilde{H}^1(\Omega)$. Existence *and* uniqueness of a trace can also be characterised in various ways.

Theorem 3. *The following are equivalent.*

- *Every element of $\widetilde{H}^1(\Omega)$ has a unique trace.*
- *There exists a $c > 0$ such that*

$$\int_\Gamma |u|^2 \leq c\Big(\int_\Omega |\nabla u|^2 + \int_\Omega |u|^2\Big)$$

  for all $u \in H^1(\Omega) \cap C(\overline{\Omega})$.
- $0 \notin \sigma_{\mathrm{ess}}(D_0)$.
- $\lim_{t\to\infty} S_t = P$ *in* $\mathcal{L}(L_2(\Gamma))$, *where P is the projection onto the constants.*
- *There exists an $\varepsilon > 0$ such that* $\|S_t - P\|_{\mathcal{L}(L_2(\Gamma))} \leq e^{-\varepsilon t}$ *for all* $t > 0$.
- *There exists a $\beta > 0$ such that $\mathfrak{a}_\beta$ is lower bounded, where*

$$\mathfrak{a}_\beta(u, v) = \int_\Omega \nabla u \cdot \overline{\nabla v} - \beta \int_\Gamma u\,\overline{v}$$

  with $D(\mathfrak{a}_\beta) = H^1(\Omega) \cap C(\overline{\Omega})$.

The second item in the previous theorem gives continuity of a trace map from $\widetilde{H}^1(\Omega)$ into $L_2(\Gamma)$. If Ω has a Lipschitz boundary then this trace map is also compact. There are counter examples ([1] Example 9.4 and [3] Subsection 5.3) that in general continuity does not imply compactness. Again a characterisation of compactness is nevertheless possible.

Theorem 4. *The following are equivalent.*

- *The Dirichlet-to-Neumann operator* D_0 *has compact resolvent.*
- *Every element in* $\widetilde{H}^1(\Omega)$ *has a unique trace and the map* $\mathrm{Tr} : \widetilde{H}^1(\Omega) \to L_2(\Gamma)$ *is compact.*
- *For all* $\beta > 0$ *the form* $\mathfrak{a}_\beta$ *is lower bounded, where*

$$\mathfrak{a}_\beta(u, v) = \int_\Omega \nabla u \cdot \overline{\nabla v} - \beta \int_\Gamma u\, \overline{v}$$

with $D(\mathfrak{a}_\beta) = H^1(\Omega) \cap C(\overline{\Omega})$.

The proofs of these theorems are in the paper [1]. We finish with some open problems.

Open problem 5. *Suppose that* Ω *has a continuous boundary. Does it follow that every element of* $H^1(\Omega)$ *has at most one trace?*

Does existence of a trace imply uniqueness? Precisely:

Open problem 6. *Suppose that every element of* $\widetilde{H}^1(\Omega)$ *has a trace. Does it follow that every element of* $H^1(\Omega)$ *has at most one trace?*

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Boundary triplets and trace formulas

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(joint work with Mark Malamud)

In [7] M.G. Kreĭn proved the existence of a real-valued function $\xi(\cdot) \in L^1(\mathbb{R})$ for pairs $\{H, H_0\}$ of self-adjoint operators which differ by a trace class operator, i.e. $V := H - H_0 \in \mathfrak{S}_1(\mathfrak{H})$, such that the following trace formula

$$\mathrm{tr}\,(\Phi(H) - \Phi(H_0)) = \frac{1}{\pi} \int_{\mathbb{R}} \Phi'(t)\xi(t)dt \tag{1}$$

holds for a sufficiently large class of functions $\Phi(\cdot)$. The function $\xi(\cdot)$ is usually called the spectral shift function (SSF) of the pair $\{H, H_0\}$. Furthermore, for the rigorous justification of the existence of the SSF he introduced the concept of perturbation determinant $\Delta_{H/H_0}(\cdot)$ and proved the inversion formula

$$\xi(t) = \lim_{y \downarrow 0} \mathrm{Im}\,(\log(\Delta_{H/H_0}(t + iy))) \quad \text{for a.e.} \quad t \in \mathbb{R}, \tag{2}$$

expressing $\xi(\cdot)$ by means of $\Delta_{H/H_0}(\cdot)$ where the branch of the logarithm is fixed by the condition $\lim_{y\to+\infty} \log(\Delta_{H/H_0}(t+iy)) = 0$. Such treatment has allowed him to show that there exists a unique SSF satisfying $\xi(\cdot) \in L^1(\mathbb{R}; dt)$. In subsequent publications M.G. Kreĭn [8, 9] extended (1) to a pair $\{H, H_0\}$ of self-adjoint *resolvent comparable operators*, i.e., operators satisfying

$$(3) \qquad (H - z)^{-1} - (H_0 - z)^{-1} \in \mathfrak{S}_1(\mathfrak{H}), \quad z \in \rho(H) \cap \rho(H_0).$$

This extension has been motivated by applications to Schrödinger operators $H = H_0 + q$ (and other differential operators). Clearly, H is not a trace class perturbation of $H_0 = -\Delta$ while the pair $\{H, H_0\}$ satisfies (3) for certain classes of decaying potentials q. For pairs satisfying (3) the spectral shift function $\xi(\cdot) = \overline{\xi(\cdot)}$ exists and belongs to $L^1(\mathbb{R}; \frac{dt}{1+t^2})$, however, it is determined up to an additive real constant.

In [10, Theorem 9.2] Kreĭn has studied the accumulative case $H := H_0 - iG$, $G \geq 0$. For a sufficiently large class of functions $\Phi(\cdot)$ holomorphic in $\mathbb{C}_-$ the following trace formula

$$(4) \qquad \operatorname{tr}(\Phi(H) - \Phi(H_0)) = -i \int_{\mathbb{R}} \Phi'(t)\, d\omega_K(t).$$

was verified, where $\omega_K(\cdot) = \overline{\omega_K(\cdot)}$ is a bounded non-decreasing function.

Finally, pairs $\{H, H_0\}$ with $H_0 = H_0^*$ and $H := H_0 - iG$ where $G \geq 0$, and $G\log(G) \in \mathfrak{S}_1(\mathfrak{H})$, were studied in [1]. It is proved in [1] that under the assumption $G\log(G) \in \mathfrak{S}_1(\mathfrak{H})$ there exists a *real-valued* function $\xi(\cdot) \in L^1(\mathbb{R}, \frac{dt}{1+t^2})$ such that in place of (4) one has

$$\operatorname{tr}(\Phi(H) - \Phi(H_0)) = \int_{\mathbb{R}} \Phi'(t)\, \xi(t)\, dt$$

for Φ from a certain class of holomorphic in $\mathbb{C}_-$ functions. Notice that $G\log(G) \in \mathfrak{S}_1(\mathfrak{H})$ is stronger than $G \in \mathfrak{S}_1(\mathfrak{H})$. Both results are improved as follows:

Theorem 1. *Let $\{H', H\}$ be a pair of m-accumulative resolvent comparable operators in $\mathfrak{H}$ and $\rho(H) \cap \mathbb{C}_- \neq \emptyset$. For a sufficiently large class of functions $\Phi(\cdot)$, which are holomorphic in neighborhood of $\sigma(H')$ and $\sigma(H)$ such that $\Phi(H') - \Phi(H) \in \mathfrak{S}_1(\mathfrak{H})$ the following holds:*
(i) *There exists a complex-valued function $\omega(\cdot) \in L^1(\mathbb{R}; \frac{dt}{1+t^2})$ called the SSF of the pair $\{H', H\}$ such that the following trace formula holds*

$$(5) \qquad \operatorname{tr}(\Phi(H') - \Phi(H)) = \frac{1}{\pi} \int_{\mathbb{R}} \Phi'(t)\omega(t)dt.$$

A complex-valued function $\widetilde{\omega} \in L^1(\mathbb{R}; \frac{dt}{1+t^2})$ is also a SSF of the pair $\{H', H\}$, i.e. (5) *holds with $\widetilde{\omega}$ in place of ω, if and only if $\widetilde{\omega}(\cdot) - \omega(\cdot) \in H^1_-(\mathbb{R}, \frac{dt}{1+t^2})$.*
(ii) *If in addition, either the resolvent difference of H' and H is finite dimensional or the imaginary part of ω satisfies the Zygmund condition*

$$\int_{\mathbb{R}} |\omega_I(t)| \log(1 + |\omega_I(t)|) \frac{dt}{1+t^2} < \infty, \qquad \omega_I(\cdot) := \operatorname{Im}(\omega(\cdot)),$$

then there exists a real-valued SSF $\xi(\cdot) \in L^1\left(\mathbb{R}; \frac{dt}{1+t^2}\right)$. *The latter happens if, in particular,* $\omega(\cdot) \in L^2\left(\mathbb{R}; \frac{dt}{(1+t^2)^{\alpha/2}}\right)$ *for some* $\alpha \in [0,2]$. *Moreover, if* $\alpha \in (0,1)$, *then there is a real SSF* $\xi(\cdot)$ *satisfying* $\xi(\cdot) \in L^2\left(\mathbb{R}; \frac{dt}{(1+t^2)^{\alpha/2}}\right)$.
(iii) *If* $H = H^*$ *(resp.* $H' = H'^*$*), then there is a SSF* $\omega(\cdot)$ *of the pair* $\{H', H\}$ *satisfying* $\mathrm{Im}\,(\omega(t)) \leq 0$ *(resp.* $\mathrm{Im}\,(\omega(t)) \geq 0$*) for a.e.* $t \in \mathbb{R}$.

In the case of additive perturbations $H := H_0 - iG$ with $G \in \mathfrak{S}_1$, Theorem 1 can be specified. Namely, in this case a *complex-valued SSF* $\omega(\cdot)$ can be chosen to be summable, i.e. $\omega(\cdot) \in L^1(\mathbb{R}; dt)$. Notice that if $H = H^*$ Theorem 1 improves Kreĭn's formula (4): the measure $d\omega_K$ becomes absolutely continuous.

We treat all the problems in the framework of extension theory by considering the operators H' and H as proper extensions of a closed symmetric operator A with equal deficiency indices. The pair $\{H', H\}$ is called *singular* if $\mathrm{dom}(H') \neq \mathrm{dom}(H)$. Notice that for singular pairs Kreĭn's definition of perturbation determinant [10] is not applicable. To overcome this difficulty we apply the boundary triplet technique elaborated in [2, 3, 4, 5, 6] and especially in our recent papers [11, 12]. In particular, it is possible to express the perturbation determinant $\Delta_{H'/H}(\cdot)$ in terms of the basic objects of the extension theory: Weyl function $M(\cdot)$ and boundary operators B, B'. For instance, if $n_\pm(A) = n < \infty$ and $H' = H'^*$, $H = H^*$, a SSF of the pair $\{H', H\}$ admits the representation

$$\xi(t) = \mathrm{Im}\,\left(\log\left(\det\left(B' - M(t+i0)\right)\right) - \log\left(\det\left(B - M(t+i0)\right)\right)\right), \tag{6}$$

where $M(t+i0) := \lim_{\varepsilon\to+0} M(t+i\varepsilon)$. Formula (6) complements Kreĭn's inversion formula (2) and makes it possible to compute the SSF explicitly for a pair of boundary value problems for certain classes of ordinary differential operators. The proof of the result above and other results can be find in [15], see also the preprints [11, 13] and brief summary [14].

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Some classes of Weyl functions of abstract boundary mappings

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(joint work with Vladimir Derkach, Mark Malamud)

During last four decades the concepts of a boundary triplet (BT) and the corresponding Weyl function have been elaborated thoroughly with numerous applications e.g. in the study of boundary value problems, extension theory of symmetric operators, and spectral theory. The concept of a BT for a densely defined symmetric operator S was introduced by A. Kochubei and V. Bruk (see e.g. [8]) as follows: a triplet $\{\mathcal{H},\Gamma_0,\Gamma_1\}$, where $\mathcal{H}$ is a Hilbert space and Γ_0 and Γ_1 are linear mappings from $\operatorname{dom} S^*$ into $\mathcal{H}$, is called (nowadays) a *boundary triplet* for S^*, if:

- for all $f,g\in\operatorname{dom} S^*$ the abstract Green's identity holds:

$$(S^*f,g)_{\mathfrak{H}}-(f,S^*g)_{\mathfrak{H}}=(\Gamma_1 f,\Gamma_0 g)_{\mathcal{H}}-(\Gamma_0 f,\Gamma_1 g)_{\mathcal{H}}.$$

- the combined mapping $\Gamma=\Gamma_0\times\Gamma_1 : \operatorname{dom} S^*\to\mathcal{H}\times\mathcal{H}$ is surjective.

In fact, using a somewhat different (geometric) approach, a more general concept of a *reduction operator* involving abstract boundary conditions has been introduced and developed already much earlier by J.W. Calkin [2].

In the beginning of 1980s V. Derkach and M. Malamud (see the concluding paper [6]) associated with a BT the *Weyl function* $M(\cdot)$ via the formula

$$M(z)\Gamma_0 f_z=\Gamma_1 f_z,\quad f_z\in\ker(S^*-z),\quad z\in\mathbb{C}\setminus\mathbb{R}.$$

In particular, they proved that $M(\cdot)$ is a Q-function. This latter concept was originally introduced by M.G. Kreĭn in 1944 in connection with his formula for resolvents of selfadjoint extensions of a symmetric operator. Later on the concept of a Q-function and Kreĭn's resolvent formula itself have been developed further and investigated in a well-known series of papers by M.G. Kreĭn and H. Langer.

In the 1990s Derkach and Malamud [7] extended the concept of a boundary triplet as follows. Let S_* be a dense linear subset of S^*. Then a triplet $\{\mathcal{H},\Gamma_0,\Gamma_1\}$, where $\mathcal{H}$ is a Hilbert space and Γ_0 and Γ_1 are (closable) mappings from S_* into $\mathcal{H}$, is called a *(B-)generalized boundary triplet* for S_* if:

(B1) for all $\widehat{f} = \{f, f'\}$, $\widehat{g} = \{g, g'\} \in S_*$ the abstract Green's identity holds:

$$(f', g)_{\mathfrak{H}} - (f, g')_{\mathfrak{H}} = (\Gamma_1 f, \Gamma_0 g)_{\mathcal{H}} - (\Gamma_0 f, \Gamma_1 g)_{\mathcal{H}};$$

(B2) $\Gamma_0 : S_* \to \mathcal{H}$ is surjective;

(B3) $A_0 := \ker \Gamma_0$ is a selfadjoint relation in $\mathfrak{H}$.

This notion made it possible to treat a pair $\{A_0, A_1\}$ of disjoint selfadjoint extensions of S ($A_0 \cap A_1 = S$), instead of a pair of transversal selfadjoint extensions of S ($A_0 \widehat{+} A_1 = S^*$), as the basic extensions $A_0 = \ker \Gamma_0$ and $A_1 = \ker \Gamma_1$ of such a generalized boundary triplet.

A general class of abstract boundary mappings for symmetric operators or relations S in a Hilbert space was introduced by Derkach, Hassi, Malamud and de Snoo [3]; see also [4, 5]. This concept of a *generalized BT* (GBT) (called a boundary relation in [3]) allowed the authors to prove the most general realization theorem: every Nevanlinna family is the Weyl function (or a Weyl family) of such a GBT.

The definition reads as follows: with $\mathcal{H}$ a Hilbert space, a linear relation $\Gamma : \mathfrak{H}^2 \mapsto \mathcal{H}^2$ is a *boundary relation* for S^*, if $\operatorname{dom} \Gamma$ is dense in S^* and

(BR1) the abstract Green's identity

$$(f', g)_{\mathfrak{H}} - (f, g')_{\mathfrak{H}} = (h', k)_{\mathcal{H}} - (h, k')_{\mathcal{H}},$$

holds for every $\{\widehat{f}, \widehat{h}\}$, $\{\widehat{g}, \widehat{k}\} \in \Gamma$;

(BR2) Γ is maximal in the sense that if $\{\widehat{g}, \widehat{k}\} \in \mathfrak{H}^2 \times \mathcal{H}^2$ satisfies the Green's identity in (BR1) for every $\{\widehat{f}, \widehat{h}\} \in \Gamma$, then $\{\widehat{g}, \widehat{k}\} \in \Gamma$.

The condition (BR1) means that in a Kreĭn space sense Γ is isometric, while the condition (BR2) guarantees that Γ is in fact unitary (in the sense of Shmul'yan) w.r.t. the underlying Kreĭn spaces. Associated with a general isometric or unitary (cf. (BR1), (BR2)) boundary mapping Γ from S^* (with dense domain in S^*) into $\mathcal{H} \times \mathcal{H}$ there are the corresponding component mappings Γ_0 and Γ_1 into $\mathcal{H}$. Our recent investigations in the theory of boundary triplets and their Weyl functions concern the following two cases:

(S) $A_0 = \ker \Gamma_0$ is selfadjoint;

(ES) $A_0 = \ker \Gamma_0$ is essentially selfadjoint.

The corresponding triplets are called *S-generalized* and *ES-generalized* boundary triplets, respectively (when Γ is single-valued). The Weyl functions associated with these classes of boundary triplets are characterized. For this purpose a new class of *form-domain invariant Nevanlinna families* is introduced. A Nevanlinna function $M(\cdot)$ is said to be form-domain invariant if:

(F) the quadratic form generated in $\mathcal{H}$ by the imaginary part of $M(\lambda)$ via

$$\mathfrak{t}_{M(\lambda)}[u, v] = \frac{1}{\lambda - \bar{\lambda}} \left[(M(\lambda)u, v) - (u, M(\lambda)v) \right],$$

is closable for all $\lambda \in \mathbb{C} \setminus \mathbb{R}$ and the closure of the form $\mathfrak{t}_{M(\lambda)}$ has a constant domain.

The first main theorem reads as follows.

Theorem 1. *Let $\{\mathcal{H}, \Gamma_0, \Gamma_1\}$ be a unitary boundary triplet for S^* and let $M(\cdot)$ and $\gamma(\cdot)$ be the corresponding Weyl function and the γ-field. Then the following statements are equivalent:*

(i) $\{\mathcal{H}, \Gamma_0, \Gamma_1\}$ *is an ES-generalized boundary triplet;*

(ii) $\gamma(\lambda)$ *admits a single-valued closure* $\overline{\gamma(\lambda)}$ *for some* $\lambda \in \mathbb{C}_+$ *and some* $\lambda \in \mathbb{C}_-$ *or, equivalently, for every* $\lambda \in \mathbb{C} \setminus \mathbb{R}$*;*

(iii) the Weyl function $M(\cdot) \in R(\mathcal{H})$ *is form-domain invariant.*

ES-generalized boundary triplets admit a natural renormalization which give rise to, in general, isometric boundary triplets which are S-generalized. In particular, it is shown that for every form-domain invariant operator-valued strict Nevanlinna function $M(\cdot) \in R^s(\mathcal{H})$ there exist a bounded operator $G \in [\mathcal{H}]$ with $\ker G = \ker G^* = \{0\}$, a closed symmetric densely defined operator E in $\mathcal{H}$, and a bounded Nevanlinna function $M_0(\cdot) \in R[\mathcal{H}]$, such that

$$M(\lambda) = G^{-*}(E + M_0(\lambda))G^{-1}, \quad \lambda \in \mathbb{C} \setminus \mathbb{R}.$$

In general, the function $E + M_0(\cdot)$ in this formula does not belong to the class of Nevanlinna functions. It can be realized as a Weyl function of an *almost B-generalized boundary triplet*. The class of almost B-generalized boundary triplets is a natural extension of the class of (B-)generalized boundary triplets, as well as of the class of quasi-boundary triplets in [1], where the conditions (B1), (B3) are satisfied and the condition (B2) is relaxed as follows:

(B2′) $\operatorname{ran} \Gamma_0$ is dense in $\mathcal{H}$.

The corresponding class of Weyl functions is characterized and a Kreĭn type resolvent formula for almost B-generalized boundary triplets is established.

Applications of ES-generalized boundary triplets in concrete boundary value problems are discussed in the PDE and ODE settings. In particular, it is shown that simple $J_{\mathcal{H}}$-unitary transforms of B-generalized boundary triplets can produce ES-generalized boundary triplets for S^*, whose Weyl functions are unbounded form-domain invariant Nevanlinna functions. This fact can be used to study e.g. the Laplacian operator Δ in (bounded) domains $\Omega(\subset \mathbb{R}^3)$ with a smooth boundary $\partial\Omega$, in which case the above transform of boundary triplets correspond to certain regularizations of the trace operators γ_D and γ_N being extensively studied in the literature; see e.g. [1, 9, 11, 13]. Another class of applications for ES-generalized boundary triplets appears naturally in the study of Sturm-Liouville operators with infinitely many point interactions, see [10], and with operator potential, see [12].

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Does Diffusion Determine the Domain?

Wolfgang Arendt

In 1966 Marc Kac asked the famous question: "Can one hear the shape of a drum?". Even though there exist counterexamples in form of polygonals Kac' proper question seems still to be wide open. We give an account and propose some new operator theoretical investigations. The talk consists of three parts:

1. An account on known positive results and counterexamples.
2. A shift of paradigm: Intertwining operators with special properties.
3. Analysis of the counterexamples in the spirit of intertwining operators.

1. The question and some answers

Given a domain in $\mathbb{R}^d$ (i.e. a bounded, open, connected set), the space $L^2(\Omega)$ has an orthonormal basis $(e_k)_{k\in\mathbb{N}}$ such that $e_k \in H_0^1(\Omega)$ and

$$-\Delta e_k = \lambda_k e_k, \quad 0 < \lambda_1 \leq \lambda_2 \leq \dots .$$

We call the series $(\lambda_k)_{k\in\mathbb{N}}$ the *Dirichlet eigenvalues*. If we want to denote the dependence on Ω we write $\lambda_k(\Omega) := \lambda_k$.

Two Lipschitz domains Ω_1, Ω_2 are called *isospectral* if $\lambda_k(\Omega_1) = \lambda_k(\Omega_2)$ for all $k \in \mathbb{N}$. A *Lipschitz domain* is a domain with Lipschitz boundary

Two domains Ω_1, Ω_2 are called *congruent* if there exists an isometry $\tau\colon \mathbb{R}^d \to \mathbb{R}^d$ such that $\tau(\Omega_2) = \Omega_1$. Recall that an *isometry* is a mapping of the form $\tau(x) = Bx + b$, where B is an orthogonal $d \times d$-matrix and $b \in \mathbb{R}^d$.

Problem 1. *If Ω_1 and Ω_2 are two isospectral Lipschitz domains, are they necessarily congruent?*

M. Kac asked this famous problem in 1966 in a more restricted form, namely supposing that the dimension d is two and Ω_1, Ω_2 have C^∞-boundary. This restricted question is still open today. No C^1-counterexamples are known in any dimension and no convex Lipschitz domains are known which give a counterexample in dimension 2 or 3.

However, Gordon, Webb and Wolpert gave a counterexample to Problem 1 in dimension 2 in form of two polygonals. In dimension 4 they modified a counterexample of Urakawa and constructed two cut convex cones which are isopectral and non-congruent (see [3, Section 1.7] for more precise statements and references).

There is an interesting positive result which is based on Weyl's formula and a version of the Faber-Kahn inequality due to D. Daners and J. Kennedy [7]. It is remarkable that it is true under the optimal regularity hypothesis at the boundary. We say that Ω is *regular in capacity* if $\mathrm{cap}(B_r(z) \setminus \Omega) > 0$ for all $z \in \partial\Omega$, $r > 0$, where $B_r(z) := \{x \in \mathbb{R}^d : |x - z| < r\}$. For each domain Ω there is a unique domain $\tilde{\Omega} \supset \Omega$ which is regular in capacity such that $\mathrm{cap}(\tilde{\Omega} \setminus \Omega) = 0$ (see [2] for more details).

Theorem. *Let Ω be a domain in $\mathbb{R}^d$ which is regular in capacity and B be a ball. If Ω and B are isospectral, then they are congruent.*

We refer to [3, Section 1.7] for more details.

2. Intertwining operators

Some time ago we proposed a shift of paradigm. Denote by Δ_Ω the Dirichlet Laplacian on $L^2(\Omega)$; i.e $D(\Delta_\Omega) = \{u \in H_0^1(\Omega) : \Delta u \in L^2(\Omega)\}$, $\Delta_\Omega u = \Delta u$. Then Δ_Ω is selfadjoint, has compact resolvent and $\lambda_k(\Omega)$ is the k-th eigenvalue of $-\Delta_\Omega$. It is easy to see that the following three assertions are equivalent

(a) Ω_1 and Ω_2 are isospectral
(b) there exists a unitary operator $U \colon L^2(\Omega_1) \to L^2(\Omega_2)$ such that $Ue^{t\Delta_{\Omega_1}} = e^{t\Delta_{\Omega_2}} U$ for all $t \geq 0$.
(c) there exists some invertible operator $\phi \colon L^2(\Omega_1) \to L^2(\Omega_2)$ such that $\phi\, e^{t\Delta_{\Omega_1}} = e^{t\Delta_{\Omega_2}} \phi$.

Here $e^{t\Delta_{\Omega_1}}$ is the semigroup generated by Δ_{Ω_1}, and $u(t) = e^{t\Delta_{\Omega_1}} u_0$ solves the diffusion equation

$$\begin{aligned} \dot{u} &= \Delta_{\Omega_1} u \\ u(0) &= u_0 \end{aligned}$$

for $u_0 \in L^2(\Omega_1)$. Thus condition (c) means that ϕ maps solutions to solutions.

If $u_0 \geq 0$, then $e^{t\Delta_{\Omega_1}} u_0 \geq 0$ for all $t \geq 0$. The positive solutions are of special interest: They describe the evolution of an initial density. This is a motivation to consider in (c) an *order isomorphism* ϕ; i.e. an isomorphism for which $\phi f \geq 0$ iff $f \geq 0$ for all $f \in L^2(\Omega_1)$.

Theorem (c.f. [2])**.** *Let ϕ be an order isomorphism from $L^2(\Omega_1)$ to $L^2(\Omega_2)$ where Ω_1 and Ω_2 are domains which are regular in capacity. If $e^{t\Delta_{\Omega_2}} \phi = \phi\, e^{t\Delta_{\Omega_1}}$ for all*

$t \geq 0$, then Ω_1 and Ω_2 are congruent and there exists an isomorphism τ such that $\phi f = c \cdot f \circ \tau$ for all $f \in L^2(\Omega_1)$ and some $c > 0$.

Given that $L^2(\Omega) = L^2(\tilde{\Omega})$ and $\Delta_\Omega = \Delta_{\tilde{\Omega}}$, the regularity assumption is optimal in this result.

Since order isomorphisms transform positive solutions into positive solutions, the result might be expressed by saying that *"diffusion determines the domain"*. It remains true for Robin and Neumann boundary conditions on Lipschitz domains but not for all boundary conditions; see [2]. It also remains true for complete Riemannian manifolds (see [4], [5]).

3. The counterexample in the spirit of intertwining operators

In view of this result we reconsider the counterexample of Gordon-Webb and Wolpert which works for Dirichlet and Neumann boundary conditions. It has been analysed and simplified by several mathematicians (among them P. Bérard, S.J. Chapman). We construct in [5] explicitly an intertwining operator ϕ instead of considering merely the eigenfunctions. It turns out that in the case of Neumann boundary conditions ϕ can even be chosen positive (i.e. $f \geq 0$ implies $\phi f \geq 0$). However, we know by the theorem above that it is not possible that $\phi^{-1} \geq 0$.

The analysis of the counterexample in terms of the intertwining operator is rewarding. We can show that for the special domains in the counterexample no such operator can exist which is intertwining for both Dirichlet and Neumann boundary conditions, even though different intertwining operators do exist. However the following problem seems to be open.

Consider for every $\beta \geq 0$ the Laplacian Δ_Ω^β with Robin boundary conditions $\partial_\nu u + \beta u = 0$.

Problem 2. *Let Ω_1, Ω_2 be two Lipschitz domains. Assume that $\Delta_{\Omega_1}^\beta$ and $\Delta_{\Omega_2}^\beta$ have the same series of eigenvalue for each $\beta \geq 0$ and also for $\beta = \infty$ (i.e. Dirichlet boundary conditions). Does it follow that Ω_1 and Ω_2 are congruent?*

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Strong coupling in leaky graphs and Robin billiards

Pavel Exner

(joint work with Michal Jex, Sylwia Kondej, Alexander Minakov, Konstantin Pankrashkin, Leonid Parnovski)

A lot of attention was paid in recent years to *leaky quantum graphs* described by singular Schrödinger operators which can be written formally as

$$H_{\alpha,\Gamma} = -\Delta - \alpha\delta(x-\Gamma)\,, \quad \alpha > 0\,,$$

where Γ is a manifold or a complex of a lower dimensionality. Such operators can be defined naturally through the associated quadratic form if $\operatorname{codim}\Gamma = 1$ and through boundary conditions involving generalized boundary values if $\operatorname{codim}\Gamma = 2$. One can derive various spectral properties of $H_{\alpha,\Gamma}$, in particular, existence of a geometrically induced discrete spectrum and its strong-coupling asymptotic behavior for a sufficiently smooth curve or surface Γ without a boundary; for results prior to 2008 we refer to [1]. At the same time, some questions remained open and have been addressed only recently. The aim of the present talk is to review briefly the corresponding new results.

Theorem 1 ([2])**.** *Suppose γ is a C^4 smooth open arc in $\mathbb{R}^2$ of length L with* regular ends*; then the strong-coupling limit of the j-th negative eigenvalue of $H_{\alpha,\Gamma}$ is*

$$\lambda_j(\alpha) = -\frac{1}{4}\alpha^2 + \mu_j + \mathcal{O}\Big(\frac{\ln\alpha}{\alpha}\Big) \quad \textit{as} \quad \alpha \to +\infty\,,$$

where μ_j is the j-th eigenvalue of the operator $-\frac{\mathrm{d}^2}{\mathrm{d}Ds^2} - \frac{1}{4}\gamma(s)^2$ on $L^2(0,L)$ with Dirichlet b.c.*, where $\gamma(s)$ is the signed curvature of Γ at the point $s \in (0,L)$.*

The technique of bracketing leading to tight two-sided estimates of $H_{\alpha,\Gamma}$ used for curves without endpoints gives here the upper bound only. To get the lower one, a Neumann bracketing on neighborhoods of an extended curve has to be used in combination with the eigenfunction expression

$$\psi_j(x) = \frac{1}{2\pi}\int_\Gamma K_0(\kappa_j|x-\Gamma(s)|)\, f_j(s)\,\mathrm{d}s$$

corresponding to the eigenvalue $\lambda_j = -\kappa_j^2$, where f_j is the eigenfunction of the corresponding Birman-Schwinger operators, and the fast decay of the Green function for large κ_j. A similar result holds for non-closed finite curves in $\mathbb{R}^3$ [3].

Another problem concerns more singular Schrödinger operators with the δ interaction replaced by the so-called δ' [4]. The corresponding operator $H_{\beta,\Gamma}$ is associated with the quadratic form

$$h_{\beta,\Gamma}[\psi] = \|\nabla\psi\|^2 - \beta^{-1}\int_\Gamma |\psi(s,0_+) - \psi(s,0_-)|^2\,\mathrm{d}s$$

defined on functions $\psi \in C(\mathbb{R}^2) \cap H^1(\mathbb{R}^2 \setminus \Gamma)$ written in the natural curvilinear coordinates as $\psi(s,u)$. For a loop-shaped curve Γ we have the following result:

Theorem 2 ([5]). *Let Γ be a C^4-smooth closed curve without self-intersections. Then $\sigma_{\rm ess}(H_{\beta,\Gamma}) = [0,\infty)$ and to any $n \in \mathbb{N}$ there is a $\beta_n > 0$ such that $\#\sigma_{\rm disc}(H_{\beta,\Gamma}) \geq n$ holds for $\beta \in (0,\beta_n)$. Denoting for such a β by $\lambda_j(\beta)$ the j-th eigenvalue of $H_{\beta,\Gamma}$, again counted with its multiplicity, we have the asymptotic expansion*

$$\lambda_j(\beta) = -\frac{4}{\beta^2} + \mu_j + \mathcal{O}\big(\beta|\ln\beta|\big)\,, \quad j = 1,\dots,n\,,$$

valid as $\beta \to 0_+$, where μ_j is the j-th eigenvalue of the comparison operator S, the same as above. *Moreover, for the counting function $\beta \mapsto \#\sigma_d(H_{\beta,\Gamma})$ we have*

$$\#\sigma_{\rm disc}(H_{\beta,\Gamma}) = \frac{2L}{\pi\beta} + \mathcal{O}(|\ln\beta|) \quad \text{as } \beta \to 0_+\,.$$

A similar result holds for infinite non-straight curves. For smooth curved surfaces, finite closed or infinite without boundaries, we have an analogous asymptotic expansion [6] in which S is replaced by

$$S = -\Delta_\Gamma + K - M^2\,,$$

where $-\Delta_\Gamma$ is the Laplace-Beltrami operator on $L^2(\Gamma)$ and K, M are the Gauss and mean curvature of Γ, respectively.

The second topic of the talk concerns the related 'one-sided' problem, namely the large-parameter asymptotics of a Robin billiard. Let Ω be an open, simply connected set in $\mathbb{R}^2$ with a closed C^4 Jordan boundary $\partial\Omega = \Gamma : [0,L] \ni s \mapsto (\Gamma_1,\Gamma_2) \in \mathbb{R}^2$, with $\gamma : [0,L] \to \mathbb{R}$ being the signed curvature of Γ. We consider the boundary-value problem

$$-\Delta f = \lambda f \;\text{ in } \Omega\,, \quad \frac{\partial f}{\partial n} = \beta f \;\text{ on } \Gamma\,,$$

with $\beta > 0$, where $\frac{\partial}{\partial n}$ is the outward normal derivative. The corresponding self-adjoint operator H_β is associated with the quadratic form

$$q_\beta[f] = \|\nabla f\|^2_{L^2(\Omega)} - \beta \int_\Gamma |f(x)|^2 {\rm d}s$$

defined on ${\rm Dom}(q_\beta) = H^1(\Omega)$. In [7, 8] asymptotics of the ground-state eigenvalue was found. In [9] we extended it for higher eigenvalues: we showed that for a fixed j we have in the asymptotic regime $\beta \to +\infty$ the relation

$$\lambda_n(\beta) = -\beta^2 - \gamma^*\beta + \mathcal{O}\big(\beta^{2/3}\big)\,, \quad \gamma^* := \max_{[0,L]} \gamma(s)\,.$$

This result was further improved and extended to higher dimensions in [10]; these authors proved that for open, connected domains $\Omega \subset \mathbb{R}^d$, $d \geq 2$, with a C^3-smooth boundary, the j-th the eigenvalue behaves in the limit $\beta \to \infty$ as

$$\lambda_j(\beta) = -\beta^2 - (d-1)H_{\max}(\Omega)\beta + \mathcal{O}\big(\beta^{2/3}\big)\,,$$

where $H_{\max}(\Omega)$ is the maximum *mean curvature* at the boundary $\partial\Omega$. Moreover, if the boundary is C^4 smooth, the error term can be replaced by $\mathcal{O}\big(\beta^{1/2}\big)$. For an infinitely smooth boundary of a planar Ω whose curvature has a single maximum the following terms of the asymptotics have been recently computed in [11].

As the last item let us mention the paper [12] in which infinite domains $\Omega \in \mathbb{R}^2$ have been considered. We found there the analogous strong coupling asymptotics, also for particular domains such as waveguides, i.e. Ω in the form of a curved strip. Moreover, we have discussed there the spectrum of H_β in the non-asymptotic regime for domains the boundary of which is straight outside a compact. In particular, we have shown that if such an Ω is concave, $\sigma_{\rm disc}(H_\beta) = \emptyset$ holds for any $\beta > 0$.

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Partially fundamentally reducible operators in Kreĭn spaces

Vladimir Derkach

(joint work with Branko Ćurgus)

Recall that a complex vector space $\mathcal{K}$ with a sesquilinear form $[\cdot,\cdot]$ is called a Kreĭn space, if there exist orthogonal subspaces $\mathcal{K}_+$, $\mathcal{K}_-$ of $\mathcal{K}$ such that

$$\mathcal{K} = \mathcal{K}_+ \dot{+} \mathcal{K}_-, \qquad \text{a direct sum,} \tag{1}$$

and $(\mathcal{K}_+, [\cdot,\cdot])$ and $(\mathcal{K}_-, -[\cdot,\cdot])$ are Hilbert spaces. The decomposition (1) is called a fundamental decomposition of $\mathcal{K}$ and the operator $J = P_+ - P_-$ is called a *fundamental symmetry* of a Kreĭn space. For the general theory of Kreĭn spaces and operators acting in them we refer to the monographs [1, 2].

The ultimate task in this note is to provide sufficient conditions for a self-adjoint operator in a Kreĭn space to be similar to a self-adjoint operator in a Hilbert space. A simple characterization of similarity is as follows. A self-adjoint operator A in a Kreĭn space $(\mathcal{K}, [\cdot,\cdot])$ is similar to a self-adjoint operator in a Hilbert space, if and only if A is fundamentally reducible in $(\mathcal{K}, [\cdot,\cdot])$; where *fundamentally reducible* means that there exists a fundamental decomposition $\mathcal{K} = \mathcal{K}_+ \dot{+} \mathcal{K}_-$ of $(\mathcal{K}, [\cdot,\cdot])$ such that A is the direct sum of its restrictions to $\mathcal{K}_+ \cap (\text{dom A})$ and $\mathcal{K}_- \cap (\text{dom A})$.

In what follows the operator A is supposed to be nonnegative self-adjoint operator with a nonempty resolvent set. The spectrum of such an operator is real and, excluding arbitrary neighborhoods of 0 and ∞, the operator A has a projector valued spectral function whose properties resemble the properties of the spectral function $E(\Delta)$ of a self-adjoint operator in a Hilbert space; for details see [12]. If the spectrum of A accumulates on both sides of 0 (∞), then 0 (∞, respectively) is called a *critical point* of A. If the spectral function of A is bounded in a neighborhood of a critical point, then that critical point is said to be *regular*. Otherwise, it is said to be a *singular* critical point. The set of all singular critical points of A is denoted by $c_s(A)$. Here, by definition, $c_s(A) \subseteq \{0, \infty\}$.

Our first step in studying the similarity question is to introduce a new concept related to the fundamental reducibility in Kreĭn spaces.

Definition 1. *We say that a self-adjoint operator A in a Kreĭn space $(\mathcal{K}, [\cdot,\cdot])$ is* partially fundamentally reducible *if there exists a fundamental decomposition $\mathcal{K} = \mathcal{K}_+ [\dot{+}] \mathcal{K}_-$ of $(\mathcal{K}, [\cdot,\cdot])$ such that the subspaces*

$$\mathcal{D}_+ = \bigl\{ f \in \mathcal{K}_+ \cap (dom\ ) : Af \in \mathcal{K}_+ \bigr\} \quad and \quad \mathcal{D}_- = \bigl\{ f \in \mathcal{K}_- \cap (dom\ ) : Af \in \mathcal{K}_- \bigr\}$$

are dense in $\mathcal{K}_+$ and $\mathcal{K}_-$ and the restrictions $S_+ = A|_{\mathcal{D}_+}$ and $S_- = -A|_{\mathcal{D}_-}$ are symmetric operators with defect numbers $(1, 1)$ in the Hilbert spaces $(\mathcal{K}_+, [\cdot,\cdot])$ and $(\mathcal{K}_-, -[\cdot,\cdot])$, respectively.

Our objective is to give sufficient conditions on a nonnegative partially fundamentally reducible operator A in a Kreĭn space under which either $0 \notin c_s(A)$, or $\infty \notin c_s(A)$, or both $0, \infty \notin c_s(A)$. The importance of the condition $0, \infty \notin c_s(A)$ lies in the fact that it is equivalent to the similarity of A to a self-adjoint operator in a Hilbert space.

To this end we will use a coupling method developed in [6] and based on the boundary triple technique (see [10, 5]). We will apply this theory to the symmetric operators S_+ and S_- associated via Definition 1 with a partially fundamentally reducible operator A. Specifically, let $\bigl(\mathbb{C}, \Gamma_0^+, \Gamma_1^+\bigr)$ be a boundary triple of the operator S_+^*, the adjoint of S_+ in the Hilbert space $(\mathcal{K}_+, [\cdot,\cdot])$, and let m_+ be the corresponding Weyl function. Then there exists a unique boundary triple $\bigl(\mathbb{C}, \Gamma_0^-, \Gamma_1^-\bigr)$ for S_-^* such that the operator A is a coupling of S_+ and S_- relative to the boundary triples $\bigl(\mathbb{C}, \Gamma_0^+, \Gamma_1^+\bigr)$ and $\bigl(\mathbb{C}, \Gamma_0^-, \Gamma_1^-\bigr)$. That is, $f \in \text{dom}\ (A)$ if and only if there exist $f_+ \in \operatorname{dom}\left(S_+^*\right)$ and $f_- \in \operatorname{dom}\left(S_-^*\right)$ such that

$$f = f_+ + f_- \quad \text{and} \quad \Gamma_0^+ f_+ = \Gamma_0^- f_-, \quad \Gamma_1^+ f_+ = -\Gamma_1^- f_-.$$

Let m_- be the Weyl function of S_- corresponding to the boundary triple $\left(\mathbb{C}, \Gamma_0^-, \Gamma_1^-\right)$. The Weyl functions m_+ and m_- belong to the class of Nevanlinna functions and completely characterize the simple (non-self-adjoint) parts of the symmetric operators S_+ and S_- acting in the Hilbert spaces $\mathcal{K}_+$ and $\mathcal{K}_-$. Therefore, it is natural to look for conditions for the fundamental reducibility of A in terms of the local behavior of the associated Weyl functions m_+ and m_- at 0 and ∞.

This approach was utilized, for example in [8, 9], where the boundedness of the function

$$y \mapsto \frac{\operatorname{Im} m_+(iy) + \operatorname{Im} m_-(iy)}{m_+(iy) + m_-(-iy)}, \quad y > 0, \tag{2}$$

on $(0,1)$ (on $(1,\infty)$) was proved to be necessary for $0 \notin c_s(A)$ ($\infty \notin c_s(A)$, respectively). Since we use these necessary conditions in an essential way, we introduce the following terminology. A pair of functions (m_+, m_-) is said to have *D_0-property* (*D_∞-property*) if the function in (2) is bounded on $(0,1)$ (on $(1,\infty)$, respectively).

Next we introduce two different kinds of local behavior of a Nevanlinna function m with the integral representation

$$m(z) = a + \int_{\mathbb{R}} \left(\frac{1}{t-z} - \frac{t}{1+t^2} \right) d\sigma(t)$$

where $a = \bar{a}$ and the measure $d\sigma(t)/(1+t^2)$ is finite. The function m is said to have B_0-property (B_∞-property, respectively) if the mapping

$$f \mapsto \int_0^{+\infty} \frac{f(x)}{x+y}\, d\sigma(x),$$

is a bounded mapping from $L^2_\sigma(\mathbb{R}_+)$ into $L^2_{w_m}(0,1)$ ($L^2_{w_m}(1,\infty)$, respectively). Here $w_m(y) = \left(\operatorname{Im} m(iy)\right)^{-1}$, the reciprocal of the imaginary part of m.

Further, we define the asymptotic class $\mathcal{A}_\infty$ which consists of all Nevanlinna functions m for which there exist $\alpha \in (0,1)$, $C > 0$ and a Möbius transformation $\mu(z) = (az+b)/(cz+d)$ with $|ad - bc| = 1$ such that the composition $\mu \circ m$ is a Stieltjes function and for all $z \in \mathbb{C} \setminus \mathbb{R}$ we have

$$(\mu \circ m)(rz) \sim \frac{C}{(-rz)^\alpha} \qquad \text{as} \qquad r \to +\infty.$$

The asymptotic class $\mathcal{A}_0$ consists of all Nevanlinna functions m for which the function $-m(1/z)$ belongs to $\mathcal{A}_\infty$. We prove that all functions in $\mathcal{A}_\infty$ satisfy B_∞-property and all pairs of functions in $\mathcal{A}_\infty$ have D_∞-property and that all functions in $\mathcal{A}_0$ satisfy B_0-property and all pairs of functions in $\mathcal{A}_0$ have D_∞-property.

With D- and B-properties our main results are as follows. If A is a nonnegative partially fundamentally reducible operator and if the associated Weyl functions m_+ and m_- have B_∞-property, then $\infty \notin c_s(A)$ if and only if the pair (m_+, m_-) has D_∞-property. Analogously, if A is a nonnegative partially fundamentally reducible operator and if the associated Weyl functions m_+ and m_- have B_0-property, then $0 \notin c_s(A)$ if and only if the pair (m_+, m_-) has D_0-property and $\ker(A) = \ker(A^2)$. Together these two results give sufficient conditions for a nonnegative partially fundamentally reducible operator in a Kreĭn space $(\mathcal{K}, [\cdot, \cdot])$ to be similar to a self-adjoint operator in a Hilbert space. The proof of the main results are based on the Veselić similarity criterion [13]. In the case when $m_+ = m_-$ these results were proved in [11].

The results are applied to indefinite Sturm-Liouville differential operators. In some cases they lead to a new point of view at some results from [3, 4, 7, 8, 9]. We also get some new results for the case of nonsymmetric coefficients and the case when A is a coupling of two differential operators of different order.

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Uniqueness results for systems of ODE and canonical systems

Mark Malamud

I. General uniqueness result for system of ODE.
Let B be a non-singular diagonal $n \times n$ matrix,

$$B = \operatorname{diag}(b_1 I_{n_1}, \ldots, b_r I_{n_r}) \in \mathbb{C}^{n\times n} \qquad n = n_1 + \ldots + n_r. \tag{1}$$

Consider on [0,1] a system of differential equations of the form

$$-\,iB^{-1}y' + Q(x)y = \lambda y, \qquad y = \operatorname{col}(y_1, \ldots, y_n), \tag{2}$$

with a summable potential matrix $Q \in L^1[0,1] \otimes \mathbb{C}^{n\times n}$.

Alongside with equation (2) we consider the vector equation

$$-\,iB^{-1}\widetilde{y}' + \widetilde{Q}(x)\widetilde{y} = \lambda \widetilde{y}, \qquad \widetilde{y} = \operatorname{col}(\widetilde{y}_1, \ldots, \widetilde{y}_n), \tag{3}$$

with summable potential matrix $\widetilde{Q} \in L^1[0,1] \otimes \mathbb{C}^{n\times n}$.

Consider block matrix representations of the potential matrices $Q(\cdot)$ and $\widetilde{Q}(\cdot)$ with respect to the orthogonal decomposition $\mathbb{C}^n = \bigoplus_{j=1}^{r} \mathbb{C}^{n_j}$.

$$Q = (Q_{jk})_{j,k=1}^{r}, \quad \widetilde{Q} = (\widetilde{Q}_{jk})_{j,k=1}^{r}, \quad Q_{jk}, \widetilde{Q}_{jk} : [0,1] \to \mathbb{C}^{n_j\times n_k}, \tag{4}$$

Assume that both potential matrices $Q(\cdot)$ and $\widetilde{Q}(\cdot)$ have zero block diagonal with respect to this decomposition

$$Q_{jj}(x) = \widetilde{Q}_{jj}(x) = 0, \quad x \in [0,1], \quad j \in \{1, \ldots, r\}. \tag{5}$$

The latter condition can be achieved by means of a respective gauge transformation

Theorem 1. *Let $Q = (Q_{jk})_{j,k}^{r}$ and $\widetilde{Q} = (\widetilde{Q}_{jk})_{j,k=1}^{r}$ be the potential matrices of the form* (4)–(5), *and let $T \in \mathbb{C}^{n\times n}$, $\det T \neq 0$. Let also $W(\cdot,\lambda)$ and $\widetilde{W}(\cdot,\lambda)$ be $n \times n$ fundamental matrices solutions of equations* (2) *and* (3), *respectively satisfying by initial conditions*

$$W(0,\lambda) = \widetilde{W}(0,\lambda) = T, \qquad \lambda \in \mathbb{C}. \tag{6}$$

If the monodromy matrices of these systems coincide, i.e.

$$W(\lambda) := W(1,\lambda) = \widetilde{W}(1,\lambda) =: \widetilde{W}(\lambda),$$

then $Q(x) = \widetilde{Q}(x)$ for a.e. $x \in [0,1]$.

Remark 2. *If the spectrum of $x\,B$ is simple, $(n_1 = \ldots = n_r = 1)$, i.e. $r = n$, the result was obtained by Z. Leibenzon* [1] *by a completely different method.*

II. Systems of ODE. Self-adjoint case.
Here we substantially improve Theorem 1 assuming that the matrix B and a potential matrix Q are self-adjoint.

Theorem 3. *Let $B = diag(b_1 I_{n_1}, \dots, b_r I_{n_r}) = B^*$ and $Q = Q^*$, $\widetilde{Q} = \widetilde{Q}^* \in L^1[0,1] \otimes \mathbb{C}^{n\times n}$. Suppose also that $n_1 = \ldots = n_r = p$ and T is a B^{-1}–unitary matrix ($T^* B^{-1} T = B^{-1}$), and*

$$W(0,\lambda) = \widetilde{W}(0,\lambda) = T = (T_{jk})_{j,k=1}^r, \qquad T_{jk} \in \mathbb{C}^{p\times p}. \tag{7}$$

Assume also that the matrices T_{jk} are non–degenerate, $\det T_{jk} \neq 0$, $j,k \in \{1,\dots,r\}$, and $\det(W_{kk}(\lambda)\widetilde{W}_{kk}(\lambda)) \neq 0$. Assume also that after certain rearrangement of the columns of the monodromy matrices $W(\cdot)$ and $\widetilde{W}(\cdot)$ the following equalities hold

$$W_{jk}(\lambda)W_{kk}^{-1}(\lambda) = \widetilde{W}_{jk}(\lambda)\widetilde{W}_{kk}^{-1}(\lambda), \quad 1 \le k \le r-1, \ j \in \{j_{k,1}, \dots, j_{k,r-k}\}. \tag{8}$$

Then $Q(x) = \widetilde{Q}(x)$ for a.a. $x \in [0,1]$.

In other words, there exists such a permutation of the columns of the monodromy matrix $W(\cdot)$ that the potential matrix $Q(\cdot)$ is uniquely determined by the family of $r(r-1)/2$ matrix–functions $M_{jk} := W_{jk}W_{kk}^{-1}$ where $k \in \{1,\dots,r-1\}$ and $j = j(k)$ takes exactly $r-k$, depending on k, values $j \in \{j_{k,1},\dots,j_{k,r-k}\}$.

Corollary 4. *Let $B = B^*$ and $Q = Q^* \in L^1[0,1] \otimes \mathbb{C}^{n\times n}$. Suppose also that $n_1 = \ldots = n_r = p$ and T is a B^{-1}–unitary matrix. Then Q is uniquely determined by $r-1$ matrix columns of the monodromy matrix $W(\cdot)$.*

Remark 5. *(i) In fact, in accordance with Theorem 3 a potential matrix Q is uniquely determined by $r(r-1)/2 + r - 1 = (r+2)(r-1)/2$ matrix entries of the monodromy matrix $W(\cdot)$. Note also that under a stronger assumption $Q = Q^* \in L^\infty[0,1]\otimes\mathbb{C}^{n\times n}$ Corollary 4 was proved in* [2] *by using of triangular transformation operators for system* (2)*. The latter were also constructed in* [2].

(ii) In the case $r = 2$ Theorem 3 was also proved in [2]*. In this case it means that a potential matrix Q is uniquely determined by the Weyl function $M(\cdot)$.*

It can be shown that it is not the case for $r \ge 3$. Moreover, the number $r(r-1)/2$ of functions $M_{jk}(\cdot)$ indicated in Theorem 3 is minimal possible for the unique determination of a potential matrix Q.

Let us clarify Theorem 3 for $r = 3$.

Theorem 6. *Let $r = 3$, $\det(W_{jj}(\lambda)\widetilde{W}_{jj}(\lambda)) \not\equiv 0$, $j \in \{1,2,3\}$, and let $\det T_{jk} \neq 0$ for $j,k \in \{1,2,3\}$. Suppose also that*

$$M_{j1}(\lambda) := W_{j1}(\lambda)W_{11}^{-1}(\lambda) = \widetilde{W}_{j1}(\lambda)\widetilde{W}_{11}^{-1}(\lambda) =: \widetilde{M}_{j1}(\lambda), \ j \in \{2,3\}, \tag{9}$$

$\lambda \in \Omega_{11} \cap \widetilde{\Omega}_{11}$, and one of the following conditions is satisfied:

(i) if $M_{12}(\lambda)M_{21}(\lambda) \not\equiv I_p$ and $\widetilde{M}_{12}(\lambda)\widetilde{M}_{21}(\lambda) \not\equiv I_p$, then

$$W_{32}(\lambda)W_{22}^{-1}(\lambda) = \widetilde{W}_{32}(\lambda)\widetilde{W}_{22}^{-1}(\lambda), \quad \lambda \in \Omega_{22} \cap \widetilde{\Omega}_{22}; \tag{10}$$

(ii) if $M_{12}(\lambda)M_{21}(\lambda) \equiv I_p$ and $\widetilde{M}_{12}(\lambda)\widetilde{M}_{21}(\lambda) \equiv I_p$, then

$$M_{23}(\lambda) := W_{23}(\lambda)W_{33}^{-1}(\lambda) = \widetilde{W}_{23}(\lambda)\widetilde{W}_{33}^{-1}(\lambda) = \widetilde{M}_{23}(\lambda), \ \lambda \in \Omega_{33} \cap \widetilde{\Omega}_{33}. \tag{11}$$

Then $Q(x) = \widetilde{Q}(x)$ for a.e. $x \in [0,1]$.

III. Hamiltonian (canonical) systems. Let us apply the previous results to the canonical systems of the form

$$J\frac{dz}{dt} = \lambda\mathcal{H}(t)z, \qquad J\frac{d\widetilde{z}}{dt} = \lambda\widetilde{\mathcal{H}}(t)\widetilde{z}, \qquad t \in [0,1]. \tag{12}$$

Here $J = -J^* = -J^{-1}$ is a diagonal signature matrix, and $\mathcal{H}(\cdot)$ and $\widetilde{\mathcal{H}}(\cdot)$ are the nonnegative Hamiltonians, $\mathcal{H}(t)$, $\widetilde{\mathcal{H}}(t) \geq 0$, $t \in [0,1]$.

Let $W_{\mathcal{H}}(\cdot,\lambda)$ and $W_{\widetilde{\mathcal{H}}}(\cdot,\lambda)$ be $n \times n$ fundamental matrix of solutions of these equations satisfying the initial condition

$$W_{\mathcal{H}}(0,\lambda) = W_{\widetilde{\mathcal{H}}}(0,\lambda) = I_n. \tag{13}$$

Finally we denote by $W_{\mathcal{H}}(\lambda) := W_{\mathcal{H}}(1,\lambda)$ and $W_{\widetilde{\mathcal{H}}}(\lambda) := W_{\widetilde{\mathcal{H}}}(1,\lambda)$ monodromy matrices of the first and the second system, respectively.

In the case $n = 2$ the problem of the unique determination of the Hamiltonian by the monodromy matrix has completely been solved by de Branges [3] (see also [4]). Let us recall his classical result.

Theorem 7 ([3])**.** *Let*

$$\mathcal{H}(t) = \begin{pmatrix} h_{11}(t) & h_{12}(t) \\ h_{12}(t) & h_{22}(t) \end{pmatrix} \quad \text{and} \quad \widetilde{\mathcal{H}}(t) = \begin{pmatrix} \widetilde{h}_{11}(t) & \widetilde{h}_{12}(t) \\ \widetilde{h}_{12}(t) & \widetilde{h}_{22}(t) \end{pmatrix}, \tag{14}$$

be real symmetric non-negative and trace normed Hamiltonians, i.e.

$$tr\mathcal{H}(t) = h_{11}(t) + h_{22}(t) = \widetilde{h}_{11}(t) + \widetilde{h}_{22}(t) = tr\widetilde{\mathcal{H}}(t) = 1, \quad t \in [0,1]. \tag{15}$$

If $W_{\mathcal{H}}(\lambda) = W_{\widetilde{\mathcal{H}}}(\lambda)$, then $\mathcal{H}(t) = \widetilde{\mathcal{H}}(t)$ for a.e. $t \in [0,1]$.

Our main uniqueness result for $n \times n$ canonical systems reads as follows.

Theorem 8. *Let $B = B^* \in \mathbb{C}^{n\times n}$ and let $-iB = J|B| = |B|^{1/2}J|B|^{1/2}$ be the polar decomposition of the matrix $-iB$ and let $\mathcal{H}(\cdot)$ and $\widetilde{\mathcal{H}}(\cdot)$ be Hamiltonians admitting representations*

$$\mathcal{H}(t) = |B|^{1/2}P^*(t)P(t)|B|^{1/2}, \quad \widetilde{\mathcal{H}}(t) = |B|^{1/2}\widetilde{P}^*(t)\widetilde{P}(t)|B|^{1/2}, \quad t \in [0,1], \tag{16}$$

where matrix–functions $P(\cdot)$, $\widetilde{P}(\cdot)$, are absolutely continuous, $P(\cdot), \widetilde{P}(\cdot) \in AC[0,1] \otimes \mathbb{C}^{n\times n}$ and B^{-1}–unitary, i.e.

$$P^*(t)B^{-1}P(t) = B^{-1}, \qquad \widetilde{P}^*(t)B^{-1}\widetilde{P}(t) = B^{-1}, \quad t \in [0,1]. \tag{17}$$

Assume also that $P(1) = \widetilde{P}(1)$ and $[P(0), B] = [\widetilde{P}(0), B] = 0$. If $W_{\mathcal{H}}(\lambda) = W_{\widetilde{\mathcal{H}}}(\lambda)$, then $\mathcal{H}(t) = \widetilde{\mathcal{H}}(t)$ for a.e. $t \in [0,1]$.

Remark 9. *(i) If $B = J = J^* = J^{-1}$, then Hamiltonians $\mathcal{H}$ and $\widetilde{\mathcal{H}}$ are symplectic. In this case Theorem 8 improves the Krein uniqueness result. Namely, he assumed in addition that both Hamiltonians $\mathcal{H}$ and $\widetilde{\mathcal{H}}$ are bounded and their entries are real-valued. Certain other uniqueness results for $n \times n$ canonical systems can be found in* [4].

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Spectral information contained in Weyl functions

Ian Wood

(joint work with B.M. Brown, J. Hinchcliffe, M. Marletta, S. Naboko)

The Weyl-Titchmarsh m-function is an important tool in the study of forward and inverse problems for ODEs, as it contains all the spectral information of the problem.

In this talk we will consider the abstract operator M-function or Weyl function which can be introduced using the abstract setting of boundary triples for an adjoint pair of operators. Our aim is to study how much spectral information is still contained in the M-function in this more general setting.

Definition 1. *Two closed, densely defined operators A, $\widetilde{A}$ in a Hilbert space H are an adjoint pair, if $A^* \supseteq \widetilde{A}$ and $\widetilde{A}^* \supseteq A$.*

It is well-known that for each adjoint pair of closed densely defined operators on H, there exist "boundary spaces" $\mathcal{H}$, $\mathcal{K}$ and "trace operators"

$$\Gamma_1 : D(\widetilde{A}^*) \to \mathcal{H}, \quad \Gamma_2 : D(\widetilde{A}^*) \to \mathcal{K}, \quad \widetilde{\Gamma}_1 : D(A^*) \to \mathcal{K} \quad \text{and} \quad \widetilde{\Gamma}_2 : D(A^*) \to \mathcal{H}$$

such that for $u \in D(\widetilde{A}^*)$ and $v \in D(A^*)$ we have an abstract Green formula

$$\left\langle \widetilde{A}^* u, v \right\rangle_H - \left\langle u, A^* v \right\rangle_H = \left\langle \Gamma_1 u, \widetilde{\Gamma}_2 v \right\rangle_{\mathcal{H}} - \left\langle \Gamma_2 u, \widetilde{\Gamma}_1 v \right\rangle_{\mathcal{K}}.$$

The trace operators Γ_1, Γ_2, $\widetilde{\Gamma}_1$ and $\widetilde{\Gamma}_2$ are bounded with respect to the graph norm. The pair (Γ_1, Γ_2) is surjective onto $\mathcal{H} \times \mathcal{K}$ and $(\widetilde{\Gamma}_1, \widetilde{\Gamma}_2)$ is surjective onto $\mathcal{K} \times \mathcal{H}$.

Definition 2. *The collection $\{\mathcal{H} \oplus \mathcal{K}, (\Gamma_1, \Gamma_2), (\widetilde{\Gamma}_1, \widetilde{\Gamma}_2)\}$ is called a boundary triple for the adjoint pair $A, \widetilde{A}$.*

We next fix a realisation of the operator by setting $A_B := \widetilde{A}^*|_{\ker(\Gamma_1 - B\Gamma_0)}$ for $B \in \mathcal{L}(\mathcal{K}, \mathcal{H})$.

Definition 3. *For $\lambda \in \rho(A_B)$, we define*

(1) the solution operator $S_{\lambda,B}$ as a mapping $S_{\lambda,B} : \mathcal{H} \to \ker(\widetilde{A}^* - \lambda)$ where $u = S_{\lambda,B} f$ solves

$$(\widetilde{A}^* - \lambda)u = 0 \quad \textit{and} \quad (\Gamma_1 - B\Gamma_0)u = f,$$

(2) the M-function $M_B(\lambda) : \mathcal{H} \to \mathcal{K}$ via $M_B(\lambda)f = \Gamma_0 S_{\lambda,B} f$.

$\widetilde{S}_{\lambda,B}$ and $\widetilde{M_B}(\lambda)$ are defined analogously using the adjoint operators.

For the symmetric case the answer to the question of how much spectral information is contained in $M_B(\lambda)$ is well-known due to results by Kreĭn and Langer and, in our setting, by Derkach and Malamud: the information contained in $M_B(\lambda)$ corresponds to the completely non-selfadjoint (or simple) part of the operator.

We will therefore be particularly interested in the non-symmetric case. Mimicking the construction of the completely non-selfadjoint subspace of a symmetric operator, we introduce the following space.

Definition 4. *For $\mu_0 \notin \sigma(A_B)$, define the space*

$$\mathcal{S} = \mathrm{Span}_{\delta \notin \sigma(A_B)} (A_B - \delta I)^{-1} \mathrm{Ran}(S_{\mu_0,B}).$$

We call its closure $\overline{\mathcal{S}}$ the detectable subspace.

Again, there is a space $\widetilde{\mathcal{S}}$, defined analogously using the adjoint.

The detectable subspace is independent of μ_0 and B and is a regular invariant subspace for the resolvent of A_B.

It turns out that under mild assumptions on $\sigma(A_B)$ the M-function $M_B(\lambda)$ is analytic at a point λ_0 if and only if $P_{m,\widetilde{\mathcal{S}}}(A_B - \lambda I)^{-1} P_{n,\mathcal{S}}$ is analytic at a point λ_0. Here, $P_{n,\mathcal{S}}$ and $P_{m,\widetilde{\mathcal{S}}}$ denote projections onto any finite-dimensional subspaces of $\mathcal{S}$ and $\widetilde{\mathcal{S}}$. Under some additional assumptions, we even have that the finite-dimensional projections bordering the resolvent can be replaced by projections onto $\overline{\mathcal{S}}$ and $\overline{\widetilde{\mathcal{S}}}$, respectively.

We will conclude the talk by considering the problem of determining the detectable subspace in several examples. The first is the very simple case of the 1d-Schrödinger operator. As expected, we have:

Proposition 5. *For the Schrödinger operator on the interval $(0, 1)$ with bounded potential we have $\overline{\mathcal{S}} = L^2(0, 1)$.*

Our second example is a matrix differential operator

$$\widetilde{A}^* = \begin{pmatrix} -\frac{d^2}{dx^2} + q(x) & \widetilde{w}(x) \\ w(x) & u(x) \end{pmatrix},$$

where q, u, w and $\widetilde{w}$ are $L^\infty(0, 1)$-functions.

In the special case when the product of the off-diagonal elements vanishes, we can determine $\overline{\mathcal{S}}$. Even for this simple special case, the answer is not straightforward.

Theorem 6. *Assume $w\widetilde{w} = 0$, that $\theta(\cdot, \lambda), \phi(\cdot, \lambda)$ are a fundamental system for $-y'' + (q - \lambda)y = 0$ and set*

$$E_{u,w} := \mathrm{Span}_{n\in\mathbb{N}} w(x)\theta(x, u(x))u^n(x) + \mathrm{Span}_{n\in\mathbb{N}} w(x)\phi(x, u(x))u^n(x).$$

Then

$$\mathcal{S}^{\perp} = \left\{ \begin{pmatrix} h \\ g \end{pmatrix} : g \perp E_{u,w}, \right.$$
$$\left. h(x) = \int_0^x (wg)(t)[\phi(t,u(t))\theta(x,u(t)) - \theta(t,u(t))\phi(x,u(t))]dt \right\}$$

In particular,

$$\overline{\mathcal{S}} = \begin{pmatrix} L^2(0,1) \\ \chi_{\{w\neq 0\}} L^2(0,1) \end{pmatrix}$$

iff $E_{u,w} = \chi_{\{w\neq 0\}} L^2(0,1)$.

Finally, we consider the Friedrichs model, which is a perturbed multiplication operator on $L^2(\mathbb{R})$: Let $\phi, \psi \in L^2(\mathbb{R})$ and

$$(Af)(x) = xf(x) + \langle f, \phi\rangle \psi(x),$$

with domain

$$D(A) = \left\{ f \in L^2(\mathbb{R}) \,\middle|\, xf(x) \in L^2(\mathbb{R}), \quad \lim_{R\to\infty} \int_{-R}^{R} f(x)dx = 0 \right\}.$$

Let $\widetilde{A}$ be defined in the same way with the roles of ϕ and ψ exchanged. Then

$$\widetilde{A}^* f = xf(x) - c_f \mathbf{1} + \langle f, \phi\rangle \psi(x),$$

with

$$D(\widetilde{A}^*) = \left\{ f \in L^2(\mathbb{R}) \,|\, \exists c_f \in \mathbb{C} : xf(x) - c_f\mathbf{1} \in L^2(\mathbb{R}) \right\}.$$

We obtain a boundary triple for the pair $A, \widetilde{A}$ by setting

$$\Gamma_1 f = \lim_{R\to\infty} \int_{-R}^{R} f(x)dx \quad \text{and} \quad \Gamma_0 f = c_f.$$

We again only consider a special case, namely when ϕ and ψ have disjoint support. In the following, we denote

$$\widehat{f}(k \pm i0) := \lim_{\epsilon\to 0} \int_{\mathbb{R}} \frac{f(x)}{x - (k \pm i\epsilon)} dx.$$

Theorem 7. *Let* $\phi \cdot \psi = 0$ *and* $\Omega_1 := \{k \in \mathbb{R} : \ \widehat{\overline{\phi}}(k - i0)\psi(k) = 1\}$.

(i) If Ω_1 *has zero measure, then* $\overline{\mathcal{S}} = L^2(\mathbb{R})$.

(ii) If Ω_1 *has non-zero measure, then* $\overline{\mathcal{S}}^{\perp} \neq \{0\}$ *and*

$$\overline{\mathcal{S}} = \{f \in L^2(\mathbb{R}) : \ f(k) = \psi(k)\widehat{\left(f\overline{\phi}\right)}(k - i0) \ \text{on } \Omega_1\}.$$

In this case, $\dim \mathcal{S}^{\perp} = \infty$.

We have the following results on complete detectability, i.e. $\overline{\mathcal{S}} = L^2(\mathbb{R})$.

Theorem 8. *Assume* $\phi \cdot \psi = 0$, *then complete detectability is generic: Replace* ψ *by* $\alpha\psi$ *for* $\alpha \in \mathbb{C}$ *and denote the corresponding detectable subspace by* $\overline{\mathcal{S}}_\alpha$. *Then*

- *for all α outside a countable set E_0 we have $\overline{\mathcal{S}}_\alpha = L^2(\mathbb{R})$,*
- *for sufficiently small $|\alpha|$ we have $\overline{\mathcal{S}}_\alpha = L^2(\mathbb{R})$.*

Many more results on the detectable subspace for the Friedrichs model can be found in [2].

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The Connection Problem for Solutions of Sturm-Liouville Problems with two singular endpoints, and its relation to m-Functions

Charles T. Fulton

(joint work with Heinz Langer, Annemarie Luger, Steven Pruess, David Pearson)

In his seminal paper [22, p. 230] Hermann Weyl made the following assertion:

> Da sich Gleichungen vom Grenzkreistypus in jeder Hinsicht wie Gleichungen ohne Singularitäten verhalten, hat man danach den Grenzkreisfall als den regulären aufzufassen.

It has since become well known that for problems with two singular endpoints, one **LC** endpoint is sufficient to guarantee a simple spectrum. Henceforth we restrict attention to **SL** equations on a doubly singular interval (a, b) which have the **LP** case at each endpoint. We show that some LP endpoints are also rather similar to regular endpoints in the sense that they also guarantee a simple spectrum, irrespective of the type of **LP** endpoint at the other singular endpoint. It is well known that every singular endpoint of a Sturm-Liouville equation must belong to exactly one of the following five mutually exclusive endpoint classifications: $\mathbf{LP/N}, \mathbf{LP/O}, \mathbf{LP/O-N}, \mathbf{LC/N}, \mathbf{LC/O}$ [21, p.144]. If the left endpoint $x = a$ is of type **LP/N** (that is, **LP** and Nonoscillatory for all real λ) then there exists a nontrivial solution $\varphi(x, \lambda)$ of the **SL** equation satisfying the following properties (see Fulton, Langer, Luger [4, Assumption A, p.1793] and Gezstesy, Zinchenko [1, Hypothesis 3.1, p.1058]): (i) φ is entire in λ for each $x \in (a, b)$, (ii) φ is square integrable near $x = a$, and (iii) φ is real for all $x \in (a, b)$ when λ is real. Under these assumptions ([4, p.1795] or [1, p.1065]) the eigenfunction expansion for $(py')' + (\lambda r - q)y = 0$ has the form

$$g(x) = \int_a^b \mathcal{F}_g(\lambda)\varphi(x, \lambda)d\rho(\lambda), \tag{1}$$

where $\rho(\lambda)$ is a scalar spectral function, and $\mathcal{F}$ is the *Fourier transform* of g.

The Sturm-Liouville Connection Problem

Under the **LP/N** assumption at $x = a$ it is possible to select a fundmental system of solutions $\{\varphi, \psi\}$ which are both entire in λ, real on the real λ-axis, with φ satisfying the above three properties at $x = a$, and which are normalized by $p(x) \cdot W_x(\varphi, \psi) = 1$. For $Im(z) \neq 0$, let $\Psi(x, z)$ be the solution which is square integrable at the **LP** endpoint $x = b$. Then the **Sturm-Liouville Connection Problem** is the problem of finding the (complex-valued) constants $C_1(z)$ and $C_2(z)$ in the relation of linear dependence (for all $x \in (a, b)$ and all $Im(z) \neq 0$),

$$\Psi(x, z) = C_1(z)\varphi(x, z) + C_2(z)\psi(x, z). \tag{2}$$

This is often a special case of the so-called **Central Connection Problem** studied by W. Balser [14, p.193-196] and many others (compare [19] for the related idea of Stokes multipliers). The Weyl Function arises from (2) as

$$m(z) = -C_1(z)/C_2(z). \tag{3}$$

For all cases investigated so far (which also have the left endpoint being a R.S.P.), the $m - functions$ arising from the above connection formula approach have been found to be generalized Nevanlinna functions (see Krein and Langer [20]), which are not Herglotz; see [2, 3, 4, 6] (Bessel, Associated Legendre, H-atom equations), K.L. Schmidt [12] (extension to a Dirac operator), and (the earliest cases) Derkach [16], Dijksma and Shondin [17, 18]) (Bessel, Laguerre equations). See also recent work on theory and application of cases with two **LP** endpoints by G. Teschl, A. Kostenko, A. Sakhnovich, P. Kurasov, A. Luger, L. Silva, and J. Toleza [7, 8, 9, 10, 11]. **Open Problem**: Formulate satisfactory normalizations of the fundamental system $\{\varphi, \psi\}$ near endpoints of **LP/N** type, which are not regular singular points, for example, Lennard-Jones potentials near $x = 0$: $q(x) = 4\epsilon\left[(\sigma/x)^{12} - (\sigma/x)^6\right], \epsilon > 0, \sigma > 0$.

The Appell Equations and Spectral Density Functions

For problems where the left endpoint is Regular or a R.S.P. of type **LC/N** or **LP/N**, and the right endpoint, $x = \infty$, is $\mathbf{LP/O-N}$ with cutoff $\Lambda = 0$, and q is absolutely integrable near $+\infty$, Fulton, Pearson and Pruess [5] have recently obtained the following characterization of the spectral density function:

$$f(\lambda) = \frac{1}{\pi[P(x,\lambda)\varphi(x,\lambda)^2 + Q(x,\lambda)\varphi(x,\lambda)\varphi'(x,\lambda) + R(x,\lambda)\varphi'(x,\lambda)^2]}, \tag{4}$$

where $(P, Q, R)^T$ is the unique solution of the I.V.P. for the $APPELL$ system

$$\frac{d}{dx}\begin{bmatrix} P \\ Q \\ R \end{bmatrix} = \begin{bmatrix} 0 & \lambda - q & 0 \\ -2 & 0 & 2(\lambda - q) \\ 0 & -1 & 0 \end{bmatrix} \cdot \begin{bmatrix} P \\ Q \\ R \end{bmatrix}, \tag{5}$$

$$\lim_{x\to\infty} (P(x,\lambda), Q(x,\lambda), R(x,\lambda)) = \left(\sqrt{\lambda}, 0, \frac{1}{\sqrt{\lambda}}\right), \qquad \lambda \in (0, \infty). \tag{6}$$

In (4) the solution $\varphi(x.\lambda)$ is a solution which is entire in λ and (i) satisfying a Regular or **LC** boundary condition at the left endpoint, or (ii) satisfying the above three properties at a left endpoint of type **LP/N**. Numerical algorithms using (4) which achieve near machine precision accuracy were given in [5]. Some applications where such computations are of importance arise in quantum chemistry, Brändas et al [15], [5, p.641-642] and in plasma physics, Wilkening and Cerfon [13].

The importance of investigations on spectral multiplicity for ordinary differential operators (particularly higher order and Hamiltonian systems, where the problems are mostly wide-open) was emphasized by Naimark in his 1952 book, Linear Differential Operators (p. 133):

> The determination of a minimal generating basis for various classes of differential operators and consequently the determination of the multiplicity of the spectrum in relation to the properties of the coefficients in the differential expression is *one of the most important problems in the theory of differential operators.*

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Dirac systems with rational data: explicit formulas and related nonlinear equations

Marinus A. Kaashoek

The research reported on has its roots in joint work with the late Israel Gohberg and Alexander Sakhnovich (Vienna), which started in the nineties. The emphasis is on "early results" rather than on the most recent ones. A main theme is the use of the state space method of mathematical system theory in solving direct and inverse problems for differential systems with rational data. The differential systems will be of Dirac type, selfadjoint or skew-selfadjoint, continuous variable or discrete variable. These systems are also known as canonical or pseudo-canonical systems, Zakharov-Shabat systems, AKNS systems (Ablowitz, Kamp, Newell, Segur). Amongst other things they serve as auxiliary systems for integrable nonlinear PDEs.

Selfadjoint continuous variable setting. Let us begin with a selfadjoint Dirac system on the positive half line $\mathbb{R}_+$:

$$\frac{d}{dx}y(x,z) = \big(ijz + ijV(x)\big)y(x,z), \quad x \in \mathbb{R}_+,\ z \in \mathbb{C},$$

$$j = \begin{bmatrix} I_m & 0 \\ 0 & -I_m \end{bmatrix}, \quad V(x) = \begin{bmatrix} 0 & v(x) \\ v(x)^* & 0 \end{bmatrix}.$$

Here I_m is the $m \times m$ identity matrix, and v is an $m \times m$ matrix function which (by some abuse of terminology) will be referred to as the *potential* of the system. Note that the $2m \times 2m$ matrix $V(x)$ is selfadjoint for each $x \in \mathbb{R}_+$, and hence the operator $H := -ij\frac{d}{dx} - V(x)$ is formally selfadjoint.

Now let us introduce the class of potentials v we shall be dealing with in this selfadjoint continuous variable setting. We start with two matrices γ_1 and γ_2, both of size $n \times m$, and a square matrix β of size $n \times n$ such that $\beta^* - \beta = i\gamma_2\gamma_2^*$. Using these three matrices we define

$$A = \begin{bmatrix} \beta^* & \gamma_2\gamma_2^* \\ 0 & \beta \end{bmatrix}, \quad B = \begin{bmatrix} \gamma_2 \\ \gamma_1, \end{bmatrix}, \quad C = \begin{bmatrix} \gamma_1^* & \gamma_2^* \end{bmatrix}, \tag{1}$$

$$P = \begin{bmatrix} I_n & -iI_n \\ 0 & 0 \end{bmatrix}, \quad A^\times = A - BC.$$

We shall assume that the potential v is given by

$$v(x) = -2C\Big(Pe^{-2ixA^\times}|_{\mathrm{Im}\ P}\Big)^{-1} PB, \quad x \geq 0. \tag{2}$$

The fact that $\beta^* - \beta = i\gamma_2\gamma_2^*$ guarantees that the linear map $Pe^{-2ixA^\times}|_{\mathrm{Im}\ P}$, acting on the range of P, is invertible, and hence the right hand side of (2) is well-defined. We call v defined by (2) the *pseudo-exponential potential generated by* γ_1, γ_2, and β, and we shall denote this class of potentials by the acronym PE1. The pseudo-exponential potentials have their roots in a 1995 paper of Alpay-Gohberg [1]. More precisely, they are generalizations of the potentials considered in [1]. The following theorem [2, Theorem 4.3] solves the direct spectral problem.

Theorem 1. *Let v be the pseudo-exponential potential generated by γ_1, γ_2, and β, and let $z_1, \ldots, z_p$ be the real eigenvalues of β. Put*

$$\begin{aligned} \nu_k &= 2\pi \mathrm{res}_{z=z_k} \gamma_1^*(zI_n - \beta)^{-1}\gamma_1, \quad k = 1, \ldots, p, \\ W(z) &= I_m + C(zI_{2n} - A)^{-1}B, \end{aligned}$$

where the matrices A, B, and C are qiven by (1). *Then the matrix function*

$$\tau(z) = \int_0^z W(t)\,dt + \sum_{z_k < z} \nu_k$$

is the nondecreasing piecewise absolutely continuous spectral function of the selfadjoint Dirac system with v as its potential. In particular, the rational matrix function W has no poles on $\mathbb{R}$ and is positive semi-definite on $\mathbb{R}$.

The representation (2) of a potential v in PE1 is very useful in the theory of integrable nonlinear equations. Indeed, as the following theorem [2, Theorem 6.1] shows, by adding an additional parameter t we obtain explicit solutions of some of the classical matrix-valued nonlinear equations.

Theorem 2. *Let γ_1 and γ_2 be $n \times m$ matrices, and let β be an $n \times n$ matrix with the additional property $\beta^* - \beta = i\gamma_2\gamma_2^*$. Put*

$$v(x,t) = 2\gamma_1^* \left([I_n \quad -iI_n] e^{-2i(xA^\times + t(A^\times)^k)} \begin{bmatrix} I_n \\ 0 \end{bmatrix} \right)^{-1} (i\gamma_1 - \gamma_2),$$

where k is a positive integer and

$$A^\times = \begin{bmatrix} \alpha^* & 0 \\ -\gamma_1\gamma_1^* & \alpha \end{bmatrix} \quad \textit{with} \quad \alpha := \beta^\times = \beta - \gamma_1\gamma_2^*.$$

Then for each t in a sufficiently small interval $0 \le t < \varepsilon$ the function $v(x,t)$ is a pseudo-exponential potential from the class PE1. Moreover, they give solutions of the matrix nonlinear Schrödinger equation

$$2\frac{\partial v}{\partial t} + i\frac{\partial^2 v}{\partial x^2} = 2ivv^* v$$

if $k = 2$, and of the matrix modified Korteweg-de Vries equation

$$4\frac{\partial v}{\partial t} + \frac{\partial^3 v}{\partial x^3} = 3\left(\frac{\partial v}{\partial x}v^* v + vv^* \frac{\partial v}{\partial x}\right)$$

if $k = 3$.

If v is a pseudo-exponential potential for a selfadjoint Dirac system generated by the matrices γ_1, γ_2, and β, then v can also be written in the form

$$v(x) = 2\theta_1^* e^{ix\alpha^*} S(x)^{-1} e^{ix\alpha}\theta_2, \quad x \ge 0, \tag{3}$$

where $\alpha = \beta - \gamma_1\gamma_2^*$, $\theta_1 = \gamma_1$, $\theta_2 = i\gamma_1 - \gamma_2$, and

(*a*) $\alpha - \alpha^* = i(\theta_1\theta_1^* - \theta_2\theta_2^*)$,

(*b*) $S(x) = I_n + \int_0^x \Lambda(t)\Lambda(t)^*\, dt$, with $\Lambda(x) = \begin{bmatrix} e^{-ix\alpha}\theta_1 & e^{ix\alpha}\theta_2 \end{bmatrix}$.

Skew-selfadjoint continuous variable setting. Next we consider a skew-selfadjoint Dirac system:

$$\frac{d}{dx}y(x,z) = \big(ijz + jV(x)\big)y(x,z), \quad x \in \mathbb{R}_+,\ z \in \mathbb{C},$$

$$j = \begin{bmatrix} I_m & 0 \\ 0 & -I_m \end{bmatrix}, \quad V(x) = \begin{bmatrix} 0 & v(x) \\ v(x)^* & 0 \end{bmatrix}.$$

We assume that the potential v belongs to the class PE2, that is, v is given by

$$v(x) = 2\theta_1^* e^{ix\alpha^*} S(x)^{-1} e^{ix\alpha}\theta_2, \quad x \ge 0. \tag{4}$$

where

(c) α is an $n \times n$ matrix, θ_1 and θ_2 are matrices of size $n \times m$ and S_0 is an $n \times n$ positive definite matrix satisfying the Lyapunov equation

$$\alpha S_0 - S_0\alpha^* = i(\theta_1\theta_1^* + \theta_2\theta_2^*);$$

(d) $S(x)$ is the $n \times n$ matrix function given by

$$S(x) = S_0 + \int_0^x \Lambda(t)j\Lambda(t)^* dt, \quad \Lambda(x) = \begin{bmatrix} e^{-ix\alpha}\theta_1 & e^{ix\alpha}\theta_2 \end{bmatrix}.$$

Although formulas (2) and (4) are identical, the classes PE1 and PE2 are quite different. This follows from the differences between items (c) and (d) on the one hand and the preceding items (a) and (b) on the other hand. In the sequel an ordered set $\Sigma := \{\alpha, S_0, \theta_1, \theta_2\}$ is called an *admissible quadruple* whenever item (c) above is fulfilled. In that case $S(x)$ in item (d) is well-defined and positive definite for each $x \geq 0$, and we refer to the function v defined by (4) as the *pseudo-exponential potential generated by the admissible quadruple* $\Sigma = \{\alpha, S_0, \theta_1, \theta_2\}$. The notion of an admissible quadruple is closely related to the notion of a so-called symmetric S-node; see Chapter 2 of [6].

It turns out the Weyl function φ of a skew-selfadjoint Dirac system with the pseudo-exponential potential v determined by an admissible quadruple is a strictly proper rational matrix function. More precisely, the following theorem [3, Theorem 2.1] holds.

Theorem 3. *Assume that the pseudo-exponential potential v of the skew-selfadjoint Dirac system is generated by the admissible quadruple $\{\alpha, S_0, \theta_1, \theta_2\}$. Then the system has a unique Weyl function φ which is given by*

$$\varphi(z) = i\theta_2^* S_0^{-1}(zI_n - \alpha^\times)^{-1}\theta_1, \quad \alpha^\times := \alpha - i\theta_1\theta_1^* S_0^{-1}.$$

In [3, Theorem 2.1] the matrix S_0 is the $n \times n$ identity matrix. This additional condition is not essential; see [5, Theorem 2.5]. The following theorem solves the corresponding inverse problem.

Theorem 4. *Let φ be a strictly proper rational $m \times m$ matrix function. Then φ is the Weyl function of a skew-selfadjoint Dirac system with a pseudo-exponential potential v. Moreover, the potential v can be be explicitly recovered from φ using the procedure described below.*

STEP 1. *Let n be the McMillan degree of φ, and construct a minimal realization of φ:*

$$\varphi(z) = i\beta_2^*(zI_n - \gamma)^{-1}\beta_1.$$

STEP 2. *Consider the algebraic Riccati equation*

$$\gamma X - X\gamma^* - iX\beta_2\beta_2^* X + i\beta_1\beta_1^* = 0,$$

and use the fact (Kalman-Falb-Arbib, 1969) that this equation has a unique positive definite solution X.

STEP 3. *Put*

$$S_0 = I_n, \quad \theta_1 = X^{-1/2}\beta_1, \quad \theta_2 = X^{1/2}\beta_2,$$

$$\alpha = X^{-1/2}\gamma X^{1/2} + i\theta_1\theta_1^*.$$

Then $\alpha - \alpha^* = i(\theta_1\theta_1^* + \theta_2\theta_2^*)$. Thus $\Sigma = \{\alpha, S_0, \theta_1, \theta_2\}$ is an admissible quadruple and φ is the Weyl function of the skew-selfadjoint Dirac system of which the pseudo-exponential potential is generated by Σ.

Theorem 2 has a natural analogue in the skew-selfadjoint case; see Theorem 4.1 in [3].

Skew-selfadjoint Dirac systems in a discrete variable setting. More recent research [4] and [5] deals with pseudo-exponential potentials for discrete variable skew-selfadjoint Dirac systems:

$$y_{k+1}(z) = \Big(I_{2m} + iz^{-1}C_k\Big)y_k(z), \quad k \in \mathbb{N}_0,\ z \in \mathbb{C},$$

$$C_k = U_k^* j U_k, \quad U_k \text{ unitary } 2m \times 2m \text{ matrix.}$$

The sequence $\{C_k\}_{k\in\mathbb{N}_0}$ is called the *potential*; in math physics the term spin sequence is used. Note that $C_k = C_k^* = C_k^{-1}$ for all $k \in \mathbb{N}_0$.

To define a pseudo-exponential potential in the discrete case we start with an admissible quadruple $\Sigma = \{\alpha, S_0, \theta_1, \theta_2\}$, which is now assumed to be strongly admissible, i.e., the pair $\{\alpha, \theta_1\}$ is required to be controllable. The latter implies that the eigenvalues of α all belong to the open upper half plane $\mathbb{C}_+$. Put

$$\begin{aligned}
\Sigma_k &= \{\alpha, S_k, (I_n + i\alpha^{-1})^k\theta_1, (I_n - i\alpha^{-1})^k\theta_2\},\\
\Lambda_k &= \big[(I_n + i\alpha^{-1})^k\theta_1 \quad (I_n - i\alpha^{-1})^k\theta_2\big],\\
S_k &= \int_{-\infty}^{\infty} (\lambda I_n - \alpha)^{-1}\Lambda_k\Lambda_k^*(\lambda I_n - \alpha^*)^{-1}\, d\lambda.
\end{aligned}$$

Then the quadrupple Σ_k is again strongly admissible. The sequence

$$C_k = j + \Lambda_k^* S_k^{-1}\Lambda_k - \Lambda_{k+1}^* S_{k+1}^{-1}, \quad k \in \mathbb{N}_0,$$

is called the *pseudo-exponential potential generated by the strongly admissible quadruple* $\Sigma = \{\alpha, S_0, \theta_1, \theta_2\}$.

It turns out (see [5, Theorems 3.8 and 3.9]) that the solutions of the direct and inverse problems for Weyl functions in the discrete variable skew-selfadjoint setting are very similar to those for the continuous variable case. Indeed (see [5, Theorem 3.8]), for a discrete variable skew-selfadjoint Dirac system with a pseudo-exponential potential generated by the strongly admissible quadruple $\Sigma = \{\alpha, S_0, \theta_1, \theta_2\}$ the Weyl function is given by the strictly proper rational matrix function

$$\varphi(z) = -i\theta_1^* S_0^{-1}(zI_n + \gamma)^{-1}\theta_2, \quad \gamma := \alpha - i\theta_2\theta_2^* S_0^{-1}.$$

Conversely (see [5, Theorem 3.9]), any strictly proper rational matrix function appears as the Weyl function of such a system, and the procedure to solve the inverse problem is analogous to the procedure describe in Theorem 4.

Analogous to the continuous variable case, skew-selfadjoint discrete variable systems with a pseudo-exponential potential depending on an additional continuous time parameter lead to explicit solutions of discrete integrable nonlinear equations. For the scalar the nonlinear equations correspond to the isotropic Heisenberg magnet model [4]. For matrix-valued, possibly non-square, pseudo-exponential potentials the equations involved are related to the generalized discrete Heisenberg model [5, Section 4].

Finally, the results presented in this talk are exemplary for a range of analogous results involving other classes of functions (e.g., scattering functions, reflection coefficients, transmission coefficients), or other classes of systems (e.g., Sturm-Liouville systems or full line systems). The references can be found in the book [7].

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On the asymptotic spectral density of one-dimensional Dirac operators

Karl Michael Schmidt

The radial Dirac operator

$$-i\sigma_2 \frac{d}{dr} + m\,\sigma_3 + \frac{k}{r}\,\sigma_1 + q(r), \qquad 0 < r < \infty,$$

with mass $m \in \mathbb{R}$, angular momentum quantum number $|k| \geq \frac{1}{2}$ and real-valued, locally integrable potential q is known to have purely absolutely continuous spectrum covering the real line if $\lim_{r\to\infty} q(r) = -\infty$ and the Erdélyi condition $\int^{\infty} \frac{|q'|}{q^2} < \infty$ is satisfied [2], [3]. In [1], the question of whether the spectral density has local maxima as the spectral parameter tends to ∞ ("high-energy points of spectral concentration") was studied, and it is the purpose of the present work to give an answer to this question.

Our approach to the spectral density does not use the Weyl-Titchmarsh m function, but is based on the oscillation method for real spectral parameter only. Although the problem has 2 singular end-points, there is a single spectral function and a spectral expansion in terms of one solution. The key is the following set of observations.

Theorem 1 (see [1], Theorem 1). *Let $k \geq \frac{1}{2}$, $q \in L^1_{\mathrm{loc}}[0,\infty)$. Then, for each $\lambda \in \mathbb{R}$,*

$$(-i\sigma_2 \frac{d}{dr} + m\,\sigma_3 + \frac{k}{r}\,\sigma_1 + q(r))\,u(r) = \lambda u(r)$$

has a unique $\mathbb{R}^2$*-valued solution* $w(r,\lambda) = \begin{pmatrix} o(1) \\ 1+o(1) \end{pmatrix} r^k$ $(r \to 0)$. *When we introduce Prüfer variables* $w = |w| \begin{pmatrix} \sin\vartheta \\ \cos\vartheta \end{pmatrix}$, *the angle satisfies* $\lim_{r\to 0} \vartheta(r,\lambda) = 0$, $\lim_{r\to 0} \frac{\partial}{\partial\lambda}\vartheta(r,\lambda) = 0$.

Lemma 2 (see [1], Corollary 1; Theorem 2)**.** *With* w, ϑ *as above,*

$$\frac{\partial\vartheta}{\partial\lambda}(r,\lambda) = \frac{1}{|w(r,\lambda)|^2} \int_0^r |w(t,\lambda)|^2\,dt \qquad (r>0, \lambda \in \mathbb{R});$$

$$\frac{\partial\vartheta}{\partial\lambda}(r,\lambda) = r(1+o(1)) \qquad (\lambda \to \infty).$$

By a spectral averaging argument, combined with the fact that, under the above assumptions on q, the limit $|w(\infty,\lambda)| := \lim_{r\to\infty} |w(r,\lambda)|$ exists, this can be used to show that the spectral expansion formula with respect to Φ, the canonical fundamental system at an intermediate point $c \in (0,\infty)$, is given by

$$f(x) = \int_{\mathbb{R}} \Phi(x,\lambda)\, \frac{w(c,\lambda)w(c,\lambda)^T}{\pi\,|w(\infty,\lambda)|^2} \int_0^\infty \Phi(y,\lambda)^T f(y)\,dy\,d\lambda \quad (f \in L^2(0,\infty)^2).$$

Hence, considering a fundamental system $\Psi = (w \mid z)$, where z is any linearly independent solution, a straightforward calculation gives

$$f(x) = \int_{\mathbb{R}} w(x,\lambda) \int_0^\infty w(y,\lambda)^T f(y)\,dy\, \frac{d\lambda}{\pi\,|w(\infty,\lambda)|^2} \quad (f \in L^2(0,\infty)^2).$$

It remains to study, for large λ, the spectral density

$$\frac{1}{\pi\,|w(\infty,\lambda)|^2} = \frac{1}{\pi} \exp\left(-2\log|w(c,\lambda)| - 2\int_c^\infty (\frac{k}{s}\cos 2\vartheta(s,\lambda) + m \sin 2\vartheta(s,\lambda))\,ds \right).$$

By repeated integrations by parts, it was shown in [1] that

$$\frac{d}{d\lambda} \int_c^\infty (\frac{k}{s}\cos 2\vartheta(s,\lambda) + m\sin 2\vartheta(s,\lambda))\,ds = -\frac{mc\sin 2\vartheta(c,\lambda) + k\cos 2\vartheta(c,\lambda)}{\lambda - q(c)} + o(\frac{1}{\lambda})$$

$(\lambda \to \infty)$, provided q satisfies the additional hypotheses

$$q, q' \in AC_{\mathrm{loc}}(0,\infty), -q(r) \geq Cr^a$$

for some $C, a > 0$, and either

$$(P) \qquad \exists \tilde{C} > 0 : |q^{(k)}(r)| \leq \tilde{C} r^{a-k} \quad (k \in \{1,2\})$$

or

$$(E) \qquad \begin{cases} \exists \delta > 0 : \frac{rq^{(k)}}{|q|^{1+\delta}} \in L^1(0,\infty) \quad (k \in \{1,2\}) \\ \forall \varepsilon > 0 : \frac{q'(r)}{|q(r)|^{1+\varepsilon}} = O(1) \quad (r \to \infty). \end{cases}$$

If q is constant (w.l.o.g., $q = 0$) on $[0, c]$, symmetries of the differential equation can be used to show that

$$\frac{\partial}{\partial\lambda} \log |w(r, \lambda)| = -\frac{k}{\lambda} + \frac{k}{\lambda} \cos 2\vartheta(r, \lambda) + \frac{rm}{\lambda} \sin 2\vartheta(r, \lambda) + O(\frac{1}{\lambda^2}) \qquad (\lambda \to \infty).$$

Combining these asymptotics, we find

$$\frac{d}{d\lambda} \frac{1}{\pi |w(\infty, \lambda)|^2} = \left(\frac{2k}{\lambda} + o(\frac{1}{\lambda})\right) \frac{1}{\pi |w(\infty, \lambda)|^2} \qquad (\lambda \to \infty),$$

which shows that the spectral density is eventually strictly monotonic, so there are no high-energy points of spectral concentration, and that the spectral density has asymptotic estimates

$$\lambda^{2k-\varepsilon} << \frac{1}{\pi |w(\infty, \lambda)|^2} << \lambda^{2k+\varepsilon} \qquad (\lambda \to \infty)$$

for all $\varepsilon > 0$.

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Remarks on the convergence of pseudospectra

Sabine Bögli

(joint work with Petr Siegl)

For $\varepsilon > 0$ the ε-pseudospectrum of a closed operator T acting in a Banach space X is defined as the set

$$\sigma_\varepsilon(T) := \sigma(T) \cup \left\{\lambda \in \rho(T) \, : \, \|(T - \lambda)^{-1}\| > \frac{1}{\varepsilon}\right\}. \tag{1}$$

While the ε-pseudospectrum of a normal operator in a Hilbert space coincides with the open ε-neighbourhood of the spectrum, the situation is more involved in the non-normal case or for operators acting in Banach spaces, see the classical book by Trefethen and Embree [8]. In this talk we address the convergence of pseudospectra for sequences of unbounded operators and the related problem when the resolvent norm of an operator T may be constant on an open subset of the resolvent set $\rho(T)$.

In view of applications in PDEs, e.g. the domain truncation method where the operators T and T_k act in different Hilbert spaces H and H_k, we employ the so-called generalised norm resolvent convergence where the spaces are assumed to be subspaces of a larger Hilbert space and the projected resolvents converge in norm,

$$\exists \lambda_0 \in \bigcap_{k\in\mathbb{N}} \rho(T_k) \cap \rho(T) : \quad \|(T_k - \lambda_0)^{-1} P_{H_k} - (T - \lambda_0)^{-1} P_H\| \longrightarrow 0. \tag{2}$$

The first main result we present is the pseudospectral convergence (in Hausdorff distance) for a sequence of operators that converges in generalised norm resolvent sense. This is a generalisation of Hansen's result [4] for a sequence of operators acting in the same space (and converging in gap topology which is equivalent to norm resolvent convergence).

The pseudospectral convergence result relies on the condition that the limiting operator T does not have constant resolvent norm on an open set; for classes of operators which do not a priori satisfy this condition, it needs to be guaranteed by assumption.

Whether the resolvent norm of a bounded operator in a Banach space can be constant on an open set was first studied by Globevnik [3]. He showed that this cannot happen in the unbounded component of the resolvent set. Since then, the occurrence of constant resolvent norm on an open set has been studied in particular by Shargorodsky et al. [5, 7, 6, 2]. The occurrence has been excluded if the Banach space X satisfies certain convexity properties (that are satisfied for Hilbert spaces and L^p-spaces with $1 \leq p \leq \infty$) and the operator T acting in X is *i*) bounded, or *ii*) the generator of a C_0-semigroup, or *iii*) densely defined with compact resolvent.

On the other hand, several examples are known in which the resolvent norm is constant on an open set; there are examples of bounded operators in carefully chosen Banach spaces that violate the convexity assumptions, but there exist also examples of unbounded operators in Hilbert spaces.

As the second main result, we prove that if a closed operator T (acting in a Hilbert space or, more generally, in a complex uniformly convex Banach space) has constant resolvent norm on an open set, then this constant is the global minimum. As a consequence, a resolvent norm decay (along some path in $\rho(T)$ tending to infinity) is a sufficient condition for excluding constant resolvent norm on an open set. This applies in particular if T is bounded or generates a C_0-semigroup, thus generalising Shargorodsky's conditions *i*) and *ii*) above.

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Some remarks on Weyl–Titchmarsh and Donoghue m-functions

Fritz Gesztesy

(joint work with S. Clark, S. Naboko, R. Nichols, R. Weikard, and M. Zinchenko)

We report on recent results on m-functions for Schrödinger operators based on [2] and [3]. For brevity, we will focus on the case of scalar potentials only.

Starting with the half-line case, we asssume $V \in L^1([0,R])$ for all $R > 0$, V real-valued, and introduce the differential expression $\tau_+ = -\frac{d^2}{dx^2} + V(x)$, $x \geq 0$. To avoid boundary conditions at ∞, we suppose that τ_+ is in the limit point case at ∞ and denote the self-adjoint operator in $L^2([0,\infty))$ associated with τ_+ and the boundary condition $\sin(\alpha)g'(0_+) + \cos(\alpha)g(0_+) = 0$ by $H_{+,\alpha}$, $\alpha \in [0,\pi)$.

Next, let $\phi_\alpha(z,\,\cdot\,)$, $\theta_\alpha(z,\,\cdot\,)$, $z \in \mathbb{C}$, be a normalized fundamental system of solutions of $\tau_+\psi = z\psi$ given by,

$$\phi_\alpha(z,0) = -\theta'_\alpha(z,0) = -\sin(\alpha), \quad \phi'_\alpha(z,0) = \theta_\alpha(z,0) = \cos(\alpha), \quad \alpha \in [0,\pi).$$

(Here $' \equiv d/dx$.) Weyl–Titchmarsh solutions $\psi_{+,\alpha}(z,\,\cdot\,)$ of $\tau_+\psi = z\psi$, $z \in \mathbb{C}\backslash\mathbb{R}$, are the unique solutions satisfying

$$\psi_{+,\alpha}(z,\,\cdot\,) \in L^2([0,\infty)), \quad \sin(\alpha)\psi'_{+,\alpha}(z,0_+) + \cos(\alpha)\psi_{+,\alpha}(z,0_+) = 1, \quad \alpha \in [0,\pi).$$

Uniqueness of $\psi_{+,\alpha}(z,\,\cdot\,)$ is a consequence of the limit point hypothesis and hence $\psi_{+,\alpha}(z,\,\cdot\,)$ is necessarily of the form

$$\psi_{+,\alpha}(z,x) = \theta_\alpha(z,x) + \phi_\alpha(z,x)\, m_{+,\alpha}(z), \quad z \in \mathbb{C}\backslash\mathbb{R},\ x \in \mathbb{R}, \quad \alpha \in [0,\pi),$$

with $m_{+,\alpha}(\cdot)$ the Weyl–Titchmarsh m-function for $H_{+,\alpha}$. $m_{+,\alpha}(\cdot)$ is well-known to be a Nevanlinna–Herglotz function (mapping $\mathbb{C}_+$ to $\mathbb{C}_+$ analytically, $\mathbb{C}_+ = \{z \in \mathbb{C} \,|\, \mathrm{Im}(z) > 0\}$), and hence there exists a (nonnegative) measure $d\rho_{+,\alpha}$ such that

$$m_{+,\alpha}(z) = \mathrm{Re}(m_{+,\alpha}(i)) + \int_{\mathbb{R}} d\rho_{+,\alpha(\lambda)} \frac{1+\lambda z}{(\lambda^2+1)(\lambda - z)}, \quad \alpha \in [0,\pi),$$

and $\int_{\mathbb{R}} d\rho_{+,\alpha}(\lambda)[|\lambda|^2+1]^{-1} < \infty$, $\alpha \in [0,\pi)$.

With $E_T(\cdot)$ the family of spectral projections for a self-adjoint operator T in the Hilbert space $\mathcal{H}$, and $\sigma(T)$ the spectrum of T, $d\rho_{+,\alpha}(\cdot) \sim dE_{H_{+,\alpha}}(\cdot)$, that is, $d\rho_{+,\alpha}(\cdot)$ is a (scalar) control measure for the operator-valued spectral measure $dE_{H_{+,\alpha}}(\cdot)$ of $H_{+,\alpha}$. In particular, $\mathrm{supp}(d\rho_{+,\alpha}(\cdot)) = \mathrm{supp}(dE_{H_{+,\alpha}}(\cdot)) = \sigma(H_{+,\alpha})$.

Due to our limit point assumption on τ_+ at infinity, it is known that

$$m_{+,\alpha}(z) = -\lim_{x\to\infty} \theta_\alpha(z,x)/\phi_\alpha(z,x), \quad z \in \mathbb{C}\backslash\mathbb{R},\ \alpha \in [0,\pi),$$

and using the elementary fact (for $z \in \mathbb{C}\backslash\mathbb{R}$)

$$\phi_\alpha(z,x) = -\tan(\alpha)\theta_\alpha(z,x) + \theta_\alpha(z,x)\int_0^x \frac{dx'}{\theta_\alpha(z,x')^2}, \quad \alpha \in [0,\pi)\backslash\{\pi/2\},$$

$$\theta_\alpha(z,x) = -\cot(\alpha) - \phi_\alpha(z,x)\int_0^x \frac{dx'}{\phi_\alpha(z,x')^2}, \quad \alpha \in (0,\pi),$$

one readily obtains for $z \in \mathbb{C}\backslash\mathbb{R}$,

$$m_{+,\alpha}(z) = \begin{cases} \lim_{x\to\infty} \left[\tan(\alpha) - \int_0^x dx'\, \theta_\alpha(z,x')^{-2}\right]^{-1}, & \alpha \in [0,\pi)\backslash\{\pi/2\}, \\ \cot(\alpha) + \lim_{x\to\infty} \int_0^x dx'\, \phi_\alpha(z,x')^{-2}, & \alpha \in (0,\pi). \end{cases}$$

If in addition, τ_+ is also nonoscillatory at ∞, then $H_{+,\alpha}$ is bounded from below (i.e., for some $c_\alpha \in \mathbb{R}$, $H_{+,\alpha} \geq c_\alpha I$), and the previous limits of integrals exist as Lebesgue integrals (employing Hartman's theory of (non)principal solutions) and they extend to all $z \in \mathbb{C}\backslash[c_\alpha, \infty)$ in the form

$$m_{+,\alpha}(z) = \begin{cases} \left[\tan(\alpha) - \int_0^\infty dx'\, \theta_\alpha(z,x')^{-2}\right]^{-1}, & \alpha \in [0,\pi)\backslash\{\pi/2\}, \\ \cot(\alpha) + \int_0^\infty dx'\, \phi_\alpha(z,x')^{-2}, & \alpha \in (0,\pi). \end{cases}$$

The case $\alpha = 0$ appeared in [4] (see also [5]); the case of matrix-valued potentials V has recently been dealt with in [2].

Turning to the half-line Donoghue m-function next, we recall the minimal and maximal operators $H_{+,min}$ and $H_{+,max}$ associated with τ_+ (the former with the boundary conditions $g(0_+) = g'(0_+) = 0$, the latter without any boundary conditions, both being closed operators), satisfying $H_{+,min} \subset H_{+,\alpha} \subset H_{+,max} = H^*_{+,min}$, the defect space for $H_{+,min}$ is spanned by $\psi_{+,\alpha}(z,\,\cdot\,)$,

$$\ker(H^*_{+,min} - zI) = \text{lin.span}\{\psi_{+,\alpha}(z,\,\cdot\,)\}, \quad z \in \mathbb{C}\backslash\mathbb{R},\ \alpha \in [0,\pi).$$

The Donoghue definition of the m-function for $H_{+,\alpha}$ then reads (cf. [1])

$$\begin{aligned} m^{Do}_{+,\alpha}(z) &= z + (z^2+1)\big(\psi_{+,\alpha}(i), (H_{+,\alpha} - zI)^{-1}\psi_{+,\alpha}(i)\big)_{L^2([0,\infty))} \|\psi_{+,\alpha}(i)\|^{-2}_{L^2([0,\infty))} \\ &= \int_{\mathbb{R}} d\omega^{Do}_{+,\alpha}(\lambda) \frac{1+\lambda z}{(\lambda^2+1)(\lambda - z)}, \quad z \in \mathbb{C}\backslash\mathbb{R},\ \alpha \in [0,\pi), \end{aligned}$$

where

$$d\omega^{Do}_{+,\alpha}(\lambda)[\lambda^2+1]^{-1} = d\|E_{H_{+,\alpha}}(\lambda)\psi_{+,\alpha}(i)\|^2_{L^2([0,\infty))} \|\psi_{+,\alpha}(i)\|^{-2}_{L^2([0,\infty))}, \quad \alpha \in [0,\pi).$$

One notes the normalization, $m^{Do}_{+,\alpha}(\pm i) = \pm i$, and since $\|\psi_{+,\alpha}(z,\,\cdot\,)\|^2_{L^2([0,\infty))} = \text{Im}\big(m_{+,\alpha}(z)\big)/\text{Im}(z)$, $z \in \mathbb{C}\backslash\mathbb{R}$, this yields the relationships

$$\begin{aligned} m^{Do}_{+,\alpha}(z) &= \big[m_{+,\alpha}(z) - \text{Re}\big(m_{+,\alpha}(i)\big)\big]\big[\text{Im}\big(m_{+,\alpha}(i)\big)\big]^{-1}, \quad z \in \mathbb{C}\backslash\mathbb{R}, \\ d\omega^{Do}_{+,\alpha}(\cdot) &= \frac{1}{\text{Im}\big(m_{+,\alpha}(i)\big)} d\rho_{+,\alpha}(\cdot), \quad \alpha \in [0,\pi), \end{aligned}$$

and hence, $d\omega^{Do}_{+,\alpha}(\cdot) \sim d\rho_{+,\alpha}(\cdot) \sim dE_{H_{+,\alpha}}(\cdot)$, and

$$\text{supp}\left(d\omega^{Do}_{+,\alpha}(\cdot)\right) = \text{supp}\left(d\rho_{+,\alpha}(\cdot)\right) = \text{supp}(dE_{H_{+,\alpha}}(\cdot)) = \sigma(H_{+,\alpha}), \quad \alpha \in [0,\pi).$$

This is in accordance with $\psi_{+,\alpha}(i,\,\cdot\,)$, and hence $\psi_{+,\alpha}(z,\,\cdot\,)$ for all fixed $z \in \mathbb{C}\backslash\mathbb{R}$, $\alpha \in [0,\pi)$, being a cyclic vector for $H_{+,\beta}$ for all $\beta \in [0,\pi)$.

Continuing with the full-line case, we assume $V \in L^1_{loc}(\mathbb{R}; dx)$, V real-valued, such that the differential expression $\tau = -\frac{d^2}{dx^2} + V(x)$, $x \in \mathbb{R}$, is in the limit point case at $\pm\infty$. The associated self-adjoint and maximally defined Schrödinger operator in $L^2(\mathbb{R};dx)$ is then denoted by H. Introducing Weyl–Titchmarsh solutions of $\tau\psi = z\psi$ and m-functions, $\psi_{\pm,\alpha}(z,\,\cdot\,)$ and $m_{\pm,\alpha}(z)$, $z \in \mathbb{C}\backslash\mathbb{R}$, corresponding to any neighborhood of $\pm\infty$, one now has

$$\psi_{\pm,\alpha}(z,x) = \theta_\alpha(z,x) + \phi_\alpha(z,x)\, m_{\pm,\alpha}(z), \quad z \in \mathbb{C}\backslash\mathbb{R},\ x \in \mathbb{R},\ \alpha \in [0,\pi),$$

and obtains for the 2×2 matrix-valued Weyl–Titchmarsh function for H,

$$M_\alpha(z) = \begin{pmatrix} \frac{1}{m_{-,\alpha}(z)-m_{+,\alpha}(z)} & \frac{1}{2}\frac{m_{-,\alpha}(z)+m_{+,\alpha}(z)}{m_{-,\alpha}(z)-m_{+,\alpha}(z)} \\ \frac{1}{2}\frac{m_{-,\alpha}(z)+m_{+,\alpha}(z)}{m_{-,\alpha}(z)-m_{+,\alpha}(z)} & \frac{m_{-,\alpha}(z)m_{+,\alpha}(z,x_0)}{m_{-,\alpha}(z)-m_{+,\alpha}(z)} \end{pmatrix}, \quad z \in \mathbb{C}\backslash\mathbb{R},\ \alpha \in [0,\pi).$$

$M_\alpha(\cdot)$ is a 2×2 matrix-valued Nevanlinna–Herglotz function with representation

$$M_\alpha(z) = \mathrm{Re}(M_\alpha(i)) + \int_{\mathbb{R}} d\Omega_\alpha(\lambda) \frac{1+\lambda z}{(\lambda^2+1)(\lambda - z)}, \quad z \in \mathbb{C}\backslash\mathbb{R},\ \alpha \in [0,\pi),$$

with $\int_{\mathbb{R}} \|d\Omega_\alpha(\lambda)\|[1+\lambda^2]^{-1} < \infty$, $\alpha \in [0,\pi)$. Thus, $d\Omega_\alpha(\cdot) \sim dE_H(\cdot)$ and $\mathrm{supp}(d\Omega_\alpha(\cdot)) = \mathrm{supp}(dE_H(\cdot)) = \sigma(H)$, $\alpha \in [0,\pi)$.

Decomposing $\mathbb{R} = (-\infty,0] \cup [0,\infty)$ and $P_\pm L^2(\mathbb{R}) = L^2([0,\pm\infty))$, one is lead to the corresponding 2×2 matrix-valued Donoghue m-function for H,

$$\begin{aligned} M_\alpha^{Do}(z) &= z\,I_2 + (z^2+1)\times \\ &\times \begin{pmatrix} \big(P_-\psi_{-,\alpha}(i),(H-zI)^{-1}P_-\psi_{-,\alpha}(i)\big)_{L^2(\mathbb{R})} & \big(P_-\psi_{-,\alpha}(i),(H-zI)^{-1}P_+\psi_{+,\alpha}(i)\big)_{L^2(\mathbb{R})} \\ \big(P_+\psi_{+,\alpha}(i),(H-zI)^{-1}P_-\psi_{-,\alpha}(i)\big)_{L^2(\mathbb{R})} & \big(P_+\psi_{+,\alpha}(i),(H-zI)^{-1}P_+\psi_{+,\alpha}(i)\big)_{L^2(\mathbb{R})} \end{pmatrix} \\ &= T_\alpha^* M_\alpha(z,x_0) T_\alpha + C_\alpha, \quad \alpha \in [0,\pi), \end{aligned}$$

where T_α and $C_\alpha = C_\alpha^*$ are z-independent 2×2 matrices, T_α is invertible and C_α is off-diagonal. Again, $M_\alpha^{Do}(\cdot)$ is a 2×2 matrix-valued Nevanlinna–Herglotz function and its associated 2×2 nonnegative matrix-valued measure $d\Omega_\alpha^{Do}(\cdot)$ contains all the spectral information for H in $L^2(\mathbb{R})$, $d\Omega_\alpha^{Do}(\cdot) \sim d\Omega_\alpha(\cdot) \sim dE_H(\cdot)$. Especially, $\{P_-\psi_{-,\alpha}(i), P_+\psi_{+,\alpha}(i)\}$ generates a cyclic subspace for any self-adjoint extension of $H_{min} = H_{-,min} \oplus H_{+,min}$, and hence for H in $L^2(\mathbb{R})$. The case of bounded operator-valued potential coefficients V is discussed in [3].

Acknowledgements. F.G., S.N., R.W. are indebted to J. Behrndt, M. Brown, M. Plum, and C. Tretter for their kind invitation to this most stimulating workshop.

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L_1-Estimates for Eigenfunctions of the Dirichlet Laplacian.

Rainer Hempel

(joint work with Michiel van den Berg and Jürgen Voigt)

For $d \in \mathbb{N}$ and $\Omega \subset \mathbb{R}^d$ open, we consider the Dirichlet Laplacian H_Ω of Ω, defined as the Friedrichs extension of $-\Delta$ on $C_c^\infty(\Omega)$. Note that Ω need not be bounded nor of finite volume; also, no boundary regularity is required. The spectrum $\sigma(H_\Omega)$ is the disjoint union of the essential spectrum, $\sigma_{\text{ess}}(H_\Omega)$, and the discrete spectrum $\sigma_{\text{discr}}(H_\Omega)$. It is well-known that H_Ω has compact resolvent if and only if $\sigma_{\text{ess}}(H_\Omega) = \emptyset$; in this case, $\sigma(H_\Omega)$ consists in a sequence of positive eigenvalues tending to infinity, each of finite multiplicity. H_Ω has compact resolvent if Ω is bounded or of bounded volume, but neither condition is necessary.

In this note, we discuss eigenfunctions Φ of H_Ω, associated with a discrete eigenvalue, where we provide upper bounds for the L_1-norm $\|\Phi\|_1$ in terms of the L_2-norm $\|\Phi\|_2$ and spectral data of H_Ω. In view of an important application in the problem of the *heat content* of Ω as discussed in van den Berg and Davies [1], the possibility and the structure of an estimate on $\|\Phi\|_1$ in terms of spectral data was recently formulated as an open problem in an Oberwolfach report [3]; here a key requirement was to find an estimate which would be independent of the volume of Ω. We are now happy to provide a solution to this problem.

For simplicity, we describe our main results only for the case where H_Ω has compact resolvent. Here the eigenvalues can be ordered according to the min-max-principle as $0 < \lambda_1 \leq \lambda_2 \leq \ldots \leq \lambda_k \leq \lambda_{k+1} \leq \ldots$, with $\lambda_k \to \infty$ as $k \to \infty$. We will also use the *eigenvalue counting function*

$$\mathcal{N}_t := \max\{k \in \mathbb{N}\,;\ \lambda_k < t\}, \qquad t > 0. \tag{1}$$

We then have:

Theorem 1 (van den Berg, Hempel, and Voigt [4], Vogt [5])**.** *For $\Omega \subset \mathbb{R}^d$ open with $\sigma_{\text{ess}}(H_\Omega) = \emptyset$, we have*

$$\|\Phi\|_1^2 \leq C\lambda_k^{-d/2}\mathcal{N}_{2\lambda_k}(H_\Omega)\|\Phi\|_2^2, \tag{2}$$

for all eigenfunctions Φ of H_Ω associated with the eigenvalue λ_k.

Remarks.

(a) The constant C in (2) can be given an explicit value which depends only on the dimension d ([5]).

(b) For Ω of bounded volume $|\Omega|$, the Schwarz inequality yields the simple estimate $\|u\|_1^2 \leq |\Omega|\|u\|_2^2$ for any $u \in L_2(\Omega)$. One could argue that $\mathcal{N}_{2\lambda_k}(H_\Omega)$ is a substitute for $|\Omega|$.

(c) The "classical" heat-kernel bound

$$\|\Phi\|_\infty \leq C_1\lambda^{d/4}\|\Phi\|_2, \tag{3}$$

valid for eigenfunctions of H_Ω associated with the eigenvalue λ, immediately gives a lower bound on the L_1-norm of the form

$$\|\Phi\|_1 \geq C_2 \lambda^{-d/4} \|\Phi\|_2. \tag{4}$$

(d) Very roughly, the basic idea of proof goes as follows: By means of an IMS-partition of unity, we find a subset $\tilde{\Omega} \subset \Omega$ which satisfies two requirements: first, the volume $|\tilde{\Omega}|$ can be estimated in terms of $\mathcal{N}_{2\lambda_k}$; second, the eigenfunctions Φ as in Thm. I decay exponentially away from $\tilde{\Omega}$ with a quantitative control of the decay. The progress in [5] as compared to [4] lies mainly in the second property.

(e) The method of proof of [4] is rather robust and flexibel and allows for generalizations in various directions, including Schrödinger operators, elliptic operators in divergence form, or Laplace-Beltrami operators as in [2]. There is ongoing research in this direction by M. Stautz (Ph.D.-thesis, TU Braunschweig) with several promising preliminary results.

We now turn to an application of Theorem 1 to the heat content and the heat trace of Ω. Suppose $0 \leq u = u(x,t)$ solves the heat equation on Ω with Dirichlet boundary conditions zero and initial condition $u(x,0) = 1$ for all $x \in \Omega$. The *heat content* of Ω at time $t \geq 0$ is

$$Q_\Omega(t) := \int_\Omega u(x,t)\mathrm{d}x = \int_\Omega \int_\Omega p_\Omega(x,y;t)\mathrm{d}x\mathrm{d}y \in [0,\infty], \tag{5}$$

with p_Ω denoting the heat kernel. If H_Ω has compact resolvent, p_Ω has an expansion in terms of an orthonormal basis of eigenfunctions $\{\Phi_k\}_{k\in\mathbb{N}}$ of H_Ω,

$$p_\Omega(x,y;t) = \sum_{k=1}^{\infty} \mathrm{e}^{-t\lambda_k} \Phi_k(x)\Phi_k(y), \tag{6}$$

and thus

$$Q_\Omega(t) \leq \sum_k \mathrm{e}^{-t\lambda_k} \|\Phi_k\|_1^2 \in [0,\infty]. \tag{7}$$

A somewhat simpler quantity is the *heat trace of* Ω, given by

$$Z_\Omega(t) = \sum_k \mathrm{e}^{-\lambda_k t} \in [0,\infty]. \tag{8}$$

It has been shown in [1] that

$$Z_\Omega(t) \leq (2\pi t)^{-d/2} Q_\Omega(t/2), \tag{9}$$

for all $t > 0$. Our Thm. 1 implies that $Q_\Omega(t)$ is finite for all $t > 0$ provided $Z_\Omega(t)$ is finite for all $t > 0$. More precisely, we have the following estimate:

Theorem 2 ([4], [5]). *Suppose H_Ω has compact resolvent. Then for all $t_0 > 0$ and $t > 2t_0$,*

$$Q_\Omega(t) \leq C_{\varepsilon,d} \lambda_1^{-d/2} Z_\Omega^2\left(\frac{t}{2+\varepsilon}\right), \qquad 0 < \varepsilon < \frac{t}{t_0} - 2; \tag{10}$$

the constant $C_{\varepsilon,d}$ in (10) depends only on d and ε.

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Spectral results for mixed problems and fractional order elliptic operators

Gerd Grubb

1. Krein formula for the mixed problem. Let Ω be bounded smooth open $\subset \mathbb{R}^n$, with $\partial\Omega = \Sigma$. Denote $\partial_n^j u|_\Sigma = \gamma_j u$, $j \in \mathbb{N}_0$. For the L_2-Sobolev space $H^s(\mathbb{R}^n)$, we denote $r_\Omega H^s(\mathbb{R}^n) = H^s(\Omega)$, $\{u \in H^s(\mathbb{R}^n) \mid \operatorname{supp} u \subset \overline{\Omega}\} = \dot{H}^s(\overline{\Omega})$.

Consider a symmetric strongly elliptic second-order differential operator on Ω

$$Au = -\sum\nolimits_{j,k=1}^{n} \partial_j (a_{jk}(x)\partial_k u) + a_0(x)u,$$

with real C^∞-coefficients. The associated sesquilinear form $a(u,v)$ is coercive on $H^1(\Omega)$, and we can assume it positive by adding a constant to a_0. Set $\nu u = \sum n_j \gamma_0(a_{jk}\partial_k u)$, the conormal derivative. Realizations of A:

The maximal realization $A_{\max}$ with $D(A_{\max}) = \{u \in L_2(\Omega) \mid Au \in L_2(\Omega)\}$.
The Dirichlet realization A_γ with $D(A_\gamma) = \{u \in H^2(\Omega) \mid \gamma_0 u = 0\}$.
The Neumann realization A_ν with $D(A_\nu) = \{u \in H^2(\Omega) \mid \nu u = 0\}$.
A mixed realization $A_{\nu,U}$. Here U is a smooth open subset of Σ, and

$$D(A_{\nu,U}) = \{u \in H^1(\Omega) \cap D(A_{\max}) \mid \nu u = 0 \text{ on } U,\ \gamma_0 u = 0 \text{ on } \Sigma \setminus U\}.$$

The latter three are defined variationally from the form considered on $\dot{H}^1(\overline{\Omega})$, $H^1(\Omega)$, resp. $H^1_U(\Omega) = \{u \in H^1(\Omega) \mid \operatorname{supp} \gamma_0 u \subset \overline{U}\}$. They are selfadjoint positive, and whereas $D(A_\gamma), D(A_\nu) \subset H^2(\Omega)$, it is known that $D(A_{\nu,U}) \subset H^{\frac{3}{2}-\varepsilon}(\Omega)$ only.

Let $Z = \ker A_{\max}$, and let K_γ denote the Poisson operator $K_\gamma \colon \varphi \mapsto u$ solving the semihomogeneous Dirichlet problem $Au = 0$ on Ω, $\gamma_0 u = \varphi$ on Σ; it maps e.g. $K_\gamma \colon H^{-\frac{1}{2}}(\Sigma) \overset{\sim}{\to} Z$, closed subset of $L_2(\Omega)$.

Let $P = \nu K_\gamma$, the **Dirichlet-to-Neumann operator**; it is known to be a symmetric pseudodifferential operator on Σ of order 1.

Proposition 1. *Let $x' \in \Sigma$ and choose coordinates such that the interior normal is $(0,\dots,0,1)$. Write the principal symbol of A at x' as $a_{nn}(x')\xi_n^2 + 2b(x',\xi')\xi_n + c(x',\xi')$, and let*

$$m(x',\xi') = a_{nn}(x')c(x',\xi') - b(x',\xi')^2,$$

it is positive for $\xi' \neq 0$ by the ellipticity of A.

Then P has principal symbol $p^0(x',\xi') = -m(x',\xi')^{\frac12}$ at x'.

Hence if M is a symmetric differential operator on Σ with principal symbol m, $P = -M^{\frac12}+$ order 0.

For the mixed problem, define on Σ the restriction operator $r^+ \colon \varphi \to \varphi|_U$, and the extension operator e^+ extending ψ on U by 0 on $\Sigma \setminus U$. When Q is an operator over Σ we denote $r^+ Q e^+ = Q_+$ (truncation).

Let $X = \dot H^{-\frac12}(\overline{U})$ (the subspace of distributions in $H^{-\frac12}(\Sigma)$ supported in $\overline{U}$). Its dual space is $X^* = H^{\frac12}(U) = r^+ H^{\frac12}(\Sigma)$. Define $V = K_\gamma(X) \subset Z$ and denote the restriction $K_\gamma|_X$ by

$$K_{\gamma,X} \colon X \overset{\sim}{\to} V, \text{ with adjoint } K^*_{\gamma,X} \colon V \overset{\sim}{\to} X^*.$$

In J.Math.An.Appl.'11, we showed:

Theorem 2. *For the mixed problem there is an operator L mapping $D(L) \subset X$ bijectively onto X^* such that the Kreĭn resolvent formula holds:*

$$A^{-1}_{\nu,U} - A^{-1}_{\gamma} = \mathrm{i}_V K_{\gamma,X} L^{-1} K^*_{\gamma,X} \mathrm{pr}_V \equiv T. \qquad (1)$$

Here L acts like $-P_+$ and has

$$D(L) = \{\varphi \in X \mid P_+\varphi \in X^*\} \subset \dot H^{1-\varepsilon}(\overline{U}).$$

In the same paper we also determined the spectral asymptotics of T, when $A = -\Delta+$ lower order terms. The crucial point is to understand L^{-1}. (It is NOT the same as $-(P^{-1})_+$.) The methods were based on Eskin '81, Birman-Solomiak '77, Laptev '81. Now a new tool is available: Boundary value theories for fractional powers of elliptic operators. This will allow general A.

2. Boundary problems for fractional order operators. A basic example of a ps.d.o. of noninteger order is the fractional Laplacian $(-\Delta)^a$, $0 < a < 1$:

$$(-\Delta)^a u = \mathcal{F}^{-1}(|\xi|^{2a}\hat u(\xi)), \quad \hat u(\xi) = \mathcal{F}u = \int_{\mathbb{R}^{n'}} e^{-ix\cdot\xi} u(x)\,dx.$$

It is currently of interest both in probability, finance, mathematical physics and geometry. More general example: M^a, where M is a 2'order strongly elliptic differential operator on $\mathbb{R}^{n'}$. M^a is a ps.d.o. of order $2a$ by Seeley '66.

Let U be bounded smooth open $\subset \mathbb{R}^{n'}$. Dirichlet problem for M^a on U?

Let $m_a(u,v) = (M^a u, v)$ for $u, v \in C_0^\infty(U)$. It satisfies

$$\operatorname{Re} m_a(u,u) \geq c\|u\|_a^2 - k\|u\|_0^2, \; c > 0, k \in \mathbb{R},$$

and its closure with domain $\dot H^a(\overline{U})$ defines an operator M^a_{Dir} in $L_2(U)$ by variational theory. It acts like M^a_+, with $D(M^a_{\mathrm{Dir}}) \subset \dot H^a(\overline{U})$. It represents the problem

$$M^a_+ u = f, \quad u \text{ sought in } \dot H^a(\overline{U}). \qquad (2)$$

Regularity results for (2) are quite recent.

Ros-Oton and Serra in J.An.Pur.Appl.'14 showed by potential theory and integral operator methods, when $M = -\Delta$ and U is $C^{1,1}$, that

$$f \in L_\infty(U) \implies u \in d^a C^\alpha(\overline{U}) \cap C^a(\overline{U}), \tag{3}$$

for some $\alpha > 0$. Here $d(x) = \operatorname{dist}(x, \partial U)$. They stated that they did not know of other regularity results for $(-\Delta)^a$ in the literature.

Ps.d.o. methods? The Boutet de Monvel calculus '71 requires integer order plus a so-called 0-transmission property at ∂U. M^a is not covered.

But we have recently developed another calculus. It is based on a more general μ-transmission property, introduced by Hörmander in his 1985 book (in fact in an unpublished lecture note from IAS Princeton 1965). Here M^a has the a-transmission property, since the symbol has even parity and is of order $2a$.

It allows to improve the information in (3) to $u \in d^a C^a(\overline{U})$ and to get higher regularity: $f \in C^t(\overline{\Omega}) \implies u \in d^a C^{a+t}(\overline{\Omega})$ for $t > 0$ (Adv.Math.'15, Anal.&PDE'14),

The results rely on constructing an approximate inverse of M^a_{Dir} (a parametrix).

Consider a localized situation where U and $\Sigma \setminus \overline{U}$ are replaced by, respectively, $\mathbb{R}^{n'}_\pm = \{x \mid \pm x_{n'} > 0\}$. There exist **order-reducing operators**:

Theorem 3. *There exist two families of ps.d.o.s* $\Lambda^{(t)}_\pm$ *of order* $t \in \mathbb{R}$*, preserving support in* $\overline{\mathbb{R}}^{n'}_\pm$*, respectively, such that for all* $s \in \mathbb{R}$*,*

$$\Lambda^{(t)}_+ : \dot{H}^s(\overline{\mathbb{R}}^{n'}_+) \xrightarrow{\sim} \dot{H}^{s-t}(\overline{\mathbb{R}}^{n'}_+), \quad (\Lambda^{(t)}_-)_+ : H^s(\mathbb{R}^{n'}_+) \xrightarrow{\sim} H^{s-t}(\mathbb{R}^{n'}_+).$$

Theorem 4. *On* $\dot{H}^a(\overline{U})$*, the operator* M^a_+ *can be written in the form*

$$M^a_+ = (\Lambda^{(a)}_-)_+ r^+ Q \Lambda^{(a)}_+, \tag{4}$$

where Q *is a ps.d.o. of order* 0 *in the Boutet de Monvel calculus, such that the problem*

$$Q_+ v = g, \quad \operatorname{supp} v \subset \overline{\mathbb{R}}^{n'}_+, \tag{5}$$

is well-posed. Here the solution to (2) *is found as* $\Lambda^{(-a)}_+ e^+ v$*, when* $g = (\Lambda^{(-a)}_-)_+ f$*.*

Theorem 5. *Let* $\widetilde{Q}_+ + G_0$ *be a parametrix for* (5) *(*G_0 *being a singular Green op. of order and class* 0 *in the B.d.M. calculus). Then the problem* (2) *has the parametrix*

$$R = (\Lambda^{(-a)}_+)_+ (\widetilde{Q}_+ + G_0)(\Lambda^{(-a)}_-)_+. \tag{6}$$

Similar results can be obtained in the situation of the manifold $\Sigma = \partial\Omega$ and its subset U (of dimension $n' = n - 1$).

For spectral asymptotics, we have a general result for truncated ps.d.o.s: When

$$\mathcal{P} = P_{1,+} \dots P_{l_0,+}(P_{0,+} + G) P_{l_0+1,+} \dots P_{l,+},$$

where P_0 is of order 0, G is a sing. Green op. on U of order and class 0, the P_j are of order $-t_j < 0$, then the singular values $s_k(\mathcal{P})$ satisfy, with $t = t_1 + \dots + t_l$,

$$s_k(\mathcal{P}) k^{t/(n-1)} \to C(\mathcal{P}) \text{ for } k \to \infty,$$

$C(\mathcal{P})$ defined from the principal symbols on U. This applies immediately to (6).

3. Application to the mixed problem. Consider (1). Since $L = -P_+$ is of the form $M_+^{\frac{1}{2}}+$ lower order terms by Proposition 1, we can show that $L^{-1} = R+$ l.o.t., where R is the parametrix of $M_{\mathrm{Dir}}^{\frac{1}{2}}$. Then

$$T = \mathrm{i}_V K_{\gamma,X} R K_{\gamma,X}^* \mathrm{pr}_V + \text{ l.o.t.}$$

The operators surrounding R are replaced by truncated ps.d.o.s on U in a similar way as in J.Math.An.Appl.'11, and we arrive at the result (in J.Math.An.Appl.'15):

Theorem 6. *The eigenvalues of T satisfy*

$$\mu_k(T)k^{2/(n-1)} \to C(T) \textit{ for } k \to \infty,$$

where $C(T)$ is an integral over U of a function defined from the principal symbols:

$$C(T) = \tfrac{1}{(n-1)(2\pi)^{n-1}} \int_U \int_{|\xi'|=1} \Big(\frac{a_{nn}(x')}{2m(x',\xi')} \Big)^{\frac{n-1}{2}} d\omega(\xi') dx'.$$

The finite section method for dissipative Jacobi and Schrödinger operators

Marco Marletta
(joint work with Sergey Naboko, Rob Scheichl)

We consider the approximation of the essential spectrum by finite section methods, for dissipative Schrödinger operators in $L^2(0,\infty)$ with a regular endpoint at 0, and dissipative Jacobi matrices in $\ell^2(\mathbb{N})$.

In the Schrödinger case, our starting point is an operator L_0 given by an expression

$$L_0 u = -u'' + q(x)u,$$

in which the real-valued potential q is in the limit-point case at infinity and integrable at 0; at 0 we impose, without loss of generality, a Dirichlet boundary condition. The domain of L_0 is thus

$$D(L_0) = \{u \in L^2(0,\infty) \mid -u'' + qu \in L^2(0,\infty),\ u(0) = 0\}$$

and L_0 is self-adjoint. We make no further assumptions about the spectrum of L_0, which may be any closed, unbounded-above subset of $\mathbb{R}$. We are interested in the finite section method applied to the dissipative operator

$$L = L_0 + is(x)\cdot,$$

in which s is a bounded, non-negative element of $L^1(0,\infty)$. In applications using dissipative barrier methods s usually has compact support - say, $\mathrm{supp}(s) \subseteq [0, N]$ for some $N > 0$ - and is often the characteristic function of some finite interval.

In the case of Jacobi operators we start with a self-adjoint operator in $\ell^2(\mathbb{N})$ represented formally by an infinite matrix

$$J_0 = \begin{pmatrix} b_1 & a_1 & 0 & 0 & 0 & \cdots \\ a_1 & b_2 & a_2 & 0 & 0 & \cdots \\ 0 & a_2 & b_3 & a_3 & 0 & \cdots \\ . & . & . & . & . & \cdots \\ . & . & . & . & . & \cdots \\ . & . & . & . & . & \cdots \end{pmatrix}$$

and we consider the finite section method applied to

$$J = J_0 + i\mathrm{diag}(s_1, s_2, \ldots),$$

in which $s_j \in \ell^1(\mathbb{N})$. In applications s is often finitely supported with, say, $s_{N-1} > 0$ and $s_j = 0$ for $j \geq N$.

We have proved the following results.

Theorem 1. *Suppose that $s \in \ell^1(\mathbb{N})$ and $s_j \geq 0$ for all j. Suppose that λ_{ess} is a point of essential spectrum of $J = J_0 + is$. Then every open neighbourhood of λ_{ess} in $\mathbb{C}$ contains eigenvalues of the leading $M \times M$ submatrix of J, for all sufficiently large M.*

For each $M > 0$ let L^M denote the dissipative Schrödinger operator in $L^2(0, M)$ equipped with Dirichlet boundary conditions $u(0) = 0 = u(M)$.

Theorem 2. *Suppose $L_0 = L_0^*$, that $\min(q, 0) \in L^\infty(0, \infty)$, $q \in L^2_{loc}$, $s \in L^1(0, \infty) \cap L^\infty(0, \infty)$, $s \geq 0$ and $s(x) \to 0$ as $x \to \infty$. Suppose that λ_{ess} is a point of essential spectrum of $L = L_0 + is$. Then every open neighbourhood of λ_{ess} in $\mathbb{C}$ contains eigenvalues of the finite-interval operators L^M, for all sufficiently large M.*

Theorem 3. *Suppose $L_0 = L_0^*$ and that there is an open interval $I = (\alpha, \beta)$ of $\sigma_{\mathrm{ess}}(L_0)$ with gaps above and below, with no nearby eigenvalues of L^M for some sequence of $M \nearrow \infty$. Then for each subinterval $\tilde{I} = (\alpha + \delta, \beta - \delta)$ and $\nu > 0$ we have the counting estimate*

$$\left\{N_P(M, \tilde{I})\right\}^{\frac{\delta^2}{2\nu^2}} \log(N_P(M, \tilde{I})) \leq \frac{C_\nu}{\delta} N_P(M, I)$$

where C_ν is independent of M and $\delta > 0$. Consequently L_0 has no integrated density of states on the interval I.

These results are contained in [1]. For the case of a Schrödinger operator with periodic potential (or compactly supported perturbations thereof) we have the following result.

Theorem 4. *Let λ_{ess} be an interior point of a spectral band. Then there exists for each $M > 0$ an eigenvalue $\lambda_M \in spec(L^M)$, such that for large M,*

$$|\lambda_{\mathrm{ess}} - \lambda_M| = O(M^{-1}).$$

This result, together with further results on spectral pollution, are in [2].

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Dispersion estimates for one-dimensional Schrödinger and Klein–Gordon equations

Gerald Teschl

(joint work with Iryna Egorova, Elena Kopylova, Vladimir Marchenko)

Let

$$H = -\frac{d^2}{dx^2} + V(x), \qquad x \in \mathbb{R},$$

be the one-dimensional Schrödinger operator with a decaying potential and let

$$\psi(t) = \mathrm{e}^{-\mathrm{i}tH}\psi_0$$

be the solution of the corresponding Schrödinger equation

$$\mathrm{i}\psi_t = H\psi.$$

In the case without potential, $V \equiv 0$, the kernel of the evolution group can be computed explicitly

$$\psi(t,x) = \frac{1}{\sqrt{4\pi \mathrm{i}t}} \int_{\mathbb{R}} \mathrm{e}^{\mathrm{i}\frac{(x-y)^2}{4t}} \psi_0(y)dy$$

and immediately gives rise to the dispersive decay estimate

$$(\star) \qquad \|\psi(t)\|_\infty \le \frac{C}{\sqrt{t}} \|\psi_0\|_1.$$

Here $\|.\|_p$ denotes the usual L^p norms. Combining this estimate with preservation of the L^2 norm further gives the usual interpolation estimates plus the well-known Strichartz estimates (cf. [5]) which play an important role in the investigation of associated nonlinear evolution equations like the nonlinear Schrödinger (NLS) or the Gross–Pitaevskii equation.

Of course, in the presence of a potential V, one expects the dispersive estimate $(\star)$ to hold true as long as V decays sufficiently fast as $|x| \to \infty$ provided ψ_0 is restricted to the absolutely continuous subspace of H. In fact, this was shown by several authors in increasing generality with respect to the assumptions on the potential V [1, 3, 4, 6]. The best result was the one by Goldberg and Schlag [4] who proved the above estimate under the assumption $\int_{\mathbb{R}}(1 + |x|^j)|V(x)|dx < \infty$ with $j = 1$ if there is no resonance at the bottom of the continuous spectrum and with $j = 2$ in case of a resonance. In [2] we give a simplified approach to this problem and remove the extra decay condition on the potential in the resonant case. Our proof is based on the novel fact that, under the above assumption with $j = 1$, the scattering matrix is in the unital Wiener algebra of functions with integrable Fourier transforms.

Furthermore, we establish a faster decay in the non-resonant case and show corresponding disperse decay estimates for the Klein–Gordon equation

$$\psi_{tt}(x,t) = -(H+m^2)\psi(x,t), \qquad m>0.$$

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Blowup and gradient growth for model equations of fluid dynamics

Vu Hoang

(joint work with A. Kiselev, M. Radosz, X. Xu)

In this talk I address a few problems of great interest concerning non-local, non-linear transport equations. The basic equation is given by

$$\theta_t(x,t) + u(x,t)\theta_x(x,t) = 0, \quad \theta(x,0) = \theta_0(x) \tag{1}$$

where the unknown function θ depends on $(x,t) \in \mathbb{R} \times (0,\infty)$. To complete the problem, one must also specify u in terms of θ. By setting $u = \theta$ we get Burgers' equation.

A basic problem for (1) is to decide whether solutions with smooth initial data (C^∞ with suitable decay at infinity) remain so for all time. One speaks of blowup, if they become singular at a finite time (for reasonable velocity field, one can usually prove local-in-time existence of a solution $\theta : \mathbb{R} \times [0,T] \to \mathbb{R}$, where the existence time $T = T(\theta_0) > 0$ depends on the initial data.

For Burgers' equation, blowup is easily proven. Inspired by the equations of fluid mechanics (see e.g. [1]), we consider nonlocal velocity fields

$$u(x,t) = \int_{\mathbb{R}} g(x-y)\theta(y)dy \tag{2}$$

given by integration over a (more or less) singular kernel. E.g. if $k(\tau) = \log|\tau|$, then global existence of a smooth solution from smooth data is easily proven. Another model is the Cordoba-Cordoba-Fontelos (CCF) model, for which

$$u(x,t) = PV\int_{\mathbb{R}} (y-x)^{-1}\theta(y)dy \tag{3}$$

the velocity field is the Hilbert transform. Blowup for this model was proven in the remarkable paper [2]. Interpolating between (2) and (3), we may also consider the kernel $g(\tau) = |\tau|^\gamma$ with $\gamma \in (0,1)$, i.e.

$$u(x,t) = \int_{\mathbb{R}} |y-x|^{-\gamma}\theta(y)dy \tag{4}$$

This is less singular than (3) (but note the difference in the sign of the kernel). For this model, blowup can also be proven. This will be published elsewhere.

There are a number of open questions.

- Numerical simulations [2] seem to indicate that in case of symmetric, non-negative initial data with maximum at $x = 0$, the singularity that occurs "first" is a cusp. The solution at the singular time behaves possibly like a square root around the origin. Can one show that a cusp always occur in the situation described, or at least construct initial data such that the resulting singularity has a square-root like behavior (see also the recent preprint [3])?
- It is true that singularities for (4) tend to form at points where $\theta' \neq 0$ and that singularities for (3) form only at points with $\theta' = 0$?
- Construct a blowup solution for (4) such that the first singularity that forms is a infinite slope of θ at the origin, θ being otherwise smooth at all other points.

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Linear systems, transfer functions, and operator theory

Jonathan R. Partington

This talk is largely of a tutorial nature, beginning with a short account of the theory of continuous-time, single-input single-output linear systems, regarded as closed linear operators on $L^2(0,\infty)$. For more details of the basic theory, as well as many supplementary references, we refer to [7]. We also discuss Hankel operators, and scaled Hankel operators, and their use in rational approximation.

As an example, consider the delay equation

$$\frac{dy}{dt} + ay(t-\tau) = u(t), \qquad y(0) = 0,$$

where $\tau > 0$, the function u is the input, and y is the output. Taking Laplace transforms $U = \mathcal{L}u$, etc., we have

$$Y(s) = \frac{U(s)}{s + ae^{-s\tau}}$$

for $s \in \mathbb{C}_+$, the complex right half-plane. Here $G(s) := 1/(s+ae^{-s\tau})$ is the *transfer function*. By the Paley–Wiener theorem the Laplace transform is $\sqrt{2\pi}$ times an isometry between $L^2(0,\infty)$ and $H^2(\mathbb{C}_+)$, the Hardy space of analytic functions in $\mathbb{C}_+$ with boundary values in $L^2(i\mathbb{R})$. Now, the operator T of multiplication by G is everywhere-defined and bounded if and only if G lies in $H^\infty(\mathbb{C}_+)$, the space of bounded analytic functions. In our example, this requires the condition $a\tau < \pi/2$.

George Weiss noted that for $1 \le p < \infty$, if $T : L^p(0,\infty) \to L^p(0,\infty)$ is bounded and shift-invariant, then there is a $G \in H^\infty(\mathbb{C}_+)$ such that for $y = Tu$ we have $Y(s) = G(s)U(s)$, $s \in \mathbb{C}_+$. Also $\|T\| \ge \|G\|_\infty$, with equality for $p = 2$.

Some cases, e.g. $p = 2$, have been known for much longer. However, Weiss also gave a counterexample for $p = \infty$ using the axiom of choice.
One way to study systems in general is by their graphs,

$$\mathcal{G}(T) := \{(u, Tu) : u \in \mathrm{Dom}(T)\},$$

allowing for closed unbounded operators defined on a proper subspace of $L^2(0,\infty)$.

We now say T is *shift-invariant*, if $\mathcal{G}(T)$ is a closed shift-invariant subspace. As observed by Georgiou and Smith we may use Beurling's theorem to describe the graph in the frequency domain, i.e., as a closed subspace of pairs $(U, Y)^T$ in $H^2(\mathbb{C}_+)^2$. In the non-degenerate case it is

$$\mathcal{G}(T) = \begin{pmatrix} M \\ N \end{pmatrix} H^2(\mathbb{C}_+),$$

where $M, N \in H^\infty(\mathbb{C}_+)$ and (M, N) is *inner*, i.e., $|M(iy)|^2 + |N(iy)|^2 = 1$ a.e. That is, the transfer function "is" $N(s)/M(s)$.

For $h \in L^1(0,\infty)$ the convolution system

$$y(t) = \int_0^t h(t-\tau)u(\tau)\, d\tau$$

has transfer function $G = \mathcal{L}h \in H^\infty(\mathbb{C}_+)$. Here h is called the *impulse response*.

As another example, we mention *diffusive systems*. Let μ be a Borel measure on $[0,\infty)$ such that the (impulse response)

$$h(t) := \int_0^\infty e^{-tx}\, d\mu(x)$$

exists for $t > 0$. The transfer function is then

$$G(s) := \mathcal{L}h(s) = \int_0^\infty \frac{d\mu(x)}{x+s},$$

a Stieltjes transform. These have been studied by Montseny, and more recently in more generality by my student Bashar Abusaksaka [1]. They can be realised in terms of PDEs such as the heat equation.

Rational approximation. For practical implementation, analysis and control design, rational H^∞ transfer functions are preferred. Consider the Hankel operator

Γ_h on $L^2(0,\infty)$, where

$$\Gamma_h u(t) = \int_0^\infty h(t+\tau)u(\tau)\,d\tau.$$

Then Γ_h has finite rank n if and only if $G = \mathcal{L}h$ is rational of degree n (Kronecker's theorem). Alternatively, for $G \in H^\infty(\mathbb{C}_+)$ we define the unitarily equivalent $\widetilde{\Gamma}_G : H^2(\mathbb{C}_-) \to H^2(\mathbb{C}_+)$ by

$$\widetilde{\Gamma}_G u = P_{H^2(\mathbb{C}_+)}(Gu),$$

noting that $L^2(i\mathbb{R}) = H^2(\mathbb{C}_-) \oplus H^2(\mathbb{C}_+)$. That is, we multiply and take an orthogonal projection.

We always have $\|\widetilde{\Gamma}_G\| \le \|G\|_\infty$, so if R is rational of degree n, then

$$\|G-R\|_\infty \ge \|\widetilde{\Gamma}_G - \widetilde{\Gamma}_R\|$$

which involves approximating a compact operator by an operator of rank n.

Write $\sigma_1 \ge \sigma_2 \ge \dots$ for the singular values (approximation numbers, s-numbers) of Γ. Then, by definition,

$$\|\widetilde{\Gamma}_G - T\| \ge \sigma_{n+1},$$

if T is any operator of rank n. In fact, this can be achieved with a Hankel operator $T = \Gamma_R$, as proved by Adamjan, Arov, and Krein. This "optimal Hankel-norm approximant" is easy to compute.

Estimates for Hankel singular values of delay systems can be found in [3]. For example, let

$$\frac{dy}{dt} + y(t-1) = u(t-\alpha),$$

where $\alpha > 0$, fixed; we have the transfer function

$$G(s) = \frac{e^{-\alpha s}}{s+e^{-s}}, \qquad \text{and}$$

$$n\sigma_n \to \frac{\alpha}{\pi} \qquad (\alpha > 0), \qquad \text{while} \quad n^2\sigma_n \to \frac{1}{\pi^2} \qquad (\alpha = 0).$$

The degree-n optimal Hankel-norm approximant R gives an error bound satisfying

$$\sigma_{n+1} \le \|G-R\|_\infty \le \sigma_{n+1} + \sigma_{n+2} + \dots$$

(Glover for $\deg G < \infty$, Glover et al [2], and recently Guiver–Opmeer [6] for the general case). Alternative "general" approximation schemes (e.g. "balanced truncation") give a similar error bound. But convergence is not guaranteed unless $\widetilde{\Gamma}_G$ is nuclear. However for delay systems, Padé approximants give the optimal convergence rate [4].

For $L^2(0,\infty)$ approximation of h or $H^2(\mathbb{C}_+)$ approximation of G, we may use properties of the scaled Hankel operator

$$(\Theta_h u)(t) = \frac{1}{\sqrt{\pi}} \int_0^\infty t^{-1/4} h(t+\tau)\tau^{-1/4} u(\tau)\,d\tau.$$

Like the Hankel operator, its rank is the degree of $G = \mathcal{L}h$. Now its Hilbert–Schmidt norm is exactly the L^2 norm of h. Thus, for rational degree-n approximants.

$$\|G - R\|_2 \geq \sqrt{2\pi}(\sigma_{n+1}^2 + \sigma_{n+2}^2 + \ldots)^{1/2},$$

where the (σ_n) are the singular values of Θ_h. These are much harder to calculate explicitly, although, e.g. for delay systems, their rate of decay is known. Again the optimal approximation rate can be achieved in this case [5].

Finally we mention again the thesis [1], where for diffusive systems a full analysis of boundedness, Hilbert–Schmidt and nuclearity properties is given, for both the Hankel and scaled Hankel operators.

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The State/Signal Resolvent Functions

Olof Staffans

(joint work with Damir Z. Arov)

In this talk I give an introduction to the spectral theory of a linear stationary s/s (state/signal) system in continuous time. A s/s system has a state space which plays the same role as the state space of an ordinary i/s/o (input/state/output) system, but it differs from an i/s/o systems in the sense that the interaction signal which connects the system to the outside world has not been divided a priori into one part which is called the "input" and another part which is called the "output". The class of s/s systems can be used to model, e.g., linear time-invariant circuits which may contain both lumped and distributed components. To each s/s system corresponds in general an infinite number of i/s/o systems which differ from each other by the choice of how the interaction signal has been divided into an input part and output part. Each such i/s/o system is called an i/s/o representation of the given s/s system.

I begin by giving an introduction to the time domain theory for i/s/o systems. Such a system can be written in the form

$$(1)\qquad \Sigma_{\text{iso}}\colon \begin{bmatrix}\dot{x}(t)\\ y(t)\end{bmatrix} = S\begin{bmatrix}x(t)\\ u(t)\end{bmatrix}, \quad \begin{bmatrix}x(t)\\ u(t)\end{bmatrix} \in \text{dom}\,(S), \quad t\in\mathbb{R}^+, \quad x(0)=x^0.$$

Here $S\colon \left[\begin{smallmatrix}\mathcal{X}\\ \mathcal{U}\end{smallmatrix}\right] \to \left[\begin{smallmatrix}\mathcal{X}\\ \mathcal{Y}\end{smallmatrix}\right]$ is a closed operator, $\mathcal{X}, \mathcal{U}$ and $\mathcal{Y}$ are Hilbert spaces, $x(t)\in\mathcal{X}$ is the state, $u(t)\in\mathcal{U}$ is the input, and $y(t)\in\mathcal{Y}$ is the output. By a classical future trajectory of Σ_{iso} we mean a triple of functions $\left[\begin{smallmatrix}x\\ u\\ y\end{smallmatrix}\right]$ which satisfies (1) for all $t\in\mathbb{R}^+$, with x continuously differentiable with values in $\mathcal{X}$ and $\left[\begin{smallmatrix}u\\ y\end{smallmatrix}\right]$ continuous with values in $\left[\begin{smallmatrix}\mathcal{U}\\ \mathcal{Y}\end{smallmatrix}\right]$. Different classes of i/s/o systems of this type are described in [Sta05].

A general i/s/o system can be seen as an extension of a standard finite-dimensional i/s/o system. If S is bounded, the S can be written in block matrix form $S = \left[\begin{smallmatrix}A & B\\ C & D\end{smallmatrix}\right]$, and (1) becomes

$$(2)\qquad \Sigma_{\text{iso}}\colon \begin{cases}\dot{x}(t) = Ax(t)+Bu(t),\\ y(t) = Cx(t)+Du(t),\end{cases} \quad t\in\mathbb{R}^+, \quad x(0)=x^0.$$

In this case we say that A is the main operator, B is the control operator, C is the observation operator, D is the feedthrough operator. The case where A generates a C_0 semigroup and B, C, and D are bounded is described in the book [CZ95].

I then move on to discuss i/s/o systems in the frequency domain. One gets the frequency domain version of (1) by taking a formal Laplace transform in (1) to get an equation of the type

$$(3)\qquad \widehat{\Sigma}_{\text{iso}}\colon \begin{bmatrix}\lambda\hat{x}(\lambda)-x^0\\ \hat{y}(\lambda)\end{bmatrix} = S\begin{bmatrix}\hat{x}(\lambda)\\ \hat{u}(\lambda)\end{bmatrix}, \quad \begin{bmatrix}\hat{x}(\lambda)\\ \hat{u}(\lambda)\end{bmatrix}\in\text{dom}\,(S), \quad \lambda\in\Omega,$$

where Ω is some open subset of the complex plane. By definition, the *resolvent set* $\rho(\Sigma_{\text{iso}})$ consists of those points $\lambda\in\mathbb{C}$ for which (3) defines a bounded linear operator from $\left[\begin{smallmatrix}x^0\\ \hat{u}(\lambda)\end{smallmatrix}\right]$ to $\left[\begin{smallmatrix}\hat{x}(\lambda)\\ \hat{y}(\lambda)\end{smallmatrix}\right]$, i.e., to each $\left[\begin{smallmatrix}x^0\\ \hat{u}(\lambda)\end{smallmatrix}\right]\in\left[\begin{smallmatrix}\mathcal{X}\\ \mathcal{U}\end{smallmatrix}\right]$ there is a unique pair $\left[\begin{smallmatrix}\hat{x}(\lambda)\\ \hat{y}(\lambda)\end{smallmatrix}\right]$ which satisfies (1) (since S is closed it will then automatically be true that $\left[\begin{smallmatrix}\hat{x}(\lambda)\\ \hat{y}(\lambda)\end{smallmatrix}\right]$ depends continuously on $\left[\begin{smallmatrix}x^0\\ \hat{u}(\lambda)\end{smallmatrix}\right]$). The bounded linear operator $\widehat{\mathfrak{S}}(\lambda)\colon \left[\begin{smallmatrix}x^0\\ \hat{u}(\lambda)\end{smallmatrix}\right]\to\left[\begin{smallmatrix}\hat{x}(\lambda)\\ \hat{y}(\lambda)\end{smallmatrix}\right]$, defined for all $\lambda\in\rho(\Sigma_{\text{iso}})$, is is called the *i/s/o resolvent matrix* of Σ_{iso}.[1]

The i/s/o resolvent matrix $\widehat{\mathfrak{S}}$ turns out to be analytic in $\rho(\Sigma_{\text{iso}})$, and it satisfies an *i/s/o resolvent identity*, which is a generalization of the standard i/s/o resolvent identity. Since $\widehat{\mathfrak{S}}(\lambda)$ maps $\left[\begin{smallmatrix}\mathcal{X}\\ \mathcal{U}\end{smallmatrix}\right]$ into $\left[\begin{smallmatrix}\mathcal{X}\\ \mathcal{U}\end{smallmatrix}\right]$ this operator can be split into four blocks $\widehat{\mathfrak{S}}(\lambda) := \left[\begin{smallmatrix}\widehat{\mathfrak{A}}(\lambda) & \widehat{\mathfrak{B}}(\lambda)\\ \widehat{\mathfrak{C}}(\lambda) & \widehat{\mathfrak{D}}(\lambda)\end{smallmatrix}\right]$, where $\widehat{\mathfrak{A}}(\lambda)$ maps x^0 into $\hat{x}(\lambda)$, $\widehat{\mathfrak{B}}(\lambda)$ maps $\hat{u}(\lambda)$ into $\hat{x}(\lambda)$, etc. The different components of $\widehat{\mathfrak{S}}(\lambda)$ are called as follows:

[1]It is, of course, possible to define $\widehat{\mathfrak{S}}(\lambda)$ also for $\lambda\notin\rho(\Sigma_{\text{iso}})$ by means of its graph determined by (3). For such λ the operator $\widehat{\mathfrak{S}}(\lambda)$ will still be closed, but it will be unbounded or multi-valued or not defined on all of $\left[\begin{smallmatrix}\mathcal{X}\\ \mathcal{U}\end{smallmatrix}\right]$.

- $\widehat{\mathfrak{A}}$ is the s/s (state/state) resolvent function,
- $\widehat{\mathfrak{B}}$ is the i/s (input/state) resolvent function,
- $\widehat{\mathfrak{C}}$ is the s/o (state/output) resolvent function,
- $\widehat{\mathfrak{D}}$ is the i/o (input/output) resolvent function.

In the literature these four operator-valued function are known under different names. The operator $\widehat{\mathfrak{A}}$ is the usual resolvent of the *main operator* A of the system, i.e., $\widehat{\mathfrak{A}}(\lambda) = (\lambda - A)^{-1}$, where $Ax = \begin{bmatrix} 1_{\mathcal{X}} & 0 \end{bmatrix} S \left[\begin{smallmatrix} x \\ 0 \end{smallmatrix}\right]$ with $\operatorname{dom}(A) = \left\{x \in \mathcal{X} \,\middle|\, \left[\begin{smallmatrix} x \\ 0 \end{smallmatrix}\right] \in \operatorname{dom}(S)\right\}$ (this is the "top left corner" of S). If Σ_{iso} has been constructed from a conservative boundary triplet as described in [AKS12a] and [AKS12b], then $\widehat{\mathfrak{B}}$ is the so called "Gamma field" and $\widehat{\mathfrak{D}}$ is the "Weyl function". Two other names for $\widehat{\mathfrak{D}}$ are the "transfer function" (used in control theory) and the "characteristic function of the main operator" (used in operator theory).

We then continue to discuss the notion of a state/signal system in time and frequency domain. In the time domain the dynamics of a s/s system Σ can be described by an equation of the type

$$\Sigma\colon \begin{bmatrix} \dot{x}(t) \\ x(t) \\ w(t) \end{bmatrix} \in V, \qquad t \in \mathbb{R}^+. \tag{4}$$

A classical future trajectory of (4) is a pair of continuous functions $\left[\begin{smallmatrix} x \\ w \end{smallmatrix}\right]$, with x continuously differentiable, which satisfies (4). By taking the formal Laplace transform of (4) we get the corresponding frequency domain equation

$$\widehat{\Sigma}\colon \begin{bmatrix} \lambda\hat{x}(\lambda) - x_0 \\ \hat{x}(\lambda) \\ \hat{w}(\lambda) \end{bmatrix} \in V, \qquad \lambda \in \Omega, \tag{5}$$

where Ω is some open subset of the complex plane $\mathbb{C}$. The equation (5) defines an analytic family $\widehat{\mathfrak{S}}(\lambda)$ of subspaces of $\left[\begin{smallmatrix} \mathcal{X} \\ \mathcal{X} \\ \mathcal{W} \end{smallmatrix}\right]$, where $\mathcal{X}$ is the state space and $\mathcal{W}$ is the signal space (which for the purpose of this talk may be taken to be the product of the input and output spaces). The family $\widehat{\mathfrak{S}}$ of subspaces is called the *characteristic node bundle*, and it is a special case of an analytic vector bundle defined in the full complex plane $\mathbb{C}$. The *resolvent set* $\rho(\Sigma)$ of the s/s system Σ is defined in such a way that $\lambda \in \rho(\Sigma)$ if and only if Σ has an i/s/o representation Σ_{iso} with $\lambda \in \rho(\Sigma_{\text{iso}})$. This is equivalent to requiring that the characteristic node bundle $\widehat{\mathfrak{S}}(\lambda)$ has a graph representation of a particular type.

In the s/s setting the i/o resolvent function $\widehat{\mathfrak{D}}$ of an i/s/o representation Σ_{iso} is replaced by the *characteristic signal bundle* $\widehat{\mathfrak{F}}$, whose fibers $\widehat{\mathfrak{F}}(\lambda)$ are obtained from the corresponding fiber of the characteristic node bundle $\widehat{\mathfrak{E}}(\lambda)$ by an intersection and a projection, i.e., we take x^0 in (5) to be zero and ignore the value of $\hat{x}(\lambda)$ to get

$$\widehat{\mathfrak{F}}(\lambda) = \left\{ w \in \mathcal{W} \,\middle|\, \begin{bmatrix} 0 \\ z \\ w \end{bmatrix} \in \widehat{\mathfrak{E}}(\lambda) \text{ for some } z \in \mathcal{X} \right\}. \tag{6}$$

A more detailed version of this abstract is found in [Sta15]. Another introduction to what I have been explaining above is written down in [Sta14]. Proofs are given in [AS14]. The connection to boundary triplets and generalized boundary triplets is explained in [AKS12a] and [AKS12b].

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Accretive closure relations for impedance passive systems nodes.

Hans Zwart

1. Introduction

Consider the linear system given by

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t) \\ y(t) &= B^*x(t), \end{aligned}$$

where we assume that A is the infinitesimal generator of a C_0-semigroup on the Hilbert space X and B is a bounded linear map from another Hilbert space U into X. If the semigroup generated by A is contractive and $S \in \mathcal{L}(U)$ is self-adjoint and non-negative, then it is easy to see that the feedback $u = -Sy$ gives the closed loop operator $A - BSB^*$ which generates a contraction semigroup as well. In this paper we show that this result can be extended to a larger class a systems, including boundary triplets as a special case.

The starting point of this research is the following result, [5, Chapter 6], [2], or [3].

Theorem 1. *Let A_{ext} be a linear operator on $X \oplus U$ of the form*

$$A_{\text{ext}} = \begin{pmatrix} A_1 \\ A_{21} \;\; 0 \end{pmatrix}.$$

If A_{ext} generates a contraction semigroup and $S \in \mathcal{L}(U)$ satisfies $Re\langle Su, u\rangle \geq \varepsilon\|u\|^2$ for all $u \in U$ and some $\varepsilon > 0$, then the operator A_S generates a contraction semigroup on X. Here A_S is defined as

$$A_S x = A_1 \begin{pmatrix} x \\ SA_{21}x \end{pmatrix}.$$

with domain $D(A_S) = \{x \in D(A_{21})$ such that $\left(\begin{smallmatrix} x \\ SA_{21}x \end{smallmatrix}\right) \in D(A_{\text{ext}})\}$

By taking $S = 0$ and

$$A_{\text{ext}} = \begin{pmatrix} 0 & \frac{d}{d\zeta} \\ \frac{d}{d\zeta} & 0 \end{pmatrix}$$

on $H^1(\mathbb{R})^2$, A_S becomes bounded, but not closed. Thus the condition that $\text{Re}S \geq \varepsilon I > 0$ cannot be weakened in general.

In the following session we show that under an extra condition on A_{ext} the condition on S may be weakened.

2. Main result

To formulate the extra condition on A_{ext} we return to the system introduced in the beginning of the previous section. However, now we allow for unbounded B's. Before doing so we have to introduce some notation. Let A be an infinitesimal generator, and let A^* denotes its adjoint. By X_1 we denote $D(A)$ with the graph norm, and by X_{-1} we denote that dual (with respect to X) of $D(A^*)$. It is well-known that A has a bounded extension from X into X_{-1}. We denote this extension by $\tilde{A}$. For $B \in \mathcal{L}(U, X_{-1})$ and $C \in \mathcal{L}(X_1, U)$ we define the system node Σ as

$$\begin{pmatrix} A\&B \\ C\&0 \end{pmatrix} \begin{pmatrix} x \\ u \end{pmatrix} = \begin{pmatrix} \tilde{A}x + Bu \\ y \end{pmatrix} \tag{1}$$

with domain those pairs (x, u) such that $\tilde{A}x + Bu \in X$. For $x \in D(A)$, y equals Cx, see also [4]. We assume that Σ has no *explicite feedthrough*, i.e., for all $(0, u) \in D(\Sigma)$ we have that

$$\begin{pmatrix} C\&0 \end{pmatrix} \begin{pmatrix} 0 \\ u \end{pmatrix} = 0.$$

Theorem 2. *Let A_{ext} be the linear operator on $X \oplus U$ defined by*

$$A_{\text{ext}} = \begin{pmatrix} A\&B \\ -C\&0 \end{pmatrix}$$

with domain $D(\Sigma)$. Assume that

- *If A_{ext} generates a contraction semigroup;*
- *There exists an s in the resolvent set of A and an $m > 0$ such that $(sI - A)^{-1}Bu\| \geq m\|u\|^2$;*
- *$S \in \mathcal{L}(U)$ satisfies $Re\langle Su, u\rangle \geq 0$ for all $u \in U$,*

then the operator A_S generates a contraction semigroup on X. Here A_S is defined as

$$A_S x = (A\&B)\begin{pmatrix} x \\ u \end{pmatrix}, \qquad \textit{with } \left(\begin{smallmatrix} x \\ u \end{smallmatrix}\right) \in D(A_S) \tag{2}$$

and $D(A_S)$ given by

$$\begin{aligned} D(A_S) = \{x \in X \mid \ & \textit{there exists a } u \in U \textit{ such that } \left(\begin{smallmatrix} x \\ u \end{smallmatrix}\right) \in D(\Sigma) \\ & \textit{and } -S(C\&0)\begin{pmatrix} x \\ u \end{pmatrix} = u\}. \end{aligned} \tag{3}$$

By our assumption, the u in (3) is unique for every x and thus $A_S x$ is well-defined.

We end with an example. Let (U, Γ_0, Γ_1) be a boundary triplet with maximal operator A_0^*. We define the system node or "equivalently" A_{ext} as

$$A_{\text{ext}}\begin{pmatrix} x \\ u \end{pmatrix} = \begin{pmatrix} iA_0^* x \\ i\Gamma_1 x \end{pmatrix} \tag{4}$$

with domain

$$D(A_{\text{ext}}) = \left\{\begin{pmatrix} x \\ u \end{pmatrix} \in D(A_0^*) \mid \Gamma_0 x = u\right\}.$$

By the properties of the boundary triplet, the conditions in the theorem are satisfied, and so for $S + S^* \geq 0$ we have that A_S generates a contraction semigroup on X. By the form of our extended operator we see that $A_S x = A_0^* x$ for those $x \in D(A_0^*)$ for which $\Gamma_0 x = -iS\Gamma_1 x$. Thus we made a restriction of the maximal operator by imposing boundary conditions. This reproves the corresponding results in [1].

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An isopectral problem for the conservative Camassa–Holm flow

Aleksey Kostenko
(joint work with Jonathan Eckhardt)

Over the last two decades, a lot of work has been devoted to the Cauchy problem for the Camassa–Holm equation, a nonlinear wave equation, given by

$$u_t - u_{xxt} = 2u_x u_{xx} - 3uu_x + uu_{xxx}, \quad u|_{t=t_0} = u_0. \tag{1}$$

The Camassa–Holm equation first appeared as an abstract bi-Hamiltonian partial differential equation in an article of Fokas and Fuchssteiner [10]. However, it did not receive much attention until Camassa and Holm [5] derived it as a nonlinear wave equation which models unidirectional wave propagation on shallow water.

One of the most eminent properties of the Camassa–Holm equation lies in the fact that it is formally integrable in the sense that there is an associated Lax pair. The isospectral problem of this Lax pair turned out to be

$$-f''(x) + \frac{1}{4} f(x) = z\,\omega(x,t) f(x), \quad x \in \mathbb{R}, \tag{2}$$

where $\omega = u - u_{xx}$ and $z \in \mathbb{C}$ is a complex spectral parameter. Of course, (inverse) spectral theory for this Sturm–Liouville problem is of peculiar interest for solving the Cauchy problem of the Camassa–Holm equation; [1, 2, 3, 9].

A particular kind of solutions of the Camassa–Holm equation are the so-called *multi-peakon solutions*. These are solutions of the form

$$u(x,t) = \sum_{n=1}^{N} p_n(t)\,\mathrm{e}^{-|x-q_n(t)|}, \tag{3}$$

where the functions on the right-hand side satisfy the following nonlinear system of ordinary differential equations:

$$q_n' = \sum_{k=1}^{N} p_k\,\mathrm{e}^{-|q_n-q_k|}, \qquad p_n' = \sum_{k=1}^{N} p_n p_k\,\mathrm{sgn}(q_n-q_k)\,\mathrm{e}^{-|q_n-q_k|}. \tag{4}$$

Note that the system (4) is Hamiltonian, that is,

$$\frac{dq_n}{dt} = \frac{\partial H(p,q)}{\partial p_n}, \qquad \frac{dp_n}{dt} = -\frac{\partial H(p,q)}{\partial q_n}, \tag{5}$$

with the Hamiltonian given by

$$H(p,q) = \frac{1}{2}\sum_{n,k=1}^{N} p_n p_k\,\mathrm{e}^{-|q_n-q_k|} = \frac{1}{4}\|u\|^2_{H^1(\mathbb{R})}. \tag{6}$$

It turns out that the behavior of multi-peakon solutions crucially depends on whether all the heights p_n of the single peaks are of the same sign or not. In the first case, all the positions q_n of the peaks stay distinct, move in the same direction and the system (4) allows a unique global solution. Otherwise, some of the positions q_n of the peaks will collide eventually, which causes the corresponding heights p_n to blow up in finite time [5]. All this happens in such a way that the solution u in (3) stays uniformly bounded in $H^1(\mathbb{R})$ but its derivative develops a singularity at the points where two peaks collide.

However, it turned out that it is always possible to continue weak solutions beyond wave-breaking. In order to end up with unique continuations, one has to impose additional constraints on the solutions. For example, if one requires the energy of the solutions to be conserved, one is led to the notion of *global conservative solutions* [4, 11, 12]. For the corresponding Cauchy problem to be well-posed, it is necessary to introduce an additional quantity, which measures the energy density of the solution (as done recently in [4, 11]). Following [11], *a global conservative solution* consist of a pair (u, μ) where μ is a non-negative Borel measure with absolutely continuous part determined by u via

$$\mu_{\mathrm{ac}}(B, t) = \int_B |u(x,t)|^2 + |u_x(x,t)|^2 dx, \quad t \in \mathbb{R} \tag{7}$$

for each Borel set $B \in \mathcal{B}(\mathbb{R})$. Within this picture, blow-up of solutions corresponds to concentration of energy (measured by μ) to sets of Lebesgue measure zero. For further details we refer to [4, 11] (see also [12], where a detailed description of global conservative multi-peakon solutions was given).

For the special case of multi-peakon solutions, the weight ω in (2) is always a finite sum of weighted Dirac measures. The corresponding spectral problem (2) is equivalent to the one for an indefinite Krein–Stieltjes string [13, §13]. This connection and the solution of the corresponding inverse problem due to Krein (employing Stieltjes theory of continued fractions) has successfully been employed by Beals, Sattinger and Szmigielski [1] in order to study multi-peakon solutions (in the sense of (4)). In particular, they noticed that in the indefinite case, the inverse problem is not always solvable within the class of spectral problems (2), which directly corresponds to the fact that the system (4) may blow up. It is the purpose of my talk to introduce a generalized isospectral problem for global conservative multi-peakon solutions of the Camassa–Holm equation. Of course, an eligible spectral problem also has to incorporate the singular part of μ in some way and indeed, it turns out that the appropriate spectral problem is given by

$$-f''(x) + \frac{1}{4}f(x) = z\,\omega(x,t)f(x) + z^2\upsilon(x,t)f(x), \quad x \in \mathbb{R}, \tag{8}$$

where $\upsilon(\,\cdot\,, t)$ denotes the singular part of $\mu(\,\cdot\,, t)$ and $z \in \mathbb{C}$ is a complex spectral parameter. The idea for considering this particular spectral problem goes back to work of Krein and Langer [14] (see also [15]) on the indefinite moment problem.

More precisely, let $N(t) \in \mathbb{N}$ and $x_1(t), \dots, x_{N(t)}(t) \in \mathbb{R}$ be a strictly increasing sequence such that

$$\omega(\,\cdot\,, t) = \sum_{n=1}^{N(t)} 2p_n(t)\delta_{x_n(t)} \quad \text{and} \quad \upsilon(\,\cdot\,, t) = \sum_{n=1}^{N(t)} \upsilon_n(t)\delta_{x_n(t)}, \tag{9}$$

where $p_n(t) \in \mathbb{R}$, $\upsilon_n(t) \geq 0$ and $|p_n(t)| + \upsilon_n(t) > 0$ for each $n = 1, \dots, N(t)$. One should note that it is always possible to go back and forth between the pair (u, μ) and the measures ω and υ. Now consider the family of spectral problems (8).

The spectrum of (8) will be denoted with $\sigma(t)$ and the corresponding norming constants with $\gamma_\lambda(t)$ for each $\lambda \in \sigma(t)$. Note that $\sigma(t)$ consists of $N(t) + \#\mathrm{supp}(\upsilon)$ simple non-zero eigenvalues. For further details concerning the direct and inverse spectral theory for the generalized indefinite spectral problem (8) we refer to [7, 8].

The connection between these spectral problems and the conservative Camassa–Holm equation now lies in the following observation.

Theorem 1 ([6]). *The pair* (u, μ) *is a global conservative multi-peakon solution of the Camassa–Holm equation if and only if the problems* (8) *are isospectral with*

$$\gamma_\lambda(t) = \mathrm{e}^{-t/2\lambda}\gamma_\lambda(0), \quad t \in \mathbb{R},\ \lambda \in \sigma(0). \tag{10}$$

Moreover, utilizing the solution of the indefinite moment problem given by M. G. Krein and H. Langer in [14], we show that the conservative Camassa–Holm equation is integrable by the inverse spectral transform in the multi-peakon case.

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The inverse spectral problem for indefinite strings

Jonathan Eckhardt
(joint work with Aleksey Kostenko)

Classical objects in spectral theory are the differential equation

$$-u'' = z\,u\,\omega \tag{1}$$

on an interval $[0, L)$ (where $L \in (0, \infty]$, ω is a non-negative Borel measure on $[0, L)$ and z is a complex spectral parameter) and the corresponding Weyl–Titchmarsh function m, which encodes all the spectral information. A remarkable and well-known result of M. G. Krein and L. de Branges identifies the totality of all possible Weyl–Titchmarsh functions for this class of spectral problems with the class of so-called Stieltjes functions in a bijective way. In other words, they were able to uniquely solve the inverse spectral problem for (1). My talk is concerned with further questions in this direction which are not far to seek: *What happens if ω is allowed to be a real-valued Borel measure on $[0, L)$ instead of just non-negative? Is there an equally concise analogue of the solution of the inverse spectral problem?*

Although there does not seem to exist simple answers to these questions, one way to overcome those problem was suggested by M. G. Krein and H. Langer [2] by means of extending the class of spectral problems. Inspired by their work, we consider the modified differential equation

$$-u'' = z\,u\,\omega + z^2 u\,\upsilon \tag{2}$$

on an interval $[0, L)$, where ω is a real-valued distribution in $H^{-1}_{\mathrm{loc}}[0, L)$ and υ is a non-negative Borel measure on $[0, L)$. Of course, this differential equation has to be understood in a weak sense: We say that a function $f \in H^1_{\mathrm{loc}}[0, L)$ is a solution of (2) if there is a constant $\Delta_f \in \mathbb{C}$ so that

$$\Delta_f h(0) + \int_0^L f'(x)h'(x)dx = z\,\omega(fh) + z^2 \int_{[0,L)} fh\,d\upsilon \tag{3}$$

for all $h \in H^1_{\mathrm{c}}[0, L)$. In this case, the constant Δ_f is uniquely determined and will henceforth always be denoted with $f'(0-)$ for apparent reasons.

Associated with the differential equation (2) (and suitable boundary conditions at the endpoints) is the Weyl–Titchmarsh function m defined on $\mathbb{C}\backslash\mathbb{R}$ by

$$m(z) = \frac{\psi'(z, 0-)}{z\psi(z, 0)}, \quad z \in \mathbb{C}\backslash\mathbb{R}, \tag{4}$$

where $\psi(z, \cdot\,)$ is a Weyl solution of the differential equation (2) (that is, a solution with a certain behaviour at L). The Weyl–Titchmarsh function m is a Herglotz–Nevanlinna function and therefore admits an integral representation of the form

$$m(z) = c_1 z + c_2 - \frac{1}{Lz} + \int_{\mathbb{R}} \frac{1}{\lambda - z} - \frac{\lambda}{1 + \lambda^2}\, d\mu(\lambda), \quad z \in \mathbb{C}\backslash\mathbb{R}, \tag{5}$$

where the measure μ turns out to be a *spectral measure* for (the operator part of) a particular self-adjoint realization of (2) in a suitable Hilbert space.

The main result of [1] and my talk is the identification of the totality of all possible Weyl–Titchmarsh functions for this class of spectral problems with the whole class of Herglotz–Nevanlinna functions in a bijective way, giving an indefinite analogue of the solution of the inverse spectral problem for definite strings.

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Numerical Evans function method for spectral stability of solitary waves in periodic media

Tomáš Dohnal

(joint work with Elisabeth Blank)

For solitary waves of PDEs in one spatial dimension the eigenvalues of the linearized stability problem can often be determined as zeros of the corresponding Evans function [2, 3]. Zeros of the Evans function detect linear dependence of the stable and unstable manifolds of the linearized problem and thus coincide with eigenvalues of the linearized operator. The analytic Evans function has been used extensively in the literature for problems with constant or asymptotically constant ($|x| \to \infty$) coefficients. In such cases the stable and unstable manifolds are spanned by exponentially decaying Fourier waves.

In problems with periodic coefficients the manifolds are spanned by Bloch waves, which can be generally computed only numerically. We have applied this method in [4] to the stability of gap solitons $u(x,t) = e^{-\mathrm{i}\omega t}\phi(x)$ of the periodic nonlinear Schrödinger equation (PNLS)

$$\mathrm{i}\partial_t u + \partial_x^2 u - V(x)u + \Gamma(x)|u|^2 u = 0, \qquad x, t \in \mathbb{R}$$

with a linear potential V that is periodic for $|x| \geq L$ for some $L \geq 0$ and with ω in any spectral gap of $-\partial_x^2 + V$. As we show, a numerically stable approximation of the manifolds requires the use of exterior algebra and Grassmanian preserving ODE integrators. Evans function zeros can then be detected via the complex argument principle.

A motivation for the Evans function method as opposed to a standard discretization of the eigenvalue problem and a numerical evaluation of its eigenvalues is the absence of spectral pollution in the former method. Also existing rigorous methods on nonlinear stability are often inapplicable in periodic problems. For instance methods of Grillakis-Shatah-Strauss type [1] apply only to gap solitons of (PNLS) with ω inside the semi-infinite gap of the spectrum of the purely periodic operator $-\partial_x^2 + V_\infty$.

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