Mathematisches Forschungsinstitut Oberwolfach

Report No. 3/2015

DOI: 10.4171/OWR/2015/3

Geometric Topology

Organised by
Martin Bridson, Oxford
Clara Löh, Regensburg
Thomas Schick, Göttingen

18 January – 24 January 2015

Abstract. Geometric topology has seen significant advances in the understanding and application of infinite symmetries and of the principles behind them. On the one hand, for advances in (geometric) group theory, tools from algebraic topology are applied and extended; on the other hand, spectacular results in topology (e.g., the proofs of new cases of the Novikov conjecture or the Atiyah conjecture) were only possible through a combination of methods of homotopy theory and new insights in the geometry of groups. This workshop focused on the rich interplay between algebraic topology and geometric group theory.

Mathematics Subject Classification (2010): 18-xx, 19-xx, 20-xx, 55-xx, 57-xx.

Introduction by the Organisers

Geometric topology has seen significant advances in the understanding and application of infinite symmetries and of the principles behind them. This workshop focused on the rich interplay between algebraic topology and geometric group theory. The research fields of the 53 participants of the workshop covered homotopy theory, manifold topology, low-dimensional topology, geometric group theory, and geometry of topological groups.

Some of the main topics of the workshop were:

- Variations of hyperbolicity
- Rigidity versus flexibility of geometry and topology
- Homological properties of manifolds
- Bounded cohomology and its applications
- Classification of groups and their representations by geometric means

This wide range of interests is united on several levels: The type of problems considered and aspired goals of research are driven by related ideas (e.g., rigidity phenomena). Also the tools and techniques are shared (e.g., the language of algebraic topology, probabilistic methods). Moreover, the topics listed above are interrelated in various ways (e.g., classically, hyperbolicity is a strong indicator for rigidity). The workshop offered the opportunity to strengthen the bonds between these fields.

The formal part of the programme consisted of twenty regular research talks and a *Gong Show* of 10 minutes talks by eleven PhD students and recent postdocs. This formal part was complemented by a variety of lively discussions in smaller groups.

On the one hand, the research talks communicated and documented the current state of the art. On the other hand, many of the talks also advertised open problems linking topology and group theory. For example, Ian Leary proposed further variations of the question by Eilenberg and Ganea on the relation between cohomological dimension and geometric dimension of groups; and Kevin Schreve proposed the *action dimension conjecture* bounding the minimal dimension of a contractible manifold on which a group acts by twice the L^2-cohomological dimension. This is a relative of the Singer conjecture, which predicts concentration of L^2-cohomology in middle degree of an aspherical manifold. Michał Marcinkowski advertised new candidates (constructed via Davis' asphericalization construction and surgery) for counterexamples of Gromov's macroscopic dimension conjecture (which bounds the macroscopic dimension of an n-dimensional manifold with positive scalar curvature by $n-2$).

We will now describe the main topics and some selected recent achievements that were discussed during this workshop in more detail:

Extended notions of hyperbolicity. Because of its strong relation with rigidity, many attempts have been made in the past to vary the notion of hyperbolicity. A particularly versatile generalisation of hyperbolicity for groups is acylindrical hyperbolicity, introduced by Denis Osin. Recently, Denis Osin (in joint work with Hull) showed that acylindrically hyperbolic groups with trivial finite radical are highly transitive, and hence do not satisfy mixed identities (a notion related to universal equivalence of groups). Another type of variation of hyperbolicity was proposed by Bogdan Nica (in joint work with Jan Spakula): They introduced superbolicity as a concept conveniently interpolating between CAT(-1)-spaces and Green metrics on hyperbolic groups on the one hand and good hyperbolicity and strong bolicity on the other hand.

Rigidity versus flexibility of geometry and topology. Rigidity phenomena in geometry, topology, and group theory have significantly shaped and fused these fields. A prominent example is the Baum-Connes conjecture connecting these fields via operator algebras. Using a graphical small cancellation technique and probabilistic tools, Damian Osajda constructed finitely generated groups that contain *isometrically* embedded (into specific Cayley graphs) expanders. Topologically, these groups lead to examples of aspherical manifolds whose fundamental groups contain quasi-isometrically embedded expanders. Groups of this type provide an interesting source of counterexamples, e.g., in the context of the Baum-Connes conjecture.

Romain Tessera studied a new geometric relation on infinite groups that lies between the (purely algebraic) commensurability and the (very flexible) quasi-isometry and showed its success on generalized Baumslag-Solitar groups.

L^2-Invariants and their applications. L^2-invariants, in particular L^2-Betti numbers, L^2-torsion, and Novikov-Shubin invariants are a powerful and well-established toolbox of invariants. Many aspects of them, on the other hand, still remain mysterious.

Łukasz Grabowski presented the first counterexamples to a conjecture of Lott and Lück; showing that there are manifolds such that some of the Novikov-Shubin invariants are equal to 0. This is particularly important in light of the fact that their positivity used to be a standard assumption in the treatment of secondary L^2-invariants, in particular the L^2-torsion.

On the other hand, by now it is known that this condition can be replaced by the much weaker condition of "L^2-determinant class", which is known to be satisfied in many cases. The development and use of L^2-torsion therefore remains meaningful and is vigorously carried out.

Wolfgang Lück reported on a new development here: the twisted L^2-torsion function, which is a powerful invariant. He, and also Stefan Friedl, presented a host of structural and computational results of this new invariant, in particular for 3-manifolds. It turns out to recover the hyperbolic volume, also the Thurston norm, and the information contained in the classical Alexander polynomial and modern twisted version of it. Stefan Friedl, on the other hand, explained an explicit and easy combinatorial algorithm to compute the Thurston norm for certain classes of 3-manifolds; This algorithm is based on associated polyhedra that also play a role in the calculation of the L^2-torsion function.

Bounded cohomology and its applications. Bounded cohomology in higher degrees has remained rather mysterious in the past decades. However, in recent years, a young community has evolved and taken first, promising steps to approach bounded cohomology in higher degrees. For example, Tobias Hartnick and Andreas Ott related problems in bounded cohomology to partial differential equations and proved that continuous bounded cohomology of $\mathrm{SL}(2, \mathbb{R})$ in degree four is trivial, thereby confirming a conjecture of Monod in a special case. Michelle Bucher-Karlsson (in joint work with Burger and Iozzi) adapted an explicit construction by Goncharov to prove that the comparison map between continuous bounded cohomology and continuous cohomology of $\mathrm{PSL}(n, \mathbb{C})$ is an isomorphism in degree 3, thereby proving a classical conjecture by Dupont in a special case. This result is not only interesting in its own right, but also has applications to rigidity of volume representations of hyperbolic 3-manifolds. Both of the above approaches to continuous bounded cohomology have the potential to generalise to further higher degrees. For discrete groups, Roberto Frigerio (in joint work with Pozzetti and Sisto) extended the framework of quasi-morphisms to higher degrees and combined this with hyperbolically embedded subgroups, which gives interesting inheritance results for bounded cohomology in higher degrees.

All of these topics are also related to the *classification of groups and their representations by geometric means*.

The Mathematische Forschungsinstitut Oberwolfach provided an excellent environment and inspiring atmosphere for this workshop and we are grateful for its hospitality.

Acknowledgement: The MFO and the workshop organizers would like to thank the National Science Foundation for supporting the participation of junior researchers in the workshop by the grant DMS-1049268, “US Junior Oberwolfach Fellows”.

Workshop: Geometric Topology

Table of Contents

Abstracts

Group laws for finite simple groups

Andreas Thom
(joint work with Gady Kozma)

We provide new bounds for the divisibility function of the free group F_2 and construct short laws for the symmetric groups $\mathrm{Sym}(n)$, see [5] for details. The construction is random and relies on the classification of the finite simple groups.

Let Γ be a finitely generated group and let us consider some subset $S \subset \Gamma$ such that $\Gamma = \langle S \rangle$ with $|S| < \infty$ and $S = S^{-1}$. Each element of Γ can be written as $s_1 \dots s_k$ in many different ways. We denote by $|\gamma|_S$ the length of the shortest such word, i.e., the distance of γ and e in the Cayley graph $\mathrm{Cay}(\Gamma, S)$.

Definition 1 (divisibility function)**.** *In the situation above, we set:*

(1) $d_\Gamma(\gamma) = \inf\{|\Gamma : \Lambda| \mid \gamma \notin \Lambda \leq \Gamma\}$, *where* $\gamma \in \Gamma$.
(2) $D_\Gamma(k) = \max\{d_\Gamma(\gamma) \mid |\gamma|_S = k\}$

An easy consequence of the Prime Number Theorem gives the following:

Observation 2. $D_{\mathbb{Z}}(m) \approx \log m$.

The study of the function D_Γ for a different finitely generated groups is usually considerably more complicated and provides a challanging task. Bogopolski [4] asked whether $D_{\mathbb{F}_2}(m) \approx \log m$ also holds. However, this was answered negatively in work of Bou-Rabee and McReynolds:

Theorem 3 (Bou-Rabee and McReynolds, [1])**.**

$$D_{\mathbb{F}_2}(m) \geq \frac{\log(m)^2}{C \log(\log(m))},$$

where C is a large positive constant.

Throughout, we use the letter C to denote a constant, which is usually large and might change from statement to statement. We are able to improve this bound considerably:

Theorem 4 (see [5])**.** *There exist $C > 0$ such that the following holds for all m:*

$$D_{\mathbb{F}_2}(m) \geq \exp\left(\left(\frac{\log(m)}{C \log(\log(m))}\right)^{\frac{1}{4}}\right).$$

Using the statement of the famous conjecture of Babai (not yet proved), that the diameter of $\mathrm{Sym}(n)$ is polynomial in n, we also proved:

Theorem 5 (see [5])**.** *Babai's conjecture implies that there exists $C > 0$ such that the following holds for all m:*

$$D_{\mathbb{F}_2}(m) \geq \exp\left(\frac{\log(m)}{C \log(\log(m))}\right) = m^{\frac{1}{C \log(\log(m))}}.$$

The study of the functions $d_{\mathbb{F}_2}$ and $D_{\mathbb{F}_2}$ is closely related to the study of laws for the symmetric group. Our main result in this direction is the following:

Theorem 6. *The length of the shortest non-trivial law for* Sym(n) *is less than*

$$\exp(C \log^4 n \log\log n).$$

A key technical ingrediant is the following result on the diameter of Cayley graphs of symmetric groups.

Theorem 7 (Helfgott-Seress, see [3])**.** *If σ and τ generate* $\mathrm{Sym}(n)$*, then*

$$\mathrm{diam}(\mathrm{Cay}(\mathrm{Sym}(n), \sigma^{\pm 1}, \tau^{\pm 1})) \leq \exp(C \log^4 n \log\log n).$$

Babai conjectured that an even better bound can be given for the diameter of the symmetric group. This polynomial bound can be used to prove our second theorem. The only nontrivial lower bound for the length of a law for $\mathrm{Sym}(n)$ is $2n$ and was proved by Buskin [2].

References

[1] K. Bou-Rabee and D. B. McReynolds, Asymptotic growth and least common multiples in groups, Bull. Lond. Math. Soc. 43 (2011), no. 6, 1059–1068.
[2] N. V. Buskin, Economical separability in free groups, Sibirsk. Mat. Zh. 50 (2009), no. 4, 765–771 (Russian, with Russian summary); English transl., Sib. Math. J. 50 (2009), no. 4, 603–608.
[3] H. Helfgott and A. Seress, On the diameter of permutation groups, Annals of Math. 179 (2014), no. 2, 611–658.
[4] V. D. Mazurov and E. I. Khukhro (eds.), The Kourovka notebook, Sixteenth edition, Russian Academy of Sciences Siberian Division Institute of Mathematics, Novosibirsk, 2006.
[5] G. Kozma and A. Thom, Divisibility and group laws for finite simple groups, arXiv:1403.2324.
[6] E. Landau, Uber die Maximalordnung der Permutationen gegebenen Grades, Archiv der Math. und Phys. (1903), 92–103.

Kazhdan projections in Banach spaces

Piotr W. Nowak
(joint work with C. Druţu)

A Kazhdan projection is a central idempotent p in the maximal group C^*-algebra $C^*_{\max}(G)$ of G such that for every unitary representation π of G on a Hilbert space H the image $\pi(p) \in B(H)$ is the orthogonal projection for H onto the subspace of invariant vectors $H^\pi \subseteq H$. It was proved by Akemann and Walters [1] that the existence of a Kazhdan projection characterizes Kazhdan's property (T). Another proof of this fact was given by Valette [7]. Kazhdan projections have many applications, in particular they are the source of the few known counterexamples to certain versions of the Baum-Connes conjecture. The reason is that K-theory classes represented by projections of Kazhdan-type usually do not live in the image of the Baum-Connes assembly map. Nevertheless, Kazhdan projections have been considered mysterious objects and no explicit constructions of such projections were known.

Our main result is a new method to construct Kazhdan projections in the general setting of uniformly convex Banach spaces. The main tools are random walks and the associated Markov operators, for which we provide new estimates and convergence results. For a locally compact group G consider an (linear) isometric representation π on a uniformly convex Banach space E. The subspace of invariant vectors E^π has a natural π-invariant complement E_π, so that $E = E_\pi \oplus E^\pi$ [2, 3]. The Kazhdan constant of π relative to a compact generating set S is the number $\inf_{v \in E_\pi} \sup_{s \in S} \|\pi_s v - v\|$ and we say that π has a spectral gap if this constant is positive. For a certain large class of admissible probability measures μ on G we consider the Markov operator $A_\pi^\mu = \int_G \pi_g v d\mu$.

We first explain the quantitative relation between projections onto invariant vectors and spectral gaps in the setting of uniformly convex spaces and in particular prove

Theorem 1. Let μ be an admissible measure. If a representation π as above has a positive Kazhdan constant then the restriction of the Markov operator to E_π satisfies $\|A_\pi^\mu|_{E_\pi}\| \leq \lambda < 1$, where λ depends only on the Kazhdan constant of π, the modulus of uniform convexity of E and the measure μ.

Moreover, the projection $\mathcal{P}_\pi : E \to E^\pi$ along E_π is given by the formula

$$\mathcal{P}_\pi = I - \left(\sum_{n=0}^{\infty} (A_\pi^\mu)^n \right) (I - A_\pi^\mu).$$

and $\mathcal{P}_\pi = \lim_{n\to\infty} (A_\pi^\mu)^n$, where the convergence is uniform for a family $\mathcal{F}$ of isometric representations as long as λ above is uniform for all representations in $\mathcal{F}$.

A quantitative converse to the above holds as well. From the above theorem we derive an explicit construction of Kazhdan projections in various group Banach algebras. Let $\mathcal{F}$ be a family of isometric representations of G on a uniformly convex family of Banach spaces. Let $C_c(G)$ denote the convolution algebra of compactly supported continuous functions on G. For $f \in C_c(G)$ define

$$\|f\|_{\mathcal{F}} = \sup_{\pi \in \mathcal{F}} \|\pi(f)\|$$

and let $C_{\mathcal{F}}(G)$ be the Banach algebra obtained as a completion of $C_c(G)$ in the above norm. A Kazhdan projection in $C_{\mathcal{F}}(G)$ is then a central idempotent $p \in C_{\mathcal{F}}(G)$ such that $\pi(p) = \mathcal{P}_\pi$ for every $\pi \in \mathcal{F}$.

Theorem 2. There exists a Kazhdan projection in $C_{\mathcal{F}}(G)$ if and only if there is a uniform positive lower bound on the Kazhdan constants for all $\pi \in \mathcal{F}$.

This construction of Kazhdan projections is new in particular in the setting of Hilbert space and property (T), where $\mathcal{F}$ is taken to be the collection of all unitary representations of G. It also allows to give a direct comparison of various versions of property (T) in the context of Banach spaces: properties (TE), FE studied in [2] and Lafforgue's reinforced Banach property (T), introduced in [5].

Kazhdan projections can be viewed as invariant means in the setting of property (T). We apply them to show a natural generalization of property (τ) to the context of a uniformly convex Banach space E and show that it is equivalent to the fact that the related family of Cayley graphs of finite quotients forms a family of E-expanders. We also obtain results in ergodic theory, where we apply Kazhdan projections to a question posed by Kleinbock and Margulis on shrinking target problems.

Finally, we show a new construction of non-compact ghost projections for warped cones, a class of metric spaces constructed by Roe using an action of group on a compact space [6]. Ghosts are certain operators on Hilbert modules, that are locally invisible at infinity, yet are not compact. Such ghost projections are known to give rise to K-theory classes that are obstructions to the coarse Baum-Connes conjecture. Their existence was previously established only for expanders, Willett and Yu asked for new examples.

Theorem 3. Let G be a finitely generated group acting ergodically on a compact metric probability space M by measure preserving Lipschitz homeomorphisms. If the corresponding unitary representation of G on $L_2(M)$ has a spectral gap then the warped cone $\mathcal{O}_G(M)$ has a non-compact ghost projection, which is a limit of finite propagation operators.

We conjecture that the coarse Baum-Connes conjecture fails for warped cones provided by the above theorem. As a particular example consider the action of certain free subgroups $G = \mathbb{F}_2$ on $M = SU(2)$. The spectral gap property for many such subgroups was established by Bourgain and Gamburd [4].

In relation to the above theorem Guoliang Yu posed the following question: *is there an action with a spectral gap such that the corresponding warped cone does not contain a coarsely embedded sequence of expanders?*

References

[1] C. A. Akemann, M. E. Walter, *Unbounded negative definite functions*, Canad. J. Math. **33** (1981), no. 4, 862–871.

[2] U. Bader, A. Furman, T. Gelander, N. Monod, *Property (T) and rigidity for actions on Banach spaces*, Acta Math. **198** (2007), no. 1, 57–105.

[3] U. Bader, C. Rosendal, R. Sauer, *On the cohomology of weakly almost periodic group representations*, J. Topol. Anal. **6** (2014), no. 2, 153–165.

[4] J. Bourgain, A. Gamburd, *On the spectral gap for finitely-generated subgroups of* SU(2). Invent. Math. 171 (2008), no. 1, 83–121.

[5] V. Lafforgue, *Un renforcement de la propriété (T)*, Duke Math. J. **143** (2008), no. 3, 559–602.

[6] J. Roe, *Warped cones and property A*. Geom. Topol. **9** (2005), 163–178.

[7] A. Valette, *Alain Minimal projections, integrable representations and property (T)*, Arch. Math. (Basel) **43** (1984), no. 5, 397–406.

Enhanced hyperbolicity: why and how

Bogdan Nica
(joint work with Ján Špakula)

Enhanced hyperbolicity. It is a well-known fact that CAT(-1) spaces are Gromov hyperbolic. Both are metric notions of negative curvature, the difference being that the CAT(-1) condition is sharp whereas hyperbolicity is coarse. A simple illustration of this difference is the following: in CAT(-1) spaces, pairs of points are joined by unique geodesics; in hyperbolic spaces, there may be several geodesics but they are uniformly close.

Enhanced hyperbolicity is an informal term denoting a middle ground between the sharp and the coarse. More specifically, though still not entirely precise, we would like hyperbolic groups to admit geometric actions on hyperbolic spaces that have additional CAT(-1) features. Our concrete motivations come from the analytic theory of hyperbolic groups, though somewhere in the background is the old and still unresolved foundational question whether hyperbolic groups admit geometric actions on CAT(-1) spaces. Below, we illustrate the analytic uses of enhanced hyperbolicity in two instances. But before we get to imposing additional demands, we should be ready to make some concessions. Namely, we give ourselves the flexibility of working with *roughly geodesic* hyperbolic spaces. A metric space X is said to be roughly geodesic if there is a constant $C \geq 0$ so that, for any pair of points $x, y \in X$, there is a (not necessarily continuous) map $\gamma : [a, b] \to X$ satisfying $\gamma(a) = x$, $\gamma(b) = y$, and $|s - s'| - C \leq |\gamma(s), \gamma(s')| \leq |s - s'| + C$ for all $s, s' \in [a, b]$.

Why? After the deep and groundbreaking work of Vincent Lafforgue, the resolution of the Baum - Connes conjecture for hyperbolic groups hinged on the following geometric ingredient: every hyperbolic group Γ admits a geometric action on a roughly geodesic, *strongly bolic* hyperbolic space. A roughly geodesic hyperbolic space is said to be strongly bolic if for every $\eta, r > 0$ there exists $R > 0$ such that $|x, y|+|z, t| \leq r$ and $|x, z|+|y, t| \geq R$ imply $|x, t|+|y, z| \leq |x, z|+|y, t|+\eta$. Mineyev and Yu [5] show that, indeed, every hyperbolic group Γ can be endowed with an 'admissible' metric - that is, a metric which is Γ-invariant, quasi-isometric to the word metric, and roughly geodesic - which is furthermore strongly bolic.

In a different direction, the geometric ingredient needed in [6] is the following: every hyperbolic group Γ admits a geometric action on a roughly geodesic, *good* hyperbolic space X. One can then obtain a proper affine isometric action of Γ on an L^p-space associated to the double boundary $\partial X \times \partial X$. Here, we say that a hyperbolic space X is good if the following two properties hold: i) the Gromov product $(\cdot, \cdot)_o$ extends continuously from X to the bordification $X \cup \partial X$ for each basepoint $o \in X$, and ii) there is some $\epsilon > 0$ such that $\exp(-\epsilon\, (\cdot, \cdot)_o)$ is a metric on the boundary ∂X, again for each basepoint $o \in X$. The concrete X used in [6] is Γ itself, equipped with an 'admissible' metric which is furthermore good. Such a metric was constructed by Mineyev in [3, 4], and it is a slightly improved version of the metric used by Mineyev and Yu in [5].

We note that CAT(−1) spaces are strongly bolic (this can be checked directly, and it even holds for CAT(0) spaces) and good (this is a theorem of Bourdon [1]).

How? Our main goal in [7] was to find a metric notion, weaker than the CAT(−1) condition, that guarantees enhanced hyperbolicity, and such that every hyperbolic group has a *natural* 'admissible' metric satisfying it. Here is our metric notion, and the results that fulfill our wishes.

Definition 1. A metric space X is *superbolic* if, for some $\epsilon > 0$, we have

$$\exp(-\epsilon\,(x,y)_o) \leq \exp(-\epsilon\,(x,z)_o) + \exp(-\epsilon\,(z,y)_o)$$

for all $x, y, z, o \in X$.

Theorem 2. *A roughly geodesic superbolic space is a good, strongly bolic hyperbolic space.*

Theorem 3. CAT(−1) *spaces are superbolic.*

Theorem 4. *The Green metric arising from a random walk on a hyperbolic group is superbolic.*

Theorem 2 is quantitative: ϵ-superbolic implies ϵ-good, $(\log 2)/\epsilon$-hyperbolic, and strongly bolic with exponential control. In Theorem 3, we show that CAT(−1) spaces are 1-superbolic; as a corollary, we recover Bourdon's theorem that CAT(−1) spaces are 1-good. We also find the best constant of hyperbolicity, in the sense of Gromov's original definition, for the hyperbolic plane $\mathbb{H}^2$. Quite surprisingly, this was not known before.

Corollary 5. $\mathbb{H}^2$ *is* $\log 2$*-hyperbolic, and this is optimal.*

In Theorem 4, the random walk is assumed to be symmetric and supported on a finite generating subset of the hyperbolic group. The random walk metric, or the *Green metric*, on a hyperbolic group is given by the formula $|x,y|_G = -\log F(x,y)$, where $F(x,y)$ is the probability that the random walk started at x ever hits y. It turns out that the Green metric is 'admissible'. A corollary of Theorem 4 is the fact that the Green metric is good, and this is a positive answer to a question raised in [6]. As another corollary, we recover the result of Haïssinsky and Mathieu [2] that the Green metric is strongly bolic. A third corollary is the following.

Corollary 6. *On the boundary of a hyperbolic group, the harmonic measure defined by a random walk equals the Hausdorff probability measure defined by any Green visual metric.*

We find the Green metric to be a simple and natural alternative to the metrics constructed in [5, 3, 4].

References

[1] M. Bourdon: *Structure conforme au bord et flot géodésique d'un* CAT(−1)*-espace*, Enseign. Math. (2) 41 (1995), no. 1-2, 63–102

[2] P. Haïssinsky, P. Mathieu: *La conjecture de Baum-Connes pour les groupes hyperboliques par les marches aléatoires*, preprint 2011

[3] I. Mineyev: *Flows and joins of metric spaces*, Geom. Topol. 9 (2005), 403–482
[4] I. Mineyev: *Metric conformal structures and hyperbolic dimension*, Conform. Geom. Dyn. 11 (2007), 137–163
[5] I. Mineyev, G. Yu: *The Baum-Connes conjecture for hyperbolic groups*, Invent. Math. 149 (2002), no. 1, 97–122
[6] B. Nica: *Proper isometric actions of hyperbolic groups on L^p-spaces*, Compos. Math. 149 (2013), no. 5, 773–792
[7] B. Nica, J. Špakula: *Superbolicity*, arXiv:1408.0250

The conjugation invariant geometry of cyclic subgroups.

Jarek Kędra

Let G be a group generated by a set S which is the union of finitely many conjugacy classes. Let $|g|$ denote the word norm with respect to S. It is conjugation invariant and its Lipschitz class does not depend on the choice of a finite set of conjugacy classes. We are interested in the geometry of cyclic subgoups, and more precisely, in the growth rate of the sequence $|g^n|$, where g is an element of G.

In the paper [1] we observed that for many classes of groups the sequence $|g^n|$ is either bounded or grows linearly. In other words, the cyclic subgroup generated by g is either bounded or undistorted (this is in contrast with classical geometric group theory where other types of growth occur). Such a dichotomy holds for many classes of groups of geometric origin (eg. braid groups, Coxeter groups, hyperbolic groups, lattices in solvable Lie groups, lattices in some higher rank semisimple groups, Baumslag-Solitar groups, right angled Artin groups etc). Since undistortedness is usually detected by evaluating a homogeneous quasimorphism there is a stronger dichotomy: a cyclic subgroup is either bounded or dectected by a homogeneous quasimorphism.

Open problem: Find a finitely presented group which violates either dichotomy.

Muranov found a group generated by two elements with an unbounded but distorted cyclic subgroup. In the paper [2] we proved that the commutator subgroup of the infinite braid group has an undistorted cyclic subgroup and it is well known that this group does not admit undbounded quasimorphisms. So this example violates the stronger dichotomy. This example is a byproduct of the existence of a quasihomomorphism from braid groups to the concordance group of knots.

The plan of the talk:

- introduction of conjugation invariant word norms and basic facts.
- examples of bounded cyclic subgroups and a trick showing their boundedness (the nicest examples are in braid groups).
- stating the dichotomies, the list of groups satisfying them and the open problem.
- definition of the quasihomomorphism $\Psi_n \colon B_n \to \mathrm{Conc}$ and its main properties (it is a quasihomomorphism with respect to the slice genus and it is Lipschitz with respect to the slice genus and the conjugation invariant word norm on B_n).

- examples and applications:
  - the example of an unbounded cyclic subgroup in the infinite braid group (with a proof);
  - replation between stable commutator length and the stable slice genus;
  - applications to knot theory.

References

[1] M.Brandenbursky, Ś.R.Gal, J.Kędra, M.Marcinkowski, *Cancelation norm and the geometry of biinvariant word metrics*, Glasgow Mathematical Journal, to appear. `arXiv:1310.2921`.

[2] M.Brandenbursky, J.Kędra, *Concordance group and stable commutator length in braid groups.* Algebraic and Geometric Topology, to appear. `arXiv:1402.3191`.

The Thurston norm via Fox calculus

Stefan Friedl

(joint work with Kevin Schreve and Stephan Tillmann)

Let N be a compact orientable 3-manifold. The *Thurston seminorm* of a class $\phi \in H^1(N;\mathbb{Z}) = H_2(N, \partial N;\mathbb{Z})$ is defined as

$$x(\phi) := \min \big\{ \chi_-(\Sigma) \,|\, \Sigma \subseteq N \text{ properly embedded surface dual to } \phi \big\}.$$

Here, given a surface Σ with connected components $\Sigma_1 \cup \dots \cup \Sigma_k$, we define its complexity to be $\chi_-(\Sigma) = \sum_{i=1}^k \max\{-\chi(\Sigma_i), 0\}$. A class $\phi \in H^1(N;\mathbb{R})$ is called *fibered* if it can be represented by a non-degenerate closed 1-form. By [19] an integral class $\phi \in H^1(N;\mathbb{Z}) = \mathrm{Hom}(\pi_1(N), \mathbb{Z})$ is fibered if and only if there exists a fibration $p\colon N \to S^1$ such that $p_* = \phi : \pi_1(N) \to \pi_1(S^1) = \mathbb{Z}$.

Thurston [18] showed that x is a seminorm on $H^1(N;\mathbb{Z})$ which extends to a seminorm on $H^1(N;\mathbb{R})$. He also considered the norm ball

$$\mathcal{N}_N := \{\phi \in H_1(N;\mathbb{R}) \,:\, x(\phi) \leq 1\}$$

and the corresponding dual norm ball

$$\mathcal{P}_N := \{v \in H_1(N;\mathbb{R}) \,:\, \phi(v) \leq 1 \text{ for all } \phi \in \mathcal{N}_N\}.$$

He showed that $\mathcal{P}_N$ is a polytope with integral vertices, i.e. with vertices in $\mathrm{Im}\{H_1(N;\mathbb{Z})/\text{torsion} \to H_1(N;\mathbb{R})\}$. Furthermore, Thurston showed that we can turn $\mathcal{P}_N$ into a marked polytope $\mathcal{M}_N$ which has the property that a cohomology class $\phi \in H^1(N;\mathbb{R})$ is fibered if and only if it pairs maximally with a marked vertex, i.e. if and only if there exists a marked vertex v of $\mathcal{M}_N$ such that

$$\phi(v) > \phi(w) \text{ for all } v \neq w \in \mathcal{P}_N.$$

Now let $\pi = \langle x, y \,|\, r\rangle$ be a presentation with two generators and one relator, such that r is cyclically reduced and such that $b_1(\pi) = 2$. In [9] we associated to such a presentation π a marked polytope $\mathcal{M}_\pi$ in $H_1(\pi;\mathbb{R})$ as follows:

(1) We start at the origin and walk across $H_1(\pi;\mathbb{Z}) = \mathbb{Z}^2$ as dictated by the word r which we start reading from the left.

(2) We take the convex hull of all the points reached in (1). We furthermore mark all vertices which get hit only once by the path in (1).
(3) We take the midpoints of all the squares in the convex hull that touch a vertex of the polytope defined in (2). We mark a midpoint if all the corresponding vertices in (2) are marked.
(4) We take the marked polytope corresponding to the set of points in (3) and denote it by $\mathcal{M}_\pi$.

An alternative, more formal definition of $\mathcal{M}_\pi$ is given in [9] in terms of the Fox derivatives of r. The main result of [10] says the following.

Theorem 1. *Let N be an irreducible, compact, orientable 3-manifold that admits a presentation $\pi = \langle x, y \,|\, r\rangle$ as above. Then*

$$\mathcal{M}_N = \mathcal{M}_\pi.$$

The following corollary gives the statement of the theorem in a slightly more informal fashion. The method for reading off the fibered classes in $H^1(N;\mathbb{R})$ from r is closely related to Brown's algorithm [5].

Corollary 2. *If N is an irreducible, compact, orientable 3-manifold that admits a presentation $\pi = \langle x, y \,|\, r\rangle$ as above, then the Thurston norm and the set of fibered classes can be read off from the relator r.*

In the proof of Theorem 1 we use the definition of $\mathcal{M}_\pi$ in terms of Fox derivatives. This makes it possible to relate the polytope $\mathcal{M}_\pi$ to the chain complex of the universal cover of the 2-complex corresponding X to the presentation π. This makes it possible to study the 'size' of $\mathcal{M}_\pi$ using twisted Reidemeister torsions of X. These twisted Reidemeister torsions agree with twisted Reidemeister torsions of N since X is simple homotopy equivalent to N.

In the following we denote by $\mathcal{P}_N$ and $\mathcal{P}_\pi$ the polytopes $\mathcal{M}_N$ and $\mathcal{M}_\pi$ without the markings. At this point the proof of Theorem 1 breaks up into three parts:

(1) We first show that $\mathcal{P}_N \subset \mathcal{P}_\pi$. Put differently, we need to show that $\mathcal{P}_\pi$ is 'big enough' to contain $\mathcal{P}_N$. We show this using the main theorem of [11] which says that twisted Reidemeister torsions detect the Thurston norm of N. This result in turn relies on the work of Agol [1], Liu [14], Przytycki-Wise [16, 15] and Wise [21] which says in this context that Agol's virtual fibering theorem [1] applies.
(2) Next we need to show the reverse inclusion $\mathcal{P}_\pi \subset \mathcal{P}_N$. This means that we need to show that $\mathcal{P}_\pi$ is 'not bigger than necessary'. At this stage it is crucial that r is cyclically reduced. By [20] this implies that all summands in the Fox derivative $\frac{\partial r}{\partial x}$ are distinct elements in the group ring $\mathbb{Z}[\pi]$. Using the fact that $\pi_1(N)$ is residually torsion-free elementary-amenable (which is a consequence of the aforementioned papers [1, 2, 14, 16, 15, 21] and a result of Linnell–Schick [13]) and using the non-commutative Reidemeister torsions of [6, 8, 12] we show that indeed $\mathcal{P}_\pi \subset \mathcal{P}_N$.
(3) Finally we need to show that the markings of $\mathcal{M}_N$ and $\mathcal{M}_\pi$ agree. We prove this using Novikov-Sikorav homology [4, 17].

References

[1] I. Agol, *Criteria for virtual fibering*, J. Topol. **1** (2008), no. 2, 269–284.
[2] I. Agol, *The virtual Haken conjecture*, with an appendix by I. Agol, D. Groves and J. Manning, Documenta Math. **18** (2013), 1045–1087.
[3] M. Aschenbrenner, S. Friedl and H. Wilton, *3-manifold groups*, Preprint (2013)
[4] R. Bieri, *Deficiency and the geometric invariants of a group*, With an appendix by Pascal Schweitzer, J. Pure Appl. Algebra **208** (2007), 951–959.
[5] K. S. Brown, *Trees, valuations, and the Bieri-Neumann-Strebel invariant*, Invent. Math. **90** (1987), 479-504.
[6] T. Cochran, *Noncommutative knot theory*, Algebr. Geom. Topol. **4** (2004), 347–398.
[7] N. Dunfield, *Alexander and Thurston norms of fibered 3–manifolds*, Pacific J. Math. **200** (2001), no. 1, 43–58.
[8] S. Friedl, *Reidemeister torsion, the Thurston norm and Harvey's invariants*, Pac. J. Math. **230** (2007), 271–296.
[9] S. Friedl and S. Tillmann, *Two-generator one-relator groups and marked polytopes*, preprint, 2015.
[10] S. Friedl, K. Schreve, S. Tillmann, *Thurston norm via Fox calculus*, preprint, 2015.
[11] S. Friedl and S. Vidussi, *The Thurston norm and twisted Alexander polynomials*, Preprint (2012), to be published by J. Reine Ang. Math.
[12] S. Harvey, *Higher–order polynomial invariants of 3–manifolds giving lower bounds for the Thurston norm*, Topology **44** (2005), 895–945.
[13] P. Linnell and P. Schick, *Finite group extensions and the Atiyah conjecture*, J. Amer. Math. Soc. **20** (2007), no. 4, 1003–1051.
[14] Y. Liu, *Virtual cubulation of nonpositively curved graph manifolds*, J. Topol. **6** (2013), no. 4, 793–822.
[15] P. Przytycki and D. Wise, *Graph manifolds with boundary are virtually special*, J. Topology **7** (2014), 419–435.
[16] P. Przytycki and D. Wise, *Mixed* 3*-manifolds are virtually special*, preprint (2012).
[17] J.–C. Sikorav, *Homologie de Novikov associée à une classe de cohomologie réelle de degré un*, thèse, Orsay, 1987.
[18] W. P. Thurston, *A norm for the homology of 3–manifolds*, Mem. Amer. Math. Soc. **59**, no. 339 (1986), 99–130.
[19] D. Tischler, *On fibering certain foliated manifolds over* S^1, Topology **9** (1970), 153–154.
[20] C. Weinbaum, *On relators and diagrams for groups with one defining relation*, Illinois J. Math. **16** (1972), 308–322.
[21] D. Wise, *From riches to RAAGs:* 3*-manifolds, right–angled Artin groups, and cubical geometry*, CBMS Regional Conference Series in Mathematics, 2012.

Extension of higher cocycles and higher degree bounded cohomology

Roberto Frigerio
(joint work with Maria Beatrice Pozzetti, Alessandro Sisto)

Bounded cohomology of discrete groups is very hard to compute. For example, as observed in [6], there is not a single countable group G whose bounded cohomology (with trivial coefficients) is known in every degree, unless it is known to vanish in all positive degrees (this is the case, for example, of amenable groups). Nevertheless, since Gromov's foundational paper [4] appeared in 1982, bounded cohomology has proven to be a powerful tool in several areas, from rigidity theory for representations to the study of topological invariants of manifolds, from the study of characteristic classes to geometric group theory (see e.g. [6] for a survey).

Let us briefly recall the definition of bounded cohomology. Let G be a (discrete) group. We denote the complex of cochains on G (with trivial coefficients) by $(C^*(G), \delta)$, where $C^n(G) = \{\varphi\colon G^{n+1} \to \mathbb{R}\}$ and δ is the usual differential. If $\varphi \in C^n(G)$, we say that φ is G-invariant if $\varphi(g_0, \ldots, g_n) = \varphi(g^{-1}g_0, \ldots, g^{-1}g_n)$ for every $(g_0, \ldots, g_n) \in G^{n+1}$, $g \in G$. Moreover, the norm of φ is defined by

$$\|\varphi\|_\infty = \sup\{|\varphi(g_0, \ldots, g_n)| \,|\, (g_0, \ldots, g_n) \in G^{n+1}\} \in [0, \infty] .$$

We denote by $C_b^n(G) \subseteq C^n(G)$ the subspace of bounded cochains, and by $C^n(G)^G$, $C_b^n(G)^G$ the subspaces of invariant (bounded) cochains. Then, the (bounded) cohomology of G is just the cohomology of the complex $C^*(G)^G$ (resp. $C_b^*(G)^G$), and it is denoted by $H^*(G)$ (resp. $H_b^*(G)$). The inclusion of bounded cochains into ordinary cochains induces a map in cohomology, whose kernel is called *exact bounded cohomology of* G, and denoted by $EH^*(G)$.

Of course $H_b^0(G) = H^0(G) = \mathbb{R}$ for every group G, while $H_b^1(G)$ may be identified with the spaces of bounded homomorphisms of G into $\mathbb{R}$, so $H_b^1(G) = 0$ for every group G. Several interesting phenomena already occur in degree 2: roughly speaking, degree-2 bounded cohomology seems to vanish in non-negative curvature, while tends to be infinite-dimensional in negative curvature. We have already mentioned that $H_b^2(G) = 0$ for every amenable group, while a recent result by Hull and Osin [5] ensures that $\dim H_b^2(G) = \infty$ whenever G is *acylindrically hyperbolic*. Notice that acylindrically hyperbolic groups provide a very large class of groups displaying a negatively curved behaviour: for example, non-elementary (relatively) hyperbolic groups, mapping class groups of compact hyperbolic surfaces and $\mathrm{Out}(F_n)$, $n \geq 2$, are acylindrically hyperbolic. We refer the reader to [7] for the definition and a discussion of the main properties of acylindrically hyperbolic groups.

In degree 2, non-trivial bounded cohomology classes may be constructed by combinatorial and geometric methods by explicitly exhibiting *quasi-morphisms* (see below). This is no more true in degree 3, where negative curvature plays an even more explicit role. Let M be a complete pinched negatively curved Riemannian n-manifold. The volume form of M defines a bounded coclass in $H_b^n(M)$, that translates in turn into an element $\omega_M \in H_b^n(\pi_1(M))$: bounded cohomology may be defined also for topological spaces, and the isomorphism $H_b^n(X) \cong H_b^n(\pi_1(X))$ still holds for any aspherical space X (thanks to the very same argument which works in the case of ordinary cohomology) and even for any possibly non-aspherical CW-complexes (thanks to a deep result by Gromov). Recall that the n-manifold M is *closed at infinity* if, for every $\varepsilon > 0$, there exists a compact n-submanifold with boundary $M_\varepsilon \subseteq M$ such that $\mathrm{vol}(\partial M_\varepsilon)/\mathrm{vol}(M) < \varepsilon$ (i.e. if the Cheeger constant of M vanishes). It is a remark by Gromov that, if M is pinched negatively curved and closed at infinity, then $\omega_M \in H_b^n(M)$ is non-trivial. This applies for example when M is the infinite cyclic covering of the figure-eight knot complement: in this case, M is 3-dimensional, negatively curved and closed at infinity, and $\pi_1(M)$ is isomorphic to the free group F_2: therefore, $H_b^3(F_2) \neq 0$. In fact, an argument by Soma [8] shows that by suitably perturbing the volume form of M many linearly independent bounded coclasses may be exhibited, so that $\dim H_b^3(F_2) = \infty$. Notice that $H_b^n(F_2)$ is unknown for every $n \geq 4$. A recent result by Bowen [1], however, shows that F_2 cannot occur as the fundamental group of any complete hyperbolic n-manifold which is closed at infinity, provided that n is even and bigger than 3. Therefore, Soma's strategy does not seem too promising in order to prove non-vanishing results for $H_b^n(F_2)$, $n \geq 4$. Our main result generalizes Soma's result to the wide class of acylindrically hyperbolic groups mentioned above:

Theorem 1 ([3]). *Let G be acylindrically hyperbolic. Then* $\dim EH_b^3(G) = \infty$.

A result by Dahmani, Guirardel and Osin ensures that any acylindrically hyperbolic group contains a hyperbolically embedded copy of $F_2 \times K$ for some finite group K (see [2] for the definition of hyperbolically embedded subgroup). Therefore, we may deduce Theorem 1 from the following:

Theorem 2 ([3]). *Let H be a hyperbolically embedded subgroup of G. Then the restriction $EH_b^n(G) \to EH_b^n(H)$ is surjective for every n.*

Let us briefly describe the proof of Theorem 2. For every group G, we define the space of *n-quasi-cocycles* as follows:

$$QZ^n(G) = \{\varphi \in C^n(G) \,|\, \delta^n\varphi \in C_b^{n+1}(G)\} .$$

Roughly speaking, quasi-cocycles are those cochains whose differential is quasi-null. By the very definitions, any element in $EH_b^n(G)$ is represented by the coboundary of an invariant *quasi-cocycle*, and it is an easy exercise to show that the kernel of the map $QZ^n(G) \to EH_b^{n+1}(G)$ is given by $Z^n(G) + C_b^n(G)$. Therefore, we say that a quasi-cocycle is *trivial* if it belongs to $Z^n(G) + C_b^n(G)$, and we observe that $EH_b^{n+1}(G)$ is isomorphic to the space of n-quasi-cocycles, modulo the trivial ones.

In degree 2, quasi-cocycles are usually called quasi-morphisms, and have been widely studied. As mentioned above, many non-vanishing results concerning the second bounded cohomology of negatively curved groups were obtained by various authors via the construction of non-trivial quasi-morphisms.

We deduce our Theorem 2 from the following result, which generalizes to higher degrees an analogous statement proved by Hull and Osin in the case of quasi-morphisms [5]:

Theorem 3 ([3]). *Let H be hyperbolically embedded in G. Then any alternating quasi-cocycle on H may be extended to an alternating quasi-cocycle on G.*

Our proof exploits several results regarding the geometric properties of suitably defined projections on the lateral classes of H in G. It is maybe worth mentioning that such projections may be reconstructed from our extension of a suitably chosen 2-quasi-cocycle on H (which has coefficients in a non-trivial Banach H-module). In a sense, this means that our extension captures the fact that H is hyperbolically embedded in G, and seems to suggest that it should be possible to provide a characterization of hyperbolically embedded subgroups in terms of bounded cohomology.

References

[1] L. Bowen, *Cheeger constants and L^2-Betti numbers*, arXiv:1303.5963, to appear in Duke Math. J.

[2] F. Dahmani, V. Guirardel, D. Osin, *Hyperbolically embedded subgroups and rotating families in groups acting on hyperbolic spaces*, arXiv:1111.7048, to appear in Mem. Amer. Math. Soc.

[3] R. Frigerio, M.B. Pozzetti, A. Sisto, *Extending higher-dimensional quasi-cocycles*, arXiv:1311.7633, to appear in J. Topol.

[4] M. Gromov, *Volume and bounded cohomology*, Inst. Hautes Études Sci. Publ. Math. **56** (1982), 5-99.

[5] M. Hull, D. Osin, *Induced quasicocycles on groups with hyperbolically embedded subgroups*, Algebr. Geom. Topol. **13** (2013), 2635-2665.

[6] N. Monod, *An invitation to bounded cohomology*, International Congress of Mathematicians. Vol. II, 1183-1211, Eur. Math. Soc., Zürich, 2006.

[7] D. Osin, *Acylindrically hyperbolic groups*, arXiv:1304.1246, to appear in Trans. Amer. Math. Soc.

[8] T. Soma, *Bounded cohomology and topologically tame Kleinian groups*, Duke Math. J. **88** (1997), no. 2, 357-370.

Geometry of random Coxeter groups

Jason Behrstock

Divergence, thickness, and relative hyperbolicity are three geometric properties which determine aspects of the quasi-isometric geometry of a finitely generated group. We start with a discussion of the basic properties of these notions and some of the relations between them. We then survey how these properties manifest in right-angled Coxeter groups and, using random graphs, describe the geometry of a "random right-angled Coxeter group".

Thickness is an $\mathbb{N}$–valued quasi-isometric invariant property due to Behrstock–Drutu–Mosher which, when it holds for a group, roughly quantifies the extent to which the group can be built out of subgroups which are direct products [1]. Two key properties of this notion for the present talk are that it is an obstruction to relative hyperbolicity and that if this invariant is n then the group has divergence at most x^{n+1}.

The first results discussed was:

Theorem 1 (Behrstock–Hagen–Sisto; [2])**.** Any right-angled Coxeter group is either thick, or else it is hyperbolic relative to a (possibly empty) collection of thick subgroups, further, such a collection is canonical.

The proof of the above them uses the following result which shows that the collection of graphs whose corresponding right-angled Coxeter group is thick admits a purely combinatorial description. Moreover, we note that this result yields a polynomial-time algorithm to check whether a right-angled Coxeter group is relatively hyperbolic or thick.

Theorem 2 (Behrstock–Hagen–Sisto; [2])**.** The collection, τ, of simplical graphs for which the corresponding right-angled Coxeter group is thick is the smallest set of graphs which contains the square and is closed under the following two constructions:

- Coning: Given any graph $\Gamma \in \tau$ and any induced subgraph $\Lambda \subset \Gamma$ which is not a clique, then the graph obtained by taking the disjoint union of Γ and a new point x together with all edges from vertices of Λ to x is in τ.
- Thick unions: Given any graphs $\Gamma_1, \Gamma_2 \in \tau$ and any graph Λ which is not a clique, and which is isomorphic to induced subgraphs $\Lambda_i \subset \Gamma_i$, then the graph Γ' obtained by taking the disjoint union of the Γ_i and identifying Λ_1 with Λ_2 is in τ. Further, any graph obtained from Γ' by adding edges connecting vertices of $\Gamma_1 \setminus \Lambda_1$ to vertices of $\Gamma_2 \setminus \Lambda_2$ is in τ as well.

Erdös–Renyi introduced a model for the systematic study of random graphs [4, 6]. The $G(n,p)$ model of random graphs is a probability space consisting of all graphs with n vertices and where each pair of vertices is independently declared to span an edge with probability p. One says that a given graph-theoretic property holds for graphs in $G(n,p)$, if as n approach infinity the probability that a random graph with n vertices possesses that property approaches 1. Propeties of graphs in $G(n,p)$ are widely studied by combinatorialists. Since to any graph Γ, one naturally associates a right-angled Coxeter group W_Γ, we use the $G(n,p)$ model to study random right-angled Coxeter groups.

We announced the following, which we recently obtained:

Theorem 3 (Behrstock–Hagen–Susse; [3])**.** For every fixed constant $0 < p < 1$ and random graph $\Gamma \in G(n,p)$, the corresponding right-angled Coxeter group W_Γ is thick of order 1.

This result, in particular, implies that the random right-angled Coxeter group has quadratic divergence. This is particularly interesting since it is known that there exist right-angled Coxeter groups exhibiting polynomial divergence of arbitrary degrees as well as ones with exponential divergence [2, 5]. It is also in contrast to the situation for "random groups" which Gromov proved are hyperbolic and, thus have, exponential divergence.

References

[1] J. Behrstock, C. Druţu, and L. Mosher. Thick metric spaces, relative hyperbolicity, and quasi-isometric rigidity. *Math. Ann.* **344**(2009), 543–595.

[2] J. Behrstock, M. Hagen, and A. Sisto. Thickness, relative hyperbolicity, and randomness in Coxeter groups. arXiv:1312.4789, 2013.

[3] J. Behrstock, M. Hagen, and T. Susse. In preparation, 2015.

[4] Béla Bollobás. *Random graphs*, volume 73 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, second edition, 2001.

[5] P. Dani and A. Thomas. Divergence in right-angled Coxeter groups. *Trans. AMS* (2012). To appear. arXiv:math/1211.4565.

[6] Paul Erdós and Alfred Rényi. The evolution of random graphs. *Magyar Tud. Akad. Mat. Kutató Int. Közl.* (1960), 17–61.

Action dimension of Right-Angled Artin Groups

Kevin Schreve

(joint work with Grigori Avramidi, Michael Davis, Boris Okun)

If a group G has a finite dimensional classifying space BG, then its *geometric dimension*, denoted $\mathrm{gd}(G)$, is the minimum dimension of a model for BG. Its *action dimension*, denoted $\mathrm{actdim}\, G$, is the minimum dimension of a contractible manifold M which admits a proper G-action. If G is torsion-free, then any proper G-action is free, so M/G is a finite dimensional model for BG.

For example, if G is the fundamental group of a closed, aspherical n-manifold, then $\mathrm{actdim}(G) = n$. Therefore, we are mostly interested in group actions where the action is not cocompact or the manifold admits some sort of boundary.

Interestingly, $\mathrm{actdim}(G)$ has an upper bound of $2\,\mathrm{gd}\, G$, at least for groups of type F. To see this, note that if some BG embeds into $\mathbb{R}^N$, then a regular neighborhood of BG in $\mathbb{R}^N$ is an aspherical manifold; its universal cover is a contractible manifold on which G acts properly. By a theorem of Stallings [8], we can choose a model for BG that embeds into $\mathbb{R}^{2\,\mathrm{gd}\, G}$.

We compute the action dimension for a large class of *right-angled Artin groups*. Recall that for any flag complex L there is a right-angled Artin group A_L which has generators corresponding to vertices and two generators commute if and only if there is an edge between them.

The standard classifying space BA_L for A_L is a subcomplex of a torus which has one S^1 factor for each vertex of L. The space BA_L is a locally CAT(0) cube complex of dimension equal to $\dim L+1$. Since this is the cohomological dimension of A_L we have $\mathrm{gd}\, A_L = \dim L + 1$.

Our main result relates $\mathrm{actdim}\, A_L$ to the homology groups of the flag complex.

Main Theorem. *Suppose L is a k-dimensional flag complex.*

(1) *If $H_k(L; \mathbb{Z}/2) \neq 0$, then* $\mathrm{actdim}\, A_L = 2k + 2 = 2\,\mathrm{gd}\, A_L$.

(2) *If $H_k(L; \mathbb{Z}/2) = 0$, and $k \neq 2$ then* $\mathrm{actdim}\, A_L \leq 2k + 1$.

Remark. For $k = 1$, this was proved previously by Droms [5].

The motivation for this result comes from L^2-cohomology. The ℓ^2-Betti numbers $b_i^{(2)}(G)$ are well-defined invariants of a group G. The *ℓ^2-dimension of G*, denoted $\ell^2\mathrm{dim}\, G$, is defined by

$$\ell^2\mathrm{dim}\, G := \sup\{i \mid b_i^{(2)}(G) \neq 0\}.$$

In [3], Davis and Okun conjectured that ℓ^2-Betti numbers of a group G should give lower bounds for its action dimension. More precisely, we have the following.

Action Dimension Conjecture. $\mathrm{actdim}\, G \geq 2\,\ell^2\mathrm{dim}\, G$.

The ℓ^2-Betti numbers of A_L were computed by Davis–Leary in [2] as follows:

$$b_{i+1}^{(2)}(A_L) = \overline{b}_i(L),$$

where $\overline{b}_i(L)$ denotes the ordinary reduced Betti number, $\dim_{\mathbb{Q}} \overline{H}_i(L; \mathbb{Q})$.

Therefore, our main theorem comes close to providing a proof of the Action Dimension Conjecture for general right-angled Artin groups.

The proof of the theorem relies on the *obstructor dimension* methods of Bestvina, Kapovich, and Kleiner [1]. This method relates the action dimension to the minimum dimension m in which certain finite simplicial complexes can embed piecewise linearly in S^m, specifically the mod 2 van Kampen obstruction to embedding K into S^m. We quickly review this obstruction.

Let $\mathcal{C}(K)$ denote the configuration space of unordered pairs of distinct points in K, i.e., if Δ denotes the diagonal in $K \times K$, then $\mathcal{C}(K)$ is the quotient of $(K \times K) - \Delta$ by the involution which switches the factors. The double cover $(K \times K) - \Delta \to \mathcal{C}(K)$ is classified by a map $c : \mathcal{C}(K) \to \mathbb{R}P^\infty$. The *van Kampen obstruction* in degree m is the cohomology class $\nu^m_{\mathbb{Z}_2}(K) \in H^m(\mathcal{C}(K); \mathbb{Z}/2)$ defined by

$$\nu^m_{\mathbb{Z}_2}(K) = c^*(w_1^m),$$

where $w_1 \in H^1(\mathbb{R}P^\infty; \mathbb{Z}/2)$ is the first Stiefel–Whitney class of the canonical line bundle over $\mathbb{R}P^\infty$. The class $\nu^m_{\mathbb{Z}_2}(K)$ is an obstruction to embedding K in S^m. We say K is an *m-obstructor* if $\nu^m_{\mathbb{Z}_2}(K) \neq 0$. The *van Kampen dimension* of K, denoted by $\operatorname{vkdim} K$, is the maximum m such that $\nu^m_{\mathbb{Z}_2}(K) \neq 0$.

Here is a special case of the main theorem from [1].

Theorem 1. *Let G be a group that admits a $\mathcal{Z}$-structure, and let Z be a $\mathcal{Z}$-boundary of G. If K embeds into Z, then* $\operatorname{actdim} G \geq \operatorname{vkdim} K + 2$.

The idea here is as follows. Suppose that G is of type F and that EG, the universal cover of BG, has a Z-set compactification. Denote the boundary of this compactification by $\partial_\infty G$. Suppose further that G acts properly on a contractible n-manifold M which has a Z-set compactification with boundary $\partial_\infty M$ and that the equivariant map $EG \to M$ extends to an inclusion of Z-set boundaries. (For example, this is the case, if M is a proper CAT(0)-space and EG is a convex subspace.) To further simplify the discussion, suppose $\partial_\infty M$ is homeomorphic to S^{n-1}. If K is a finite complex embedded in $\partial_\infty G$, then $K \subset \partial_\infty G \subset \partial_\infty M = S^{n-1}$. So, one expects $\operatorname{actdim} G \geq \operatorname{vkdim} K + 2$.

We concentrate on computing the vkdim of a certain finite simplicial complex OL called the *octahedralization* of L. The complex OL is constructed by "doubling the vertices of L." Essentially, OL replaces each n-simplex of L with an n-octahedron.

It turns out that $OL \subset \partial_\infty A_L$, so we can use $\operatorname{vkdim} OL$ to give a lower bound on $\operatorname{actdim}(A_L)$. On the other hand, we show that if OL piecewise linearly embeds in S^m (and if the codimension is not 2), then A_L acts on a contractible $(m+1)$-manifold, which means that computing $\operatorname{vkdim} OL$ also gives us an upper bound on $\operatorname{actdim} A_L$.

References

[1] M. Bestvina, M. Kapovich, B. Kleiner, *Van Kampen's embedding obstruction for discrete groups*, Invent. Math **150** (2002), 219-235.
[2] M.W. Davis, I.J. Leary, *The l^2-cohomology of Artin groups*, J. London Math. Soc. (2) **68** (2003), 493–510.
[3] M.W. Davis, B.Okun *Vanishing theorems and conjectures for the L^2-homology of right-angled Coxeter groups*, Geom. Topol. **5** (2001), 7-74.
[4] A.N. Dranišnikov, D. Repovš, *Embeddings up to homotopy type in Euclidean space*, Bull. Austral. Math. Soc. **47** (1993), 145-148.
[5] C. Droms, *Graph groups, coherence, and three-manifolds*, J. Algebra **106** (1987), 484-489.
[6] C. McA. Gordon, *Artin groups, 3-manifolds and coherence*, Bol. Soc. Mat. Mexicana (3) **10** (2004), 193-198.
[7] E. R. van Kampen *Komplexe in euklidischen Räumen*, Abh. Math. Sem. Univ. Hamburg **9** (1933), 72-78.
[8] J. Stallings *Embedding homotopy types into manifolds*, unpublished (1965).

Exotic l^2-invariants

Łukasz Grabowski

In the talk I described some examples of CW-complexes (or, equivalently, group ring elements which appear as Laplacians in such CW-complexes) with interesting asymptotics of the number of eigenvalues around 0. They are taken from the preprints [3] and [4].

The most important technical problem in the general theory of l^2-invariants is establishing bounds on the spectral density of group ring elements. Let us illustrate it with three examples. For an introduction to l^2-invariants see [2] or [6] for a more comprehensive treatment.

(i) The celebrated *Lück approximation theorem* states that the l^2-Betti numbers of a normal cover of a finite CW-complex are limits of the ordinary Betti numbers of intermediate finite covers, normalized by the cardinality of the fibers. This is an easy corollary of the following statement. For every $C > 0$ there exists a function $f: \mathbb{R}_+ \to \mathbb{R}_+$ such that $f(\varepsilon) \to 0$ when $\varepsilon \to 0$, and such that for every residually finite group G, and every self-adjoint T in the integral group ring $\mathbb{Z}[G]$ whose l^1-norm is bounded by C, we have

$$\mu_T((0, \varepsilon)) < f(\varepsilon). \tag{1}$$

In other words, Lück approximation follows from *having any uniform bound at all on the spectral density around* 0. Lück [7] showed that one can take $f(\varepsilon) := \frac{C'}{|\log(\varepsilon)|}$, where C' depends on C.

(ii) Another l^2-invariant, the l^2*-torsion*, is not known to be well-defined for arbitrary normal covers. It is well-defined for all normal covers with a given deck transformation group G if and only if for every self-adjoint $T \in \mathbb{Z}[G]$ the integral

$$\int_{0^+}^{1} \log(x)\, d\mu_T(x)$$

is convergent. This is clearly a statement about the density of μ_T around 0. The convergence of the integral above was established by Clair [1] and Schick [8] for a large class of groups G, including all residually-finite ones. As a consequence, using the little-o notation, we have

$$\mu_T((0,\varepsilon)) = o\left(\frac{1}{|\log(\varepsilon)|}\right), \tag{2}$$

which is the best *general* upper bound known.

(iii) A major open problem, known as the *determinant approximation conjecture*, is the analog of Lück approxination for the l^2-torsion. It is currently known only when G has the infinite cyclic group $\mathbf{Z}$ as a subgroup of finite index (see [6, Lemma 13.53]). It is not difficult to show that if, under the assumption stated in the example (i), for some $\delta > 0$ we had

$$\mu_T((0,\varepsilon)) < \frac{C'}{|\log^{1+\delta}(\varepsilon)|},$$

then the determinant approximation conjecture would be true.

For a long time it has not been known if there actually exist group ring elements whose spectral density is as large as the best known general bound (2) suggests. This is reflected in the following conjecture made by Lott and Lück.

Conjecture 1 (Lott-Lück [5])**.** Let G be a discrete group. For a self-adjoint $T \in \mathbb{Z}[G]$ there exist $C, \eta > 0$ such that for sufficiently small ε we have

$$\mu_T((0,\varepsilon)) < C\varepsilon^\eta.$$

Note that the bound in Conjecture 1 is very far away from the best known bound (2): for every $\eta > 0$ and sufficiently small ε we have $\varepsilon^\eta < \frac{1}{|\log(\varepsilon)|}$. However, in this note we show that (2) is not too far away from an optimal bound.

Theorem 2. For every $\delta > 0$ there is a group G_δ and a self-adjoint element $S_\delta \in \mathbb{Z}[G_\delta]$ such that for some constant $C > 0$ we have

$$\mu_{S_\delta}((0,\varepsilon_i)) > \frac{C}{|\log(\varepsilon_i)|^{1+\delta}} \tag{3}$$

for some sequence of positive ε_i converging to 0. In particular, Conjecture 1 is false for S_δ.

However, in view of how we construct S_δ, we state the following conjecture, whose content essentially is that *the Clair-Schick bound* (2) *is optimal.* We say a function $g\colon \mathbb{R}_+ \to \mathbb{R}_+$ is *computable* if there is an algorithm which given $\varepsilon \in \mathbb{Q}$ computes $g(\varepsilon)$.

Conjecture 3. For every continuous computable function $g\colon \mathbb{R}_+ \to \mathbb{R}_+$ such that $g(\varepsilon) \to 0$ when $\varepsilon \to 0$ there exists a group G and $S \in \mathbb{Z}[G]$ such that

$$\mu_S((0,\varepsilon_i)) > \frac{g(\varepsilon_i)}{|\log(\varepsilon_i)|}$$

for some sequence of positive ε_i converging to 0.

For our second result let us recall the definition of the *Novikov-Shubin invariant*. It measures the growth of the number of eigenvalues around 0 of a given group ring element T. More precisely, given a self-adjoint $T \in \mathbb{C}[\Gamma]$, the Novikov-Shubin invariant of T is defined as

$$\alpha(T) := \liminf_{\lambda \to 0^+} \frac{\log(\mu_T((0,\lambda]))}{\log(\lambda)}, \tag{4}$$

where μ_T is the spectral measure of T (see [6, Chapter 2] for more details).

Remarks 4. (i) It is irrelevant whether we take $\mu_T((0,\lambda])$ or $\mu_T((0,\lambda))$ in (4). However, it is important that we do not include 0, since otherwise $\alpha(T)$ would be equal to 0 whenever the spectral measure of T has an atom at 0.

(ii) Both the numerator and the denominator are negative when λ is sufficiently small, so $\alpha(T) \in [0,\infty]$.

It is easy to see that Conjecture 1 is equivalent to saying that $\alpha(T) > 0$. However, Lott and Lück [5] proposed also the following conjecture.

Conjecture 5. When $T \in \mathbb{Q}[\Gamma]$ then $\alpha(T) \in \mathbb{Q}$.

We show the following.

Theorem 6. There is a family $T(b) \in \mathbb{R}[\mathbf{Z}_2 \wr \mathbf{Z}]$, $b \in (1,\infty)$ such that for $b \in \mathbb{Q}$ we have $T(b) \in \mathbb{Q}[\mathbf{Z}_2 \wr \mathbf{Z}]$ and $\alpha(T(b)) = \frac{1}{2\log_2(b)}$. In particular Conjecture 5 is false.

Note that the Novikov-Shubin invariant of T and kT is the same for $k > 0$, and so we also obtain examples of $T \in \mathbb{Z}[\mathbf{Z}_2 \wr \mathbf{Z}]$ with irrational Novikov-Shubin invariants.

Thereom 6 has an interesting consequence that the set of the Novikov-Shubin invariants of all the elements of $\mathbb{Q}[\mathbf{Z}_2 \wr \mathbf{Z}]$, which is countable, is different than the set of the Novikov-Shubin invariants of all the elements of $\mathbb{R}[\mathbf{Z}_2 \wr \mathbf{Z}]$. The analogous question has been asked among the experts for l^2-Betti numbers, since there are classes of torsion-free groups for which the Atiyah conjecture is known for $\mathbb{Q}[\Gamma]$ but not for $\mathbb{R}[\Gamma]$.

References

[1] Clair, B.: Residual amenability and the approximation of L^2-invariants. Michigan Math. J. **46**(2), 331–346 (1999)

[2] Eckmann, B.: Introduction to l_2-methods in topology: reduced l_2-homology, harmonic chains, l_2-Betti numbers. Israel J. Math. **117**, 183–219 (2000). Notes prepared by Guido Mislin

[3] Grabowski, Ł.: Irrational l^2-invariants arising from the lamplighter group, 2010; arXiv:1009.0229.

[4] Grabowski, Ł.: Group ring elements with large spectral density, 2014; arXiv:1409.3212.

[5] Lott, J., Lück, W.: L^2-topological invariants of 3-manifolds. Inventiones mathematicae **120**(1), 15–60 (1995)

[6] Lück, W.: L^2-invariants: theory and applications to geometry and K-theory, *Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics [Results in Mathematics and Related Areas. 3rd Series. A Series of Modern Surveys in Mathematics]*, vol. 44. Springer-Verlag, Berlin (2002)

[7] Lück, W.: Approximating L^2-invariants by their finite-dimensional analogues. Geom. Funct. Anal. **4**(4), 455–481 (1994)
[8] Schick, T.: L^2-determinant class and approximation of L^2-Betti numbers. Trans. Amer. Math. Soc. **353**(8), 3247–3265 (2001)

Volume and homology growth

Roman Sauer

Let M be a manifold whose fundamental group $\Gamma = \pi_1(M)$ is *residually finite*. That is, Γ possesses a decreasing sequence – called a *residual chain* – of normal subgroups $\Gamma_i < \Gamma$ of finite index whose intersection is trivial. By covering theory there is an associated sequence of finite regular coverings $\ldots \to M_2 \to M_1 \to M$ of M such that $\pi_1(M_i) \cong \Gamma_i$ and $\deg(M_i \to M) = [\Gamma : \Gamma_i]$, which we call a *residual tower of finite covers*.

How does the size of the homology of M_i grow as $i \to \infty$?

What do we mean by size? If we measure size by Betti numbers $b_k(M_i) = \operatorname{rk}_{\mathbb{Z}} H_k(M_i;\mathbb{Z})$, there is a general answer: the limit of $b_k(M_i)/[\Gamma : \Gamma_i]$ is the k-th ℓ^2-Betti number of M by a result of Lück [2]. If we measure size by mod p Betti numbers or in terms of the cardinality of the torsion subgroups $\operatorname{tors} H_k(M_i;\mathbb{Z}) \subset H_k(M_i;\mathbb{Z})$, no general answer is available but a conjectural picture was presented for arithmetic locally symmetric spaces in the work of Bergeron-Venkatesh [1].

Our aim is to establish upper bounds for the homology and the homology growth of aspherical Riemannian manifolds under geometric conditions. We prove the following two statements.

Theorem. For every $n \in \mathbb{N}$ and $V_0 > 0$ there is $C(n, V_0) > 0$ with the following property: Let M be an n-dimensional closed connected aspherical Riemannian manifold such that every 1-ball of the universal cover $\widetilde{M}$ has volume at most V_0. Assume that the fundamental group is residually finite, and let (M_i) be a residual tower of finite covers. Then for every $k \in \mathbb{N}$

$$\limsup_{i\to\infty} \frac{\dim_{\mathbb{F}_p} H_k(M_i;\mathbb{F}_p)}{\deg(M_i \to M)} < C(n, V_0)\operatorname{vol}(M) \text{ and}$$
$$\limsup_{i\to\infty} \frac{\log\bigl(|\operatorname{tors} H_k(M_i;\mathbb{Z})|\bigr)}{\deg(M_i \to M)} < C(n, V_0)\operatorname{vol}(M).$$

Theorem. For every $n \in \mathbb{N}$ there is a constant $\epsilon(n) > 0$ with the following property: Let M be a closed connected aspherical n-dimensional Riemannian manifold M such that $\operatorname{Ricci}(M) \geq -1$ and the volume of every 1-ball in M is at most $\epsilon(n)$. Assume that the fundamental group is residually finite, and let (M_i) be a residual tower of finite covers. Then for every $k \in \mathbb{N}$

$$\lim_{i\to\infty} \frac{\dim_{\mathbb{F}_p} H_k(M_i;\mathbb{F}_p)}{\deg(M_i \to M)} = 0 \text{ and}$$

$$\lim_{i\to\infty} \frac{\log\big(|\operatorname{tors} H_k(M_i;\mathbb{Z})|\big)}{\deg(M_i \to M)} = 0.$$

References

[1] N. Bergeron and A. Venkatesh *The asymptotic growth of torsion homology of arithmetic groups.* Journal of the Institute of Mathematics of Jussieu, 12, pp 391-447.

[2] W. Lück. *L^2-Invariants: Theory and Applications to Geometry and K-Theory*, volume 44 of *Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics.* Springer-Verlag, Berlin, 2002.

[3] R. Sauer, *Volume and homology growth of aspherical manifolds.* Preprint, arXiv:1403.7319.

On the definition of the volume of a representation of hyperbolic 3-manifolds

Michelle Bucher

(joint work with Marc Burger and Alessandra Iozzi)

We study the volume of a representation $\rho : \Gamma \to \mathrm{PSL}(n,\mathbb{C})$ that we will rename as the *Borel invariant of* ρ. Indeed, the continuous cohomology of $\mathrm{PSL}(n,\mathbb{C})$ in degree 3 is generated by a specific class called the *Borel class* $\beta(n)$. When M is compact, the definition of the Borel invariant of ρ is straightforward as it is the evaluation on the fundamental class $[M]$ of the pullback by ρ of the Borel class. If M has cusps, the definition of this invariant presents interesting difficulties which we overcome by the use of bounded cohomology. More precisely, $\beta(n)$ can be represented by a bounded cocycle, which gives rise to a bounded continuous class

$$\beta_b(n) \in H^3_{c,b}(\mathrm{PSL}(n,\mathbb{C}),\mathbb{R}).$$

The Borel invariant of $\rho : \Gamma \to \mathrm{PSL}(n,\mathbb{C})$ is then defined as

$$\mathcal{B}(\rho) = \langle \rho^*(\beta_b(n)), [N,\partial N]\rangle,$$

where N is a compact core of M. This definition does not use any triangulation, it is independent of the choice of compact core and can be made for any compact oriented 3-manifold whose boundary has amenable fundamental group.

The bounded cocycle entering the definition of $\beta_b(n)$ is constructed by means of an invariant

$$B_n : \mathcal{F}(\mathbb{C}^n)^4 \longrightarrow \mathbb{R}$$

of 4-tuples of complete flags, which on generic 4-tuples has been defined and studied by A.B. Goncharov, [4]. It generalizes the volume function in the case $\mathcal{F}(\mathbb{C}^2) = P^1\mathbb{C} = \partial\mathbb{H}^3$. This invariant can also be used to give an efficient formula for $\mathcal{B}(\rho)$. To this end assume that M has toric cusps. Let $\varphi : \mathcal{C} \to \mathcal{F}(\mathbb{C}^n)$ be a decoration, that is any Γ-equivariant map from the set of cusps $\mathcal{C} \subset \partial\mathbb{H}^3$ into $\mathcal{F}(\mathbb{C}^n)$, and let

$P_1, \ldots, P_r$ be a family of oriented ideal tetrahedra with vertices in $\mathcal{C}$ forming an ideal triangulation of M. If $(P_i^0, P_i^1, P_i^2, P_i^3)$ are the vertices of P_i, then

$$\mathcal{B}(\rho) = \sum_{i=1}^{r} B_n(\varphi(P_i^0), \varphi(P_i^1), \varphi(P_i^2), \varphi(P_i^3)). \tag{1}$$

The right hand side of the previous equation is the definition of the volume in [3, 1] upon passing to a barycentric subdivision of the ideal triangulation or restricting to generic decorations.

Our main result is that on the character variety Γ into $\mathrm{PSL}(n, \mathbb{C})$, the invariant $\mathcal{B}$ attains a unique maximum at $[\pi_n|_\Gamma]$.

Theorem 1. *Let $\Gamma = \pi_1(M)$ be the fundamental group of a finite volume real hyperbolic 3-manifold and let $\rho : \Gamma \to \mathrm{PSL}(n, \mathbb{C})$ be any representation. Then*

$$|\mathcal{B}(\rho)| \leq \frac{n(n^2-1)}{6} \mathrm{Vol}(M)\,,$$

with equality if and only if ρ is conjugate to $\pi_n|_\Gamma$ or to its complex conjugate $\overline{\pi}_n|_\Gamma$.

We refer to [2] for more details.

References

[1] N. Bergeron, E. Falbel, and A. Guilloux, *Tetrahedra of flags, volume and homology of SL(3)*, Geom. Topol. 18 (2014), no. 4, 19111971.

[2] M. Bucher, M. Burger, and A. Iozzi, *Rigidity of representations of hyperbolic lattices* $\Gamma < \mathrm{PSL}(2, \mathbb{C})$ *into* $\mathrm{PSL}(n, \mathbb{C})$, http://arxiv.org/abs/1412.3428.

[3] S. Garoufalidis, D. Thurston, and C. Zickert, *The complex volume of SL(n,C)-representations of 3-manifolds*, http://arxiv.org/abs/1111.2828.

[4] A. B. Goncharov, *Explicitc onstruction of characteristic classes*, I.M. Gel?fand Seminar, Adv. Soviet Math., vol. 16, Amer. Math. Soc., Providence, RI, 1993, 169210.

Bordism of L^2-bettiless manifolds

JIM DAVIS

(joint work with Sylvain Cappell and Shmuel Weinberger)

Let X be a finite complex and Γ a discrete group. Then one can define the L^2-betti numbers $b_i^{(2)}(X \to B\Gamma) \in \mathbb{R}$. A map $X \to B\Gamma$ is L^2-*bettiless* (or L^2-acyclic) if $b_i^{(2)}(X \to B\Gamma) = 0$ for all i. L^2-bettiless manifolds are of interest analytically. Here are some properties of L^2-betti numbers (see [2]):

- $\chi(X) = \sum (-1)^i b_i^{(2)}(X \to B\Gamma)$.
- $b_0^{(2)}(X \to B\Gamma) = 0$ when Γ is infinite.
- If H is a subgroup of Γ of finite index, then

$$b_i^{(2)}(X \to B\Gamma) = \frac{b_i^{(2)}(X \to BH)}{|\Gamma \colon H|}.$$

- Let $F = \mathbb{Q}(t_1, \dots, t_n)$ be the quotient field of the integral group ring $\mathbb{Z}[\mathbb{Z}^n]$.
$$b_i^{(2)}(X \to B\mathbb{Z}^n) = \dim_F H_i(X; F)$$
Here $H_i(X; F)$ is homology with local coefficients $H_i(S_*(\overline{X}) \otimes_{\mathbb{Z}[\mathbb{Z}^n]} F)$ where $\overline{X}$ is the induced $\mathbb{Z}^n$-cover of X. (This fits with the still open Atiyah conjecture that when Γ is torsion free the L^2-betti numbers are integers.)

We are interested in computing the bordism group $\Omega_k^{(2)}(B\Gamma)$ of closed oriented k-manifolds mapping to $B\Gamma$ which are L^2-bettiless. This seems fruitless in the case of the trivial group (where there are no L^2-bettiless manifolds) and in the case of the free group on two letters (where $b_1^{(2)}(X \to B\Gamma) \neq 0$ if $\pi_1 X \to B\Gamma$ is an epimorphism), but seems quite interesting in the case where Γ is virtually abelian. In particularly we prove:

Theorem 1. *Let $n \geq 1$. There is a long exact sequence*
$$\cdots \to \Omega_k^{(2)}(B\mathbb{Z}^n) \to \Omega_k(B\mathbb{Z}^n) \xrightarrow{\sigma} L_k(\mathbb{Q}(t_1, \dots, t_n)) \xrightarrow{\partial} \Omega_{k-1}^{(2)}(B\mathbb{Z}^n) \to \dots$$
where $\Omega_k(B\mathbb{Z}^n)$ is the bordism group of closed oriented k-manifolds mapping to $B\Gamma$, where $L_k(\mathbb{Q}(t_1, \dots, t_n))$ is the algebraic L-group which vanishes for k not divisible by four and which is the Witt group of nonsingular Hermitian forms when k is divisible by four. Furthermore σ is given by the Witt class of the intersection form with $\mathbb{Q}(t_1, \dots, t_n)$-coefficients.

Corollary 2. *Let $n \geq 1$. A manifold $M \to B\mathbb{Z}^n$ is cobordant to an L^2-bettiless manifold if and only if its signature is zero.*

To prove the theorem and corollary one follows the classical surgery program of Milnor [3] and Kervaire-Milnor [1] together with two extra tricks. In the situation of the corollary, one first represents a bordism class by $M \to B\mathbb{Z}^n$ with an isomorphism on the fundamental group. One then inductively uses the technique of surgery (together with the two tricks), to represent a bordism class by $M \to B\mathbb{Z}^n$ inducing an isomorphism on the fundamental group and so that $H_i(M; \mathbb{Q}(t_1, \dots, t_n))$ vanishes for $i < [n/2]$. One then uses the signature hypothesis to do surgery so that $H_i(M; \mathbb{Q}(t_1, \dots, t_n))$ vanishes for $i \leq [n/2]$. Poincaré duality then guarantees that M is $\mathbb{Q}(t_1, \dots, t_n)$-acyclic as desired.

The tricks are needed to deal with two well-known obstructions to surgery – one needs the relative Hurewicz Theorem to represent homology classes by spheres and one need spheres with trivial normal bundle to do surgery on. There was not time enough in the talk to cover the first trick (the "kill and kill again trick"), but second trick is given by the Embedding Lemma below.

Lemma 3 (Embedding Lemma)**.** *Let M be a connected n-manifold, $1 \leq p < n-1$, and $g \in \pi_1 M$. Suppose $\alpha \in \pi_p M$ is represented by an embedded sphere with a nowhere zero normal vector field. If $p > 1$, then $(g - e)\alpha \in \pi_p M$ is represented by an embedding $S^p \times D^{n-p} \hookrightarrow M$. If $p = 1$, then $g\alpha g^{-1}\alpha^{-1} \in \pi_1 M$ is represented by an embedding $S^1 \times D^{n-1} \hookrightarrow M$.*

Finally the talk discussed the case of a virtually abelian group. Let Γ be a group with a finite index subgroup of the form $\mathbb{Z}[\mathbb{Z}^n]$ with $n \geq 1$. Let $S = \mathbb{Z}[\mathbb{Z}^n] - 0$. The properties of L^2-betti numbers mentioned above show that $X \to B\Gamma$ is L^2-acyclic if and only if $H_*(X; S^{-1}\mathbb{Z}[\Gamma]) = 0$.

Theorem 4. *There is a long exact sequence*

$$\cdots \to \Omega_k^{(2)}(B\Gamma) \to \Omega_k(B\Gamma) \xrightarrow{\sigma} L_k(S^{-1}\mathbb{Z}[\Gamma]) \xrightarrow{\partial} \Omega_{k-1}^{(2)}(B\Gamma) \to \dots$$

Here the L-groups are more interesting, for example they can be nonzero in every dimension and they can have arbitrarily large torsion.

The audience had intriguing suggestions for extending these results to more general groups.

References

[1] M. A. Kervaire, J. W. Milnor, *Groups of homotopy spheres. I.*, Ann. of Math. **77** (1963), 504–537.

[2] W. Lück, *L^2-invariants: theory and applications to geometry and K-theory*, Springer-Verlag (2002).

[3] J. Milnor, *A procedure for killing homotopy groups of differentiable manifolds*, Proc. Sympos. Pure Math., Vol. III, American Mathematical Society, (1961), 39–55.

Transitivity degrees of countable groups and acylindrical hyperbolicity

Denis Osin

An action of a group G on a set Ω is *k-transitive* if $|\Omega| \geq k$ and for any two k-tuples of distinct elements of Ω, $(a_1, ..., a_k)$ and $(b_1, ..., b_k)$, there exists $g \in G$ such that $ga_i = b_i$ for $i = 1, ..., k$. The *transitivity degree* of a countable group G, denoted $\mathrm{td}(G)$, is the supremum of all $k \in \mathbb{N}$ such that G admits a k-transitive faithful action. For finite groups, this notion is classical and fairly well understood. It is easy to see that $\mathrm{td}(S_n) = n$, $\mathrm{td}(A_n) = n - 2$, and it is a consequence of the classification of finite simple groups that any finite group G other than S_n or A_n has $\mathrm{td}(G) \leq 5$. Moreover, if G is not S_n, A_n, or one of the Mathieu groups M_{11}, M_{12}, M_{23}, M_{24}, then $\mathrm{td}(G) \leq 3$ (see [2]).

For infinite groups, however, very little is known. For example, we do not know the answer to the following basic question: Does there exist an infinite countable group of transitivity degree k for every $k \in \mathbb{N}$? There are examples for $k = 1, 2, 3$, and ∞, but the problem seems open even for $k = 4$. There is also a new phenomenon, which does not occur in the finite world: highly transitive actions. Recall that an action of a group is *highly transitive* if it is k-transitive for all $k \in \mathbb{N}$. We say that a group is *highly transitive* if it admits a highly transitive faithful action; it is easy to see that a countably infinite group is highly transitive if and only if it embeds as a dense subgroup in the infinite symmetric group $Sym(\mathbb{N})$ endowed with the topology of pointwise convergence. Obviously $\mathrm{td}(G) = \infty$ whenever G is highly transitive, but we do not know if the converse is true. Yet another interesting question is whether there exists a reasonable classification of highly transitive groups (or, more generally, groups of high transitivity degree). The main goal of this paper is to address these questions in certain geometric and algebraic settings.

We prove that every countable acylindrically hyperbolic group admits a highly transitive action with finite kernel. This theorem uniformly generalizes many previously known results and allows us to answer a question of Garion and Glasner on the existence of highly transitive faithful actions of mapping class groups. It also implies that in various geometric and algebraic settings, the transitivity degree of an infinite group can only take two values, namely 1 and ∞. Here by *transitivity degree* of a group we mean the supremum of transitivity degrees of its faithful permutation representations. For the definition and details about acylindrically hyperbolic groups we refer to [1, 3].

Further, for any countable group G admitting a highly transitive faithful action, we prove the following dichotomy: Either G contains a normal subgroup isomorphic to the infinite alternating group or G resembles a free product from the model theoretic point of view. We apply this theorem to obtain new results about universal theory and mixed identities of acylindrically hyperbolic groups. Finally, we discuss some open problems.

References

[1] F. Dahmani, V. Guirardel, D. Osin, Hyperbolically embedded subgroups and rotating families in groups acting on hyperbolic spaces, arXiv:1111.7048, to appear in *Memoirs. Amer. Math. Soc.*

[2] J.D. Dixon, B. Mortimer, Permutation groups. Spinger, 1996.

[3] D. Osin, Acylindrically hyperbolic groups, arXiv:1304.1246; to appear in *Trans. Amer. Math. Soc.*

Universal L^2-torsion, Twisted L^2-Euler characteristic, Thurston norm and higher order Alexander polynomials

Wolfgang Lück
(joint work with Stefan Friedl)

We want to investigate and compare the following four invariants of 3-manifolds which are of rather different nature: the Thurston norm, the degree of higher order Alexander polynomials in the sense of Cochrane and Harvey, see [1, 3], the degree of the L^2-torsion function and a version of the L^2-Euler characteristic. We explain that the L^2-Euler characteristic encompasses the degree of higher order Alexander polynomials. We relate all these invariants by inequalities and equalities. In particular we show that they agree for the universal coverings and (for many other coverings) of a compact connected irreducibel orientable 3-manifold with infinite fundamental group and empty or toroidal boundary. We will explain universal L^2-torsion which encompasses all the invariants above and is based on localizations techniques applied to group rings and K_1. Some of these results have been conjectured in [2]. For basic introduction to L^2-invarians we refer to [4].

Behind all these invariants is the universal L^2-torsion $\rho_u^2(\overline{M};\mathcal{N}(G)) \in \mathrm{Wh}^w(G)$ of a G-covering $\overline{X} \to X$ of a finite connected CW-complex X such that all its L^2-Betti numbers $b_n^{(2)}(\overline{X};\mathcal{N}(G))$ vanish. Here $\mathrm{Wh}^w(G)$ is a variation of the classical Whitehead group, where one considers instead of matrices $A \in M_{n,n}(\mathbb{Z}G)$, which are invertible, those ones, for which the induced G-operator $r_A \colon L^2(G)^n \to L^2(G)^n$ is a weak isomorphism. In the sequel we assume that the torsionfree group G satisfies the Atiyah Conjecture about the integrality of L^2-Betti numbers. This is for instance the case if G is residually torsionfree elementary amenable or the fundamental group of an irreducible 3-manifold which is not a closed graph manifold.

If $\mathcal{D}(G)$ is the division closure of $\mathbb{Z}G$ in the algebra $\mathcal{U}(G)$ of operators affiliated to the group von Neumann algebra $\mathcal{N}(G)$, then $\mathcal{D}(G)$ is a skewfield and there is an isomorphism

$$\mathrm{Wh}^w(G) \cong \mathrm{Wh}(\mathcal{D}(G)) = K_1(\mathcal{D}(G))/\{\pm g \mid g \in G\}.$$

The Dieudonne determinant yields an isomorphism

$$\mathrm{Wh}^w(\mathcal{D}(G)) \cong \mathcal{D}(G)^\times/[\mathcal{D}(G)^\times, \mathcal{D}(G)^\times] \cdot \{\pm g \mid g \in G\}.$$

Let $\mathcal{P}(H_1(G;\mathbb{R}))$ be the Grothendieck group of the abelian monoid of polytopes in $H_1(G;\mathbb{R})$ under the Minkowski sum. We define a group homomorphism

$$P' \colon \mathcal{D}(G)^\times \to \mathcal{P}(H_1(G;\mathbb{R})).$$

From these data we obtain a homomorphism

$$P \colon \mathrm{Wh}^w(G) \to \mathcal{P}(H_1(G;\mathbb{R})).$$

Hence we can consider $P(\rho_u^{(2)}(\overline{X})) \in \mathcal{P}(H_1(G;\mathbb{R}))$.

One of our main theorems says

Theorem. Let M be a compact connected orientable irreducible 3-manifold with infinite fundamental group π and empty or incompressible torus boundary which is not a closed graph manifold.

Then there is a virtually finitely generated free abelian group Γ, and a factorization $\pi_1(M) \xrightarrow{\alpha} \Gamma \xrightarrow{\beta} H_1(M)_f := H_1(M)/\operatorname{tors}(H_1(M))$ of the canonical projection into epimorphisms such that the following holds:

For any factorization of $\alpha \colon \pi \to \Gamma$ into group homomorphisms $\pi \xrightarrow{\mu} G \xrightarrow{\nu} \Gamma$ for a torsionfree group G satisfying the Atiyah Conjecture the composite

$$\mathrm{Wh}^w(G) \xrightarrow{P} \mathcal{P}(H_1(G;\mathbb{R})) \xrightarrow{\mathcal{P}(H_1(\beta\circ\nu;\mathbb{R}))} \mathcal{P}(H_1(M;\mathbb{R}))$$

sends $\rho_u^{(2)}(\overline{M};\mathcal{N}(G))$ to the class of the Thurston polytope of M.

Notice that it applies in particular to the universal covering, i.e., $G = \pi_1(M)$, $\mu = \mathrm{id}$ and $\overline{M} = \widetilde{M}$.

References

[1] T. D. Cochran. Noncommutative knot theory. *Algebr. Geom. Topol.*, 4:347–398, 2004.
[2] J. Dubois, S. Friedl, and W. Lück. The L^2–Alexander torsion of 3-manifolds. preprint, arXiv:1410.6918 [math.GT], 2014.

[3] S. L. Harvey. Higher-order polynomial invariants of 3-manifolds giving lower bounds for the Thurston norm. *Topology*, 44(5):895–945, 2005.
[4] W. Lück. *L^2-Invariants: Theory and Applications to Geometry and K-Theory*, volume 44 of *Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics [Results in Mathematics and Related Areas. 3rd Series. A Series of Modern Surveys in Mathematics]*. Springer-Verlag, Berlin, 2002.

Topological characterization of boundaries of free products of groups

Jacek Świątkowski

This report describes some results from [1].

Recall that Gromov boundary of a hyperbolic group is a compact metrisable space. Not much is known about explicit topological spaces that can occur as Gromov boundary of a hyperbolic group.

Each hyperbolic group splits over finite subgroups into a graph of groups with all vertex groups finite or 1-ended hyperbolic. Such a splitting is called *terminal* since its factors do not split further. In view of this, the problem of understanding Gromov boundaries of hyperbolic groups consists of the following two parts:

(1) to understand boundaries of 1-ended hyperbolic groups (this class coincides with trhe class of hyperbolic groups whose Gromov boundaries are connected);
(2) to understand boundaries of ∞-ended hyperbolic groups in terms of boundaries of factors (i.e. vertex groups) in their terminal splittings.

The results presented below provide a satisfactory answer to part (2) of the above problem.

Theorem 1. *Let $\mathcal{X} = (X_i)_{i\in I}$ be a nonempty countable (finite or infinite) family of nonempty metric compacta. Suppose that a space Y satisfies the following conditions:*

(1) *Y is compact metrisable;*
(2) *Y contains a family of pairwise disjoint subspaces $X_{i,\lambda} : i \in I, \lambda \in \Lambda_i$ such that each index set Λ_i is countable infinite, and for each $i \in I$ and any $\lambda \in \Lambda_i$ the subspace $X_{i,\lambda}$ is homeomorphic to X_i;*
(3) *the family $(X_{i,\lambda})_{i,\lambda}$ is null, i.e. diameters of the sets converge to 0;*
(4) *each subspace $x_{i,\lambda}$ is boundary in Y, i.e. its complement is dense;*
(5) *any two distinct points of Y not contained in the same $X_{i,\lambda}$ can be separated by a closed-open subset $H \subset Y$ which is $(X_{i,\lambda})_{i,\lambda}$-saturated, i.e. for each $i \in I$ and each $\lambda \in \Lambda_i$ either $X_{i,\lambda\subset H}$ or $X_{i,\lambda} \cap H = \emptyset$.*

Then Y exists, and is unique up to homeomorphism.

Notation: denote the unique space Y as above with $\widetilde{\sqcup}\mathcal{X}$ or $\widetilde{\sqcup}(X_i : i \in I)$ and call it the *dense amalgam* of the family $\mathcal{X}$.

Theorem 2. *Let $\mathcal{G}$ be a graph of groups with finite edge groups and with hyperbolic vertex groups, at least one of which is infinite. Suppose also that the fundamental* *group $\Gamma = \pi_1(\mathcal{G})$ is ∞-ended. Then Γ is hyperbolic, and $\partial\Gamma \cong \widetilde{\sqcup}\mathcal{X}$, where $\widetilde{\sqcup}\mathcal{X}$ is the family of Gromov boundaries of infinite vertex groups of $\mathcal{G}$.*

References

[1] J. Świątkowski, *The dense amalgam of metric compacta and topological characterization of boundaries of free products of groups*, preprint 2014, arXiv:1410.4989.

Embedding expanders into groups and applications

Damian Osajda

I describe here my recent construction of finitely generated groups containing an infinite family of finite connected graphs of bounded degree [10]. It provides first examples of groups containing isometric copies of expanding families of graphs, and other exotic finitely generated groups.

Let $\Theta = (\Theta_n)_{n\in\mathbb{N}}$ be a family of disjoint finite connected graphs of bounded degree. We assume that there exists a constant $A > 0$ such that $\operatorname{diam}\Theta_n \leq A \operatorname{girth}\Theta_n$, where diam denotes the diameter, and girth is the length of the shortest simple cycle. We fix a small cancellation constant $\lambda \in (0, 1/6]$, and we assume that $1 < \lfloor\lambda \operatorname{girth}\Theta_n\rfloor < \lfloor\lambda \operatorname{girth}\Theta_{n+1}\rfloor$.

Theorem 1. *There exists a $C'(\lambda)$–small cancellation labeling of $(\Theta_n)_{n\in\mathbb{N}}$ over a finite set S of labels.*

With such labelled graph family Θ we associate a graphical small cancellation presentation:

$$\mathcal{P} = \langle S \mid \Theta\rangle. \tag{1}$$

Theorem 2. *For every n, the graph Θ_n embeds isometrically into the Cayley graph Cay(G, S) of the group G defined by the presentation (1).*

For Θ being an expander family, as an immediate corollary we obtain the following.

Corollary 1. *There exist finitely generated groups with expanders embedded isometrically into Cayley graphs.*

These are the first examples of such groups. In particular, they are not coarsely embeddable into a Hilbert space and do not satisfy the Baum-Connes conjecture with coefficients. The only other groups with such properties are the Gromov monsters [7] (see [1] for an explanation of the construction). The Gromov construction uses a graphical presentation with much weaker 'small cancellation' properties. Consequently, only a weak embedding of expanders is established for those examples. The isometric embedding of expanders for the groups from Corollary 1 is useful for analyses of the failure of the Baum-Connes conjecture – see e.g. [13],[5],[6],[8].

Using Sapir's [12] version of Higman embedding we obtain the first examples of groups as follows.

Corollary 2. *There exist closed aspherical manifolds with expanders embedded quasi-isometrically into their fundamental groups.*

The group G defined by the presentation (1) is the limit of finitely presented groups G_i defined by presentations $\langle S \mid (\Theta_n)_{n=1}^{i} \rangle$. For Θ being a family of d–regular graphs with $d > 2$, we obtain the first examples of groups as follows.

Corollary 3. *There exists a sequence $G_1 \twoheadrightarrow G_2 \twoheadrightarrow G_3 \twoheadrightarrow \cdots$ of finitely presented groups with the following properties. For all i, asdim$(G_i) = 2$, and the asymptotic dimension of the limit group G is infinite.*

Note that despite the group G above has infinite asymptotic dimension, it behaves in many ways as a two-dimensional group – see e.g. [11].

Using the construction of the small cancellation presentation (1) provided by Theorem 1, and the method of constructing walls for small cancellation groups developed in [14], [15], [3], and [4], we obtain the following.

Theorem 3. *There exist finitely generated groups acting properly on CAT(0) cubical complexes and not having property A.*

In particular, such groups have the Haagerup property, and thus admit an equivariant coarse embedding into a Hilbert space. This answers in the negative the well known question whether, for groups, Yu's property A (equivalent e.g. to the exactness of the reduced C^*–algebra of the group) implies coarse embedding into a Hilbert space. For spaces, the answer to the corresponding question was already known by [9] and [2].

References

[1] G. Arzhantseva, T. Delzant, *Examples of random groups*, preprint (2008).
[2] G. Arzhantseva, E. Guentner, J. Špakula, *Coarse non-amenability and coarse embeddings*, Geom. Funct. Anal. **22** (2012), 22–36.
[3] G. Arzhantseva, D. Osajda, *Infinitely presented small cancellation groups have the Haagerup property*, J. Topol. Anal. (2015).
[4] G. Arzhantseva, D. Osajda, *Graphical small cancellation groups with the Haagerup property*, preprint (2014).
[5] P. Baum, E. Guentner, R. Willett, *Expanders, exact crossed products, and the Baum-Connes conjecture*, preprint (2013).
[6] M. Finn-Sell, *On the Baum-Connes conjecture for Gromov monster groups*, preprint (2014).
[7] M. Gromov, *Random walk in random groups*, Geom. Funct. Anal. **13** (2003), 73–146.
[8] D. Hume, *A continuum of expanders*, preprint (2014).
[9] P. Nowak, *Coarsely embeddable metric spaces without Property A*, J. Funct. Anal. **252** (2007), 126–136.
[10] D. Osajda, *Small cancellation labellings of some infinite graphs and applications*, preprint (2014).
[11] D. Osajda, J. Świątkowski, *On asymptotically hereditarily aspherical groups*, preprint (2013).
[12] M. Sapir, *A Higman embedding preserving asphericity*, J. Amer. Math. Soc. **27** (2014), 1–42.
[13] R. Willett, G. Yu, *Higher index theory for certain expanders and Gromov monster groups, I*, Adv. Math. **229** (2012), 1380–1416.
[14] D. Wise, *Cubulating small cancellation groups*, Geom. Funct. Anal. **14** (2004), 150–214.
[15] D. Wise, *The structure of groups with quasiconvex hierarchy*, preprint (2011).

Brown's question for cocompact actions

Ian Leary
(joint work with Nansen Petrosyan)

The question of K. S. Brown referred to in the title concerns properly discontinuous actions of virtually torsion-free discrete groups [3, page 32]. There are a number of slightly different versions of this question; we use right-angled Coxeter groups to give answers to two variants.

For a flag complex L, the associated right-angled Coxeter group W_L is the group given by the presentation in which the generators are the vertices of L, subject only to the relations that each generator has order two, and each pair of generators that spans an edge in L commutes. Denote this generating set for W_L by S_L. The Davis complex $\Sigma(W_L, S_L)$ is a CAT(0) cube complex on which W_L acts properly discontinously by cellular isometries [4]. W_L acts vertex transitively, and the link of each vertex in Σ is isomorphic to L.

For a discrete group G, a model for $\underline{E}G$ is a G-CW-complex X in which all stabilizer subgroups are finite and such that for every finite $F \leq G$, the fixed point set X^F is contractible. The minimal dimension of any model for $\underline{E}G$ is denoted by $\underline{\mathrm{gd}}G$. There is an algebraic analogue $\underline{\mathrm{cd}}G$ of $\underline{\mathrm{gd}}G$. It is known that $\underline{\mathrm{cd}}G = \underline{\mathrm{gd}}G$, except that for some groups $\underline{\mathrm{cd}}G = 2 < \underline{\mathrm{gd}}G = 3$ [9, 7, 2]. The virtual cohomological dimension $\mathrm{vcd}G$ of a virtually torsion-free group G is by definition the cohomological dimension of a finite-index torsion-free subgroup of G.

For the first theorem, we let L be a finite contractible 3-dimensional flag complex, with an admissible action of a cyclic group C_p such that the fixed point set L^{C_p} is a mod-q Moore space; here p and q are distinct primes. Let G be either the semidirect product of W_L and the cyclic group C_p, or its normal subgroup consisting of the semidirect product of the commutator subgroup W_L and the cyclic group C_p. The action of W_L on the Davis complex $\Sigma = \Sigma(W_L, S_L)$ extends to an action of G, and Σ is a cocompact model for $\underline{E}G$.

Theorem 1. For this G, $\mathrm{vcd}G = 3$ while $\underline{\mathrm{cd}}G = 4$. Similarly, for any $n \geq 1$, $\mathrm{vcd}G^n = 3n$ while $\underline{\mathrm{cd}}G^n = 4n$.

In contrast, it has been shown that $\mathrm{vcd}W = \underline{\mathrm{cd}}W$ for any Coxeter group W [6]. Examples of groups for which $\mathrm{vcd}G < \underline{\mathrm{cd}}G$ were known previously [8, 10, 5]. However, the earlier examples are all based on Bestvina-Brady groups [1], rather than Coxeter groups, and they do not admit cocompact models for $\underline{E}G$.

For our second theorem, we let L be a flag triangulation of a finite acyclic 2-dimensional complex such that the fundamental group $\pi_1(L)$ admits a non-trivial unitary representation $\rho : \pi_1(L) \to U(n)$. For example L could be the 2-skeleton of the Poincaré homology sphere, whose fundamental group has order 120 and admits a faithful representation into $U(2)$. We let G be the right-angled Coxeter group W_L.

Theorem 2. For this G, $\mathrm{vcd}G = \underline{\mathrm{cd}}G = 2$ but there does not exist any contractible 2-dimensional proper G-CW-complex.

It was known previously that for this group $\underline{\mathrm{cd}}G = 2$ but $\underline{\mathrm{gd}}G = 3$ [2]; what is new is an argument that rules out *all* equivariant homotopy types of contractible proper G-CW-complex.

Any torsion-free finite-index subgroup H of this G will have cohomological dimension two. M. Bestvina and M. Davis proposed these groups H as potential counterexamples to the Eilenberg-Ganea conjecture. Our work sheds no new light on this question.

References

[1] M. Bestvina and N. Brady, *Morse theory and finiteness properties of groups*, Invent. Math. **129** (1997), 445-470.

[2] N. Brady, I. J. Leary and B. E. A. Nucinkis, *On algebraic and geometric dimensions for groups with torsion* J. London Math. Soc. **64** (2001) 489–500.

[3] K. S. Brown, *Groups of virtually finite dimension*, in *Homological group theory* London Math. Soc. Lecture Note Series **36** Springer (1979) 27–70.

[4] M. W. Davis, *The geometry and topology of Coxeter groups* London Math. Soc. Monographs **32** Princeton (2008).

[5] D. Degrijse and N. Petrosyan, *Geometric dimension of groups for the family of virtually cyclic subgroups*, J. Topol. **7** (2014), 697–726.

[6] D. Degrijse and C. Martínez-Pérez, *Dimension invariants for groups admitting a cocompact model for proper actions*, preprint arXiv:1311.6419.

[7] M. J. Dunwoody, *Accessibility and groups of cohomological dimension one*, Proc. London Math. Soc. **38** (1979) 193–215.

[8] I. J. Leary and B. E. A. Nucinkis, *Some groups of type VF*, Invent. Math. **151** (2003) 135–165.

[9] W. Lück, *Transformation groups and algebraic K-theory*, Lecture Notes in Math. **1408**, Springer (1989).

[10] C. Martínez-Pérez, *Euler classes and Bredon cohomology for groups with restricted families of finite subgroups*, Math. Z. **275** (2013) 761–780.

The rational homology of generalised Thompson groups

Brita Nucinkis

(joint work with Conchita Martínez-Pérez and Marco Varisco)

Thompson's group V is an infinite group of homeomorphisms of the Cantor-set that is finitely presented and simple. K.S. Brown [3] showed that Thompson's group V is rationally acyclic.

Generalisations of this group are due to Higman [5], Stein [9] and Brin [1] and these were shown to be of type F_∞ by Brown [2], Stein [9] and Fluch-Marschler-Witzel-Zaremsky [4] respectively. Using a description analogously to Higman via Cantor-algebras, one can further generalise these constructions to automorphism groups $V_r(\Sigma)$ of these Cantor-algebras, and show that under some mild hypotheses on the Cantor algebra, these are also of type F_∞, see [6, 7]. The proofs all use very similar methods by constructing a contractible complex, on which the groups act with finite stabilisers, and which has a filtration by G-finite complexes. One now needs to show that these complexes are highly connected. The strongest such construction is due to Stein [9], called the Stein-complex, and using Morse-theoretic methods [4, 7] one can show that the Stein-complex has the desired connectedness properties.

In his proof that V is rationally acyclic, Brown [3] used a truncated version of the Stein-complex, $X_{p,q}$, and showed that for each n and $q \geq p + n$, $X_{p,q}$ is an n-dimensional, $(n-1)$-connected simplicial complex such that V acts with finite stabilisers and with an n-simplex as a fundamental domain.

We use a similar construction for the general case. In particular, for Brin's group sV, where $s \geq n$ is an integer we have:

Proposition.[8] *Let $G = sV$. For all integers $n \geq 1$ there is an integer p_0 depending on n such that $X_{p,q}$ is n-connected for $p \geq p_0$ and $q \geq p + 2^s n$.*

Let $Y_{p,q} = X_{p,q}/G$. We show that there is a long exact sequence in homology:

$$\cdots \to H_j(Y_{p+1,q}) \to H_j(Y_{p,q}) \to \tilde{H}_{j-1}(Z_{p+1,q}) \to \dots,$$

where $Z_{p+1,q}$ is a complex which is contractible for $q-(p+1)$ big enough. This now yields that the rational homology of $V_r(\Sigma)$ vanishes in sufficiently high dimensions. For example:

Theorem.[8] *Let $s \geq 2$ be an integer and let $G = sV$. Then for all $n \geq 3^s - 2^s$,*

$$H_n(G, \mathbb{Q}) = 0.$$

References

[1] M. G. Brin. *Higher dimensional Thompson groups*, Geom. Dedicata, **108**, (2004), 163–192.

[2] K. S. Brown, *Finiteness properties of groups*, Journal of Pure and Applied Algebra, **44** (1987), 45–75.

[3] K. S. Brown. *The geometry of finitely presented infinite simple groups*, Algorithms and classification in combinatorial group theory (Berkeley, CA, 1989), Math. Sci. Res. Inst. Publ., 23, Springer, New York, (1992), 121–136.

[4] M. Fluch, M. Marschler, S. Witzel, and M. C. B. Zaremsky. *The Brin-Thompson groups sV are of type F_∞*, to appear in Pacific J. Math.

[5] Graham Higman. *Finitely presented infinite simple groups*, Department of Pure Mathematics, Department of Mathematics, I.A.S. Australian National University, Canberra, 1974. Notes on Pure Mathematics, No. 8 (1974).

[6] C. Martínez-Pérez, and B. E. A. Nucinkis, *Bredon cohomological finiteness conditions for generalisations of Thompson's groups*, Groups Geom. Dyn. **7** (2013), 931–959,

[7] C. Martínez-Pérez, F. Matucci and B. E. A. Nucinkis, *Cohomological finiteness conditions and centralisers in generalisations of Thompson's group V* preprint (2014).

[8] C. Martínez-Pérez, B. Nucinkis and M. Varisco, *Conjugacy and homology in generalised Thompson groups*, in preparation.

[9] M. Stein. *Groups of piecewise linear homeomorphisms*, Trans. Amer. Math. Soc., **332**(2), (1992), 477–514.

A Dehn-Nielsen-Baer theorem for surfaces with boundary

Tobias Hartnick

(joint work with Gabi Ben Simon, Marc Burger, Alessandra Iozzi, Anna Wienhard)

The classical Dehn-Nielsen-Baer theorem states that if Σ_1, Σ_2 are closed oriented surfaces with respective fundamental groups Γ_1, Γ_2, then every isomorphism $\Gamma_1 \to \Gamma_2$ is induced by a homeomorphism $\Sigma_1 \to \Sigma_2$. We establish a variant for surfaces with boundary as follows:

Let Σ be a compact surface with boundary, $\Gamma = \pi_1(\Sigma)$ and $\rho : \Gamma \to \mathrm{PSL}_2(\mathbb{R})$ the holonomy representation of a complete hyperbolic structure on the interior of Σ. Then ρ induces an action of Γ on the circle, which (since Γ is free) in turn lifts to an action on the real line. If Λ denotes the commutator subgroup of Γ, then the action of Λ on $\mathbb{R}$ is independent of the choice of lift, and we obtain a bi-invariant partial order on Λ by declaring that

$$g > h :\Leftrightarrow \forall x \in \mathbb{R} : g.x > h.x. \tag{1}$$

It turns out that the order on Λ depends only on Σ, but not on the choice of hyperbolization ρ, whence we will denote it by $<_\Sigma$. We refer to the triple $(\Gamma, \Lambda, <_\Sigma)$ as the *ordered fundamental group* of Σ. A *morphism* $(\Gamma_1, \Lambda_1, <_{\Sigma_1}) \to (\Gamma_2, \Lambda_2, <_{\Sigma_2})$ between ordered fundamental groups is defined as a group homomorphsm $\Gamma_1 \to \Gamma_2$ which is order preserving when restricted to $\Lambda_1 \to \Lambda_2$.

Theorem 1 (Dehn-Nielsen-Baer theorem for surfaces with boundary, [1])**.** *Every homeomorphism $\Sigma_1 \to \Sigma_2$ of compact surfaces with boundary induces an isomorphism $(\Gamma_1, \Lambda_1, <_{\Sigma_1}) \to (\Gamma_2, \Lambda_2, <_{\Sigma_2})$ of ordered fundamental groups, and conversely every isomorphism of ordered fundamental groups is induced from a homeomorphism.*

Theorem 1 is a geometric consequence of the following purely representation theoretic result. Every representation $\rho : \Gamma \to G$ gives rise to an action of the commutator subgroup Λ on the real line, and hence defines a partial order $<_\rho$ on Λ via (1).

Theorem 2 (Characterization of Teichmüller space, [1])**.** *Let $\Gamma := \pi_1(\Sigma)$. A homomorphism $\Gamma \to \mathrm{PSL}_2(\mathbb{R})$ is the holonomy representation of a complete hyperbolic structure on the interior of Σ if and only if $<_\rho = <_\Sigma$. In this case, ρ is faithful with discrete image.*

The proof of Theorem 2 is based on a geometric study of the bounded Euler class. The full relevance of the order $<_\Sigma$ becomes clear in the study of representations of $\Gamma := \pi_1(\Sigma)$ into more general Hermitian Lie groups. To formulate a result analogous to Theorem 2 for representations into a more general target group we introduce the following notation. Let

$$\mathrm{Homeo}^+_{\mathbb{Z}}(\mathbb{R}) := \{f : \mathbb{R} \to \mathbb{R} \mid f \text{ or.-pr. homeomorphism}, f(x+1) = f(x) + 1\}$$

and recall that the translation number $T : \mathrm{Homeo}^+_{\mathbb{Z}}(\mathbb{R}) \to \mathbb{R}$ is defined by

$$Tf := \lim_{n\to\infty} \frac{f^n(0)}{n}.$$

Then the condition $g > h$ as defined by (1) can be written as $T(h^{-1}g) > 0$. This motivates the following definition. Let Σ, Γ, Λ as in Theorem 1 and let $\rho : \Gamma \to \mathrm{PSL}_2(\mathbb{R})$ be a hyperbolization as above. As before this induces an action of Λ on $\mathbb{R}$ and we define

$$g >_{\Sigma,n} h :\Leftrightarrow T(h^{-1}g) > n,$$

so that $<_\Sigma = <_{\Sigma,0}$. The whole family of orders $<_{\Sigma,n}$ depends only on Σ. It can be used to detect higher Teichmüller representations in the sense of the following theorem:

Theorem 3 (Discrete-faithfulness criterion, [1])**.** *Let G be a Hermitian simple Lie group with finite center, $\widetilde{G}$ its unique cyclic covering and $\preceq$ a continuous bi-invariant partial order on $\widetilde{G}$. Every homomorphism $\rho : \Gamma \to G$ induces a unique homomorphism $\widetilde{\rho} : \Lambda \to \widetilde{G}$. If $\widetilde{\rho}$ is order-preserving with respect to one of the orders $\leq_{\Sigma,n}$ on Λ and the order $\preceq$ on $\widetilde{G}$, then ρ is discrete and faithful.*

It turns out that Theorem 2 can be deduced from Theorem 3. The proof of the latter is again based on techniques from continuous bounded cohomology.

References

[1] G. Ben Simon, M. Burger, T. Hartnick, A. Iozzi, A. Wienhard, *On order preserving representations*, in preparation.

Gong show for junior participants

PhD students and recent postdocs were offered the opportunity to present themselves and their research in a 10 minutes talk, as part of a so-called *Gong Show*. Eleven junior participants of the workshop engaged in the Gong Show and created a mathematical mosaic ranging from index theory to quasi-morphisms. The Gong Show was well received by junior and senior participants and helped the different scientific generations to grow closer together.

In the gong show, the following speakers presented their work as specified.

- **Rudolf Zeidler (Göttingen)**: *Secondary index theory and product formulas.* We exhibit product formulas for higher ρ-classes associated to metrics of positive scalar curvature (PSC). These are spin bordism invariants of PSC metrics. On the product of a closed spin PSC manifold with an arbitrary closed spin manifold, the higher ρ-invariant is computed as an external product of the higher ρ-invariant on the first factor and the K-homological fundamental class of the second factor.

- **Christopher Wulff (Augsburg)**: *Ring and module structures in K-theory of leaf spaces.* A new K-theory model for the leaf space of a foliation is introduced. It is a ring and Connes's K-theory model is a module over this ring. Ring and module structure are motivated by longitudinal index theory of twisted operators.
- **Markus Steenbock (Wien)**: *Rips construction without unique products.* Given a finitely presented group Q, we produce a short exact sequence $1 \to N \hookrightarrow G \twoheadrightarrow Q \to 1$ such that G is a torsion-free *hyperbolic* group without the unique product property and N has Property (T). Varying Q, we obtain a wide diversity of *new* concrete examples of groups without the unique product property. Kaplansky's zero-divisor conjecture is open for our groups. This is a joint work with Goulnara Arzhantseva. M. Steenbock is recipient of the DOC fellowship of the Austrian Academy of Sciences, and partially supported by the ERC grant ANALYTIC no. 269527 of Goulnara Arzhantseva.
- **Henrik Rüping (Bonn)**: *On the Farrell-Jones conjecture.* In my talk I mentioned the Definition of the Farrell-Jones conjecture, stated the properties of the class of Groups for which the Farrell-Jones conjecture is known and asked whether specific Groups are in this class.
- **Christina Pagliantini (Zürich)**: *Quasi-morphisms and bounded cohomology.* We discuss the second bounded cohomology $H^2_b(\Gamma, H; \mathbb{R})$ of a free group Γ of finite rank relative to a subgroup H of finite rank by means of the theory of relative quasimorphisms. This is a joint work with P. Rolli.
- **Andreas Ott (Heidelberg)**: *Bounded cohomology and partial differential equations.* We present a new technique that employs partial differential equations in order to compute bounded group cohomology. This is joint work with Tobias Hartnick.
- **Christoforos Neofytidis (Binghamton)**: *Groups presentable by products and maps of non-zero degree.* We discuss obstructions to the existence of maps of non-zero degree from direct products to rationally essential manifolds, with special emphasis to aspherical manifolds whose fundamental groups have non-trivial center. As an application, we obtain an ordering of all non-hyperbolic 4-manifolds possessing a Thurston aspherical geometry.
- **Michal Marcinkowski (Wroclaw)**: *Gromov's positive scalar curvature conjecture and macroscopically large rationally inessential manifolds.* We show examples of macroscopically large (in the sense of Gromov) but rationally inessential manifolds. In spin case they do not admit positive scalar curvature metrics, thus support a conjecture of Gromov.
- **Robert Kropholler (Oxford)**: *Non-hyperbolic subgroups of hyperbolic groups.* In this talk I give an infinite family of hyperbolic groups each of which has a finitely presented not hyperbolic subgroup. The key technique used is Bestvina-Brady Morse theory.

- **David Kielak (Bonn)**: *Nielsen realisation for right-angled Artin groups.* We will discuss the recent development in realising finite subgroups of outer automorphism groups of RAAGs as groups acting on cube complexes.
- **Matthias Blank (Regensburg)**: *Relative bounded cohomology for groupoids.* We discuss bounded cohomology for (pairs of groupoids). First, we extend results about amenable groups to the groupoid case. Second, we present an extension of Gromov's mapping theorem, relating under some conditions the bounded cohomology of a pair of (not necessarily connected spaces) to the bounded cohomology of the pairs of fundamental groupoids.

Reporter: Rudolf Zeidler

Participants

Prof. Dr. Goulnara N. Arzhantseva
Fakultät für Mathematik
Universität Wien
Oskar-Morgenstern-Platz 1
1090 Wien
AUSTRIA

Prof. Dr. Arthur Bartels
Fachbereich Mathematik
Universität Münster
Einsteinstr. 62
48149 Münster
GERMANY

Dr. Jason Behrstock
Department of Mathematics
Herbert H. Lehman College
CUNY
Bedford Park, Blvd. West
Bronx, NY 10468
UNITED STATES

Matthias Blank
Fakultät für Mathematik
Universität Regensburg
93040 Regensburg
GERMANY

Prof. Dr. Martin R. Bridson
Mathematical Institute
Oxford University
Andrew Wiles Bldg.
Woodstock Road
Oxford OX2 6GG
UNITED KINGDOM

Prof. Dr. Michelle Bucher-Karlsson
Université de Geneve
Section de Mathématiques
2-4, rue du Lievre
Case Postale 64
1211 Geneve 4
SWITZERLAND

Prof. Dr. James F. Davis
Department of Mathematics
Indiana University
Bloomington, IN 47405
UNITED STATES

Prof. Dr. Michael W. Davis
Department of Mathematics
The Ohio State University
100 Mathematics Building
231 West 18th Avenue
Columbus, OH 43210-1174
UNITED STATES

Prof. Dr. Johannes Ebert
Fachbereich Mathematik
Universität Münster
Einsteinstr. 62
48149 Münster
GERMANY

Dr. Eugenia Ellis Raggio
Instituto de Matematica y Estadistica
Facultad de Ingenieria
Julio Herrera y Reissig 565
Montevideo 11300
URUGUAY

Prof. Dr. Stefan Klaus Friedl
Universität Regensburg
Fakultät für Mathematik
93040 Regensburg
GERMANY

Dr. Roberto Frigerio
Dip. di Matematica Applicata
Universita di Pisa
Via F. Buonarroti 1C
56127 Pisa
ITALY

Dr. Giovanni Gandini
Dept. of Mathematical Sciences
University of Copenhagen
Universitetsparken 5
2100 Copenhagen
DENMARK

Prof. Dr. Sebastian Goette
Mathematisches Institut
Universität Freiburg
Eckerstr. 1
79104 Freiburg i. Br.
GERMANY

Dr. Lukasz Grabowski
Mathematical Institute
Oxford University
24-29 St. Giles
Oxford OX1 3LB
UNITED KINGDOM

Prof. Dr. Bernhard Hanke
Institut für Mathematik
Universität Augsburg
86135 Augsburg
GERMANY

Prof. Dr. Tobias Hartnick
Department of Mathematics
TECHNION
Israel Institute of Technology
Haifa 32000
ISRAEL

Prof. Dr. Tadeusz Januszkiewicz
Institute of Mathematics of the Polish
Academy of Sciences
ul. Sniadeckich 8
00-656 Warszawa
POLAND

Dr. Holger Kammeyer
Mathematisches Institut
Universität Bonn
Endenicher Allee 60
53115 Bonn
GERMANY

Dr. Jarek Kedra
Dept. of Mathematical Sciences
University of Aberdeen
Fraser Noble Bldg.
Aberdeen AB24 3UE
UNITED KINGDOM

Dr. Qayum Khan
Saint Louis University
Dept. of Mathematics & Computer
Science
203 Ritter Hall
St. Louis, MO 63103
UNITED STATES

Dr. Dawid Kielak
Mathematisches Institut
Universität Bonn
Endenicher Allee 60
53115 Bonn
GERMANY

Prof. Dr. Linus Kramer
Mathematisches Institut
Universität Münster
Einsteinstr. 62
48149 Münster
GERMANY

Dr. Robert Kropholler
Mathematical Institute
Oxford University
Andrew Wiles Bldg.
Woodstock Road
Oxford OX2 6GG
UNITED KINGDOM

Dr. Jean-Francois Lafont
Department of Mathematics
The Ohio State University
100 Mathematics Bldg.
231 West 18th Avenue
Columbus, OH 43210-1174
UNITED STATES

Prof. Dr. Ian J. Leary
School of Mathematics
University of Southampton
Highfield Campus
Southampton SO17 1BJ
UNITED KINGDOM

Prof. Dr. Clara Löh
Fakultät für Mathematik
Universität Regensburg
93040 Regensburg
GERMANY

Prof. Dr. Wolfgang Lück
Mathematisches Institut
Universität Bonn
Endenicher Allee 60
53115 Bonn
GERMANY

Dr. Michal Marcinkowski
Institute of Mathematics
University of Wroclaw
50-384 Wroclaw
POLAND

Dr. Christoforos Neofytidis
Dept. of Mathematical Sciences
University of Binghamton
Binghamton NY 13902-6000
UNITED STATES

Dr. Bogdan Nica
Mathematisches Institut
Georg-August-Universität Göttingen
Bunsenstr. 3-5
37073 Göttingen
GERMANY

Prof. Dr. Piotr Nowak
Institute of Mathematics
University of Warsaw
Banacha 2
02-097 Warszawa
POLAND

Dr. Brita E.A. Nucinkis
Department of Mathematics
Royal Holloway College
University of London
London TW20 0EX
UNITED KINGDOM

Prof. Dr. Boris L. Okun
Department of Mathematics
University of Wisconsin-Milwaukee
P.O. Box 413
Milwaukee, WI 53201
UNITED STATES

Dr. Damian L. Osajda
Institute of Mathematics
Wroclaw University
pl. Grunwaldzki 2/4
50-384 Wroclaw
POLAND

Prof. Dr. Denis Osin
Mathematics Department
Vanderbilt University
1326 Stevenson Center
Nashville, TN 37240
UNITED STATES

Dr. Andreas Ott
Mathematisches Institut
Universität Heidelberg
Im Neuenheimer Feld 288
69120 Heidelberg
GERMANY

Viktoriya Ozornova
FB 3 - Mathematik & Informatik
Universität Bremen
28359 Bremen
GERMANY

Dr. Cristina Pagliantini
Departement Mathematik
ETH-Zentrum
Rämistr. 101
8092 Zürich
SWITZERLAND

Dr. Christophe Pittet
Centre de Mathématiques &
d'Informatique
Aix-Marseille Université
39, Rue Joliot-Curie
13453 Marseille Cedex 13
FRANCE

Dr. Henrik Rüping
Department of Mathematics
The University of British Columbia
Room 121
1984 Mathematics Road
Vancouver, B.C. V6T 1Z2
CANADA

Prof. Dr. Roman Sauer
Institut für Algebra & Geometrie
Fakultät für Mathematik (KIT)
Kaiserstr. 89-93
76133 Karlsruhe
GERMANY

Prof. Dr. Thomas Schick
Mathematisches Institut
Georg-August-Universität Göttingen
Bunsenstr. 3-5
37073 Göttingen
GERMANY

Dr. Kevin Schreve
Department of Mathematics
University of Wisconsin-Milwaukee
P.O. Box 413
Milwaukee, WI 53201
UNITED STATES

Dr. Markus Steenbock
Fakultät für Mathematik
Universität Wien
Oskar-Morgenstern-Platz 1
1090 Wien
AUSTRIA

Jun.-Prof. Dr. Wolfgang Steimle
Mathematisches Institut
Universität Leipzig
Postfach 10 09 20
04009 Leipzig
GERMANY

Prof. Dr. Jacek Swiatkowski
Institute of Mathematics
Wroclaw University
pl. Grunwaldzki 2/4
50-384 Wroclaw
POLAND

Dr. Romain A. Tessera
Département de Mathématiques
Université de Paris-Sud
Bat. 425
91405 Orsay Cedex
FRANCE

Prof. Dr. Andreas B. Thom
Mathematisches Institut
Universität Leipzig
Augustusplatz 10
04103 Leipzig
GERMANY

Dr. Christopher Voll
Fakultät für Mathematik
Universität Bielefeld
Postfach 100131
33501 Bielefeld
GERMANY

Dr. Xiaolei Wu
Institut für Mathematik
Freie Universität Berlin
14195 Berlin
GERMANY

Christopher Wulff
Institut für Mathematik
Universität Augsburg
86135 Augsburg
GERMANY

Rudolf Zeidler
Mathematisches Institut
Georg-August-Universität Göttingen
Bunsenstr. 3-5
37073 Göttingen
GERMANY