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Mini-Workshop: Coideal Subalgebras of Quantum Groups

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Abstract. Coideal subalgebras of quantized enveloping algebras appear naturally if one considers quantum group analogs of Lie subalgebras. Examples appear in the theory of quantum integrable systems with boundary and in harmonic analysis on quantum group analogs of Riemannian symmetric spaces. Recently, much progress has been made to develop a deeper representation theoretic understanding of these examples. On the other hand, coideal subalgebras play a fundamental role in the theory of Nichols algebras. The workshop aimed to discuss these theories in view of the recent developments.

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Introduction by the Organisers

Quantum groups are well-known objects in representation theory, in the theory of integrable systems, and in the theory of Hopf algebras. Quantum group analogs of homogeneous spaces were much slower to develop, mainly because the immediate definition ((co)invariants with respect to a Hopf subalgebra) turned out to be too rigid to allow for interesting examples. By now, however, there is ample evidence that one-sided coideal subalgebras of quantum groups provide a suitable concept of quantum homogeneous spaces.

The relation between quantum integrable systems with boundary and comodule algebras for quantum groups was apparent early on. About 15 years ago, explicit examples of coideal subalgebras of quantum groups in Drinfeld-Jimbo realization appeared in the investigation of quantum integrable systems with boundary. These coideal subalgebras provided a tool to construct solutions of the so called reflection equation, an integrability condition for systems with boundary. In the 90s, a theory of quantum group analogs of symmetric spaces was developed by M. Noumi et al. and independently by G. Letzter. The aim of their program was to provide new interpretations of Macdonald-Koornwinder polynomials as zonal spherical functions on quantum symmetric spaces. The construction of quantum symmetric pairs has recently been extended to involutive automorphisms of symmetrizable Kac-Moody algebras by S. Kolb. It includes the examples obtained from integrable systems.

Very recently, H. Bao & W. Wang and M. Ehrig & C. Stroppel indicated that much of modern representation theory for quantum groups (Schur-Jimbo duality, canonical bases, Kazhdan-Lusztig theory, categorification) extends to quantum symmetric pairs. From the algebraic side, coideal subalgebras of quantum groups have been classified within the wider context of Nichols algebras, a development which provides ample new technology.

The workshop brought together experts on coideal subalgebras from different backgrounds (integrable systems, special functions, quantum symmetric pairs, representation theory, Nichols algebras). The aim was to get to know each others perspective, to analyze the present state of the art, and to pursue avenues of cross-fertilization.

The workshop started out with three mini lecture series. N. Reshetikhin gave an introduction to the physics origins of the reflection equation. Starting from the 6-vertex model in statistical mechanics he introduced the boundary q-Knizhnik-Zamolodchikov equations as the consistency conditions for correlation functions with reflecting boundary. C. Stroppel gave an overview of categorification of quantum groups in view of recent developments for coideal subalgebras. H.-J. Schneider delivered a crash course on Nichols algebras. He explained how right coideal subalgebras form a potent tool within this general theory and outlined their classification in terms of the Weyl groupoid.

The talks by the remaining 13 participants gave insight into different aspects of coideal subalgebras of quantum groups. The recent developments around Bao & Wang's program of canonical bases and Ehrig & Stroppel's categorification for quantum symmetric pairs generated a lot of enthusiasm. One central ingredient here is a new bar involution for these coideal subalgebras. This has already been very fruitful in the construction of a universal K-matrix in the finite setting, see M. Balagović's talk. Several talks highlighted the recent important developments regarding the role of coideal subalgebras in integrable systems with boundaries.

We left the workshop with the impression that this research area has made a big leap forward and that further developments are to be expected soon.

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Mini-Workshop: Coideal Subalgebras of Quantum Groups

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Abstracts

Universal solutions of the reflection equation

Martina Balagovic
(joint work with Stefan Kolb)

The aim of this project is to construct universal solutions of the reflection equation (RE) using quantum symmetric pair coideal subalgebras, in a way analogous to the well known construction of universal solutions of the quantum Yang Baxter equation (QYBE) using quantum groups.

The QYBE describes constraints in the scattering of three particles. For a quantum enveloping algebra $U_q\mathfrak{g}$, and a pair of representations V_1, V_2 , the aim is to find an operator $R_{V_1,V_2} : V_1 \otimes V_2 \to V_2 \otimes V_1$ which commutes with the action of $U_q\mathfrak{g}$. The constraint is $R_{V_2,V_3} \circ R_{V_1,V_3} \circ R_{V_1,V_2} = R_{V_1,V_2} \circ R_{V_1,V_3} \circ R_{V_2,V_3}$ as operators $V_1 \otimes V_2 \otimes V_3 \to V_3 \otimes V_2 \otimes V_1$.

The RE describes constraints in the interaction of two particles and a wall. For V a representation of $U_q\mathfrak{g}$, the aim is to find an operator $K_V : V \to V$ or $K_V : V \to V^{\mathrm{twisted}}$, such that $R_{V_2,V_1} \circ K_{V_2} \circ R_{V_1,V_2} \circ K_{V_1} = K_{V_1} \circ R_{V_2,V_1} \circ K_{V_2} \circ R_{V_1,V_2}$ as operators $V_1 \otimes V_2 \to V_1 \otimes V_2$. Additionally, K_V should commute with the action of some coideal subalgebra of $U_q\mathfrak{g}$.

These questions can be stated for any Hopf algebra, but quantum enveloping algebras allow for a construction of a universal solution of the QYBE in the following way (see [Lus94]):

(1) Construct a bar involution $x \mapsto \overline{x}$ on $U_q\mathfrak{g}$.
(2) Find the quasi R-matrix $\mathcal{R}_0$ in (a certain completion of) $U_q\mathfrak{g} \otimes U_q\mathfrak{g}$, such that $\mathcal{R}_0\Delta(\overline{x}) = \overline{\Delta(x)}\mathcal{R}_0$.
(3) For κ a certain operator acting as a constant on every tensor product of weight spaces, consider the operators $\mathcal{R} = \mathcal{R}_0 \circ \kappa$ and $\check{\mathcal{R}} = \mathcal{R}_0 \circ \kappa \circ \mathrm{flip}$.

The operator $\check{\mathcal{R}}$ satisfies:

a) $\check{\mathcal{R}}$ commutes with $U_q\mathfrak{g}$;
b) For every pair of (integrable, category $\mathcal{O}$) representations V_1, V_2 of $U_q\mathfrak{g}$, its action induces a linear operator $R_{V_1,V_2} = \mathcal{R}|_{V_2 \otimes V_1} \circ \mathrm{flip}$ on $V_1 \otimes V_2$;
c) This is consistent with the tensor product of modules, in the sense that $(\Delta \otimes 1)(\mathcal{R}) = \mathcal{R}_{13} \circ \mathcal{R}_{23}$ and $(1 \otimes \Delta)(\mathcal{R}) = \mathcal{R}_{13} \circ \mathcal{R}_{12}$.

From these properties it quickly follows that

d) $\check{\mathcal{R}}$ is a universal solution of the QYBE, in the sense that $\mathcal{R}$ satisfies

$$\mathcal{R}_{23} \circ \mathcal{R}_{13} \circ \mathcal{R}_{12} = \mathcal{R}_{12} \circ \mathcal{R}_{13} \circ \mathcal{R}_{23},$$

and consequently $\check{\mathcal{R}}$ induces a solution of the QYBE for any triple of modules V_1, V_2, V_3.

The work of [BW13] and [ES13] gave an indication that an analogously defined bar involution should exist for quantum symmetric pairs. Moreover, [BW13] performed a part of this program (steps (1),(2),(3),(a),(b)) for quantum symmetric pair coideal subalgebras corresponding to $\mathfrak{gl}_n \times \mathfrak{gl}_n \subseteq \mathfrak{gl}_{2n}$ and $\mathfrak{gl}_{n+1} \times \mathfrak{gl}_n \subseteq \mathfrak{gl}_{2n+1}$. The natural question then is whether it is possible to extend this program to other quantum symmetric pairs, and whether it is possible to prove (c) and (d) in those cases as well.

Let $B_{\mathbf{c},\mathbf{s}}$ be a quantum symmetric pair coideal subalgebra of $U_q\mathfrak{g}$, as defined in [Let02] and [Kol14]. It depends on the choice of a finite type subalgebra $U_q\mathfrak{m}_X$, an involution τ of the Dynkin diagram, and some parameters $\mathbf{c}, \mathbf{s}$. We proceed in several steps, corresponding to the steps in solving the QYBE above:

(1) In the paper [BK14] we describe the exact values of parameters $\mathbf{c}, \mathbf{s}$ for which there is a natural bar involution $x \mapsto \overline{x}^B$ on $B_{\mathbf{c},\mathbf{s}}$. The proof uses the presentation of $B_{\mathbf{c},\mathbf{s}}$ from [Let02], [Kol14], and extends their description of the relations to some new cases.

(2) We construct a quasi K-matrix $\mathfrak{X}$, which intertwines two different bar involutions $x \mapsto \overline{x}^B$ on $B_{\mathbf{c},\mathbf{s}}$ and $x \mapsto \overline{x}$ on $U_q\mathfrak{g}$, in the sense that $\mathfrak{X}\overline{x} = \overline{x}^B\mathfrak{X}$ for every $x \in B_{\mathbf{c},\mathbf{s}}$. (A special case of $\mathfrak{X}$ was constructed in [BW13]).

(3) We construct an element $\mathcal{K}$ of a certain completion of $U_q\mathfrak{g}$. For instance, when $U_q\mathfrak{g}$ is of finite type, $\mathcal{K}$ is given by

$$\mathcal{K} = \mathfrak{X} \circ \xi \circ T_{w_0}^{-1} \circ T_{w_X}^{-1},$$

for ξ a certain operator acting as a constant on every weight space, and T_{w_0}, T_{w_X} are the Lusztig braid group operators associated to the longest element of the Weyl group of $U_q\mathfrak{g}$ and of $U_q\mathfrak{m}_X$. (Again, this is extending the work of [BW13]).

The operator $\mathcal{K}$ satisfies:

a) $\mathcal{K}$ commutes with $B_{\mathbf{c},\mathbf{s}}$.

b) For every (integrable, category $\mathcal{O}$) representation V of $U_q\mathfrak{g}$, its action induces a linear operator K_V, which is an isomorphism between certain twists of V. For instance, in finite type,

$$K_V : V \to V^{\tau\tau_0},$$

with τ_0 the involution of the Dynkin diagram induced by w_0 and $V^{\tau\tau_0}$ is the module with the $U_q\mathfrak{g}$-action twisted by the automorphism $\tau\tau_0$.

c) In finite type, this is consistent with the tensor product of modules, in the sense that

$$\Delta(\mathcal{K}) = (\mathcal{K} \otimes 1) \circ \check{\mathcal{R}}^{\tau\tau_0 \otimes 1} \circ (\mathcal{K} \otimes 1) \circ \check{\mathcal{R}}.$$

From these properties it quickly follows that, for a finite type $U_q\mathfrak{g}$,

d) $\mathcal{K}$ is a universal solution of the RE, in the sense that it satisfies

$$(\mathcal{K} \otimes 1) \circ \check{\mathcal{R}}^{\tau\tau_0 \otimes 1} \circ (\mathcal{K} \otimes 1) \circ \check{\mathcal{R}} = \check{\mathcal{R}} \circ (\mathcal{K} \otimes 1) \circ \check{\mathcal{R}}^{\tau\tau_0 \otimes 1} \circ (\mathcal{K} \otimes 1),$$

and induces a solution K_V of the reflection equation on every finite dimensional representation V.

References

[BK14] M. Balagovic and S. Kolb, *The bar involution for quantum symmetric pairs*, preprint, `arXiv:1409.5074v1` (2014), 22 pp.

[BW13] H. Bao and W. Wang, *A new approach to Kazhdan-Lusztig theory of type B via quantum symmetric pairs*, preprint, `arXiv:1310.0103v1` (2013), 89 pp.

[ES13] M. Ehrig and C. Stroppel, *Nazarov-Wenzl algebras, coideal subalgebras and categorified skew Howe duality*, preprint, `arXiv:1310.1972v2` (2013), 76 pp.

[Kol14] S. Kolb, *Quantum symmetric Kac-Moody pairs*, Adv. Math. **267** (2014), 395–469.

[Let02] G. Letzter, *Coideal subalgebras and quantum symmetric pairs*, New directions in Hopf algebras (Cambridge), MSRI publications, vol. 43, Cambridge Univ. Press, 2002, pp. 117–166.

[Lus94] G. Lusztig, *Introduction to quantum groups*, Birkhäuser, Boston, 1994.

Canonical bases arising from quantum symmetric pairs

Huanchen Bao

The discovery of the canonical bases on quantum groups and their integral modules by Lusztig was a breakthrough in the study of quantum groups. Recently, a theory of canonical bases arising from quantum symmetric pairs was initiated in joint work with Weiqiang Wang [2], motivated by a formulation of Kazhdan-Lusztig theory for the ortho-symplectic Lie superalgebras. The relevant quantum symmetric pair is $(\mathbf{U}_q(\mathfrak{sl}_r), \mathbf{U}^\imath_q(\mathfrak{sl}_r))$. At $q = 1$, the relevant involution of the Lie algebra $\mathfrak{sl}_r$ is the rotation of the matrices by 180 degrees.

M. Jimbo generalized the classical Schur duality to a duality between $\mathbf{U}_q(\mathfrak{sl}_r)$ and $\mathcal{H}_{A_{n-1}}$ (the Hecke algebra of type A). We generalize the Jimbo-Schur duality using the coideal subalgebra $\mathbf{U}^\imath_q(\mathfrak{sl}_r)$.

Theorem 1. *Let $\mathbb{V}$ be the natural representation of $\mathbf{U}_q(\mathfrak{sl}_r)$. The coideal subalgebra $\mathbf{U}^\imath_q(\mathfrak{sl}_r)$ and the Hecke algebra $\mathcal{H}_{B_n}$ of type B form double centralizers when acting on $\mathbb{V}^{\otimes n}$.*

The actions of the generators of $\mathcal{H}_{A_{n-1}}$ on $\mathbb{V}^{\otimes n}$ are realized by $\mathcal{R}$ matrices. Recall that the Hecke algebra $\mathcal{H}_{B_n}$ is generated by the subalgebra $\mathcal{H}_{A_{n-1}}$ together with one additional generator T. We have constructed in [2] an isomorphism $\mathcal{T}$ of $\mathbf{U}^\imath_q(\mathfrak{sl}_r)$-modules as an analog of the $\mathcal{R}$ matrix, whose action on $\mathbb{V}^{\otimes n}$ realizes the action of the additional generator T of $\mathcal{H}_{B_n}$.

Both dualities can be described in the following diagram:

$$\begin{tikzcd}
\mathbf{U}_q(\mathfrak{sl}_r) \arrow[dr, bend left] & & \mathcal{H}_{A_{n-1}} \arrow[dl, bend right, "\mathcal{R}"'] \arrow[dd, hook] \\
& \mathbb{V}^{\otimes n} & \\
\mathbf{U}^\imath_q(\mathfrak{sl}_r) \arrow[uu, hook] \arrow[ur, bend left] & & \mathcal{H}_{B_n} \arrow[ul, bend right, "\mathcal{R},\mathcal{T}"]
\end{tikzcd}$$

Our presentation of the coideal subalgebra $\mathbf{U}^\imath_q(\mathfrak{sl}_r)$ (different from existing ones in the literature [5, 6]) exhibits manifestly a bar involution ($q \mapsto q^{-1}$) on $\mathbf{U}^\imath_q(\mathfrak{sl}_r)$.

(The bar involution has been observed independently in [4].) The bar involution on $\mathbf{U}_q^\imath(\mathfrak{sl}_r)$ allows us to construct a new bar involution (different from Lusztig) on any integral $\mathbf{U}_q(\mathfrak{sl}_r)$-modules (considered as $\mathbf{U}_q^\imath(\mathfrak{sl}_r)$-modules by restriction), denoted by $\psi_\imath$.

Theorem 2. *Simple integrable $\mathbf{U}_q(\mathfrak{sl}_r)$-modules and their tensor products admit $\psi_\imath$-invariant bases, whose transition matrices with respect to Lusztig's canonical bases are uni-upper-triangular with coefficients in $q\mathbb{Z}[q]$.*
(These new bases are called the $\imath$-canonical bases.)

The theory of canonical basis arising from quantum symmetric pairs allows us to formulate and establish the KL theory for the ortho-symplectic Lie superalgebra $\mathfrak{osp}(2n+1|2m)$ for the first time. Set $r=\infty$. Let $\mathbb{V}$ the natural representation of $\mathbf{U}_q(\mathfrak{sl}_\infty)$ and $\mathbb{V}^*$ be its restricted dual. Let $\mathcal{O}_{n|m}$ be the BGG category of $\mathfrak{osp}(2n+1|2m)$-modules of integer (or half-integer) weights and denote its Grothendieck group by $[\mathcal{O}_{n|m}]$. There exists a natural isomorphism $\Phi : \mathbb{V}^{\otimes n} \otimes (\mathbb{V}^*)^{\otimes m}|_{q=1} \rightarrow [\mathcal{O}_{n|m}]$ matching the standard basis (consists of simple tensors) with the basis of Verma modules. By Theorem 2, there exist an $\imath$-canonical basis (and a dual $\imath$-canonical basis constructed in a similar way) on the tensor space $\mathbb{V}^{\otimes n} \otimes (\mathbb{V}^*)^{\otimes m}$.

Theorem 3. *We have the following correspondence of bases via the map Φ:*

$$\begin{array}{rrcl}
\Phi: & \mathbb{V}^{\otimes n} \otimes (\mathbb{V}^*)^{\otimes m}|_{q=1} & \xrightarrow{\cong} & [\mathcal{O}_{n|m}] \\
& \textit{Standard basis} & \longmapsto & \textit{Verma} \\
& \imath\textit{-CB} & \longmapsto & \textit{Tilting} \\
& \textit{Dual } \imath\textit{-CB} & \longmapsto & \textit{Simple}
\end{array}$$

In other words, the entries of the transition matrix from the standard basis to the $\imath$-canonical basis play the role of the KL polynomials for $\mathfrak{osp}(2n+1|2m)$. This settles the basic open problem since 1970's of determining the irreducible characters for $\mathfrak{osp}(2n+1|2m)$.

When setting $m=0$, thanks to the $\imath$Schur duality in Theorem 1, Theorem 3 reformulates the classical KL theory for the Lie algebra $\mathfrak{so}(2n+1)$ (and similarly for the Lie algebra $\mathfrak{sp}(2n)$).

In joint work with Jonathan Kujawa, Yiqiang Li and Weiqiang Wang [3], we also provided a geometric realization of the modified coideal subalgebra $\dot{\mathbf{U}}_q^\imath(\mathfrak{gl}_r)$ and the $\imath$Schur duality using the partial flag varieties of type B/C. Moreover $\dot{\mathbf{U}}_q^\imath(\mathfrak{gl}_r)$ admits a canonical basis. This generalizes the influential geometric construction of $\dot{\mathbf{U}}_q(\mathfrak{gl}_r)$ by Beilinson, Lusztig and MacPherson [1].

References

[1] A. Beilinson, G. Lusztig and R. MacPherson, *A geometric setting for quantum deformations of GL_n*, Duke Math. J. **61** (1990), 655–677.
[2] H. Bao, W. Wang, *A new approach to Kazhdan-Lusztig theory of type B via quantum symmetric pairs*, arXiv:1310.0103.
[3] H. Bao, J. Kujawa, Y. Li, W. Wang, *Geometric Schur duality of classical type*, arXiv:1404.4000v2 with Appendix by H. Bao, Y. Li and W. Wang.
[4] M. Ehrig, C. Stroppel, *Nazarov-Wenzl algebras, coideal subalgebras and categorified skew Howe duality*, arXiv:1310.1972.
[5] S. Kolb, *Quantum symmetric Kac-Moody pairs*, arXiv:1207.6036, Adv. Math. (to appear).
[6] G. Letzter, *Coideal subalgebras and quantum symmetric pairs*, New directions in Hopf algebras (Cambridge), **43** (2002), 117–166, arXiv:math/0103228v2.

The algebra $\mathcal{A}_q$, $q-$Onsager algebras and coideal subalgebras: two open problems

Pascal Baseilhac

Introduced in the mathematical physics literature, the Onsager algebra (OA) and its representation theory has been used to solve different types of quantum integrable systems, i.e. deriving explicit spectrum and eigenstates of the Hamiltonian for instance. Among these, one finds the Ising, superintegrable chiral Potts, XY models,... From the point of view of algebra and representation theory, the OA admits two presentations. The first presentation is given in terms of two generators A_0, A_1 which satisfy a pair of relations, the so-called Dolan-Grady relations [DG]. They read:

$$[A_0,[A_0,[A_0,A_1]]] = 16[A_0,A_1], \qquad [A_1,[A_1,[A_1,A_0]]] = 16[A_1,A_0].$$

The second presentation which first appeared in Onsager's work on the exact solution of the two-dimensional Ising model [Ons] is given in terms of generators $\{A_k, G_l | k,l \in \mathbb{Z}\}$ and relations:

$$\big[A_k, A_l\big] = 4G_{k-l}, \quad \big[G_l, A_k\big] = 2A_{k+l} - 2A_{k-l}, \quad \big[G_k, G_l\big] = 0.$$

In the 90's, Davies [Dav]: showed that the OA is isomorphic with a fixed-point subalgebra of $\widehat{sl_2}$ under the action of a certain automorphism of $\widehat{sl_2}$; established the isomorphism between the first and second presentation. Using this isomorphism, generators $\{A_k, G_l\}$ were systematically written as polynomials in A_0, A_1. For its usefulness in mathematical physics, finite dimensional (evaluation) modules of the OA were constructed. Fundamental generators then satisfy a set of linear relations, the so-called Davies' relations [Dav]. According to these, quotients of the Onsager algebra can be defined.

In the context of quantum integrable systems with boundaries and related spectral parameter dependent reflection equation algebra [Sk], an algebra which can be seen as a $q-$deformed analog of the OA ($q-$OA) with generators $\mathsf{W}_0, \mathsf{W}_1$ appeared

[Bas]. Corresponding defining relations are given by[1]:

$$\big[\mathsf{W}_0,\big[\mathsf{W}_0,\big[\mathsf{W}_0,\mathsf{W}_1\big]_q\big]_{q^{-1}}\big]=\rho\big[\mathsf{W}_0,\mathsf{W}_1\big],\ \big[\mathsf{W}_1,\big[\mathsf{W}_1,\big[\mathsf{W}_1,\mathsf{W}_0\big]_q\big]_{q^{-1}}\big]=\rho\big[\mathsf{W}_1,\mathsf{W}_0\big]$$

where ρ is a scalar. These relations describe a special case the so-called tridiagonal algebra that previously appeared in the mathematical literature [Ter2] in the context of $P-$ and $Q-$polynomial schemes. Studying in details the algebraic structure of the most general (non-scalar) solution of the reflection equation algebra - the so-called Sklyanin operator [Sk] - an infinite dimensional current algebra called $\mathcal{A}_q$ was identified in 2005 [BK], which first modes satisfy the above pair of relations. Precisely, $\mathcal{A}_q$ is defined in terms of generators $\{\mathsf{W}_k,\mathsf{W}_{k+1},\mathsf{G}_{k+1},\tilde{\mathsf{G}}_{k+1}|k\in\mathbb{Z}_+\}$ and defining relations given in [BK]. Now, considering a vector space of finite dimension on which $\mathcal{A}_q$'s generators act, using Sklyanin's $q-$determinant it is possible to show that $\mathcal{A}_q$'s generators are polynomials in $\mathsf{W}_0,\mathsf{W}_1$ of the $q-$OA [BB1]. In the limiting case $q=1$, the connection with the polynomials for related the OA first and second presentation has been checked. This strongly suggests that the algebra $\mathcal{A}_q$ is one candidate for a second presentation of the $q-$OA, by analogy with the case $q=1$. In addition to these results, an explicit homomorphism from the $q-$OA to a certain coideal subalgebra of $U_q(\widehat{sl_2})$ has been exhibited in 2004 [Bas]. Let $\{e_i,f_i,q^{h_i}|i=0,1\}$ be $U_q(\widehat{sl_2})$ Chevalley generators and $k_\pm,\epsilon_\pm$ be scalars. According to a certain choice of coproduct structure for $U_q(\widehat{sl_2})$, it reads

$$\mathsf{W}_0\mapsto k_+e_1+k_-q^{-1}f_1q^{h_1}+\epsilon_+q^{h_1},\mathsf{W}_1\mapsto k_-e_0+k_+q^{-1}f_0q^{h_0}+\epsilon_-q^{h_0},\rho\mapsto(q+q^{-1})^2k_+k_-.$$

For $q=1$, one recovers Davies' homomorphism mapping OA to $\widehat{sl_2}$ [Dav]. Studying finite dimensional modules of the algebra $\mathcal{A}_q$, $q-$analogs of Davies' ones were derived explicitly [BK]. All these facts together with other strong evidences suggest the following first problem for mathematicians:

Problem 1-a: Show that the infinite dimensional algebra $\mathcal{A}_q$, the $q-$Onsager algebra and the above coideal subalgebra of $U_q(\widehat{sl_2})$ are isomorphic.

More recently, another object, called the 'augmented $q-$OA', independently appeared in the mathematics [IT] an physics [BB2] literature. This algebra is generated by $\mathsf{K}_0,\mathsf{K}_1,\mathsf{Z}_1,\tilde{\mathsf{Z}}_1$ subject to the defining relations:

$$[\mathsf{K}_0,\mathsf{K}_1]=0,\ \mathsf{K}_0\mathsf{Z}_1=q^{-2}\mathsf{Z}_1\mathsf{K}_0,\ \mathsf{K}_0\tilde{\mathsf{Z}}_1=q^2\tilde{\mathsf{Z}}_1\mathsf{K}_0,\ \mathsf{K}_1\mathsf{Z}_1=q^2\mathsf{Z}_1\mathsf{K}_1,\ \mathsf{K}_1\tilde{\mathsf{Z}}_1=q^{-2}\tilde{\mathsf{Z}}_1\mathsf{K}_1,$$

$$\big[\mathsf{Z}_1,\big[\mathsf{Z}_1,\big[\mathsf{Z}_1,\tilde{\mathsf{Z}}_1\big]_q\big]_{q^{-1}}\big]=\frac{(q^3-q^{-3})(q^2-q^{-2})^3}{q-q^{-1}}\mathsf{Z}_1(\mathsf{K}_1\mathsf{K}_1-\mathsf{K}_0\mathsf{K}_0)\mathsf{Z}_1,$$

$$\big[\tilde{\mathsf{Z}}_1,\big[\tilde{\mathsf{Z}}_1,\big[\tilde{\mathsf{Z}}_1,\mathsf{Z}_1\big]_q\big]_{q^{-1}}\big]=\frac{(q^3-q^{-3})(q^2-q^{-2})^3}{q-q^{-1}}\tilde{\mathsf{Z}}_1(\mathsf{K}_0\mathsf{K}_0-\mathsf{K}_1\mathsf{K}_1)\tilde{\mathsf{Z}}_1.$$

Remarquably, considering a certain quotient of the reflection equation algebra[2] one obtains a new current algebra [BB2]. Let us denote $\mathcal{A}_q^{diag}$ as the algebra generated by the currents' modes $\mathsf{K}_k,\mathsf{K}_{k+1},\mathsf{Z}_{k+1},\tilde{\mathsf{Z}}_{k+1}$. The defining relations can be obtained in a straighforward manner. Importantly, an explicit homomorphism from the augmented $q-$Onsager algebra to another coideal subalgebra of $U_q(\widehat{sl_2})$

[1] $[X,Y]_q=qXY-q^{-1}YX$, q is assumed not to be a root of unity.

[2] The structure of the spectral parameter's power serie expansion of the entries of the Sklyanin operator are slightly restricted.

is known [IT, BB2]. It reads:

$$\mathsf{K}_0 \mapsto \epsilon_+ q^{h_1},\ \mathsf{K}_1 \mapsto \epsilon_- q^{h_0},$$
$$\mathsf{Z}_1 \mapsto (q^2-q^{-2})\big(\epsilon_+ q^{-1} e_0 q^{h_1} + \epsilon_- f_1 q^{h_1+h_0}\big),\ \tilde{\mathsf{Z}}_1 \mapsto (q^2-q^{-2})\big(\epsilon_- q^{-1} e_1 q^{h_0} + \epsilon_+ f_0 q^{h_1+h_0}\big).$$

Problem 1-b: Show that the infinite dimensional algebra $\mathcal{A}_q^{diag}$, the augmented $q-$OA and the above coideal subalgebra of $U_q(\widehat{sl_2})$ are isomorphic; Find the polynomial formulae which relates $\mathcal{A}_q^{diag}$ to the augmented $q-$OA.

Remarks: For the $U_q(\widehat{sl_2})$ quantum Kac-Moody algebra, two of the coideal subalgebras considered in [Kol] generate the $q-$Onsager and augmented $q-$Onsager algebra, respectively. For $U_q(\widehat{g})$ with $\widehat{g}$ a simply or non-simply laced affine Lie algebra, the defining relations associated with one of the coideal were proven in [BB3].

Finally, recall that for finite dimensional irreducible modules of $\mathcal{A}_q^{diag}$, $q-$analogs of Davies' relations were exhibited in some examples [BK]. From a general point of view, these relations are such that certain polynomials of total degree $2N+1$ in $\mathsf{W}_0, \mathsf{W}_1$ are vanishing on the module. Let $\mathcal{A}_q^{[2N+1]}$ denote the quotient of $\mathcal{A}_q^{diag}$ by such relations. For $N=1$, it is isomorphic with the Askey-Wilson algebra $AW(3)$ [Zhedanov,1991]. In addition, recall that there exists an homomorphism from $AW(3)$ to the double affine Hecke algebra of type $C_1, C_1^\vee$ [Terwilliger,2010].

Problem 2-a: Is there an homomorphism from $\mathcal{A}_q^{[2N+1]}$ to an algebra that generalizes the double affine Hecke algebra of $C_1, C_1^\vee$?

Recall that the Askey-Wilson polynomials provide an infinite dimensional basis of $AW(3)$. Recently, it is shown that some mutlivariable polynomials introduced by Gasper and Rahman in 2006 provide an infinite dimensional basis of the $q-$OA [BM].

Problem 2-b: Classify finite dimensional irreducible modules of $\mathcal{A}_q^{[2N+1]}$; Study in details finite dimensional modules associated with Gasper-Rahman multivariable polynomials.

References

[Bas] P. Baseilhac, *An integrable structure related with tridiagonal algebras*, Nucl.Phys. **B 705** (2005) 605-619, `arXiv:math-ph/0408025`.

[BB1] P. Baseilhac and S. Belliard, *A note on the $O_q(\widehat{sl_2})$ algebra*, `arXiv:1012.5261`; (—-), 2010 unpublished notes.

[BB2] P.Baseilhac and S.Belliard, *The half-infinite XXZ chain in Onsager's approach*, Nucl.Phys. **B 873** (2013) 550-583, `arXiv:math-ph/1211.6304`.

[BB3] P. Baseilhac and S. Belliard, *Generalized q-Onsager algebras and boundary affine Toda field theories*, Lett. Math. Phys. **93** (2010) 213-228, `arXiv:0906.1215`.

[BK] P. Baseilhac and K. Koizumi, *A new (in)finite dimensional algebra for quantum integrable models*, Nucl. Phys. **B 720** (2005) 325-347, `arXiv:math-ph/0503036`.

[BM] P. Baseilhac and X. Martin, *A $q-$hypergeometric formulation for a class of quantum integrable systems*, in preparation.

[Dav] B. Davies, *Onsager's algebra and superintegrability*, J. Phys. **A 23** (1990) 2245-2261; B. Davies, *Onsager's algebra and the Dolan-Grady condition in the non-self-dual case*, J. Math. Phys. **32** (1991) 2945-2950.

[DG] L. Dolan and M. Grady, *Conserved charges from self-duality*, Phys.Rev. **D 25** (1982) 1587.

[IT] T. Ito and P. Terwilliger, *The augmented tridiagonal algebra*, Kyushu Journal of Mathematics, **64** (2010) No. 1, 81-144, `arXiv:0904.2889v1`.

[Kol] S. Kolb, *Quantum symmetric Kac-Moody pairs*, `arXiv:12067.6036`.

[Ons] L. Onsager, *Crystal Statistics. I. A Two-Dimensional Model with an Order-Disorder Transition*, Phys. Rev. **65** (1944) 117-149.

[Ter2] P. Terwilliger, *Two relations that generalize the q−Serre relations and the Dolan-Grady relations*, Proceedings of the Nagoya 1999 International workshop on physics and combinatorics. Editors A. N. Kirillov, A. Tsuchiya, H. Umemura. 377-398, `math.QA/0307016`.

[Sk] E.K. Sklyanin, *Boundary conditions for integrable quantum systems*, J. Phys. **A 21** (1988) 2375-2389.

Coideal subalgebras and three dualities

Michael Ehrig

(joint work with Catharina Stroppel)

In this talk we want to investigate the coideal subalgebra $B(m)$ of type AIII (see [4] and [5]) from the viewpoint of categorification and the ideas of Schur-Weyl duality and skew Howe duality. It is based on the papers [2] and [3]. The main tool for this is the graded version of the BGG category $\mathcal{O}$ for the classical semi-simple Lie algebra of type D as well as its parabolic analogues.

Denote by $\mathfrak{g} = \mathfrak{so}_{2n}$ the even orthogonal complex Lie algebra. Furthermore denote by X the lattice of integral weights, which in this case can be decomposed into a copy of $\mathbb{Z}^n$ and $(\mathbb{Z}+\frac{1}{2})^n$. By $\mathcal{O}(\mathfrak{g})$ we denote the graded BGG category $\mathcal{O}$ of $\mathfrak{g}$, assuming that the weights of all modules are integral and inside $(\mathbb{Z}+\frac{1}{2})^n$. The graded version is defined via the results from [1]. It decomposes into blocks $\mathcal{O}(\mathfrak{g}) = \bigoplus_\chi \mathcal{O}_\chi$, where the blocks are labelled by orbits of the Weyl group W of $\mathfrak{g}$ inside $(\mathbb{Z}+\frac{1}{2})^n$. In this case W acts by permutations as well as even number of sign changes on the elements in X. We denote by $X(n,m) = \{\lambda \in (\mathbb{Z}+\frac{1}{2})^n | \frac{1}{2} \leq |\lambda_i| \leq m - \frac{1}{2}\}$ a subset of X which is stable under the action of the Weyl group and by $\mathcal{O}_{\leq m}$ the sum of those blocks of $\mathcal{O}(\mathfrak{g})$ whose W-orbits are contained in $X(n,m)$.

The Grothendieck group of $\mathcal{O}_{\leq m}$ extended to the field $\mathbb{Q}(q)$ for generic q is naturally isomorphic to the vectorspace $V_{2m}^{\otimes n}$, where V_{2m} is a vector space of dimension $2m$.

After a choice of a basis in V_{2m} we identify the standard basis in $V_{2m}^{\otimes n}$ with the classes of Verma modules. We define an endofunctor $\mathcal{B}$ of $\mathcal{O}_{\leq m}$ as the graded analogue of the functor $\mathrm{pr}_{\leq m} \circ (? \otimes L(\omega_1))$ which first takes the tensor product with the vector representation of $\mathfrak{so}(2n)$ and afterwards projects onto the blocks in $\mathcal{O}_{\leq m}$. By pre- and postcomposing with suitable projections onto blocks one can decompose the functor $\mathcal{B} = \mathcal{B}_0 \oplus \bigoplus_{i=1}^{m-1} \mathcal{B}_{\pm i}$. Using these functor one obtains:

The endofunctors $\mathcal{B}_i$ of the category $\mathcal{O}_{\leq m}$ give a categorification of the action of $B(m) \subset \mathcal{U}_q(\mathfrak{gl}_{2m})$ on $V_{2m}^{\otimes n}$.

Using in addition derived Zuckerman or twisting functors one obtains a categorification of the action of the Hecke algebra $\mathcal{H}_q(D_n)$ of type D on $D^b(\mathcal{O}_{\leq m})$. Adding in an additional easily defined autoequivalence this extends to an action of $\mathcal{H}_{q,1}(B_n)$, a specialization of the two parameter version of the Hecke algebra of type B. This categorifies a Schur-Weyl type duality between this algebra and the coideal subalgebra $B(m)$. The set-up gives a natural notion of a bar involution compatible with the $B(m)$-action on the module $V_{2m}^{\otimes d}$ via the graded duality of the category, as well as a notion of a canonical respectively dual canonical basis in the form of the classes of simple respectively indecomposable projective modules. This is further extended to a skew Howe type duality on the vector space $\bigwedge^n(V_{2m} \otimes V_r) \cong \bigwedge^n(V_m \otimes V_{2r})$ between $B(m)$ and $B(r)$. This is accomplished by decomposing this vector space in two different ways, with respect to the action of both $\mathcal{U}_q(\mathfrak{gl}_{2m})$ and $\mathcal{U}_q(\mathfrak{gl}_{2r})$. Each of these decompositions in turn is categorified by parabolic versions of category $\mathcal{O}$ as in the set-up above. It is argued that, after passing to the bounded derived categories on both sides, the categorifications are equivalent via Koszul duality, allowing us to transfer functors from one categorification to the other. Let us denote by $\mathcal{C}$ the category used for the categorification with respect to the decomposition for $\mathcal{U}_q(\mathfrak{gl}_{2m})$.

There are families of endofunctors $\mathcal{B}_i$, $-(m-1) \leq i \leq m-1$, and $\mathcal{B}'_j$, $-(r-1) \leq j \leq r-1$, of the category $D^b(\mathcal{C})$ giving a categorification of the commuting actions of $B(m)$ and $B(r)$ on $\bigwedge^n(V_{2m} \otimes V_r)$.

The endofunctors are defined in complete analogy to the Schur-Weyl duality case. Finally we discuss how the Koszul duality can be described combinatorially to check whether the categorified action coincide with the restriction of the actions of the quantum groups.

References

[1] A. Beilinson, V. Ginzburg, and W. Soergel. Koszul duality patterns in representation theory. *J. Amer. Math. Soc.*, 9(2):473–527, 1996.

[2] M. Ehrig and C. Stroppel. Diagrams for perverse sheaves on isotropic Grassmannians and the supergroup $SOSP(m|2n)$. Available from: http://arxiv.org/abs/1306.4043.

[3] M. Ehrig and C. Stroppel. Nazarov-Wenzl algebras, coideal subalgebras and categorified skew Howe duality. Available from: http://arxiv.org/abs/1310.1972.

[4] G. Letzter. Coideal subalgebras and quantum symmetric pairs. In *New directions in Hopf algebras*, volume 43 of *Math. Sci. Res. Inst. Publ.*, pages 117–165. Cambridge Univ. Press, Cambridge, 2002.

[5] G. Letzter. Quantum symmetric pairs and their zonal spherical functions. *Transform. Groups*, 8(3):261–292, 2003.

Matrix-valued orthogonal polynomials associated to an explicit quantum symmetric pair

Erik Koelink
(joint work with Noud Aldenhoven, Pablo Román)

It is well-known that several families of orthogonal polynomials of hypergeometric type, i.e. polynomials in the so-called Askey scheme, occur as zonal spherical functions on compact symmetric spaces of rank 1. Matrix-valued orthogonal polynomials were originally introduced by M.G. Kreĭn, see e.g. references in [3], and matrix-valued orthogonal polynomials have been studied from the analytic point of view in several contexts. It is natural to ask for a group-theoretic interpretation of matrix-valued orthogonal polynomials and use group theory to derive explicit families of matrix-valued orthogonal polynomials and its properties. In [1] a special class of matrix-valued orthogonal polynomials has been introduced using the compact symmetric space $(SU(3), U(2))$, where the derivation heavily relies on differential operators.

In [3, 4] another approach has been developed for the group $G = SU(2)\times SU(2)$ with the diagonal subgroup $K = SU(2)$ motivated by [5]. Then the zonal spherical functions are the characters, which are Chebyshev polynomials of the second kind $U_n(\cos\theta) = \sin((n+1)\theta)/\sin\theta$. In particular, the matrix-valued orthogonal polynomials introduced and studied in [3, 4] can be considered as matrix-valued analogues of the Chebyshev polynomials.

The Chebyshev polynomials can also be considered as a special case of the Askey-Wilson polynomials, and they occur as characters on the quantum $SU(2)$ group [7]. More generally, the zonal spherical functions for rank 1 quantum symmetric spaces can be expressed in terms of Askey-Wilson polynomials [6].

In studying the quantum analogue of [3, 4], we consider the quantum symmetric pair $(U_q(\mathfrak{g}), \mathcal{B})$ with $\mathfrak{g} = \mathfrak{sl}(2)\oplus\mathfrak{sl}(2)$ and $\mathcal{B}$ the right coideal subalgebra generated by $K^{\pm1}, B_1, B_2$, essentially following [2]. Using the results of [2] we see that $\mathcal{B} \cong U_q(\mathfrak{sl}(2))$ as an algebra, so that we can use standard results on representation theory. In particular, the type 1 representations of $U_q(\mathfrak{g})$, respectively $\mathcal{B}$, are labeled by two spins (ℓ_1, ℓ_2), respectively one spin ℓ, for $\ell, \ell_1, \ell_2 \in \frac{1}{2}\mathbb{N}$. Using this identification, the Clebsch-Gordan decomposition governs the branching rule: the multiplicity $[\pi^{(\ell_1,\ell_2)}|_{\mathcal{B}} \colon \pi^{\ell}] = 1$ if and only if $|\ell_1 - \ell_2| \le \ell \le \ell_1 + \ell_2$ and $\ell_1 + \ell_2 - \ell \in \mathbb{Z}$. Moreover, the intertwiners are explicitly known. Let $\mathcal{H}^{\ell} \cong \mathbb{C}^{2\ell+1}$ be the representation space of π^{ℓ} and define for (ℓ_1, ℓ_2) satisfying the condition above, the matrix-valued spherical function

$$\Phi^{\ell}_{\ell_1,\ell_2} \colon U_q(\mathfrak{g}) \to \mathrm{End}(\mathcal{H}^{\ell}), \quad X \mapsto \beta^* \circ \pi^{(\ell_1,\ell_2)}(X) \circ \beta$$

where β is a $\mathcal{B}$-intertwiner from $\mathcal{H}^{\ell}$ into the representation space of $\pi^{(\ell_1,\ell_2)}$. By construction, $\Phi^{\ell}_{\ell_1,\ell_2}(XYZ) = \pi^{\ell}(X)\Phi^{\ell}_{\ell_1,\ell_2}(Y)\pi^{\ell}(Z)$ for $X, Z \in \mathcal{B}$, $Y \in U_q(\mathfrak{g})$. Then the case $\ell = 0$, implying $\ell_1 = \ell_2$, corresponds to the spherical case, and we put $\varphi = \Phi^{0}_{1/2,1/2}$. Using the coideal property of $\mathcal{B}$ one can show that $\varphi\,\Phi^{\ell}_{\ell_1,\ell_2}$ satisfies the same transformation property. Here $\big(\varphi\Phi^{\ell}_{\ell_1,\ell_2}\big)(X) = \sum_{(X)} \varphi(X_{(1)})\,\Phi^{\ell}_{\ell_1,\ell_2}(X_{(2)})$ by definition. By uniqueness and tensor product decomposition we obtain $\varphi\,\Phi^\ell_{\ell_1,\ell_2} = \sum_{i,j=\pm 1/2} A_{i,j}\Phi^\ell_{\ell_1+i,\ell_2+j}$, where $A_{1/2,1/2}\neq 0$. This gives an explicit recursion relation where the coefficients $A_{i,j}$ can be given explicitly in terms of Clebsch-Gordan coefficients, i.e. the matrix entries of β.

We now fix ℓ. We label $(\ell_1,\ell_2) = \xi(n,k) = (\frac{1}{2}(n+k), \ell+\frac{1}{2}(n-k))$ for the representation labels for $U_q(\mathfrak{g})$ for which the multiplicity in the branching rule is 1. The recursion can be iterated, and this proves the existence of polynomials $r^{\ell,k}_{n,m}$ with $n\in\mathbb{N}$, $0\leq m,k\leq 2\ell+1$ so that $\Phi^\ell_{\xi(n,m)} = \sum_{k=0}^{2\ell+1} r^{\ell,k}_{n,m}(\varphi)\Phi^\ell_{\xi(0,k)}$. We then define P_n to be the matrix-valued polynomial with entries $\big(P_n(x)\big)_{i,j} = r^{\ell,i}_{n,j}(x)$. So $P_n(x)$ is a $(2\ell+1)\times(2\ell+1)$-matrix.

Now that the matrix-valued polynomials have been introduced we can exploit the quantum group theoretic interpretation in order to obtain various properties of the matrix-valued polynomials. First of all, the Schur orthogonality relations give rise to the matrix-valued orthogonality relations

$$\int_{-1}^{1} (P_n(x))^* W(x) P_m(x)\sqrt{1-x^2}\,dx = \delta_{n,m}G_n$$

where for $-1<x<1$ the matrix $W(x)$ is strictly positive definite. The integrand is a matrix and the integral is taken entrywise. The 'squared norm' matrix G_n is strictly positive definite. Up to a simple action, the matrix entries of W can be expressed as a spherical function, hence as an expansion in Chebyshev polynomials. Moreover, the P_n's satisfy a matrix-valued three-term recurrence relation from the recursion relation for the matrix-valued spherical function. They are also eigenfunctions for two different Askey-Wilson-type matrix-valued q-difference operators, by considering Casimir operators in $U_q(\mathfrak{g})$. Using analytic techniques we can also obtain the LDU-decomposition of W and study the entries of the matrix-valued polynomials in terms of scalar valued orthogonal polynomials from the q-Askey scheme.

All the results mentioned above will appear in a forthcoming preprint.

References

[1] F.A. Grünbaum, I. Pacharoni, J. Tirao, *Matrix valued spherical functions associated to the complex projective plane*, J. Funct. Anal. **188** (2002), 350–441.

[2] S. Kolb, *Quantum symmetric Kac-Moody pairs*, Adv. Math. **267** (2014), 395–469.

[3] E. Koelink, M. van Pruijssen, P. Román, *Matrix-valued orthogonal polynomials related to* $(SU(2)\times SU(2), diag)$, Int. Math. Res. Not. IMRN **2012**, no. 24, 5673–5730.

[4] E. Koelink, M. van Pruijssen, P. Román, *Matrix-valued orthogonal polynomials related to* $(SU(2)\times SU(2), diag)$ *II*, Publ. Res. Inst. Math. Sci. **49** (2013), 271–312.

[5] T.H. Koornwinder, *Matrix elements of irreducible representations of* $SU(2)\times SU(2)$ *and vector-valued orthogonal polynomials*, SIAM J. Math. Anal. **16** (1985), 602–613.

[6] G. Letzter, *Quantum zonal spherical functions and Macdonald polynomials*, Adv. Math. **189** (2004), 88–147.

[7] S.L. Woronowicz, *Compact matrix pseudogroups*, Comm. Math. Phys. **111** (1987), 613–665.

Coideal subalgebras of quantum groups

Stefan Kolb

The talk aims to present main classes of examples of right coideal subalgebras (RCSA) of $U_q(\mathfrak{g})$ for semisimple $\mathfrak{g}$. Here k is a field, $q \in k$ is not a root of unity, and $U_q(\mathfrak{g})$ is a Hopf algebra over k with generators $E_i, F_i, K_i^{\pm 1}$ for $i \in I$ and coproduct $\Delta(E_i) = E_i \otimes 1 + K_i \otimes E_i$.

1. Koornwinder's twisted primitive element in $U_q(\mathfrak{sl}_2)$. For $\alpha, \beta \in k$ the element $B = F + \alpha E K^{-1} + \beta K^{-1}$ generates a RCSA of $U_q(\mathfrak{sl}_2)$. This was the first example for which the program of harmonic analysis on quantum symmetric spaces was performed [Koo93].

2. Noumi's construction. We restrict to the symmetric pair $(\mathfrak{g}, \mathfrak{k})$ with $\mathfrak{g} = \mathfrak{gl}(2m)$ and $\mathfrak{k} = \mathfrak{gl}(2m) \times \mathfrak{gl}(2m)$ which has been considered in the recent work by Bao & Wang and Ehrig & Stroppel. Set $N = 2m$ and $J = \sum_{i=1}^{N} E_{i,N-i+1}$ where $E_{i,j}$ denotes the elementary $N \times N$ matrix with entry 1 in the (i,j)-th position. For $E = \sum_{i,j=1}^{N} E_{i,j} \otimes E_{i,j} \in \mathfrak{g} \otimes \mathrm{End}(\mathbb{C}^N)$ the Lie subalgebra $\mathfrak{k}$ is generated by the coefficients of $EJ + JE$ where J is multiplied in the second tensor entry.

This construction translates to $U_q(\mathfrak{g})$ for $k = \mathbb{C}$ using L-operators $L^-, L^+ \in U_q(\mathfrak{g}) \otimes \mathrm{End}(\mathbb{C}^N)$, see [FRT89]. A q-analog of $\mathfrak{k}$ is obtained as the span of the matrix coefficients of

$$L^- J_q + J_q L^+ \in U_q(\mathfrak{g}) \otimes \mathrm{End}(\mathbb{C}^N)$$

for a suitable q-deformation J_q of J which satisfies the reflection equation. The subalgebra of $U_q(\mathfrak{g})$ generated by the matrix coefficients of

$$S(L^-) J_q L^+$$

is a RCSA of $U_q(\mathfrak{g})$ which we call the Noumi coideal subalgebra corresponding to the symmetric pair $(\mathfrak{g}, \mathfrak{k})$. See [DS99, Section 6] for details of this example and [NS95] for other classical symmetric pairs.

3. Letzter's quantum symmetric pairs. A comprehensive theory of quantum symmetric pairs was developed by G. Letzter. Her theory includes classification results and subsumes Noumi's coideal subalgebras. See Letzter's talk for details.

4. Quantum homogeneous spaces. A RCSA C of a Hopf algebra H is called a quantum homogeneous space for H if H is faithfully flat as a right C-module. This notion goes back to work by M. Takeuchi, A. Masuoka, and H.-J. Schneider. Masuoka showed that if H is pointed then C is a quantum homogeneous space for H if and only if the intersection $C \cap H^0$ of C with the coradical H^0 is a Hopf algebra. All examples of RCSA appearing in Sections 1, 2, 3 are quantum homogeneous spaces.

The classification of quantum homogeneous spaces for $U_q(\mathfrak{g})$ is still open, but important subclasses of RCSAs of $U_q(\mathfrak{g})$ have been classified. In a much broader context of bozonisations of Nichols algebras, I. Heckenberger and H.-J. Schneider established a one-to-one correspondence between elements of the Weyl group W

and RCSAs of $U_q(\mathfrak{g})$ which contain the coradical U^0, [HS13]. See [HK11] for subsequent classification results. The following problem seems natural.

Problem. Relate the classification results for quantum homogeneous spaces and for quantum symmetric pairs to the classification of Poisson homogeneous spaces in terms of Lagrangian Lie subalgebras of the double $D(\mathfrak{g})$ given in [Dri93].

5. Shift of base point. Let $C \subseteq H$ a RCSA and let $\chi : C \to k$ be a one-dimensional representation. Then $C_\chi = \{\chi(c_{(1)})c_{(2)} \mid c \in C\}$ is a RCSA of H. We say that C_χ is obtained from C by shift of base point. Essentially all quantum homogeneous spaces for $U_q(\mathfrak{b}^+)$ are obtained in this way, see [HK11].

6. The locally finite part $F_r(U_q(\mathfrak{g}))$. Any Hopf algebra H acts on itself by the right adjoint action given by $(ad)(h)(x) = S(h_{(1)})xh_{(2)}$. The right locally finite part of H is defined by $F_r(H) = \{x \in H \mid \dim(\mathrm{ad}_r(H)(x)) < \infty\}$. The proof of the following result is a recommendable exercise in the use of the Sweedler notation.

Proposition. The right locally finite part $F_r(H)$ is a RCSA of H.

It follows from results by Joseph and Letzter that $F_r(U_q(\mathfrak{g}))$ is not a quantum homogeneous space as defined in Section 4. However, shift of base point can be used to obtain the Noumi coideal subalgebra in Section 2 from the locally finite part. Indeed, as an algebra $F_r(U_q(\mathfrak{sl}_N))$ is generated by the matrix entries of $S(L^-)L^+$. Suitably normalized, the solution J_q of the RE gives rise to a one-dimensional representation χ_J of $F_r(U_q(\mathfrak{sl}_N))$. It follows from $\Delta(L^\pm) = L^\pm \otimes L^\pm$ that $F_r(U_q(\mathfrak{sl}_N))_{\chi_J}$ is the Noumi coideal subalgebra described in Section 2. Details can be found in [KS09].

References

[Dri93] V.G. Drinfeld, *On Poisson homogeneous spaces and Poisson-Lie groups*, Theo. Math. Phys. **95** (1993), no. 2, 226–227.

[DS99] M.S. Dijkhuizen and J.V. Stokman, *Some limit transitions between BC type orthogonal polynomials interpreted on quantum complex Grassmannians*, Publ. Res. Inst. Math. Sci. **35** (1999), 451–500.

[FRT89] L.D. Faddeev, N.Yu. Reshetikhin, and L.A. Takhtajan, *Quantization of Lie groups and Lie algebras*, Algebra and Analysis **1** (1989), 178–206.

[HK11] I. Heckenberger and S. Kolb, *Right coideal subalgebras of the Borel part of a quantized enveloping algebra*, Int. Math. Res. Not. IMRN **2011** (2011), no. 2, 419–451.

[HS13] I. Heckenberger and H.-J. Schneider, *Right coideal subalgebras of Nichols algebras and the Duflo order on the Weyl groupoid.*, Israel J. Math. **197** (2013), 139–187.

[Koo93] T. Koornwinder, *Askey-Wilson polynomials as zonal spherical functions on the $SU(2)$ quantum group*, SIAM J. Math. Anal. **24** (1993), no. 3, 795–813.

[KS09] S. Kolb and J. Stokman, *Reflection equation algebras, coideal subalgebras, and their centres*, Selecta Math. (N.S.) **15** (2009), 621–664.

[NS95] M. Noumi and T. Sugitani, *Quantum symmetric spaces and related q-orthogonal polynomials*, Group theoretical methods in physics (Singapore) (A. Arima et. al., ed.), World Scientific, 1995, pp. 28–40.

Different types of quantum groups and the Frobenius homomorphism

Simon D. Lentner

In this talk I review the definition of Lusztig's quantum group of divided powers in [Lusz90a], accompanied by some own results that complete the picture. Then I discuss briefly the representation theoretic significance of this Hopf algebra and discuss applications to the construction of logarithmic conformal field theories:

Lusztig's construction consist of several steps and associates quantum groups over various rings to a (semi-)simple finite-dimensional complex Lie algebra $\mathfrak{g}$:

a) Already the *rational form* of the quantum groups $U_q^{\mathbb{Q}(q)}(\mathfrak{g},\Lambda)$ comes with a choice of a lattice $\Lambda_R \subset \Lambda \subset \Lambda_W$, so the coradical of the quantum group is the group algebra $\mathbb{Q}(q)[\Lambda]$ (actually the relations even allow to take up to $\subset \Lambda W^\vee$). The significance of the choice Λ is as follows:
   - From a *Hopf algebra perspective*, the choice of Λ is almost trivial, the usual case $\Lambda = \Lambda_R$ is "link-indecomposable", larger Λ are just group extensions.
   - From a *geometric perspective*, the choice corresponds to the choice of a complex Lie group associated to $\mathfrak{g}$, which is precisely parametrized by a subgroup of the fundamental group $\pi_0 := \Lambda_W/\Lambda_R$. For example, the usual choice $\Lambda = \Lambda_R$ is the "adjoint form" (e.g. $PSL_n(\mathbb{C})$), while the maximal choice Λ_W is "simply-connected form" (e.g. $SL_n(\mathbb{C})$).
   - From a *representation theory perspective*, the choice of Λ makes a huge difference, especially for the later-on considered small quantum group $u_q(\mathfrak{g},\Lambda)$: E.g. it has already been known to several authors, that $u_q(\mathfrak{sl}_2,\Lambda_R)$ does not have a braided representation category, while some quadratic extension has. In [LN14b] I have with a PhD student constructed braided structures (sometimes several different!) for suitable choices of Λ. The message of this calculation is: While the most difficult part of the braiding is encoded in Lusztig's "quasi-R-matrix" Θ from the Drinfel'd double construction $D(U_q^{\geq 0}(\mathfrak{g}))$, in the end also the toral part $R_0 \in \mathbb{K}[\Lambda] \otimes \mathbb{K}[\Lambda]$ matters: We parametrize possible R_0-matrices by subgroups $|H_1| = |H_2|$ of Λ/Λ_R and a pairing ω, then $R_0 = \sum_{\bar\mu \in H_1, \bar\nu \in H_2} \omega(\mu,\nu) q^{(\mu,\nu)} K_\mu \otimes K_\nu$ This is proven using additive combinatorics in [LN14a]. Which choices of H_1, H_2, ω then really admit an R-matrix depends on the root of unity. On the other hand, choosing a too large Λ usually destroys modularity!

b) Then, one chooses an *integral form* $U_q^{\mathbb{Z}[q,q^{-1}]}(\mathfrak{g},\Lambda)$ such that $\otimes_{\mathbb{Z}}[q,q^{-1}]$ is an isomorphism to the rational form. We take Lusztig's choice of divided powers. Another choice due to DeConcini-Kac-Procesi appears more directly for Nichols algebras (see below) and is related by a simple dualization procedure.

c) Finally the *specialized form* $U_q^{\mathcal{L}}(\mathfrak{g},\Lambda)$, a complex Hopf algebra for a concrete value $q \in \mathbb{C}^\times$ is obtained by tensoring the integral form with the 1-dimensional $\mathbb{Z}[q,q^{-1}]$-module $\mathbb{C}_q$, where the the indeterminant q acts by the concrete value.

Let especially q be a primitive ℓ-th root of unity and $\ell_\alpha := \mathrm{ord}(q^{(\alpha,\alpha)})$. Note that we have a vector space basis of PBW-type also in any specialization, but the algebra structure can be modified significantly. For example $E_\alpha^{\ell_\alpha} = 0$, and the specialization includes a finite-dimensional Hopf algebra $u_q^{\mathcal{L}}$.

Under the nowadays common restriction $\mathrm{ord}(q^2) > (\alpha,\alpha)/2$, Lusztig proves that $u_q^{\mathcal{L}}$ is generated by the E_i, F_i and Λ (and the usual relations) and there is a short exact sequence of Hopf algebras:

$$u_q(\mathfrak{g}) \to U_q^{\mathcal{L}}(\mathfrak{g}) \to U(\mathfrak{g})$$

In [Len14a][Len14b] I have completed this picture by calculating the remaining degenerate cases, reversing Lusztig's proof order: First we identify $u_q^{\mathcal{L}}$, which may not be generated by the E_i, F_i, but for some other $u_q(\mathfrak{g}')$. Then we show it is normal and the quotient must be some universal enveloping algebra $U(\mathfrak{g}'')$, which we finally identify case-by-case. So we find in general short exact sequences, e.g.

$$\begin{aligned} q = \sqrt[4]{1}: \qquad & u_q(A_1 \times A_1 \times \cdots)^+ \to U_q^{\mathcal{L}}(B_n)^+ \to U(C_n)^+ \\ & u_q(D_n)^+ \to U_q^{\mathcal{L}}(C_n)^+ \to U(B_n)^+ \\ & u_q(A_3)^+ \to U_q^{\mathcal{L}}(G_2)^+ \to U(G_2)^+ \end{aligned}$$

We conclude by explaining several intriguing conjectures by Feigin, Semikhatov, Tipunin relating $u_q \to U_q^{\mathcal{L}} \to U$ to new logarithmic conformal field theories, and more general approaches for arbitrary Nichols algebras, see e.g. [FT10, ST12]. Our results above seem to give new cases, that nicely show the different roles of the sequence terms. For example, for $B_n, q = \sqrt[4]{1}$ we expect to yield well-known vertex algebra of *n pairs of symplectic fermions*, with global symplectic symmetry $U(C_n)$, invariant sub-VOA $\hat{B}_n$ and representation category equivalent to that of $u_q(A_1 \times A_1 \times \cdots)$, which is essentially a Clifford algebra. Hence we repeat a question from Oberwolfach workshop 2014 (Infinite-dimensional Hopf algebras) and [Len14a] Problem 7.4, where we give several sources of examples:

Find all extensions of finite-dimensional pointed Hopf algebras by universal envelopings $u \to H \to U(\mathfrak{g}'')$ and then corresponding CFT's!

References

[Lusz90a] G. Lusztig: Finite dimensional Hopf algebras arising from quantized universal enveloping algebras, J. Amer. Math. Soc. 3 (1990), 257-296.

[Len14a] S. Lentner: A Frobenius homomorphism for Lusztig's quantum groups over arbitrary roots of unity, to appear in Commun. Contemp. Math. arXiv:1406.0865.

[Len14b] S. Lentner: Quantum affine algebras at small root of unity, Preprint (2014) arXiv:1411.2959.

[LN14a] S. Lentner, D. Nett: A theorem on roots of unity and a combinatorial principle, Preprint (2014) arXiv:1409.5822.

[LN14b] S. Lentner, D. Nett: New R-matrices for small quantum groups, Preprint (2014) arXiv:1409.5824.

[FT10] B. L. Feigin, I. Yu. Tipunin: Logarithmic CFTs connected with simple Lie algebras, Preprint (2010) arXiv:1002.5047.

[ST12] A. M. Semikhatov, I. Yu. Tipunin: The Nichols algebra of screenings, Commun. Contemp. Math. 14, (2012) 66 pages.

An Overview of Quantum Symmetric Pairs

Gail Letzter

The theory of quantum symmetric pairs began shortly after the discovery of quantum groups in the 1980's. Since then, many interesting applications to mathematical physics, q-special functions and representation theory have emerged. We present here an overview of the special one-sided coideal subalgebras used to form quantum symmetric pairs.

Our starting point is a simple complex Lie algebra $\mathfrak{g}$ with triangular decomposition $\mathfrak{g} = \mathfrak{n}^- \oplus \mathfrak{h} \oplus \mathfrak{n}^+$. All results carry over easily to the semisimple case. There are also generalizations to Kac Moody Lie algebras (see [6]). Let $e_1, \ldots, e_n$, $h_1, \ldots, h_n$, $f_1, \ldots, f_n$ denote a standard Chevalley basis with respect to this triangular decomposition. A classical (infinitessimal) symmetric pair is a pair $\mathfrak{g}, \mathfrak{g}^\theta$ where $\mathfrak{g}^\theta$ is a Lie subalgebra of $\mathfrak{g}$ fixed by an involution θ. Conjugating by a Lie algebra automorphism if necessary we may assume that θ is maximally split with respect to the given triangular decomposition ([10]). The Lie algebra $\mathfrak{g}^\theta$ is generated by the two Lie subalgebras $\mathfrak{m} = \langle e_i, f_i, h_i | \ \theta(h_i) = h_i \rangle, \mathfrak{h}^\theta = \langle h \in \mathfrak{h} | \ \theta(h) = h \rangle$ and the elements $f_i + \theta(f_i)$ for $f_i \notin \mathfrak{m}$. Note that $\theta(f_i)$ is a root vector in $\mathfrak{n}^+$.

The quantized enveloping algebra $U = U_q(\mathfrak{g})$ is the Hopf algebra generated over $\mathbb{C}(q)$ (q an indeterminate) by $E_i, K_i^{\pm 1}, F_i$, $i = 1, \ldots, n$ with relations and Hopf structure as in [10]. A quantum symmetric pair is a pair U, B_θ where B_θ is a left coideal subalgebra of U that specializes to $U(\mathfrak{g}^\theta)$ as q goes to 1. As an algebra, B_θ has generators very similar to those of $\mathfrak{g}^\theta$. In particular, B_θ is generated by the two Hopf subalgebras $\mathcal{M} = U_q(\mathfrak{m})$, $\mathbb{C}(q)[K_1^{u_1} \cdots K_n^{u_n} | \sum_i u_i h_i \in \mathfrak{h}^\theta]$ and the elements $B_i = F_i K_i + \tilde{\theta}(F_i) K_i$ for $F_i \notin \mathcal{M}$. Here, $\tilde{\theta}$ is a lift of θ to the quantum setting (see for example [10]) such that $\tilde{\theta}(F_i)$ is a quantum version of the root vector $\theta(f_i)$.

When $\mathfrak{g}^\theta$ has a nontrivial center, there is actually a one parameter family of coideal analogs of $U(\mathfrak{g}^\theta)$ inside of U. The extra degree of freedom shows up in the definition of one of the B_i and is based on two distinguished subsets S and D (which depend on θ) of the set of simple roots of $\mathfrak{g}$ (see [10], [11]). Both S and D are empty if $\mathfrak{g}^\theta$ is semisimple. On the other hand, if $\mathfrak{g}^\theta$ has a nontrivial center, then either $|S| = 1$ or $|D| = 1$ and the other set is empty. The quantum analog $B_{\theta,s,d}$ of $U(\mathfrak{g}^\theta)$ inside U is the left coideal subalgebra generated by the same two Hopf subalgebras stated for B_θ above, the elements $B_i = F_i K_i + \tilde{\theta}(F_i) K_i$ for $F_i \notin \mathcal{M}$ and $\alpha_i \notin S \cup D$, $B_i = F_i K_i + \tilde{\theta}(F_i) K_i + s K_i$ for $\alpha_i \in S$, and $B_i = F_i K_i + d\tilde{\theta}(F_i) K_i$ for $\alpha_i \in D$. Note that $B_\theta = B_{\theta,s,d}$ where $s = 0$ and $d = 1$.

Relations satisfied by the generators of $B_{\theta,s,d}$ are given in [10] and [11] (see also [6]). Perhaps the only complicated relations are those for the $B_i, i = 1, \ldots, n$ which look like the quantum Serre relations up to terms of "lower degree" where $B_i = F_i K_i$ for $F_i \in \mathcal{M}$. For example, if $a_{ij} = -1$, $\theta(\alpha_i) = -\alpha_i$ and $\theta(\alpha_j) = -\alpha_j$, then

$$B_i^2 B_j - (q_i + q_i^{-1}) B_i B_j B_i + B_j B_i^2 = q_i^{-1} B_j$$

where $q_i = q^{(\alpha_i,\alpha_i)/2}$.

Using a quantum version of the Iwasawa decomposition, we obtain a uniqueness result ([10]): Any maximal left coideal subalgebra of U that specializes to $U(\mathfrak{g}^\theta)$ as q goes to 1 is isomorphic to $B_{\theta,s,d}$ for some choice of s and d up to a Hopf algebra automorphism of U. As a consequence, the NSD (Noumi-Sugitani-Dijkhuizen [14], [16], [17]) construction of quantum symmetric pairs via reflection equations yields the same left coideal subalgebras. It should be noted that some authors prefer to use right coideal subalgebras instead of left ones. The theories are equivalent. Indeed, it is easy to convert a left coideal subalgebra into a right one via an algebra automorphism of U that transforms the Hopf structure appropriately.

For good choices of s and d, one checks that $B = B_{\theta,s,d}$ is preserved by a conjugate linear antiautomorphism of U that gives U the structure of a Hopf $*$ algebra. It follows that B acts semisimply on any finite-dimensional simple U-module. A general theory of finite-dimensional simple B-modules is yet to be developed. A smattering of partial results can be found in [9], [3], [4], and [15]. New constructions of B canonical bases compatible with Lusztig's basis for finite-dimensional U-modules and special choices of θ ([2], [1]) suggest this is an interesting avenue for further exploration.

There has been extensive work analyzing various algebras of B-invariants which we summarize here. The center Z of a slight extension of B is a polynomial ring ([7]). Certain central elements yield solutions to reflection equations, thus providing another connection to the NSD construction of quantum symmetric pairs ([5], see also [8]). Quantum zonal spherical functions, which are B-biinvariants of the quantized function algebra, also form a polynomial ring and can be identified with Macdonald polynomials ([11], [12]). Using a Harish-Chandra type projection map, one obtains a quantum version of a theorem of Helgason: B-invariants of the simply connected quantized enveloping algebra project onto the restricted Weyl group invariants of a particular polynomial ring ([13]).

References

[1] H. Bao and W. Wang, *A new approach to Kazhdan-Lusztig theory of type B via quantum symmetric pairs*, arXiv:1310.0103, October 2013.

[2] M. Ehrig and C. Stroppel, *Nazarov-Wenzl algebras, coideal subalgebras and categorified skew Howe duality*, arXiv:1310.1972v2, October 2013.

[3] A. M. Gavrilik and A. U. Klimyk, *q-Deformed orthogonal and pseudo-orthogonal algebras and their representations*, Lett. Math. Phys. **21** (1991) 215-220.

[4] N. Z. Iorgov and A. U. Klimyk, *Classification theorem on irreducible representations of the q-deformed algebra* $U_q'(\mathrm{so}_n)$, Int. J. Math. Math. Sci. (2005), no. 2, 225-262.

[5] S. Kolb, *Quantum symmetric pairs and the reflection equation*, Algebr. Represent. Theory **11** (2008), no. 6. 519-544.

[6] S. Kolb, *Quantum symmetric Kac-Moody pairs*, Adv. Math. **267** (2014), 395-469.

[7] S. Kolb and G. Letzter, *The center of quantum symmetric pair coideal subalgebras*, Represent. Theory **12** (2008), 294-326.

[8] S. Kolb and J. V. Stokman, *Reflection equation algebras, coideal subalgebras, and their centers*, Sel. Math. New Ser. **15** (2009) 621-664.

[9] G. Letzter, *Harish-Chandra modules for quantum symmetric pairs*, Represent. Theory **4** (1999) 64-96.

[10] G. Letzter, *Coideal subalgebras and quantum symmetric pairs*, New directions in Hopf algebras, Math. Sci. Res. Inst. Publ. **43** (2002), 117-166.
[11] G. Letzter, *Quantum symmetric pairs and their zonal spherical functions*, Transform. Groups **8** (2003), 261-292.
[12] G. Letzter, *Quantum zonal spherical functions and Macdonald Polynomials*, Adv. Math. **189** (2004), no.1, 88-147.
[13] G. Letzter, *Invariant Differential Operators for Quantum Symmetric Spaces*, Mem. Amer. Math. Soc. **193**, no. 903.
[14] M. Noumi, *Macdonald's symmetric polynomials as zonal spherical functions on some quantum homogeneous spaces*, Adv. Math. **123** (1996), no. 1, 16-77.
[15] A. I. Molev, *Representations of the twisted quantized envloping algebra of type C_n* Mosc. Math. J. **6** (2006) 531-551.
[16] M. Noumi and T. Sugitani, *Quantum symmetric spaces and related q-orthogonal polynomials*, In: Group Theoretical Methods in Physics, World Scientific Publishing (1995) 28-40.
[17] M. S. Dijkhuizen, *Some remarks on the construction of quantum symmetric spaces*, Acta Appl. Math **44** (1996), no. 1-2, 59-80.

K-matrices: from twisted Yangians to quantum loop algebras

Vidas Regelskis
(joint work with Bart Vlaar)

K-matrices are solutions of the matrix reflection equation (RE) that play a key role in quantum integrable models with open boundary conditions [Sk, GZ]. Two classes will address in this talk are *rational* and *trigonometric* K-matrices.

Rational K-matrices describe a reflection process in quantum integrable models with the underlying symmetry of the twisted Yangian type (see [Ma2] and references therein). Let us recall that the Yangian $\mathcal{Y}(\mathfrak{g})$ is a flat deformation of the current algebra $\mathfrak{U}\mathfrak{g}[\lambda]$, where $\mathfrak{g}$ is a simple, complex Lie algebra and λ is a formal parameter ([Dr1, Dr2], see also [MNO], Section 1). Let θ be an involution $\theta : \mathfrak{g} \to \mathfrak{g}$ and let $\mathfrak{g}^\theta$ denote the θ-fixed subalgebra of $\mathfrak{g}$. The pair of Lie algebras $(\mathfrak{g}, \mathfrak{g}^\theta)$ is called a symmetric pair. Set $\theta : \lambda \to -\lambda$ and let $\mathfrak{g}[\lambda]^\theta$ denote the θ-fixed subalgebra of $\mathfrak{g}[\lambda]$; its enveloping algebra $\mathfrak{U}\mathfrak{g}[\lambda]^\theta$ is called a twisted current algebra. The twisted Yangian $\mathcal{Y}(\mathfrak{g}, \mathfrak{g}^\theta)$ is a flat deformation of $\mathfrak{U}\mathfrak{g}[\lambda]^\theta$ and is a coideal subalgebra of the Yangian $\mathcal{Y}(\mathfrak{g})$. Twisted Yangians are known to have two different presentations: the Drinfeld J-basis ([Ma1, BeRe]) and the RTT-presentation ([FRT, Ol], see also [MNO], Section 3). Constructing twisted Yangians in the Drinfeld J-basis (also called MacKay twisted Yangians) requires only a knowledge of the symmetric pair $(\mathfrak{g}, \mathfrak{g}^\theta)$ and the structure constants of $\mathfrak{g}$. However this is not true for the RTT-presentation: twisted Yangians in the RTT-presentation are defined by the reflection equation and its rational solution, the K-matrix $K(u)$. In particular, for Lie algebras of classical types, they are in one-to-one correspondence with the compact symmetric pairs (see e.g. [He], Section X.6). Let us give the explicit form of the corresponding K-matrices.

We must first introduce the necessary notations and definitions. Choose $(N \geq 2) \in \mathbb{N}$ and let $n \in \mathbb{N}$ be such that $N = 2n$ or $N = 2n + 1$. Let $\mathfrak{g}_N$ denote the general Lie algebra $\mathfrak{gl}_N$, the orthogonal Lie algebra $\mathfrak{so}_N$ or the symplectic Lie algebra $\mathfrak{sp}_N$ (only when $N=2n$). Let $i,j\in\{-n,\dots,-1,1,\dots,n\}$ if $N=2n$ and $i,j\in\{-n,\dots,-1,0,1,\dots,n\}$ if $N=2n+1$. Let V be an N-dimensional vector space and let $E_{ij}\in \mathrm{End}V$ denote the usual matrix units. Set $I=\sum_{i,j=-n}^{n}E_{ii}\otimes E_{jj}$ to be the identity operator, $P=\sum_{i,j=-n}^{n}E_{ij}\otimes E_{ji}$ to be the permutation operator and $Q=P^{t_1}=P^{t_2}$ to be the one-dimensional projector satisfying $PQ=QP=\pm Q$ and $Q^2=NQ$, where t_i denotes the transposition in the i-th tensor space given by $(E_{ij})^t=\theta_{ij}E_{-j,-i}$; $\theta_{ij}=\mathrm{sign}(i)\cdot\mathrm{sign}(j)$ if $\mathfrak{g}_N=\mathfrak{sp}_N$, and $\theta_{ij}=1$ otherwise, for all i,j; similarly, the lower sign in $\pm$ corresponds to the symplectic case and the upper sign otherwise.

It is known that the R-matrices $R(u)\in\mathrm{End}(V^{\otimes 2})$ with $u\in\mathbb{C}\backslash\{0\}$ given by [AMR, MNO]

$$R(u)=I-\frac{P}{u}\qquad\text{and}\qquad R(u)=I-\frac{P}{u}+\frac{Q}{u-\kappa},\tag{1}$$

where $\kappa=N/2\mp 1$, are rational solution of the (quantum) Yang Baxter equation

$$R_{12}(u)\,R_{13}(u+v)\,R_{23}(v)=R_{23}(v)\,R_{13}(u+v)\,R_{12}(u).\tag{2}$$

Here $R_{ij}(u)\in\mathrm{End}(V^{\otimes 3})$ with legs in i-th and j-th tensor spaces. The first R-matrix in (1) is called of $\mathfrak{gl}_N$ type, the second one is of $\mathfrak{so}_N$ or $\mathfrak{sp}_N$ type, depending on the choice of the transposition t defining Q.

Consider the following REs (the second equation is called the twisted RE):

$$R_{12}(u-v)K_1(u)R_{21}(u+v)K_2(v)=K_2(v)R_{21}(u+v)K_1(u)R_{12}(u-v),\tag{3}$$

$$R_{12}(u-v)K_1(u)R_{21}^t(-u-v)K_2(v)=K_2(v)R_{21}^t(-u-v)K_1(u)R_{12}(u-v),\tag{4}$$

where $K_1(u)=K(u)\otimes I$, $K_2(v)=I\otimes K(v)$, $R_{21}(u)=PR_{12}(u)P$ and $R^t(u)=R^{t_1}(u)=R^{t_2}(u)$. An important question to ask is how to find K-matrices $K(u)$ satisfying (one of the) equations above? There is no universal method known to find general solutions of the REs given above. One could restrict to K-matrices associated with the compact symmetric pairs of classical types in which case the answer is known. Consider matrix $\mathcal{G}$ given by:

- AI $(\mathfrak{gl}_N,\mathfrak{so}_N)$ and AII $(\mathfrak{gl}_N,\mathfrak{sp}_N)$: $\mathcal{G}=I$;
- AIII $(\mathfrak{gl}_N,\mathfrak{gl}_p\oplus\mathfrak{gl}_{N-p})$, $p<N/2$: $\mathcal{G}=I-2\sum_{i=n-p}^{n}E_{ii}$;
- CI $(\mathfrak{sp}_N,\mathfrak{gl}_{N/2})$, N even: $\mathcal{G}=\sum_{i=1}^{\frac{N}{2}}(E_{ii}-E_{-i,-i})$;
- DIII $(\mathfrak{so}_N,\mathfrak{gl}_{N/2})$, N even: $\mathcal{G}=\sum_{i=1}^{\frac{N}{2}}(E_{ii}-E_{-i,-i})$;
- CII $(\mathfrak{sp}_N,\mathfrak{sp}_p\oplus\mathfrak{sp}_q)$, N, p and q are even and >0, $N=p+q$: $\mathcal{G}=-\sum_{i=1}^{\frac{q}{2}}(E_{ii}+E_{-i,-i})+\sum_{i=\frac{q}{2}+1}^{\frac{N}{2}}(E_{ii}+E_{-i,-i})$;
- BDI $(\mathfrak{so}_N,\mathfrak{so}_p\oplus\mathfrak{so}_q)$: $\mathfrak{g}=\mathfrak{so}_N$, $\mathfrak{g}^\rho=\mathfrak{so}_p\oplus\mathfrak{so}_q$ where $p>q>0$ if N is odd, and $p\geq q>0$ if N is even. (If $q=1$, then $\mathfrak{so}_q$ is the zero Lie algebra.) When N is even, p and q have the same parity and $\mathcal{G}$ is given by $\mathcal{G}=\sum_{i=1}^{\frac{p-q}{2}}(E_{ii}+E_{-i,-i})+\sum_{i=\frac{p-q}{2}+1}^{\frac{N}{2}}(E_{-i,i}+E_{i,-i})$. When N is odd, $p-q$ is odd and $\mathcal{G}=\sum_{i=-\frac{p-q-1}{2}}^{\frac{p-q-1}{2}}E_{ii}+E_{-i,-i})+\sum_{i=\frac{p-q+1}{2}}^{\frac{N-1}{2}}(E_{-i,i}+E_{i,-i})$.

Then the K-matrix satisfying (3) (or (4) for AI and AII cases) is:

- $K(u) = \mathcal{G}$ for cases AI, AII, AIII, CI, DIII and DI, CII when $p = q$;
- $K(u) = (I - c\,u\,\mathcal{G})(1 - c\,u)^{-1}$ with $c = \frac{4}{p-q}$ for cases BDI, CII when $p > q$.

Knowing this data one can construct twisted Yangians in the RTT-presentation [Ol, MNO, MR, GR]. Moreover, one can also construct boundary KZ equations [EFK], or solve the spectral problem for the corresponding integrable model with open boundaries [BeRa]. (Note that there are many more rational K-matrices known that do not fit into this classification.)

We will obtain an analogous classification of trigonometric K-matrices that are solutions of the trigonometric REs and are in one-to-one correspondence with quantum symmetric pairs of the non-twisted affine type classified by Kac [Kac]. Many trigonometric K-matrices are already known (see e.g. [DG, LS, NS]). However, to our knowledge, prior to this work there was no attempt to give a uniform classification. We want to explain a method that allows for the possibility of obtaining such a classification.

Quantum symmetric pairs can be graphically represented by Satake diagrams [BBBR]. Moreover, for each quantum symmetric pair one can construct an associated coideal quantum group [Ko, Le]. As explained in the talks by Stefan Kolb and Gail Letzter, these quantum groups (denoted by $\mathcal{B}_{\mathbf{d},\mathbf{s}}$ with $\{\mathbf{d}, \mathbf{s}\}$ being a set of parameters) are coideal subalgebras of quantum groups of Drinfeld-Jimbo type. Given a Satake diagram it is then straightforward to "read" the associated coideal quantum group. The corresponding trigonometric K-matrix is then obtained by solving the boundary intertwining equation

$$K(u)\,T_{\eta u}(b) = (T_{\eta u^{-1}}(b))^{inv}\,K(u) \tag{5}$$

for the matrix entries $k_{ij}(u)$ of $K(u)$ with all $b \in \mathcal{B}_{\mathbf{d},\mathbf{s}}$. Here $T_{\eta u}$ is the N-dimensional vector representation of the corresponding Drinfeld-Jimbo quantum group, u is the evaluation parameter, $\eta \in \mathbb{C}\backslash\{0\}$ is the reflection phase, and inv denotes either the identity map id or a quantum analogue τ of the transposition t. Solving (5) defines $K(u)$ uniquely up to an overall constant (this follows from the fact that $T_{\eta u}$ is an irreducible representation of $\mathcal{B}_{\mathbf{d},\mathbf{s}}$) and defines η in terms of the parameters in the set $\{\mathbf{d}, \mathbf{s}\}$. Let us give a simple example:

Consider the following Satake diagram:  . The corresponding coideal subalgebra $\mathcal{B}_{\mathbf{d},\mathbf{s}}$ is generated by the elements

$$b_0 = x_0^- k_0^+ + q d_0 k_1^- x_1^+ k_0^+, \;\; b_1 = x_1^- k_1^+ + d_1 k_0^- x_0^+ k_1^+, \;\; t^\pm = k_0^\pm k_1^\mp, \tag{6}$$

where $x_i^\pm, k_i^\pm$ are the standard Chevalley generators of $\mathfrak{U}_q\hat{\mathfrak{sl}}_2$ and $d_i \in \mathbb{C}\backslash\{0\}$. Solving (5) with $inv = id$ gives

$$K(u) = (u + \rho)E_{-1,-1} + (u^{-1} + \rho)E_{11}, \quad \rho = (d_0/d_1)^{1/2}, \quad \eta = (d_0 d_1)^{1/2}.$$

This is a solution of the trigonometric analogue of (3). The algebra $\mathcal{B}_{\mathbf{d},\mathbf{s}}$ generated by elements (6) is also known as the augmented tridiagonal algebra [IT] or the augmented q-Onsager algebra [BB].

The complete classification of trigonometric K-matrices associated with quantum symmetric pairs will be presented in [RV]. This data is the first step towards a classification of the RTT-presentation of coideal quantum groups in the spirit of [MRS, CGM]. Moreover, this will allow for the future studies of the q-KZ equations for generic boundaries [RSV], boundary TQ-relations [BR], and many other unanswered questions related to quantum integrable systems with open boundaries.

References

[AMR] D. Arnaudon, A. Molev, E. Ragoucy, *On the R-matrix realization of Yangians and their representations*, Ann. Henri Poincaré **7** (2006), no. 7-8, 1269–1325. arXiv:math/0511481.

[BB] P. Baseilhac, S. Belliard, *The half-infinite XXZ chain in Onsager's approach* Nucl. Phys. B **873** (2013), 550–583, arXiv:1211.6304.

[BBBR] V. Back-Valente, N. Bardy-Panse, H. Ben Messaoud, and G. Rousseau, *Formes presque-déloyées des algèbres de Kac-Moody: Classification et racines relatives*, J. Algebra **171** (1995), 43–96.

[BeRa] S. Belliard, E. Ragoucy, *Nested Bethe ansatz for 'all' open spin chains with diagonal boundary conditions*, J. Phys. A **42** (2009), 205203, arXiv:0902.0321.

[BeRe] S. Belliard, V. Regelskis, *A new basis of twisted Yangians*, preprint, arXiv:1401.2143.

[BR] R. Baiyasi, R. Murgan, *Generalized T-Q relations and the open spin-s XXZ chain with nondiagonal boundary terms* J. Stat. Mech. **1210** (2012), P10003, arXiv:1206.0814.

[CGM] H. Chen, N. Guay, X. Ma, *Twisted Yangians, twisted quantum loop algebras and affine Hecke algebras of type BC*, Trans. Amer. Math. Soc. **366** (2014), no. 5, 2517–2574.

[DG] G. W. Delius, A. George, *Quantum affine reflection algebras of type $D_n^{(1)}$ and reflection matrices*, Lett. Math. Phys. **62** (2002), 211–217, arXiv:math/0208043.

[Dr1] V. Drinfel'd, *Hopf algebras and the quantum Yang-Baxter equation*, Soviet Math. Dokl. **32** (1985), 254–258.

[Dr2] V. G. Drinfel'd, *Quantum groups*, Proceedings of the International Congress of Mathematicians, Berkeley, 1986, A. M. Gleason (ed), 798–820, Amer. Math. Soc., Providence, RI.

[EFK] P. Etingof, I. Frenkel, A. Kirillov, *Lectures on Representation Theory and Knizhnik-Zamolodchikov Equations*, Math. Surveys and Monographs **58**, Amer. Math. Soc., (1998).

[FRT] L. D. Faddeev, N. Y. Reshetikhin, L. A. Takhtajan, *Quantization of Lie Groups and Lie Algebras*, Leningrad Math. J. **1** (1990), no. 1, 193–225. [Alg. Anal. **1** (1989), no. 1, 178–206.]

[GZ] S. Ghoshal, A. B. Zamolodchikov, *Boundary S-matrix and boundary state in two-dimensional integrable quanutm field theory*, Int. J. Mod. Phys. A **9** (1994), 3841 arXiv:hep-th/9306002.

[GR] N. Guay, V. Regelskis, *Twisted Yangians for symmetric pairs of types B, C, D*, preprint, arXiv:1407.5247.

[He] S. Helgason, *Differential Geometry, Lie Groups, and Symmetric Spaces*, Graduate Studies in Mathematics **34**, American Mathematical Society, Providence, RI (2001).

[IT] T. Ito, P. Terwilliger, *The augmented tridiagonal algebra*, Kyushu J. of Math. **64** (2010), no. 1, 81–144, arXiv:0904.2889.

[Kac] V. G. Kac, *On automorphisms of Kac-Moody algebras and groups*, Adv. Math. **92** (1992), no. 2, 129–195.

[Ko] S. Kolb, *Quantum symmetric Kac-Moody pairs*, Adv. Math. **267** (2014), 395–469, arXiv:1207.6036.

[Le] G. Letzter, *Coideal subalgebras and quantum symmetric pairs*, in 'New directions in Hopf algebras', Math. Sci. Res. Inst. Publ. **43**, Cambridge Univ. Press, Cambridge (2002), 117, `math/0103228`.

[LS] A. Lima-Santos, *$B_n^{(1)}$ and $A_{2n}^{(2)}$ reflection K-matrices* Nucl. Phys. B **654** (2003), 466–482; *$A_{n-1}^{(1)}$ Reflection K-Matrices*, Nucl. Phys. B **644** (2002), 568–584; *$D_{n+1}^{(2)}$ Reflection K-matrices*, Nucl. Phys. B **612** (2001), 446–460.

[Ma1] N. J. Mackay, *Rational K-matrices and representations of twisted Yangians*, J. Phys. A **35** (2002), no. 37, 7865–7876, `arXiv:math/0205155`.

[Ma2] N. J. Mackay, *Introduction to Yangian symmetry in integrable field theory*, Internat. J. Modern Phys. A **20** (2005), no. 30, 7189–7217, `arXiv:hep-th/0409183`.

[MNO] A. Molev, M. Nazarov, G. Olshanskii, *Yangians and classical Lie algebras*, Russ. Math. Surv. **51** (1996), no. 2, 205–282, `arXiv:hep-th/9409025`.

[MR] A. Molev, E. Ragoucy, *Representations of reflection algebras*, Rev. Math. Phys. **14** (2002), no. 3, 317–342. `arXiv:math/0107213`.

[MRS] A. Molev, E. Ragoucy, P. Sorba, *Coideal subalgebras in quantum affine algebras*, Rev. Math. Phys. **15** (2003), no. 8, 789–822, `arXiv:math/0208140`.

[NS] M. Noumi, T. Sugitani *Quantum symmetric spaces and related q-orthogonal polynomials*, Group Theoretical Methods in Physics (ICGTMP), Toyonaka, Japan (1994), World Science Publishing, River Edge, NJ (1995), 28–40, `arXiv:math/9503225`.

[Ol] G. Olshanskii, *Twisted Yangians and infinite-dimensional classical Lie algebras*, *Quantum groups (Leningrad, 1990)*, 104–119, Lecture Notes in Math. **1510**, *Springer*, *Berlin*, 1992.

[RV] V. Regelskis, B. Vlaar, *Classification of trigonometric reflection matrices for quantum affine Kac-Moody pairs of classical types*, in preparation.

[RSV] N. Reshetikhin, J. Stokman, B. Vlaar, *Boundary quantum Knizhnik-Zamolodchikov equations and Bethe vectors*, Comm. Math. Phys. (2014), doi:10.1007/s00220-014-2227-2, `arXiv:1305.1113`.

[Sk] E. K. Sklyanin, *Boundary conditions for integrable quantum systems*, J. Phys. A **21** (1988), no. 10, 2375–2389.

Boundary reflection equation (physics origins)

Nicolai Reshetikhin

The talk is an overview of works in statistical mechanics of the 6-vertex models. The first part is an overview of the 6-vertex model and of how the q-Knizhnik-Zamolodchikov (qKZ) equation appears in the analysis of correlation functions. In the second part the boundary qKZ equation is introduced as the equation for correlation functions with reflecting boundary conditions.

Nichols algebras and their right coideal subalgebras

Hans-Jürgen Schneider

(joint work with István Heckenberger)

I first gave a survey on Nichols algebras. The idea of Nichols algebras goes back to Nichols (1978) and Woronowicz (1983). Let H be a Hopf algebra with bijective antipode over a field k. The Drinfeld center of the monoidal category of left H-modules is a braided monoidal category which can be described as the category ${}^H_H\mathcal{YD}$ of Yetter-Drinfeld modules over H. An object of ${}^H_H\mathcal{YD}$ is a left H-module

V which is a left H-comodule with coaction $\delta : V \to H \otimes V$ such that $\delta(hv) = h_{(1)}v_{(-1)}S(h_{(3)}) \otimes h_{(2)}v_{(0)}$ for all $h \in H, v \in V$ (Sweedler notation). If H is finite-dimensional, then ${}^H_H\mathcal{YD}$ can be identified with the modules over the Drinfeld double of H. Let $V \in {}^H_H\mathcal{YD}$. The tensor algebra $T(V)$ is a Hopf algebra in the braided category ${}^H_H\mathcal{YD}$ where elements of V are primitive. Let $I(V)$ be the largest coideal of $T(V)$ which is contained in $\bigoplus_{n\geq 2} T^n(V)$. The Nichols algebra of V is defined as $\mathcal{B}(V) = T(V)/I(V)$. It is an $\mathbb{N}_0$-graded Hopf algebra quotient of $T(V)$ whose primitive elements are exactly the elements in degree 1. Nichols algebras have been used intensively in the classification theory of (finite-dimensional) pointed Hopf algebras (a Hopf algebra is pointed if its simple subcoalgebras are 1-dimensional), see for example [2]. It follows from the construction in Lusztig's book on quantum groups that $U_q^+(\mathfrak{g}), \mathfrak{g}$ a Kac-Moody algebra, q generic, is a Nichols algebra. Also the plus-parts of the small quantum groups and of the multiparameter variations of quantum groups are Nichols algebras.

In the second part of my talk I discussed some recent results of joint work with I. Heckenberger. Let $\theta \geq 1$, and $M = (M_1, \dots, M_\theta)$ a tuple of finite-dimensional and irreducible objects $M_i \in {}^H_H\mathcal{YD}$, and $\mathcal{B}(M) = \mathcal{B}(M_1 \oplus \cdots \oplus M_\theta)$. Under some finiteness assumptions the Weyl groupoid $\mathcal{W}(M)$ is defined. This was done in the diagonal case in [1], and in the general case in [3]. Its points are the isomorphism classes $[N]$ of all $(N_1, \dots, N_\theta)$ of the form

$$R_{i_1} \cdots R_{i_m}(M), m \geq 0, 1 \leq i_1, \dots, i_m \leq \theta,$$

where R_i is a reflection operator on the tuples M. Assume that $\mathcal{W}(M)$ is finite. The Hopf algebra $\mathcal{B}(M)$ has a natural $\mathbb{N}_0^\theta$ -gradation. In this general context we show in [5] that there is a bijection between the morphisms in the Weyl groupoid ending in $[M]$ and the set of all $\mathbb{N}_0^\theta$-graded right coideal subalgebras of $\mathcal{B}(M)$. In particular, we prove that there are finite-dimensional and irreducible subobjects $M_{\beta_l} \subseteq \mathcal{B}(M)$ of degree $\beta_l \in \mathbb{N}_0^\theta$, $1 \leq l \leq m$, such that the multiplication map

$$k[M_{\beta_m}] \otimes \cdots \otimes k[M_{\beta_1}] \to \mathcal{B}(M)$$

is bijective, and $k[M_{\beta_l}] \cong \mathcal{B}(M_{\beta_l})]$ for all l. The M_{β_l} generalize Lusztig's root vectors. A similar isomorphism holds for the right coideal subalgebras. In the special case of $U_q^+(\mathfrak{g})$, $\mathfrak{g}$ a semisimple Lie algebra, q generic, we obtain Lusztig's PBW-basis without case by case considerations, and we prove a conjecture of Kharchenko [4] saying that the number of right coideal subalgebras of $U_q^{\geq}(\mathfrak{g})$ is the order of the Weyl group of $\mathfrak{g}$. In very recent work (which will appear in a book we are writing), we gave more transparent proofs of some results in [3], [5] and [6] in the framework of abstract braided monoidal categories.

References

[1] I. Heckenberger, *The Weyl groupoid of a Nichols algebra of diagonal type*, Inventiones Math. **164** (2006), 175–188.

[2] N. Andruskiewitsch and H.-J. Schneider, *On the classification of finite-dimensional pointed Hopf algebras*, Ann. Math. **171** (2010), 375–417.

[3] N. Andruskiewitsch, I. Heckenberger and H.-J. Schneider, *The Nichols algebra of a semisimple Yetter-Drinfeld module*, Amer. J. Math. **132** (2010), 1493–1547.
[4] V. K. Kharchenko, *Right coideal subalgebras of $U_q^+(\mathfrak{so}_{2n+1})$*, J. European Math. Soc.**13** (2011), 1677–1735.
[5] I. Heckenberger and H.-J. Schneider, *Right coideal subalgebras of Nichols algebras and the Duflo order on the Weyl groupoid*, Israel J. of Math. **197** (2013), 139–187.
[6] I. Heckenberger and H.-J. Schneider, *Yetter-Drinfeld modules over bosonizations of dually paired Hopf algebras*, Advances in Math. **244** (2013), 354–394.

Integrable systems from the classical reflection equation.

Gus Schrader

A large and well-studied class of integrable Hamiltonian systems consists of those whose phase space can be realized as a Poisson submanifold of a coboundary Poisson-Lie group G. In this situation, the conjugation invariant functions $I_G \subset C(G)$ form a Poisson commutative subalgebra, and particular integrable systems arise by restricting these functions to symplectic leaves in G, see [RSTS], [STS].

In our talk, based on the paper [Sc], we outlined a construction of integrable systems on Poisson homogeneous spaces of the form G/K, where (G, r) is a coboundary Poisson Lie group and K is a Lie subgroup of G which arises as the fixed point set of a Lie group automorphism $\sigma : G \to G$. In this setting, the condition for G/K to inherit a Poisson structure from G is equivalent to the requirement that the Lie subalgebra $\mathfrak{k} = \mathrm{Lie}(K)$ be a Lie bialgebra coideal in the Lie bialgebra $\mathfrak{g}$, i.e.

$$\delta(\mathfrak{k}) \subset \mathfrak{g} \otimes \mathfrak{k} + \mathfrak{k} \otimes \mathfrak{g} \tag{1}$$

where $\delta : \mathfrak{g} \to \mathfrak{g} \otimes \mathfrak{g}$, $\delta(x) = x \cdot r$ is the cobracket defining the Lie bialgebra structure on $\mathfrak{g}$. In terms of the r-matrix, the coideal condition is equivalent to the requirement that the quantity

$$C_\sigma(r) = (\sigma \otimes \sigma)(r) + r - (\sigma \otimes 1 + 1 \otimes \sigma)(r) \tag{2}$$

be a $\mathfrak{k}$-invariant in $\mathfrak{g} \otimes \mathfrak{g}$. In the special case $C_\sigma(r) = 0$, we say that the triple $(\mathfrak{g}, r, \sigma)$ is a solution of the *classical reflection equation* (CRE). In this case, one can construct a *classical reflection monodromy matrix* $\mathcal{T}$ with the property that the classical reflection transfer matrices obtained by taking the trace of $\mathcal{T}$ in finite dimensional representations of G form a Poisson commuting family of functions in $C(G/K) \subset C(G)$. These functions are no longer Ad_G-invariant, but are instead bi-invariant under the action of $K \times K$ on G by left and right translations.

At the quantum level, one may pass to the quantized enveloping algebra $U_q(\mathfrak{g})$ together with its dual Hopf algebra $C_q(G)$ of quantized regular functions on G. Then the quantum version of the situation discussed above amounts to the fact given a subalgebra $\mathcal{A} = C_q(G/K)$ in $C_q(G)$, its annihilator $\mathcal{A}^\perp$ in $U_q(\mathfrak{g})$ is a (double sided) Hopf coideal in $U_q(\mathfrak{g})$.

The motivation for our construction comes from the quantum spin chains with reflecting boundary conditions introduced by Sklyanin [Sk]. We show that the semiclassical limit of Sklyanin's quantum reflection equation coincides with the CRE for an appropriate choice of group G and automorphism σ, and explain how to derive local Hamiltonians for the corresponding homogeneous classical spin chain.

We conclude by mentioning some possible directions for future work. Firstly, it would be interesting to understand the quantization problem for solutions of the classical reflection equation,in the sense of Etingof-Kazhdan [EK]: given a solution $(\mathfrak{g}, \sigma, r)$ of the CRE, it is possible to lift it to construct a corresponding coideal in $U_q(\mathfrak{g})$? Secondly, in our construction, there is no requirement that the automorphism σ be an involution. It will be interesting to study the integrable systems obtained from arbitrary finite order solutions of the CRE (in particular when $\mathfrak{g}$ is of affine type), and to understand whether, at the quantum level, one can obtain quantum symmetric pair-type coideals generalizing those constructed by Letzter [L] in this fashion.

References

[EK] Etingof, P. and Kazhdan, D.: *Quantization of Poisson algebraic groups and Poisson homogeneous spaces*, arxiv: 9510020

[RSTS] Reyman, A. and Semenov-Tian-Shanksy, M.: *Group-theoretical methods in the theory of finite dimensional integrable suystems.* In *Dynamical Systems VII: Integrable systems, nonholonomic dynamical systems* volume 1 of *Encyclopedia of Mathematical Sciences*, page 341. Springer 1994.

[Sc] G. Schrader: *Integrable systems from the classical reflection equation* arXiv:1405.5506

[STS] Semenov-Tian-Shansky, M.A. : *Dressing transformations and Poisson group actions*, Publ. RIMS. 21 (1985), 1237–1260.

[Sk] Sklyanin, E.: *Boundary conditions for integrable quantum systems*, J. Phys. A: Math. Gen. **21** (1988) 2375–2389.

[L] Letzter, G.: *Coideal subalgebras and quantum symmetric pairs.* New directions in Hopf algebras (Cambridge), vol. 43, pp. 117–166. MSRI Publications, Cambridge University Press, Cambridge (2002)

Boundary quantum Knizhnik-Zamolodchikov equations

Jasper V. Stokman

Boundary quantum Knizhnik-Zamolodchikov (bqKZ) equations were introduced by Cherednik [3, §4] in 1992 as a particular class of consistent systems of difference equations

$$A_i(\mathbf{z})f(\mathbf{z}+\epsilon_i) = f(\mathbf{z}), \qquad i = 1, \ldots, n \tag{1}$$

for $V^{\otimes n}$-valued meromorphic functions $f(\mathbf{z})$ in $\mathbf{z} \in \mathbb{C}^n$. Here V is a complex vector space, ϵ_i is the ith standard basis vector of $\mathbb{C}^n$, and the $\mathrm{End}(V^{\otimes n})$-valued meromorphic functions $A_i(\mathbf{z})$ in $\mathbf{z} \in \mathbb{C}^n$, called transport operators, are determined by factorized scattering data for particles on a line segment. The scattering data are described by a solution $R(x) : V \otimes V \to V \otimes V$ of the quantum Yang-Baxter equation and solutions $\underline{K}(x), K(x) : V \to V$ of the associated left and right reflection equation respectively.

BqKZ equations have important applications in the theory of integrable one-dimensional quantum spin chains with boundaries (see, e.g., [6, 11]) and are closely related to quantum harmonic analysis [10].

Important examples of quantum integrable spin chains with boundaries can be described in terms of representation theory of quantum affine symmetric pairs (U, B) [7]. The quantum affine algebra U plays the role of the quantum symmetry algebra at the bulk sites of the quantum spin chain, while the right coideal subalgebra B of U, called the boundary quantum group, encodes the quantum symmetries at the boundary (see, e.g., [5, §2.2]). In this setup the vector space $V = V(z)$ is an evaluation representation of U with evaluation parameter $z \in \mathbb{C}^\times$, and intertwiners

$$\check{R}(z_1/z_2) \in \mathrm{Hom}_U\big(V(z_1) \otimes V(z_2), V(z_2) \otimes V(z_1)\big), \quad \underline{K}(z) \in \mathrm{Hom}_B\big(V(z), V(z^{-1})\big)$$

produce the scattering data $(R(x), \underline{K}(x), K(x))$, where $R(x) = P\check{R}(x)$ with P the permutation operator, and with $K(x)$ a solution of the right reflection equation naturally associated to $\underline{K}(x)$, see [6, §2.1]. The bqKZ equations appear as consistency conditions for the associated quantum correlation functions (see, e.g., [6] in case of the semi-infinite Heisenberg XXZ spin-$\frac{1}{2}$ chain).

The connection to quantum harmonic analysis arises as follows. Cherednik [3] attached to a double affine Hecke algebra and a representation M of the underlying (extended) affine Hecke algebra a consistent system of difference equations for M-valued meromorphic functions, called quantum affine KZ equations. BqKZ equations arise as special examples of quantum affine KZ equations [12, 10]. Asymptotic techniques have led to the construction of a basis of solutions of the quantum affine KZ equations for principal series modules M [11]. The difference Cherednik-Matsuo correspondence relates these solutions to solutions of the spectral problem of the family of commuting scalar Macdonald-Koornwinder difference operators, whose Laurent polynomial eigenfunctions are the celebrated Macdonald-Koornwinder polynomials (see, e.g., [4, 11]). The spectral problem of this family of commuting scalar difference operators naturally appears in the harmonic analysis of Letzter's quantum symmetric pairs [8].

In particular cases the bqKZ equations can be studied from both the quantum group and the Hecke algebra perspective. In these cases the two representation theoretic contexts are related by a Schur-Weyl type duality. The resulting possibility to mix the insights and techniques from the two different viewpoints provides a particularly rich playground for obtaining new insights.

This happens for instance for the quantum affine symmetric pair (U, B) with U the quantum affine algebra associated to $\widehat{\mathfrak{sl}(2)}$, $B \subseteq U$ the q-Onsager algebra [1], and $V = \mathbb{C}^2(z)$ the two-dimensional evaluation representation. The underlying quantum spin chain is the Heisenberg XXZ spin-$\frac{1}{2}$ chain with reflecting boundary [12, 11]. The associated bqKZ equations are quantum affine KZ equations associated to the double affine Hecke algebra of type $C^\vee C_n$ and the spin representation $\big(\mathbb{C}^2\big)^{\otimes n}$ of the underlying affine Hecke algebra of type C_n (see, e.g., [12]). For this example the quantum group approach has led to special solutions of the bqKZ equations as quantum correlation functions [6] and as (q-)integrals for special choices of K-matrices $\underline{K}(x), K(x)$ (see, e.g., [2, 9]), while the affine Hecke algebra perspective has led to a basis of solutions of the bqKZ equations defined in terms of their asymptotic behaviour deep in a fixed Weyl chamber. Understanding the relation between these different types of solutions is an interesting open problem.

References

[1] P. Baseilhac, *An integrable structure related with tridiagonal algebras*, Nucl. Phys. B **705** (2005), no. 3, 605–619.

[2] P. Baseilhac, T. Kojima, *Correlation functions on the half-infinite XXZ spin chain with a triangular boundary*, arXiv:1309.7785.

[3] I. Cherednik, *Quantum Knizhnik-Zamolodchikov equations and affine root systems*, Comm. Math. Phys. **150**, 109–136 (1992).

[4] I. Cherednik, *Induced representations of double affine Hecke algebras and applications*, Math. Res. Lett. **1** (1994), 319–337.

[5] G.W. Delius, N.J. MacKay, *Quantum group symmetry in sine-Gordon and affine Today field theories on the half line*, Comm. Math. Phys. **233** (1995), 173–190.

[6] M. Jimbo, R. Kedem, H. Konno, T. Miwa, R. Weston, *Difference equations in spin chains with a boundary*, Nucl. Phys. B **448** (1995), 429–456.

[7] S. Kolb, *Quantum symmetric Kac-Moody pairs*, Adv. Math. **267** (2014), 395–469.

[8] G. Letzter, *Quantum zonal spherical functions and Macdonald polynomials*, Adv. Math. **189** (2004), 88–147.

[9] N. Reshetikhin, J.V. Stokman, B.H.M. Vlaar, *Boundary quantum Knizhnik-Zamolodchikov equations and Bethe vectors*, arXiv:1305.1113. Comm. Math. Phys. (to appear).

[10] J.V. Stokman, *The c-function expansion of a basic hypergeometric function associated to root systems*, Ann. of Math. (2) **179** (2014), no. 1, 253–299.

[11] J.V. Stokman, *Connection problems for quantum affine KZ equations and integrable lattice models*, arXiv:1410.4383. Comm. Math. Phys. (to appear).

[12] J.V. Stokman, B.H.M. Vlaar, *Koornwinder polynomials and the XXZ spin chain*, arXiv:1310.5545. J. Approx. Th. (to appear).

Quantum groups and coideals and categorification

Catharina Stroppel

Lusztig's theory of canonical bases and integral forms of quantum groups allows the question if there is a categorification of these structures and of the finite dimensional representations. We review Lusztig's construction, connect it with representations of Hecke algebras and their Kazhdan-Lusztig bases. Finally we describe some examples of categorified representations. The second talk deals with quantum symmetric pairs, certain coideal subalgebras of quantum groups and canonical bases. We provide categorification of certain representations. All this relies on an interesting Schur-Weyl duality between the coideal subalgebra (of type AIII) and the type B or D Hecke algebra. As an application we mention new results on characters of Lie superalgebras and (non-semismiple) branching rules for Brauer algebras. In this context representations of quantum symmetric pairs appear naturally.

Boundary quantum KZ equations – integral solutions

Bart Vlaar

(joint work with Nicolai Reshetikhin and Jasper V. Stokman)

Let $N \in \mathbb{Z}_{\geq 1}$. The quantum Knizhnik-Zamolodchikov (qKZ) equations are linear difference equations for functions meromorphically depending on N parameters and taking values in $V^{\otimes N}$ for some complex vector space V. Cherednik [2] introduced them depending on so-called quantum R-matrix datum (solutions to the quantum Yang-Baxter equation and associated identities), associated to an arbitrary affine root system. If this is of type A one recovers difference equations which were already known and motivated in terms of mathematical physics [11] and representation theory [4]. Taking instead an affine root system of type B, C or D one obtains the *boundary qKZ equations*, satisfied by matrix elements of vertex operators with respect to so-called boundary states [6]. In this case the R-matrix datum contains up to two K-matrices, solutions of quantum reflection equations.

In [10] Sklyanin used R- and K-matrices in a different way, namely to construct (double-row) boundary monodromy matrices. In terms of these the associated commuting transfer matrices are defined, of importance in 2-dimensional statistical models and 1-dimensional quantum integrable systems with reflecting boundary conditions. In the corresponding version of the algebraic Bethe ansatz, one may use these monodromy matrices also to define *boundary Bethe vectors*, distinguished elements of $V^{\otimes N}$ depending on a tuple $(x_1, \ldots, x_M) \in \mathbb{C}^M$ with $0 \leq M \leq N$. These can be made into eigenvectors of the boundary transfer matrices by imposing equations on the x_i, known as the (boundary) Bethe ansatz equations. We briefly discuss a recent research result [12] establishing a direct connection between boundary transfer matrices and the boundary qKZ equations, thus providing another motivation for the study of these equations.

The main topic of the talk is the construction of solutions of the boundary qKZ equations as bilateral series [8] and integrals [9] of weighted boundary Bethe vectors, which is part of joint work with N. Reshetikhin and J. Stokman. Instead of satisfying Bethe ansatz equations, the x_i play the role of integration/summation variables. What makes this work in essence is that both the boundary qKZ equations and the boundary Bethe vectors are given in terms of R- and K-matrices, so that we have the Yang-Baxter equation and the reflection equation at our disposal to simplify expressions. The construction pertains to a special case when the R-matrix itself is the image of the universal R-matrix of $U_q(\hat{\mathfrak{sl}}_2)$ acting in the tensor square of its fundamental (2-dimensional) representation $V = \mathbb{C}^2$, and the K-matrices are the general diagonal solutions to the reflection equation for this R-matrix. These K-matrices are associated to the coideal subalgebra of $U_q(\hat{\mathfrak{sl}}_2)$ prescribed by the admissible pair $(\emptyset, 0 \leftrightarrow 1)$ in the framework of [7], also known as the augmented q-Onsager algebra [1, 5]. In this setting it is possible to make everything explicit: the domain of integration/summation, the weight function and, we expect, the boundary Bethe vector (this latter point is work in progress). In the case of integrals, it leads to a basis of solutions of the boundary qKZ equations in $(\mathbb{C}^2)^{\otimes N}$.

A natural open question would be whether the coideal subalgebra may be replaced by the q-Onsager algebra, which is the one given by the admissible pair $(\emptyset, \mathrm{id})$, yielding the general non-diagonal solution of the reflection equation for the R-matrix under consideration, and see if and how the construction of the solutions could be modified - this would likely require the use of certain transformations to return the K-matrix to a diagonal form (thus allowing Sklyanin's boundary Bethe vectors to remain available, in modified form, for our construction) at the cost of introducing a dynamical parameter, cf. [3].

References

[1] P. Baseilhac, S. Belliard, *Generalized q-Onsager algebras and boundary affine Toda field theories*, Lett. Math. Phys. **93** (2010) 213, arXiv:0906.1215.

[2] I. Cherednik, *Quantum Knizhnik-Zamolodchikov equations and affine root systems*, Comm. Math. Phys. **150** (1992), 109–136; I. Cherednik, *A unification of Knizhnik-Zamolodchikov and Dunkl operators via affine Hecke algebras*, Invent. Math. **106** (1991), 411–431.

[3] G. Filali, N. Kitanine, *Spin chains with non-diagonal boundaries and trigonometric SOS model with reflecting end*, SIGMA **7** (2011), 012, 22pp.

[4] I. Frenkel, N. Reshetikhin, *Quantum affine algebras and holonomic difference equations*, Comm. Math. Phys. **146** (1992), 1–60.

[5] T. Ito, P. Terwilliger, *The augmented tridiagonal algebra*, Kyushu J. Math. **64** (2010) 81144, arXiv:0904:2889.

[6] M. Jimbo, R. Kedem, T. Kojima, H. Konno, T. Miwa, *XXZ chain with a boundary*, Nucl. Phys. B **441** (1995), no. 3, 437–470; M. Jimbo, R. Kedem, H. Konno, T. Miwa, R. Weston, *Difference equations in spin chains with a boundary*, Nucl. Phys. B **448** (1995), no. 3, 429–456.

[7] S. Kolb, *Quantum symmetric Kac-Moody pairs*, Adv. Math. **267** (2014), 395-469.

[8] N. Reshetikhin, J. Stokman, B. Vlaar, *Boundary quantum Knizhnik-Zamolodchikov equations and Bethe vectors*, Comm. Math. Phys., doi:10.1007/s00220-014-2227-2, arXiv:1305.1113.

[9] N. Reshetikhin, J. Stokman, B. Vlaar, *Integral solutions to boundary quantum Knizhnik-Zamolodchikov equations*, in preparation.

[10] E. Sklyanin, *Boundary conditions for integrable quantum systems*, J. Phys. A: Math. Gen. **21** (1988), 2375–2389.

[11] F. Smirnov, *A general formula for soliton form factors in the quantum sine-Gordon model*, J. Phys. A **19**, no. 10 (1986), L575–L578.

[12] B. Vlaar, *Boundary transfer matrices and boundary quantum KZ equations*, preprint at arXiV:1408.3364.

Borelsubalgebras of quantum groups

Karolina Vocke

This talk is about goal and results of my PhD thesis: The classification of all right coideal subalgebras $C \subset U_q(\mathfrak{g})$ of a quantum group with generic q, where C has the additional property that all irreducible representations are 1-dimensional and C is maximal with this property.

We call such a right coideal subalgebra a *Borel subalgebra*. This is due to a theorem of Sophus Lie stating that the Borel subalgebras of a semisimple Lie algebra have only 1-dimensional representations and are maximal with this property. We shall see that indeed there are the so-called *standard Borel subalgebras*, among them $U_q^{\geq 0}(\mathfrak{g})$ and $U_q^{\leq 0}(\mathfrak{g})$, and their reflections which are parametrized by an element of the Weyl group. But there are more examples, already in $U_q(sl_2)$ appears a family of *Weyl algebras* generated by two elements $EK^{-1}+\lambda K^{-1}$ and $F+\lambda' K^{-1}$ with $\lambda\lambda' = \frac{q^2}{(1-q^2)(q-q^{-1})}$. So the question arises, which other kinds of Borel subalgebras exist. My goal is to give a complete classification of Borel subalgebras for q generic.

Borel subalgebra and subgroups are in the theory of algebraic groups, semisimple Lie algebras and representation theory the basic components of many standard constructions (flag varieties, spherical varieties, Verma modules and their irreducible quotient, etc.). The interesting question was discussed during the workshop to understand the induced Verma modules for $U_q(\mathfrak{g})$ obtained from these alternative Borel subalgebras C, maybe even study a respective category $\mathcal{O}$.

In the following my approach and results are discussed in more detail:

The classification of all *right coideal subalgebras* (RCSA) $C \subset U_q(\mathfrak{g})$ made significant progress in the recent years. A major result in [HS09] (more generally for Nichols algebras), classifies all RCSA with the additional properties $U^0 \subset C$ (called *homogeneous*) and $C \subset U_q(\mathfrak{g})^{\geq 0}$ to be in $1:1$ correspondence to elements in the *Weyl group* $C = U^+[x]U^0,\ x \in W$. By the results in [HK11a] all homogeneous RCSA have a triangular decomposition, hence $C = U^+[x]U^0U^-[y]$ for some $x, y \in W$. On the other hand, all possibly inhomogeneous RCSA that still satisfy $C \subset U_q(\mathfrak{g})^{\geq 0}$ (assume the additional technical condition that $C \cap U^0 =: L$ is a Hopf algebra) were classified in [HK11b] to be so-called *character shifts*

$$C = (U^+[x]L)_\chi := \{a_{(1)}\chi(a_{(2)}) \mid a \in U^+[x]L\}$$

where χ is any 1-dimensional representation of the already known RCSA $U^+[x]L$, which they describe explicitly. The general case is open:

RCSA	$C \subset U_q(\mathfrak{g})^{\geq 0}$	$C \subset U_q(\mathfrak{g})$
$U^0 \subset C$	$C = U^+[x]U^0$	$C = U^+[x]U^0U^-[y]$
$U^0 \cap C =: L$	$(U^+[x]L)_\chi$	general case ?

Now we turn our attention to RCSA which are Borel subalgebras: First, C cannot contain both E_α, F_α for some root α, since this implies a (suitably nice contained) subalgebra $U_q(sl_2)$ which has irreducible representations of dimension > 1. Second, the maximality forces any Borel subalgebra B to automatically satisfy the condition $B \cap U$ is a Hopf algebra, hence any Borel subalgebra lies in the diagram above. Also the maximality forces any $C \subset U_q(\mathfrak{g})^{\geq 0}$ to already be the full positive part. Hence in the three solved cases above, the Borel subalgebras are only the full positive part $U_q(\mathfrak{g})^{\geq 0}$ resp. all Borel subalgebras obtained by reflections $T_x(U_q(\mathfrak{g})^{\geq 0})$.

Our classification for Borel subalgebras of $C \subset U_q(\mathfrak{g})$ in the general case proceeds now as follows: We first assume that C is *triangular* i.e. $C = C^+LC^-$, use the classification of C^+, C^- by character shifts and work out the observations above:

Theorem 4. *Each triangular Borel subalgebra of $U_q(\mathfrak{g})$ is as an algebra isomorphic to (more precisely a reflection of) some $C = U^+[w_0]_{\phi^+}LU^-[s_{\alpha_{i_1}}, \ldots s_{\alpha_{i_k}}]_{\phi^-}$. Here w_0 is the maximal element in the Weylgroup, $\alpha_{i_1}, \ldots \alpha_{i_k}$ is some coclique in the Dynkin diagram of $\mathfrak{g}$, the characters ϕ^+,ϕ^- vanish outside a support $supp(\phi^-) = \{\alpha_1, \ldots \alpha_i\} = supp(\phi^+)$ and L is generated by the K_α with $\alpha \in supp(\phi^-)^\perp$.*

For example, in $U_q(sl_3)$ we find apart from standard Borel subalgebras and up to reflections and diagram automorphisms a single family of Borel subalgebras for $\alpha_{i_1} = \alpha_1$ with a free parameter, containing a Weyl algebra for α_1 as well as E_{α_2}.

Then we want to show that each Borel subalgebra of $U_q(\mathfrak{g})$ *is* in fact triangular. This is much harder than for homogeneous RCSA and failes without the assumed maximality. For the proof I developed the following much more general theorem to construct generating elements of an arbitrary RCSA. The proof of this theorem consists of an elaborate induction in an explicit PBW-basis

Theorem 5. *Let $C \subset U_q(\mathfrak{g})$ be a RCSA such that $U^0 \cap C =: L$ is a Hopf algebra. Then we can choose for C a generating set consisting of elements of the form*

$$E_\alpha^{\phi_E} K_\alpha^{-1} + \lambda_F K_{\alpha-\beta}^{-1} F_\beta^{\phi_F} + \lambda_K K_\alpha^{-1}$$

For roots $\alpha \in \Phi$ and constants $\lambda_F, \lambda_K \in k$ and some characters ϕ_E, ϕ_F on subalgebras of U^- resp. U^+, such that $E_\alpha^{\phi_E}, F_\beta^{\phi_F}$ are some kind of character shifts of E_α resp. F_β.

References

[AHS10] N. Andruskiewitsch, I. Heckenberger, H.-J. Schneider: The Nichols algebra of a semisimple Yetter-Drinfeld module, Amer. J. Math. 132 (2010), 1493–1547

[HK11a] I. Heckenberger and S. Kolb, *Homogeneous right coideal subalgebras of quantized enveloping algebras*, Preprint arXiv:1109.3986 (2011), 13 pp.

[HK11b] I. Heckenberger and S. Kolb, *Right coideal subalgebras of the Borel part of a quantized enveloping algebra*, Int. Math. Res. Not. IMRN **2011** (2011), no. 2, 419–451.

[HS09] I. Heckenberger and H.-J. Schneider, *Right coideal subalgebras of Nichols algebras and the Duflo order on the Weyl groupoid*, Preprint (2009), 43 pp.

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