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Mini-Workshop: Ideals of Linear Subspaces, Their Symbolic Powers and Waring Problems

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Abstract. It is a fundamental challenge for many problems of significant current interest in algebraic geometry and commutative algebra to understand symbolic powers $I^{(m)}$ of homogeneous ideals I in polynomial rings, particularly ideals of linear varieties. Such problems include computing Waring ranks of polynomials, determining the occurrence of equality $I^{(m)} = I^m$ (or, more generally, of containments $I^{(m)} \subseteq I^r$), computing Waldschmidt constants (i.e., determining the limit of the ratios of the least degree of an element in $I^{(m)}$ to the least degree of an element of I^m), and studying major conjectures such as Nagata's Conjecture and the uniform SHGH Conjecture (which respectively specify the Waldschmidt constant of ideals of generic points in the plane and the Hilbert functions of their symbolic powers).

Mathematics Subject Classification (2010): Primary: 14N20, 11P05; Secondary: 13F20, 13P10.

Introduction by the Organisers

The mini-workshop, *Ideals of Linear Subspaces, Their Symbolic Powers and Waring Problems*, involved 11 men and 7 women (one of whom, due to medical issues, did not attend in person but presented her talk by skype). The participants represented 6 different countries, and were drawn from all career ranks (2 postdocs, 3 early career researchers, 6 midcareer researchers and 7 senior researchers), covering a range of specialties and expertise. This variety of expertise not only generated stimulating discussions during the workshop, but the working group sessions have led to at least three on-going research collaborations which are expected to be the basis for a number of research articles in the near future.

The theme of the workshop

Ideals of linear subspaces, and points in particular, have long held a prominent position in algebraic geometry. They have, in particular, played a prime role in recent progress on the Waring Problem for forms, which deals with power sum representations of forms, i.e., expressions of the type $F = L_1^d + \ldots + L_r^d$, where F is a form of degree d and the L_i are forms of degree 1. A crucial quantity for this problem is the Waring rank $\mathrm{rk}(F)$ of F, defined as the least r for which F can be written as such a sum of powers. In the 90s, results of Alexander and Hirschowitz [1] for ideals of points in projective space gave the dimension of all secant varieties of Veronese varieties, which in turn determined $\mathrm{rk}(F)$ for generic forms F of any degree in any number of variables. But the Waring rank for a specific form can be larger than this generic value; obtaining bounds for $\mathrm{rk}(F)$ is an active area of research for which ideals of points have played a crucial role. For example, Carlini, Catalisano and Geramita [9] use the geometry of reduced points to compute the Waring rank of monomials and of the sum of coprime monomials. Before this result, the Waring rank was explicitly known only for quadratic forms, binary forms and cubic ternary forms.

Ideals of linear subspaces have also been a focus of attention in recent research on the question of which symbolic powers of an ideal are equal to or at least contained in specific ordinary powers of the ideal. Interest in which powers are symbolic goes back at least to work of Hochster [28] and more recently has gotten attention in the work of Morey [32] and Li and Swanson [31]. In a talk in the late 00s, Huneke asked whether $I^{(m)} = I^m$ for all $m \geq 1$ if $I^{(c)} = I^c$, given any homogeneous ideal I of big height c. Recent work of Guardo, Harbourne and Van Tuyl [21, 23] has exploited the fact that ideals of arrangements of points in $\mathbb{P}^1 \times \mathbb{P}^1$ can be regarded as defining arrangements of lines in $\mathbb{P}^3$ to give a negative answer to Huneke's question.

The question of containment of symbolic powers of an ideal in specific ordinary powers of the ideal has seen even more explosive growth, starting with a paper of Swanson [34] which prompted the seminal papers of Ein-Lazarsfeld-Smith and Hochster-Huneke [16, 29] showing (as one minor consequence) that the symbolic fourth power $I^{(4)}$ of any radical ideal of points in the projective plane is contained in I^2. (In the case of the radical ideal I of points $p_1, \ldots, p_s \in \mathbb{P}^n$, we note that $I^{(m)} = \cap_{1\leq i\leq s}(I(p_i)^m)$ where $I(p_i)$ is the ideal generated by all forms that vanish at p_i.) Further stimulation came from the following question of Huneke.

Question: If I is the radical ideal of points in the projective plane, must it be true that $I^{(3)} \subseteq I^2$?

While a number of important basic results are now available (see [6, 7, 8, 11, 12, 13, 21]), it was only two years ago that a special configuration of points was discovered giving a negative answer to Huneke's question [14]. Since then additional mainly sporadic examples have been found [27, 2, 33] giving counterexamples to the containment in Huneke's question (and in the case of [27], also to certain related containment conjectures posed in [3, 26]), but many questions and conjectures remain. In addition, new avenues of research have opened up (see, e.g., [26]) and old problems that had become quiescent have been given new life.

One such problem that has been resurrected is that of computing Waldschmidt constants. In the late 70s, in work related to transcendence questions in number theory, Waldschmidt introduced an asymptotic quantity $\widehat{\alpha}(I)$ [36] for homogeneous ideals I, now known as a Waldschmidt constant [15]. It is defined as $\lim_{m\to\infty} \frac{\alpha(I^{(m)})}{m}$, where for any homogeneous ideal $J \neq (0)$, $\alpha(J)$ is the degree of a generator of J of least degree. Efforts to compute or estimate Waldschmidt constants started soon after its introduction [10, 17]. The fact that it, and variants of it, are closely related to the problem of which symbolic powers of an ideal I are contained in given ordinary powers of I [7, 8, 22] has caused a resurgence of interest in computing Waldschmidt constants; see for example [3, 15, 19, 18]. Additional related work [22, 23] which became a focus of discussion at the workshop used the connection between points in multi-projective spaces and higher dimensional linear varieties in single projective spaces to study Waldschmidt constants. The foundation for these papers was understanding points in $\mathbb{P}^1 \times \mathbb{P}^1$; previous work on points in $\mathbb{P}^1 \times \mathbb{P}^1$, such as, for example, [20, 24, 25, 35] provided important tools relied on in [21]. Moreover, the recent attention given to Waldschmidt constants has led to additional new questions, starting with the paper [5] of Bocci and Chiantini, related to fattenings of linear subvarieties in projective space (see [4, 30]).

The structure of the workshop

The design of the workshop was successful in prompting a lot of research interaction. Short talks (35 minutes each) by participants were scheduled for the mornings, with afternoons and evenings reserved for working on specific problems raised by the participants. Potential problems for workshopping were solicited from the participants in advance of the meeting. On the first day of the workshop, the participants by acclimation settled on three main problems to focus on during the workshop. Participants were free to move from one discussion group to another and to change the focus of the discussions, as warranted by individual interest and by the potential for progress.

The topics selected for focused discussions in the discussion groups were as follows:

- H-constants and ideal containments;
- Waring rank problems; and
- computing Waldschmidt constants and stability questions (how many powers of an ideal must be symbolic for all of them to be symbolic).

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Abstracts

Zariski decomposition on surfaces, and its connection to bounded negativity

Thomas Bauer
(joint work with Mirel Caibăr, Gary Kennedy, Piotr Pokora and David Schmitz)

The Zariski decomposition of divisors plays a well-known fundamental role in the theory of algebraic surfaces. In the talk I focused on three aspects:

- a simple proof for their existence and uniqueness on surfaces [1],
- a general abstract Zariski decomposition type result [2] valid in the setting of linear algebra that contains the geometric Zariski decomposition as a special case, and
- a new surprising connection [4] to the Bounded Negativity Conjecture.

Zariski's result [6] from 1962 states that for every effective $\mathbb{Q}$-divisor D on a smooth projective surface X, there are uniquely determined effective (possibly zero) $\mathbb{Q}$-divisors P and N with

$$D = P + N$$

such that

(i) P is nef,
(ii) N is zero or has negative definite intersection matrix,
(iii) $P \cdot C = 0$ for every irreducible component C of N.

Fujita [5] extended the result to pseudo-effective divisors. The geometric significance of Zariski decompositions lies in the fact that, given a pseudo-effective integral divisor D on X with Zariski decomposition $D = P + N$, one has for every sufficiently divisible integer $m \geq 1$ the equality

$$H^0(X, \mathcal{O}_X(mD)) = H^0(X, \mathcal{O}_X(mP)) .$$

In other words, all sections of $\mathcal{O}_X(mD)$ come from the nef line bundle $\mathcal{O}_X(mP)$.

A simple proof, and an abstract version. While Zariski's original proof uses an inductive procedure to construct the negative part N, our simple proof [1] is based on the idea that the positive part P can be constructed as the unique maximal nef subdivisor of D. In joint work [2] with M. Caibăr and G. Kennedy we showed that the same basic idea can be used to prove a very general abstract version of the Zariski decomposition theorem – its setup is as follows: Let V be a vector space over $\mathbb{Q}$, equipped with a symmetric bilinear form $\langle\cdot|\cdot\rangle$. (In the geometric model, V is the vector space $\mathrm{Div}_{\mathbb{Q}}(X)$ of $\mathbb{Q}$-divisors, and $\langle\cdot|\cdot\rangle$ is the intersection product.) Let $E = (e_i)$ be a fixed basis of V such that $\langle e_i|e_j\rangle \geq 0$ for $i \neq j$. (In the geometric model E is the set of irreducible curves, where this condition is clearly satisfied.) Surprisingly, one can prove in this abstract setup – where so little geometry is encoded in the data – that every vector has a unique Zariski decomposition:

Theorem on Zariski Decomposition [2]**.** *Every effective vector $v \in V$ has a unique decomposition into effective elements p and n*

$$v = p + n$$

with

(i) *p is nef*
(ii) *$n = 0$ or $\langle \cdot | \cdot \rangle$ is negative definite on* $\operatorname{supp}(n)$
(iii) *$\langle p | e \rangle = 0$ for every e in the support of n*

Here, the notions *effective* and *nef* are defined as follows (again, following the geometric model): Writing $E = \{e_i \mid i \in I\}$, a vector $v = \sum v_i e_i \in V$ is *effective*, if $v_i \geq 0$ for all i. Its *support* $\operatorname{supp}(v)$ is the set of all e_i occurring with $v_i > 0$. And v is *nef*, if $\langle v | w \rangle \geq 0$ for all effective elements $w \in V$ (or, equivalently, if $\langle v | e_i \rangle \geq 0$ for all i). The classical theorem on Zariski decomposition is then clearly a special case of this abstract version.

It is a quite surprising feature of this result that no further assumptions on the bilinear form are needed – for instance its signature need not coincide with that of an intersection form occurring in geometry (in other words, there is no assumption of Hodge index type).

Bounded Zariski denominators and bounded negativity. It is an interesting problem, posed by A. Küronya, to find out whether on a given surface X, the denominators appearing in the Zariski decompositions $D = N + P$ of all *integral* divisors D (i.e., in the coefficients of the $\mathbb{Q}$-divisors P and N) are bounded. If that is the case on X, then we say for short that *X has bounded Zariski denominators.* In joint work with P. Pokora and D. Schmitz we recently showed that there is a surprising connection between this condition and the condition that X has *bounded negativity*:

Theorem [4]**.** *For a smooth projective surface X over an algebraically closed field the following two statements are equivalent:*

(i) *X has bounded Zariski denominators.*
(ii) *X has bounded negativity, i.e., there is a bound $b(X)$ such that for every irreducible curve C on X one has*

$$C^2 \geq -b(X)$$

The *Bounded Negativity Conjecture* (see [3]) is the conjecture that condition (ii) is satisfied for every smooth projective surface over the field of complex numbers. The exact origin of this conjecture is unclear, but it has a long oral tradition that can be traced back via Ciro Ciliberto and Alfredo Franchetta to Federigo Enriques. The conjecture is open in general. By contrast, it is known that bounded negativity does not hold in general in positive characteristics; so, according to the theorem, unbounded Zariski denominators must appear in positive characteristics – and in fact examples of those can be explicitly constructed (see [4]).

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Hadamard Powers of Points and Lines

Cristiano Bocci

(joint work with Enrico Carlini, Joe Kileel)

The concept of Hadamard product, as matrix entry-wise multiplication, is well known in linear algebra: it has nice properties in matrix analysis ([3, 4]) and has applications in both statistics and physics ([3, 5]). Recently, in the papers [1, 2], the authors use this entry-wise multiplication to define a Hadamard product between projective varieties.

Definition 1. *Let $p, q \in \mathbb{P}^n$ be two points of coordinates respectively $[a_0 : a_1 : \ldots : a_n]$ and $[b_0 : b_1 : \ldots : b_n]$. If $a_ib_i \neq 0$ for some i, their Hadamard product $p \star q$ of p and q, is defined as*

$$p \star q = [a_0b_0 : a_1b_1 : \ldots : a_nb_n].$$

If $a_ib_i = 0$ for all $i = 0, \ldots, n$ then we say $p \star q$ is not defined. Let $X, Y \subset \mathbb{P}^n$ be two varieties, then their Hadamard product $X \star Y$ is

$$X \star Y = \overline{\{p \star q : p \in X, q \in Y, p \star q \text{ is defined}\}}.$$

For any projective variety X, we may consider its Hadamard square $X^{\star 2} = X \star X$ and its higher Hadamard powers $X^{\star r} = X \star X^{\star(r-1)}$. In [1], the authors use this definition to describe the algebraic variety associated to the restricted Boltzmann machine which is the undirected graphical model for binary random variables specified by the bipartite graph $K_{r,n}$. This variety is the $r-$th Hadamard power of the first secant variety of $(\mathbb{P}^1)^n$. Note that [2] concerns the case $r = 2, n = 4$.

Since in [1] and [2] only the definition of Hadamard product of varieties X, Y is given, it is surely interesting to study properties of $X \star Y$, also in terms of the properties of X and Y. This paper is a first step in that direction.

We start by giving a different definition of the Hadamard product of varieties in terms of projections of Segre products.

Definition 2. *Given varieties $X, Y \subset \mathbb{P}^n$ we consider the usual Segre product*

$$X \times Y \subset \mathbb{P}^N$$
$$([a_0 : \cdots : a_n], [b_0 : \cdots : b_n]) \mapsto [a_0 b_0 : a_0 b_1 : \ldots : a_n b_n]$$

and we denote with z_{ij} the coordinates in $\mathbb{P}^N$. Let $\pi : \mathbb{P}^N \dashrightarrow \mathbb{P}^n$ be the projection map from the linear space defined by equations $z_{ii} = 0, i = 0, \ldots, n$. The Hadamard product *of X and Y is*

$$X \star Y = \overline{\pi(X \times Y)},$$

where the closure is taken in the Zariski topology.

Before two state our results we need a preliminary definition.

Definition 3. *Let $H_i \subset \mathbb{P}^n, i = 0, \ldots, n$, be the hyperplane $x_i = 0$ and set*

$$\Delta_i = \bigcup_{0 \leq j_1 < \ldots < j_{n-i} \leq n} H_{j_1} \cap \ldots \cap H_{j_{n-i}}.$$

In other words, Δ_i is the $i-$dimensional variety of points having *at most* $i+1$ non-zero coordinates. Thus Δ_0 is the set of coordinates points and Δ_{n-1} is the union of the coordinate hyperplanes. Note that elements of Δ_i have *at least* $n-i$ zero coordinates. We have the following chain of inclusions:

$$\Delta_0 = \{[1 : 0 : \ldots : 0], \ldots, [0 : \ldots : 0 : 1]\} \subset \Delta_1 \subset \ldots \subset \Delta_{n-1} \subset \Delta_n = \mathbb{P}^n.$$

As a first important result we give a Hadamard version of Terracini's Lemma.

Lemma 4. *Consider varieties $X, Y \subset \mathbb{P}^n$. If $p \in X$ and $q \in Y$ are general points, then*

$$T_{p \star q}(X \star Y) = \langle p \star T_q(Y), q \star T_p(X) \rangle.$$

Moreover, if $p_1, \ldots, p_r \in X$ are general points and $p_1 \star \ldots \star p_r \in X^{\star r}$ is a general point, then

$$T_{p_1 \star \ldots \star p_r}(X^{\star r}) = \langle p_2 \star \ldots \star p_r \star T_{p_1}(X), \ldots, p_1 \star \ldots \star p_{r-1} \star T_{p_r}(X) \rangle.$$

Then we fix our attention on two distinct cases. First, we study the Hadamard products of linear spaces, analyzing, in particular, the case of a line, where we find explicit equations for all Hadamard powers.

Let $L \subset \mathbb{P}^n$ be a linear space of dimension m. If $p \in \mathbb{P}^n$, then $p \star L$ is either empty or it is a linear space of dimension at most m. If $p \notin \Delta_{n-1}$, then $\dim(p \star L) = m$.

Theorem 5. *Let $L \subset \mathbb{P}^n$, $n > 1$, be a line. If $L \cap \Delta_{n-2} = \emptyset$, then $L^{\star r} \subset \mathbb{P}^n$ is a linear space of dimension* $\min\{r, n\}$.

Second, we study the r-th square-free Hadamard power $Z^{\underline{\star} r}$ of a finite set Z of projective points.

Definition 6. *Let $Z \subset \mathbb{P}^n$ be a finite set of points. The $r-th$* square-free *Hadamard power of Z is*

$$Z^{\underline{\star} r} = \{p_1 \star \ldots \star p_r : p_i \in Z \text{ and } p_i \neq p_j \text{ for } i \neq j\}.$$

The two cases are glued together in the following theorem which gives a complete classification of the $r-$th square-free Hadamard power of a set of collinear points.

Theorem 7 *Let $L \subset \mathbb{P}^n$ be a line, $Z \subset L$ be a set of m points and $r \leq \min\{m, n\}$. If $L \cap \Delta_{n-2} = \emptyset$ and $Z \cap \Delta_{n-1} = \emptyset$, then $Z^{\underline{\star}r}$ is a star configuration in $M = L^{\star r}$.*

In contrast with the standard approach to construct star configurations as intersections of a set of randomly choosen planes, which can give points with complicated coordinates, Theorem 7 permits a different construction, easily implementable in computer algebra software.

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Waringology

Enrico Carlini

Let F be an homogeneous degree d element in $S = \mathbb{C}[x_0, \ldots, x_n]$, also called a *degree d form*. A *sum of powers decomposition of F* is an expression of the following type

$$F = L_1^d + \ldots + L_r^d$$

where the forms L_i have degree one. The *Waring rank of F*, or simply the *rank of F*, is denoted with $\mathrm{rk}(F)$ and denotes the smallest r for which there exists a sums of powers decomposition of F.

The rank of a form is an important, but elusive quantity, and a great deal of research focuses on efficient way to characterize, and hopefully to compute, $\mathrm{rk}(F)$. At the moment, we know the Waring rank of binary forms, quadratic forms, monomials, the sum of coprime monomials, cubic in three variable and reducible cubics in any number of variables. Beside these families, there are only sparse examples for which we know the rank.

The most relevant tool in the study of the Waring rank is the *Apolarity Lemma*, which states that,

$$F = L_1^d + \ldots + L_r^d$$

if and only if

$$F^{\perp} \supset I_{\mathbb{X}},$$

where $I_{\mathbb{X}}$ is the ideal of r distinct points and $F^{\perp}$ is the ideal of differential operators vanishing on F; usually $F^{\perp}$ is considered as an ideal in $T = \mathbb{C}[X_0, \ldots, X_n]$, where $X_i = \frac{\partial}{\partial x_i}$.

The Apolarity Lemma exposes the deep geometrical nature of the Waring rank and of sums of powers decompositions. Thus, it is not a surprise that geometric tools can be successfully employed. One of the most recent and promising geometrical ideas is the one of *e-computable forms*; this is contained in a joint work with M.V.Catalisno, L.Chaintini, A.V.Geramita and Y.Woo, see [1]. A form F is said to be e-computable, if there exists an ideal I and a general degree e form $q \in I$ such that

$$\mathrm{rk}(F) = \frac{1}{e}\,\ell\left(\frac{T}{F^{\perp} : I + (q)}\right)$$

where ℓ denotes the length of an Artinian ring. We can prove that several families of forms are e-computable, and hence we can compute their rank, using subtle geometric argument based on the inequality

$$\mathrm{rk}(F) \geq \frac{1}{e}\,\ell\left(\frac{T}{F^{\perp} : I + (q)}\right).$$

The notion of e-computable forms also plays a role in the study of *Strassen's additivity conjecture*. The conjecture states that, if we are given forms F_i in independent sets of variable, than the Waring rank is additive, that is

$$\mathrm{rk}(\sum_i F_i) = \mathrm{rk}(F_1) + \ldots + \mathrm{rk}(F_r).$$

This conjecture is known since 1973 and not much progress has been made on it till recently when we proved that Strassen's additivity conjecture holds for e-computable forms in [1].

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Waring Rank of Forms and Strassen's Additivity Conjecture

Maria Virginia Catalisano

(joint work with Enrico Carlini, Luca Chiantini, Anthony V.Geramita, and Youngho Woo)

The problem of determining the Waring rank of homogeneous polynomials (forms) is motivated by questions arising in signal processing, computational complexity, and other areas. The Waring rank of a degree d form $F \in S = \mathbb{C}[x_0, \ldots, x_n]$ is

$$\mathrm{rk}\ (F) = \min\{r \in \mathbb{N} \mid F = L_1^d + \cdots + L_r^d\},$$

where the L_i are linear forms of S.

From a geometric point of view, if $V \subset \mathbb{P}^N$, $(N = \binom{n+d}{n} - 1)$, is the Veronese variety, since $F \in S_d$ corresponds to a point $[F]$ in $\mathbb{P}^N$, then the rank of F is the minimum value of r for which $[F]$ is in the r-secant variety of V.

The problem of finding the rank of F is solved in few cases: if F has degree two; if F is a binary form (Sylvester 1886; Comas Seiguer 2001; Brachat, Comon, Mourren,Tsigaridas 2009); if F is a generic form (Alexander - Hirschowitz , 1995); if some algorithms work (Buczynska, Buczynski; Brachat, Comon, Mourrain, Tsigaridas; Oeding, Ottaviani).

In [1] we compute the rank of any polynomial which is the sum of pairwise coprime monomials. In particular, we determine the rank of any monomial. In [3] we introduce the notion of linear computability. By this notion we find the rank of infinitely many examples of reducible forms.

Our main tools are the Apolarity Lemma and a new method to find a lower bound for the rank of F.

The Apolarity Lemma says that, given $L_1, ..., L_r$ pairwise linearly independent linear forms, with L_i corresponding to the point P_i, and $\mathbb{X} = \{P_1, ..., P_r\}$, then $F = L_1^d + \cdots + L_r^d$ if and only if $I_{\mathbb{X}} \subset F^{\perp} \subset T = \mathbb{C}[X_0, ..., X_n]$.

Now, if $\mathbb{X}$ is a set of $\text{rk}(F)$ points apolar to F, in [3] we prove that

$$|\mathbb{X}| \geq \ell(T/F^{\perp} : I + (q)),$$

where $I \subset T$ is the ideal of a linear space and $q \in I$ is a generic linear form. This inequality gives us a lower bound for the rank of F. If $|\mathbb{X}| = \ell(T/F^{\perp} : I + (q))$ we say that F is *linearly computable.*

We show that the following forms are linearly computable: monomials, forms in two variables, and forms of the following types: $F = x_0^a(x_1^b + \cdots + x_m^b)$ with $a + 1 \geq b$, $F = x_0^a(x_0^b + x_1^b + \cdots + x_m^b)$ with $a + 1 \geq b$, $F = x_0^a(x_1^b + x_2^b)$, $F = x_0^a(x_0^b + x_1^b + x_2^b)$.

Moreover, we consider the Strassen additivity conjecture for forms, which states that if F and G are forms in disjoint sets of variables, then $\text{rk}(F + G) = \text{rk}(F) + \text{rk}(G)$. We show that the Strassen additivity conjecture is satisfied for several classes of forms (see [2] and [3]). In particular Strassen's conjecture holds for linearly computable forms, that is, if $F_1, ..., F_m$ are linearly computable forms in different sets of variables, then $\text{rk}(F_1 + \cdots + F_m) = \text{rk}(F_1) + \cdots + \text{rk}(F_m)$.

Natural questions arise.

Are all forms linearly computable? Unfortunately the answer is no, and we show that $F = x(y^3 + z^3 + w^3)$ is a non-linearly computable form.

There are examples of forms $F + G$ for which the Strassen conjecture holds, even if F is non-linearly computable ? The answer is yes. For instance, for $F(x_0, ..., x_n) + y^d$ the Strassen conjecture holds, even if F is non-linearly computable.

The following question remains open: is the sum of two linearly computable forms, still linearly computable?

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Simplifying Fat Points via Partial Intersections

Susan M. Cooper
(joint work with Elena Guardo)

The behaviour of fat point schemes plays a key role in many unanswered questions. However, properties of non-reduced fat point schemes in $\mathbb{P}^n$ can be difficult to characterize. In particular, characterizing the graded Betti numbers and Hilbert functions of non-reduced fat points are open and very difficult problems. One approach is to compare the properties of these non-reduced schemes to those of well-known families of reduced point sets. In this project we connect minimal free resolutions of fat points supported inside *grid complete intersections* with those of reduced point sets called *partial intersections*.

Our project focuses on a very structured setting. A grid complete intersection of type $\{a, b\}$, denoted $\mathbb{X}^{a,b}_{grid}$, is a complete intersection in $\mathbb{P}^2$ supported on the intersection of two sets of generically chosen lines (where there are a lines in one set and b lines in the other aand we assume $1 \leq a \leq b$). We can then consider the homogeneous fat point scheme $\mathbb{X}^{a,b}_{grid,m}$ obtained by adding multiplicity m to each point. Buckles-Guardo-Van Tuyl [1] showed that $\mathbb{X}^{a,b}_{grid,m}$ has the same minimal free resolution as a partial intersection in $\mathbb{P}^2$ of type $\{p, q\}$, where $p = (mb, (m-1)b, (m-2)b, \ldots, b)$ and $q = (a, a, \ldots, a)$. This partial intersection is a special reduced set of points which consists of mb points each on a lines, $(m-1)b$ points each on a lines, etc. In turn, this fact gives us the minimal free resolution of $\mathbb{X}^{a,b}_{grid,m}$ via a theorem of Maggioni-Ragusa [2] which characterizes the minimal free resolutions of partial intersections in $\mathbb{P}^2$ in terms of the type $\{p, q\}$.

It is natural to want to take this program further. To this end, we consider fat point schemes supported on a grid complete intersection minus a point, with multiplicity m attached to each point. We denote this scheme by $\{\mathbb{X}^{a,b}_{grid,m} \setminus P; m\}$ where P is the point removed from $\mathbb{X}^{a,b}_{grid}$. Our main result is:

Theorem: Let $b \geq a + 1$ and $m \leq b - 1$. Then $\{\mathbb{X}^{a,b}_{grid,m} \setminus P; m\}$ and the partial intersection of type $\{p, q\}$ where

$$p = (mb, mb - m, (m-1)b, (m-1)b - (m-1), (m-2)b, \ldots, b, b-1)$$
$$q = (a-1, 1, a-1, 1, a-1, \ldots, a-1, 1)$$

have the same minimal free resolution.

Using the characterization of the minimal free resolutions of partial intersections, this gives us an explicit formula for the minimal free resolution (and hence the Hilbert function) of the fat point scheme $\{\mathbb{X}^{a,b}_{grid,m} \setminus P; m\}$. It should be noted that this was proven for $m = 2$ by Buckles-Guardo-Van Tuyl [1]. Our work in progress includes: (1) generalizing the above theorem by relaxing the constraints on the multiplicity m and the degree b; and (2) applying the above theorem and its generalizations to connect the minimal socle degree of $\{\mathbb{X}^{a,b}_{grid,m} \setminus P; m\}$ with the minimum Hamming distance of an associated linear code.

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Naive, homogeneous, geometric and real Waldshmidt constants

Marcin Dumnicki

Let L be a set of s generic lines in $\mathbb{P}^3$. By $\alpha(s,m)$ we denote the least degree of a non-zero form in $I_L^{(m)}$. Define Waldshmidt constant to be

$$\widehat{\alpha}(s) = \lim_{m\to\infty} \frac{\alpha(s,m)}{m}.$$

This invariant is very hard to compute, so we define several bounds. Let $e\alpha$ denote the expected "alpha",

$$e\alpha(s,m) = \min\{d : \binom{d+3}{3} - s\frac{(3d-2m+5)m(m+1)}{6} > 0\}.$$

This invariant can be computed as the (positive) root of a polynomial $t^3 - 3st + 2s$. Define also the expected Waldshmidt constant and the homogeneous Waldshmidt constant to be:

$$\widehat{e\alpha}(s) = \lim_{m\to\infty} \frac{e\alpha(s,m)}{m},$$

$$\widehat{h\alpha}(s) = \inf_{m} \frac{e\alpha(s,m)}{m}.$$

Considering also arbitrary multiplicities of lines, we can define

$$e\alpha(m_1,\ldots,m_s) = \min\{d : \binom{d+3}{3} - \sum_{j=1}^{s} \frac{(3d-2m_j+5)m_j(m_j+1)}{6} > 0\}.$$

The above helps us to define the geometric Waldshmidt constant:

$$\widehat{g\alpha}(s) = \inf_{m_1,\ldots,m_s} \frac{s\cdot e\alpha(m_1,\ldots,m_s)}{m_1+\ldots+m_s}.$$

The values for few lines are presented in table:

s	$\widehat{\alpha(s)}$	$\widehat{g\alpha}(s)$	$\widehat{h\alpha}(s)$
1	1	1	1
2	2	2	2
3	2	2	2
4	8/3	8/3	3
5	10/3	10/3	45/13
6	72/19	42/11	27/7
7	21/5	$e\alpha(7)$	$e\alpha(7)$

The value 72/19 is computed by the hypersurface of degree 12 vanishing along lines with multiplicities $4, 3, 3, 3, 3, 3$, while 21/5 by a hypersurface of degree 12 vanishing with multiplicities $3, 3, 3, 3, 3, 3, 2$. Both of these hypersurfaces are irreducible. I will also explain the existence of the second hypersurface.

Unions of linear subspaces

Giuliana Fatabbi

(joint work with Brian Harbourne and Anna Lorenzini)

We work in the $n-$ dimensional projective space over an arbitrary field K but some results will require the characteristic to be 0. We will consider a collection of linear subvarieties $L_0, L_1, \ldots, L_r, H_0, H_1, \ldots, H_s \subset P^n$ such that the following conditions hold:

(C1) $H_0, H_1, \ldots, H_s$ are distinct hyperplanes;
(C2) $L_i \subseteq H_0$ for $i > 0$, but $L_0 \nsubseteq H_0$;
(C3) if $L_i \subseteq L_j$, then $i = j$; and
(C4) for all $i \geq 0$ and $j > 0$ we have $L_i \nsubseteq H_j$.

If $s = 0$ and each L_i is a point, then we have r points L_i, $0 < i \leq r$, on the hyperplane H_0 and one point L_0 that is not in H_0.

Another special case is related to what we call a *galaxy*. We start with a star configuration $S(n, e, u)$. We recall that the star configuration $S(n, e, u)$ is defined by a set of $u \geq n$ distinct hyperplanes $A_1, \ldots, A_u \cong P^{n-1}$ in P^n such that, for each $1 \leq i \leq n$, the intersection of any i of the hyperplanes has dimension at most $n - i$. The star configuration of codimension $e \leq n$ is the set $S(n, e, u)$ of the $\binom{u}{e}$ linear varieties arising as intersections of e arbitrary distinct choices $A_{i_1}, \ldots, A_{i_e}$ of the hyperplanes. Let $N \geq 1$ be an integer and regard P^n as a linear subvariety of P^{n+N}. The galaxy $\mathcal{G} = \mathcal{G}(n, N, e, h) = \mathcal{G}(n, N, e, h; S(n, e, u), \mathcal{H})$ consists of $S(n, e, u)$ and a choice of h general points $\mathcal{H} = \{P_1, \ldots, P_h\} \in P^{n+N}$; in particular, for each i, P_{i+1} is not in the span of P^n and $P_1, \ldots, P_i$.

Assigning a multiplicty to each subspace we can consider schemes of the form $X = \sum_{i\geq 0} l_i L_i + \sum_{j>0} h_j H_j$, by which we mean the scheme defined by the ideal $I_X = (\cap_{i\geq 0} I_{L_i}^{l_i}) \bigcap (\cap_{j>0} I_{H_j}^{h_j})$, where l_i and h_j are non-negative integers.

We are interested in the study of:

- Hilbert Function of X
- $\alpha(I_X)$ which is defined to be the least degree t such that I_X is not 0.
- A minimal free resolution of I_X
- The Waldscmidt constant $\hat{\alpha}(I_X) = \lim_{m\to\infty} \frac{\alpha(I_X^{(m)})}{m}$.

In [1] we prove that it is possible to recover the Hilbert Function and the minimal free resolution of I_X from those of several ideals of subschemes whose supports are contained in H_0 . We also prove that if $W = \sum_{i\geq 0} l_i L_i$ and $X = \sum_{i\geq 0} l_i L_i + \sum_{j>0} h_j H_j$ for non-negative integers l_i and h_i, $l' = \max(l_1, \ldots, l_r)$ and $h = h_1 + \cdots + h_s$, then $\alpha(X) = h + \alpha(W)$ and $\max(l', l_0) \leq \alpha(W) \leq l' + l_0$. Moreover, we can suitably define an integer d such that

$$\alpha(W) \leq l_0 + d,$$

with $\alpha(W) = l_0 + d$ if $\mathrm{char}(K) = 0$. Furthermore, we apply these results to the case of fat points with all but one point having support in a hyperplane and to compute galactic Waldschmidt constants in the reduced case.

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Symbolic powers versus regular powers of ideals of points in $\mathbb{P}^1 \times \mathbb{P}^1$

Elena Guardo

(joint work with Brian Harbourne, Adam Van Tuyl)

Recent work of Ein-Lazarsfeld-Smith and Hochster-Huneke raised the problem of which symbolic powers of an ideal are contained in a given ordinary power of the ideal. Most of the work done up to now has been done for ideals defining 0-dimensional subschemes of projective space.

We focus on certain subschemes given by a union of lines in $\mathbb{P}^3$ which can also be viewed as points in $\mathbb{P}^1 \times \mathbb{P}^1$. We work over an algebraically closed field k of arbitrary characteristic.

The multi-homogeneous coordinate ring $k[\mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_t}]$ of $\mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_t}$ is

$$k[x_{1,0}, \ldots, x_{1,n_1}, \ldots, x_{t,0}, \ldots, x_{t,n_t}].$$

It has a multi-grading given by

$$\deg(x_{i,j}) = e_i = (0, \ldots, 0, 1, 0, \ldots, 0) \in \mathbb{N}^t,$$

where the 1 is in the ith position. The ring $k[\mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_t}]$ is a direct sum of its multi-homogeneous components $k[\mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_t}]_{(a_1,\ldots,a_t)}$, where $k[\mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_t}]_{(a_1,\ldots,a_t)}$ is the k-vector space span of the monomials of multi-degree $(a_1, \ldots, a_t)$.

An ideal $I \subseteq k[\mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_t}]$ is multi-homogeneous if it is the direct sum of its multi-homogeneous components (i.e., of $k[\mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_t}]_{(a_1,\ldots,a_t)} \cap I$). A multi-homogeneous ideal I can be regarded as a homogeneous ideal in $k[\mathbb{P}^N]$, $N = n_1 + \cdots + n_t + t - 1$, where a monomial of multi-degree $(a_1, \ldots, a_t)$ has degree $d = a_1 + \cdots + a_t$ and the homogeneous component of I of degree d is $I_d = \bigoplus_{\sum_i a_i = d} I_{(a_1,\ldots,a_t)}$.

When $t > 1$, a multi-homogeneous ideal I when regarded as being homogeneous never defines a 0-dimensional subscheme of $\mathbb{P}^N$, even if I defines a zero-dimensional subscheme of $\mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_t}$.

Example: the multi-homogeneous ideal I of a finite set of points in $\mathbb{P}^1 \times \mathbb{P}^1$ defines a finite set of lines in $\mathbb{P}^3$, which are skew (and thus not a cone) if no two of the points lie on the same horizontal or vertical rule of $\mathbb{P}^1 \times \mathbb{P}^1$ and not a complete intersection unless the points comprise a rectangular array in $\mathbb{P}^1 \times \mathbb{P}^1$.

Let I be the ideal of a set Z of s distinct reduced points of $\mathbb{P}^1 \times \mathbb{P}^1$, i.e., $Z = \{P_1, \ldots, P_s\}$. A point has the form $P = [a_0 : a_1] \times [b_0 : b_1] \in \mathbb{P}^1 \times \mathbb{P}^1$ and its defining ideal $I(P) = (F, G)$ where $\deg F = (1, 0)$ and $\deg G = (0, 1)$. The ideal of a set of points is $I(Z) = \bigcap_{i=1}^{s} I(P_i)$.

The m-th symbolic power of $I(Z)$ has the form $I(Z)^{(m)} = \bigcap_{i=1}^{s} I(P_i)^m$. The scheme defined by $I(Z)^{(m)}$ is sometimes referred to as a *(homogeneous) fat point scheme*, and denoted $mP_1 + \cdots + mP_s$.

Correspondence between points in $\mathbb{P}^1 \times \mathbb{P}^1$ and lines in $\mathbb{P}^3$.
Consider a bigraded ideal I of a point in $\mathbb{P}^1 \times \mathbb{P}^1$. Since $k[\mathbb{P}^1 \times \mathbb{P}^1] = k[\mathbb{P}^3]$ as rings, it defines a line in $\mathbb{P}^3$ when regarded as a singly graded ideal in the usual grading on $k[\mathbb{P}^3]$. Thus the ideal of a finite set of points in $\mathbb{P}^1 \times \mathbb{P}^1$ is simultaneously (but with respect to a different grading) the ideal of a finite set of lines $\mathbb{P}^3$.

The point $P = [a_0 : a_1] \times [b_0 : b_1] \in \mathbb{P}^1 \times \mathbb{P}^1$ corresponds to a pair of points $P_1 = [a_0 : a_1] \in L_1$ and $P_2 = [b_0 : b_1] \in L_2$. The ideal $I(P)$ defines the line L_P in $\mathbb{P}^3$ through the points P_1 and P_2. Given distinct points $P, Q \in \mathbb{P}^1 \times \mathbb{P}^1$, the lines L_P and L_Q meet if and only if either $P_1 = Q_1$ or $P_2 = Q_2$; i.e., if and only if P and Q are both on the same horizontal rule or both on the same vertical rule of $\mathbb{P}^1 \times \mathbb{P}^1$.

Vice versa, given any single line $L \subset \mathbb{P}^3$, we can found lines $L_1 \cong \mathbb{P}^1$ and $L_2 \cong \mathbb{P}^1$ in $\mathbb{P}^3$ such that $I(L)$ is the ideal of a single point in $\mathbb{P}^1 \times \mathbb{P}^1$.

Problems involving fat points $Z = \sum_i m_i P_i$ with support at distinct points $P_i \in \mathbb{P}^1 \times \mathbb{P}^1$ can be translated into problems involving divisors on X.

- Given $I = I(Z)$ and (i, j), then as a vector space $I(Z)_{(i,j)}$ can be identified with $H^0(X, iH + jV - \sum_i m_i E_i)$, which itself can be regarded as a vector subspace of the space of sections $H^0(\mathbb{P}^1 \times \mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(i, j))$.

Reinterpretation of the problems involving points of $\mathbb{P}^1 \times \mathbb{P}^1$ as problems involving points of $\mathbb{P}^2$.

- $H^0(X, aH + bV - m(E_1 + \cdots + E_s)) = H^0(X, (a+b)L - m(E_1 + \cdots + E_s) - aE_{s+1} - bE_{s+2})$.
- If I is the ideal of the fat points $mP_1 + \cdots + mP_s$, we note that $\alpha(I^{(m)})$ is then the least t such that $t = a + b$ and $h^0(X, (a+b)L - m(E_1 + \cdots + E_s) - aE_{s+1} - bE_{s+2}) > 0$.

On the Waldschmidt constant

Let $k[\mathbb{P}^N]$ denote the polynomial ring $k[x_0, \ldots, x_N]$ with the standard grading (so each variable has degree 1). Given any homogeneous ideal $(0) \neq I \subseteq k[\mathbb{P}^N]$, then $\alpha(I)$ denotes the least degree of a nonzero form (i.e., homogeneous element) in I. The limit $\lim_{m\to\infty} \alpha(I^{(m)})/m$ is known to exist and is denoted by $\gamma(I)$ or $\hat{\alpha}(I)$.

- Let I be an ideal of a complete intersection. If $I^{(m)} = I^m$ for all $m \geq 1$ then $\alpha(I^{(m)}) = \alpha(I^m) = m\alpha(I)$, hence $\gamma(I) = \alpha(I)$.

When I is not a complete intersection?

- We showed that $I^{(m)} = I^m$ for all $m \geq 1$ whenever I is the ideal of two skew lines in $\mathbb{P}^N$.
- Let $I = I(X)$ be the ideal of a finite reduced ACM subscheme X in $\mathbb{P}^1 \times \mathbb{P}^1$. Then $I^{(m)} = I^m$ for all $m \geq 1$ if and only if $I^{(3)} = I^3$ (see [2]).
- Let I be the ideal of a set Z of s general points in $\mathbb{P}^1 \times \mathbb{P}^1$. Then $I^m = I^{(m)}$ for all $m > 0$ if and only if s is $1, 2, 3$ or 5. Moreover, $I^{(3)} \neq I^3$ if $s = 4$ and $I^{(2)} \neq I^2$ if $s \geq 6$ (see [3]).

The Waldschmidt constant for general points of $\mathbb{P}^1 \times \mathbb{P}^1$

Theorem. [1] Let I be the ideal of $s \geq 1$ general points of $\mathbb{P}^1 \times \mathbb{P}^1$. Let $\gamma(I) = \hat{\alpha}(I) = \lim_{m\to\infty} \alpha(I^{(m)})/m$ be the Waldschmidt constant of I. Then

- If $s = 1$, then $\gamma(I) = 1$.
- If $s = 2$ or 3, then $\gamma(I) = 2$.
- If $s = 4$, then $\gamma(I) = 8/3$.
- If $s = 5$, then $\gamma(I) = 3$.
- If $s = 6$, then $\gamma(I) = 24/7$.
- If $s = 7$, then $\gamma(I) = 56/15$.
- If $s = 8$, then $\gamma(I) = 4$.
- If $9 \leq s$, then $\sqrt{s} - 1 < \alpha(I)/2 \leq \gamma(I) \leq \sqrt{2s}$ and also $4 \leq \gamma(I)$.

Asymptotic resurgence of I

Bocci and Harbourne introduced $\rho(I) = sup\{m/r : I^{(m)} \not\subseteq I^r\}$ called the resurgence of I. For a homogeneous ideal $(0) \neq I \subset k[\mathbb{P}^N]$, we define the *asymptotic resurgence* of I $\hat{\rho}(I) = \rho_a(I) = \sup\{m/r : I^{(mt)} \not\subseteq I^{rt}$ for all $t \gg 0\}$,

Theorem. [1] Consider a homogeneous ideal $(0) \neq I \subset k[\mathbb{P}^N]$. Let $h = \min(N, h_I)$ where h_I is the maximum of the heights of the associated primes of I.

(1) We have $1 \leq \alpha(I)/\gamma(I) \leq \rho_a(I) \leq \rho(I) \leq h$.

(2) If I is the ideal of a (non-empty) smooth subscheme of $\mathbb{P}^N$, then

$$\rho_a(I) \le \frac{\omega(I)}{\gamma(I)} \le \frac{\mathrm{reg}(I)}{\gamma(I)}$$

where $\omega(I)$ is the largest degree in a minimal homogeneous set of generators of I and where $\mathrm{reg}(I)$ is the Castelnuovo-Mumford regularity of I

Corollary. [1] Let I be the ideal of s general lines in $\mathbb{P}^N$ for $N \ge 3$, where $s = \binom{t+N}{N}/(t+1)$ for any integer $t \ge 0$ such that s is an integer (there are always infinitely many such t; for example, let $t = p-1$ for a prime $p > N$). Then $\rho_a(I) = (t+1)/\gamma(I)$.

Theorem. [1] Let I be the ideal of s general lines in $\mathbb{P}^N$ for $N \ge 2$ and $s \le (N+1)/2$. Then $\rho(I) = \rho'_a(I) = \rho_a(I) = \max(1, 2\frac{s-1}{s})$. Moreover, if $2s < N+1$, then $\gamma(I) = 1$, while if $2s = N+1$, then $\gamma(I) = \frac{N+1}{N-1}$.

Open problems

(1) What can one say for ideals of other arrangements of lines?
(2) Can one extend this to higher dimensional linear spaces using products of other projective spaces?
(3) What kinds of upper bounds are there for resurgences (asymptotic or not) for ideals of particular subschemes in multiprojective spaces?

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H-constants and the containment problem

Brian Harbourne

The point of this talk is to provide background and context for working on two seemingly unrelated problems. For simplicity, assume the ground field is the complex numbers.

The Containment Problem

Recall that the symbolic power $I^{(m)}$ of an ideal I of points $p_1, \dots, p_s \in \mathbf{P}^2$ in the projective plane is defined by $I^{(m)} = (I(p_1)^m) \cap \cdots \cap (I(p_s)^m)$. In this situation, as a special case of a much bigger theorem we have:

Theorem([6, 7]): Let I be the ideal of distinct points $p_1, \ldots, p_s \in \mathbf{P}^2$. Then $I^{(4)} \subseteq I^2$.

This raises the following question [3]:

Question (C. Huneke, 2003): Let I be the ideal of distinct points $p_1, \ldots, p_s \in \mathbf{P}^2$. Is it always true that $I^{(3)} \subseteq I^2$?

After 10 years an answer was found:

Answer ([4]): No!

But examples of points whose ideal I gives $I^{(3)} \not\subseteq I^2$ so far are rare.

H-constants

Now we recall H-constants [2, 8]: given a reduced plane curve C of degree d with s distinct points $p_1, \ldots, p_s \in C$, we define

$$H(C; p_1, \ldots, p_s) = \frac{d^2 - \sum_i m_i^2}{s},$$

where $m_i = \mathrm{mult}_{p_i}(C)$ is the multiplicity of C at p_i. We write $H(C)$ when the points p_i are the singular points of C.

Examples with $H(C) \leq -2$ are rare: none are yet known with C irreducible, and none of any kind are known with $H(C) \leq -4$. If C is a union of a finite number of lines we say C is a *nontrivial configuration* of lines if not all of the lines go through the same point, and we say it has no *simple crossings* if there is no point which is on exactly two lines.

Observed Fact: Every known nontrivial example C of a finite configuration of lines in the plane with no simple crossings has both $H(C) < -2$ and $I^{(3)} \not\subseteq I^2$ where I is the ideal of the singular points of C.

Question: Is this a coincidence or an indication of a deeper connection?

Example 1: Let C be $(x-y)(x-z)(y-z) = 0$ where $\mathbf{C}[\mathbf{P}^2] = \mathbf{C}[x, y, z]$. Take the points p_1, p_2, p_3, p_4 to be the coordinate vertices $[0:0:1], [0:1:0], [1:0:0]$ and the point $[1:1:1]$. This has $H(C; p_1, p_2, p_3, p_4) = \frac{3^2 - 3(1^2) - 1(3^2)}{4} = -\frac{3}{4}$ and $I^{(3)} \subseteq I^2$. We get what is known as the the *Fermat configuration* [11] of lines by mapping $\mathbf{P}^2$ to $\mathbf{P}^2$ via $[a:b:c] \mapsto [a^n : b^n : c^n]$. Pull C back via this map to get C' defined by $(x^n - y^n)(x^n - z^n)(y^n - z^n) = 0$. There are $n^2 + 3$ singular points of C'; they are the inverse images of the points p_i. Then $H(C') = \frac{(3n)^2 - n^2 3^2 - 3n^2}{n^2+3} = \frac{-3n^2}{n^2+3}$ and $I^{(3)} \not\subseteq I^2$ holds for each $n \geq 3$ [4, 5]. The first known example of $I^{(3)} \not\subseteq I^2$ [4], was this one with $n = 3$.

Example 2: The Fermat configuration of lines has n^2 triple points and 3 additional points of multiplicity n. The following construction was considered in a working group at the *Workshop on Recent advances in Linear series and Newton-Okounkov bodies*, February 9–14, 2015 at the University of Padua (i.e., last week). Perform a linear change of coordinates so that the coordinate vertices are on the branches of one of these new triple points and the triple point is $[1:1:1]$. The

pull-back the Fermat configuration under the map $\mathbf{P}^2 \to \mathbf{P}^2$ where $[a : b : c] \mapsto [a^k : b^k : c^k]$ for some k to get C''. For $k \geq 2$, let t_k denote the number of points of C of multiplicity exactly k. Then the data for C, C' and C'' are:
C: $\deg(C) = 3$, $t_1 = 3$, $t_3 = 1$, $H(C) = \frac{-3}{4}$;
C': $\deg(C') = 3n$, $t_n = 3$, $t_3 = n^2$, $H(C') = \frac{-3n^2}{3+n^2}$, thus $H(C') < H(C)$ for $n > 1$;
C'': $\deg(C'') = 3nk$, $t_n = 3k^2$, $t_3 = n^2k^2$, $t_k = 3$, $H(C'') = \frac{-3n^2-3}{3+n^2+\frac{3}{k^2}}$. Thus $H(C'') < H(C')$ for $k \gg 0$.

Question: If I is the ideal of the singular points of C'', does $I^{(3)} \not\subseteq I^2$ hold?

Example 3: There are configurations of cubics, due to Roulleau-Urzúa [9, 10], depending on a parameter n. In the case that 3 divides n, the Roulleau-Urzúa configuration of cubics is a configuration of $4(n^2-3)/3$ cubics. If C_n is the union of the cubics, then we have: $\deg(C_n) = 4(n^2-3)$, $t_{n^2-3} = 12$ (these are the 12 singular points of the 9 lines of the $n = 3$ Fermat configuration), $t_4 = (n^2 - 3)(n^2 - 9)/3$, $t_3 = 4(n^2-3)$ and otherwise $t_k = 0$. Note that the $4(n^2-3)$ triple points come in 12 sets of $(n^2-3)/3$ triple points each, corresponding to the 12 points of multiplicity $n^2 - 3$, and the points in each set are infinitely near the corresponding point of multiplicity $n^2 - 3$, thus accounting for the 12 points of multiplicity $n^2 - 3$. We have $-2 = H(C_3) \geq H(C_n) = \frac{-4(n^4-3n^2)}{n^4+27}$, and $H(C_n)$ decreases to -4 as n grows.

The $4(n^2-3)/3$ cubics consist of 4 sets of $(n^2-3)/3$ cubics, each set consisting of members of a different isotrivial linear pencil of cubics whose smooth elements are isomorphic to $x^3 + y^3 + z^3 = 0$. Each of these four isotrivial linear pencils of cubics has 3 singular elements, all isomorphic to xy(x-y)=0. The union over all four pencils of the components of the singular elements are the 9 lines of the $n = 3$ Fermat configuration. In fact, there are 4 ways to choose 3 of the 12 points of multiplicity $n^2 - 3$ so that there is exactly one point on each of the 9 lines. For each such way, we get a set of $9 = 12 - 3$ points complementary to the 3 chosen points. Each such set of 9 points defines a pencil of cubics from which a fourth of the cubics for the configuration are chosen, and each of the four pencils of cubics arises this way.

Question: If I is the ideal of the singular points of C_n, does $I^{(3)} \not\subseteq I^2$ hold?

[Interpolated comment: By the end of the MFO workshop for which this talk was given the only case worked out was C_3. In this case it turns out that $I^{(3)} \subseteq I^2$ holds. Here the only points are the 24 infinitely near triple points and the 12 proper points of multiplicity 6. Thus I is the ideal of the 12 points taken with multiplicity 2 and the 24 infinitely near points taken with multiplicity 1. By direct calculation we get $\alpha(I^{(3)}) = 24$ but $\operatorname{reg}(I) = 12$, where α denotes the degree of a nonzero element of least degree and reg is Castelnuovo-Mumford regularity, so by a criterion of [3], we have $I^{(3)} \subseteq I^2$.]

Question (Szemberg): Are there interesting somehow analogous configurations of conics?

Question: Are there irreducible C with $H(C) \leq -2$? (The general rational plane curve of degree d is an example pointed out by J. Roé at an MFO Bounded Negativity workshop in 2010 which gives $H(C)$ approaching -2 as d increases.)

Question: What about other surfaces? (It was noted at the Padua workshop mentioned above that [1] gives a construction of a curve C on a degree d surface in $\mathbf{P}^3$ with $H(C) = \frac{d-(d-1)^2}{1} = -d^2 + 3d - 1$.)

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On the fattening of lines in $\mathbb{P}^3$

Michael Janssen

Let $Z \subseteq \mathbb{P}^N$ be a finite set of points. The celebrated result of [7] classifies the possible Hilbert functions of Z. Little, in fact, is known about the so-called double point scheme, $2Z$ (but see [6, 8]), which is defined by the symbolic square of the ideal $I = I(Z)$, denoted $I^{(2)}$. For a general homogeneous $I \subseteq k[\mathbb{P}^N] = k[x_0, \dots, x_N]$, $I^{(m)}$ is defined to be

$$I^{(m)} = R \cap \left(\bigcap_{P \in \mathrm{Ass}(I)} (I^m R_P) \right).$$

However, when I is the ideal of points $p_1, \ldots, p_r \in \mathbb{P}^N$, this simplifies to $I^{(m)} = \cap_{i=1}^r I(p_i)^m$, where $I(p_i)$ denotes the ideal generated by all homogeneous polynomials (forms) vanishing at p_i. The m-th symbolic power $I^{(m)}$ is then the ideal generated by all forms vanishing to order at least m at the points of Z.

We also define the *initial degree of* I, denoted $\alpha(I)$, to be the degree of a nonzero form of least degree in I, and define $\alpha(I^{(2)})$ analogously. The invariant $\alpha(I)$ is closely related to the Hilbert function of I; in fact, for subsets of double points Z of $\mathbb{P}^N$ supported at generic points, the problem of determining the Hilbert function of the points is equivalent to determining the initial degree $\alpha(I^{(2)})$. Additionally, the invariant α is related to questions of containment of symbolic powers $I^{(m)}$ in ordinary powers I^r, a question of study which has received a great deal of attention recently (see [9, 5, 3]).

Recent results of [1, 2, 4] take a different approach to such questions. They determine the geometry of a set of points Z by examining the growth of the sequence $(\alpha(I^{(m)}))_m$. This study was initiated in [2], where the authors postulate values for the initial difference $t = \alpha(I^{(2)}) - \alpha(I)$ and classify the geometry of the underlying points set Z. Specifically, when $t = 1$ and $Z \subseteq \mathbb{P}^2$, they use Bezout's Theorem to show that Z is either collinear or a star configuration (i.e., a configuration of all $\binom{d}{2}$ pairwise intersection points of d lines, no three of which meet in a point).

We instead consider the geometric impact of minimal growth in the first step of the sequence $(\alpha(I^{(m)}))_m$ when I is the ideal of a configuration of lines in $\mathbb{P}^3$. Additionally, as the definition of a codimension 2 star configuration in $\mathbb{P}^3$ as defined in [6] is too restrictive for our needs, we relax the definition and consider pseudostar configurations of lines; that is, we consider $\binom{d}{2}$ lines formed by pairwise intersections of d planes such that no three planes contain a line. As all sets of points in $\mathbb{P}^2$ are arithmetically Cohen-Macaulay (ACM; i.e., $k[x,y,z]/I(Z)$ is a Cohen-Macaulay ring), we only consider arithmetically Cohen-Macaulay arrangements of lines in $\mathbb{P}^3$.

Our main result is the following.

Theorem ([10])**.** Let $\mathbb{L}$ be a union of lines $\ell_1, \ell_2, \ldots, \ell_s$ and let $I = I(\mathbb{L})$.

(1) If $\mathbb{L}$ is ACM with $\alpha(I^{(2)}) - \alpha(I) = 1$, then either $\mathbb{L}$ is a pseudostar or coplanar.
(2) If $\mathbb{L}$ is either a pseudostar or coplanar, then $\mathbb{L}$ has the property that $\alpha(I^{(2)}) - \alpha(I) = 1$.

The general idea of the proof is to make use of the ACM hypothesis to reduce to the theorem of Bocci and Chiantini [2]. Our methods raise the following questions regarding additional work in this direction:

Question. Does there exist a non-ACM configuration of lines $\mathbb{L}$ in $\mathbb{P}^3$ with $I = I(\mathbb{L})$ and $\alpha(I^{(2)}) - \alpha(I) = 1$?

Question. More generally, which configurations of lines in $\mathbb{P}^3$ are ACM?

Question. Which reduced (possibly irreducible) curves C in $\mathbb{P}^3$ have $I = I(C)$ and $\alpha(I^{(2)}) - \alpha(I) = 1$?

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The Weak Lefschetz property and ideals generated by powers of general linear forms

Juan Migliore

(joint work with Rosa Miró-Roig, Uwe Nagel)

Let $A = R/I$ be a standard artinian graded algebra, where $R = k[x1, \dots, x_r]$ is the homogeneous polynomial ring over a field of characteristic zero and I is a homogeneous ideal. Let ℓ be a linear form. For each i, ℓ induces a homomorphism

$$A_i \xrightarrow{\times \ell} A_{i+1}.$$

For a general choice of ℓ, one might hope that this homomorphism is either injective or surjective, i.e. that $\times\ell$ always has maximal rank:

$$\text{rk } (\times\ell) = \min\{\dim A_i, \dim A_{i+1}\}.$$

We say that A has the *Weak Lefschetz Property (WLP)* if this condition holds for all i. We say, furthermore, that A has the *Strong Lefschetz Property (SLP)* if the homomorphisms

$$(\times\ell^d) : A_i \to A_{i+d}$$

induced by powers of a general linear form all have maximal rank, for all choices of i and d. This talk will describe some known results on the Weak Lefschetz Property, state some of the important open problems in the field, and then describe some results obtained jointly with Rosa Miró-Roig and Uwe Nagel [6] on when the WLP property holds for an ideal generated by powers of general linear forms; this latter class of ideals is of great interest in this workshop. The bridge between the two parts of the talk will be via the famous Fröberg conjecture for the Hilbert function of general forms of prescribed degrees.

In the paper [4], the authors provided some of the first important facts about the WLP in general. Among these results are a complete classification of the possible Hilbert functions of algebras with the WLP, the fact that *all* artinian quotients of $k[x, y]$ have the WLP, and the fact that all complete intersection quotients of $k[x, y, z]$ have the WLP. This leads to the first open question:

Question 1: *Do all complete intersection quotients of* $k[x_1, \ldots, x_r]$ *have the WLP for* $r \geq 4$*?*

The next two questions concern Gorenstein algebras.

Question 2: *Do all Gorenstein quotients of* $k[x, y, z]$ *have the WLP?*

It is known that not all Gorenstein quotients of $k[x_1, x_2, x_3, x_4]$ have the WLP, but their Hilbert functions are all of the sort arising from Gorenstein algebras with the WLP, namely the so-called *Stanley-Iarrobino (SI) sequences*.

Question 3: *Do all Gorenstein quotients of* $k[x_1, x_2, x_3, x_4]$ *have Hilbert functions that are SI-sequences?*

Part of the importance of the WLP lies in its amazing and surprising connections to unexpected mathematical topics. These include a connection to the dimension of osculating planes of certain projective varieties, to the g-theorem in combinatorics, to certain enumerative problems in combinatorics, and to Fröberg's conjecture. We now consider the latter.

Fröberg's conjecture can be stated inductively by saying that if $I = (F_1, \ldots, F_{t-1})$ is an ideal generated by general forms of prescribed degrees $d_1, \ldots, d_{t-1}$, and if F_t is a general form of degree d_t then the map

$$[R/I]_i \xrightarrow{\times F_t} [R/I]_{i+d_t} \tag{1}$$

has maximal rank for all i. This allows for a formula for the Hilbert function of any ideal generated by general forms of prescribed degrees. We call this *the Hilbert function predicted by Fröberg's conjecture*. A result from [6] is that if Fröberg's conjecture holds for ideals generated by general forms in r variables then ideals generated by general forms in $r + 1$ variables have the WLP.

Note that in the multiplication (1), if we replace F_t by $L_t^{d_t}$, where L_t is a general linear form, then this multiplication was part of the definition of the SLP. Extending this idea, we can ask:

Question 4: *Does an ideal generated by powers of general linear forms have the Hilbert function predicted by Fröberg for general forms of the same degrees?*

A result from [7] and [8] implies that the answer is yes for general *almost complete intersections*, i.e. for ideals generated by $r + 1$ powers of general linear forms. What about more than $r + 1$ forms? It has been known for many years that the answer is not always yes, with the simplest example (due to A. Iarrobino) being an ideal of cubes of five general linear forms in $r = 3$ variables.

The paper [6] gave results that can be applied to give a large class of new examples of this phenomenon. This is enabled by the following result.

Theorem. *Let $R = k[x_1, \dots, x_{r+1}]$, let $\ell \in R$ be a general linear form, and let $S = R/(\ell) \cong k[x_1, \dots, x_r]$. Fix integers $d_1, \dots, d_{r+2}$. If an ideal of powers of general linear forms $(L_1^{d_1}, \dots, L_{r+2}^{d_{r+2}}) \subset R$ fails to have the WLP then the restricted ideal of powers of general linear forms $(\bar{L}_1^{d_1}, \dots, \bar{L}_{r+2}^{d_{r+2}}) \subset S$ fails to have the Hilbert function predicted by Fröberg's conjecture.*

So our focus now is to study ideals of general linear forms and ask when they have the WLP. In two variables it was already noted above that all ideals have the WLP, so there is nothing to check. A surprising result of [9] is that for three variables, *all* ideals of powers of linear forms have the WLP. This is in stark contrast to the situation for SLP, where Iarrobino's example above provides a counterexample.

This brings us to four or more variables. An example in [5] gave computer evidence that the ideals $I_N = (x_1^N, x_2^N, x_3^N, x_4^N, L^N)$ fail the WLP for $N = 3, \dots, 12$ and L a general linear form. (Higher exponents were not checked.) This clearly showed that something interesting is going on, and this example inspired the papers [3] and [6] at the same time.

The paper [3] gave results about *uniform* powers of general linear forms, but they allowed more than $r + 1$ forms. The results of [6] instead allowed mixed exponents, but focused on the case of $r + 1$ forms. In this way the papers nicely complement each other.

The best result from [6] gives an almost complete classification of when an ideal $(L_1^{a_1}, L_2^{a_2}, L_3^{a_3}, L_4^{a_4}, L_5^{a_5})$ generated by powers of five general linear forms in four variables has the WLP, in terms of the degrees of the exponents. Another result gives a complete description of when an almost complete intersection ideal in five variables generated by "almost uniform powers" of general linear forms, i.e. an ideal of the form $(L_1^d, L_2^d, L_3^d, L_4^d, L_5^d, L_6^{d+e})$, has the WLP. Finally, the paper gives a collection of results about almost complete intersection ideals of uniform powers, i.e. ideals of the form $(L_1^d, \dots, L_r^d, L_{r+1}^d) \subset k[x_1, \dots, x_r]$. The bottom line is that more often than not, the WLP fails.

The main tools used in [6] are:

1. A theorem of Emsalem and Iarrobino from [2] that translates the Hilbert function of an ideal of powers of general linear forms to the dimensions of linear systems in projective space defined by certain unions of fat points.
2. A reduction to a smaller projective space.
3. Cremona transformations to compute the dimensions of the linear systems mentioned above.

Hiding in the background, one of the most difficult parts of the proofs is the determination of the degree in which to prove that R/I fails the WLP. This was made possible by extensive computations on the computer system CoCoA [1].

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Dimensions of Some Secant Varieties

Uwe Nagel

(joint work with M.V. Catalisano, A.V. Geramita, A. Gimigliano, B. Harbourne, J. Migliore, and Y.S. Shin)

Recently, secant varieties have attracted lot of attention. This is partially due to a variety of applications. However, often the most basic invariant of a secant variety, its dimension, is not known. This is even true for classical examples.

Let $X \subset \mathbb{P}^n$ be an irreducible and reduced variety, where $\mathbb{P}^n = \mathbb{P}^n_K$ is the n-dimensional projective space over an algebraically closed field K of characteristic zero. The *secant variety* of $(\ell - 1)$-dimensional spaces to X is the closure of the union of the linear spans of all subsets of ℓ distinct points of X. It is denoted by $\sigma_\ell(X)$. Thus, $\sigma_1(X) = X$, and $\sigma_2(X)$ is called the secant line variety of X.

Since ℓ points span at most a linear space of dimension $\ell - 1$, one obtains the following estimate

$$\dim \sigma_\ell(X) \leq \min\{n,\ \ell \cdot \dim X + \ell - 1\}. \tag{1}$$

One typically refers to this upper bound as the *expected dimension* of $\sigma_\ell(X)$, that is,

$$\text{exp.dim } \sigma_\ell(X) = \min\{n,\ \ell \cdot \dim X + \ell - 1\}.$$

If Inequality (1) is strict, then $\sigma_\ell(X)$ is called *defective.* One says that $\sigma_\ell(X)$ is *filling its ambient space* if $\sigma_\ell(X) = \mathbb{P}^n$.

We often expect equality in Estimate (1) and that $\sigma_\ell(X)$ is filling if ℓ is sufficiently large. However, this is certainly not always the case.

Example 1. (i) If X is a line, then $\sigma_\ell(X) = X$ for all ℓ. In particular, $\sigma_\ell(X)$ is defective for all $\ell \geq 2$, unless $n = 1$.

(ii) Consider the Veronese suface $X \subset \mathbb{P}^5$. It is defined by the ideal that is generated by the 2-minors of the generic symmetric matrix

$$M = \begin{bmatrix} x_0 & x_1 & x_2 \\ x_1 & x_3 & x_4 \\ x_2 & x_4 & x_6 \end{bmatrix}.$$

Note that X is parametrized by squares of linear forms. Thus, the secant line variety $\sigma_2(X)$ is parametrized by sums of two squares of linear forms. Since their ranks are at most two, $\sigma_2(X)$ is defined by $\det M$, that is, $\sigma_2(X)$ is a cubic hypersurface. In particular,

$$\dim \sigma_2(X) = 4 < 5 = \min\{5,\ 2 \cdot 2 + 2 - 1\} = \text{exp.dim } \sigma_2(X),$$

i.e., $\sigma_2(X)$ is defective.

The basic tool for the calculation of dimensions of secant varieties is the following well-known result:

Proposition 1. (Terracini's Lemma [6]) *Let $P_1, \ldots, P_\ell$ be general points on X, and let $T_{P_i}(X)$ be the projectivized tangent space to X at the point P_i.*

The dimension of $\sigma_\ell(X)$ is the dimension of the linear span of $\bigcup_{i=1}^{\ell} T_{P_i}(X)$.

Consider now a Segre variety $X = \mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_t} \subset \mathbb{P}^N$, where $N = (n_1 + 1) \cdots (n_t + 1) - 1$. Recently, there have been many attempts to determine the dimensions of the secant varieties to X. Using vector spaces of suitable dimensions, X is the image of the map

$$\begin{aligned} \mathbb{P}(V_1) \times \cdots \times \mathbb{P}(V_t) &\to \mathbb{P}(V_1 \otimes \cdots \otimes V_t), \\ [v_1] \times \cdots \times [v_t] &\mapsto [v_1 \otimes \cdots \otimes v_t]. \end{aligned}$$

Thus, X is parametrized by decomposable tensors, and $\sigma_\ell(X)$ is the Zariski closure of the set of sums of ℓ decomposable tensors. The existence of such decompositions is of interest in various applications.

The case of $t = 2$ factors is well understood. To this end observe that each decomposable tensor $v_1 \otimes v_2$ can be identified with a $(n_1 + 1) \times (n_2 + 1)$ matrix of rank one. Hence, in this case $\sigma_\ell(X)$ is parametrized by sums of ℓ rank one matrices. The rank of each such sum is at most ℓ. It follows that $\sigma_\ell(X)$ is defined by the ideal that is generated by the $(\ell+1)$-minors of a generic $(n_1+1) \times (n_2+1)$ matrix. Thus, the dimension of $\sigma_\ell(X)$ is known for all ℓ. It is often defective.

The problem of finding $\dim \sigma_\ell(X)$ becomes much more difficult if X has $t \geq 3$ factors. The problem is that then the Zariski closure of the set of sums of ℓ decomposable tensors can contain vectors that do not arise as such a sum. Indeed, there are rather few general results.

Remark.

(i) The dimension of $\sigma_\ell(X)$ is known if $n_1 = \cdots = n_t = 1$ by [3];

(ii) Assume $n_1 \geq \cdots \geq n_t \geq 2$ and $n_1 > (n_2+1)\cdots(n_t+1) - (n_2 + \cdots + n_t)$. Then $\sigma_\ell(X)$ is defective for some ℓ (see [2] or [3]).

If the assumptions in (ii) are satisfied, then X is called *unbalanced*; otherwise *balanced*. Abo, Ottaviani, and Peterson proposed the following intriguing conjecture:

Conjecture 1 ([2])**.** If X is balanced, then $\sigma_\ell(X)$ is not defective for all $\ell \geq 2$.

Let us now turn attention to a different class of varieties. In [5], secants to a variety of hypersurfaces with a fixed factorization pattern have been investigated. More precisely, fix a partition $\lambda = [d_1, d_2, \ldots, d_r]$ of d with $r \geq 2$ positive parts, that is, $d_1 \geq \cdots d_r \geq 1$ and $d = d_1 + \cdots + d_r$. Forms of degree d in $S = K[x_1, \ldots, x_n]$ are parametrized by $\mathbb{P}([S]_d) = \mathbb{P}^N$, where $N = \binom{d+n-1}{d} - 1$. The *variety* $\mathbb{X}_{n-1,\lambda} \subset \mathbb{P}([S]_d) = \mathbb{P}^N$ *of* λ*-reducible forms* is

$$\mathbb{X}_{n,\lambda} = \{[f] \in \mathbb{P}^N \mid f = f_1 \cdots f_r \text{ for some } f_i \in [S]_{d_i}\}.$$

More precisely, $\mathbb{X}_{n,\lambda}$ is the image of the map

$$\begin{aligned} \mathbb{P}([S]_{d_1}) \times \cdots \times \mathbb{P}([S]_{d_r}) &\to \mathbb{P}([S]_d), \\ [f_1] \times \cdots \times [f_r] &\mapsto [f_1 \cdots f_r]. \end{aligned}$$

This is a finite morphism. Thus,

$$\dim \mathbb{X}_{n,\lambda} = \sum_{i=1}^{r} \left[\binom{d_i + n - 1}{n-1} - 1 \right].$$

Example 2.

(i) If $n = 2$, then $\mathbb{X}_{n,\lambda} = \mathbb{P}^N$.

(ii) The case $d = 2$ corresponds to reducible quadrics and forces $\lambda = [1, 1]$. Thus, $\mathbb{X}_{n,\lambda}$ is defined by the ideal that is generated by the 3-minors of a generic symmetric $n \times n$ matrix.

Excluding these well-known cases, we assume $n \geq 3$ and $d \geq 3$. Besides some sporadic results, the dimension of the secant variety $\sigma_\ell(\mathbb{X}_{n,\lambda})$ was known only for two broad families of examples until very recently:

(i) $n = 3, \lambda = [1, \ldots, 1]$, and ℓ arbitrary. The dimensions are due to Abo [1]. In particular, $\sigma_\ell(\mathbb{X}_{n,\lambda})$ is not defective for all ℓ.

(ii) $n = 3, \ell = 2$, and λ arbitrary. In this case, the dimension is given by a rather complicated formula in [4].

There was not even a conjecture for the dimension of $\sigma_\ell(\mathbb{X}_{n,\lambda})$ in other cases, nor was it clear if the two above results share a common explanation. This changed with the appearance of [5]. The main conjecture in this paper proposes a formula for $\dim \sigma_\ell(\mathbb{X}_{n,\lambda})$ for all n, ℓ, and λ. It gives a unifying framework subsuming all known results. In particular, it suggests:

Conjecture 2.

(a) If $d_1 < d_2 + \cdots + d_r = s$, then $\sigma_\ell(\mathbb{X}_{n-1,\lambda})$ is not defective.
(b) If $d_1 \geq s$, then the secant variety $\sigma_\ell(\mathbb{X}_{n-1,\lambda})$ is defective if and only if it does not fill its ambient space.

Several special cases of the main conjecture are established in [5]. In each of these cases Conjecture 2 is true as well.

Theorem ([5])**.** *The main conjecture is true if:*

(a) $\ell \leq \frac{n}{2}$ *or* $\ell \geq \binom{s+n-1}{n-1}$;
(b) $r = 2$ *and either*
  (i) $\ell \leq \frac{n+1}{2}$, *or*
  (ii) $\lambda = [d-1, 1]$, *or*
  (iii) $n = 3$; *or*
(c) $r \geq 3$ *and* $n \leq \ell \leq 1 + \frac{d_1+n-1}{s}$.

This result has also consequences for Conjecture 1. These are based on the following result:

Proposition 2 ([5])**.** *Set* $n_i = \binom{d_i+n-1}{n-1} + 1$ *and* $Y_{n,\lambda} = \mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_t}$.

If $\sigma_\ell(\mathbb{X}_{n,\lambda}) = \ell \cdot \dim \mathbb{X}_{n,\lambda} + \ell - 1$ *(so, in particular,* $\sigma_\ell(\mathbb{X}_{n,\lambda})$ *is not defective), then* $\sigma_\ell(Y_{n,\lambda})$ *is not defective.*

This statement implies Conjecture 1 in many new instances.

Our approach in [5] starts with a description of the tangent spaces to $\mathbb{X}_{n,\lambda}$. They are given by Cohen-Macaulay codimension two ideals. In order to study the Hilbert function of the sum of these ideals we modify a method from intersection theory, the so-called diagonal trick. It gives us complete results if the intersections are proper. In order to study improper intersections we suggest to establish Lefschetz properties. We conjecture that the relevant algebras always have enough Lefschetz elements.

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Curves with imposed gonality in linear systems

Joaquim Roé
(joint work with Xavier Xarles)

Given an algebraic curve C defined over the perfect field k, the gonality of C is the minimal integer γ such that there is a morphism $C \to \mathbb{P}^1_k$ of degree γ. Brill-Noether theory guarantees that, if k is algebraically closed and the genus of C is $g \geq 1$,

$$2 \leq \gamma \leq \frac{g+3}{2}, \tag{1}$$

and all values of γ allowed by (1) are obtained as gonalities of curves of genus g defined over k. However, for k non algebraically closed, the situation is very different, and we are very far from knowing which pairs (g, γ) are possible over a given field k (where the cases $k = \mathbb{Q}$ and $k = \mathbb{F}_q$ are the most interesting) or even which pairs (g, γ) are possible over *some* field. It is clear that for $g > 1$, $\gamma \leq 2g-2$ (thanks to the existence of the canonical linear series) and it is known that over appropriate fields, curves of gonality $2g-2$ exist.

Denoting $\bar{k}$ the algebraic closure of k, one can bound the gonality of a curve C below by the gonality $\bar{\gamma}$ of its base change $C_{\bar{k}}$, which must satisfy (1). In [1] we proved that, when $\bar{\gamma}$ is small with respect to the genus, there are strong constraints on γ:

(1) If there is only one morphism $C_{\bar{k}} \to \mathbb{P}^1_{\bar{k}}$ of degree $\bar{\gamma}$ then there is a morphism $C \to D$ of degree $\bar{\gamma}$ with D a genus 0 curve, and $\gamma \leq 2\bar{\gamma}$.
(2) If $(\bar{\gamma}-1)^2 < g$ and one morphism $C_{\bar{k}} \to \mathbb{P}^1_{\bar{k}}$ of degree $\bar{\gamma}$ is simple (i.e., does not factor as the composition of two maps of degree > 1) then it is the only morphism of its degree.
(3) If $(\bar{\gamma}-1)^2 < g$, one morphism $C_{\bar{k}} \to \mathbb{P}^1_{\bar{k}}$ of degree $\bar{\gamma}$ is simple, and $\bar{\gamma} \equiv g \pmod 2$, then $\gamma = \bar{\gamma}$.

We are interested in giving sufficient conditions for the pair (g, γ) to be realizable as the genus and gonality for some curve, and if possible, give explicit examples of such curves. A few facts are known, which we summarize next:

- As a corollary of the results above, if p is a prime, there are over every field curves of genus g and gonality p as soon as $g > (p-1)^2$.
- In genus 1, there are curves of gonality 2 over all fields, and of all gonalities ≥ 2 over $\mathbb{Q}$ and over finite fields. Over finite fields, the gonality is bounded in terms of the size of the field. In genus 2, all curves have gonality 2.
- For $3 \leq g \leq 5$, the possible gonalities (over some field) are as follows:

g	3		4		5		
$\bar{\gamma}$	2	3	2	3	2	3	4
possible γ	2,4	3,4	2	3,4,5,6	2,4	3	4,5,6,8

- For $g \leq 15$, there are over every infinite field curves with maximal geometric gonality $\bar{\gamma} = \lfloor (g+3)/2 \rfloor$ (but it is not known whether they can be constructed with $\gamma = \bar{\gamma}$ over $\mathbb{Q}$, for instance).

This last result follows from the fact that the corresponding moduli space is unir-uled, and so there are "general" curves over infinite fields. If $g \leq 10$ the proof is due to Severi, and it leads to a general method using linear systems on curves which can be used to obtain curves with smaller gonalities as well:

Fixing $d \geq 2g/3+2$, all curves of genus g can be obtained as plane nodal curves of degree d and $\delta = (d-1)(d-2)/2 - g$ nodes. If $3\delta < (d+1)(d+2)/2$, the nodes can be chosen in general position (in particular, they can be chosen over a given infinite field); both inequalities match for $g \leq 10$. If $g < 10$, the gonal maps to $\mathbb{P}^1$ are obtained by projection from one node, and so they are defined over the base field (so $\gamma = \lfloor (g+3)/2 \rfloor$ is achieved over the rational field for $g < 10$).

Similar methods, exploiting plane curves with higher singularities, or general curves on toric surfaces, possibly with nodes, allow to construct curves with given genus and gonality, as long as $g \geq f(\gamma)$, where f is a function of quadratic growth.

Open problem. Is there an affine linear function f such that for all $g \geq f(\gamma)$ there are algebraic curves defined over the rational numbers with genus g and gonality γ? Note that necessarily $f(\gamma) \leq 2\gamma - 3$ by (1).

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A homological criterion for the (failure of) containment between symbolic and ordinary powers of ideals of points

Alexandra Seceleanu

Containments between ordinary and symbolic powers remain a source for many open questions, despite the recent surge of interest the topic has received. The starting point for our report is the first (no longer open) question in this series, proposed by Craig Huneke:

Question 1. *Does the containment $I^{(3)} \subseteq I^2$ hold for any radical ideal I defining a finite set of points in $\mathbb{P}^2$?*

It is now well known that the answer to the question above is negative. Several examples of configurations of points have arisen that exhibit a non-containment $I^{(3)} \not\subseteq I^2$, where I is the defining ideal of the set of points. The point configurations presently known to witness the non-containment are: the Fermat family (indexed by integers $s \geq 3$), given by s^2+3 points that arise as the singular locus of an arrangement of $3s$ lines [4, 5], the Klein configuration consisting of 49 points which form the singular locus of an arrangement of 21 lines, the Wiman configuration which consists of 201 points that arise as the singular locus of an arrangement of 45 lines [1], and the orchard configuration of 19 points that occur at the intersection of the 12 lines in a certain arrangement in the real projective plane [2]. Understanding these counterexamples constitutes the main motivation of this note.

We propose below a homological approach meant to verify the failures of containment mentioned above from a theoretical perspective. It is a common feature of the point configurations mentioned above that their defining ideals are three-generated. Motivated by this, we restrict our attention in Theorem 4 to three-generated ideals of height two, with minimal generators of the same degree. However, as pointed out by the workshop participants, it is not the case that every ideal that satisfies $I^{(3)} \not\subseteq I^2$ is there-generated [3].

The first step in our approach to the study of containments of the form $I^{(m)} \subseteq I^r$ is to translate them in the language of homological functors:

Proposition 2. *Let I be a homogeneous ideal and let $m \geq r > 0$ be integers. Consider the canonical surjection $R/I^m \to R/I^r$. Applying the local cohomology and extension functors respectively to this homomorphism yields natural induced maps. The following statements are equivalent:*

(1) $I^{(m)} \subseteq I^r$;
(2) *the induced map* $H^0_m(R/I^m) \to H^0_m(R/I^r)$ *is the zero map;*
(3) *the induced map* $\mathrm{Ext}^3_R(R/I^r, R) \to \mathrm{Ext}^3_R(R/I^m, R)$ *is the zero map.*

Next, we give a more concrete interpretation of the homological criterion above in the terms of the differentials in the resolutions of I^m and I^r.

Let $I \subset K[x, y, z]$ be a homogeneous ideal and let $m \geq r > 0$ be integers. Consider the minimal free resolutions for I^m and I^r and the map between these two complexes induced by the natural inclusion $I^m \hookrightarrow I^r$. These maps are depicted below by vertical arrows:

$$\begin{array}{ccccccccc} 0 \longrightarrow & R^{b_2} & \xrightarrow{X} & R^{b_1} & \longrightarrow & R^{b_0} & \longrightarrow & I^m & \longrightarrow 0 \\ & \downarrow^{Y} & & \downarrow & & \downarrow & & & \\ 0 \longrightarrow & R^{b'_2} & \longrightarrow & R^{b'_1} & \longrightarrow & R^{b'_0} & \longrightarrow & I^r & \longrightarrow 0 \end{array}$$

Let X^T and Y^T denote the dual maps to the homomorphisms X and Y appearing in the diagram above, obtained by applying the $\mathrm{Hom}(-, R)$ functor, i.e. $X^T = \mathrm{Hom}(X, R)$ and $Y^T = \mathrm{Hom}(Y, R)$. With this notation and writing $\mathrm{Im}(\phi)$ for the image of a homomorphism ϕ, we have:

Proposition 3. *Let I be a homogeneous ideal and let $m \geq r > 0$ be integers. The containment $I^{(m)} \subseteq I^r$ holds if and only if* $\mathrm{Im}(Y^T) \subseteq \mathrm{Im}(X^T)$.

Finally, applying this version of our criterion to the particular case of three-generated ideals defining points yields the most concrete version of our result.

Theorem 4. *Let $I = (f, g, h) \subset K[x, y, z]$ be a three-generated reduced ideal defining a finite sets of points in $\mathbb{P}^2$ over a field K of characteristic not equal to 3. Consider the minimal free resolutions for I^3. This is a complex of the form :*

$$0 \longrightarrow R^3 \xrightarrow{X} R^{12} \longrightarrow R^{10} \longrightarrow I^3 \longrightarrow 0$$

Then the contanment $I^{(3)} \subseteq I^2$ *holds if and only if* $\begin{bmatrix} f \\ g \\ h \end{bmatrix} \in \mathrm{Im}(\mathrm{X}^{\mathrm{T}})$.

Furthermore, the homomorphism X in Theorem 4 can be described explicitly in terms of the Hilbert-Burch matrix of I, rendering the criterion in Theorem 4 effective. Proofs of the results above can be found in [6], along with applications to the case of the Fermat and Klein configurations mentioned in the introduction. To our knowledge, this provides the only theoretical proof for the failure of containment of the symbolic cube in the ordinary square for the case of the Klein configuration.

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On the effect of points fattening on surfaces and in higher dimensions

Tomasz Szemberg

(joint work with Thomas Bauer, Sandra Di Rocco, Elena Guardo, Zach Teitler)

Let X be a smooth variety and let L be a very ample (or just ample and spanned) line bundle on X. For a suvariety (reduced scheme) $Z \subset X$ we define $(\alpha(mZ))_{m\geq 1}$ the *initial sequence of* Z as

$$\alpha(mZ) := \min\left\{d : \text{ there exists } s \in H^0(X, dL) \text{ vanishing to order } \geq m \text{ along } Z\right\}.$$

Bocci and Chiantini initialized in [3] the program of studying how the growth of the initial sequence influences the geometry of Z. Passing from $(m-1)Z$ to mZ is the fattening mentioned in the title, and its effect is expressed in the difference between $\alpha((m-1)Z)$ and $\alpha(mZ)$. Bocci and Chiantini worked with Z being a finite set of points in the projective plane $\mathbb{P}^2$. Their results have been generalized and extended in several ways.

In [6] we studied further properties of the initial sequence of points in $\mathbb{P}^2$. The case of $X = \mathbb{P}^1 \times \mathbb{P}^1$ and $L = \mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1}(1,1)$ has been investigated in [1]. In [4] we studied the effect of fattening for points on Hirzebruch surfaces. This effect for higher dimensional varieties has been studied by Jannsen in [8] and somewhat indirectly in [5]. Points in $\mathbb{P}^3$ have been studied in turn in [2].

In my talk I have presented two open conjectures in the field motivated by results from [2] and [1]. See [7] for a very enjoyable introduction to star configurations.

Conjecture 1.
Let Z be a finite set of points in projective space $\mathbb{P}^n$. If

$$d := \alpha(nZ) = \alpha(Z) + n - 1\,,$$

then either

$\alpha(Z) = 1$, i.e., Z is contained in a single hyperplane H in $\mathbb{P}^n$

or

Z consists of all intersection points (i.e., points where n hyperplanes meet) of a general configuration of d hyperplanes in $\mathbb{P}^n$, i.e., Z is a *star configuration*. Moreover for any polynomial of degree d vanishing along Z to order $\geq n$, the corresponding hypersurface decomposes into d such hyperplanes.

In the multiprojective setting we have the following.

Conjecture 2.
Let Z be a finite set of points in $(\mathbb{P}^1)^n$. If

$$d = \alpha(nZ) = \alpha(Z)\,,$$

then Z is a grid of size $d \times \ldots \times d$, i.e. there are subsets $Z_i \subset \mathbb{P}^1$ consisting of d points each such that

$$Z = Z_1 \times \ldots \times Z_n.$$

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On linear Harbourne constants

JUSTYNA SZPOND

In recent years, there has been growing interest in negative curves on algebraic surfaces. The Bounded Negativity Conjecture (BNC for short) is probably the most interesting open question in this area. The BNC predicts that for any smooth complex surface X there exists a lower bound for the selfintersection of reduced divisors on X. It is not known if the Bounded Negativity property is invariant in the birational class of a surface, i.e. given birational surfaces X and Y, it is not known if curves on X have bounded negativity if and only if they do on Y. As the first step towards understanding this question, in [1] the authors introduced and studied Harbourne constants. The purpose of this talk is to compute these constants for a low number of lines (up to 10) defined over an *arbitrary* field. This is a problem of combinatorial flavor and we hope that these results might be of interest also from this point of view.

Let $\mathbb{K}$ be an arbitrary field. Let $\mathcal{L} = \{L_1, \dots, L_d\}$ be a configuration of lines in the projective plane $\mathbb{P}^2(\mathbb{K})$, let $\mathcal{P}(\mathcal{L}) = \{P_1, \dots, P_s\}$ be the set of all singular points of the configuration. Then the *linear Harbourne constant of* $\mathcal{L}$ *at* $\mathcal{P}$ is defined as

$$H_L(\mathbb{K}, \mathcal{L}) = \frac{d^2 - \sum_{k=1}^{s} m_{\mathcal{L}}(P_k)^2}{s}$$

Similarly, we define the *linear Harbourne constant of configurations of* d $\mathbb{K}$*-lines* as the minimum

$$H_L(\mathbb{K}, d) := \min H_L(\mathbb{K}, \mathcal{L})$$

where the minimum is taken over all configurations of d lines in $\mathbb{P}^2(\mathbb{K})$.
Going over all fields $\mathbb{K}$, we introduce the *absolute linear Harbourne constant* as

$$H_L(d) := \min_{\mathbb{K}} H_L(\mathbb{K}, d).$$

As the main result of this talk we establish the values of the absolute linear Harbourne constants of up to ten lines. Since the case $\mathbb{K} = \mathbb{C}$ is of particular interest from the point of view of BNC, we compute separately also the complex Harbourne constants.

Theorem [2]
The values of the absolute Harbourne constants are

d	2	3	4	5	6	7	8	9	10
$H_L(d)$	0	-1	$-1\frac{1}{3}$	$-1,5$	$-1\frac{5}{7}$	$\mathbf{-2}$	-2	$-2,25$	$\mathbf{-2\frac{5}{12}}$

Over $\mathbb{C}$ we obtain the following values

d	2	3	4	5	6	7	8	9	10
$H_L(\mathbb{C}, d)$	0	-1	$-1\frac{1}{3}$	$-1,5$	$-1\frac{5}{7}$	$\mathbf{-1\frac{8}{9}}$	-2	$-2,25$	$\mathbf{-2\frac{4}{15}}$

References

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[2] Szpond, J.: On linear Harbourne constants, arXiv:1502.04334

Fat lines in $\mathbb{P}^3$

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(joint work with Elena Guardo, Brian Harbourne)

Given a homogeneous ideal I of a polynomial ring $R = k[x_0, \ldots, x_n]$, there has been a lot of interest in determining when $I^m = I^{(m)}$, where $I^{(m)}$ denotes the m-th symbolic power of I. It has long been known that if I is a complete intersection, then $I^m = I^{(m)}$ for all $m \geq 1$. However, there are examples of ideals I that are not complete intersections, but also have the property that $I^m = I^{(m)}$ for all $m \geq 1$. It is then natural to ask if there is an integer M such that if $I^m = I^{(m)}$ if $m \leq M$, then $I^m = I^{(m)}$ for all $m \geq 1$. In other words, how many comparisons do we need to make to guarantee that all regular powers of an ideal always equals its symbolic powers?

For some classes of ideals, we can determine M. For example, let $J(G)$ be the cover ideal of a finite simple graph $G = (V_G, E_G)$. That is,

$$J(G) := \bigcap_{\{x_i,x_j\} \in E_G} \langle x_i, x_j \rangle.$$

Then it follows from [1, 6] that $J(G)^m = J(G)^{(m)}$ for all $m \geq 1$ if and only if $J(G)^m = J(G)^{(m)}$ for $1 \leq m \leq 2$, i.e., $M = 2$. Susan Morey [5] also was able to determine M for perfect ideals in a local ring R that were also generically a complete intersection and under some codimension hypotheses.

In a talk in Lincoln around 2008, Craig Huneke suggested that M might be the *big height* of I, denoted $\text{bight}(I)$, where $\text{bight}(I) := \max\{\text{ht}(P) \mid P \in \text{Ass}(I)\}$. Note that for any cover ideal $J(G)$, $\text{bight}(J(G)) = 2$, so M is indeed the big height of the ideal in this situation.

It turns out that there exists ideals I with $\text{bight}(I) < M$. The first example of such an ideal was found in [3] by considering a union of lines in $\mathbb{P}^3$. In particular, we build our union of lines as follows. Let $R = k[x_0, x_1, y_0, y_1] = k[\mathbb{P}^3]$, and let L_1 be the line defined by $I(L_1) = \langle y_0, y_1 \rangle$ and L_2 the line defined by $I(L_2) = \langle x_0, x_1 \rangle$. Note that L_1 and L_2 are two skew lines in $\mathbb{P}^3$. On the line L_1, pick any h points $A_1, \ldots, A_h$ and on L_2 pick any v points $B_1, \ldots, B_v$. We then set $L_{i,j} = \overline{A_i B_j}$. The ideals considered in [3] where ideals of the form

$$I(X) = \bigcap_{(i,j) \in D} I(L_{i,j})$$

for some subset $D \subseteq \{(i,j) \mid 1 \le i \le h, 1 \le j \le v\}$. That is, X is a union of lines in $\mathbb{P}^3$ where every line of X meets the two skew lines L_1 and L_2.

If we give $R = k[x_0, x_1, y_0, y_1]$ a bigraded structure by setting $\deg x_i = (1,0)$ and $\deg y_i = (0,1)$ for $i = 0,1$, then the ideal $I(X)$ is also a bihomogeneous ideal in this new grading. In fact, $I(X)$ is the defining ideal of a set of points in $\mathbb{P}^1 \times \mathbb{P}^1$. So, instead of studying lines in $\mathbb{P}^3$, we can take the point-of-view that $I(X)$ defines a set of points $X \subseteq \mathbb{P}^1 \times \mathbb{P}^1$ (we abuse notation to let X also represent the points). By taking this perspective, we can exploit some known results about points in $\mathbb{P}^1 \times \mathbb{P}^1$. For example, due to work of Giuffrida, Maggioni, and Ragusa [2], we can determining when a set of points in $\mathbb{P}^1 \times \mathbb{P}^1$ is Cohen-Macaulay. Also key for our work is a result of the author and Guardo [4] that if $X \subseteq \mathbb{P}^1 \times \mathbb{P}^1$ has the property that $R/I(X)$ is Cohen-Macaulay, then $I(X)^2 = I(X)^{(2)}$.

For these special unions of lines in $\mathbb{P}^3$, or equivalently for points in $\mathbb{P}^1 \times \mathbb{P}^1$, we were able to prove the following result (see [3]):

Theorem. *Let $X \subseteq \mathbb{P}^1 \times \mathbb{P}^1$ be an arithmetically Cohen-Macaulay set of points. Then $I(X)^m = I(X)^{(m)}$ for all $m \ge 1$ if and only if $I(X)^3 = I(X)^{(3)}$.*

Because the big height of $I(X)$ is two, this theorem gives a negative answer to Huneke's question because you need to check the third regular and symbolic power of $I(X)$ to determine if $I(X)^m = I(X)^{(m)}$ for all $m \ge 1$. As a specific example, consider the ideal of points in $\mathbb{P}^1 \times \mathbb{P}^1$ (or union of lines in $\mathbb{P}^3$) given by

$$I(X) = I(L_{1,1}) \cap I(L_{1,2}) \cap I(L_{1,3}) \cap I(L_{2,1}) \cap I(L_{2,2}) \cap I(L_{3,1}).$$

This ideal has $I(X)^2 = I(X)^{(2)}$, but $I(X)^3 \neq I(X)^{(3)}$.

There are at least two questions that are suggested by the above theorem. First, is there a geometric characterization of the set of points X that are Cohen-Macaulay in $\mathbb{P}^1 \times \mathbb{P}^1$ (alternatively, lines in $\mathbb{P}^3$) that satisfy the condition $I(X)^3 = I(X)^{(3)}$. A conjectured answer is given in [3], along with some supporting evidence. Second, for any positive integer t, does there exist an ideal I_t such that $I_t^m = I_t^{(m)}$ for all $m \le \operatorname{bight}(I_t) + (t-1)$ but $I_t^m \neq I_t^{(m)}$ for $m = \operatorname{bight}(I_t) + t$. In other words, can we show that we may need to check powers arbitrarily larger than the big height of an ideal. The main example of [3] can been seen as an example for the case $t = 1$.

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